The story of i

Robin Wilson

George Airy: ‘I have not the smallest confidence in any result which is essentially obtained by the use of imaginary symbols.’

Augustus De Morgan: ‘We have shown the symbol $\sqrt{-1}$ to be void of meaning, or rather self-contradictory and absurd.’
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imaginary symbols,  

Augustus De Morgan  
We have shown the symbol \(\sqrt{-1}\) to be void of  
meaning, or rather self-contradictory and  
absurd.  

Euler  
All such expressions as \(\sqrt{-1}, \sqrt{-2}\), etc., are  
consequently impossible or imaginary  
numbers, since they represent roots of  
negative quantities; and of such numbers we  
may truly assert that they are neither nothing,  
nor greater than nothing, nor less than  
nothing, which necessarily constitutes them  
imaginary or impossible.
Some Numbers

Natural numbers:
1, 2, 3, 4, 5, ...

Integers:
..., -2, -1, 0, 1, 2, 3, ...

Rational numbers:
\( \frac{5}{7}, \frac{13}{3}, -\frac{1}{7}, \ldots \)

Real numbers:
\( \sqrt{2}, \sqrt{7}, \sqrt{2} + \sqrt{3}, \ldots \)
\( \pi, e, \ldots \)

Complex numbers:
\( \sqrt{-1}, 3 - 4\sqrt{-1}, \ldots \)
Complex Numbers

\[ a + b\sqrt{-1}, \text{ or } a + bi, \quad i^2 = -1. \]

**Addition**

\[
(2 + 3\sqrt{-1}) + (4 + 5\sqrt{-1})
\]

\[
= (2 + 4) + (3 + 5)\sqrt{-1} = 6 + 8\sqrt{-1};
\]

or: \((2+3i) + (4+5i) = 6 + 8i.\)

**Multiplication**

\[
(2 + 3\sqrt{-1}) \times (4 + 5\sqrt{-1})
\]

\[
= (2\times 4) + (3 \times 4)\sqrt{-1} + (2\times 5)\sqrt{-1} + (3\times 5)(-1)
\]

\[
= (8 - 15) + (12+10)\sqrt{-1} = -7 + 22\sqrt{-1};
\]

or: \((2+3i) \times (4+5i) = -7 + 22i.\)
Complex Numbers

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**Multiplication**

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\[ = (2 \times 4) + (3 \times 4)\sqrt{-1} + (2 \times 5)\sqrt{-1} + (3 \times 5)(-1) \]
\[ = (8 - 15) + (12 + 10)\sqrt{-1} = -7 + 22\sqrt{-1}; \]
\[ \text{or: } (2 + 3i) \times (4 + 5i) = -7 + 22i. \]
Three Quadratic Equations

- \( x^2 - 4x + 3 = 0 \)
  \[x^2 - 4x + 3 = (x-3)(x-1) = 0,\]
  so \( x = 3 \) or 1.

- \( x^2 - 4x + 4 = 0 \)
  \[x^2 - 4x + 4 = (x-2)^2 = 0,\]
  so \( x = 2 \) or 2.

- \( x^2 - 4x + 5 = 0 \)
  \[x^2 - 4x + 5 = (x-2-\sqrt{-1})(x-2+\sqrt{-1}) = 0,\]
  so \( x = 2+\sqrt{-1} \) or \( 2-\sqrt{-1} \).

Check: \((2+\sqrt{-1})^2 - 4(2+\sqrt{-1}) + 5\)
\[= (4+4\sqrt{-1}-1) - (8+4\sqrt{-1}) + 5 = 0.\]
Solving Quadratic Equations

\[ a x^2 + b x + c = 0 \] has solutions
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

