Mimetic Spectral Element Methods in CFD
Workshop: The use of high order discretization methods in CFD

Marc Gerritsma
M.I.Gerritsma@TUDelft.nl
Department of Aerospace Engineering
Delft University of Technology

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Why spectral element methods

**Exponential convergence**

Spectral element methods are based on **orthogonal polynomials** for the basis functions.

Instead of $h$-refinement (a finer mesh), spectral methods use $p$-enrichment, i.e. leave the elements unaltered, but increase the polynomial degree.

If the exact solution is sufficiently smooth we have exponential convergence

$$
\| u - u_N \|_{H^1(\Omega)} \leq C \exp(-\gamma N),
$$

where $N$ is the polynomial degree and $\gamma$ only depends on the solution $u$. 

![Graph showing exponential convergence](image-url)
Why spectral element methods

Why mimetic methods?

Despite the exponential convergence rate of spectral methods, we never resolve the solution up to machine precision.

Mimetic methods aim to preserve structures at the discrete level, such as conservation laws, definitions and equilibrium conditions.

In that sense, mimetic spectral element methods have similar properties as finite volume methods.

In addition, well-posedness is easier to establish, the resulting system is much sparser than conventional spectral element methods and the condition number is lower than conventional spectral element methods.

In curved geometries many matrices remain unchanged.
Following Ciarlet (1978) the finite element is defined by the triplet \((K, P, N)\), see Brenner & Scott (2008), Ern & Guermond (2004), where

1. \(K \subseteq \mathbb{R}^d\) is a bounded closed set with non-empty interior and piecewise smooth boundary (the element);
2. \(P\) is a finite dimensional linear function space defined on \(K\) (usually polynomials);
3. \(N = (N_1, \ldots, N_k)\) is a basis for the dual space of \(P\). The \(N_i\) will be the degrees of freedom of the finite element.
One dimensional example I

i. $K \subseteq \mathbb{R}^d$ is a bounded closed set with non-empty interior and piecewise smooth boundary (the element);

ii. $\mathcal{P}$ is a finite dimensional linear function space defined on $K$ (usually polynomials);

iii. $\mathcal{N} = (\mathcal{N}_0, \ldots, \mathcal{N}_k)$ is a basis for the dual space of $\mathcal{P}$. The $\mathcal{N}_i$ will be the degrees of freedom of the finite element.

Let $d = 1$ and set $K = [-1, 1]$, for $\mathcal{P}$ we choose the space of polynomials of degree $N$, i.e. $\mathcal{P} = \mathbb{P}^N$.

Let $-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$ be a partitioning of the domain $K$ and set $k = N$. Define the $\mathcal{N}_i^0$ by:

For all $p \in \mathcal{P}$

$$\mathcal{N}_i^0(p) = p(\xi_i), \quad i = 0, \ldots, N.$$ 

Since we sample the polynomials in the nodes we refer to these degrees of freedom as nodal degrees of freedom, (measurements).
One dimensional example II

\[ N_i^0(p) = p(\xi_i), \quad i = 0, \ldots, N, \quad \forall p \in P. \]

Since we sample the polynomials in the nodes we refer to these degrees of freedom as nodal degrees of freedom.

The nodal basis functions, \( h_i(\xi) \), are defined by \( N_i^0(h_j(\xi)) = \delta_{ij} \).

\[ p(\xi) = \sum_{i=0}^{N} N_i^0(p) h_i(\xi). \]
Instead of nodal sampling, we can also use integral degrees of freedom. Let $\mathcal{P} = P^{N-1}$ and the define $\mathcal{N}_i^1$, $i = 1, \ldots, N$ by

$$
\mathcal{N}_i^1(p) = \int_{\xi_{i-1}}^{\xi_i} p(s) \, ds , \quad i = 1, \ldots, N , \quad \forall p \in \mathcal{P} .
$$

Since we sample the polynomials along edges we refer to these degrees of freedom as edge degrees of freedom. The edge basis functions, $e_i(\xi)$, are defined by $\mathcal{N}_i^1(e_j(\xi)) = \delta_{ij}$.

$$
p(\xi) = \sum_{i=1}^{N} \mathcal{N}_i^1(p) e_i(\xi) .
$$
Differentiation of nodal representation

Let $p \in \mathbb{P}^N$ be expanded in terms of nodal degrees of freedom

$$p(\xi) = \sum_{i=0}^{N} \mathcal{N}_i^0(p) h_i(\xi) = [h_0(\xi), \ldots, h_N(\xi)] \begin{bmatrix} \mathcal{N}_0^0(p) \\ \vdots \\ \mathcal{N}_N^0(p) \end{bmatrix}.$$ 

Then the derivative can be expressed in terms of edge degrees of freedom

$$\frac{dp}{d\xi} = \sum_{i=1}^{N} \left( \mathcal{N}_i^0(p) - \mathcal{N}_{i-1}^0(p) \right) e_i(\xi)$$

$$= [e_1(\xi) \ldots e_N(\xi)] \begin{pmatrix} -1 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 1 & 0 \\ 0 & \ldots & \ldots & 0 & -1 & 1 \end{pmatrix} \begin{bmatrix} \mathcal{N}_0^0(p) \\ \vdots \\ \mathcal{N}_N^0(p) \end{bmatrix}.$$ 

