SQUARING
THE CIRCLE
AND OTHER
IMPOSSIBILITIES

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Circles to square,
and Cubes to double,
Would give a Man
excessive Trouble.

Matthew Prior (1718)
The Three Classical Problems

1. Doubling the Cube
   Given a cube, construct another with twice the volume.

2. Trisecting the Angle
   Given any angle, construct two lines that divide it into three equal parts.

3. Squaring the Circle
   Given a circle, construct a square with the same area.
# Three Periods

<table>
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<th>Period</th>
<th>Mathematician</th>
<th>Date</th>
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</thead>
<tbody>
<tr>
<td><strong>Early:</strong></td>
<td>Thales</td>
<td>600 BC</td>
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<tr>
<td></td>
<td>Pythagoras</td>
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<td><strong>Athens:</strong></td>
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<td></td>
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<td><strong>Alexandria:</strong></td>
<td>Euclid</td>
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<td>(Archimedes)</td>
<td>250 BC</td>
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<td></td>
<td>Apollonius</td>
<td>220 BC</td>
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<td></td>
<td>Ptolemy</td>
<td>150 AD</td>
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<td></td>
<td>Diophantus</td>
<td>250 AD?</td>
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<tr>
<td></td>
<td>Pappus</td>
<td>320 AD</td>
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<td></td>
<td>Hypatia</td>
<td>400 AD</td>
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Map of Greece (c. 300 BC)
The famous Thales is said to have been the first to demonstrate that the circle is bisected by the diameter. If you wish to demonstrate this mathematically, imagine the diameter drawn and one part of the circle fitted upon the other. If it is not equal to the other, it will fall either inside or outside it, and in either case it will follow that a shorter line is equal to a longer. For all the lines from the centre to the circumference are equal, and hence the line that extends beyond will be equal to the line that falls short, which is impossible.
Aristophanes’ *The Birds* (414 BC)

**METON**

These are my special rods for measuring the air. You see, the air is shaped — how shall I put it? — like a sort of extinguisher; so all I have to do is to attach this flexible rod at the upper extremity, take the compasses, insert the point here, and — you see what I mean?

**PEISTHETAERUS**

No.

**METON**

Well, I now apply the straight rod — so — thus squaring the circle; and there you are . . .

**PEISTHETAERUS**

Brilliant — the man’s a Thales.
Lines and Circles

Euclid’s postulates 1-3:

1. Draw a straight-line segment between any two points.
2. Extend a straight-line segment.
3. Draw a circle with given centre and radius.

Straight-edge (ruler) and compasses (no measuring allowed)
Euclid Book I, Prop. I

On a given straight line to construct an equilateral triangle

Let AB be the line.

With centre A and distance AB, draw the circle BCD. [Post. 3]

With centre B and distance BA, draw the circle ACE. [Post. 3]

Join AC and BC. [Post. 1]

Then the triangle ABC is equilateral.

Proof ...
Bisecting a line segment

Let \( AB \) be the line segment.

Draw:
- the circle centre \( A \), radius \( AB \),
- the circle centre \( B \), radius \( BA \).

These circles intersect at \( C \) and \( D \).
Draw the line \( CD \), and let it meet \( AB \) at \( E \). Then \( E \) is the midpoint of \( AB \).
Trisecting a line segment

Let \( AB \) be the line segment.

Draw any line \( AC \) (\( C \) not on \( AB \))

Extend it, and mark off \( AC : CD : DE \)
(using the compasses)

Draw \( EB \), and then parallel to it, \( CF, DG \),
where \( F, G \) lie on \( AB \).

Then \( AF : FC : CB \).

Can be extended to a division of \( AB \) into any number of parts...
Doubling a square

Plato's 'Meno';
Socrates and the slave boy

double the size of the square

2 \times 2 = 4
4 \times 4 = 16
3 \times 3 = 9

Take the square on the diagonal ...

(2\sqrt{2}) \times (2\sqrt{2}) = 8 \checkmark
Mean proportionals

A mean proportional of \(a\) and \(b\) is a number \(x\) such that

\[ a : x = x : b, \text{ so } x = \sqrt{ab} \]

Euclid: (IV, Prop13)

\[ BP = \sqrt{(AB)(BC)} \]

\[ \frac{x}{a} = \frac{b}{x} \]

So: a mean proportional of 1 and 2 is \(\sqrt{2}\).

Finding the mean proportional of \(a\) and \(b\) corresponds to ‘squaring’ the rectangle with sides \(a\) and \(b\).
Theon of Smyrna (2nd century AD)

In his work entitled *Platonicus*, Eratosthenes says that, when the god announced to the Delians by oracle that to get rid of a plague they must construct an altar double of the existing one, their craftsmen fell into great perplexity in trying to find how a solid could be made double of another solid, and they went to ask Plato about it.

