The 18th Century: Crossing Bridges

Robin Wilson
The Open University
and Gresham College
PHILOSOPHIAE
NATURALIS
PRINCIPIA
MATHEMATICA.

Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, REG. SOC. PRÆSES.
Juli 5. 1686.

LONDINI,
Jussu Societatis Regiae ac Typis Iosephi Sweeter. Prostat apud
plures Bibliopolas. ANNO MDCLXXXVII.
The Royal Mint
Gresham College in the 18th Century

View of Gresham College as it appeared before it was taken down
The Second Royal Exchange

Royall Exchange of
The
London
Newton at the Royal Society
OPTICKS:
OR, A TREATISE OF THE REFLEXIONS, REFRACTIONS, INFLEXIONS and COLOURS OF LIGHT.
ALSO TWO TREATISES OF THE SPECIES and MAGNITUDE OF CURVILINEAR FIGURES.

LONDON,
Printed for SAM. SMITH. and BENJ. WALFORD.
Printers to the Royal Society, at the Prince's Arms in St. Paul's Church-yard. MDCCIV.
Gottfried Wilhelm Leibniz
(1646 - 1716)
<table>
<thead>
<tr>
<th>Britain vs. the Continent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newton</td>
</tr>
<tr>
<td>Halley</td>
</tr>
<tr>
<td>Taylor</td>
</tr>
<tr>
<td>Maclaurin</td>
</tr>
<tr>
<td>Whiston</td>
</tr>
<tr>
<td>Saunderson</td>
</tr>
<tr>
<td>Bradley</td>
</tr>
<tr>
<td>...</td>
</tr>
</tbody>
</table>
The Death of Isaac Newton, 1727
Newtonian Philosophy

EDMOND HALLEY (1656-1742)

- Savilian Professor of Geometry
- while an undergraduate, sailed to St. Helena to observe southern stars
- FRs at age 22
- persuaded Newton to write ‘Principia’
- sea captain (52° south)
- edition of Apollonius’ ‘Conics’
- Astronomer Royal, 1720-1742
- predicted return of ‘Halley’s comet’

Newtonian philosophy advanced in the OLD ASHMOLEAN (1683)
in the lectures by the Savilian professors

DAVID GREGORY, JAMES BRADLEY

Radcliffe Observatory built
Halley's Edition of Apollonius's 'Conics' (1710)
Newton's calculus

Variables: changing with time
- 'flowing'

Derivatives: based on velocity
(tangents) - notation \( \dot{x}, \dot{y} \)

Example: \( y = x^2 \)

Substitute \( x + \Delta x \) for \( x \)
\n\[ y + \Delta y = (x + \Delta x)^2 \]

\[ y + \Delta y = x^2 + 2x \Delta x + \Delta x^2 \]

Cancel \( \Delta x \):
\[ \dot{y} = 2x \dot{x} + \dot{x}^2 \]

Ignore \( \Delta x \):
\[ \dot{y} = 2x \dot{x}, \text{ or } \frac{\dot{y}}{\dot{x}} = 2x \]

Integrals: find anti-derivatives
(areas) (fundamental theorem)
Bishop Berkeley's 'The Analyst'

THE ANALYST;
OR, A DISCOURSE
Addressed to an
Infidel Mathematician.
WHEREIN
It is examined whether the Objects, Principles, and Inferences of the modern Analysts are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

By the Author of The Minute Philosopher.

THE SECOND EDITION.

LONDON:
Printed for J. and R. Tonson and S. Draper in the Strand.

MDCCCLV.
Bishop Berkeley’s *The Analyst* (1734)

The Analyst, or a discourse addressed to an infidel mathematician wherein it is examined whether the Object, Principles and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

*He used these fluxions like the scaffold of a building, as things to be laid aside or got rid of.*

*What are these fluxions?*

*The velocities of evanescent increments?*

*And what are these same evanescent increments?*

*They are neither finite quantities, nor quantities infinitely small, nor yet nothing.*

*May we not call them the Ghosts of departed quantities?*
Colin Maclaurin (1698-1746)
Principia Mathematica (1687)

Laws of Motion

Book I : The Motion of Bodies
- Inverse-square law of gravity
- Kepler's laws

Book II : Motion in Resisting Media
- Wave motion for light and sound
- Vortices : Descartes' theory

Book III : The System of the World
- Orbits of comets
- Motion of the moon
- Theory of tides
- Flattening of the Earth at the poles
- Precession of the equinoxes
René Descartes

Theory of vortices

Tourbillons de Des Cartes

Segnius infans animae demissa
per aures
Quam qua sunt saevis. Subjecta fidelibus
Prop. XVIII, Theorem XVI:

That the axes of the planets are less than the diameters drawn perpendicular to the axes.