So differentiation consists of two steps:

1. Take the difference between consecutive nodal degrees of freedom;
2. Express in new basis functions, i.e. edge basis functions.
Let \( p, q \in \mathcal{P} \) be both represented by nodal degrees of freedom

\[
p(\xi) = \sum_{i=0}^{N} \mathcal{N}_i^0(p) h_i(\xi) \quad \text{and} \quad q(\xi) = \sum_{i=0}^{N} \mathcal{N}_i^0(q) h_i(\xi),
\]

then the \( L^2 \) inner product is given by

\[
(p, q)_{L^2} = \int_{-1}^{1} p(\xi)q(\xi) \, d\xi = \mathcal{N}_{}^0(p)^T \mathcal{M}^{(0)} \mathcal{N}_{}^0(q).
\]

The algebraic dual degrees of freedom of \( q \) denoted by \( \tilde{\mathcal{N}}_i^1(q) \) are then defined by, Jain et al., (2017)

\[
\mathcal{N}_{}^0(p)^T \tilde{\mathcal{N}}_i^1(q) := \mathcal{N}_{}^0(p)^T \mathcal{M}^{(0)} \mathcal{N}_{}^0(q) \implies \tilde{\mathcal{N}}_i^1(q) = \mathcal{M}^{(0)} \mathcal{N}_{}^0(q).
\]

The associated basis functions are then given

\[
[\tilde{e}_0(\xi) \ldots \tilde{e}_N(\xi)] = [h_0(\xi) \ldots h_N(\xi)] \mathcal{M}^{(0)^{-1}}.
\]
The dual basis functions have the property that
\[
(\tilde{e}_i(\xi), h_j(\xi))_{L^2} = \delta_{ij}.
\]

If we denote the mass matrix associated with the dual polynomials by \(\tilde{M}^{(1)}\) then we have
\[
\tilde{M}^{(1)} \cdot M^{(0)} = I.
\]
We can do the same with the edge polynomials. Let $u$ and $v$ be expanded as

$$u(\xi) = \sum_{i=1}^{N} \mathcal{N}_i^1(u)e_i(\xi) \quad \text{and} \quad v(\xi) = \sum_{i=1}^{N} \mathcal{N}_i^1(v)e_i(\xi) ,$$

then

$$(u, v)_{L^2} = \int_{-1}^{1} u(\xi)v(\xi) \, d\xi = \mathcal{N}^1(u)^T \mathcal{M}^{(1)} \mathcal{N}^1(v) .$$

Then the dual degrees of freedom are defined by

$$\mathcal{N}^1(u)^T \tilde{\mathcal{N}}^0(v) := \mathcal{N}^1(u)^T \mathcal{M}^{(1)} \mathcal{N}^1(v) \implies \tilde{\mathcal{N}}^0(v) = \mathcal{M}^{(1)} \mathcal{N}^1(v) .$$

The associated basis functions are then given

$$[\tilde{h}_1(\xi) \ldots \tilde{h}_N(\xi)] = [e_1(\xi) \ldots e_N(\xi)]^{\mathcal{M}^{(1)}}^{-1} .$$
The dual basis functions have the property that

\[(\tilde{h}_i(\xi), e_j(\xi))_{L^2} = \delta_{ij}.\]

If we denote the mass matrix associated with the dual polynomials by \(\tilde{M}^{(0)}\) then we have

\[\tilde{M}^{(0)} \cdot M^{(1)} = I.\]
Differentiation of dual representations I

Consider a nodal expansion for $q$ and an edge expansion for $u$

$$
q(\xi) = \sum_{i=0}^{N} \mathcal{N}_i^0(q) h_i(\xi) \quad \text{and} \quad u(\xi) = \sum_{i=1}^{N} \mathcal{N}_i^1(u) e_i(\xi)
$$

Then

$$
\int_{-1}^{1} \frac{dq}{d\xi} u d\xi = \mathcal{N}_0^0(q)^T \mathbb{E}^{1,0}^T \mathbb{M}^{(1)}_0 \mathcal{N}_1^1(u) = \mathcal{N}_0^0(q)^T \mathbb{E}^{1,0}^T \mathcal{N}_0^0(u)
$$

Therefore

$$
\int_{-1}^{1} \frac{du}{d\xi} q d\xi = qu|_{\xi=-1}^{\xi=1} - \int_{-1}^{1} \frac{dq}{d\xi} u d\xi
$$

$$
= \mathcal{N}_0^0(q)^T \mathcal{N}_0^0_{N+1}(u) - \mathcal{N}_0^0(q)^T \mathcal{N}_0^0_0(u) - \mathcal{N}_0^0(q)^T \mathbb{E}^{1,0}^T \mathcal{N}_0^0(u).
$$

