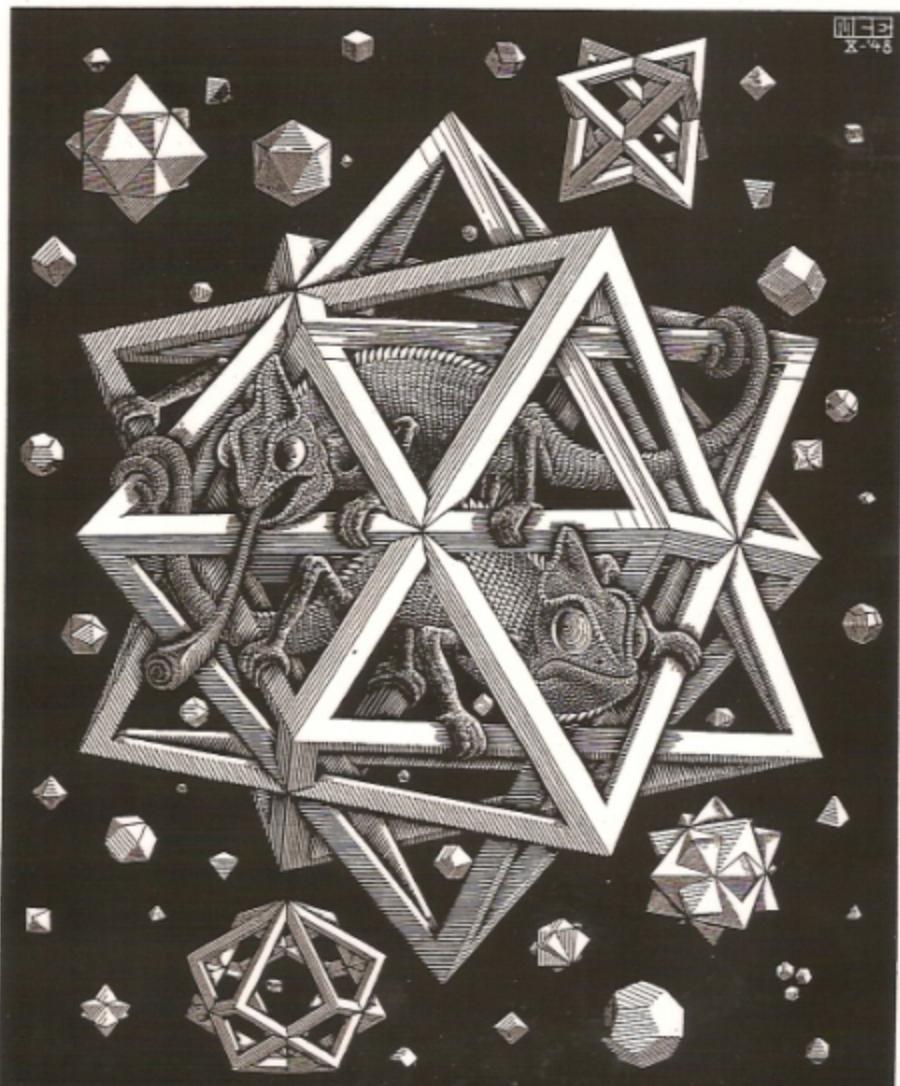


4000 YEARS OF GEOMETRY

Robin Wilson



Egyptian pyramids

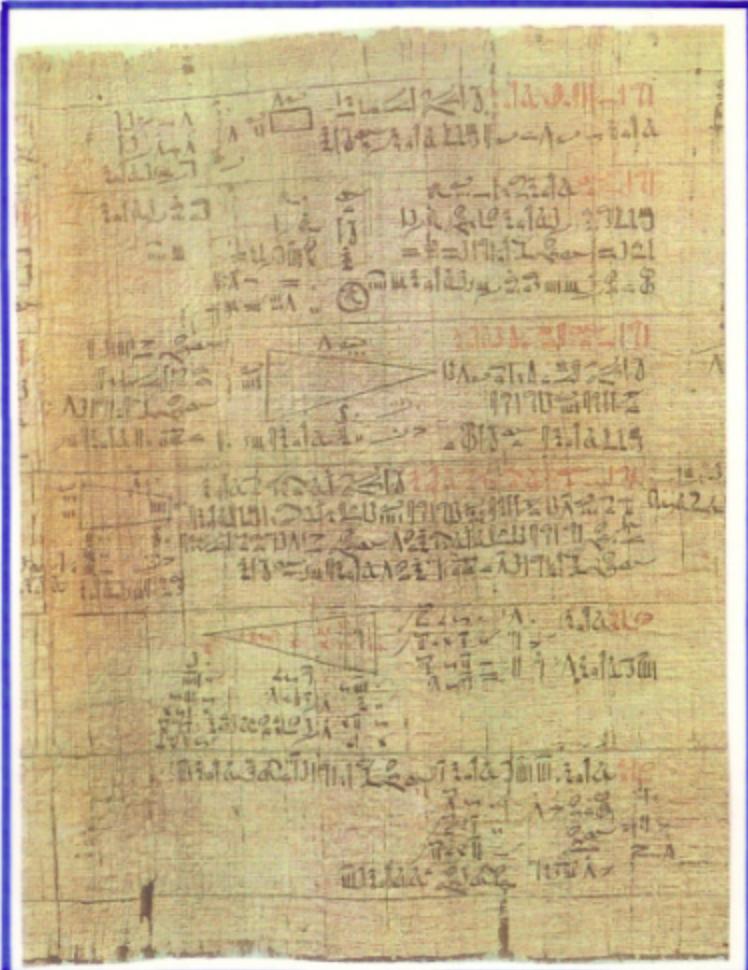
Saqqara
pyramid →

Pyramids
of Giza ↓



Rhind Papyrus

(c. 1650 BC)



An Egyptian Geometry Problem

Problem 48. Compare the areas of a circle and its circumscribing square.



A Problem in Geometry (c. 1650 BC)

Problem 48. Compare the areas of a circle and its circumscribing square.

The circle of diameter 9 The square of side 9

1 8 setat

2 16 setat

4 32 setat

8 64 setat

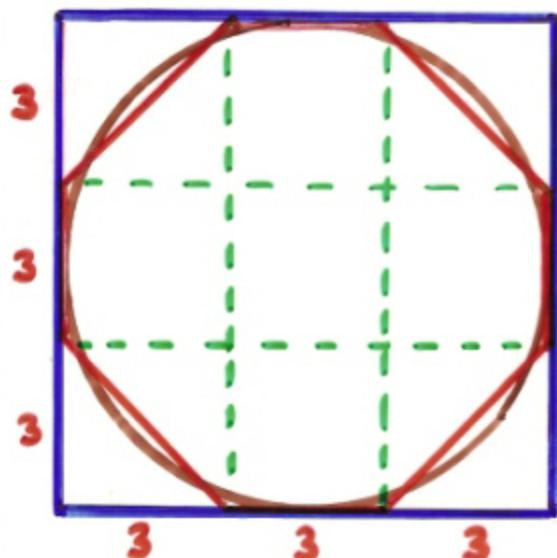
1 9 setat

2 18 setat

4 36 setat

8 72 setat

Total 81 setat



$$\text{Area} = \left(d - \frac{d}{q}\right)^2$$

$$= \frac{256}{81} r^2 \approx 3.16 r^2$$

Mesopotamian Mathematics

clay tablets - cuneiform writing

place-value system based on 60: <, ,

$$\text{Diagram of cuneiform symbols} = 41(60) + 40, \text{ or } 41\frac{40}{60}, \text{ or } \dots$$



A remarkable tablet



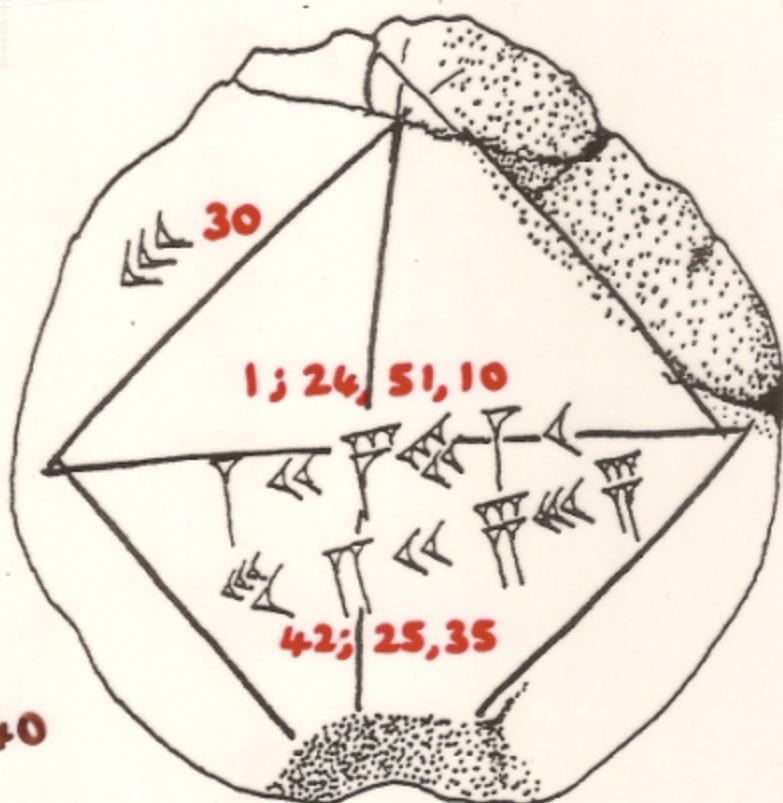
$30 = \text{side of square}$

$1; 24, 51, 10 \approx \sqrt{2}$

$42; 25, 35 \approx 30\sqrt{2}$

$$(1; 24, 51, 10)^2$$

$$= 1; 59, 59, 59, 38, 1, 40$$



Three Periods

Early: Thales 600 BC

Pythagoras 520 BC

Athens: Plato 387 BC

Aristotle 350 BC

Eudoxus 370 BC

Alexandria: Euclid 300 BC

(Archimedes) 250 BC

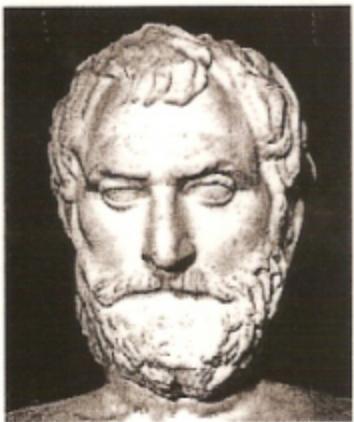
Apollonius 220 BC

Ptolemy 150 AD

Diophantus 250 AD?

Pappus 320 AD

Hypatia 400 AD



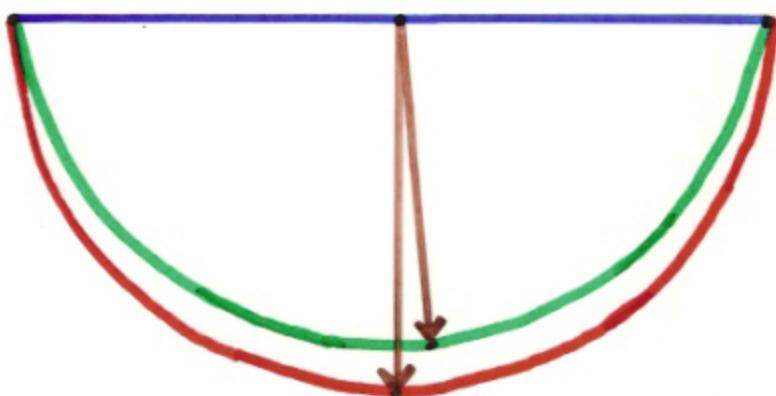
Thales of Miletus

(c. 624 – 547 BC)

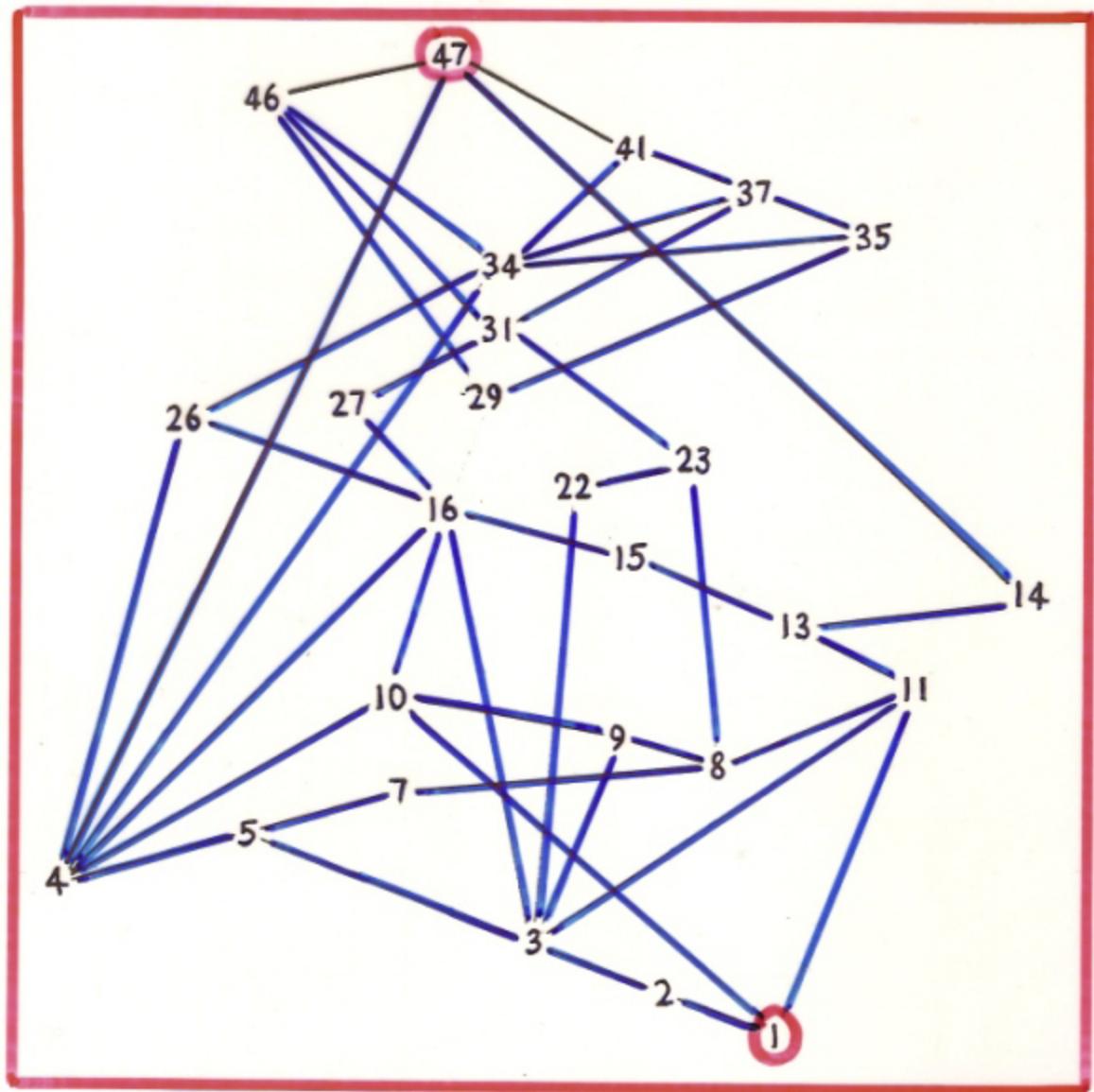
The famous Thales is said to have been the first to demonstrate that the circle is bisected by the diameter.

If you wish to demonstrate this mathematically, imagine the diameter drawn and one part of the circle fitted upon the other.

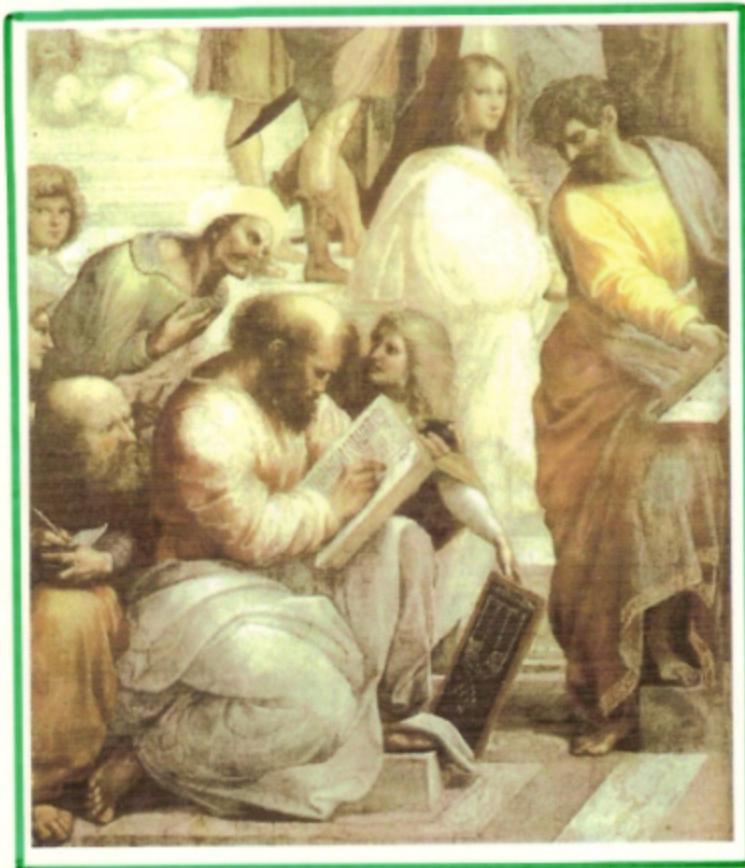
If it is not equal to the other, it will fall either inside or outside it, and in either case it will follow that a shorter line is equal to a longer. For all the lines from the centre to the circumference are equal, and hence the line that extends beyond will be equal to the line that falls short, which is impossible.



Hierarchies in Euclid Book I



Pythagoras (c. 572 - 497 BC)



YOU MAY BE RIGHT, PYTHAGORAS,
BUT EVERYBODY'S GOING TO LAUGH
IF YOU CALL IT A "HYPOTENUSE."



Pythagoras's Theorem (I. 47)

PROPOSITION 47

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right;

I say that the square on BC is equal to the squares on BA, AC .

For let there be described on BC the square $BDEC$, and on BA, AC the squares GB, HC ;

[I. 46]

through A let AL be drawn parallel to either BD or CE , and let AD, FC be joined.

Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA , and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles; therefore CA is in a straight line with AG .

[I. 14]

For the same reason

BA is also in a straight line with AH .

And, since the angle DBC is equal to the angle FBA : for each is right:

let the angle ABC be added to each;

therefore the whole angle DBA is equal to the whole angle FBC . [C.N. 2]

And, since DB is equal to BC , and FB to BA ,

the two sides AB, BD are equal to the two sides FB, BC respectively;

and the angle ABD is equal to the angle FBC ;

therefore the base AD is equal to the base FC ,

and the triangle ABD is equal to the triangle FBC . [I. 4]

Now the parallelogram BL is double of the triangle ABD , for they have the same base BD and are in the same parallels BD, AL . [I. 41]

And the square GB is double of the triangle FBC , for they again have the same base FB and are in the same parallels FB, GC .

[I. 41]

[But the doubles of equals are equal to one another.]

Therefore the parallelogram BL is also equal to the square GB .

Similarly, if AE, BK be joined,

the parallelogram CL can also be proved equal to the square HC ;

therefore the whole square $BDEC$ is equal to the two squares GB, HC . [C.N. 2]

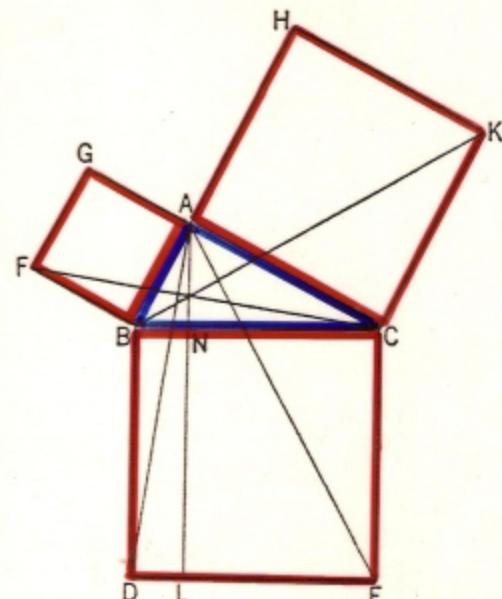
And the square $BDEC$ is described on BC ,

and the squares GB, HC on BA, AC .

Therefore the square on the side BC is equal to the squares on the sides BA, AC .

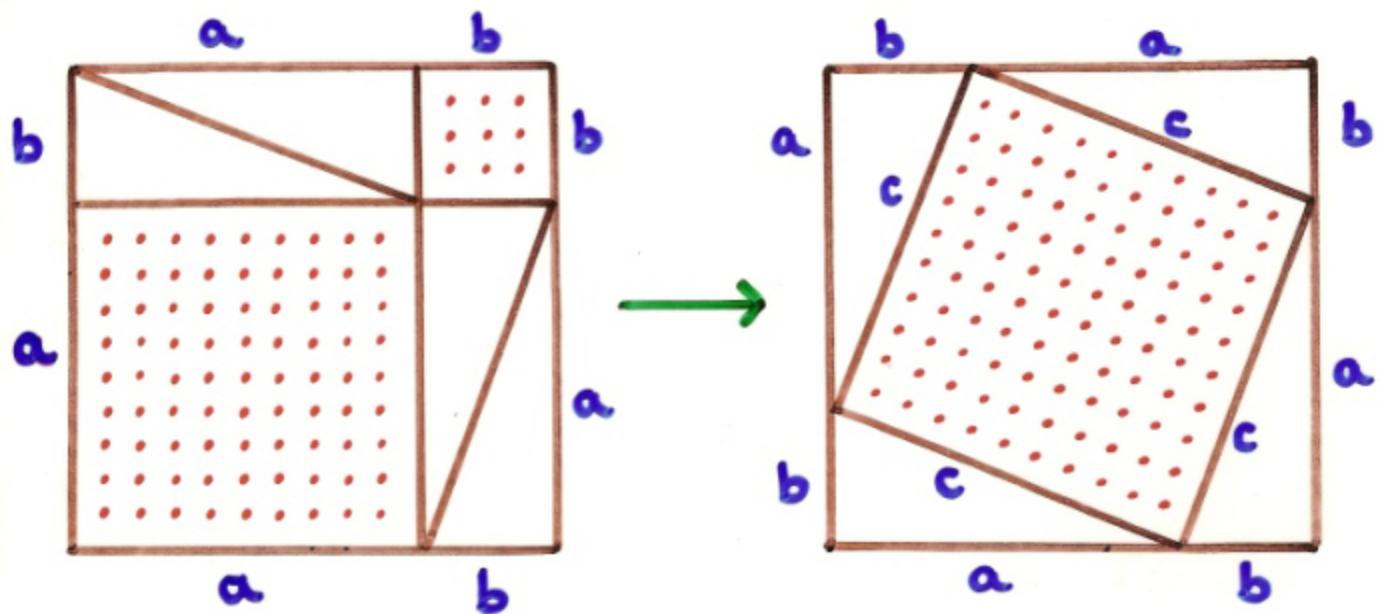
Therefore etc.

Q. E. D.



Pythagoras's Theorem

In any right-angled triangle,
the square on the hypotenuse is equal to
the sum of the squares on the other two sides



Zhou-bei suanjing

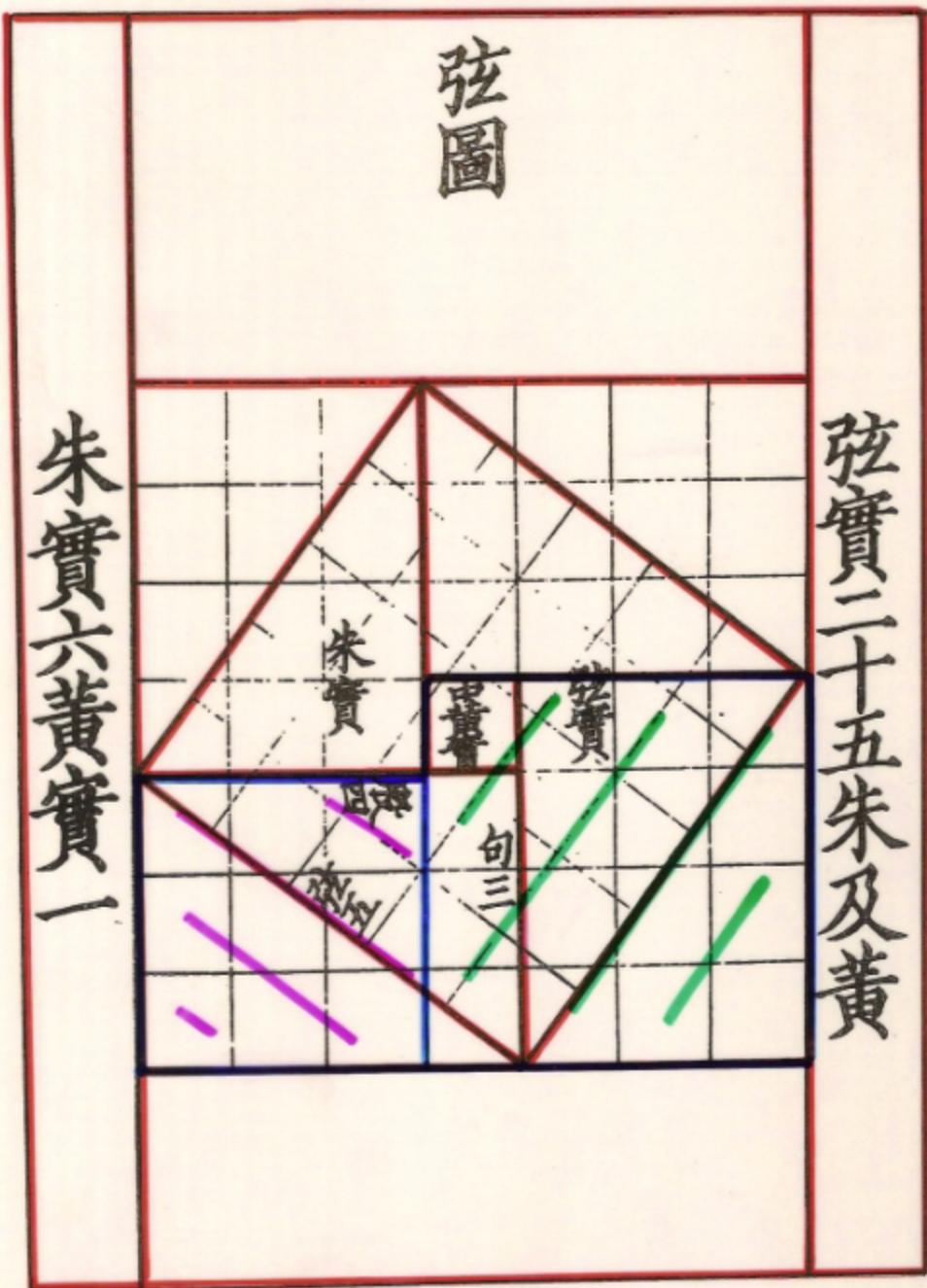
(The arithmetical classic of the gnomon...)