... if our earth was not higher about the equator than at the poles, the sea would subside about the poles, and, rising towards the equator, would lay all things there under water.

... the earth will be higher at the equator than at the poles by 854 72 feet, or 17 ½0 miles. And its height at the equator will be about 19,658,600 feet, and at the poles 19,573,000 feet.
Success of the Principia

- Immediate in Britain and the Netherlands
  'the greatest Discovery in Nature that ever was since the World's Creation'.

- France: de l'Hopital, Voltaire, Mme du Chatelet

- Maupertuis:
  The shape of the Earth: geodetic missions to Lapland and Peru to measure pendulum swings;
  Newton correct: Earth is flatter at the poles.
What shape is the Earth?

In 1672, Mr Richer, in a voyage to Cayenna, near the equator, found that the pendulum of this clock no longer made its vibrations so frequently as in the latitude of Paris. In consequence of this it was discovered that, whereas the gravity of bodies is by so much the less powerful, as these bodies are farther removed from the centre of the earth, the region of the equator must absolutely be much more elevated than that of France; and so must be further removed from the centre; and, therefore, that the earth could not be a sphere.

Voltaire

It is evident in general that Sir Isaac Newton's Figure of a flat Spheroid, and Mr Cassini's of a long one, will give very different Distances of Places that have the same Longitude and Latitude. ... In a course of 100 Degrees Longitude, there might be a Mistake of more than two Degrees, if sailing really upon Sir Isaac Newton's Earth one should imagine himself to be upon Mr Cassini's. And how many Ships have perished by smaller mistakes?

Pierre Maupertuis
Maupertuis's Mission to Lapland
Letters between Euler and Clairaut

Clairaut to Euler: 11 Sept 1747
Pending something better, the law for all nature should be as $1/d^2$ + a small function of distances detectable for the moon and almost zero for great distances...

Euler to Clairaut: 30 Sept 1747
- disagreed - Mercury doesn't fit the explanation

Clairaut to Euler: 7 December 1747
Mercury can be dealt with by adding $l^2/357d^2$ ($d$ = distance between earth and moon) - but this doesn't seem right - and $1/d^4$ is too strong

Clairaut - general announcement: 17 May 1749
Having considered anew from a new viewpoint, I have been led to reconcile observations with a force of the inverse square of the distance.
225 Mécanique analytique.
9. De cette manière la formule générale du mouvement
\( r + s + t = 0 \) (art. 1) sera transformée en celle-ci,
\[ x = d - \frac{d^2}{2} + \frac{d^3}{3} + \frac{d^4}{4} + \frac{d^5}{5} + \cdots, \]
dans laquelle en aura
\[ x = d, \quad \frac{d}{dt} = \frac{d^2}{2} + \frac{d^3}{3} + \frac{d^4}{4} + \cdots, \]
\[ \frac{d^2}{dt^2} = \frac{d^3}{2} + \frac{d^4}{3} + \frac{d^5}{4} + \cdots, \]
\[ \frac{d^3}{dt^3} = \frac{d^4}{2} + \frac{d^5}{3} + \frac{d^6}{4} + \cdots, \]
\[ \text{etc.} \]
en supposant
\[ I = S \left( \frac{d^2}{dt^2} + \frac{d^3}{2} + \frac{d^4}{3} + \cdots \right) dt, \quad P = S \frac{d}{dt}, \]
\[ q \frac{d^2}{dt^2} = P \frac{d^2}{dt^2} + Q \frac{d}{dt} + R dt + \text{etc.} \]
Si donc dans le choix des nouvelles variables \( \xi, \eta, \theta, \text{etc.} \), on a eu égard aux équations de condition données par la nature du système proposé, enfin que ces variables soient maintenues mutuellement indépendantes les unes des autres, & que par conséquent leurs variations \( \xi, \eta, \theta, \text{etc.} \), demeurent absolument indéterminées ; en sera sur le champ les équations particulières \( \xi = \xi_0, \eta = \eta_0, \theta = \theta_0, \text{etc.} \), laquelle serviront à déterminer le mouvement du système ; puisque ces équations sont en même nombre que les variables \( \xi, \eta, \theta, \text{etc.} \), d'où dépend la position du système à chaque instant.
Mais quelques fois toujours remuer la question à cet égard, puisqu'il ne s'agit que d'intégrer par les équations de condition, autour de variables quelles que soient celles de faire, & de prendre ensuite pour \( \xi, \eta, \theta, \text{etc.} \), les variables