So the derivative of the dual representation (in terms of the degrees of freedom) is

$$
\mathcal{N}_0^0 \left( \frac{du}{d\xi} \right) = \mathcal{N}_0^0_{N+1}(u) - \mathcal{N}_0^0_0(u) - \mathbb{E}^{1,0}^T \mathcal{N}_0^0(u).
$$
Differentiation of dual representations II

\[ \tilde{N}_0^0 \left( \frac{du}{d\xi} \right) = \tilde{N}_{N+1}^0(u) - \tilde{N}_0^0(u) - E_{1,0}^T \tilde{N}_0^0(u). \]

We see that

- We see that we need to supplement \( \tilde{N}_0^0(u) \) and \( \tilde{N}_{N+1}^0(u) \) for this derivative. These degrees of freedom were not part of the original dual representation. They only show up when we differentiate;
- The incidence matrix \( E_{1,0}^T \) once again plays the role of differentiation matrix applied to the degrees of freedom;
- Usually when we take the derivative of a polynomial, its degree goes down, but here we start with a polynomial of degree \( (N - 1) \) and its derivative is a polynomial of degree \( N \).
So differentiation is accomplished by the incidence matrix and conversion to the algebraic dual by the mass matrices.

Only the mass matrices depend on the basis functions and size and shape of the elements.

The incidence matrices always remain unchanged.
Calculus with degrees of freedom I

- Consider the 2D grid shown on the right.
- Number all the vertices in the mesh
- Number all edges in the mesh
- Number all volumes in the mesh
- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes
Calculus with degrees of freedom I

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- Give the vertices a default orientation
- Give a default orientation to all edges
- Give default orientation to the volumes
Incidence matrices

Calculus with degrees of freedom I

\[ \mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \]
Incidence matrices

Calculus with degrees of freedom I

\[
\mathbb{E}^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 \\
\end{pmatrix}
\]
Incidence matrices

Calculus with degrees of freedom 1

\[ E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 \\
\end{pmatrix} \]
Calculus with degrees of freedom I

$$\mathbb{E}^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0
\end{pmatrix}$$
Incidence matrices

Calculus with degrees of freedom I

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-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
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\end{pmatrix} \]
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0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix} \]
Incidence matrices

Calculus with degrees of freedom I

\[ E^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \]
Calculus with degrees of freedom I

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Calculus with degrees of freedom I

\[ E^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \]

\[ E^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & -1 & 1 \end{pmatrix} \]
Incidence matrices

Calculus with degrees of freedom I

\[ E^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \]

\[ E^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix} \]
Incidence matrices

Calculus with degrees of freedom I

\[ E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 \\
\end{pmatrix} \]

\[ E^{2,1} = \begin{pmatrix}
-1 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 1 \\
\end{pmatrix} \]
Incidence matrices

Calculus with degrees of freedom I

$E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix}$

$E^{2,1} = \begin{pmatrix}
-1 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & -1 \\
0 & 0 & -1 & 1 & 1 & 0 & -1
\end{pmatrix}$
Incidence matrices

Calculus with degrees of freedom I

\[ E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix} \]

\[ E^{2,1} = \begin{pmatrix}
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\end{pmatrix} \]
Calculus with degrees of freedom I

\[ E^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \]

\[ E^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix} \]
Calculus with degrees of freedom I

$$E^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$$E^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$
Incidence matrices

Calculus with degrees of freedom I

\[
\begin{align*}
\mathbb{E}^{1,0} &= \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix} \\
\mathbb{E}^{2,1} &= \begin{pmatrix}
-1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 1 & 1
\end{pmatrix}
\end{align*}
\]
Calculus with degrees of freedom I

\[ E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix} \]

\[ E^{2,1} = \begin{pmatrix}
-1 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 1 \\
0 & 0 & -1 & 1 & 0 & -1 & 1
\end{pmatrix} \]
Incidence matrices

Calculus with degrees of freedom I

\[ E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix} \]

\[ E^{2,1} = \begin{pmatrix}
-1 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 1
\end{pmatrix} \]

The matrices \( E^{1,0} \) and \( E^{2,1} \) are called incidence matrices. The incidence matrices only contain entries \(-1, 0\) and \(1\).
Incidence matrices

Calculus with degrees of freedom I

$$E^{1,0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$$E^{2,1} = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

The matrices $E^{1,0}$ and $E^{2,1}$ are called incidence matrices. The incidence matrices only contain entries $-1, 0$ and $1$.

The incidence matrices are independent of the shape and coarseness of the grid.
Calculus with degrees of freedom II

\[
E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 \\
\end{pmatrix}
\]

\[
E^{2,1} = \begin{pmatrix}
-1 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 1 \\
\end{pmatrix}
\]

This implies that \( R(E^{1,0}) \subseteq N(E^{2,1}) \). In this particular case \( R(E^{1,0}) \equiv N(E^{2,1}) \).

