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# *GEOMETRY*

## PROBLEMS WITH SCHOOLGIRLS

by

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# PROBLEMS WITH SCHOOLGIRLS

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This, the last of my three talks on discrete mathematics, concerns the topic known as *design theory*, or *block designs*. In this talk I shall start with latin squares and their uses in the design of experiments; then I consider Steiner triple systems and Kirkman's famous 'schoolgirls problem'; finally I link these topics in various ways with finite projective planes and some music.

### Latin squares

A *latin square* is a square array of symbols (numbers, letters, etc.) arranged so that each symbol appears exactly once in each row and each column. An example of a 4 × 4 latin square is

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 1 | 2 |
| 4 | 1 | 2 | 3 |
| 2 | 3 | 4 | 1 |

and sudoku patterns are examples of 9 × 9 latin squares. In general, latin square arrangements have been used widely in agricultural experiments when crops (such as wheat) planted in a field are being compared, since such arrangements tend to separate the varieties of wheat so that they are less susceptible to local influences (such as shade from trees or underground streams). Latin squares were investigated by Leonhard Euler, and the earliest known example of a latin square arrangement used in an agricultural experiment is due to M. Cretté de Palluel in 1788.

### Balanced block designs

However, it is usually impossible to plant all the varieties of wheat in one field, for reasons of space, so we resort to designs such as the following, in which we can compare any two varieties either directly or indirectly. Here there are nine varieties of wheat, arranged (vertically) in twelve triples of three varieties, with each variety appearing in exactly four triples:

| **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *A* | *A* | *A* | *A* | *B* | *B* | *B* | *C* | *C* | *C* | *D* | *G* |
| *B* | *D* | *E* | *F* | *D* | *E* | *F* | *D* | *E* | *F* | *E* | *H* |
| *C* | *G* | *I* | *H* | *I* | *H* | *G* | *H* | *G* | *I* | *F* | *I* |

Note that the total number of entries in the design is 9 × 4 = 12 × 3. More generally, if we have $v$ varieties arranged in $b$ blocks of $k$ elements, with each variety appearing $r$ times, then $vr = bk$.

In the above design, each pair of v*arieties can be compared directly in exactly one block – for example, varieties F and H* appear together in block 4. More generally, if each pair of varieties appear together in $\lambda$ blocks, then it is simple to show that $\lambda(v - 1) = r(k - 1)$: here, $1 \times (9 - 1) = 4 \times (3 - 1)$. Such a design is called a *balanced incomplete-block design*, abbreviated to BIBD. A BIBD with $k = 3$ (so that each block is a triple), is called a *Steiner triple system.*

### Triple systems

In the 1840s the annual *Lady's and Gentleman's Diary* was 'designed principally for the amusement and instruction of students in mathematics, comprising many useful and entertaining particulars, interesting to all persons engaged in that delightful pursuit'. In 1844 the editor Wesley Woolhouse presented Prize Question No. 1733:

*Determine the number of combinations that can be made out of n symbols, p symbols in each; with this limitation, that no combination of q symbols, which may appear in any one of them shall be repeated in any other.*

The question was interpreted by the readers in different ways and no-one produced a satisfactory answer, so in 1846 a simplified version appeared, corresponding to $p = 3$ and $q = 2$: 'How many triads can be made out of $n$ symbols,

so that no pair of symbols shall be comprised more than once amongst them?' If we replace 'more than one' by 'exactly one', we get a Steiner triple system:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 4 | 5 | 6 | 7 | 1 | 2 | 3 |

Here $n = 7$ (the numbers 1 – 7), with three numbers in each triple ($k = 3$) and each pair of numbers occurring together just once ($\lambda = 1$).

But although it's now called a Steiner triple system, the credit should probably go to a Lancashire clergyman called Thomas Penyngton Kirkman, whose light parochial duties gave him time to make substantial contributions to combinatorics and to early group theory and knot theory – and also to have seven children.

We have seen triple systems with $n = 7$ and $n = 9$. In a ground-breaking paper of 1847, Kirkman showed by a simple counting argument that triple systems can occur only when $n$ has the form $6t + 1$ or $6t + 3$, where $t$ is an integer, and showed by explicit constructions that each such $n$ is possible. For $n = 7$ and $n = 9$ there is essentially just one system, while for $n = 13$ there are two systems, and for $n = 15$ there are 80 – the 'combinatorial explosion' takes over.

The reason for the name *Steiner triple systems* is that in 1853 the Swiss mathematician Jakob Steiner asked whether such systems exist for all appropriate $n$, completely unaware that Kirkman had constructed them all six years earlier.

