The 19th Century: Revolution or Evolution?

Robin Wilson

- Algebra
- Geometry
- Calculus / Analysis
- Foundations
Gaspard Monge
(1746 - 1818)
'descriptive geometry'

representing buildings
on a plane
The École Polytechnique, Paris

Monge, Laplace, Lagrange, Cauchy
Dürer on perspective drawing
Desargues' theorem
Poncelet: projective transformation

duality (Gergonne) (1827)

point $\leftrightarrow$ line

pole $\leftrightarrow$ polar
Gergonne on duality, 1827

Let there be a plane figure composed in any way one wishes of points, lines and curves ...

One then constructs in the same plane another figure ...

1. If there is a system of a certain number of points on a line, then in the other figure there will be a system of exactly as many lines meeting in a point.

1. If there is a system of a certain number of lines meeting in a point, then in the other figure there will be a system of exactly as many points lying on a line.

Any two points determine a line.

Any two lines determine a point.
The Barycentric Calculus

1827

P has barycentric coordinates

\([a, b, c]\)

Note: \([a, b, c] = [\lambda a, \lambda b, \lambda c]\) for any \(\lambda \neq 0\)

‘homogeneous coordinates’

Duality: points ↔ lines

\([a, b, c] \leftrightarrow ax + by + cz = 0\)
Solving Quadratic Equations

Six types: (a, b, c all positive)

\[ ax^2 = bx, \quad ax^2 = b, \quad ax = b, \]
\[ ax^2 + bx = c, \quad ax^2 + c = bx, \quad ax^2 = bx + c \]

'Roots and squares are equal to numbers'

\[ x^2 + 10x = 39 \]

\[(x + 5)^2 = 39 + 25 = 64\]

So \(x + 5 = 8\), \(x = 3\)
QVESITI,
ET INVENTIONI
DIVERSE
DE NICOLÒ TARTAGLIA,
Di nuo testampati con vna Giunta al festo libro, nella quale si
mostra duo i modi di redur vna Città inespugnabile.
La divisione, ò continentia di tutta l'opra nel segnente foglio si
trouarà notata.
Solving a cubic equation

\[ x^3 + 6x = 20 \]

Find \( u \) and \( v \) so that

\[ u - v = 20 \quad \text{and} \quad uv = \left(\frac{6}{3}\right)^3 = 8. \]

Since \( v = u - 20 \), we have

\[ uv = u(u - 20) = u^2 - 20u = 8. \]

Solving this quadratic equation:

\[ u = \sqrt{108} + 10. \]

So \( v = u - 20 = \sqrt{108} - 10. \)

So

\[ x = \sqrt[3]{u} - \sqrt[3]{v} \]

\[ = \sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)} \]

\[ = 2. \]
Solving a quartic equation

\[ x^4 = px^2 + qx + r \]

\[ (x^2 + y)^2 = (p+2y)x^2 + qx + (r+y^2) \]

For RHS a square we need:

\[ (p+2y)(r+y^2) = q^2 \]

\[ b^2 = 4ac \]

(4) \[ 2y^3 + py^2 + 2ry + (pr - q^2) = 0 \]

(5) then reduces to two quadratics.

E.g.

\[ x^4 = x + 2 \]

\[ x = \sqrt[3]{\frac{2075}{6422368}} + \frac{1}{128} - \sqrt[3]{\frac{2075}{6422368}} - \frac{1}{128} \]

\[ + \sqrt[3]{\frac{1051}{3456}} + \sqrt[3]{\frac{2075}{6422368}} + \sqrt[3]{\frac{1051}{3456}} - \frac{2075}{6422368} \]

\[ + \frac{2}{3} - \sqrt[3]{\frac{2075}{6422368}} + \frac{1}{128} - \sqrt[3]{\frac{2075}{6422368}} - \frac{1}{128} \]
Lagrange's reduction methods

\[ x^3 + nx + p = 0 \]
put \[ x = y - \frac{n}{3y} \]
\[ y^6 + py^3 - \frac{1}{27} n^3 = 0 \]

quadratic in \( y^3 \)
six \( y \)'s \( \rightarrow \) three \( x \)'s

Quartic: \[ x^4 + nx^3 + px^2 + q = 0 \]
reduces to \[ y^3 - \frac{n}{2} y^2 - qy + \frac{4nq - p^2}{8} = 0 \]
cubic in \( y \), ...
Permuting the solutions

\[ x^3 + ax^2 + bx + c = 0 \]

solutions \( p, q, r \)

\[ x^3 + ax^2 + bx + c = (x-p)(x-q)(x-r) \]

\[ c = -pqr, \quad b = pq + pr + qr, \quad a = -(p+q+r) \]

Permute the solutions - no change

\[ pq + 2r : \quad pr + 2q \quad qr + 2p \quad (3 \text{ values}) \]

\[ p + 2q + 5r : \quad p + 2r + 5q, \ldots \quad (6 \text{ values}) \]

\[ \ldots \quad (\text{always divides 6}) \]
Paolo Ruffini

(1765 - 1822)

The algebraic solution of equations of degree greater than 4 is always impossible.