He told them that the god had given this oracle, not because he wanted an altar of double the size, but because he wished, in setting this task before them, to reproach the Greeks for their neglect of mathematics and their contempt for geometry.
Eutocius (6th century AD)

The story goes that one of the ancient tragic poets represented Minos having a tomb built for Glaucus, and that when Minos found that the tomb measured a hundred feet on every side, he said:

‘Too small is the tomb you have marked out as the royal resting place. Let it be twice as large. Without spoiling the form quickly double each side of the tomb.’

This was clearly a mistake. For if the sides are doubled, the surface is multiplied fourfold and the volume eightfold. Now geometers, too, sought a way to double the given solid without altering its form.

This problem came to be known as the duplication of the cube, for, given a cube, they sought to double it.
Eutocius: Hippocrates of Chios found that to double a cube, we must find two mean proportionals between given straight lines—one double the other.

So: given $a$ and $b$, find $x$ and $y$ so that

$$a : x = x : y = y : b, \text{ with } b = 2a$$

Now \( \frac{x}{a} = \frac{y}{x} = \frac{b}{y} \)

so \( \left( \frac{x}{a} \right)^3 = \frac{x}{a} \times \frac{y}{x} \times \frac{b}{y} = \frac{b}{a} = 2, \)

giving \( x^3 = 2a^3. \)

This corresponds to taking a cube with side $a$ and doubling its volume.
Conic Sections

cone

ellipse

circle

hyperbola

parabola

Menaechmus (4th century BC)

Apollonius (‘Conics’: 250 BC)
Menaechmus:

if \( a : x = x : y = y : b \), then:

\[
\frac{p}{x} = \frac{y}{x^2}, \text{ so } ay = x^2 \quad \text{(parabola)}
\]

\[
\frac{y}{y} = \frac{x}{b}, \text{ so } y^2 = bx \quad \text{(parabola)}
\]

\[
\frac{x}{y} = \frac{b}{y}, \text{ so } xy = ab \quad \text{(hyperbola)}
\]
Bisecting an angle

Let the angle be PAQ.

Draw a circle with centre A, any radius. This crosses AP and AQ at B and C.

Draw circles, centres B and C, with the same radius. These meet at D.

The line segment AD bisects PAQ.

Euclid: after finding B and C, join them, and draw the perpendicular bisector of BC.
Problem.

C.L.D
Dec:
1844

To trisect a right angle, that is, to divide it into three equal parts.

Let there be a right angle ABC, it is required to trisect it.

Produce AB to D and make BD equal to AB, and make BC equal to AB and produce CB to E and make EB equal to BC, and join AE, ED, DC, CA. Because AB is equal to BD, and BE is common to the two triangles ABE, DBE, and the angle ABE is equal to the angle DBE, therefore the base AE is equal to the base ED; and in like manner it may be proved that all the four AE, ED, DC, CA are equal, therefore AEDC is equilateral, and because the three angles of a triangle are equal to two right angles, and that the angle ABE is a right angle (for ABC is a right angle, and BC is a straight line) therefore the angles BAE, BEA are equal to one right angle and because BA is equal to BE, therefore the angle BAE is a right angle and in like manner it may be proved that the angle BAC is a right angle, therefore the angle BAC is a right angle, and in like manner it may be proved that the angles AED, EDC, DCA are also right angles, therefore AEDC is equilateral, that is, has all its angles right angles, and it was proved.
Trisecting an angle

Use a quadratrix (= trisectrix): two moving lines -
AB sweeps round a quarter-circle from B
BC moves downwards, ending along AD.

\[
QR = \frac{1}{3} PR
\]

\[
\angle SAX = \frac{1}{3} \times \angle PAAX.
\]

Similarly, by dividing PR into 4, 5, ... parts, we can divide the angle into any number of parts.
To construct a square

Start with a line segment AB.
Construct AC and BE perpendicular to AB.
With compasses at A, mark X on AC equal to AB.
Construct XZ perpendicular to AC.

Let XZ meet BE at Y: \[ \text{ABYX is a square.} \]

Euclid I, 46

Euclid also shows how to construct a regular pentagon, hexagon and 15-gon. (Book IV)
Constructing a hexagon
Fermat Prime Numbers

Fermat: \( F_n = 2^{2^n} + 1 \) is prime (?)

