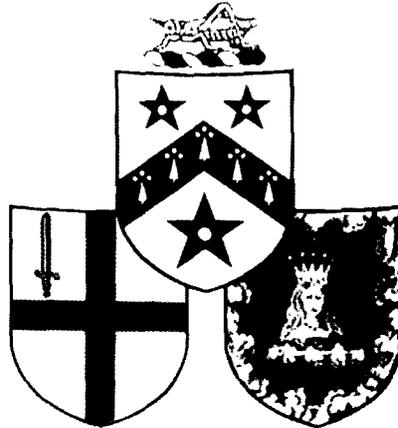


*G R E S H A M*  
*C O L L E G E*



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**FROM FOOTBALLS TO FULLERENES**

A Lecture by

**PROFESSOR IAN STEWART MA PhD FIMA CMath**  
**Gresham Professor of Geometry**

14 May 1996

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## Gresham Geometry Lecture 14 May 1996

### From Footballs to Fullerenes

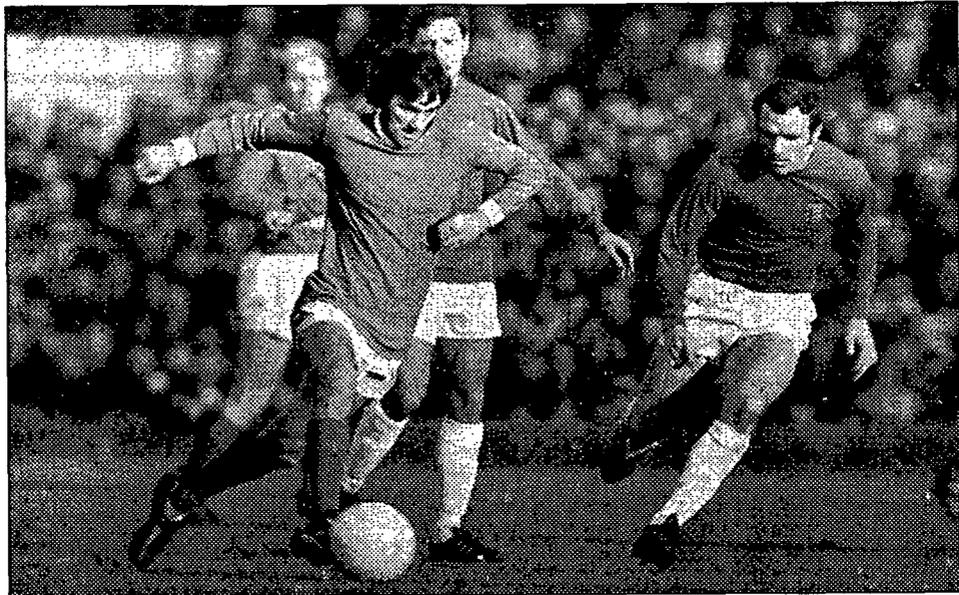
#### *the forbidden fivefold symmetry of quasicrystals*

This lecture examines two different, but related, instances of fivefold symmetry in the molecular structure of matter: **quasicrystals** and **fullerenes**.

- Perfect crystals cannot have fivefold symmetry (though they can have 2-, 3-, 4-, and 6-fold symmetry). Some years ago the mathematical physicist Roger Penrose invented a series of tilings of the plane that possessed *approximate* fivefold symmetry. It turned out that similar structures do occur in nature, leading to the discovery of quasicrystals, a new form of organization in solid matter.

- Soccer balls are made by sewing together hexagonal and pentagonal panels, in the form of a truncated icosahedron. The same structure was discovered in a new form of carbon,  $C_{60}$ , usually known as Buckminsterfullerene.

The lecture will discuss the relations between symmetry and molecular structure, and explain how these two discoveries have changed our views about fivefold symmetry in nature.



United 1970 vintage . . . George Best in his prime giving Cup rivals Ipswich the run around

Fig.1 What's football got to do with it?