So \( x^2 - 4x + c = 0 \) has solutions
\[ x = \frac{1}{2} \left( 4 \pm \sqrt{16 - 4c} \right) = 2 \pm \sqrt{4 - c} \]

\[ c = 3: \quad x = 2 \pm 1 = 3 \text{ or } 1 \]
\[ c = 4: \quad x = 2 \pm 0 = 2 \text{ or } 2 \]
\[ c = 5: \quad x = 2 \pm \sqrt{-1} = 2 + i \text{ or } 2 - i, \text{ where } i = \sqrt{-1} \]
A Sextic Polynomial

\[x^6 - 12x^5 + 60x^4 - 160x^3 + 239x^2 - 188x + 60 = 0\]

\[= (x^2 - 4x + 3)(x^2 - 4x + 4)(x^2 - 4x + 5)\]

\[= (x-1)(x-3)(x-2)^2 \times (x^2 - 4x + 5)\]

\[= (x-1)(x-3)(x-2)^2 \times (x-2-i)(x-2+i)\]

so \(x = 1, 3, 2\) (twice) and \(2 \pm i\).
Four scenarios

- We can solve all equations using only real and complex numbers;
- We may need to bring in some new ‘hyper-complex’ numbers to solve certain equations;
- Some equations may have solutions that are not numbers and don’t behave like them;
- Some equations may not have solutions of any kind.
What is the square root of $i$?

If $x^2 = i$ and $x = a + bi$, then $(a + bi)^2 = i$, so $(a^2 - b^2) + 2abi = i$. So $a^2 - b^2 = 0$ and $2ab = 1$. This has solutions $a = b = \pm \frac{1}{\sqrt{2}}$ so $x = \pm \frac{1}{\sqrt{2}} (1 + i)$. 
The Fundamental Theorem of Algebra

- Every polynomial $p(x)$ can be factorized into linear and quadratic polynomials with real coefficients.
- Every polynomial $p(x)$ can be factorized completely into linear factors with complex coefficients.
- Every polynomial equation $p(x) = 0$ of degree $n$ has at least one complex solution.
- Every polynomial equation $p(x) = 0$ of degree $n$ has exactly $n$ complex solutions (as long as we count them appropriately).
Solving a 'Quadratic Equation'

I have subtracted the side of my square from the area: 14, 30.

You write down 1, the coefficient.

You break off half of 1. 0; 30 and 0; 30 you multiply. You add 0; 15 to 14, 30.

Result 14, 30; 15. This is the square of 29; 30.

You add 0; 30, which you multiplied, to 29; 30.

Result: 30, the side of the square.

\[ x^2 - x = 870; \]

\[ 1 \to \frac{1}{2} \to \left(\frac{1}{2}\right)^2 = \frac{1}{4} \to 870 \frac{1}{4} \to 29 \frac{1}{2} \to 30. \]

\[ x^3 - bx = c: \]

\[ b \to \frac{1}{2} \to \left(\frac{1}{2}\right)^2 \to \left(\frac{1}{2}\right)^2 + c \to \sqrt{\left(\frac{1}{2}\right)^2 + c} \]

\[ \to \frac{b}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + c}. \]
al-Khwarizmi (c. 825 AD)
Solving Quadratic Equations

Six types: \( (a, b, c \text{ all positive}) \)
\[ ax^2 = bx, \ ax^2 = b, \ ax = b, \]
\[ ax^2 + bx = c, \ ax^2 + c = bx, \ ax^2 = bx + c \]

'Roots and squares are equal to numbers'

\[ x^2 + 10x = 39 \]
\[ (x + 5)^2 = 39 + 25 = 64 \]
so \( x + 5 = 8, \ x = 3 \)

\[ x^2 + 10x = 39 \]
\[ (x + 2\frac{1}{2} + 2\frac{1}{2})^2 = 39 + (4 \times 6\frac{1}{4}) = 64 \]
so \( x + 5 = 8, \ x = 3 \)
Heron becomes frustrated

frustrum of a pyramid

\[ a = 28, \ b = 4, \ c = 15 \]

height = \[ \sqrt{c^2 - 2\left(\frac{a-b}{2}\right)^2} \]

= \[ \sqrt{15^2 - 2\left(\frac{28-4}{2}\right)^2} \]

= \[ \sqrt{225 - (2\times144)} \]

= \[ \sqrt{225 - 288} = \sqrt{-63} \]

- but it appears as \( \sqrt{63} \) !
Hieronymi Car Danii, Praestantissimi Mathematici, Philosophi, ac Medicis, Artis Magnae. Siue De Regulis Algebraicis.