Behold a very important theorem which I believe I am able to assert...
Niels Henrik Abel
(1802-1829)
proved impossibility
of solving quintic equations

Evariste Galois
(1811-1832)
developed criteria for
deciding which equations
can be solved
Galois to Chevalier, 29 May 1832

My dear friend,

I have done several new things in analysis. Some concern the theory of equations; others, integral functions.

In the theory of equations I have found out in which cases the equations are solvable by radicals, which has given me the occasion to deepen the theory and to describe all the transformations admitted by an equation, even when it is not solvable by radicals.

... You will publicly beg Jacobi or Gauss to give their opinion not of the truth but of the importance of the theorems.

After this, there will, I hope, be people who will find it to their advantage to decipher all this mess.
Euclid's Postulates

Let the following be postulated:

1. To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any centre and distance.

4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
Euclid's Fifth Postulate

$\alpha + \beta < 180^\circ \Rightarrow \text{lines meet}$

Euclid I. 29:

‘parallel postulate’

Alhazen (1000 AD)

G. Saccheri (1733)

J. Wallis (1663)

Sum = $180^\circ$

Sum $> 180^\circ$ (impossible)

Sum $< 180^\circ$?
Non-Euclidean Geometry (c. 1830)

Nikolai Lobachevskii
János Bolyai
(C. F. Gauss)
(Wolfgang Bolyai)

- The angle sum of every triangle < 180°.
- Given any line \( l \) and any point \( p \) not on \( l \), there are infinitely many lines through \( p \) parallel to \( l \).
- Any two similar figures must be congruent.
Letter from Gauss:
Another theme which is almost 40 years old with me that I have been thinking about now and again in a few free hours; I mean the foundations of geometry. ... 
My opinion that we cannot establish geometry completely a priori is, if possible, much firmer. Meanwhile I will still not get round to it for some time and work up my very extensive researches for publication, and perhaps they will never appear in my lifetime, for I fear the howl of the Boeotians if I speak my opinion out loud.

Abel, from Berlin in 1825:
Crelle says that all Gauss writes is gruel since it is so obscure that it is almost impossible to understand.
Letter from Gauss to F. Bolyai, 1831:

If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it would be to praise myself.

Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years.
Poincaré’s non-Euclidean geometry
Kleinian View of Geometry (1872)

Erlanger Programm

1. A geometry is a space and a group of transformations on that space.

2. We are concerned with geometrical properties: which are unchanged by elements of the group?

3. The larger the group, the fewer geometrical properties are unchanged.
David Hilbert:

‘Foundations of Geometry’

axiomatised Euclidean and projective geometry (1899)

2nd edition (1903) axiomatised non-Euclidean geometry
Newton's Calculus

Variables: changing with time - 'flowing'

Derivatives: based on velocity - notation \( \dot{x}, \dot{y} \)

Example: \( y = x^2 \)

Substitute \( x + \Delta x \) for \( x \)
\( y + \Delta y \) for \( y \):

\[
y + \Delta y = (x + \Delta x)^2 = x^2 + 2x \Delta x + (\Delta x)^2
\]

Cancel \( \Delta x \): \( \dot{y} = 2x \dot{x} + \dot{x}^2 \Delta
\]

Ignore \( \Delta \): \( \dot{y} = 2x \dot{x}, \) or \( \dot{y}/\dot{x} = 2x \)

Integrals: find anti-derivatives (fundamental theorem)
Bernard Bolzano
(1781–1848)

Rein analytischer Beweis des Lehrsatzes, das zwischen je zwei Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege;

von
Bernard Bolzano,
Mitglied der Philosoph. r. r. Professoren der Religionswissenschaft, und ordentlichem Mitglied der r. Gesellschaft der Wissenschaften zu Prag.

Für die Abhandlungen der Gesellschaft der Wissenschaften.

Prag. 1817, gedruckt bei Gottlob Haase.