### Quasicrystals

Crystals provide one of the most striking examples of symmetry in nature (although historically there was a great deal of controversy about the apparent mathematical regularities of crystals forms, because these are much less obvious in natural samples found by field geologists). In 1922 Haüy suggested that crystals are formed from identical units arranged in patterns, like this:

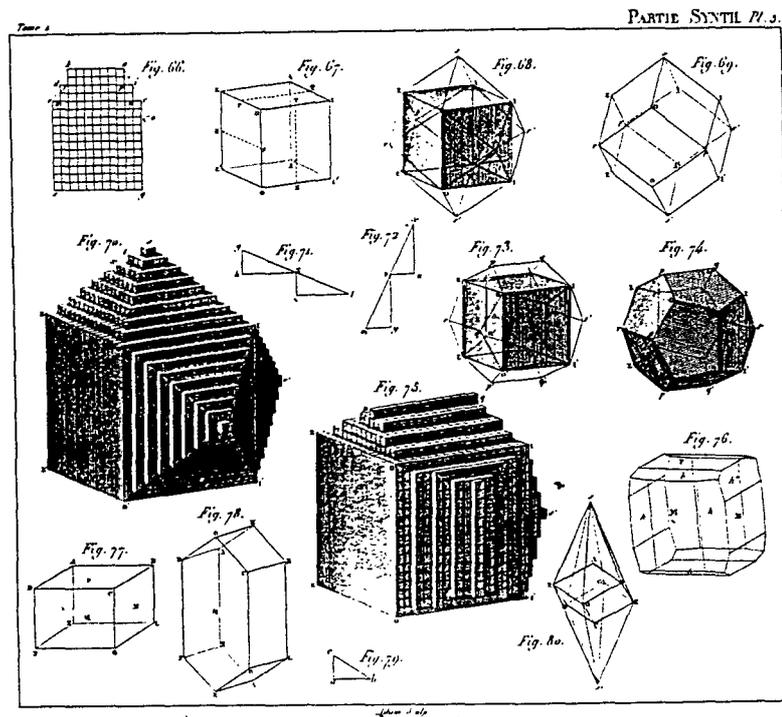


Fig.2 Haüy's idea of crystal structure

He was heading in the right direction. It has now been known for over a century that crystals have regular structure because their atoms are arranged in a lattice. Lattices in the plane (space) are determined by choosing two (three) independent directions and spacing dots at regular intervals along those directions.

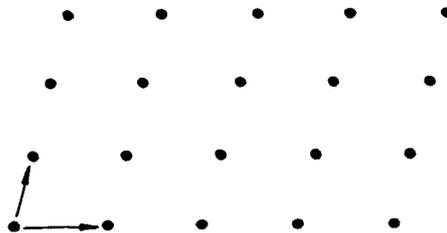


Fig.3 Lattice in the plane

Lattices have two basic types of symmetry. The obvious symmetries are the translations of the lattice, sliding it sideways along the chosen directions. But there are also *point symmetries* that rotate the lattice around a fixed lattice point, and there can also be mirror symmetries — reflections in a line or plane. In this lecture it is the point symmetries that matter. Lattices in the plane (and space) can have 2-fold, 3-fold, 4-fold, and 6-fold rotational symmetries; there are corresponding *tilings* of the plane by rectangles, equilateral triangles, squares, and hexagons. However, no lattice in the plane or space can have 5-fold point symmetry (but it *can* occur in a 4-dimensional lattice).

To prove that 5-fold point symmetry is impossible in a plane lattice, we use two facts about lattices:

- 1 They are *discrete*: there is a definite minimum distance between distinct points.
- 2 They are *uniform*: the structure is the same when viewed from any lattice point.