Lib. unus. Qui & totius operis de Arithmetica, quod Opus Perfectum inscriptum est in ordine Decimus.
Solving a cubic equation

\[ x^3 + 6x = 20 \]

Find \( u \) and \( v \) so that

\[ u - v = 20 \quad \text{and} \quad uv = \left(\frac{6}{13}\right)^3 = 8. \]

Since \( v = u - 20 \), we have

\[ uv = u(u - 20) = u^2 - 20u = 8. \]

Solving this quadratic equation:

\[ u = \sqrt{108} + 10. \]

So

\[ v = u - 20 = \sqrt{108} - 10. \]

So

\[ x = \sqrt[3]{u} - \sqrt[3]{v} \]

\[ = \sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)} \]

\[ = 2. \]
Cardano’s problem

Divide 10 into two parts whose product is 40.

If the parts are $x$ and $10 - x$, then $x(10 - x) = 40$.

Cardano: $x = 5 + \sqrt{-15}$ or $5 - \sqrt{-15}$

‘Nevertheless we will operate, putting aside the mental tortures involved.’

$x(10 - x) = (5 + \sqrt{-15})(5 - \sqrt{-15})$

$= 25 - (-15) = 40$. 
L'ALGEBRA
OPERA
Di RAPHAEL BOMBEILLI da Bologna
Diuita in tre Libri.
Con la quale ciascuno da se potrà venire in perfetta
cognizione della teorica dell'Arimeita.
Con una Tabola copiosa delle materie, che
in ella li contengono.
Post a hora in luce a beneficio dell'istudiosi di
detta professione.

IN BOLOGNA,
Per Giovanni Rosi. MDLXXIX.
Con licenza de Superiori.
Bombelli and complex numbers

$$x^3 = 15x + 4$$

Solutions: $4, -2 + \sqrt{3}, -2 - \sqrt{3}$

Cardano's method yields

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}$$

first appearance of complex numbers

Bombelli calculated that:

$$(2 + \sqrt{-1})^3 = 2 + \sqrt{-121},$$

$$(2 - \sqrt{-1})^3 = 2 - \sqrt{-121}.$$ 

So $x = (2 + \sqrt{-1}) - (-2 + \sqrt{-1}) = 4.$
Aora

Se ci si saesce ad aggiugliare i 15°

E i suoi fisica 

ce questo i 15° del qui

resta 0 m 121, sta di questo pigliata la

Eguale a 11 p +

s +

124

+ 15

s

ins

ins

Ins

fara 21 p 54

e tanto uale la

y lom 121

possi pri

com

Soma 21 lom 121 Rue 21 lom 121

p 15 lom 121 1 p 15 lom 121

Cicatore 15 lom 11 lom 11

m 15 lom 11

Somma 4 e tanto uale la cosa

p 15 lom 121 sarà 21 p 51 lom

c 15 lom 11 se aggiunto insieme fanno 4.
Descartes' constructions

The square root of $AB$ is $AD$.

$AC$ is the positive root of

$$x^2 = ax + b^2.$$
CHAP. LXVII.

The same Exemplified in Geometry.

What hath been already said of $\sqrt{} - dc$ in Algebra, (as a Mean Proportional between a Positive and a Negative Quantity:) may be thus Exemplified in Geometry.

If (for instance,) Forward from $A$, I take $AB = \frac{1}{2}dc$; and Forward from the same, $BC = \frac{1}{2}dc$ (making $AC = \frac{1}{2}AB - \frac{1}{2}BC = \frac{1}{2}dc$, the Diameter of a Circle:) Then is the Sine, or Mean Proportional $BP = \sqrt{} - dc$.

But if Backward from $A$, I take $AB = -\frac{1}{2}dc$; and then Forward from the same $BC = \frac{1}{2}dc$ (making $AC = -\frac{1}{2}AB - \frac{1}{2}BC = -\frac{1}{2}dc$, the Diameter of the Circle:) Then is the Tangent, or Mean Proportional $BP = \sqrt{} - dc$.