---

**Celestial Mechanics**

MARQUIS DE LA PLACE.

VOLUME 1.

CHICAGO PUBLISHING COMPANY, INC.

**Celestial Mechanics**

By the Marquis de la Place.

Translated into English by Nathaniel Bowditch, LL.D.

VOLUME 1.

Translation by C. F. Millikan.

CHICAGO PUBLISHING COMPANY, INC.

---

**Mécanique Céleste**

By the Marquis de la Place.

VOLUME 1.

Translation by C. F. Millikan.

CHICAGO PUBLISHING COMPANY, INC.
Nature, and Nature’s Laws
lay hid in Night.

God said, Let Newton be!
and All was Light.

A. POPE
LINEÆ
TERTII ORDINIS
NEUTONIANÆ,
SIVE
Illustratio Tractatus D. Neutoni
De Enumeratione Linearum
Tertii Ordinis,
Cum Subjungitur,
Solutio Trium Problematum.


OXONIAE,
Ex Theatro Sheldoniano, Imensis Eduardi Whistler
Bibliopolæ Oxoniensis, MDCCXVII.
Hornsby's Radcliffe Observatory
A New METHOD
For Discovering the
LONGITUDE
BOTH AT
SEA and LAND,
Humbly Proposed to the Consideration
of the Publick.

BY
William Whiston, M.A.
Humphry Dixon, Master
of the New Mathematics
School in
London.

LONDON:
Printed for John Phillips, at the Black
Mr. Saunderson's Mechanicks

1. Mechanics is that Art of Natural Philosophy which treats of Motion & Laws of Motion.

2. What we have to say on this subject shall be as follows: Where a Body shall remain in a State of Motion, contrary to its Nature & Quantity, it shall remain.

3. And we shall speak of this Law of Motion, of the Effects of moving one upon another, as in a Line, in a Plane, &c.

Rest of Motion in general.

4. Motion is a continual slipping out of one place into another, without returning in the same place for any Time, that is, the effect of Motion, by its way through a certain Space, will, in an equal Time, be in the same Proportion, or Motion, as in the same Proportion of Time, a Body is moved by the pull of another Body proportional to the Body & that Force a Body is moved by the pull of another Body proportional to the Body & Time is moved by the pull of another Body.
De L'Sphere
Septentrione

[Diagram of a spherical projection with grid lines and labeled points]
The Ballad of Gresham College

If to be rich and to be learn’d

Be every Nation’s cheifest glory,

How much are English men concern’d,

Gresham to celebrate thy story

Who built th’Exchange t’enrich the Citty

And a Colledge founded for the witty.

. . .

The College will the whole world measure;

Which most impossible conclude,

And Navigation make a pleasure

By finding out the Longitude.

Every Tar shall then with ease

Sayle any ship to the Antipodes.
The Wreck of the 'Association' (1707)
Leibniz's calculus

Variables: no concept of motion  
- sequences of close values

Derivatives: differences of successive values - infinitely small  
- notation $\frac{dy}{dx}$

Integrals: area = sum of lines  
- notation: all lines = omn. $\int l$

\[ y \]  
\[ x \]
Jakob Bernoulli (1654-1705)

polar coordinates  catenary

logarithmic spiral

probability ('Ars Conjectandi')

word "integral"
Jakob Bernoulli's 'Ars Conjectandi' (1713)

JACOBI BERNOULLI,
Gall. & Pruss. Sodal.
MATHEMATICI CELEBRERINI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Acad.
TRACTATUS
DE SERIEBUS INFINITIS,
EtEpistola Gallicæscripta
DE LUDO PILE
RETICULARIS.