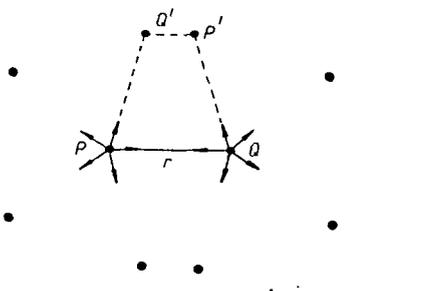
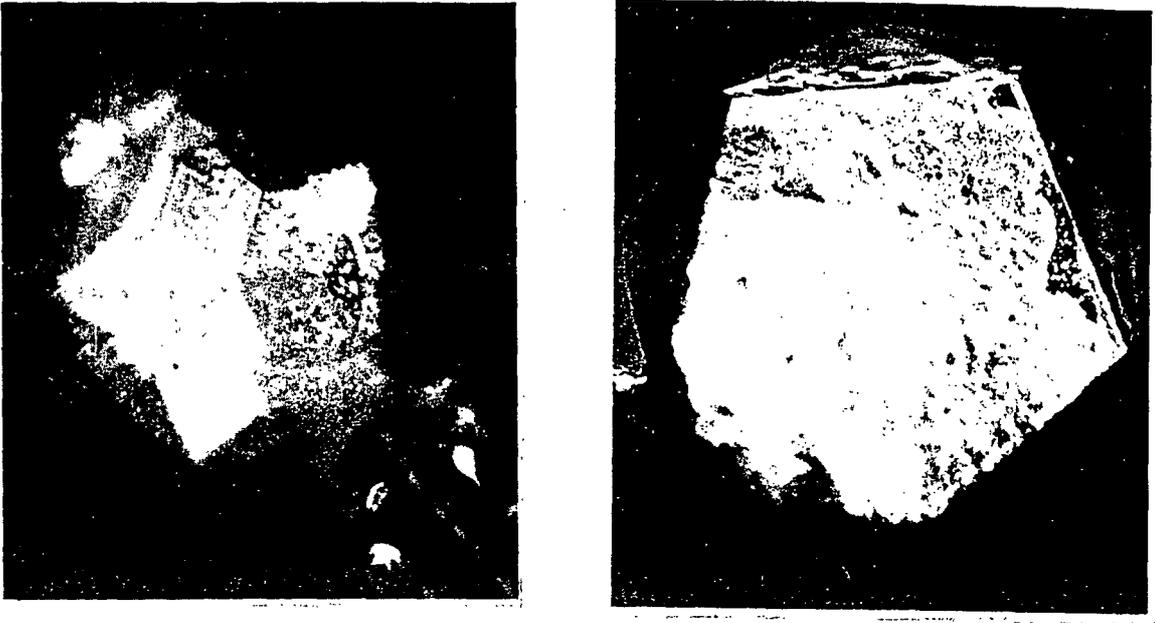


Fig.4 Impossibility proof for 5-fold point symmetry.

Suppose, for a contradiction, that a plane lattice with 5-fold symmetry exists. By property 1 we can find points P and Q of the lattice, separated by the minimum possible distance  $r$ . By property 2 the lattice has 5-fold point symmetry around either P or Q. So there are lattice points surrounding P formed by rotating Q through multiples of  $72^\circ$ , and similarly there are lattice points surrounding Q formed by rotating P through multiples of  $72^\circ$ . However, two of these points, P' and Q', are closer together than P and Q, contradicting minimality.

A similar proof works in three dimensions, but in four dimensions the geometry is sufficiently different for such proofs not to work, and indeed a lattice with 5-fold point symmetry is fairly easily found. In 5 dimensions it is *very* easy to find 5-fold point symmetry.

The absence of 5-fold symmetry is called the *crystallographic restriction*, and it has long been taken to imply that crystals cannot have 5-fold symmetries. However, recently many alloys were found that produced apparently 5-fold symmetric crystals.



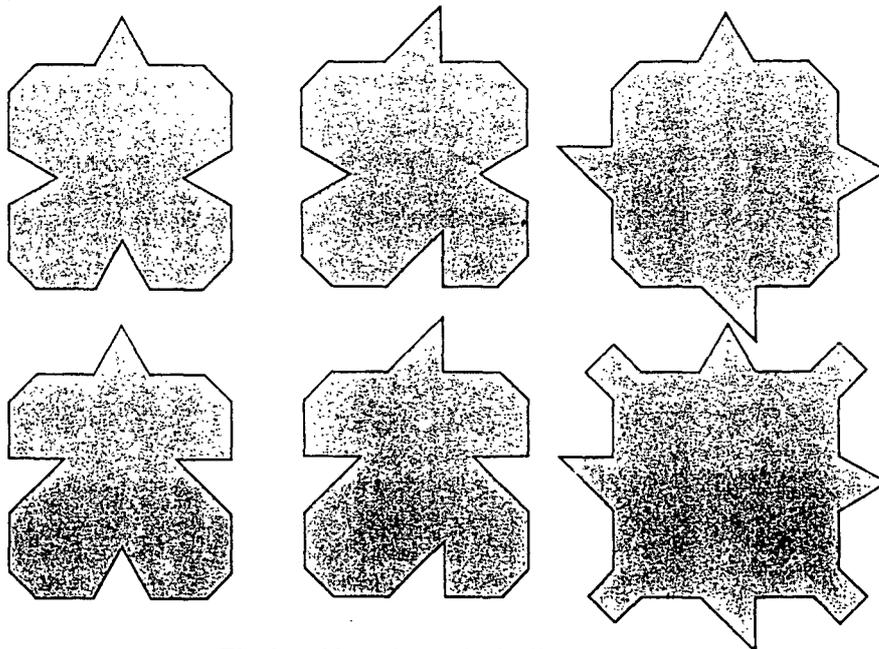
**Fig.5** What about these, then?