We also have that \( E^{1,0} \cdot E^{2,1} = 0 \) which implies \( R(E^{2,1}) \equiv N(E^{1,0}) \).
Incidence matrices

Calculus with degrees of freedom II

\[
E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 \\
\end{pmatrix}
\]

\[
E^{2,1} = \begin{pmatrix}
-1 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 1 \\
\end{pmatrix}
\]

\[
E^{2,1} \cdot E^{1,0} = 0
\]

This implies that \( \mathcal{R}(E^{1,0}) \subseteq \mathcal{N}(E^{2,1}) \).
In this particular case \( \mathcal{R}(E^{1,0}) \equiv \mathcal{N}(E^{2,1}) \).
Incidence matrices

Calculus with degrees of freedom II

\[
E^{1,0} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix}
\]

\[
E^{2,1} = \begin{pmatrix}
-1 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 1
\end{pmatrix}
\]

\[E^{2,1} \cdot E^{1,0} = 0\]

This implies that \(R(E^{1,0}) \subseteq N(E^{2,1})\).

In this particular case \(R(E^{1,0}) \equiv N(E^{2,1})\).

We also have that \(E^{1,0^T} \cdot E^{2,1^T} = 0 \implies R(E^{2,1^T}) \equiv N(E^{1,0^T})\).
What does the vectors $\mathbf{a} \in \mathbb{R}^6$ actually represent?

Suppose I assign to the vertices the value $\psi = (3, -1, 2, 3, 6, 0)^T$. Physically this could represent the stream function in the points. Then $E_{1,0}\psi = \dot{m} = (-1, 4, -1, 6, 4, 3, -4)^T$ denotes the mass flux over the edges in the direction of the default orientation. Then $E_{2,1}\dot{m} = E_{2,1}E_{1,0}\psi = (0, 0)^T$ denotes conservation of mass in each cell. So $\mathbf{a}$ could represent the stream function and $E_{1,0}\mathbf{a}$ its associated divergence-free velocity field.
What does the vectors $\mathbf{a} \in \mathbb{R}^6$ actually represent?

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So $\mathbf{a}$ could represent the stream function and $E^{1,0} \mathbf{a}$ its associated divergence-free velocity field.
What does the vectors $b \in \mathbb{R}^2$ then represent?

Suppose I assign to the cells the value $\phi = (2, 1)^T$. Physically this could represent the velocity potential in the cells. Then $E_{2,1}^T \phi = \dot{u} = (-2, 2, 1, 1, -2, -1, 1)^T$ denotes the circulation along the edges. Then $E_{1,0}^T \dot{u} = E_{1,0}^T E_{2,1}^T \phi = (0, 0, 0, 0, 0, 0)^T$ denotes vorticity in the vertices. So $b$ could represent the velocity potential and $E_{2,1}^T b$ its associated irrotational flow.

I tacitly drawn dashed lines in the figure. This a dual mesh. The combination of primal and dual grids leads to staggered schemes. Note that we assumed that $\phi = 0$ at the boundary.
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Incidence matrices

Calculus with degrees of freedom IV

What does the vectors \( \mathbf{b} \in \mathbb{R}^2 \) then represent?

Suppose I assign to the cells the value \( \phi = (2, 1)^T \). Physically this could represent the velocity potential in the cells.

Then \( \mathbb{E}^{2,1}^T \phi = \dot{\mathbf{u}} = (-2, 2, 1, 1, -2, -1, 1)^T \) denotes the circulation along the edges.

Then \( \mathbb{E}^{1,0}^T \dot{\mathbf{u}} = \mathbb{E}^{1,0}^T \mathbb{E}^{2,1}^T \phi = (0, 0, 0, 0, 0, 0)^T \) denotes vorticity in the vertices.

So \( \mathbf{b} \) could represent the velocity potential and \( \mathbb{E}^{2,1}^T \mathbf{b} \) its associated irrotational flow.

I tacitly drawn dashed lines in the figure. This a dual mesh. The combination of primal and dual grids leads to staggered schemes. Note that we assumed that \( \phi = 0 \) at the boundary.
Two dimensional expansion – nodal

Consider \([-1, 1]^2 \subset \mathbb{R}^2\). We will use tensor products of nodal expansions to construct a finite dimensional subspace. Consider of the span of \(\{h_i(\xi)h_j(\eta)\}, \ i, j = 0, \ldots, N\). Any element \(\psi^h(\xi, \eta)\) can be written as

\[
\psi^h(\xi, \eta) = \sum_{i=0}^{N} \sum_{j=0}^{N} \psi_{i,j} h_i(\xi) h_j(\eta).
\]

with \(\psi_{i,j} = \psi(\xi_i, \eta_j)\).