BASILEÆ,
Impensis THURNISIORUM, Fratrum.
clo lecc xiii.
(1713)

probability:
'law of large numbers'
limit theorems
binomial distribution

Combinatorics:
figurate numbers
combinations
'Bernoulli numbers'

preceded by a hymn on the infinite variety of
nature - this variety stems from the combinations
and arrangements of its parts, and the
combinatorial art helps us to enumerate these.
Brachistochrone Problem

1697: Johann Bernoulli:

Find the 'curve of quickest descent'

' I recognize the lion by his claws'
Johann Bernoulli (1667-1748)
calculus of variations
brachistochrone
teacher of L'Hôpital,...

Daniel Bernoulli (1700-1782)
probability
hydrodynamics
Published the first book on the calculus (1696):

‘Analyse des infiniment petits’
Leonhard Euler

(1707 - 1783)
INSTITUTIONES CALCULI DIFFERENTIALIS
CUM EIUS USU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM

AUCTORE
LEONHARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTOR
PROP. HONOR. ACAD. IMP. SCIENT. PETROP. ET ACADEMIAE REGIRUM PARISINAE ET LONDONIENSIS SOCIO.

IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM PETROPOLITANAE
1755.
INSTITUTIONVM
CALCVLI INTEGRALIS
VOLUMEN PRIMUM
IN QVO METHODVS INTEGRANDI A PRIMIS PRIN-
CIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFE-
RENTIALVM PRIMI GRADVS PERTRACTATVR.

AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LONDIN.

PETROPOLI
Impensis Academiae Imperialis Scientiarum
1768.
Joseph-Louis Lagrange (1736-1813)

\[ f(x) = a + bx + cx^2 + dx^3 + \ldots \]
\[ f'(x) = b + 2cx + 3dx^2 + \ldots \]

\[ \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots \]

so \[ \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots \]
MAP OF KÖNIGSBERG

From M. Zeiller, Topographia Prussiae et Pomerilliae, Frankfurt, c. 1650.
SOLUTIO PROBLEMATIS
AD GEOMETRIAM SITUS PERTINENTIS

Commentatio 53 indicis ENESTROMIANI
Commentorum academiae scientiarum Petropolitanae 8 (1736), 1741, p. 128—140

1. Praeter illam geometricae partem, quae circa quantitates versatur et
omni tempore summo studio est exculta, aliterius partis etiamnum admodum
ignotae primus mentionem fecit LEHRNZIUS), quam Geometricam situs
vocavit. Ista pars ab ipso in solo situ determinando situisque proprietatibus erudendis
occupata esse statuitur; in quo negotio neque ad quantitates respicientium
neque calculo quantitatum utendum sit. Cuiusmodi autem problemata ad
hanc situs geometricam pertinente et quali methodo in iis resolvendis uti
oporeat, non satis est definitum. Quamobrem, cum nuper problematis cuius-
dam mentio esset facta, quod quidem ad geometricam pertinere videbatur, at
ita erat comparatum, ut neque determinationem quantitatum requireret neque
solutionem calculi quantitatum ope admitteret, id ad geometricam situs referre
haud dubitavi, praesertim quod in eius solutione solus situs in considerationem
veniat, calculus vero nullius prorsus sit usus. Methodum ergo meam, quam
ad huius generis problemata solvenda inveni, tanquam specimen Geometricae
situs hic exponere constituui.