\( F_0 = 2^1 + 1 = 3, \quad F_1 = 2^2 + 1 = 5, \)
\( F_2 = 2^4 + 1 = 17, \quad F_3 = 2^8 + 1 = 257, \)
\( F_4 = 2^{16} + 1 = 65,537 \)
\( F_5 = 4,294,967,297 \quad ? \quad 2^{32} + 1 \)

Euler: \( F_5 \) is divisible by \( 641 \)

Proof: \( 641 = 5^4 + 2^4 = (5 \times 2^7) + 1. \)

So \( 2^{32} + 1 = 2^{28} (5^4 + 2^4) - (5 \times 2^7)^4 + 1 \)
\( = 2^{28} \times 641 - (641 - 1)^4 + 1 \)
\( = 641 \times K, \quad \text{so} \quad 641 \mid 2^{32} + 1. \)
Constructing Polygons

Gauss: A regular polygon with \( n \) sides can be constructed with straight-edge and compasses if and only if \( n \) has the form

\[
n = 2^k \times p_1 \times p_2 \times \ldots \times p_r, \quad 2^{2^k} + 1 \text{ distinct Fermat primes}
\]

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, \ldots, 257, \ldots, 65537, \ldots
Squaring things

Euclid:

I, 44  Given a triangle, we can construct a parallelogram of equal area.

I, 45  Given any polygon, we can construct a parallelogram of equal area.

II, 14  Given any polygon, we can construct a square of equal area.

We can square any polygon!

Hippocrates gave a construction for squaring "lunes".

Archimedes: the area of a parabolic segment

= \( \frac{4}{3} \times (\text{area of enclosed } \triangle) \)

→ quadrature of the parabola
An Egyptian Geometry Problem

Problem 48. Compare the areas of a circle and its circumscribing square.
**Problem 48.** Compare the areas of a circle and its circumscribing square.

<table>
<thead>
<tr>
<th>The circle of diameter 9</th>
<th>The square of side 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>9 setat</td>
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<tr>
<td>4</td>
<td>16 setat</td>
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<td>8</td>
<td>32 setat</td>
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<td></td>
<td>64 setat</td>
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<td>8</td>
<td>9 setat</td>
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<td>18 setat</td>
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<td></td>
<td>36 setat</td>
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<td></td>
<td>72 setat</td>
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<td><strong>Total 81 setat</strong></td>
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</tbody>
</table>

Area: \[ \left( d - \frac{d}{9} \right)^2 \]

\[ = \frac{256}{81} r^2 \approx 3.16 r^2 \]
Archimedes' polygons

\[ n = 6 \]
\[ n = 12 \]
\[ n = 24 \]
\[ n = 48 \]

Semi-perimeters

\[ L = 3 \]
\[ L = 3.464 \]
\[ L = 3.105 \]
\[ L = 3.215 \]
\[ L = 3.133 \]
\[ L = 3.160 \]
\[ L = 3.139 \]
\[ L = 3.146 \]
The Value of \( \pi \)

perimeter of inscribed 6-gon < circumference of circle < perimeter of exscribed 6-gon
double the number of sides:
6, 12, 24, 48, 96.

Archimedes obtained the estimates:

\[
3 \frac{10}{71} < \pi < 3 \frac{1}{7}
\]

3.14084 \hspace{1cm} 3.14286
Using algebra

Two lines:
\[ \begin{align*}
\{ & y = 2x + 3 \\
& y = 5x - 1 \\
\end{align*} \]

\[ (4/3, 17/3) \]

 fractions

Line + circle:
\[ x = 2 \]
\[ (x-1)^2 + y^2 = 4 \]

\[ (2, \sqrt{3}) \]

\[ (2, -\sqrt{3}) \]

 square roots

Two circles:

again: fractions and square roots
Constructible numbers

integers, fractions, square roots, and any number arising from adding, subtracting, multiplying, dividing, or taking square roots of these ...

Doubling a square: \( x^2 = 8 \),
so \( x = 2\sqrt{2} \)

Constructing a 17-gon:

\[
\begin{align*}
x &= \frac{-1 + \sqrt{17} - \sqrt{34 - 2\sqrt{17}}}{4}, \ldots
\end{align*}
\]
The three Classical Problems

Doubling the cube:
\[ x^3 = 2, \text{ so } x = \sqrt[3]{2} - \text{ impossible} \]

Trisecting the angle:
\[ \cos 3\theta = 4\cos^3 \theta - 3\cos \theta \]

Take \( x = \cos 20^\circ \) \[ (\cos 60^\circ = \frac{1}{2}) \]
\[ \frac{1}{2} = 4x^3 - 3x, \text{ so } 8x^3 - 6x - 1 = 0 \]

If this factorizes into a linear and a quadratic, the linear factor is:
\[ 8x \pm 1, 4x \pm 1, 2x \pm 1, x \pm 1 \}\] none works

Squaring the circle:

Is \( \pi \) constructible?

F. Lindemann (1882): - no,
so squaring the circle is impossible.