Let \(\varphi^h\) in the same space, then the \(L^2\)-inner product is given by

\[
\left( \varphi^h, \psi^h \right) = \int_{-1}^{1} \int_{-1}^{1} \varphi^h \psi^h \, d\xi d\eta
\]

\[
= \begin{bmatrix} \varphi_{0,0} & \cdots & \varphi_{N,N} \end{bmatrix} \begin{bmatrix} \psi_{0,0} \\ \vdots \\ \psi_{N,N} \end{bmatrix},
\]

where \(\mathbb{M}^{(0)}\) is the nodal mass matrix given by

\[
\mathbb{M}^{(0)} = \int_{-1}^{1} \int_{-1}^{1} h_i(\xi) h_j(\eta) h_k(\xi) h_l(\eta) \, d\xi d\eta.
\]
Two dimensional expansion – primal and dual

The primal basis functions, $N = 3$

The dual basis functions, $N = 3$. 
If we apply the perpendicular gradient $\nabla \perp$ to this nodal expansion of $\psi^h$ we obtain

$$\nabla \perp \psi^h = \left( \sum_{i=0}^{N} \sum_{j=1}^{N} (\psi_{i,j} - \psi_{i,j-1}) h_i(\xi) e_j(\eta) \right)$$

$$= \left[ \begin{array}{ccc} h_0(\xi)e_1(\eta) & \ldots & h_N(\xi)e_N(\eta) \\ 0 & \ldots & 0 \end{array} \right] \left[ \begin{array}{c} \psi_{0,0} \\ \vdots \\ \psi_{N,N} \end{array} \right]$$
Two dimensional basis functions

Two dimensional expansion – edge

If we apply the perpendicular gradient $\nabla^\perp$ to this nodal expansion of $\psi^h$ we obtain

$$
\nabla^\perp \psi^h = \left( \begin{array}{c}
\sum_{i=0}^N \sum_{j=1}^N (\psi_{i,j} - \psi_{i,j-1}) h_i(\xi) e_j(\eta) \\
\sum_{i=1}^N \sum_{j=0}^N (\psi_{i-1,j} - \psi_{i,j}) e_i(\xi) h_j(\eta)
\end{array} \right)
$$

$$
= \left[ \begin{array}{cccc}
h_0(\xi) e_1(\eta) & \cdots & h_N(\xi) e_N(\eta) & 0 & \cdots & 0 \\
0 & \cdots & 0 & e_1(\xi) h_0(\eta) & \cdots & e_N(\xi) h_N(\eta)
\end{array} \right] \left[ \begin{array}{cccc}
\psi_{0,0} \\
\vdots \\
\psi_{N,N}
\end{array} \right]
$$

If $v^h$ can be expanded as

$$
v^h = \left( \begin{array}{c}
\sum_{i=0}^N \sum_{j=1}^N u_{i,j} h_i(\xi) e_j(\eta) \\
\sum_{i=1}^N \sum_{j=0}^N v_{i,j} e_i(\xi) h_j(\eta)
\end{array} \right)
$$

$$
= \left[ \begin{array}{cccc}
h_0(\xi) e_1(\eta) & \cdots & h_N(\xi) e_N(\eta) & 0 & \cdots & 0 \\
0 & \cdots & 0 & e_1(\xi) h_0(\eta) & \cdots & e_N(\xi) h_N(\eta)
\end{array} \right] \left[ \begin{array}{cccc}
u_{0,1} \\
\vdots \\
u_{N,N} \\
v_{1,0}
\vdots \\
v_{N,N}
\end{array} \right]
$$
Two dimensional basis functions

Two dimensional expansion – edge

If we apply the perpendicular gradient $\nabla \perp$ to this nodal expansion of $\psi^h$ we obtain

$$\nabla \perp \psi^h = \left( \sum_{i=0}^{N} \sum_{j=1}^{N} (\psi_{i,j} - \psi_{i,j-1}) h_i(\xi) e_j(\eta) \right) \sum_{i=1}^{N} \sum_{j=0}^{N} (\psi_{i-1,j} - \psi_{i,j}) e_i(\xi) h_j(\eta)$$

$$= \begin{bmatrix}
  h_0(\xi)e_1(\eta) & \cdots & h_N(\xi)e_N(\eta) & 0 & \cdots & 0 \\
  0 & \cdots & 0 & e_1(\xi)h_0(\eta) & \cdots & e_N(\xi)h_N(\eta)
\end{bmatrix} \begin{bmatrix}
  \psi_{0,0} \\
  \vdots \\
  \psi_{N,N}
\end{bmatrix}$$

Then the equation $\mathbf{v}^h = \nabla \perp \psi^h$ implies

$$\begin{bmatrix}
  \psi_{0,0} \\
  \vdots \\
  \psi_{N,N}
\end{bmatrix} \begin{bmatrix}
  u_{0,1} \\
  \vdots \\
  u_{N,N} \\
  v_{1,0} \\
  \vdots \\
  v_{N,N}
\end{bmatrix} \leftarrow \text{Same as Slide 6}$$

So we preserve the discrete structure between stream function and mass fluxes! This result is independent of the basis functions, shape and coarseness of the grid!
Two dimensional expansion – edge