Dissection proof of 'Pythagoras's Theorem'

Zhou-bei suanjing

(The arithmetical classic of the gnomon...)



Dissection proof of 'Pythagoras's Theorem'

Chinese bamboo problem

Broken bamboo 10 feet high.

Top end is 3 feet from the stem.

$$x+y=10$$



$$x^2 + 3^2 = y^2$$

高數也

折抵地爲弦以句及股弦并求股故先令句自乘見矩
幕令如高而一凡爲高一丈爲股弦并之以除此幕得
差所得以減竹高而半其餘卽折者之高也此率與係
索之類更相返覆也亦可如上術令高自乘爲股弦并
幕去本自乘爲矩幕減之餘爲實倍高爲法則得折之

股弦和與勾求股法曰勾自乘爲實變股弦較乘股弦
和如股弦和而一正除得股弦較以減股弦和餘二段

Raphael's 'School of Athens'

→
Plato and
Aristotle

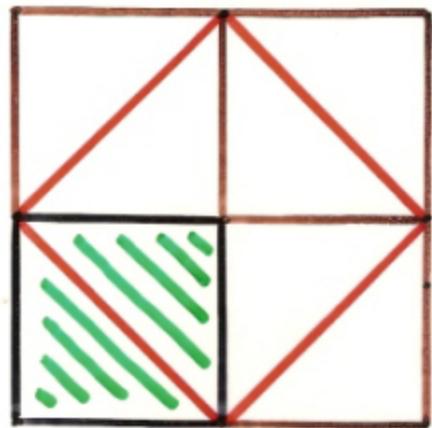


Plato's Academy (387 BC)

'Let no-one ignorant of geometry enter these doors.'

Meno:

Socrates and the slave boy : double the size of the square ...

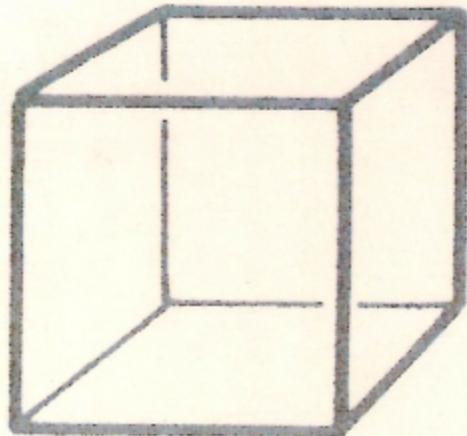


Republic:

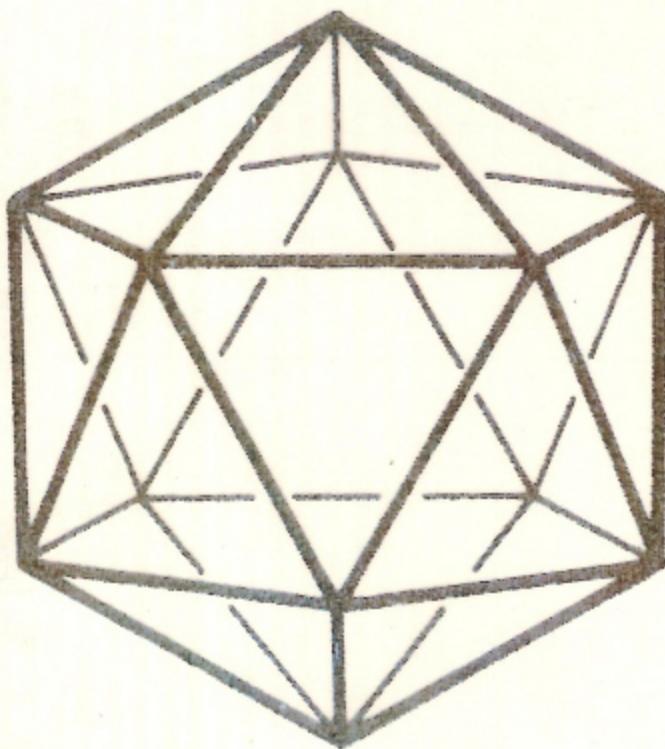
Mathematical studies for the philosopher ruler :

Arithmetic - Plane Geometry -
Solid Geometry - Astronomy - Harmonics

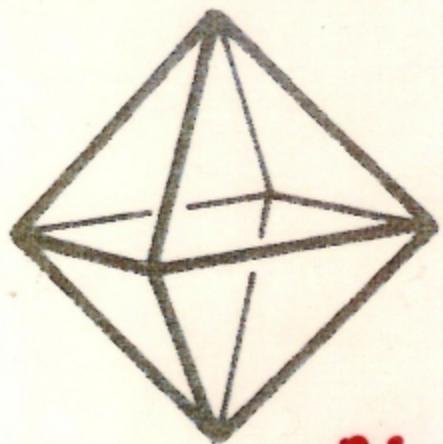
Plato's
'Timaeus'



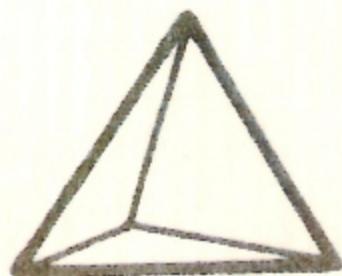
Earth



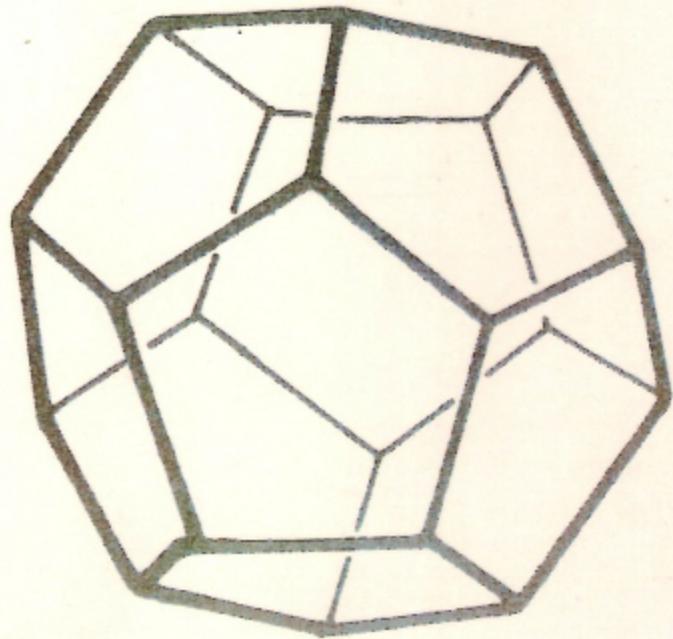
Water



Air

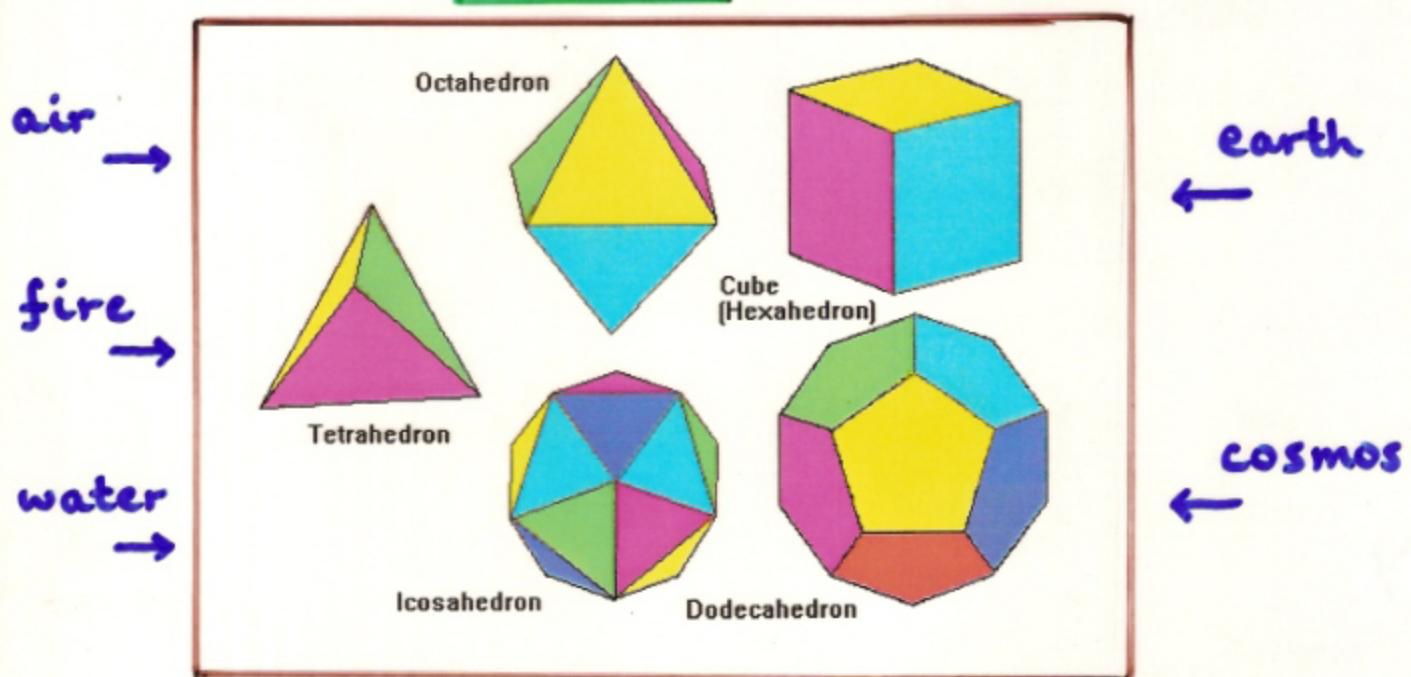


Fire

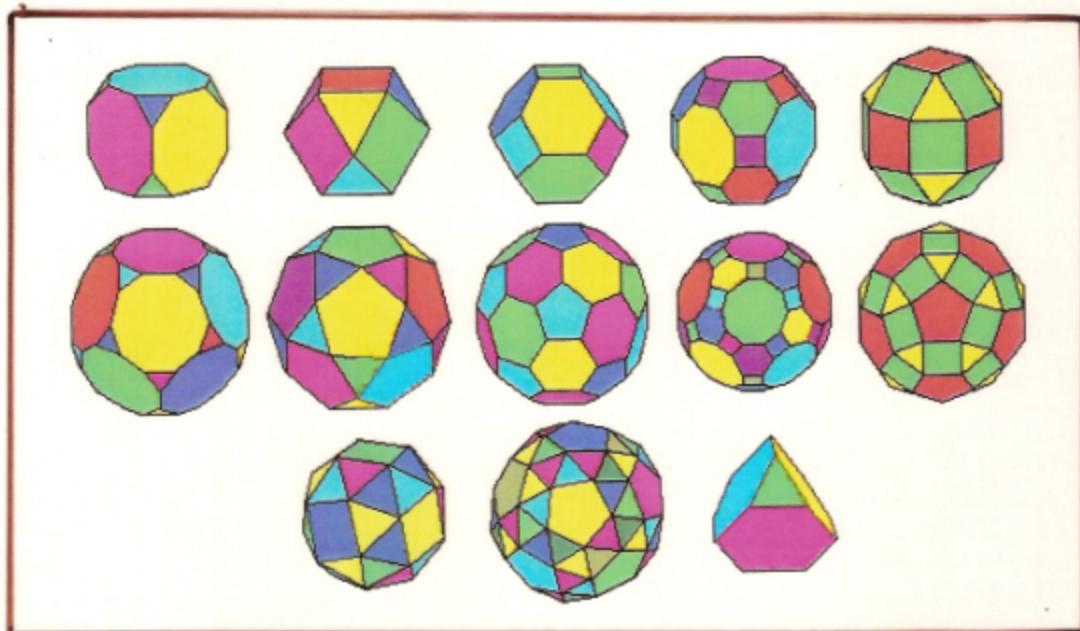


Cosmos

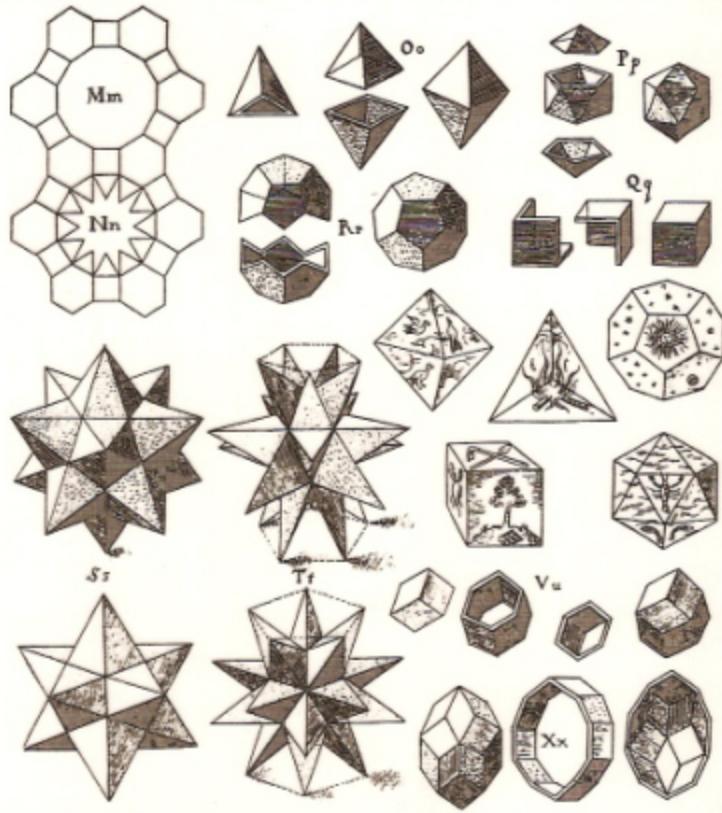
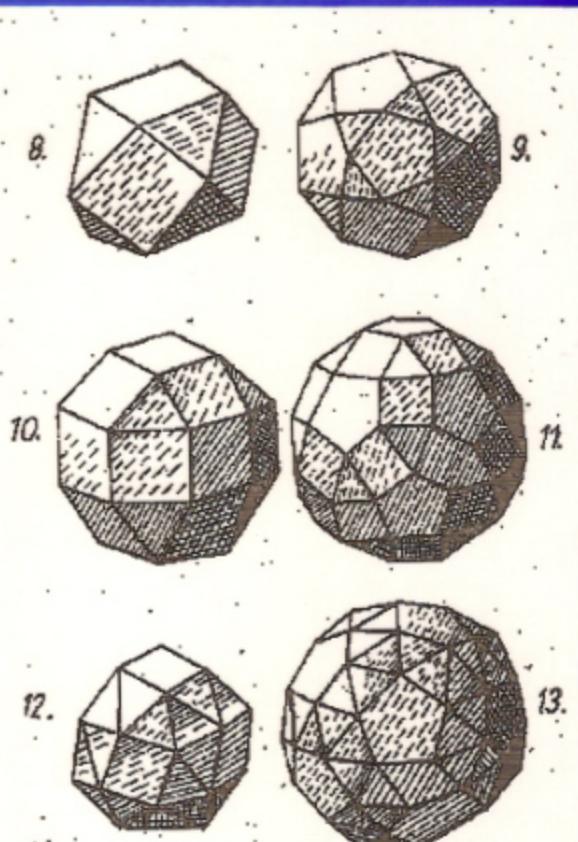
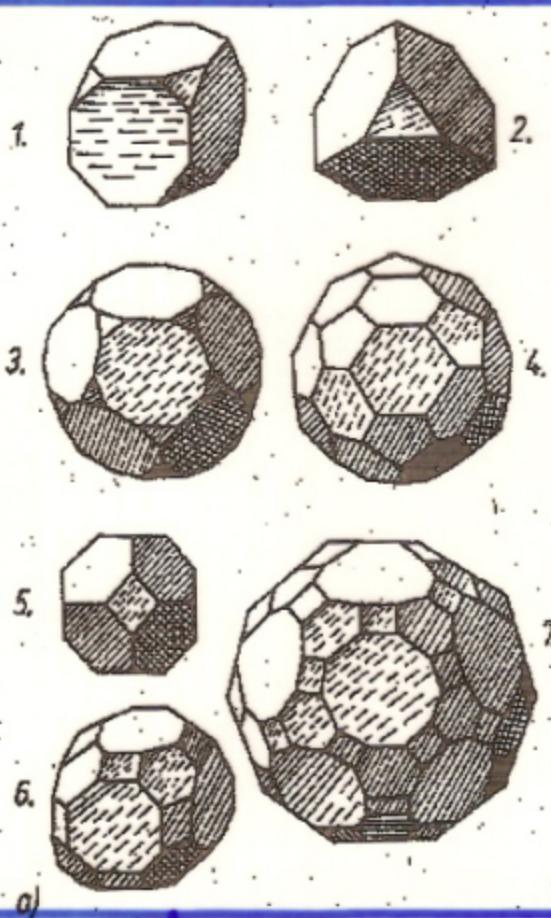
Platonic (regular) solids [Timaeus]



Archimedean (semi-regular) solids



Kepler's Polyhedra

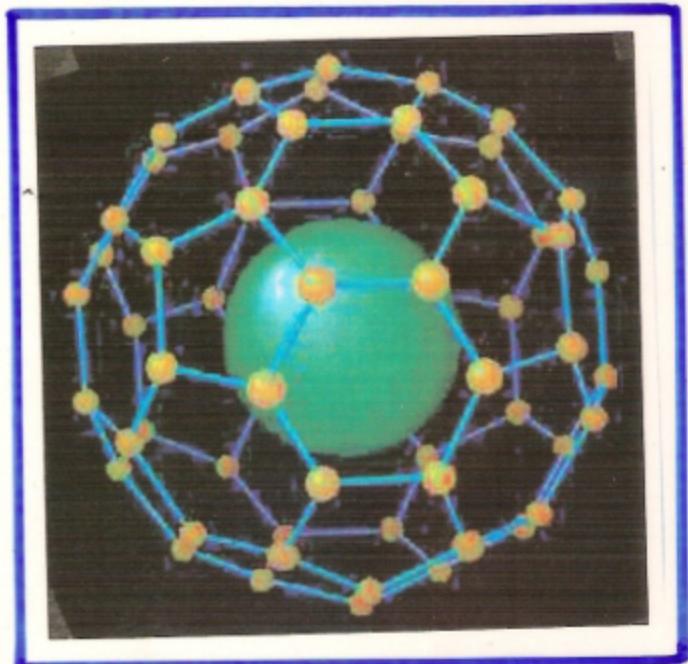
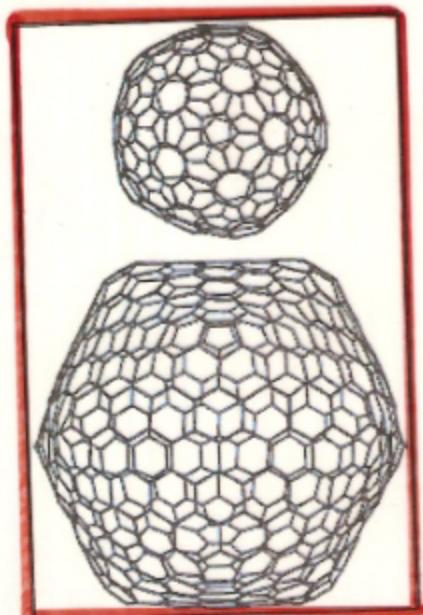


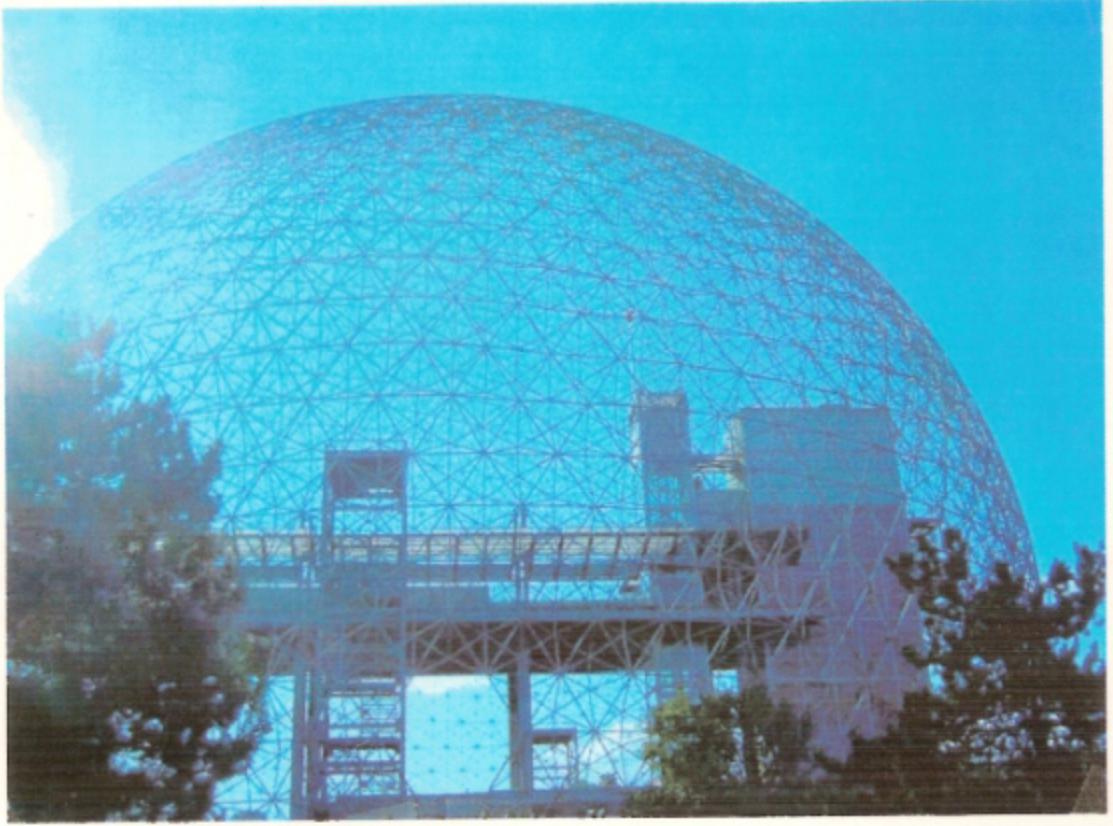
Fullerenes

(Buckyballs)

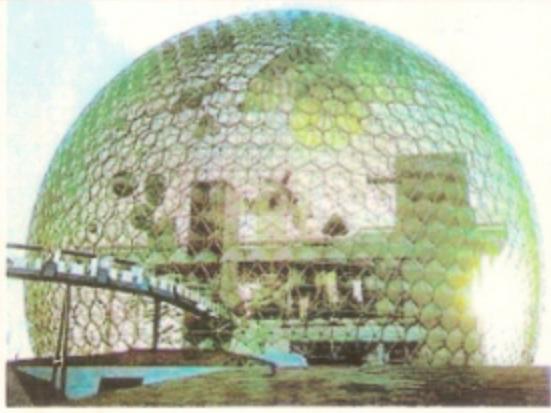
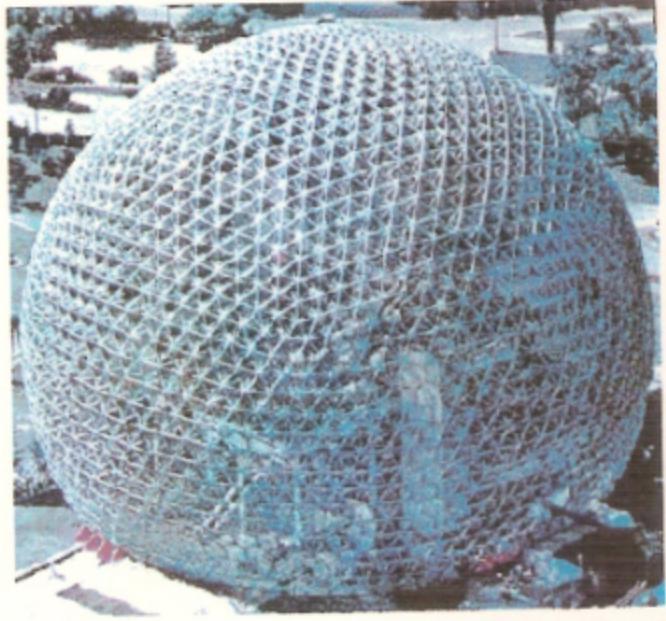
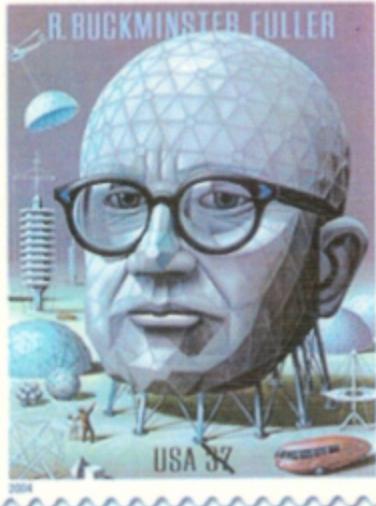


truncated
icosahedron



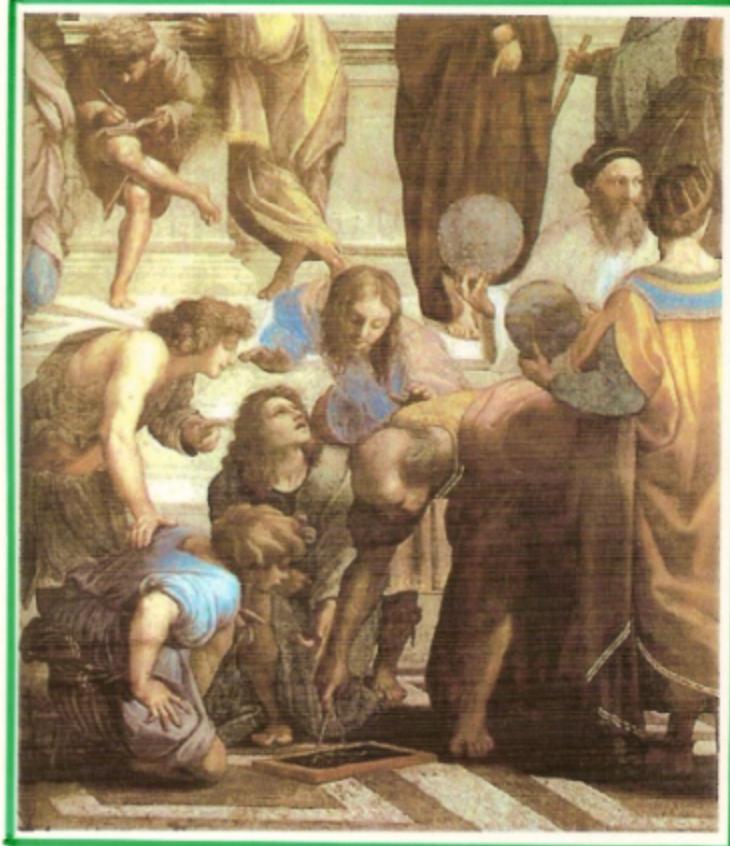
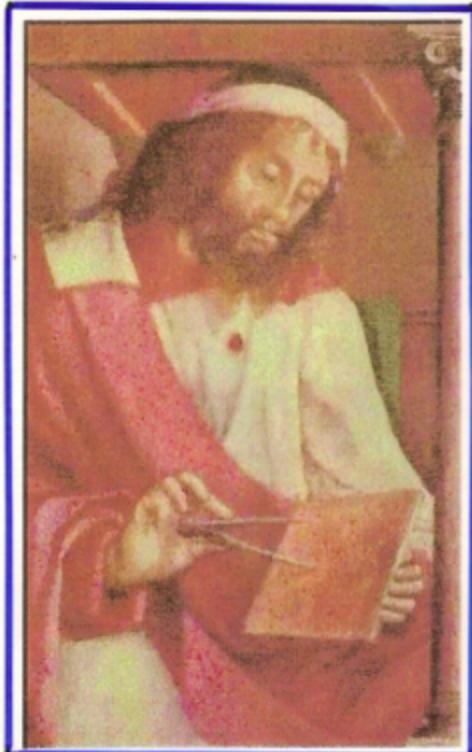


geodesic dome



Expo 67

Montreal



The image shows the title page and the beginning of Euclid's Megaren. The title 'EVCLIDIS MEGAREN' is at the top, followed by a detailed description of the work. Below this is a large initial 'P' with a decorative border. The text continues with definitions of geometric terms like 'Angulus planus', 'Angulus rectus', and 'Angulus obtusus'. There are also diagrams of various geometric shapes and figures.