So that where $\sqrt{} + dc$ signifies a Sine; $\sqrt{} - dc$ shall signify a Tangent, to the same Arch (of the same Circle,) $AP$, from the same Point $P$, to the same Diameter $AC$. 
Wallis's constructions

\[ \frac{x}{p} = \frac{x}{b} \]
so \( x = \sqrt{ab} \)

\[ BP = \sqrt{AB \cdot BC} \]

\[ x = \sqrt{(-a)b} \]

\[ B'P = \sqrt{AB' \cdot B'C} \]
Dit akademiske analytiske Betegning,
et Forsøg, 
ansende forømmelig 
til 
plane og sfæriske Polygoners Oplosning. 

af 
Caspar Bessel, Landmaaler.
Fig. 1

Fig. 2

Fig. 3

horisont og lyser ud at det vil af $y = x^3$ i Fig. 3, se at $a = 6$ for-

fæller ud, de forgængere er $a = 6$, se Fig. 2. 

Fig. 1

Fig. 2

Fig. 3
The complex plane

represent the point \(a + bi\) by \((a, b)\)

Addition: \((3 + i) + (1 + 2i) = 4 + 3i\)

Multiplication: \([r, \theta] \times [s, \phi] = [rs, \theta + \phi]\)

So, to multiply by \(i\), rotate \(\theta\) through 90°

Do so twice: \(i \times i = -1\).
Wessel and De Moivre's Theorem

\[ [r, \theta] \times [s, \phi] = [rs, \theta + \phi] \]

Take \( r = s = 1 \), \( \phi = \theta \):

\[
\begin{align*}
(\cos \theta + i \sin \theta)^2 &= (\cos 2\theta + i \sin 2\theta) \\
(\cos \theta + i \sin \theta)^3 &= (\cos 3\theta + i \sin 3\theta) \\
\cdots \\
(\cos \theta + i \sin \theta)^{n} &= (\cos n\theta + i \sin n\theta) \\
\cdots
\end{align*}
\]
may apply the idea of direction, so that having chosen two opposite directions, one for positive and one for negative values, there shall exist a third—such that the positive direction shall stand in the same relation to it that the latter does to the negative.

4. If now we assume a fixed point $K$ (Fig. 1) and the line $KA$ be taken as positive unity, and we also regard its direction, from $K$ to $A$, and write $KA$ to distinguish it from the line $KA$ as simply an absolute distance, negative unity will be $\overline{KI}$, the vinculum having the same meaning as before, and the condition to be satisfied will be met by $KE$, perpendicular to the above and with a direction from $K$ to $E$, expressed in like manner by $\overline{KE}$. For the direction of $KA$ is to that of $KE$ as is the latter to that of $KI$. Moreover we see that this same condition is equally met by $KN$, as well as by $KE$, these two last quantities being related to each other as $+1$ and $-1$. They are, therefore, what is ordinarily expressed by $+\sqrt{-1}$ and $-\sqrt{-1}$. In an analogous manner we may insert other mean proportionals between the quantities just considered.

Thus to construct the mean proportional between $KA$ and $KE$, the line $CKL$ must be drawn so as to bisect the angle $AKE$, and the required mean will be $\overline{KC}$ or $KL$. So the line $GKP$ gives in like manner the means between $KE$ and $KI$, or between $\overline{KA}$ and $KN$. $\ldots$
biquadraticorum sstet, quam ab omni parte perfectam reddere in continuatione subsequentе suscipiēmus *).

31.

Ante omnia quasdam denominationes praemittimus, per quorum introductionem breuitati et perspicuitati consuletur.

Campus numerorum complexorum $a + bi$ continet

I. numeros reales, vbi $b = 0$, et, inter hos, pro indole ipsius $a$
   1) cifram
   2) numeros positivos
   3) numeros negativos

II. numeros imaginarios, vbi $b$ cifrae inaequalis. Hic iterum distinguuntur
   1) numeri imaginarii absque parte reali, i.e. vbi $a = 0$
   2) numeri imaginarii cum parte reali, vbi neque $b$ neque $a = 0$.

Prioris si placet numeri imaginarii puri, posterioris numeri imaginarii mixti vocari possunt.

Unitatis in hac doctrina vtinur quaternis, $+1$, $-1$, $+i$, $-i$ quae simpliciter positiva, negativa, positiva imaginaria, negativa imaginaria andient.

Producta terna cuiuslibet numeri complexi per $-1$, $+i$, $-i$ illius socios vel numeros illi associatos appellabimus. Excepta itaque cifra (quae sibi ipsa associata est), semper quaterni numeri inaequales associati sunt.