Bolzano's 1817 pamphlet

Intermediate Value Theorem
The ‘number’ \( \sqrt{2} \)

\[ f(x) = x^2 - 2 \]

on the interval \([0, 2]\)

\[ f(0) = -2 \]
\[ f(2) = 2 \]

so there is a number \( x \) such that \( x^2 - 2 = 0 \).
Augustin-Louis Cauchy
(1789 - 1857)
Lorsque les valeurs successivement attribuées à une même variable s'approchent indéfiniment d'une valeur fixe, de manière à finir par en différer aussi peu que l'on voudra, cette dernière est appelée la limite de toutes les autres.

Ainsi, par exemple, un nombre irrationnel est la limite des diverses fractions qui en fournissent des valeurs de plus en plus approchées. En géométrie, la surface du cercle est la limite vers laquelle convergent les surfaces des polygones inscrits, tandis que le nombre de leurs côtés croît de plus en plus; &c. &c. &c.
Limits and Continuity

Bolzano (1817):

A function \( f(x) \) varies according to the law of continuity... if \( x \) has some such value, the difference \( f(x + w) - f(x) \) can be made smaller than any given quantity provided \( w \) can be taken as small as we please.

Cauchy (1821):

When the values successively attributed to the same variable approach a fixed value indefinitely, in such a way as to end up by differing from it as little as we could wish, this last value is called the limit of all the others...

Examples: irrationsals (limit of fractions) circles (limit of polygons)
The blancmange function

Eventually we obtain the following graph of $B$: 
Some Numbers

Natural numbers:
1, 2, 3, 4, 5, ...

Integers:
..., -2, -1, 0, 1, 2, 3, ...

Rational numbers:
5/7, 11/3, -1/7, ...

Real numbers:
\sqrt{2}, \sqrt{7}, \sqrt{2} + \sqrt{3}, ...
π, e, ...

Complex numbers:
\sqrt{-1}, 3 - 4\sqrt{-1}, ...

x + 3 = 7
x + 7 = 3
7x = 5
x^3 = 7
x^3 = -1
Rationals and Irrationals

\[
\frac{3}{4} = 0.75
\]
\[
\frac{1}{3} = 0.3333\ldots
\]
\[
\frac{2}{7} = 0.28571428571428\ldots
\]

Add:
\[
\frac{115}{84} = 1.3690476190476190\ldots
\]

Every rational number has a finite or recurring decimal, and every finite or recurring decimal can be written as a fraction.

\[
x = 0.24242424\ldots
\]
\[
100x = 24.24242424\ldots
\]
\[
99x = 24,
\]
\[
\text{so } x = \frac{24}{99} = \frac{8}{33}
\]
Defining real numbers

Define a real number as an infinite decimal - for example:

\[ \sqrt{2} = 1.414213 \ldots \]

\[ \pi = 3.1415926 \ldots \]

Problem: how do we prove that

\[ \sqrt{2} \times \sqrt{2} = 2 \]
Richard Dedekind
(1831-1916)

Georg Cantor
(1845-1918)
Dedekind cuts

1858, 1872: Dedekind cut: definition of a real number

Basic idea: the real line and the rationals differ - the latter has gaps (e.g. at $\sqrt{2}, \pi, \ldots$)

Aim: Fill the gaps with numbers

Dedekind: each gap is a number

It is defined by all rationals less than it, and greater than it:

\[
\begin{array}{c}
\sqrt{2} \\
\hline
\text{all rationals} \quad \text{all rationals} \\
\text{less than } \sqrt{2} \quad \text{greater than } \sqrt{2}
\end{array}
\]
How big is a set?

- \( \{100, 101\} \) \( \{1, 2, 3, 4\} \)
- \( \{1, 2, 3, 4, 5, 6, \ldots\} \)
- \( \{1, 4, 9, 16, 25, 36, \ldots\} \)
- \( \{\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \ldots\} \)
- \( \{0, 1, -1, 2, -2, 3, \ldots\} \)

A set is countable if we can list them all — for example, the rationals:
The real numbers are not countable

Suppose that the real numbers between 0 and 1 are countable: let’s list them all—

0 . a, a₂ a₃ a₄ a₅ ...
0 . b₁ b₂ b₃ b₄ b₅ ...
0 . c₁ c₂ c₃ c₄ c₅ ...
0 . d₁ d₂ d₃ d₄ d₅ ...
...

Now choose numbers \( X₁, X₂, X₃, \ldots \) so that

\[
X₁ \neq a₁, \ X₂ \neq b₂, \ X₃ \neq c₃, \ X₄ \neq d₄, \ldots
\]

Then the number \( 0 \cdot X₁X₂X₃X₄ \ldots \) is not in the above list — contradiction.
So they are NOT countable.