Euclid

HERE'S LOOKING
AT EUCLID



Euclid's 'Elements'

- I Foundations of plane geometry
- II The geometry of rectangles
- III The geometry of the circle
- IV Regular polygons in circles
- V Magnitudes in proportion
- VI Geometry of similar figures

- VII Basic arithmetic
- VIII Numbers in continued proportion
- IX Even and odd numbers / perfect numbers
- X Incommensurable line segments

- XI Foundations of solid geometry
- XII Areas and volumes (Eudoxus)
- XIII The Platonic solids

Euclid's Postulates

Let the following be postulated :

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



$$\alpha + \beta < 180^\circ \Rightarrow \text{lines meet}$$

Euclid Book I, Prop. I

On a given straight line
to construct an equilateral triangle

Let AB be the line.

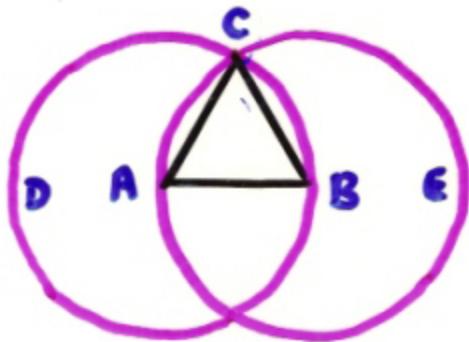
With centre A and
distance AB, draw the
circle BCD. [Post. 3]

With centre B and distance BA,
draw the circle ACE. [Post. 3]

Join AC and BC. [Post. 1]

Then the triangle ABC is equilateral.

Proof . . .



Euclid Book I

I.5



The base angles of an isosceles triangle are equal.

I. 32

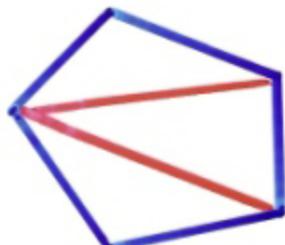


The angles of a triangle add up to 2 right angles.

I. 45



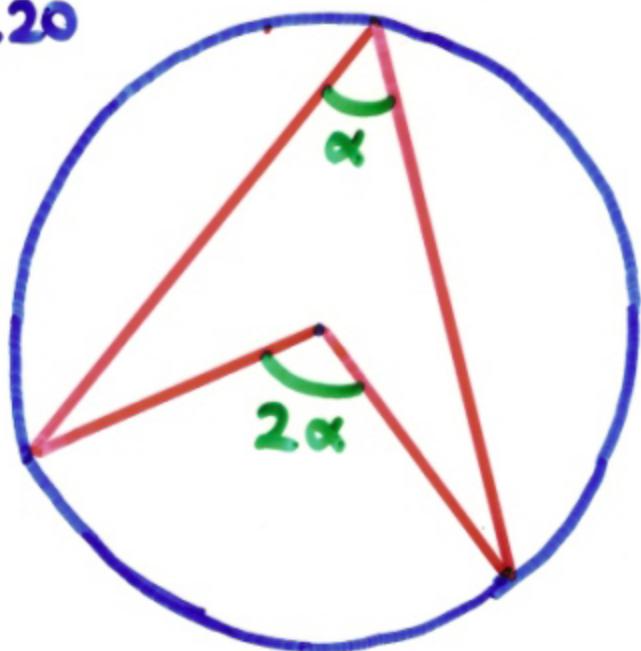
Given any triangle, construct a rectangle with the same area.



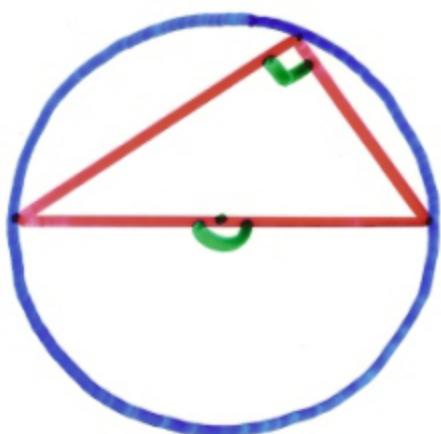
Similarly for any polygon...

Euclid : Book III

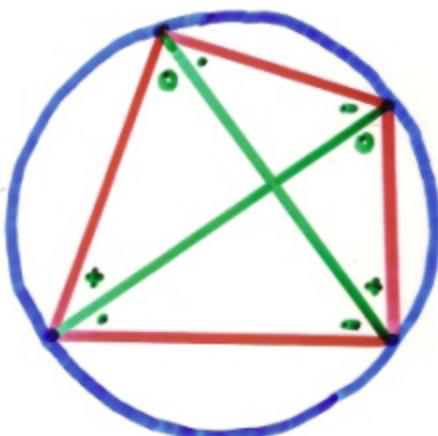
III.20



III.31



III.22

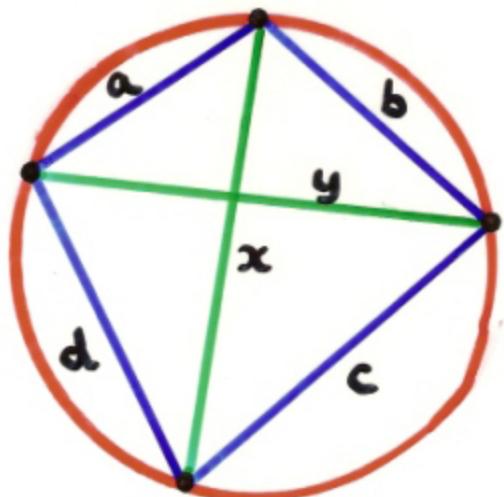


Brahmagupta : Cyclic Quadrilaterals

Area A :

If $s = (a + b + c + d)/2$, then

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$



Diagonals :

$$x^2 = \frac{(ad+bc)(ac+bd)}{ab+cd}, \quad y^2 = \frac{(ab+cd)(ac+bd)}{ab+cd}.$$

Pythagorean triples :

If $a^2 + b^2 = c^2$ and $A^2 + B^2 = C^2$,

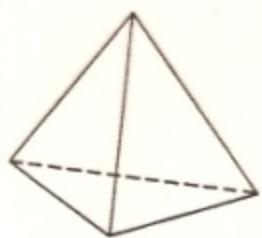
then aC, cB, bC and cA are the lengths of the sides of a cyclic quadrilateral;

e.g. $3^2 + 4^2 = 5^2$ and $7^2 + 24^2 = 25^2$

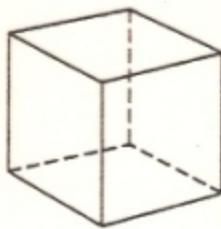
give 75, 120, 100, 35, or 15, 24, 20, 7.

Also, the diagonals are perpendicular.

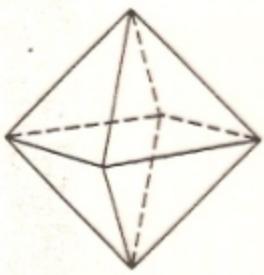
Five regular solids (Book XIII)



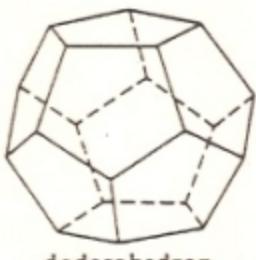
tetrahedron



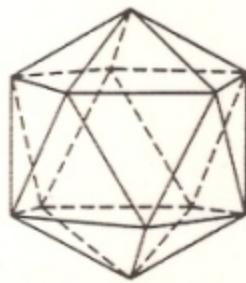
cube



octahedron

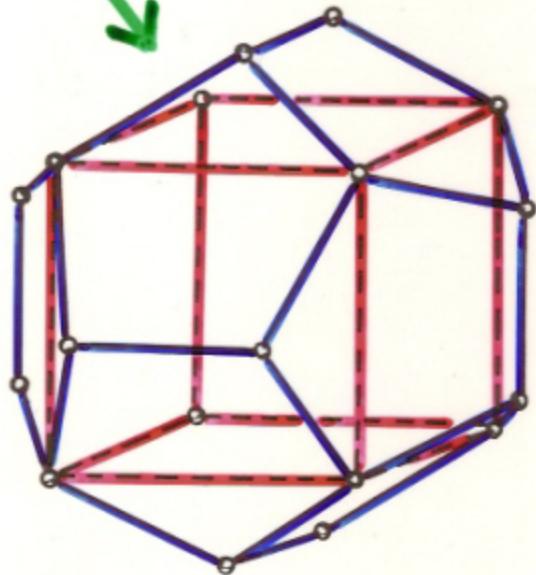
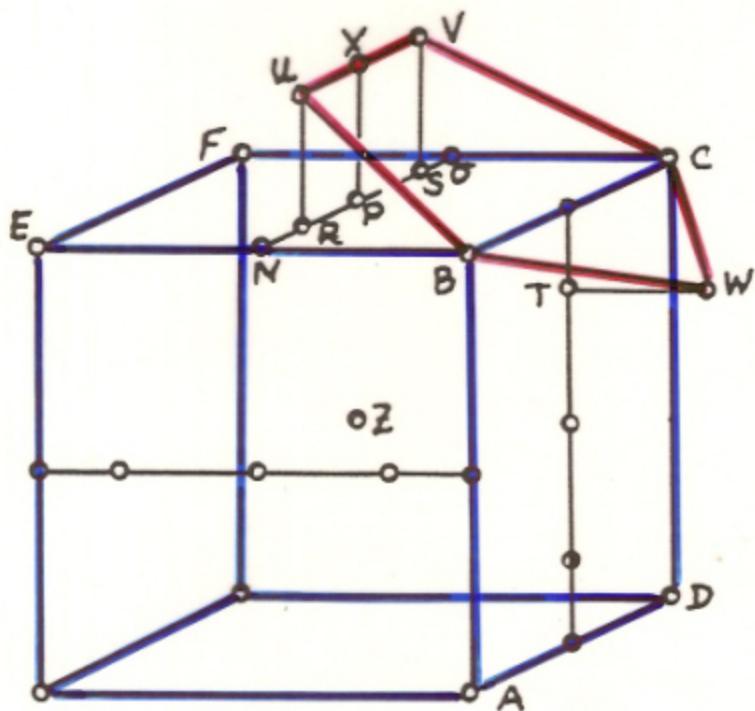


dodecahedron



icosahedron

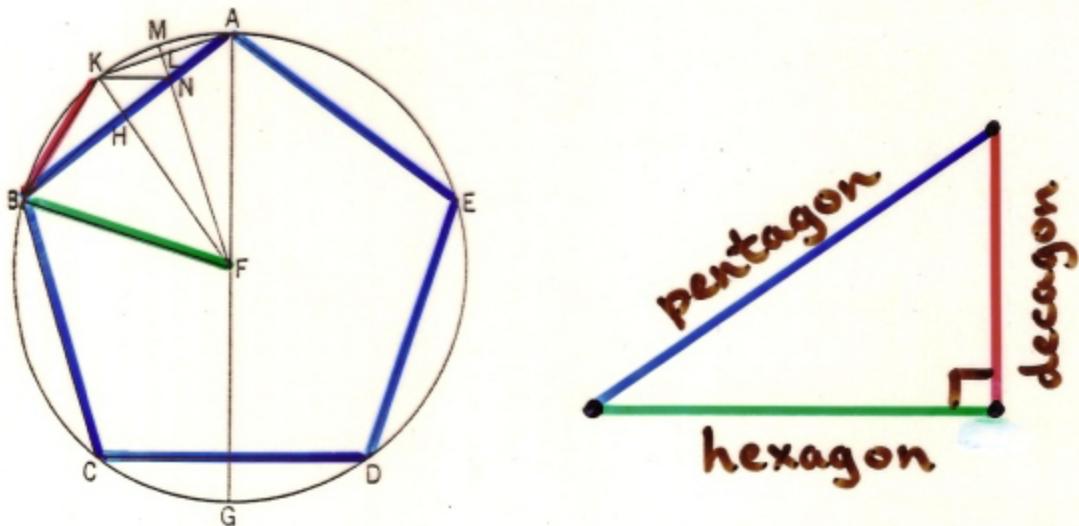
Constructing a dodecahedron from a cube



Results from Book XIII

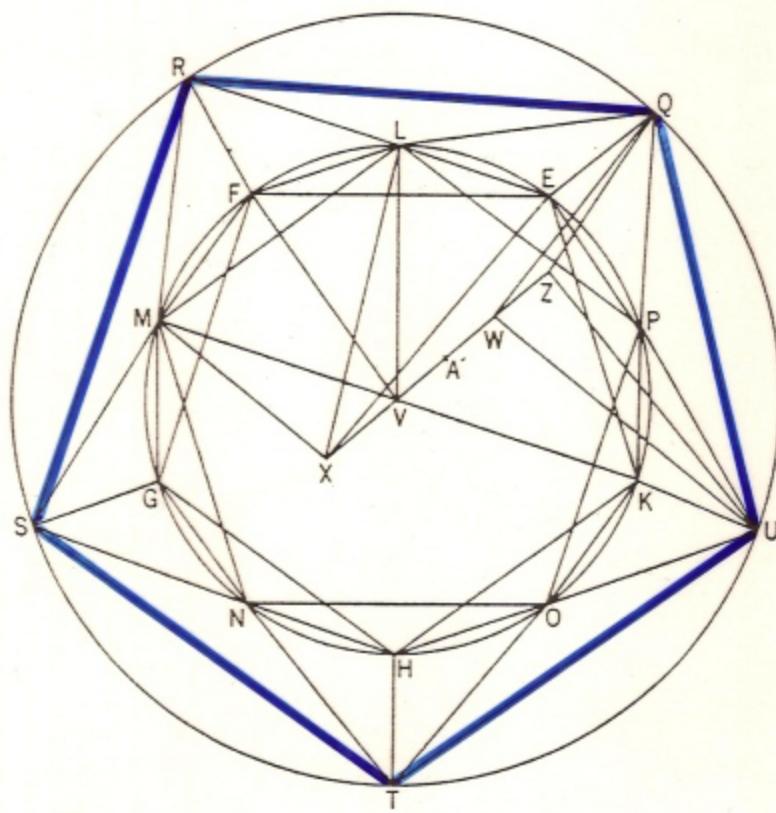
PROPOSITION 10

If an equilateral pentagon be inscribed in a circle, the square on the side of the pentagon is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the same circle.



PROPOSITION 16

To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures: and to prove that the side of the icosahedron is the irrational straight line called minor.



Euclid's 'Elements'

First printed edition, Venice, 1482

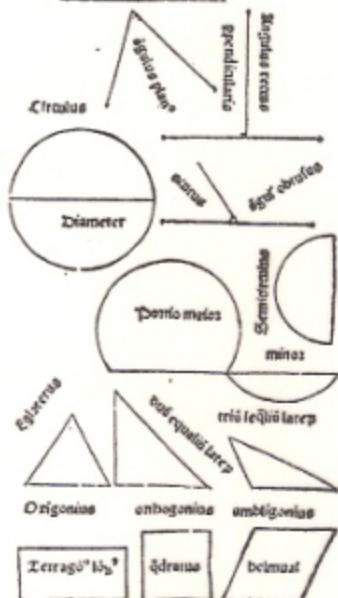
Preciarissimus liber elementorum Euclidis perspicacissimum artem Geometrie incipit quia Euclidis libri:
 Unus est cuius ps nō est. **L**inea est lōgitudo sine latitudine cui⁹ quidē extremitates sī duo pūcta. **L**inea recta ē ab uno pūcta ad aliū bīcūllima extēsio i extremitates suas vtrūqz eoz recipiens. **S**upficie ē q̄ lōgitudinē & latitudinē tm̄ bīcūlī termi quidē sūt linee. **S**upficies plana ē ab una linea ad aliam extēsio i extremitates suas recipiens. **A**ngulus planus ē duarū linearū alterius pūctus: quaz expāsio ē sup supficē applicatioqz nō directa. **A**Quādo aut angulum pīnct̄ due lincee recte rectilineū angulus noīat. **R**ēcta linea sup rectā sicuterit duoqz anguli vtrōbiqz fuerit cīles: eoz vierqz rectū crit. **L**ineaqz lincee supstās cī cui supstat ppendicularis vocat. **A**n gulis vō qui recto maior ē obtusus dicit. **A**ngulūqz minor recto acutū appellat. **C**terminū ē qd ymūlicūlīqz hīnis ē. **F**igura ē q̄ mino vī terminū pīnct̄. **C**irculū ē figura plana una qdcm linea pītē: q̄ circūferentia nota: in cui⁹ mediu pūctū ē: a quo oēs lincee recte ad circūferētiā exētes sibiūces sūt equalēs. Et bic quidē pūctū cītrū circuli dī. **D**iameter circuli ē linea recta que sup ei cīnt̄ trāiens extremitatēqz suas circūferētiā applicans circulū i duo media diuidit. **S**emicirculus ē figura plana dia/ metro circuli & mediate circūferētiā pīnct̄a. **P**ortio circuli ē figura plana recta linea & parte circūferētiā pītē: lemicirculo quidē aut maior aut minor. **R**ectilinee figure sūt q̄ rectis lincis cōtinent quarū quedā trilaterū q̄ trib⁹ rectis lincis: quedā quadrilaterū q̄ quatuor rectis lincis. qdā m̄bilatere que pluribus qz quatuor rectis lincis continent. **F**igurarū trilaterarū: alia est triangulus hīis tria latera equalia. Alia triangulus duo hīis cīlia latera. Alia triangulus triū inequalium laterū. Hāz iterū alia est orthogonū: vñ. I. rectum angulum habens. Alia ē am̄bigonū aliquem obtusum angulum habens. Alia est oxigonū unū: qua tres anguli sunt acuti. **F**igurarū autē quadrilaterarū: Alia est qdramū quod est equilaterū atqz rectangularū. Alia est tetragonū longū: q̄ est figura rectangularia: sed equilateralia non est. Alia est belunaym: que est equilatera: sed rectangularia non est.

De pīncipijs p se notia: et pīmo de diffīlētib⁹ carandēm.

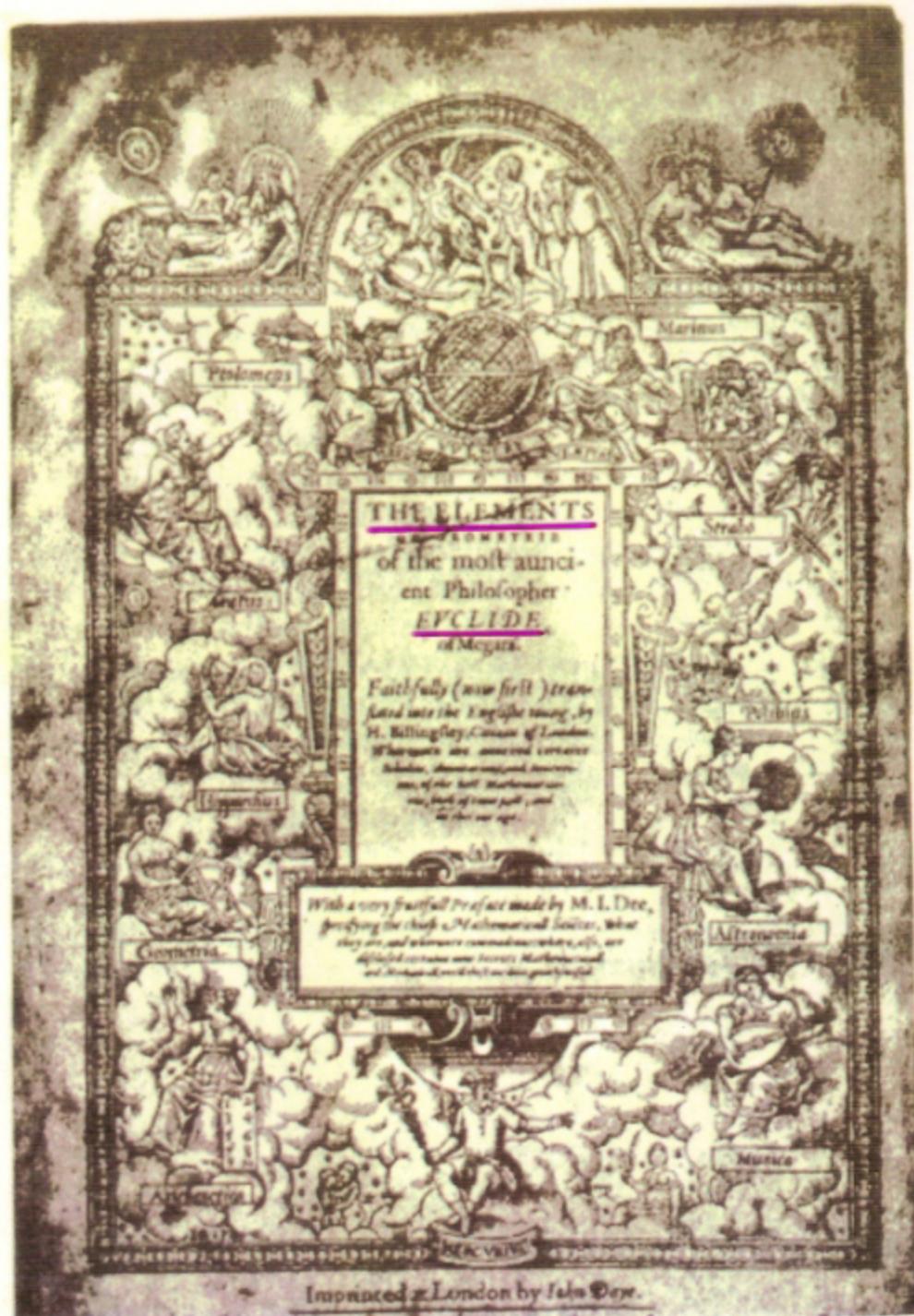
Linea

Punctus

Supficies plana.