Contra numero complexo coniunctum vocamus eum, qui per permutationem ipsius $i$ cum $-i$ inde oritur. Inter numeros imaginarios itaque bini inaequales semper coniuncti sunt, dum numeri reales sibi ipsi sunt coniuncti, sicquidem denominationem ad hos extendere placet.

*) Obiter saltem hic scribe monere content, campum ita definitum imprimis theoriae residuorum biquadraticorum accommodatum esse. Theoria residuorum cubicorum simili modo superstrenuanda est considerationi numerorum formae $a + bh$, vbi $h$ est radix imaginaria aequationis $h^3 - 1 = 0$, puta $h = -1/2 + 3i$; et perinde theoria residuorum potestatem aliquorum introductionem aliarum quantitatum imaginariarum postulabit.
2. POLYHEDRA

Sir William Rowan Hamilton
(1805 - 1865)
Hamilton explains imaginaries

Define complex numbers as pairs

\[ a + bi \rightarrow (a, b) \]

which are combined as follows:

**Addition:**

\[ (a, b) + (c, d) = (a+c, b+d) \]

\[ [ (a+bi) + (c+di) = (a+c) + (b+d)i ] \]

**Multiplication**

\[ (a, b) \times (c, d) = (ac-bd, ad+bc) \]

\[ [ (a+bi) \times (c+di) = (ac-bd) + (ad+bc)i ] \]

Note that \((a, 0)\) corresponds to a (real)
and \((0, 1)\) corresponds to \(i\),

and \((0, 1) \times (0, 1) = (-1, 0) \quad [i^2 = -1]\)
Hamilton writes to his son

Every morning, on my coming down to breakfast, your little brother William Edwin and yourself used to ask me, ‘Well Papa, can you multiply triples? Where to I was obliged to reply, with a shake of the head: ‘No, I can only add and subtract them’.
Hamilton takes a walk

As I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations exactly as I have used them ever since.

I pulled out on the spot a pocket book and made an entry ... it is fair to say that this was because I felt a problem to have been at that moment solved — an intellectual want relieved which had haunted me for at least fifteen years since.
Hamilton's Quaternions

\[ a + bi + cj + dk, \quad i^2 = j^2 = k^2 = -1. \]

**Addition:** \[ \checkmark \]

**Multiplication:**

\[ ij = k, \quad jk = i, \quad bi = j \]
\[ ji = -k, \quad kj = -i, \quad ik = -j \]

More concisely:

\[ i^2 = j^2 = k^2 = ijk = -1. \]

Non-commutative system

**Pauli matrices:**

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & 0 \\
i & 0
\end{pmatrix}
\]

\[ ] \quad i \quad j \quad k \]
Hamilton at Brougham Bridge

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
\[ i^2 = j^2 = k^2 = ijk = -1 \]
& cut it on a stone of this bridge

plaque on bridge

pocket book
Octonions

\[ \alpha + \beta i + \gamma j + \delta k + \epsilon l + \zeta m + \eta n + \Theta o, \]

where \( i^2 = j^2 = k^2 = \ldots = o^2 = -1. \)

Addition as before

Multiplication:

\[ ni = 0 \]
\[ mk = n \]
\[ in = -o \]
\[ mj = -o, \text{ etc} \]

Three systems: \( xy = yx \quad (xy)z = x(yz) \)

Complexes: commutative, associative

Quaternions: not commutative, associative

Octonions: neither
Hamilton 1856

Icosian calculus:

\[ l^2 = k^3 = l^5 = 1, \text{ where } l = i k \]

Let \( m = i k^2 = l k \); then

\[ l^3 m^3 l m l m l^3 m^3 l m l m = 1 \]

\( l = \text{ right} \)

\( m = \text{ left} \)

Hamiltonian cycle on a dodecahedron
G. H. Hardy’s only geometrical result

**If a rectangular hyperbola is a parabola,**

**then it is also an equiangular spiral.**

The curve

\[(x + iy)^2 = \lambda (x - iy)\]

is (i) a parabola,

(ii) a rectangular hyperbola,

and (iii) an equiangular spiral.

The first two statements are evidently true. The polar equation is

\[r = \lambda e^{-3i\theta},\]

the equation of an equiangular spiral. ...