2. Problema autem hoc, quod mihi satis notum esse perhibebatur, erat
sequens: Regiomonti in Borussia esse insulam A, der Kneiphof dictam, flu-
viumque aam cingentem in duos dividit: ramos, quemadmodum ex figura (Fig. 1)
videre licet; ramos vero huius fluvii septem instructos esse pontibus a, b, c, d,
e, f et g. Circa hos pontes iam ista proponebat quae siq, num quis cursum
ita instituere quaret, ut per singulos pontes semel et non plus quam semel
transeat. Hocque fieri posse, mihi dictum est, alios negare alios dubitare;
neminem vero affirmare. Ego ex hoc mihi sequens maxime generale formavi
problema: quaecunque sit fluvii figura et distributio in ramos atque qui-
cunque fuerit numerus pontium, inventa, utrum per singulos pontes semel
tantum transiri quceat an vero secus.

Fig. 1.
Letter from Leonhard Euler
to Giovanni Marinoni (Vienna)
13 March 1736

See image for handwritten text.
16. Sint quae insulae $A$ et $B$ aqua circumdatae, qua cum aqua commincent quattuor fluvii, quemadmodum figura (Fig. 3) repraesentat. Traiecto porro sint super aquam insulas circumdantem et fluvios quindecim pontes $a$, $b$, $c$, $d$ etc. et quaeritur, num quis cursum ita instituere queat, ut per

omnes pontes transeat, per nullum autem plus quam semel. Designo ergo primum omnes regiones, quae aqua a se invicem sunt separatae, litteris $A$, $B$, $C$, $D$, $E$, $F$, cuiusmodi ergo sunt sex regiones. Dein numerum pontium 15 unitate augeo et summam 16 sequenti operationi praefigo.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^s$</td>
<td>8</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$B^s$</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$C^s$</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$E$</td>
<td>5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$F^s$</td>
<td>6</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

20. Casu ergo quocunque proposito statim facillime poterit cognoscii, utrum transitus per omnes pontes semel institui queat an non, ope huius regulae:

- Si fuerint plures duabus regiones, ad quas ducentium pontium numerus est impar, tum certo affirmari potest talem transitum non dari.
- Si autem ad duas tantum regiones ducentium pontium numerus est impar, tum transitus fieri poterit, si modo cursus in altera harum regionum incipiatur.
- Si denique nulla omnino fuerit regio, ad quam pontes numero impares conducant, tum transitus desiderato modo institui poterit, in quacunque regione ambulandi initium ponatur.

Hac igitur data regula problemati proposito plenissime satisfit.
The BRIDGES of KÖNIGSBERG

Solved by EULER in 1736

whenever you enter a vertex, you must be able to leave it

So 0 or 2 odd degrees. Konigsberg has degrees 3, 3, 3, 5, and so is impossible.
Euler's Polyhedron Formula

8 vertices
6 faces
12 edges
8 + 6 = 12 + 2

V + F = E + 2

20 + 12 = 30 + 2
12 + 20 = 30 + 2
Neulich kam mir in Sinn die allgemeinen Eigenschaften der Körper, welche hedris planis eingeschlossen sind, zu bestimmen, weil kein Zweifel ist, dass sich in denselben nicht eben dergleichen allgemeine Eigenschaften finden sollten, als in den figuris planis rectilineis, deren Eigenschaften darin bestehen, dass 1. in einer jeglichen figura plana der numerus laterum dem numero angulorum gleich ist, hernach 2. dass die summa angulorum omnium gleich ist bis tot rectis quot sunt latera, deditis quatuor. Wie aber in den figuris planis nur latera und anguli zu betrachten vorkommen, so müssen bei den Körpern mehr Stücke in Betrachtung gezogen werden, als

1. die hedrae, deren Anzahl sey \( H \);
2. die anguli solidi, deren Anzahl sey \( S \);
3. die Fügungen, wo zwey hedrae secundum latera zusammenkommen, so ich aus Mangel eines recipirten Wortes, actis nenne, deren Anzahl sey \( A \);
4. die latera singularum hedrarum, quorum omnium simul summorum numerus sit \( L \);
5. die anguli plani singularum hedrarum, quorum omnium numerus sit \( P \).

1. Bei diesen fünf Stücken ist nun erstlich klar, dass \( P = L \), weil in allen hedris der numerus angulorum = numerus laterum.
2. Ist auch immer \( A = \frac{1}{2} L \), oder \( A = \frac{1}{2} P \), weil immer zwey latera diversarum hedrarum zusammenkommen, um eine aciem zu formiren.
3. Daher ist der numerus laterum seu angulorum planorum omnium hedrarum corpus includentium alzeit par.
4. Semper est vel \( L = 3 H \) vel \( L > 3 H \),
5. Semper est vel \( P = 3 S \) vel \( P > 3 S \).