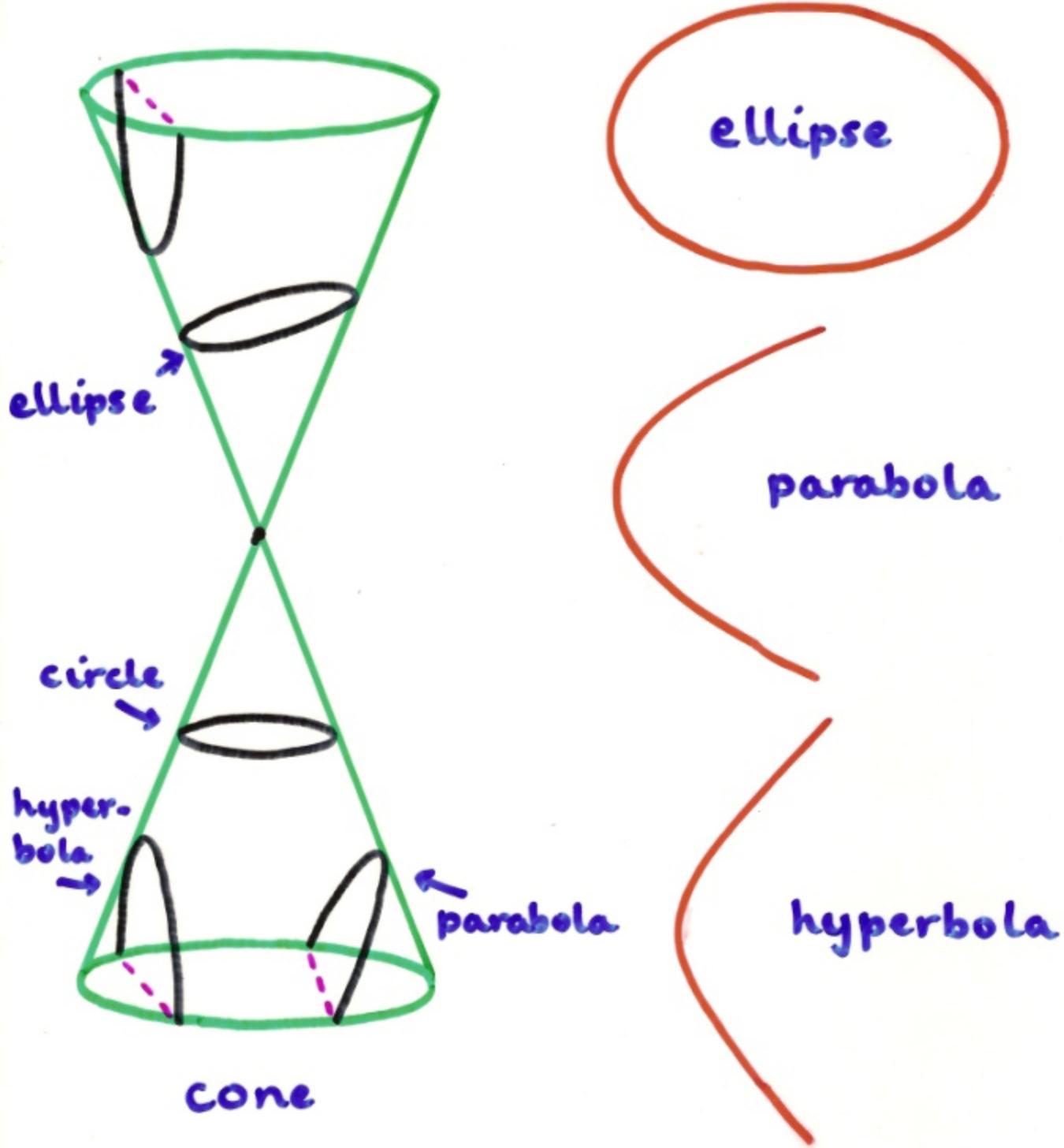


First English translation



Henry Billingsley (1570)

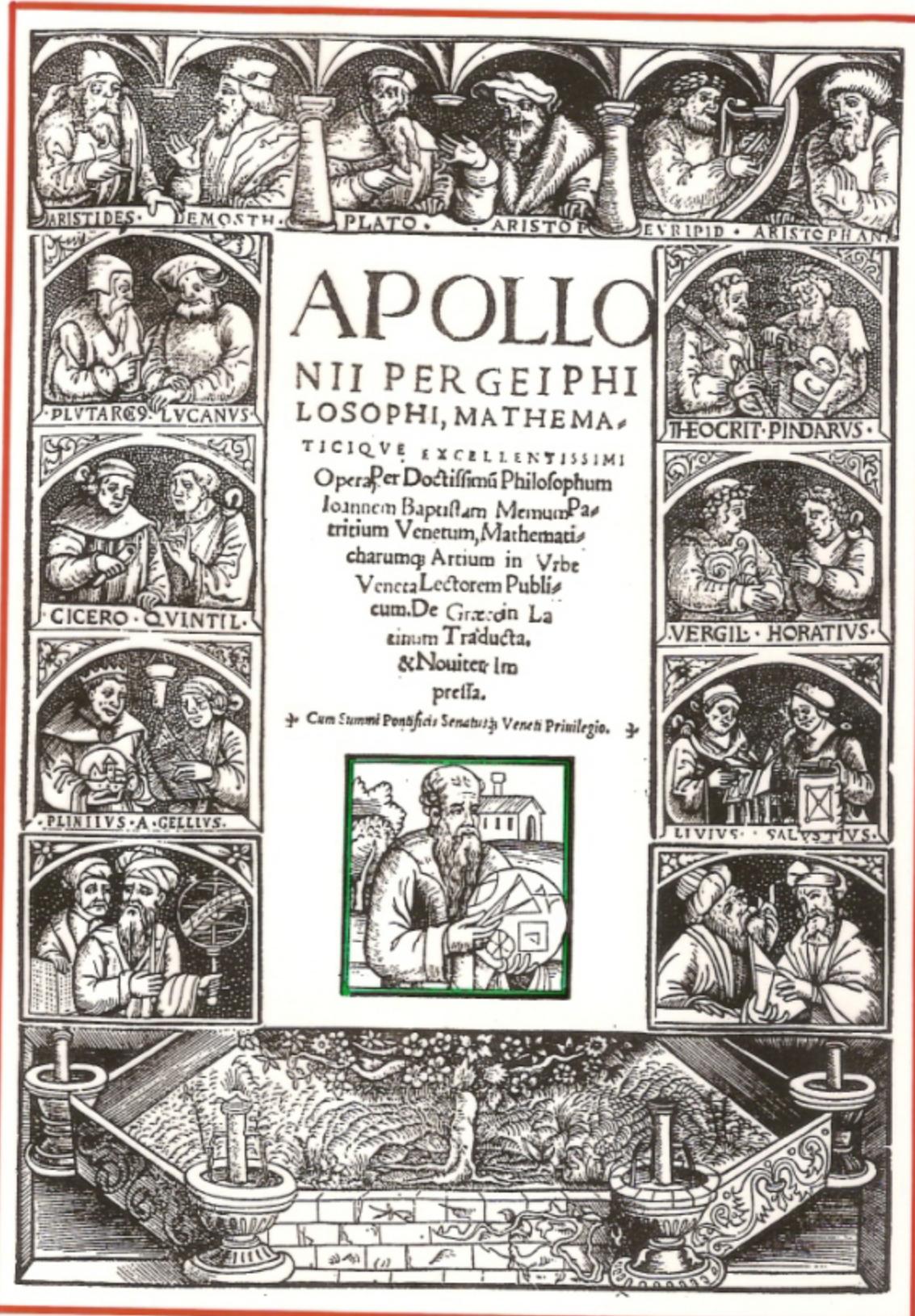
Conic Sections



Menaechmus (4th century BC)

Apollonius ('Conics': 250 BC)

Apollonius' Conics (1537)



Apollonius' Conics (Halley, 1710)



*Aristippus Philosophus Socratus, naufragio cum ejectus ad Rhodiensem
litus animadvertisset Geometrica schemata descripta, exclamavisse ad
comites ita dicitur, Bene speremus, Hominum enim vestigia video.*

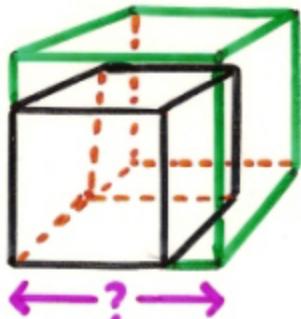
Vitruv. Architect. lib. 6. Prit.

John Bulwer sculpsit. Oxon.

The Three Classical Problems

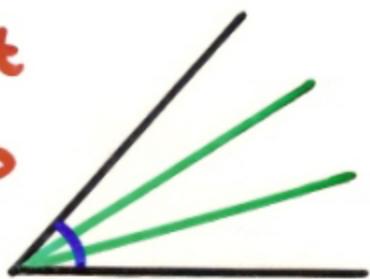
1. Doubling the Cube

Given a cube, construct another with twice the volume.



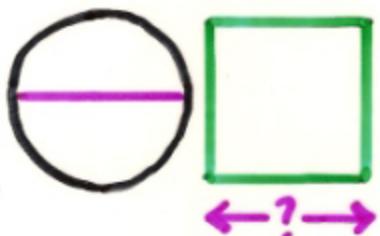
2. Trisecting the Angle

Given any angle, construct two lines that divide it into three equal parts.



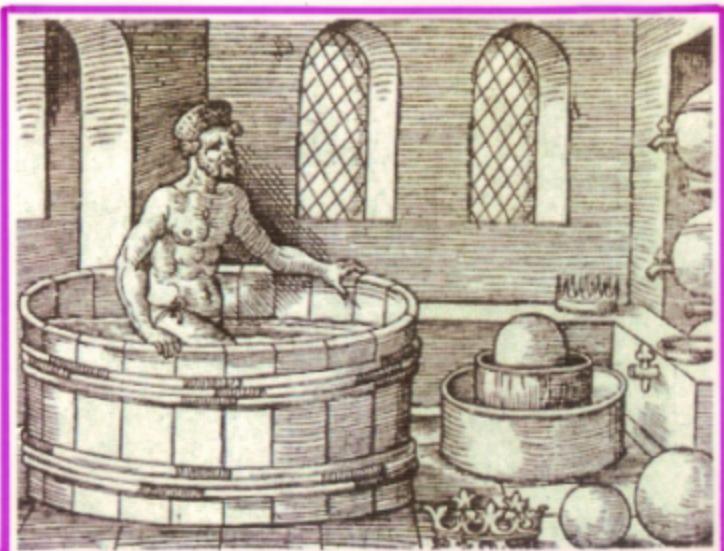
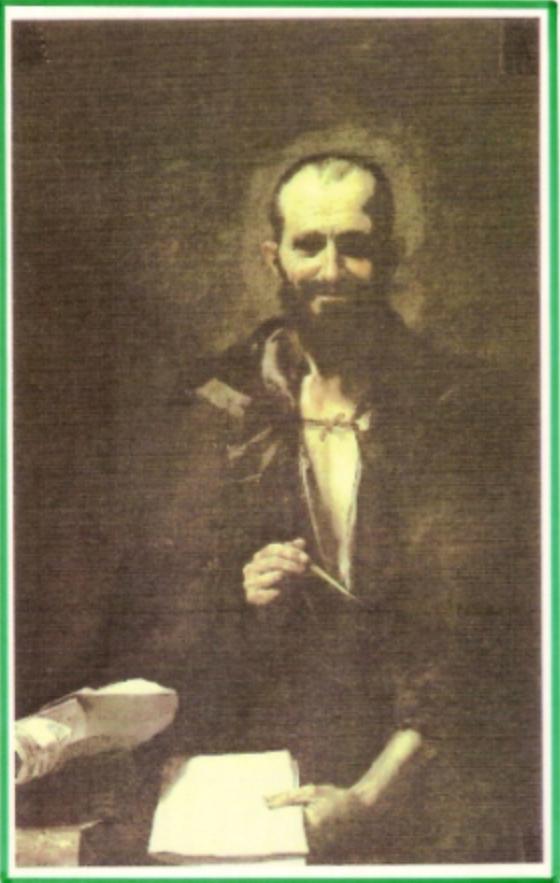
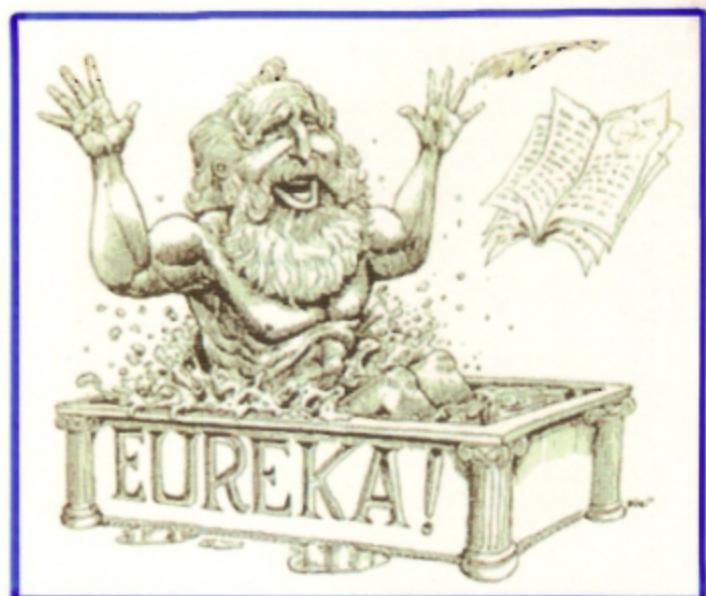
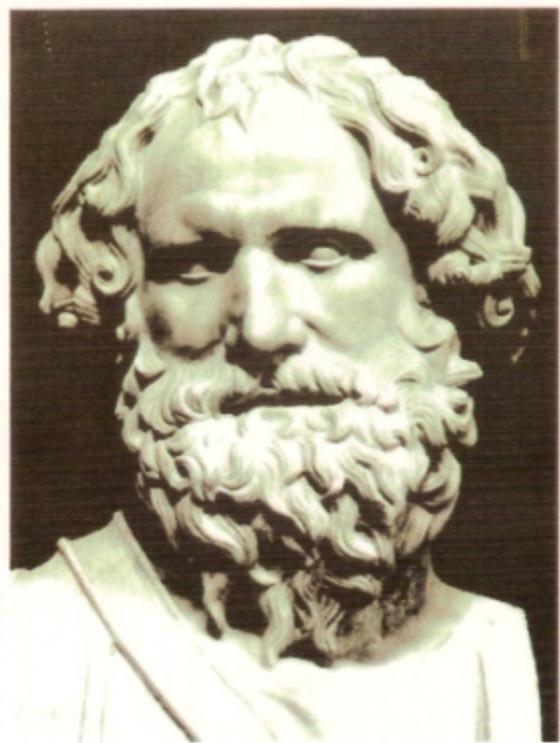
3. Squaring the Circle

Given a circle, construct a square with the same area.



Archimedes

(c. 287 - 212 BC)



Archimedes' Geometry

Centres of gravity:

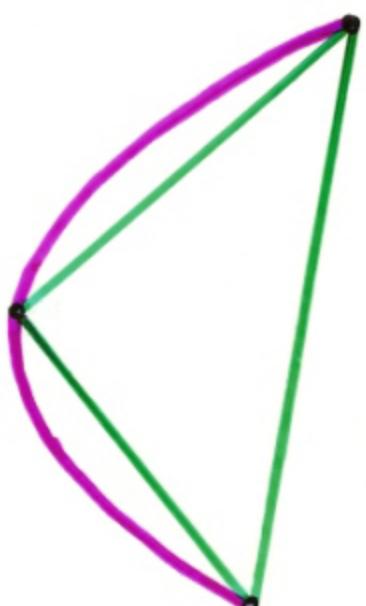
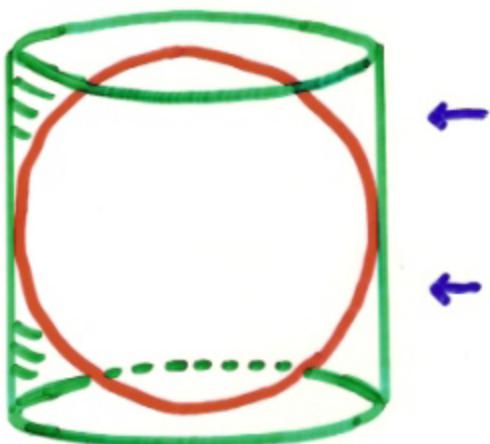
triangle, hemisphere, parallelogram

Volumes:

sphere, paraboloid

Surface areas:

cone, sphere

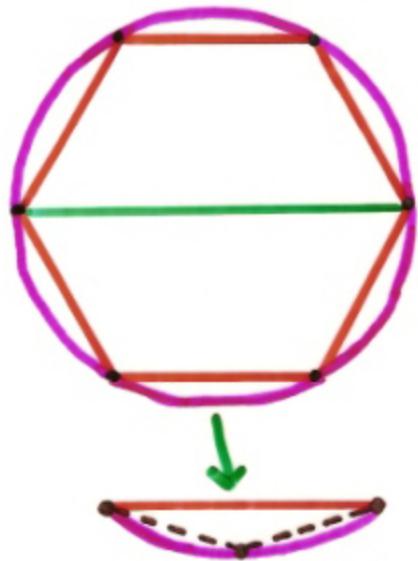


area of parabolic segment

$$= \frac{4}{3} \times (\text{area of triangle})$$

[quadrature of the parabola]

The Value of π



perimeter of inscribed 6-gon
< circumference of circle
< perimeter of exscribed 6-gon

double the number of sides:

6, 12, 24, 48, 96.

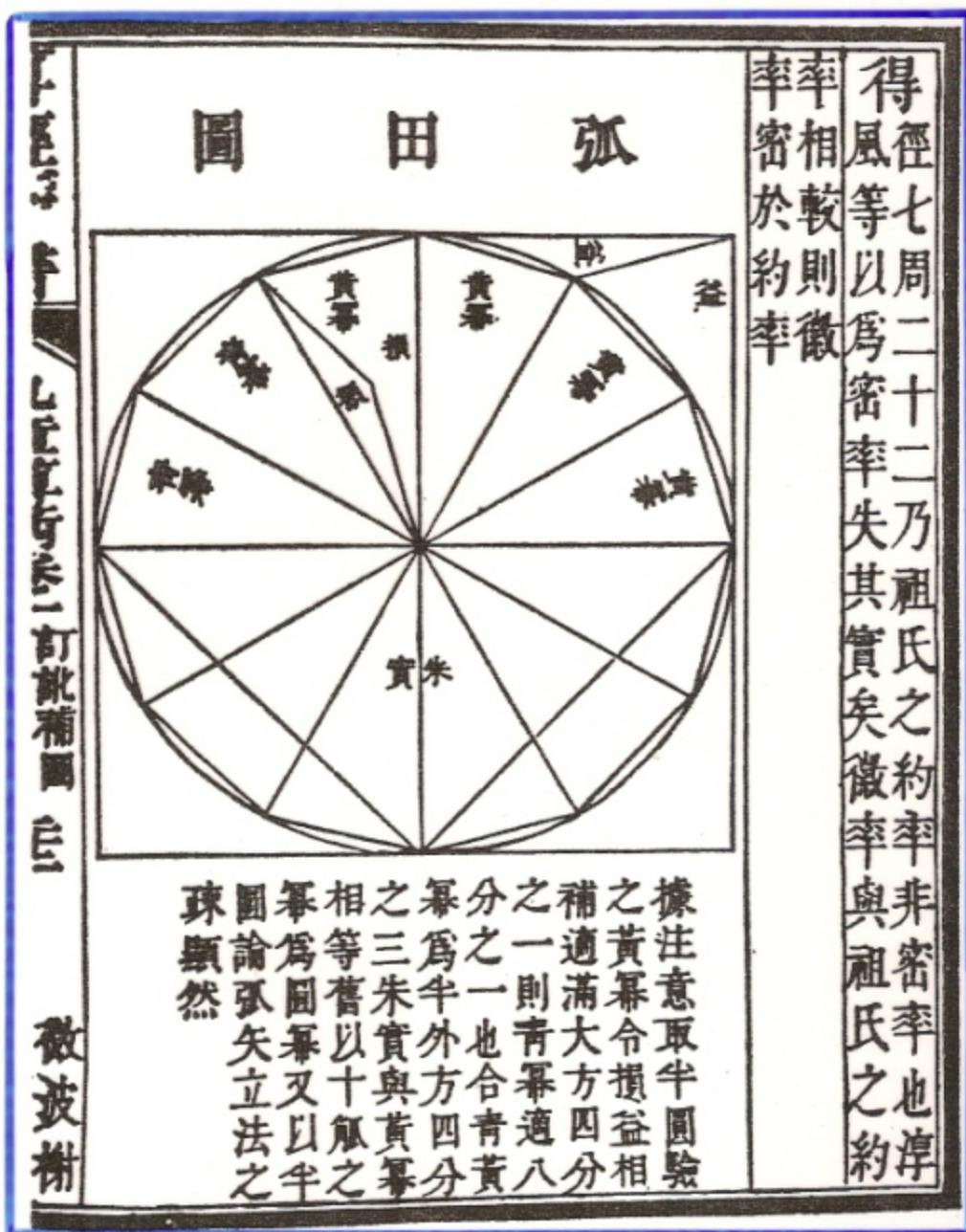
Archimedes obtained the estimates:

$$\underline{3\frac{10}{71} < \pi < 3\frac{1}{7}}$$

3.14084

3.14286

Chinese Values for π



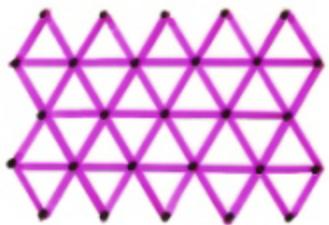
Zhang Heng (100 AD) : $\pi = \sqrt{10}$

Liu Hui (263 AD) : $\pi = 3.14159$ 3072

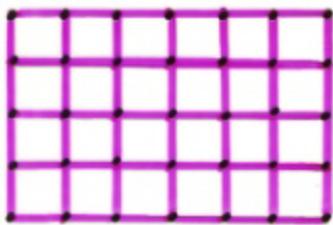
Zu Changzhi (500 AD) : $\pi = 3.1415926$ 24576
and $\pi \approx 355/113$

Pappus (early 4th century AD)

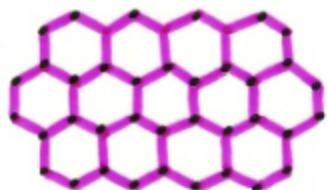
1. On the sagacity of bees



triangles

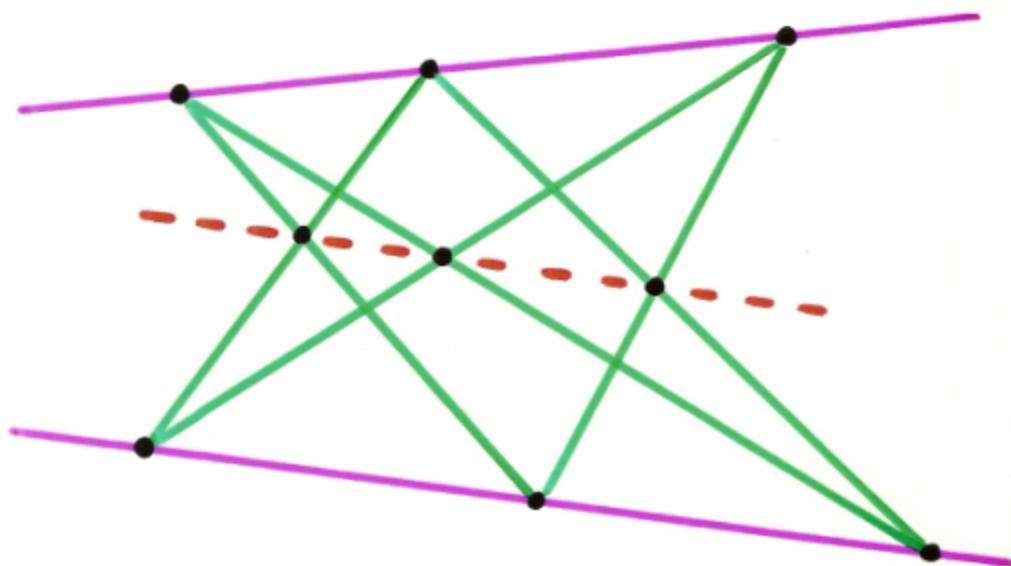


squares



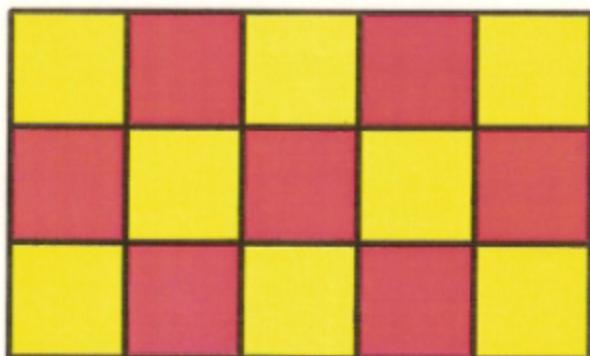
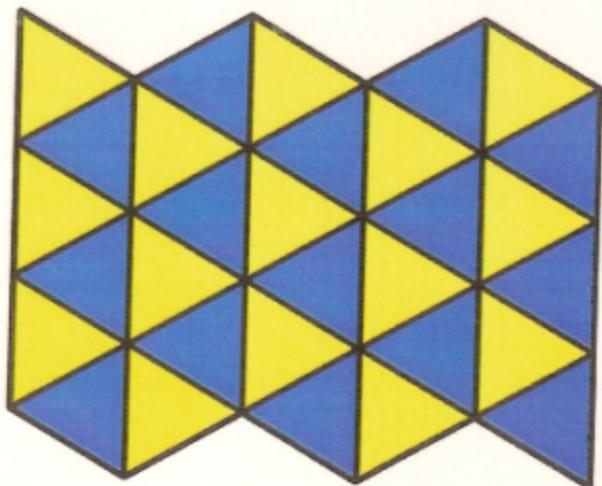
hexagons ✓

2. Pappus's theorem



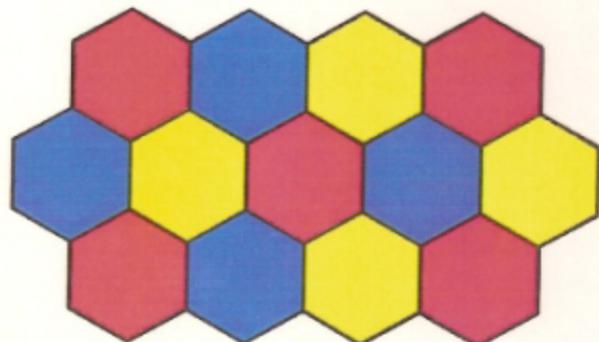
The three regular tilings

triangular

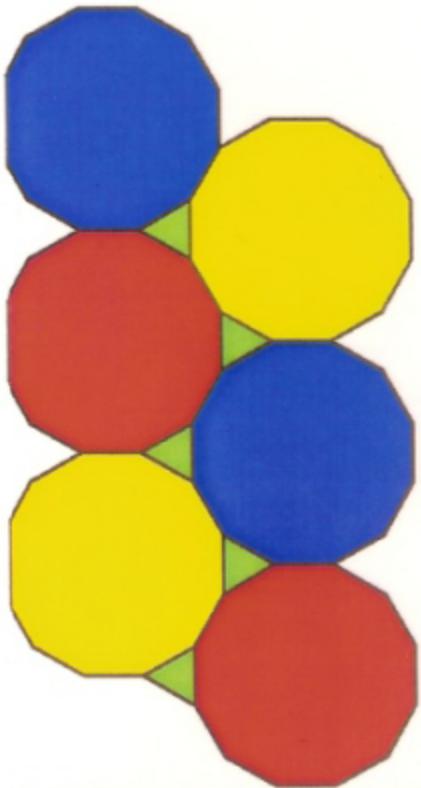
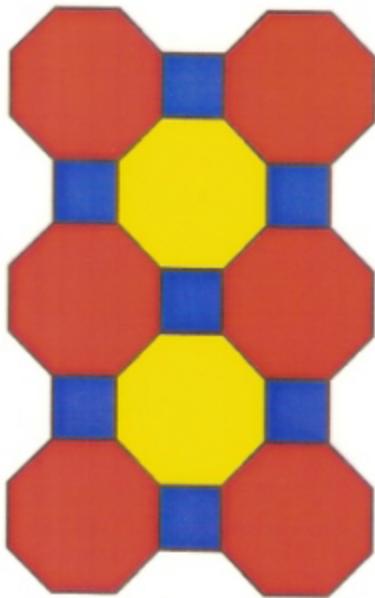
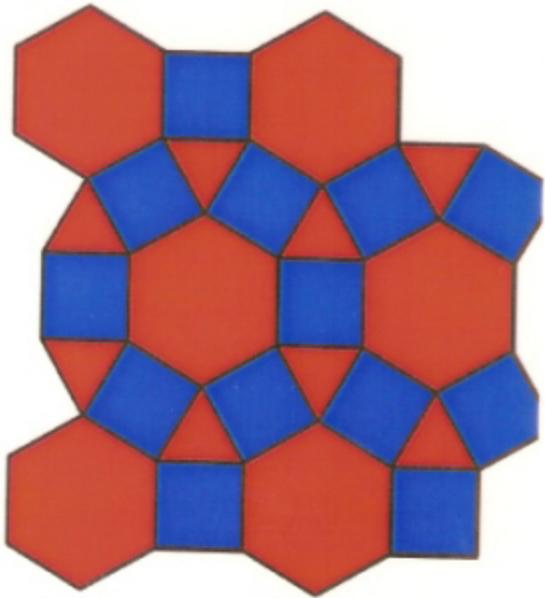
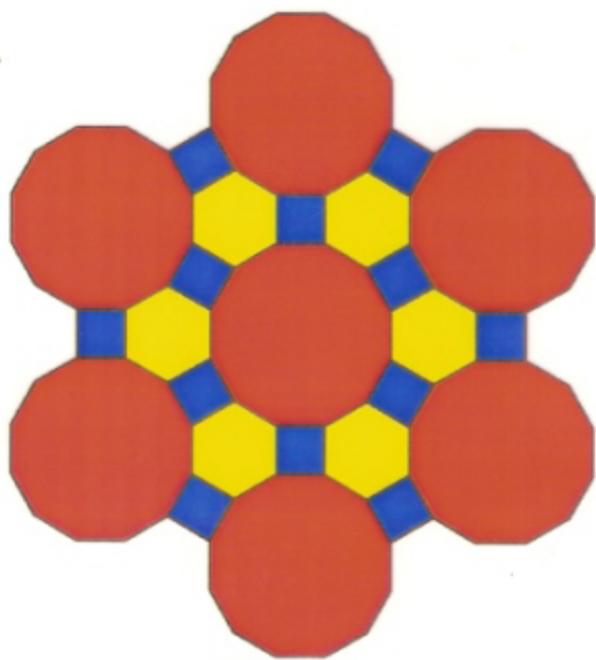


square

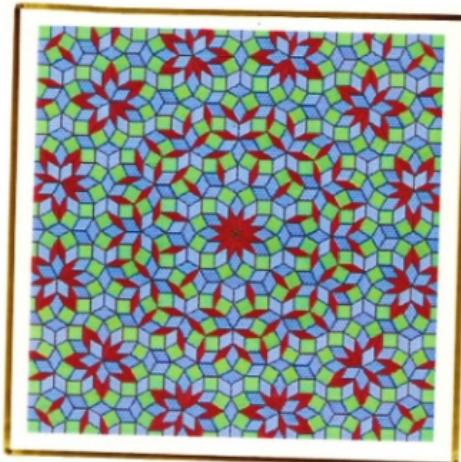
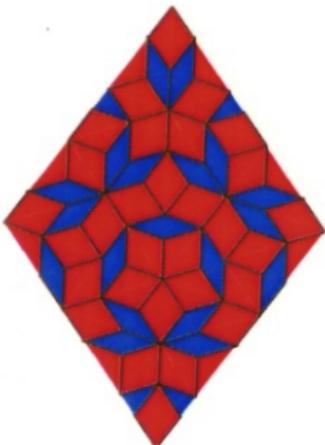
hexagonal