## Kirkman's schoolgirls' problem

The above design with $n = 9$ has the property that we can split it into four parts with all numbers from 1 to $n$ appearing in each part – systems with this property are called *resolvable*.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 7 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 2 | 5 | 8 | 4 | 5 | 6 | 6 | 4 | 5 | 5 | 6 | 4 |
| 3 | 6 | 9 | 7 | 8 | 9 | 8 | 9 | 7 | 9 | 7 | 8 |

In general we can split up a design in this way only when $n = 6t + 3$, and the idea is central to the famous *Kirkman schoolgirls problem*. This problem appeared in the *Lady's and Gentleman's Diary* for 1850, together with problems on April Fool's Day, the sons of Noah, and the saltiness of the sea:

*Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.*

Here's a re-statement of the problem, in verse:

A governess of great repute,
Young ladies had *fifteen*,
Who took their walks along the shore,
Or in the meadows green.

But as they walked they tattled and talked
In chosen *groups of three*,
Until their governess resolved,
Such trifling should not be.

For she would try for *one whole week*,
So to arrange them all,
That *no two girls a second time*
*In the same rank should fall.*

A solution of this problem is as follows:

| | | | | | |
|---|---|---|---|---|---|
| *Monday:* | *1-2-3;* | *4-5-6;* | *7-8-9;* | *10-11-12;* | *13-14-15* |
| *Tuesday:* | *1-4-7;* | *2-5-8;* | *3-12-15;* | *6-10-14;* | *9-11-13* |
| *Wednesday:* | *1-10-13;* | *2-11-14;* | *3-6-9;* | *4-8-12;* | *5-7-15* |
| *Thursday:* | *1-5-11;* | *2-6-12;* | *3-7-13;* | *4-9-14;* | *8-10-15* |
| *Friday:* | *1-8-14;* | *2-9-15;* | *3-4-10;* | *6-7-11;* | *5-12-13* |

| | | | | | |
|---|---|---|---|---|---|
| *Saturday:* | *1-6-15;* | *2-4-13;* | *3-8-11;* | *5-9-10;* | *7-12-14* |
| *Sunday:* | *1-9-12;* | *2-7-10;* | *3-5-14;* | *6-8-13;* | *4-11-15* |

Kirkman's solution appeared in the *Diary* for 1851. He incorrectly claimed that it was 'the symmetrical and the only possible solution', as in the meantime Arthur Cayley had produced a different solution, still with a lot of symmetry. Also, in the 1851 *Diary*, a single solution was given by Mr Bills of Newark-on-Trent, Mr Jones of Chester, Mr Wainman of Leeds and Mr Levy of Hungerford – it is not known how they all supposedly arrived at the same solution.

Kirkman later presented variations of the schoolgirls problem, solving the corresponding problem for nine young ladies, believing it to be impossible for 21 young ladies, and asserting the following results:
*Sixteen young ladies can all walk out four abreast, till every three have once walked abreast; so can thirty-two, and so can sixty-four young ladies; so can $4^n$ young ladies.*

The first person to treat the schoolgirls problem algebraically, rather than by organized guesswork, was another Victorian clergyman, the Revd. Robert Anstice, who presented a *cyclic* solution in 1852. Here the fifteen schoolgirls can be written as *0*, *1*, *2*, *3*, *4*, *5*, *6*, *0*, *1*, *2*, *3*, *4*, *5*, *6* and ∞, and each successive day's arrangement is obtained by adding 1 to each entry of the previous day, as follows:

| | | | | | |
|---|---|---|---|---|---|
| *Monday:* | *∞-0-0;* | *1-2-3;* | *1-4-5;* | *3-5-6;* | *4-2-6* |
| *Tuesday:* | *∞-1-1;* | *2-3-4;* | *2-5-6;* | *4-6-0;* | *5-3-0* |
| *Wednesday:* | *∞-2-2;* | *3-4-5;* | *3-6-0;* | *5-0-1;* | *6-4-1* |
| *Thursday:* | *∞-3-3;* | *4-5-6;* | *4-0-1;* | *6-1-2;* | *0-5-2* |
| *Friday:* | *∞-4-4;* | *5-6-0;* | *5-1-2;* | *0-2-3;* | *1-6-3* |
| *Saturday:* | *∞-5-5;* | *6-0-1;* | *6-2-3;* | *1-3-4;* | *2-0-4* |
| *Sunday:* | *∞-6-6;* | *0-1-2;* | *0-3-4;* | *2-4-5;* | *3-1-5* |

There are actually three possible cyclic systems, and these were later found by the famous American mathematician Benjamin Peirce: two of them turn out to be essentially the systems of Cayley and Kirkman. It is now known that there are seven essentially different solution to Kirkman's schoolgirls problem.

At this point we meet the brilliant but eccentric Sylvester, whose chequered career culminated in his becoming Savilian professor of geometry in Oxford. Sylvester believed that he had originated the schoolgirls problem, and said so in his own inimitable way ...