If we take an arbitrary $a^h$ expanded as

$$
a^h = \left( \sum_{k=0}^{N} \sum_{l=1}^{N} a_{k,l} h_k(\xi) e_l(\eta), \sum_{k=1}^{N} \sum_{l=0}^{N} b_{k,l} e_k(\xi) h_l(\eta) \right),
$$

and take the $L^2$ inner product with $\nabla \perp \psi^h$ we obtain

$$
\left( a^h, \nabla \perp \psi^h \right)_D^h = \int_{-1}^{1} \int_{-1}^{1} (a^h, \nabla \perp \psi^h) \, d\xi \, d\eta
$$

\begin{equation}
= \begin{bmatrix}
    a_{0,1} & \ldots & a_{N,N} & b_{1,0} & \ldots & b_{N,N}
\end{bmatrix} \mathbf{M}^{(1)} \mathbf{E}^{1,0} \begin{bmatrix}
    \psi_{0,0} \\
    \vdots \\
    \psi_{N,N}
\end{bmatrix}
\end{equation}

where the mass matrix $\mathbf{M}^{(1)}$ is given by

$$
\mathbf{M}^{(1)} = \begin{pmatrix}
    \int_{-1}^{1} \int_{-1}^{1} h_i(\xi) e_j(\eta) h_k(\xi) e_l(\eta) \, d\xi \, d\eta & 0 \\
    0 & \int_{-1}^{1} \int_{-1}^{1} e_p(\xi) h_q(\eta) e_r(\xi) h_s(\eta) \, d\xi \, d\eta
\end{pmatrix}.
$$
Two dimensional basis functions

Two dimensional expansion – primal and edge

The $\xi$-component of the $H(\text{div})$-field

The primal basis functions, $N = 3$

The dual basis functions, $N = 3$. 
Two dimensional expansion – primal and edge

The $\eta$-component of the $H(\text{div})$-field

The primal basis functions, $N = 3$

The dual basis functions, $N = 3$. 
Two dimensional basis functions

Two dimensional expansion – surface

Let $u^h$ be expanded as

$$u^h = \left( \begin{array}{c}
\sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi)e_j(\eta) \\
\sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi)h_j(\eta)
\end{array} \right)$$

Then $\nabla \cdot u^h$ is given by

$$\nabla \cdot u^h = \sum_{i=1}^{N} \sum_{j=1}^{N} (u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}) e_i(\xi)e_j(\eta)$$

$$= \begin{bmatrix} e_1(\xi)e_1(\eta) & \cdots & e_N(\xi)e_N(\eta) \end{bmatrix} \in \mathbb{R}^{2,1}$$

$$\begin{bmatrix}
u_{0,1} \\
\vdots \\
u_{N,N}
\end{bmatrix}.$$
Two dimensional basis functions

Two dimensional expansion – surface

Then $\nabla \cdot \mathbf{u}^h$ is given by

$$\nabla \cdot \mathbf{u}^h = \sum_{i=1}^{N} \sum_{j=1}^{N} \left( u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right) e_i(\xi)e_j(\eta)$$

$$= \begin{bmatrix} e_1(\xi)e_1(\eta) & \ldots & e_N(\xi)e_N(\eta) \end{bmatrix}_{\mathbb{R}^{2,1}} \begin{bmatrix} u_{0,1} \\ \vdots \\ u_{N,N} \\ v_{1,0} \\ \vdots \\ v_{N,N} \end{bmatrix}.$$ 

Since the basis functions $e_i(\xi)e_j(\eta)$ are linearly independent, $\nabla \cdot \mathbf{u}^h = 0$ reduces to

$$\begin{bmatrix} u_{0,1} \\ \vdots \\ u_{N,N} \\ v_{1,0} \\ \vdots \\ v_{N,N} \end{bmatrix} \in \mathbb{R}^{2,1} = 0 \underset{\text{Independent of basis functions}}{\leftarrow}$$
Two dimensional basis functions

Two dimensional expansion – surface

Let $q^h$ can be expanded as

$$q^h(\xi, \eta) = \sum_{k=1}^{N} \sum_{l=1}^{N} q_{k,l} e_k(\xi) e_l(\eta).$$

With this expansion we can write $(q^h, \nabla \cdot u^h)_S$ as

$$(q^h, \nabla \cdot u^h)_S = [q_{1,1} \ldots q_{N,N}] \mathcal{M}^{(2)} E^2,1$$

where $\mathcal{M}^{(2)}$ is the mass matrix on $S^h$ given by

$$\mathcal{M}^{(2)} := \int_{-1}^{1} \int_{-1}^{1} e_i(\xi) e_j(\eta) e_k(\xi) e_l(\eta) \, d\xi \, d\eta$$
Two dimensional basis functions

Two dimensional expansion – primal and dual surface

The primal basis functions, $N = 3$

The dual basis functions, $N = 3$. 
Consider the scalar Laplace problem for $\psi \in H_0^1(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta \psi = f \quad \text{in } \Omega$$
Nodal Laplace problem

Consider the scalar Laplace problem for $\psi \in H_0^1(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta \psi = f \quad \text{in } \Omega$$