Semi-regular tilings

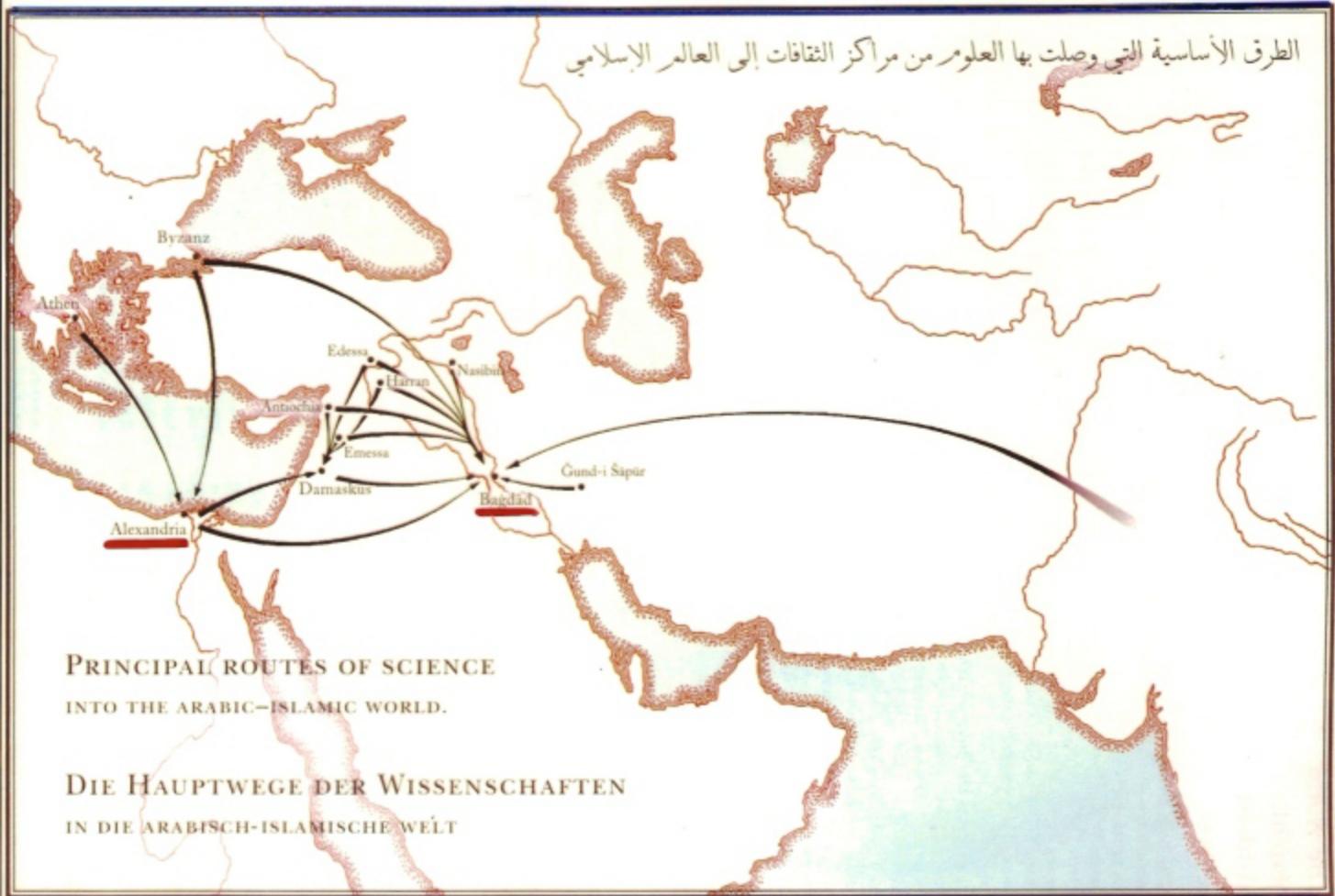


Penrose tilings

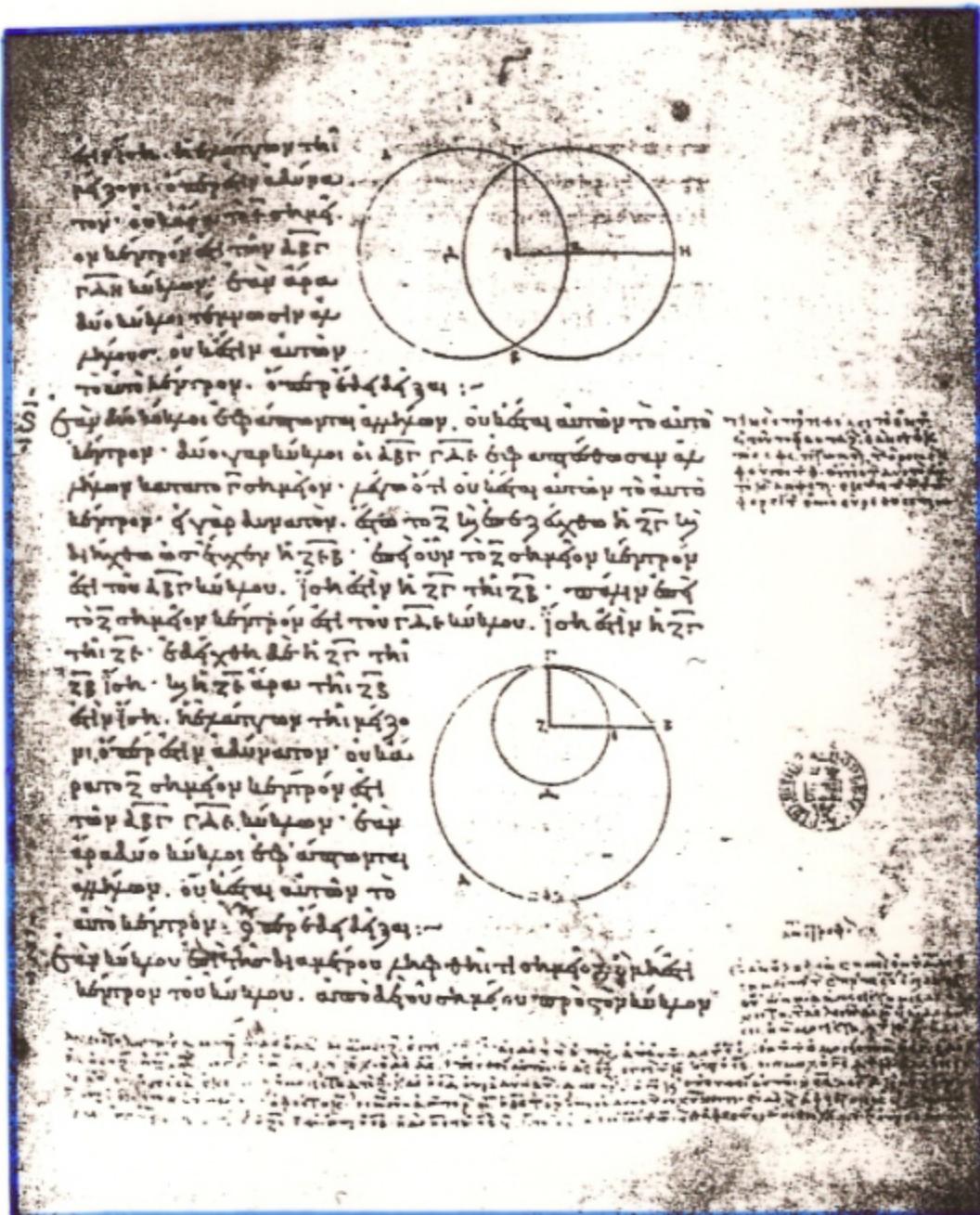


Influences on Baghdad

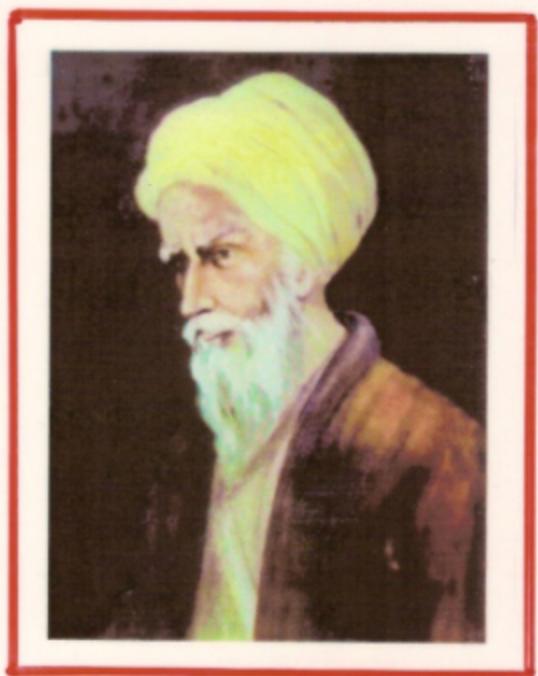
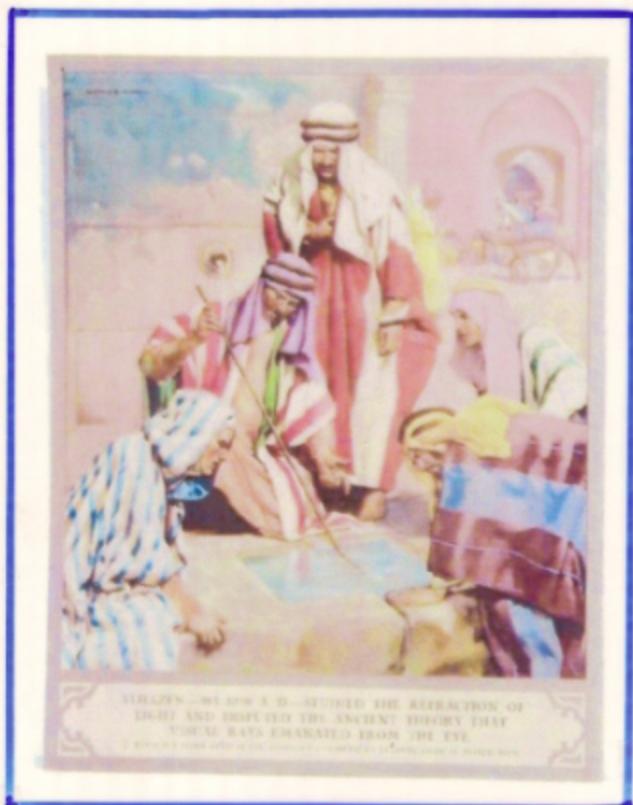
الطرق الأساسية التي وصلت بها العلوم من مراكز الثقافات إلى العالم الإسلامي



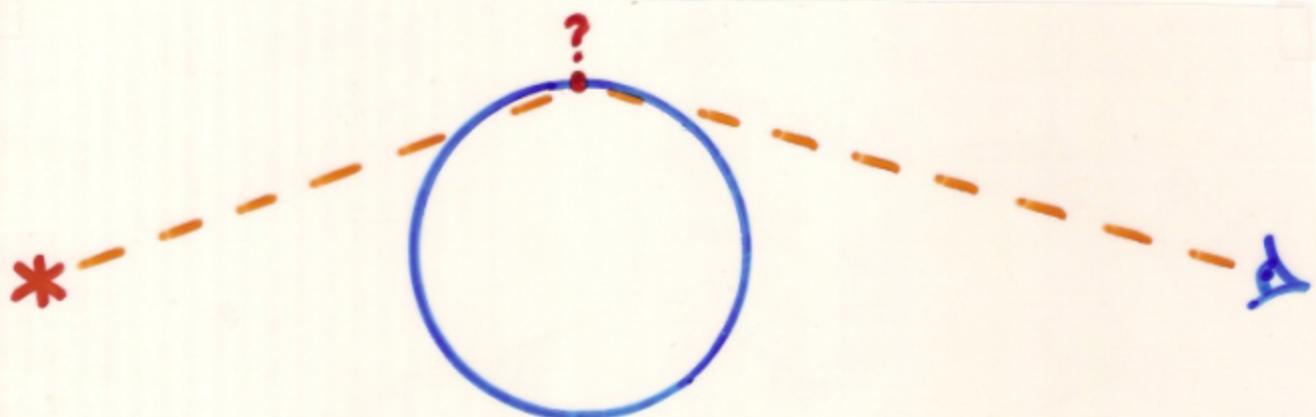
The earliest surviving Euclid (888 AD)



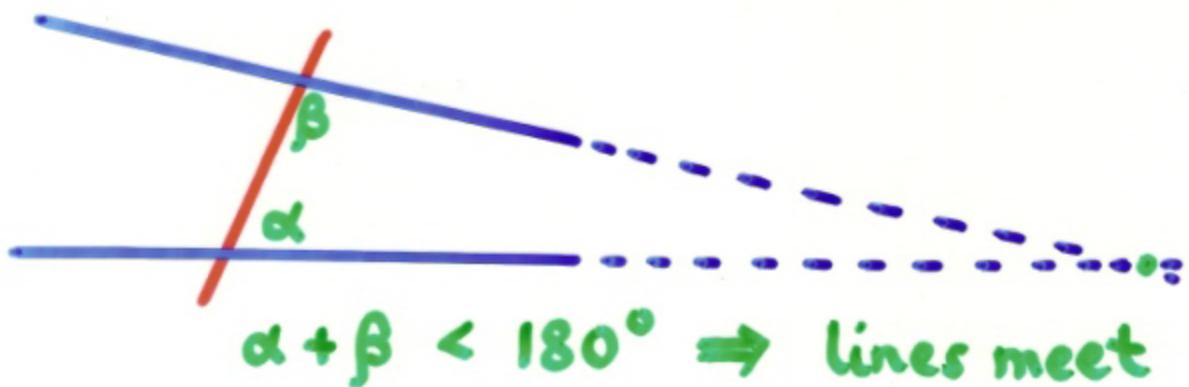
ibn al-Haitham (Alhazen) (965-1039)



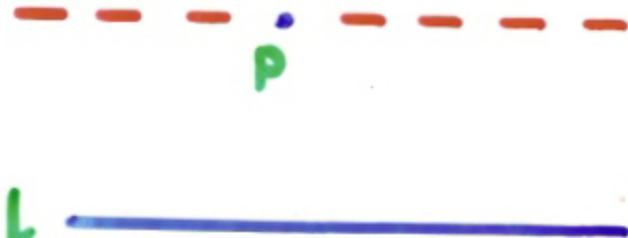
Alhazen's Problem



Euclid's Fifth Postulate

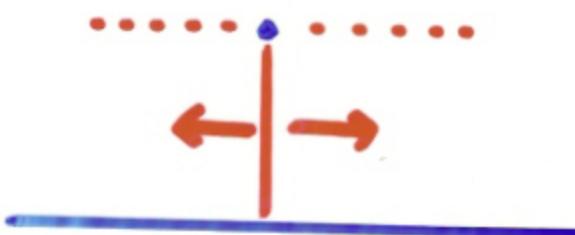


Euclid
I 29:



'parallel
postulate'

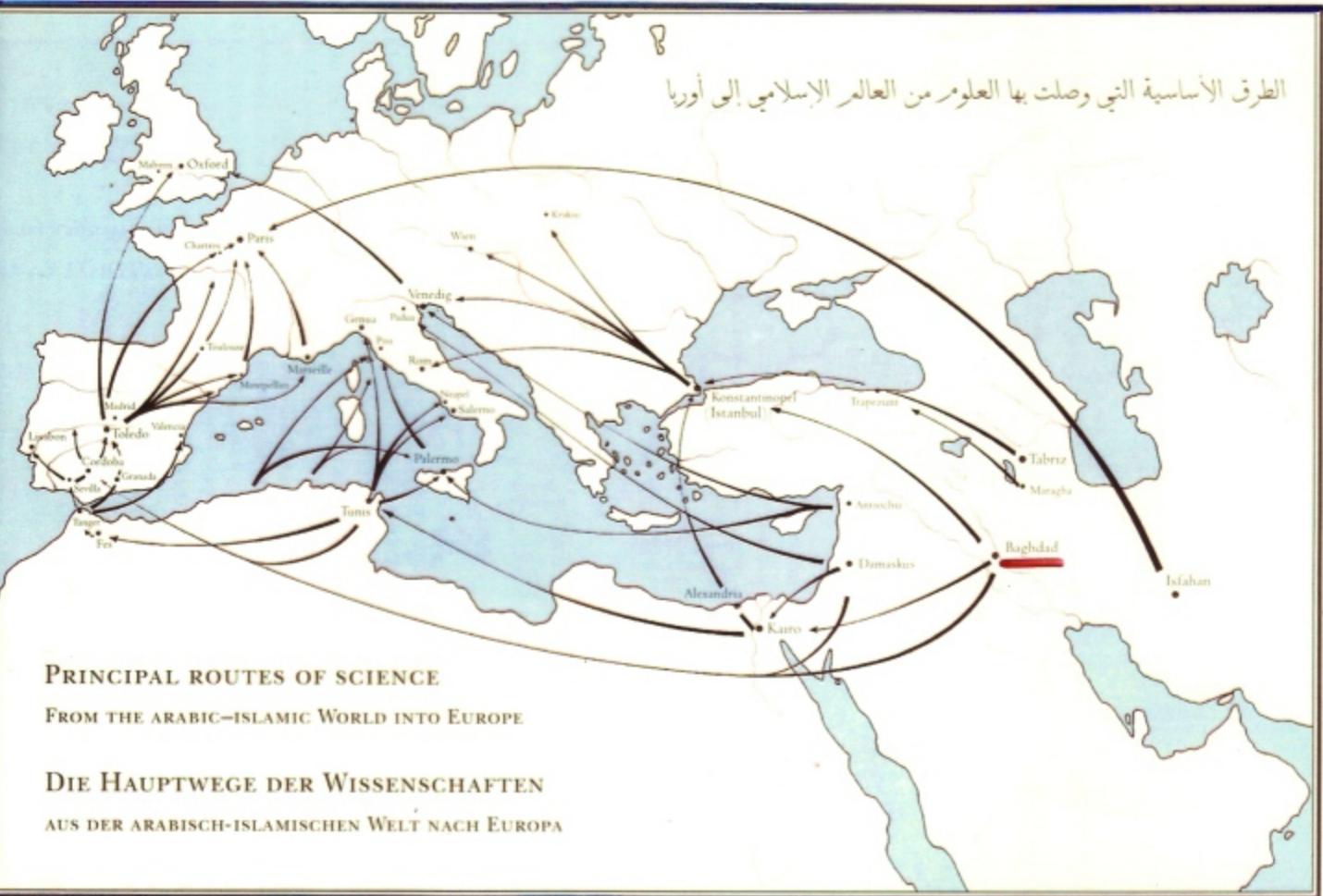
Alhazen
(1000 AD)



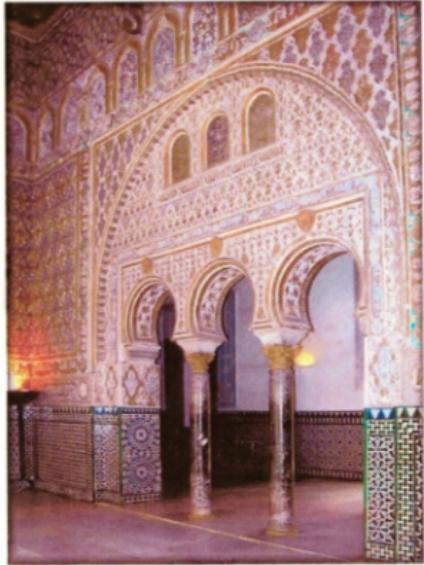
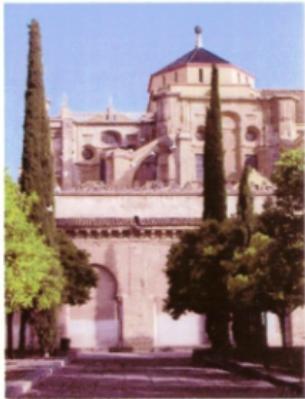
criticized by Omar Khayyam

From Baghdad to Europe

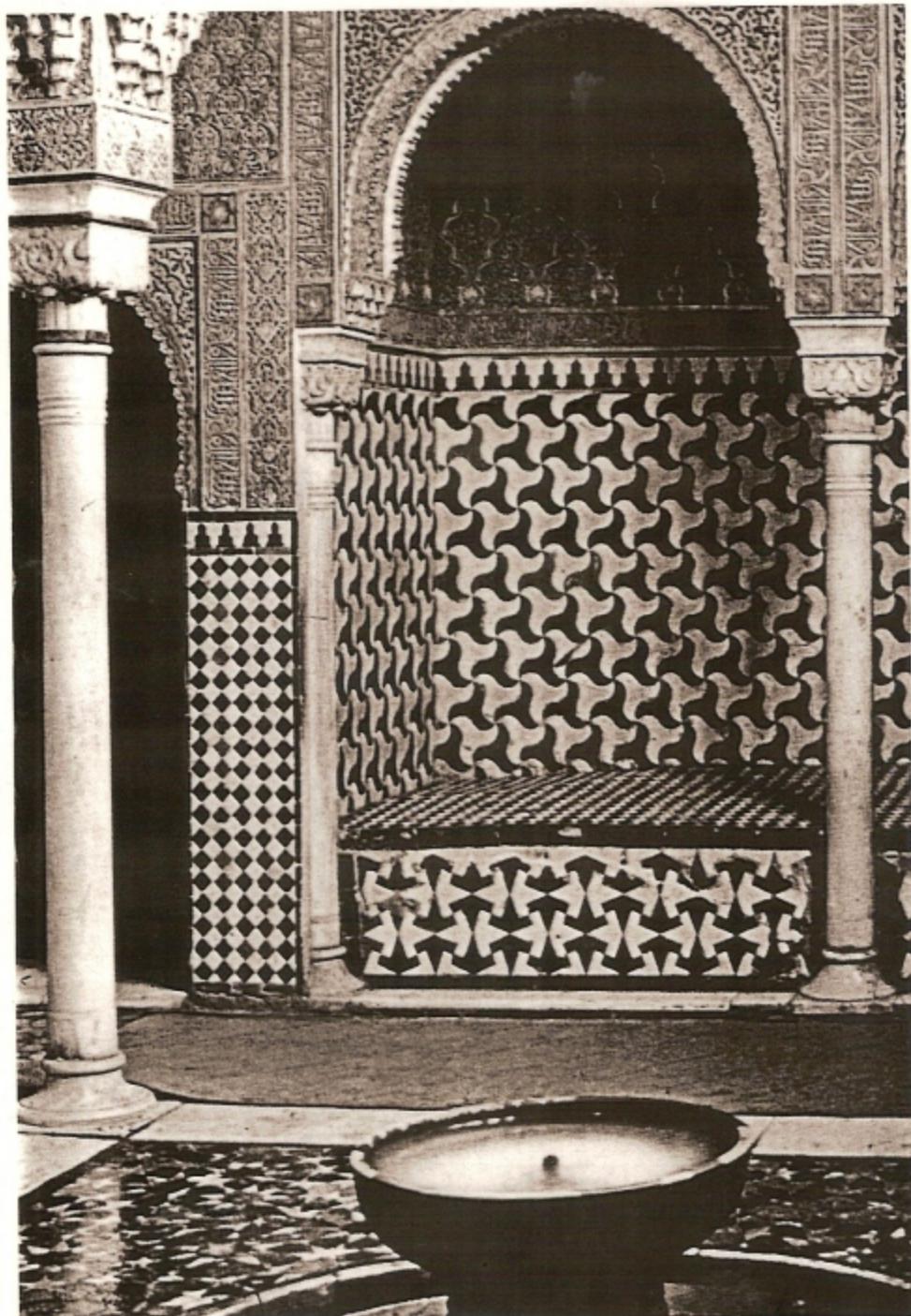
الطرق الأساسية التي وصلت بها العلوم من العالم الإسلامي إلى أوروبا