*... in connexion with my researches in combinatorial aggregation ... I had fallen upon the question of forming a heptatic aggregate of triadic synthemes comprising all duads to the base 15, which has since become so well known, and fluttered so many a gentle bosom, under the title of the fifteen schoolgirls' problem; and it is not improbable that the question ... may have originated through channels which can no longer be traced in the oral communications made by myself to my fellow-undergraduates at the University of Cambridge.*

As you can imagine, Kirkman thought little of this priority claim, dismissing it as no more than a 'guess' on Sylvester's part.

But Sylvester did have an interesting extension of the schoolgirls problem. The number of possible triples is $^{15}C_3$ = 455, which is 13 times 35. Can we arrange them all into 13 *disjoint* solutions to the problem? That is,
*can we arrange 13 weekly schedules so that each triple occurs just once in the quarter-year?*

Kirkman wrongly claimed a solution in 1850, and a correct solution was not found until 1974, by Denniston. Meanwhile, the general problem for $6n + 3$ schoolgirls remained unsolved until, around the same time, it was settled by Ray-Chaudhuri and Wilson [22], and independently about eight years earlier by Lu Xia Xi, a schoolteacher from Inner Mongolia. The corresponding Sylvester problem for disjoint solutions for $6n + 3$ schoolgirls remains unsolved to this day.

**Finite projective planes**

The Steiner triple system with seven points can be considered as a *finite projective plane*. This is a geometrical object, now known as the *Fano plane*, with seven points and seven lines, and with three points lying on each line and three lines passing through each point; the seven triples give us the points lying on the seven lines. There are musical compositions based on the mathematics of the Fano plane.

More generally, the *finite projective plane of order n* is a geometric object containing $n^2 + n + 1$ points and $n^2 + n + 1$ lines, with $n + 1$ points lying on each line and $n + 1$ lines passing through each point. There are finite projective planes with 7 points (order 2), 13 points (order 3), 21 points (order 4), 31 points (order 5), and whenever $n$ is a power of a prime number. There is no projective plane of order 6 or 10, and for all other values it is not known whether a finite projective plane exists.

**Orthogonal latin squares**

In 1782 Euler posed the following question:

*Arrange 36 officers, one of each of six ranks and from six regiments, in a square array so that each row and each column has one officer of each rank and one from each regiment.*

More generally, given two latin squares we can superimpose them, as follows:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *A* | *B* | *C* | *D* | *A* | *B* | *C* | *D* | *AA* | *BB* | *CC* | *DD* |
| *B* | *A* | *D* | *C* | *C* | *D* | *A* | *B* | *BC* | *AD* | *DA* | *CB* |
| *C* | *D* | *A* | *B* | *D* | *C* | *B* | *A* | *CD* | *DC* | *AB* | *BA* |
| *D* | *C* | *B* | *A* | *B* | *A* | *D* | *C* | *DB* | *CA* | *BD* | *AC* |

If in the resulting latin square all the pairs are different, then the original latin squares are *orthogonal*. Euler's conjecture is that there is no pair of orthogonal 6 × 6 latin squares.

Euler exhibited orthogonal pairs of $n \times n$ latin squares for all positive integers $n$ ($\geq 2$) other than 6, 10, 14, 18, 22, . . . , and conjectured that no such pair exists for these numbers. In 1900 Gaston Tarry showed that the original problem, with $n = 6$, has no solution. However, the rest of Euler's conjecture was proved wrong in spectacular fashion around 1960 when Bose, Shrikhande and Parker constructed an example for *all* these values from 10 upwards – this feat was so amazing that it was celebrated on the front cover of the *New York Times*.

We can use orthogonal latin squares to construct designs for agricultural experiments. For example, starting with the above 4 × 4 orthogonal latin squares we write

- The numbers 1 to 16 in rows
- The numbers 1 to 16 in columns
- The positions of the letters *A*, *B*, *C*, *D* in the first 4 × 4 latin square
- The positions of *A*, *B*, *C*, *D* in the second latin square

This gives the following design with sixteen blocks:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 1 | 5 | 9 | 13 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 2 | 6 | 10 | 14 | 6 | 5 | 8 | 7 | 7 | 8 | 5 | 6 |
| 9 | 10 | 11 | 12 | 3 | 7 | 11 | 15 | 11 | 12 | 9 | 10 | 12 | 11 | 10 | 9 |
| 13 | 14 | 15 | 16 | 4 | 8 | 12 | 16 | 16 | 15 | 14 | 13 | 14 | 13 | 16 | 15 |

In fact, there is a third latin square, orthogonal to the two above, which can be used to give an extra set of four blocks in the above table – we leave the finding of it as an exercise.

Finally, we mention a link between orthogonal latin squares and finite projective planes. It can be proved that there is a projective plane of order $n$ if and only if there exist $n - 1$ mutually orthogonal $n \times n$ latin squares. The impossibility of Euler's 36 officers problem shows that there are no two orthogonal 6 × 6 latin squares, let alone five, and we deduce that there is no projective plane of order 6.