Multiply this equation with any $\tilde{\psi} \in H_0^1(\Omega)$ and integrate over the domain $\Omega$

$$\int_{\Omega} -\Delta \psi \tilde{\psi} \, d\Omega = \int_{\Omega} (\nabla \psi, \nabla \tilde{\psi}) \, d\Omega = \int_{\Omega} f \tilde{\psi} \, d\Omega$$

Note that the discrete Laplacian only consists of a mass matrix and incidence matrix.
Nodal Laplace problem

Consider the scalar Laplace problem for \( \psi \in H^1_0(\Omega) \) with \( \Omega = [-1, 1]^2 \)

\[-\Delta \psi = f \quad \text{in } \Omega\]

Multiply this equation with any \( \tilde{\psi} \in H^1_0(\Omega) \) and integrate over the domain \( \Omega \)

\[
\int_{\Omega} -\Delta \psi \tilde{\psi} \, d\Omega = \int_{\Omega} (\nabla \psi, \nabla \tilde{\psi}) \, d\Omega = \int_{\Omega} f \tilde{\psi} \, d\Omega
\]

If we restrict ourselves to the conforming subspace spanned by the nodal functions

\[\psi^h = \sum_{i=0}^{N} \sum_{j=0}^{N} \psi_{i,j} h_i(\xi) h_j(\eta)\]

we obtain the discrete Laplace equation

\[E^{1,0\, T} M^{(1)} E^{1,0} \tilde{\psi} = M^{(0)} \tilde{f}\]
Nodal Laplace problem

Consider the scalar Laplace problem for \( \psi \in H^1_0(\Omega) \) with \( \Omega = [-1, 1]^2 \)

\[
-\Delta \psi = f \quad \text{in } \Omega
\]

Multiply this equation with any \( \bar{\psi} \in H^1_0(\Omega) \) and integrate over the domain \( \Omega \)

\[
\int_{\Omega} -\Delta \psi \bar{\psi} \, d\Omega = \int_{\Omega} (\nabla \psi, \nabla \bar{\psi}) \, d\Omega = \int_{\Omega} f \bar{\psi} \, d\Omega
\]

If we restrict ourselves to the conforming subspace spanned by the nodal functions

\[
\psi^h = \sum_{i=0}^N \sum_{j=0}^N \psi_{i,j} h_i(\xi) h_j(\eta)
\]

we obtain the discrete Laplace equation

\[
E^{1,0}^T M^{(1)} E^{1,0} \bar{\psi} = M^{(0)} \bar{f}
\]

Note that the discrete Laplacian only consists of a mass matrix and incidence matrix.
Volumetric Laplace problem I

Consider the scalar Laplace problem for $\phi \in L^2(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta \phi = f \quad \text{in } \Omega$$

with $\phi = 0$
Volumetric Laplace problem I

Consider the scalar Laplace problem for \( \phi \in L^2(\Omega) \) with \( \Omega = [-1, 1]^2 \)

\[-\Delta \phi = f \quad \text{in} \ \Omega\]

with \( \phi = 0 \)

If we \( \phi^h \) in terms of dual edge functions and \( u \) in terms of primal edge functions, like

\[\phi^h = \sum_{i=1}^{N} \sum_{j=1}^{N} \phi_{i,j} \tilde{h}_i(\xi) \tilde{h}_j(\eta) \quad \text{and} \quad u = \left( \begin{array}{c} \sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta) \\ \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \end{array} \right)\]

Now the derivative of an edge function \( \tilde{h}_i(\xi) \) does not exist. We therefore have to go to the **mixed formulation**

\[
\begin{cases}
\nabla \cdot u - \nabla \phi & = 0 \\
\n\nabla \cdot u & = f
\end{cases}
\]
Volumetric Laplace problem I

Consider the scalar Laplace problem for \( \phi \in L^2(\Omega) \) with \( \Omega = [-1, 1]^2 \)

\[-\Delta \phi = f \quad \text{in} \ \Omega\]

with \( \phi = 0 \)

If we \( \phi^h \) in terms of dual edge functions and \( u \) in terms of primal edge functions, like

\[
\phi^h = \sum_{i=1}^{N} \sum_{j=1}^{N} \phi_{i,j} \tilde{h}_i(\xi) \tilde{h}_j(\eta) \quad \text{and} \quad u = \left( \begin{array}{c} \sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta) \\ \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \end{array} \right)
\]

Now the derivative of an edge function \( \tilde{h}_i(\xi) \) does not exist. We therefore have to go to the mixed formulation

\[
\begin{cases}
  u - \nabla \phi &= 0 \\
  \nabla \cdot u &= f
\end{cases}
\]

If we multiply the first equation by \( v \in H_0(\text{div}; \Omega) \) and the second equation by \( q \in L^2(\Omega) \), we obtain

\[
(v, u) - (v, \nabla \phi) = 0
\]

\[
(q, \nabla \cdot u) - (q, f) = 0
\]
Laplace problems

Volumetric Laplace problem I

Consider the scalar Laplace problem for \( \phi \in L^2(\Omega) \) with \( \Omega = [-1, 1]^2 \)

\[-\Delta \phi = f \quad \text{in} \ \Omega\]

with \( \phi = 0 \)