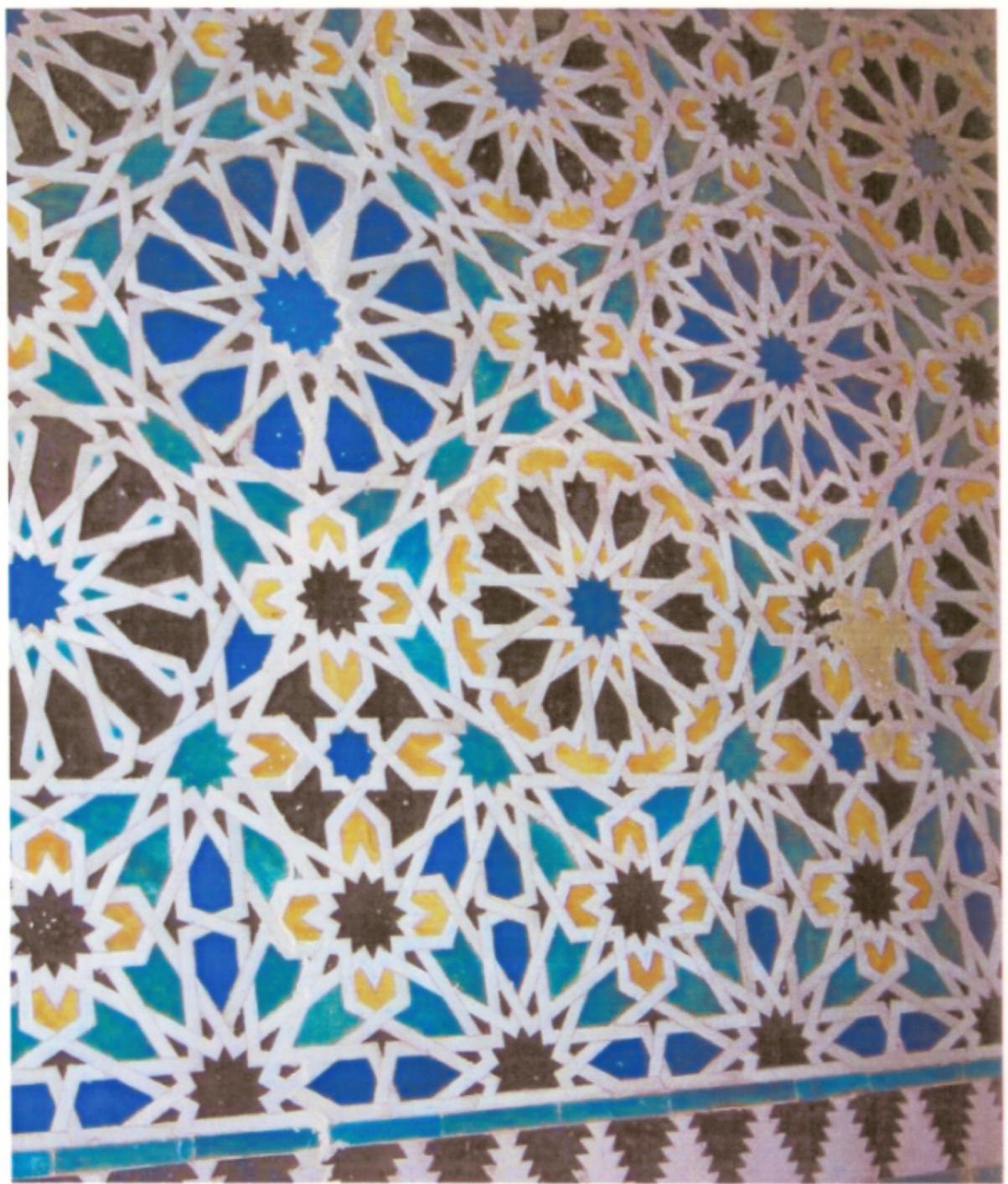
Cordoba and Seville

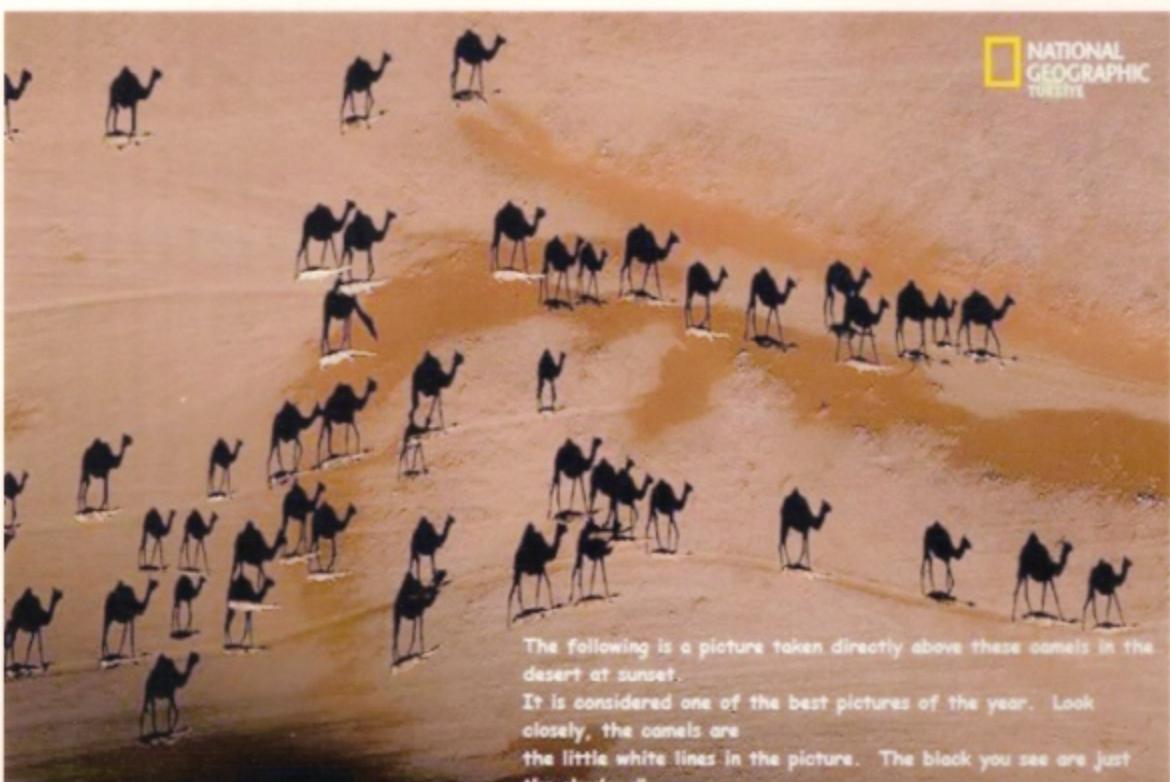


Alhambra tilings









NATIONAL
GEOGRAPHIC
TÜRKİYE

The following is a picture taken directly above these camels in the desert at sunset.

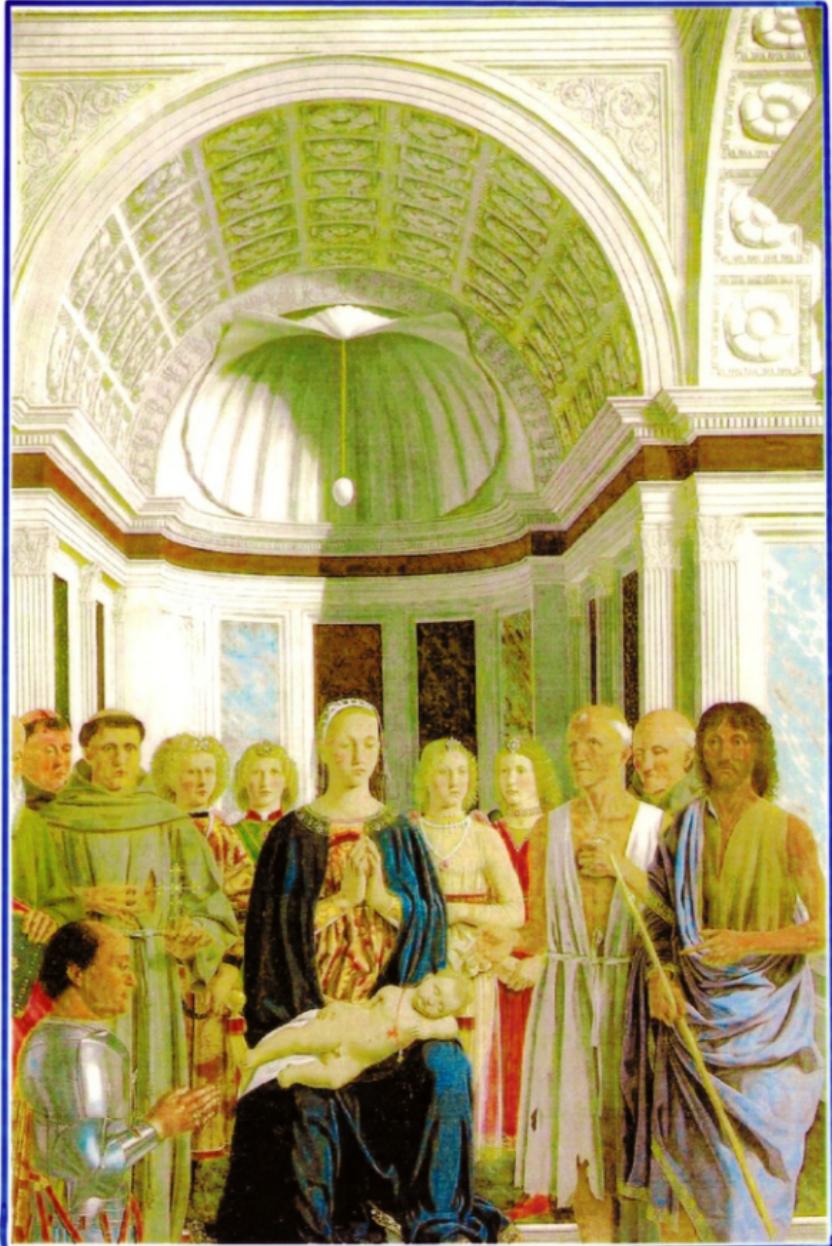
It is considered one of the best pictures of the year. Look closely, the camels are the little white lines in the picture. The black you see are just the shadows!!

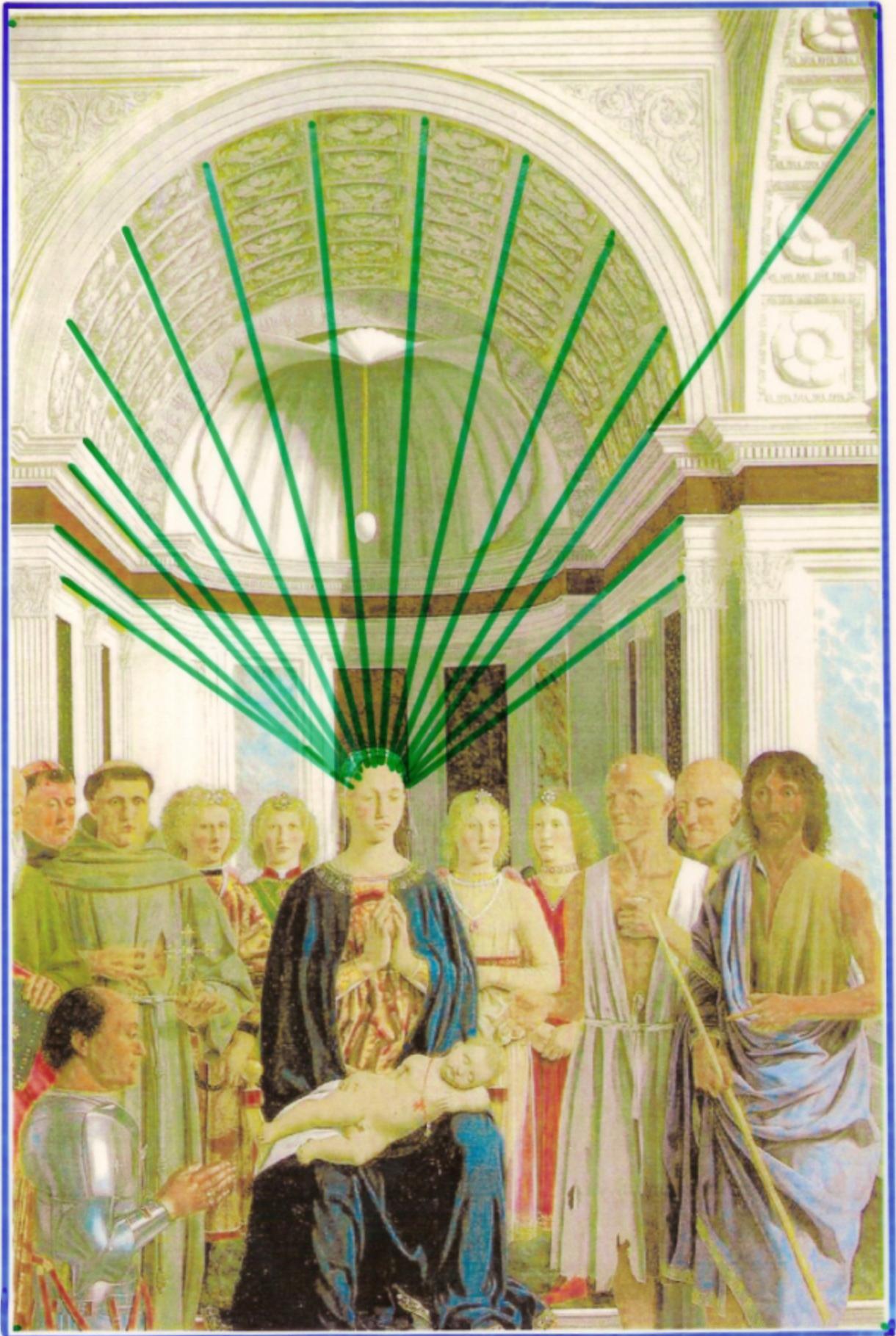
Fotoğraf: George Steinmetz

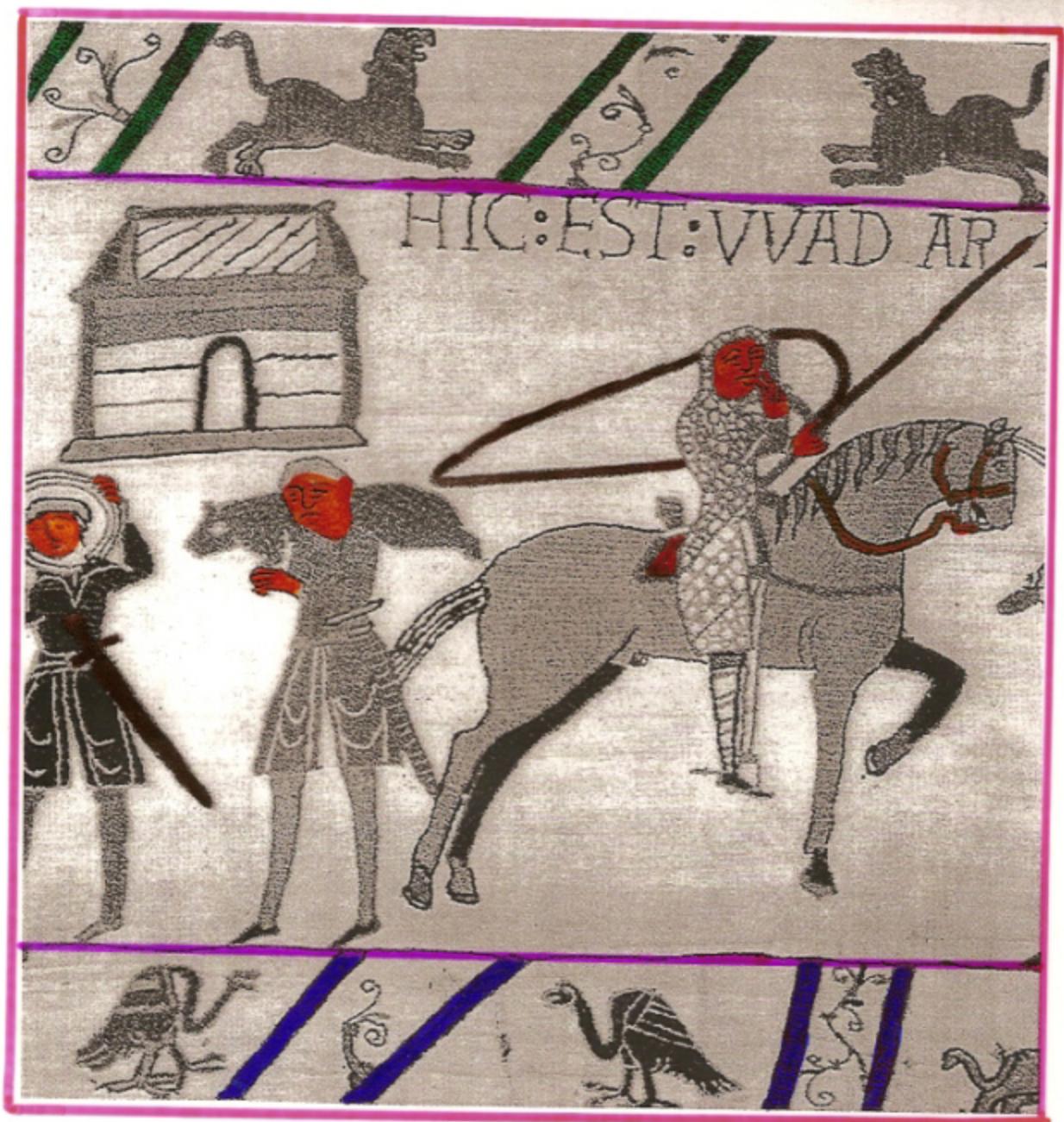
© 2005 National Geographic Society. Her hakkı saklıdır.

Dev Developér

National Geographic Türkiye, Şubat 2005



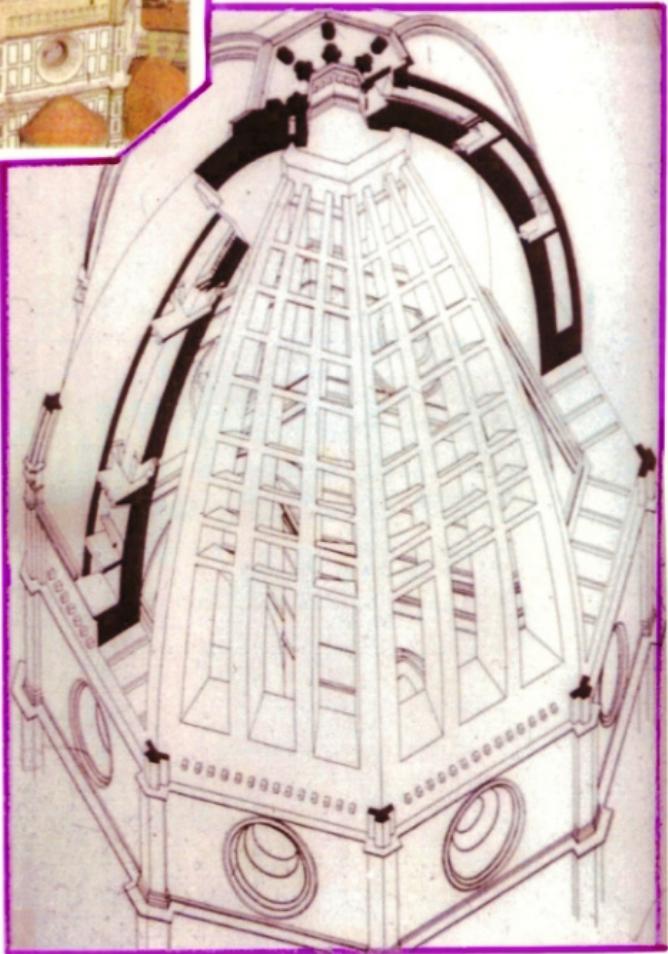
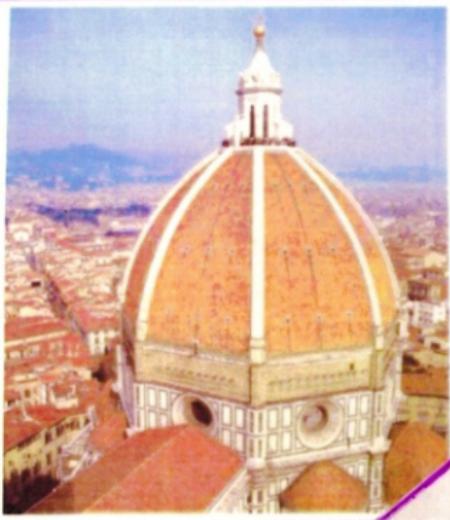




Bayeux tapestry

Brunelleschi:

Florence
Cathedral

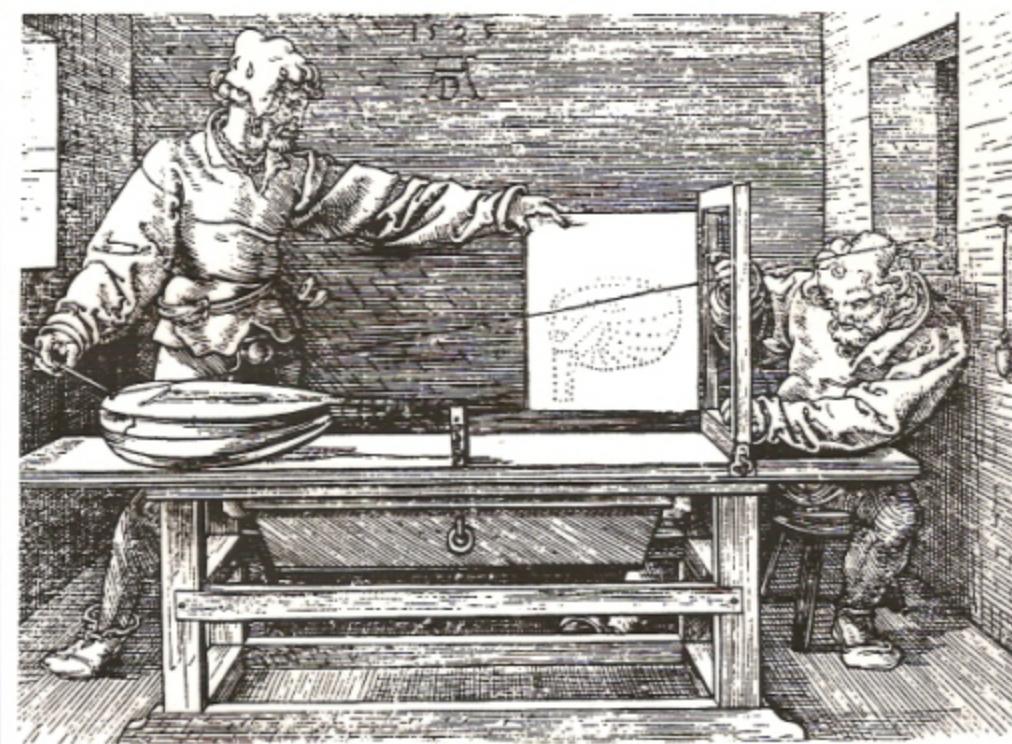


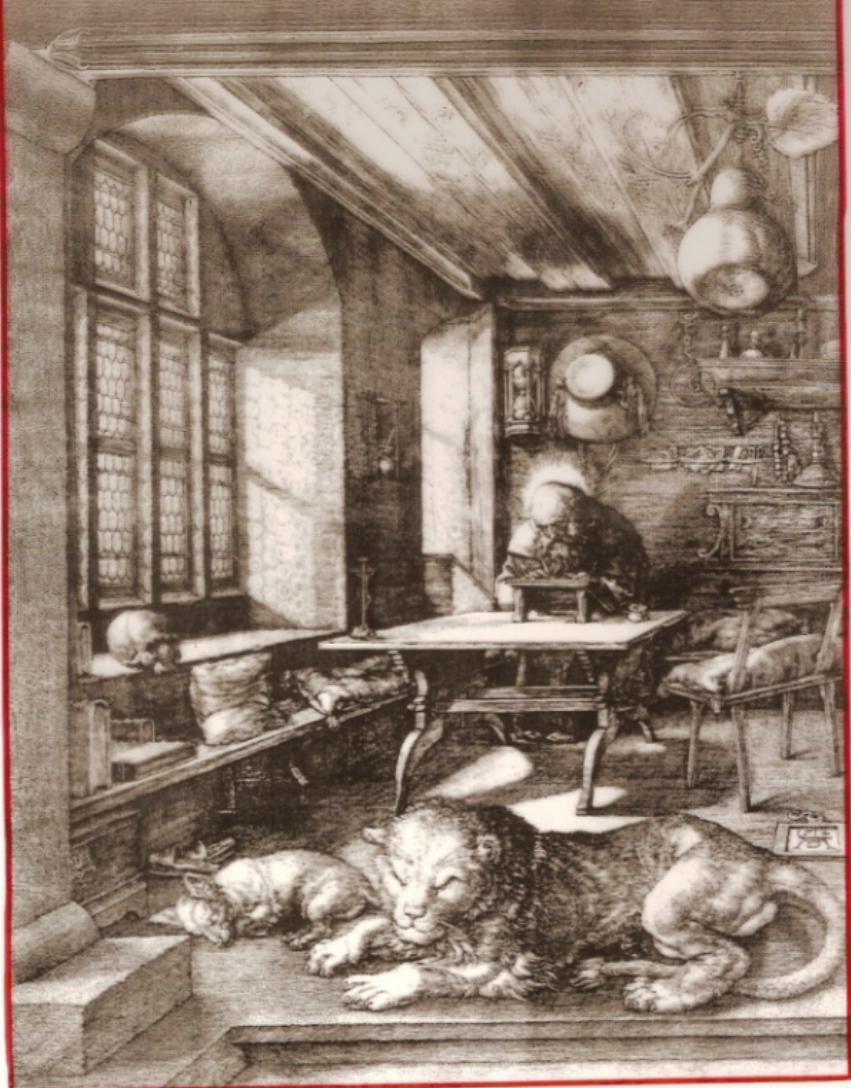
Alberti:

Santa Maria Novella

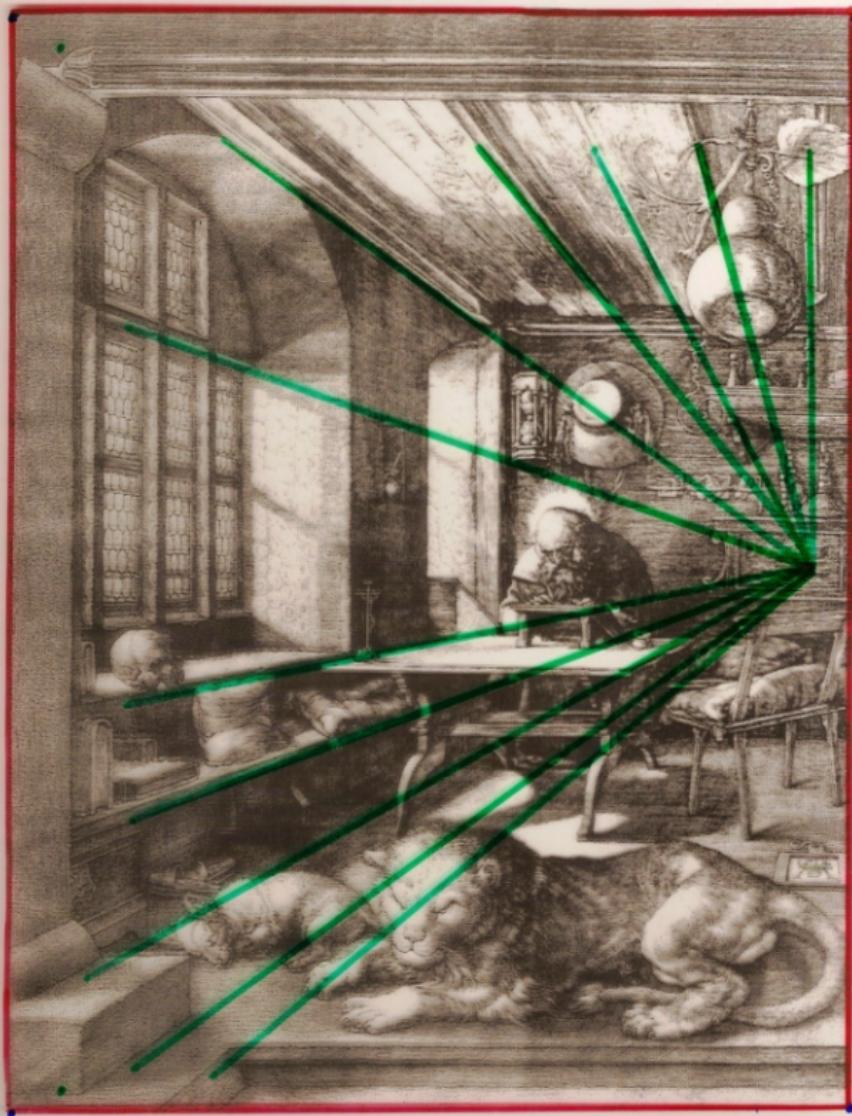


Dürer on perspective drawing





Dürer : St Jerome in his Study



Dürer : St Jerome in his Study

Robert Grosseteste (c. 1168 - 1253)



The usefulness of considering lines, angles and figures is the greatest, because it is impossible to understand natural philosophy without these ...