Dieses ist klar, weil keine hedra aus weniger als drey Seiten, und kein angulus solidus aus weniger als drey angulis planis bestehen kann. Folgende Proposition aber kann ich nicht recht rigorose demonstriren:

6. In omni solido hedris planis inclusu aggregatum ex numero hedrarum et numero angulorum solidorum binario superat numero acierum, seu est \( H + S = A + 2 \), seu \( H + S = \frac{1}{2} L + 2 = \frac{1}{2} P + 2 \).
Euler's Polyhedron Letter (1750)

1. \( F + V - E = 2 \) for any convex polyhedron.
2. Let \( a, b, c, d \) be the lengths of the edges of a polyhedron.
3. Let \( V \) be the number of vertices, \( E \) the number of edges, and \( F \) the number of faces of a polyhedron.
4. Let \( A \) be the number of angles of a polyhedron.
5. Let \( B \) be the number of planes of a polyhedron.
6. Let \( S \) be the area of a plane.
7. Let \( T \) be the volume of a tetrahedron.
8. Let \( P \) be the pressure.
9. Let \( \phi \) be the angle.
10. Let \( \Theta \) be the dihedral angle.
11. Let \( \Omega \) be the solid angle.

Example: For a regular tetrahedron with side length 1, the properties are:

- \( F = 4 \)
- \( V = 4 \)
- \( E = 6 \)
- \( S = \frac{\sqrt{3}}{4} \)
- \( T = \frac{\sqrt{2}}{12} \)
- \( P = \frac{1}{2} \)
- \( \phi = \frac{\pi}{3} \)
- \( \Theta = \frac{\pi}{2} \)
- \( \Omega = \frac{\pi}{2} \)


\( a + b + c = \) circumference of a triangle

\[ a + b + c = \text{circumference} \]
Geometry of a Triangle

Orthocentre $O$

Centroid $C$

Circumcentre $M$

Euler line
$(OC = 2\text{ cm})$
**Fermat Prime Numbers**

Fermat: \( F_n = 2^{2^n} + 1 \) is prime (?)

\[
\begin{align*}
F_0 &= 2^1 + 1 = 3, \\
F_1 &= 2^2 + 1 = 5, \\
F_2 &= 2^4 + 1 = 17, \\
F_3 &= 2^8 + 1 = 257, \\
F_4 &= 2^{16} + 1 = 65,537 \\
F_5 &= 4,294,967,297 \quad ? \quad 2^{32} + 1
\end{align*}
\]

Euler: \( F_5 \) is divisible by \( 641 \)

Proof: \( 641 = 5^4 + 2^4 = (5 \times 2^7) + 1. \)

So \( 2^{32} + 1 = 2^{28} (5^4 + 2^4) - (5 \times 2^7)^4 + 1 \)

\[
= 2^{28} \cdot 641 - (641 - 1)^4 + 1
= 641 \times K, \text{ so } 641 \mid 2^{32} + 1.
\]
Euclid Book I, Prop. I

On a given straight line to construct an equilateral triangle

Let AB be the line.

With centre A and distance AB, draw the circle BCD. [Post. 3]

With centre B and distance BA, draw the circle ACE. [Post. 3]

Join AC and BC. [Post. 1]

Then the triangle ABC is equilateral.

Proof ...
Regular Polygons

- Triangle: 60°
- Square: 90°
- Pentagon: 108°
- Hexagon: 120°
- Heptagon: ~129°
- Octagon: 135°
Constructing Polygons

Gauss: A regular polygon with \( n \) sides can be constructed with straight-edge and compasses if and only if \( n \) has the form

\[ n = 2^k \times p_1 \times p_2 \times \ldots \]

where \( k \) is a non-negative integer and \( p_1, p_2, \ldots \) are distinct Fermat primes.

3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, \ldots, 257, \ldots, 65537, \ldots