If we \( \phi^h \) in terms of dual edge functions and \( u \) in terms of primal edge functions, like

\[\phi^h = \sum_{i=1}^{N} \sum_{j=1}^{N} \phi_{i,j} \hat{h}_i(\xi) \hat{h}_j(\eta) \quad \text{and} \quad u = \left( \sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} \hat{h}_i(\xi) e_j(\eta), \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) \hat{h}_j(\eta) \right)\]

Now the derivative of an edge function \( \hat{h}_i(\xi) \) does not exist. We therefore have to go to the mixed formulation

\[
\begin{cases}
 u - \nabla \phi = 0 \\
 \nabla \cdot u = f
\end{cases}
\]

If we multiply the first equation by \( v \in H^1_0(\text{div}; \Omega) \) and the second equation by \( q \in L^2(\Omega) \), we obtain

\[
 \begin{align*}
 (v, u) &+ (\nabla \cdot v, \phi) = 0 \\
 (q, \nabla \cdot u) &+ (q, f) = 0
\end{align*}
\]
Consider the scalar Laplace problem for $\phi \in L^2(\Omega)$ with $\Omega = [-1, 1]^2$

$$-\Delta \phi = f \quad \text{in } \Omega$$

with $\phi = 0$

If we $\phi^h$ in terms of dual edge functions and $u$ in terms of primal edge functions, like

$$\phi^h = \sum_{i=1}^{N} \sum_{j=1}^{N} \phi_{i,j} \tilde{h}_i(\xi) \tilde{h}_j(\eta) \quad \text{and} \quad u = \left( \begin{array}{c} \sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) e_j(\eta) \\ \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) h_j(\eta) \end{array} \right)$$

Now the derivative of an edge function $\tilde{h}_i(\xi)$ does not exist. We therefore have to go to the mixed formulation

$$\begin{cases}
\quad u - \nabla \phi = 0 \\
\quad \nabla \cdot u = f
\end{cases}$$

If we multiply the first equation by $v \in H_0(\text{div}; \Omega)$ and the second equation by $q \in L^2(\Omega)$, we obtain

$$\begin{align*}
(v, u) + (\nabla \cdot v, \phi) &= 0 \\
(q, \nabla \cdot u) &= (q, f)
\end{align*}$$

Inserting our expansions gives

$$\mathbb{M}^{(1)} \ddot{u} + \mathbb{E}^{2,1}^T \dot{\phi} = 0$$

$$\ddot{u} + \dot{\phi} = \ddot{f}$$
Volumetric Laplace problem II

\[ \mathbf{M}^{(1)} \ddot{\mathbf{u}} + \mathbf{E}^{2,1} \mathbf{T} \phi = 0 \]

If we eliminate \( \mathbf{u} \) again, we obtain

\[ \mathbf{E}^{2,1} \mathbf{M}^{(1)}^{-1} \mathbf{E}^{2,1} \mathbf{T} \phi = \mathbf{f} \]

Again, this system matrix only consists of mass matrices and an incidence matrix.
Volumetric Laplace problem II

$$\mathbf{M}^{(1)} \mathbf{\bar{u}} + \mathbf{E}^{2,1}^T \mathbf{\bar{\phi}} = 0$$

If we eliminate $\mathbf{u}$ again, we obtain

$$\mathbf{E}^{2,1} \mathbf{M}^{(1)^{-1}} \mathbf{E}^{2,1}^T \mathbf{\bar{\phi}} = \mathbf{\bar{f}}$$

Again, this system matrix only consists of mass matrices and an incidence matrix.
Stokes

\[
\begin{cases}
\nabla \times \mathbf{u} - \omega = 0 & \text{in } \Omega \\
\nabla \times \omega + \nabla p = \mathbf{f} & \text{in } \Omega \\
\n\nabla \cdot \mathbf{u} = 0 & \text{in } \Omega
\end{cases}
\]

Using the same approach, we can discretize this as

\[
\begin{pmatrix}
M^{(0)} \\
M^{(1)} E^{1,0} \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
\mathbf{E}^{1,0}^T M^{(1)} \\
0 \\
\mathbf{E}^{2,1} \\
0
\end{pmatrix}
\begin{pmatrix}
\vec{\omega} \\
\vec{u} \\
\vec{p}
\end{pmatrix}
- 
\begin{pmatrix}
0 \\
M^{(1)} \vec{f}
\end{pmatrix}
\]
Stokes flow

Stokes flow around cylinder

[Jasper Kreeft, JCP 2014]
Final Remarks

- Further reading: see arXiv:
  https://arxiv.org/find/math/1/au:+Gerritsma_M/0/1/0/all/0/1
- Further information: Send email: M.I.Gerritsma@tudelft.nl
Final Remarks

- Further reading: see arXiv:
  https://arxiv.org/find/math/1/au:+Gerritsma_M/0/1/0/all/0/1
- Further information: Send email: M.I.Gerritsma@tudelft.nl

Tak og sjov med de andre præsentationer!