The Pythagorean School

ALL IS NUMBER

Arithmetic, Music
Geometry, Astronomy } Quadrivium



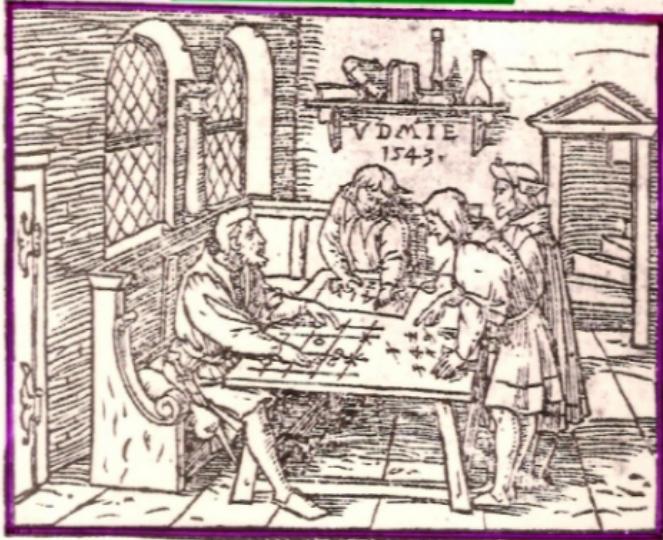
TYPVIS GEOOMETRIS



The groud of artes

teachyng the worke and ma-
rctise of Arithmetike, moch necessary
for all states of men. After a more
easier & exacter sorte, then any
lyke hath hytherto ben set
forth: with dyuers newe
additions, as by the
table doth partly
appeare.

ROBERT RECORDE.



Record's terminology

Right
angle.

Obtuse
angle.

DEFINITIONS

do precisely understand those definitions, so it shall be sufficient
and for those men, which like the 'ye' of the same tongue,
as few may durst judge them, and apply to handy work,
if they understand them fit to be true, that our warden
comes finde none errors therein.

Of lines there be two principal kinds. The one is call'd
a right or straight line, and the other a crooked line.

A straight line is the straight that may be drawn
between two points, And all other lines, that go not right from you prick to
prick, but bowereth ther selfe are call'd Crooked lines
as in these examples following, ye may see, whereof 7 lines set
but one line of a straight line, the more firmes they be
not, but of crooked lines there be no unmerciful lawes, wherof for examples I have sette here.

Crooked a right line.



Crooked lines.

So now you must understand, that
every line is drawn between
two prickes, wherof the one is at
the beginning, and the other at the end.

Therefore whensoever you see any
former of lines to touche at one notable
prickle, as in this example, then fall you



GEOMETRICAL

not call it one or shod line, but rather two lines : in as
much as there is a notable and sensible angle by A, which e
evermore is made by the meeting of two several lines. And
therewer fall you judge of this figure,  which is made of two lines, and not of
one only.

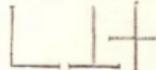
So that whenso ever any such meeting of lines doth happen
in the place of their meeting is called an Angle or corner.

Of angles there be three general kinds: a sharp angle, a
square angle, and a blunt angle. The square angle, which
is commonly named a right corner, is made of two lines, if
they meet together in suene of a square, whiche two lines, if
they be drawn forth to length, will croffe one an other as in
this example following you may see.

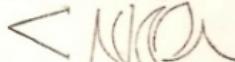
A Right angle is so call'd, because it is lesser than a  square angle, and the lines that make it do not open so wide in
their departing as in a square corner, and if they be drawn
wylle all four corners will not be equal.

A blunt or broad corner, is greater then a square angle,
of which all take these examples.

Right angles.



Sharp angles.



I have put none example of crooked lines, because it would
make

A

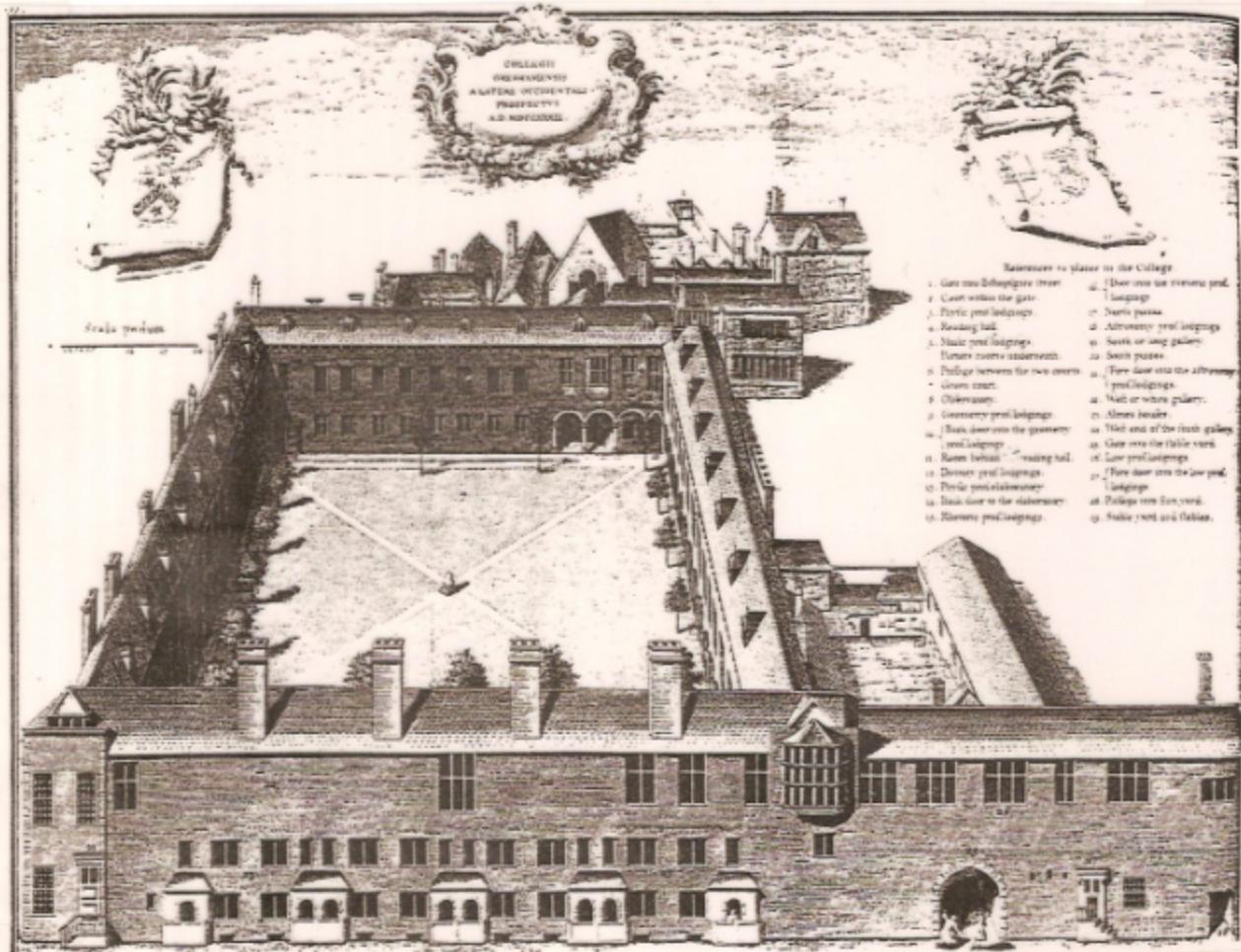
straight line

sharp angle, blunt angle

touch line (tangent)

three like (equilateral triangle)

The original Gresham College



The seven Gresham Chairs

- **Divinity**
- **Astronomy**
- **Music**
- **Geometry**
- **Law**
- **Physic**
- **Rhetoric**
- **[Commerce]**



Roberval and the Cycloid

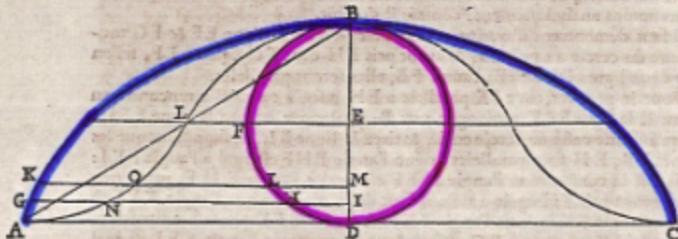
108 DES MOUVEMENTS COMPOSÉZ.

touchante par le point E, car ces positions de cercles étant parallèles, & le point E étant aussi élevé sur la base A C, que le point F, les touchantes des cercles sont parallèles, & partant l'une peut servir aussi-bien que l'autre, pour en mesurer un mouvement droit, puisque l'une & l'autre rencontrent la ligne EF, qui est la direction de ce mouvement droit. C'est pourquoi si l'on vouloit décrire le cercle de la Roulette en la position qu'il est lors que le point qui la décrit est arrivé en E, ayant premièrement décris le cercle BFD autour de l'axe BD, & tiré la ligne FI parallèle à ADC, prenez EM dans EI égale à FI, qui est comprise entre la circonference & le diamètre du cercle qui est perpendiculaire à la base AC, vous aurez le point M par où doit passer ce diamètre perpendiculaire. Et partant si vous tirez MN perpendiculaire à AC, & si vous la prolongez vers M en O en sorte que NM soit égale au diamètre du cercle de la Roulette, vous aurez le diamètre dudit cercle en la position requise, ce qui est facile.

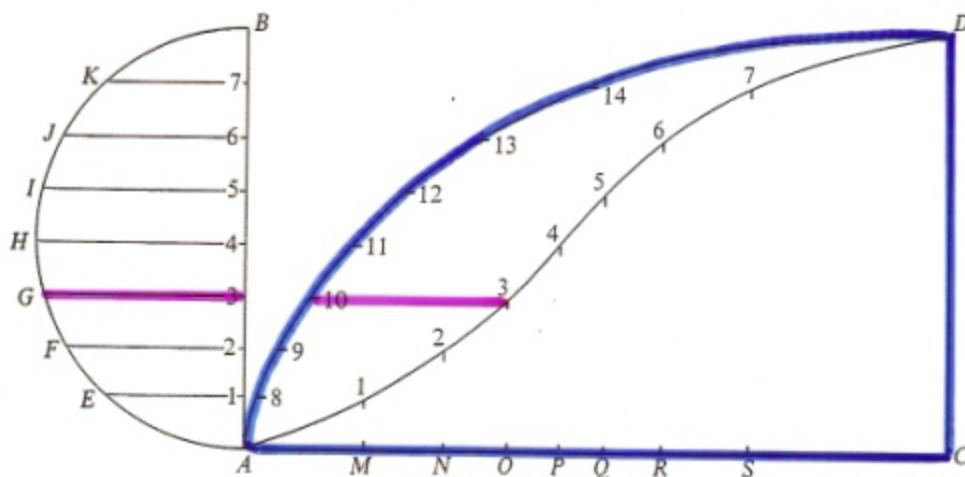
Je ne vous diray rien des propriétés de la Roulette, comme que la ligne droite EF est à l'arc FB, en même raison que la base AC à toute la circonference du cercle &c. M. de Roberval ne m'a pas encore fait voir le Traité qu'il en a fait, où après en avoir démontré cette propriété & un grand nombre d'autres, il compare ces lignes les unes aux autres, les semblables, celles de divers genres, les égales, les inégales, leurs ordonnées, leurs espaces &c. ce qu'il a expliqué dans un si bel ordre, qu'il m'a dit que son Traité estoit aussi bien comme s'il eust été sur le point de le faire imprimer.

Douzième exemple, de la compagnie de la Roulette.

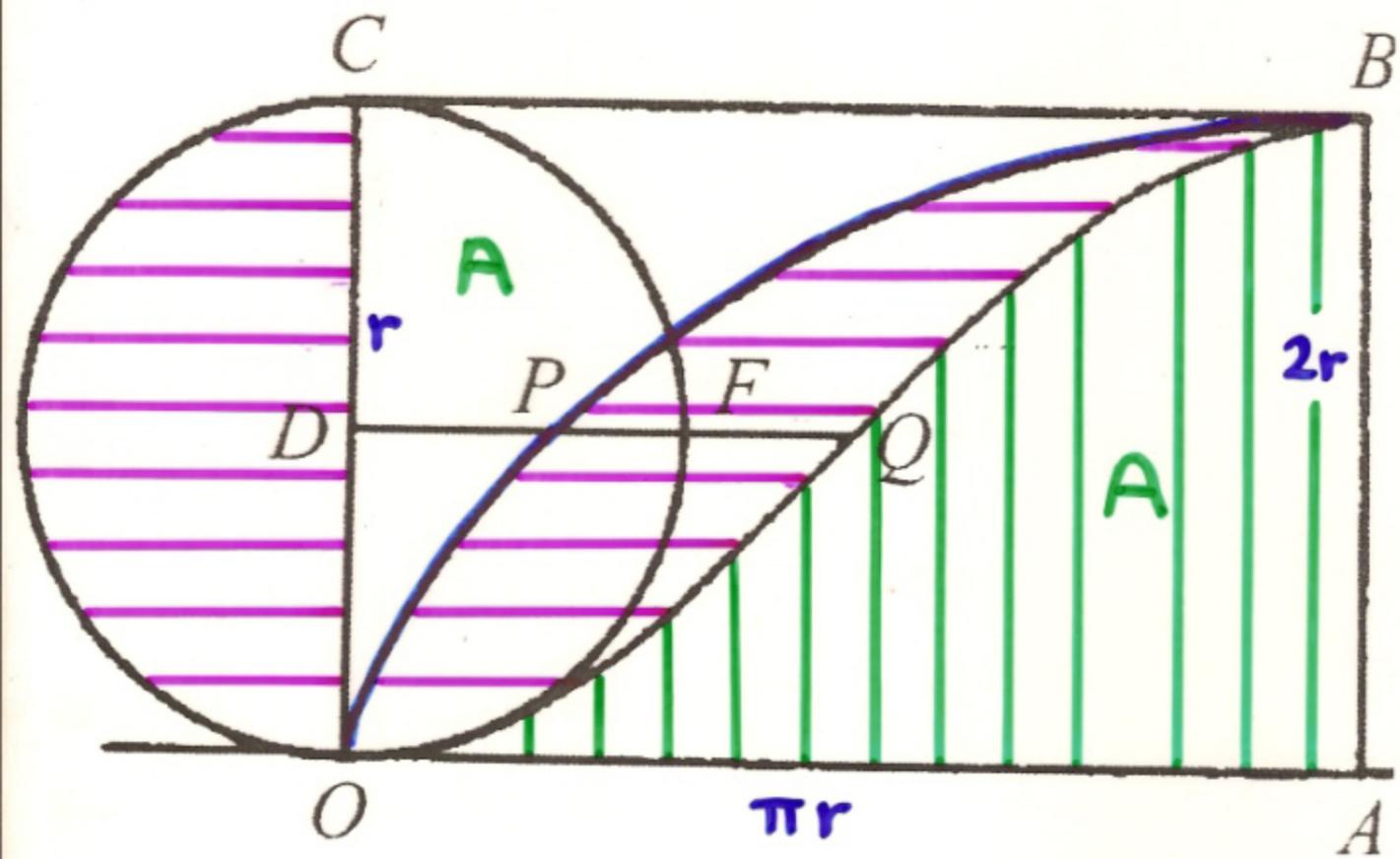
C'est ainsi que l'a voulu nommer M. de Roberval qui l'a inventée, & qui en a imaginé l'hipothèse & la description en cette sorte.



Soit proposé la Roulette ABC de laquelle la base est AC l'axe BD, le centre du cercle dans l'axe est E, & le cercle de la Roulette BFD à l'entour de l'axe. Entendez que la Roulette est décrite par la seconde façon qui en a été donnée dans l'exemple précédent; c'est à savoir que pendant que le cercle de la Roulette glisse depuis A jusques en C, en sorte que son centre E décrit d'un mouvement uniforme une ligne parallèle & égale à AC, en même temps le point mobile A parcourt par un mouvement uniforme la circonference de ce cercle, & décrit la Roulette par le mouvement composé de ces deux; imaginez maintenant que pendant que ce point parcourt ainsi la circonference DFB, un autre point A ou D mobile dans le diamètre du cercle, qui est toujours perpendiculaire à AC, monte le long de ce diamètre de D vers B d'un mouvement inégal, en sorte qu'il soit toujours également élevé sur la base AC, comme est le point qui décrit la Roulette, c'est-à-dire qu'ayant



Area under a cycloid

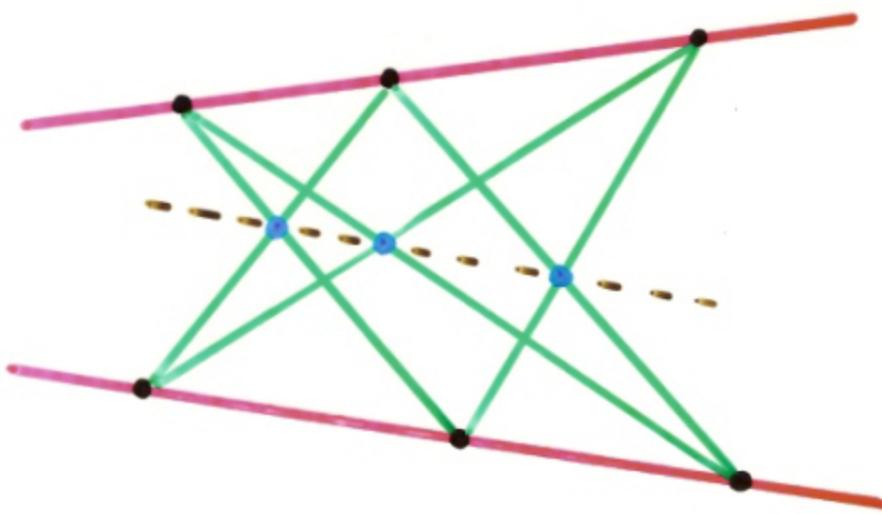


$$\frac{1}{2}(\text{area}) = \frac{1}{2}(2r)(\pi r) + \frac{1}{2}(\pi r^2)$$

$$\text{so total area} = 3\pi r^2$$

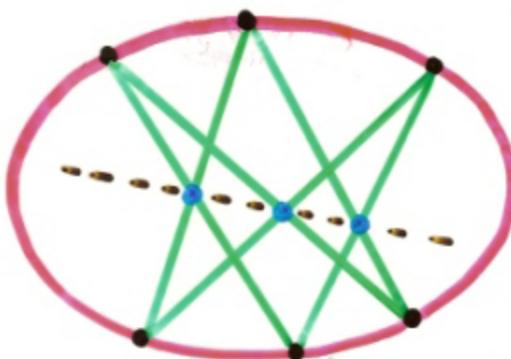
$$= \underline{\underline{3 \times (\text{area of circle})}}$$

The theorems of Pappus and Pascal



Pappus

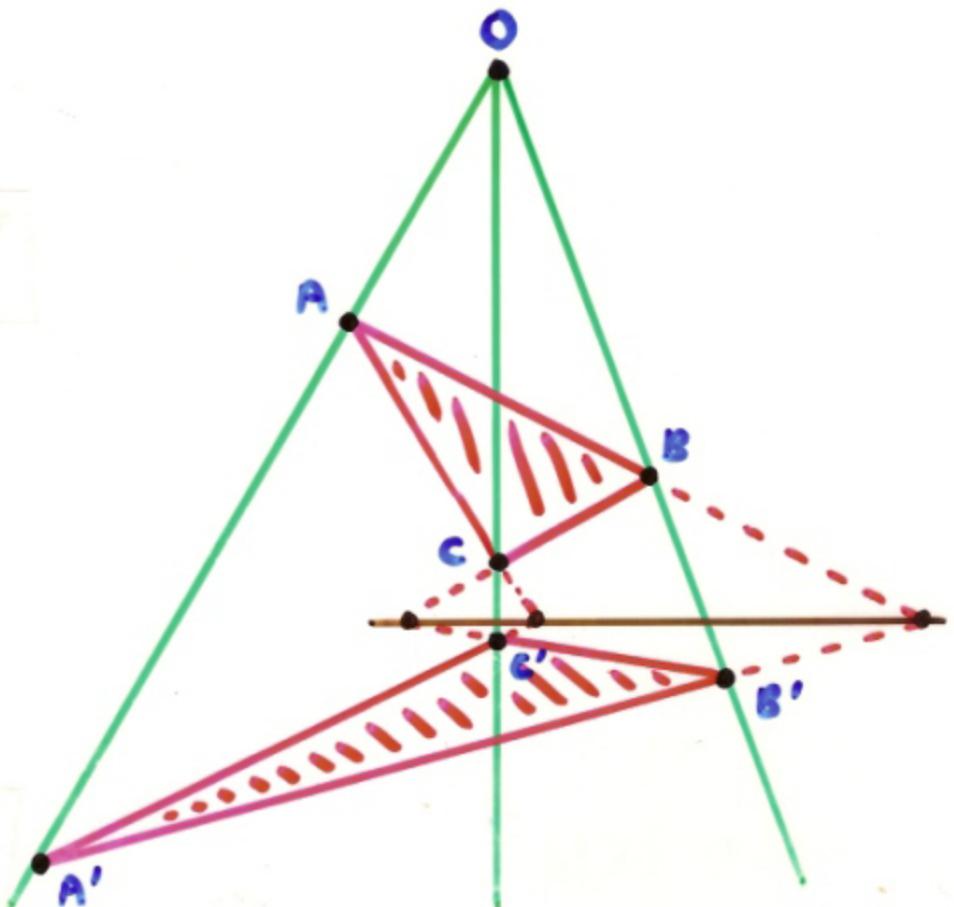
(3rd c AD)



Pascal

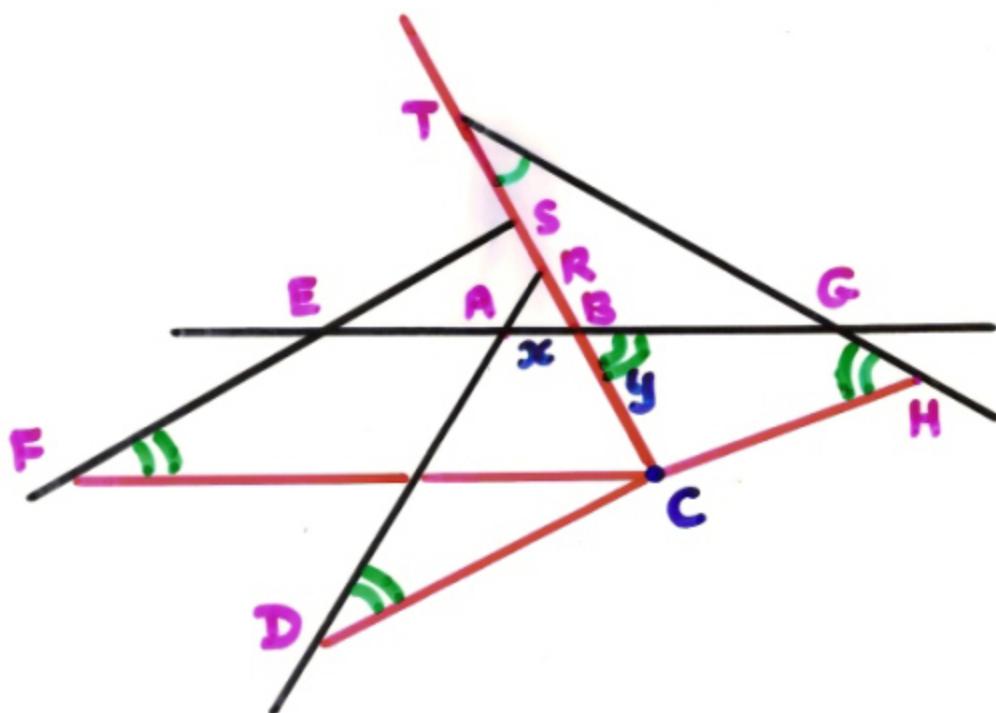
(17th c)

Desargues' theorem



Pappus's Problem

Given four lines and four angles,
find the locus of all points C



such that the red lines meet the four
black lines at these angles, and

$$CD \times CF = (\text{const}) \times CB \times CH$$

Answer : a conic

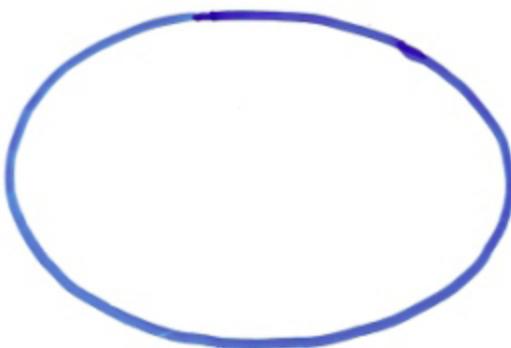
Conics

Equation: $yy = \alpha + \beta x + \gamma xx$

$$\text{or } y^2 = \alpha + \beta x + \gamma x^2$$

γ negative:

ellipse



$\gamma = 0$:

parabola



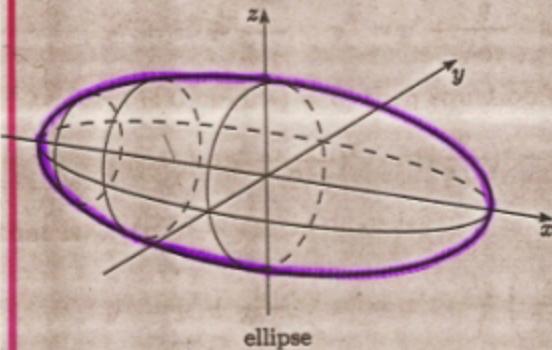
γ positive

hyperbola

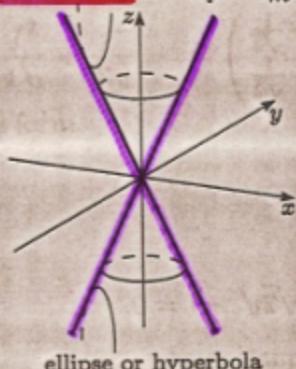


QUADRICS

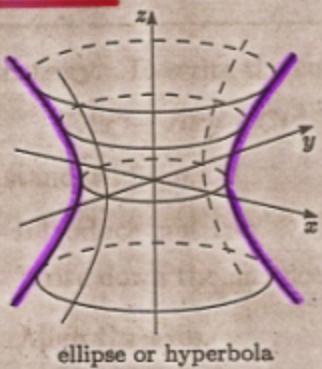
Ellipsoid $\frac{x^2}{l^2} + \frac{y^2}{m^2} + \frac{z^2}{n^2} = 1$



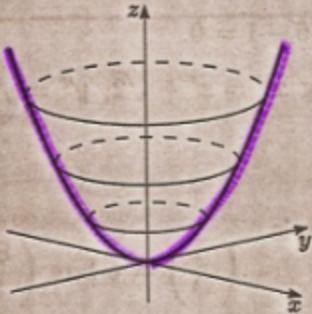
Elliptic cone $z^2 = \frac{x^2}{l^2} + \frac{y^2}{m^2}$



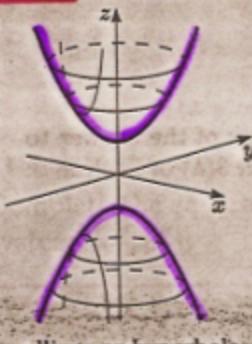
Hyperboloid of one sheet $\frac{x^2}{l^2} + \frac{y^2}{m^2} - \frac{z^2}{n^2} = 1$



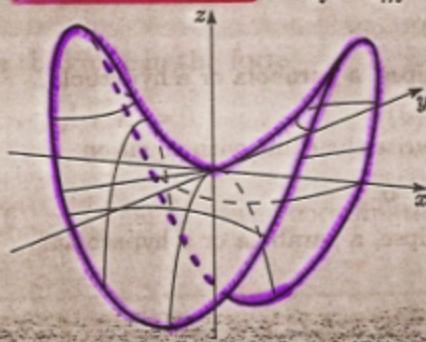
Elliptic paraboloid $z = \frac{x^2}{l^2} + \frac{y^2}{m^2}$



Hyperboloid of two sheets $\frac{x^2}{l^2} + \frac{y^2}{m^2} - \frac{z^2}{n^2} = -1$

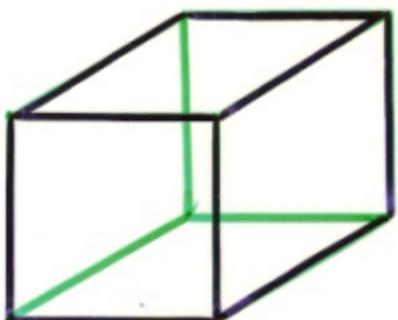


Hyperbolic paraboloid $z = \frac{x^2}{l^2} - \frac{y^2}{m^2}$



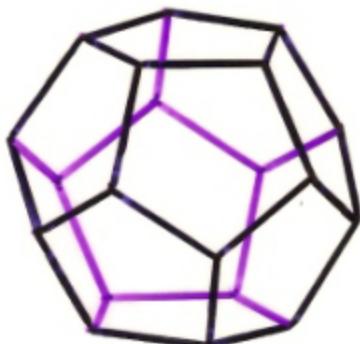
Euler's Polyhedron Formula

$$(\text{faces}) + (\text{vertices}) = (\text{edges}) + 2$$



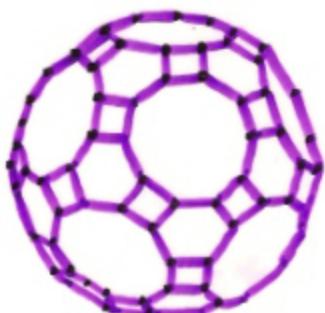
cube :

$$6 + 8 = 12 + 2$$



dodecahedron :

$$12 + 20 = 30 + 2$$



great rhombicosidodecahedron

$$62 + 120 = 180 + 2$$

Letter from Euler to Goldbach, 1750

Proved by Adrien-Marie Legendre, 1794

Euler's Polyhedron Letter (1750)

2. $\text{G} = \frac{1}{2}d$ unde $A = \frac{1}{2}P$. Und in his latere
 diceatur G pars longior, in pars aciem G formans.
 3. Dicitur P de numeris laterum seu angulorum planorum ordinum
 hedronum corporis videndentium aliquid par.
 4. Semper est vel $L = 3H$ vel $L > 3H$ } at est P d.
 5. Semper est vel $P = 3S$ vel $P > 3S$ } at est P d.
 Dicitur P illa pars H etiam H dicitur G , quia, etiam
 angulus solidus et longius G est angulus planus G super hoc
 fidei proportione abebo, non si G est etiam rigore demonstratur
 6. In solidis hedronis planis videtur generatio ex numero hedronum
 et numero angulorum solidorum binaria figurae figurari ex numero hedronum
 seu est $H + S = A + 2$ seu $H + S = \frac{1}{2}d + 2 = \frac{1}{2}P + 2$.
 7. Impossibile est ut sit $A + 6 > 3H$ vel $A + 6 > 3S$.
 8. Possibile est ut sit $H + A > S$ et $S + A > 2H$.
 9. Nullum formam potest solidum cuius omnes latera sunt 6 planorum
 laterorum, nec cuius omnes anguli solidi ex sex planis solidi anguli plani
 sunt confecti.
 10. Semper omnium angulorum planorum, qui in ambito solidi conjugae
 recurrent, tot anguli recti equales sunt, sicut initates in $2A - 2H$.
 11. Semper omnium angulorum planorum, qui sunt quatuor tot anguli
 recti, sunt sunt anguli solidi dentes octo, seu est $A - 8$ sectos.

Exemplo sit prisma triangulare ubi est

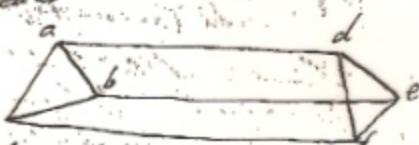
$$\begin{array}{l} \text{1. numerus hedronum } H = 5 \\ \text{2. numerus ang. pl. } S = 6 \end{array}$$

$$\begin{array}{l} \text{3. numerus laterum (ab, ac, bc, ad, bd, cd)} \\ \text{4. numerus laterum et angulorum planorum } L = P = 18. \quad \text{Inclusis enim unus} \end{array}$$

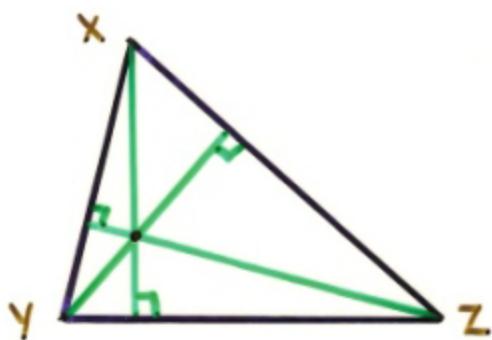
duobus triangulis et binis quadrilateris, unde $L = P = 2.3 + 3.4 = 18$.

$$\text{Ita } A = 9 \text{ secundum Theor. 6: } H + S \text{ (ii)} = A + 2 \text{ (ii)}$$

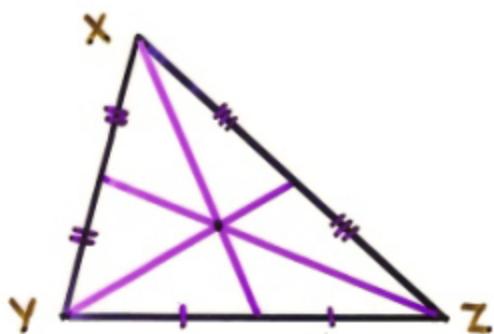
semper semper omnium angulorum planorum (autem in solidis $D = 4$ recti, unde $A = 12$) $\square = 12$ recti) $18 + 12 = 30$ recti $= 4(A - H) = 4(9 - 6) = 12$ recti.



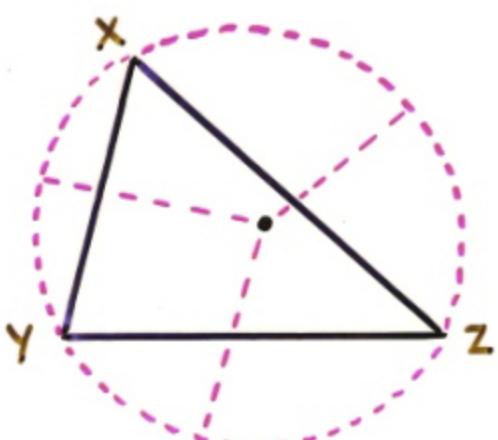
Geometry of a Triangle



orthocentre O

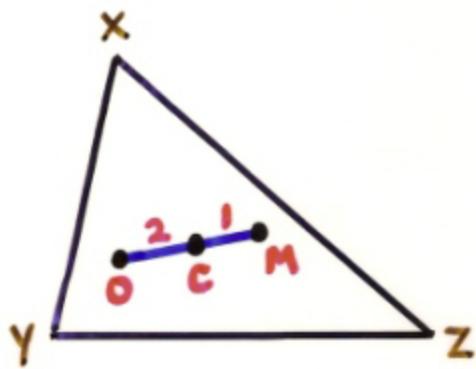


centroid C



circumcentre M

Euler Line
($OC = 2CM$)



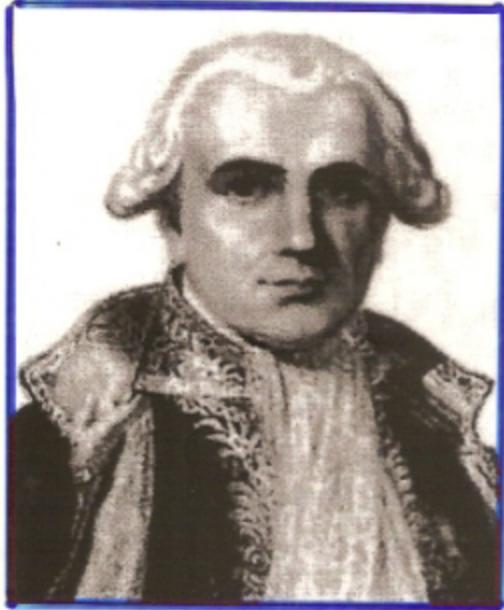
Constructing Polygons

Gauss: A regular polygon with n sides can be constructed with straight-edge and compasses if and only if n has the form

$$n = 2^k \times p_1 \times p_2 \times \dots$$

$2^{2^b} + 1$
distinct 'Fermat
primes'

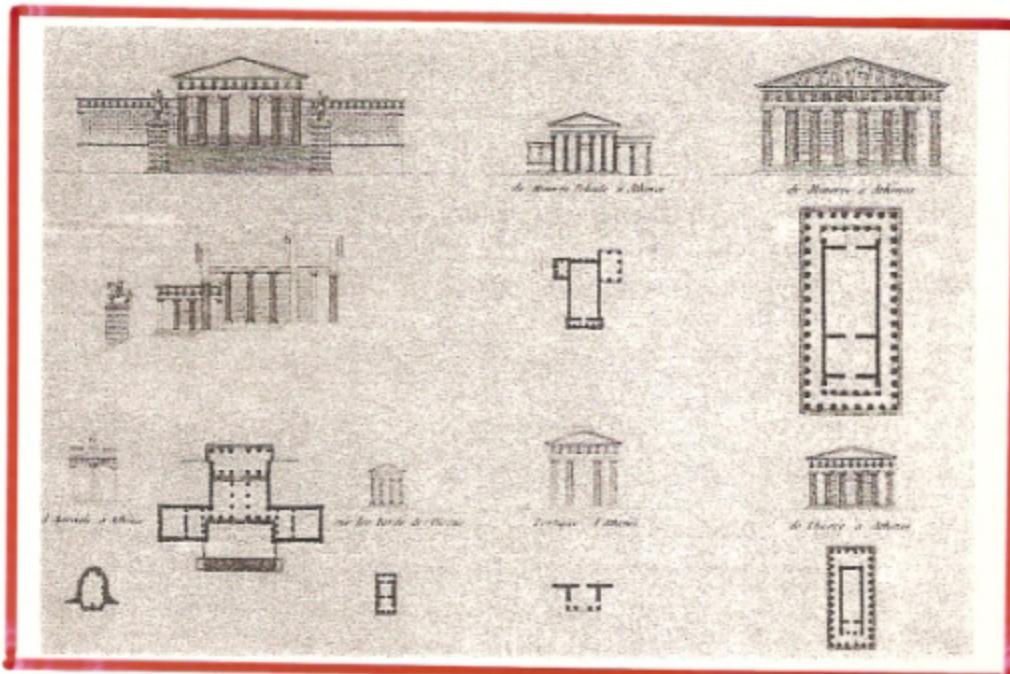
3, 4, 5, 6, 8, 10, 12, 15, 16,
17, 20, 24, 30, 32, 34, 40, 48,
51, 60, 64, 68, 80, 85, 96,
102, ..., 257, ..., 65537, ...



Gaspard Monge

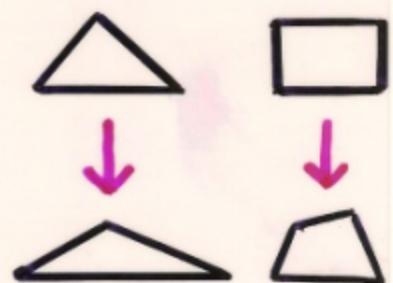
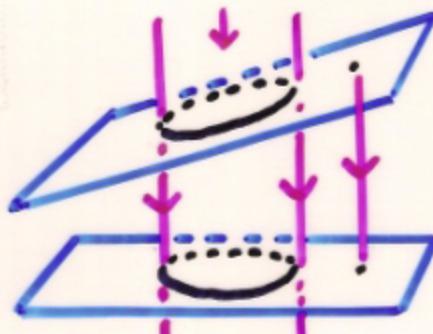
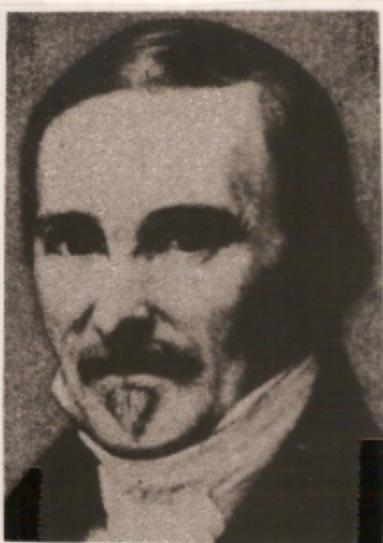
(1746 - 1818)

'descriptive
geometry'



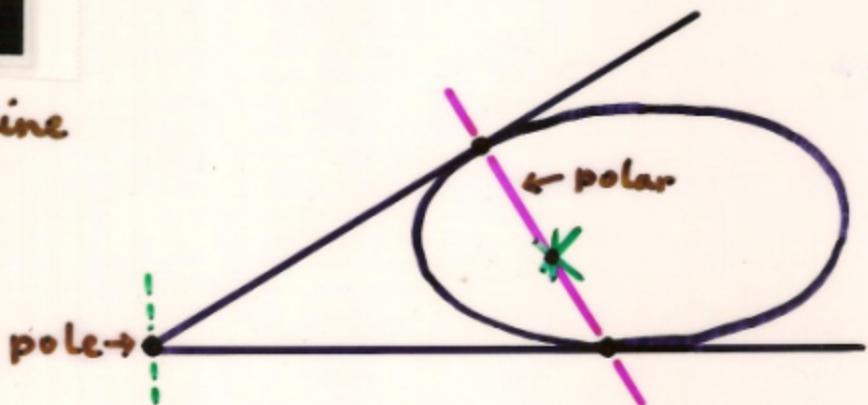
representing buildings
on a plane

Poncelet : projective transformation



duality (Gergonne) (1827)

point \leftrightarrow line



points lying on lines



'duality'

lines passing through points

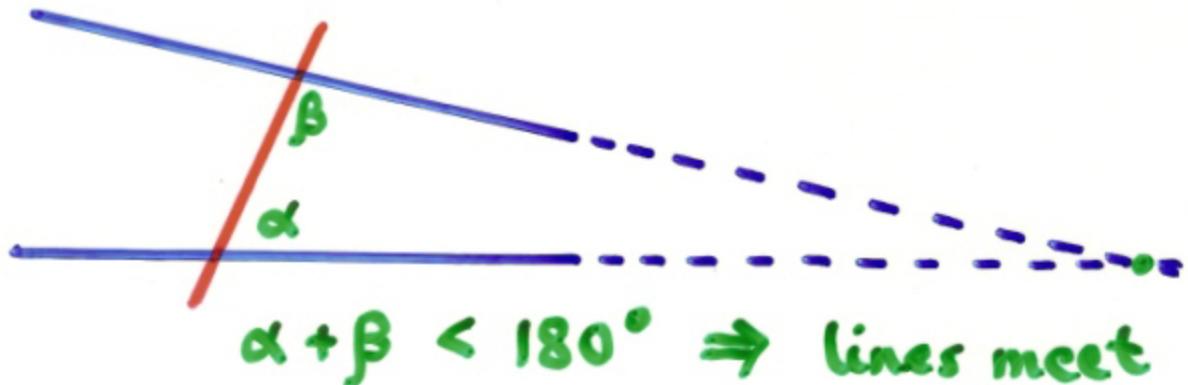
- { Any two points determine a line.
- { Any two lines determine a point.

Möbius : points $[a, b, c]$ ($= [\lambda a, \lambda b, \lambda c]$)

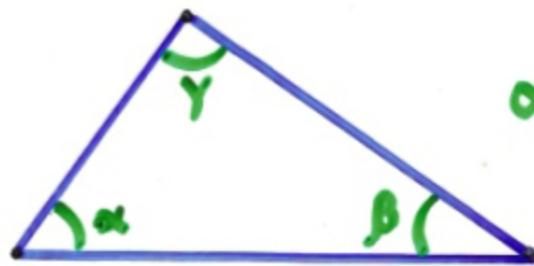
lines $ax + by + cz = 0$

duality : $[a, b, c] \leftrightarrow ax + by + cz = 0$

Euclid's Fifth Postulate



Nasir Eddin
(c. 1250)



$$\alpha + \beta + \gamma = 180^\circ$$

Euclid I.32

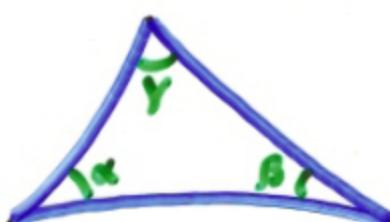
Euclidean

G. Saccheri
(1733)



$$\alpha + \beta + \gamma > 180^\circ$$

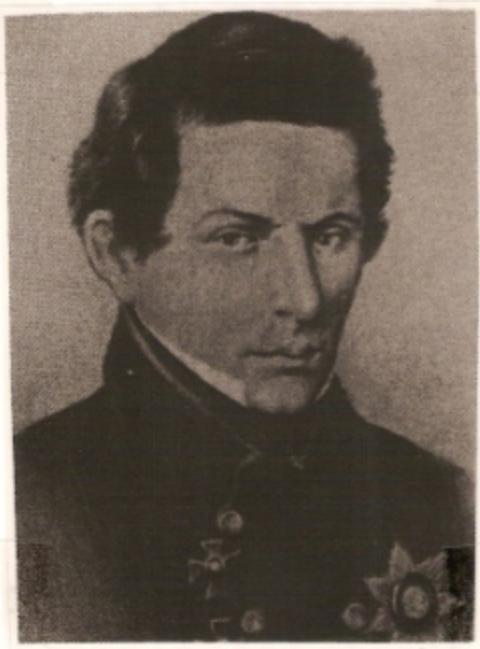
impossible



$$\alpha + \beta + \gamma < 180^\circ$$

???

Non - Euclidean Geometry (c. 1830)



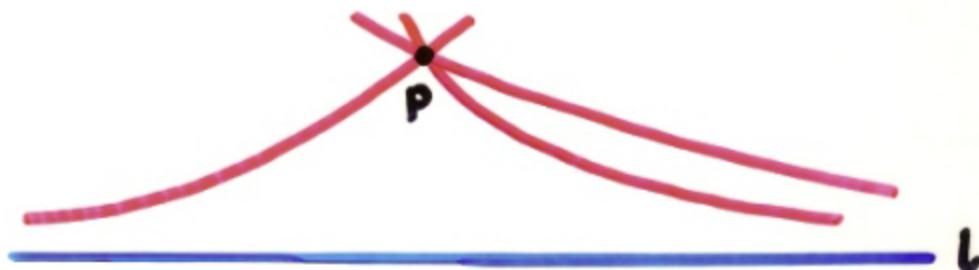
Nikolai Lobachevskii

János Bolyai

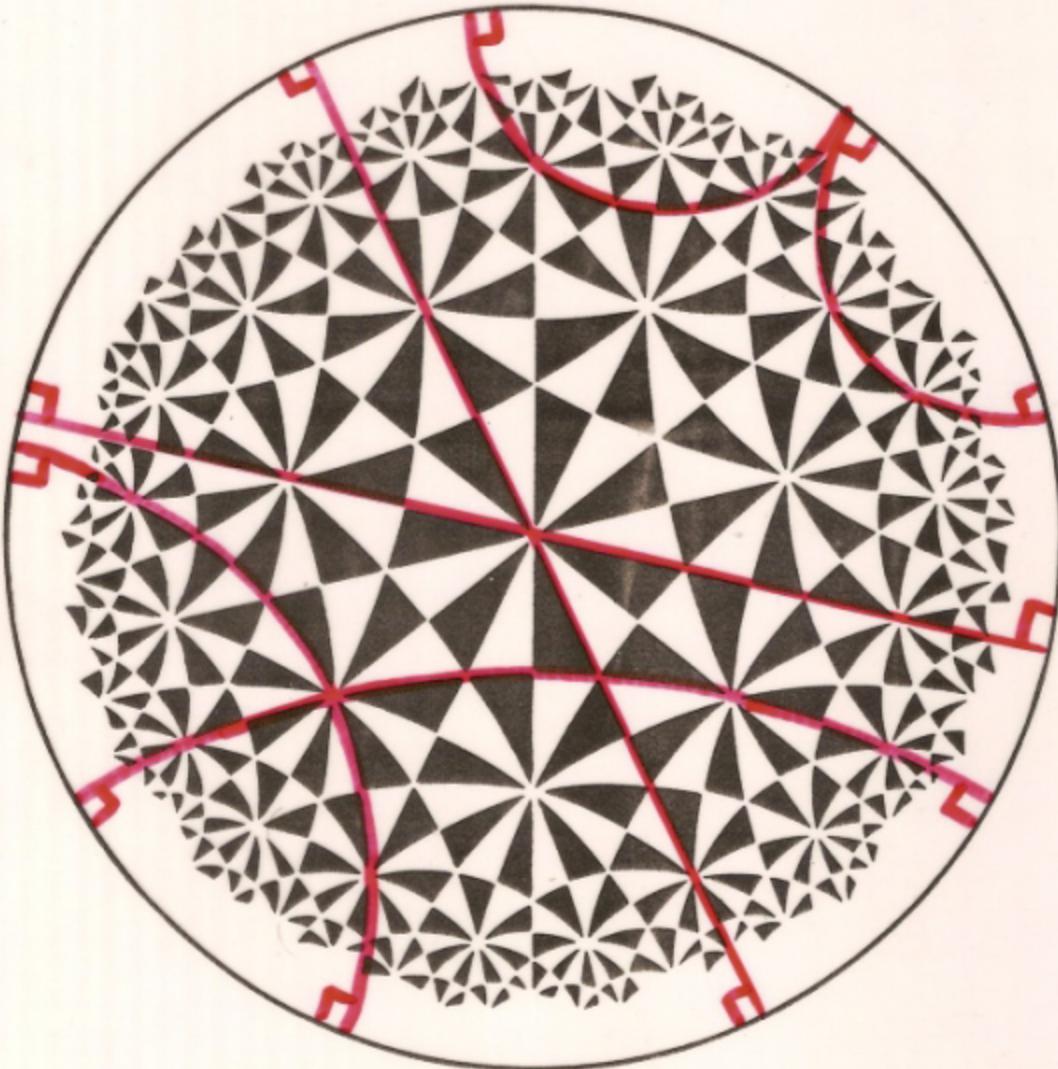
(C. F. Gauss)

(Wolfgang Bolyai)

- The angle sum of every triangle $< 180^\circ$.
- Given any line l and any point p not on l , there are infinitely many lines through p parallel to l .
- Any two similar figures must be congruent.



Poincaré's non-Euclidean geometry



Utter confusion

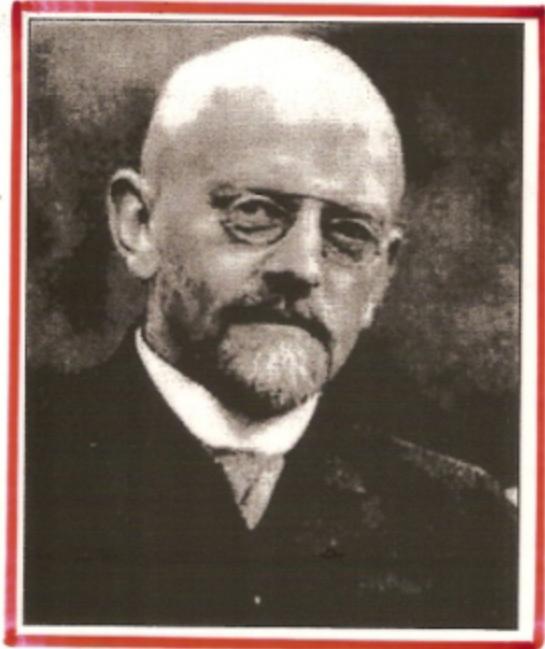
1860s : Riemann

There are infinitely many geometries - equally valid. Euclidean geometry is only one candidate for the space in which we live.

1870s : Klein's 'Erlanger Programm'

A geometry consists of a set of points with transformations acting on them.

We can classify all geometries : every geometry is contained inside projective geometry.



David Hilbert :
‘Foundations of
Geometry’

axiomatised Euclidean and
projective geometry (1899)

2nd edition (1903) axiomatised
non-Euclidean geometry

Euclid's student:

Student: What advantage shall I get by learning these things?

Euclid (to his slave):

Give him threepence -
for he must needs make profit
out of what he learns.