(Left)  $\text{Al}_6\text{Li}_3\text{Cu}$  alloy. (Right)  $\text{Al}_{63.5}\text{Cu}_{24}\text{Fe}_{12.5}$  alloy.

[Al = aluminium, Li = lithium, Cu = copper, Fe = iron.]

These 5-fold symmetric structures are not in fact crystals in the usual sense of crystallography — they do not have atomic *lattices*. They are *quasicrystals*, with atomic *quasilattices*. They were discovered on the basis of mathematical predictions that tiling patterns in the plane and space exist with striking elements of 5-fold symmetry, but possessing no lattice periodicity.

A set of 20,000 tiles without lattice periodicity was found by Robert Berger in 1964: the number was later reduced to 104. Raphael Robinson found a set with six members:



**Fig.6** Robinson's aperiodic tiles.

In 1974 Roger Penrose reduced the size of the set to two, which he called *darts* and *kites*. They are formed by cutting up a rhombus with angles  $72^\circ$  and  $108^\circ$ . (Strictly, to get down to two tiles they must be 'decorated', say with coloured curves that must join together, or their edges must be fitted with tiny bumps and dents to force the same kind of fit.)

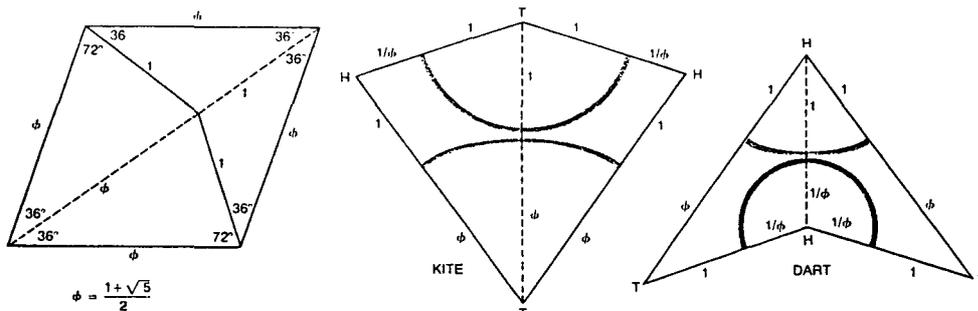
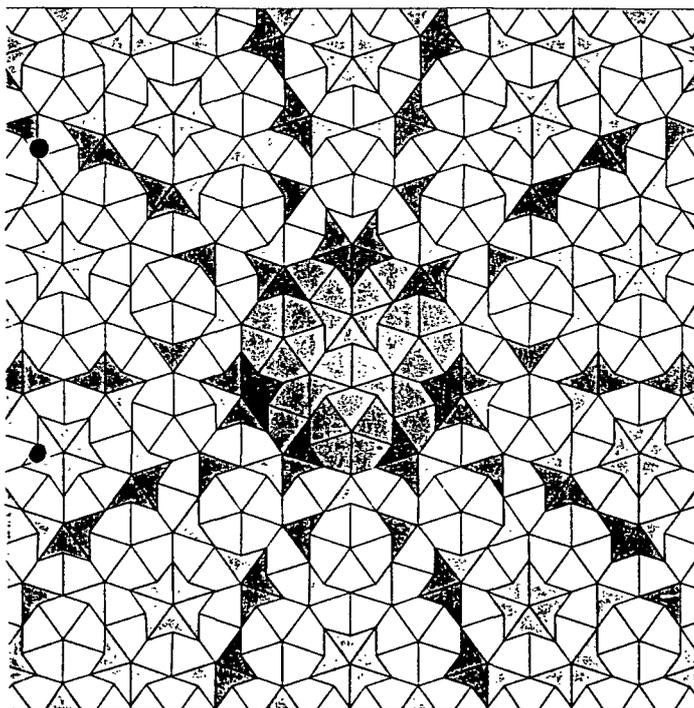


Fig.7 Penrose's darts and kites.

They were brought to public attention in 1977 by Martin Gardner:

# SCIENTIFIC AMERICAN



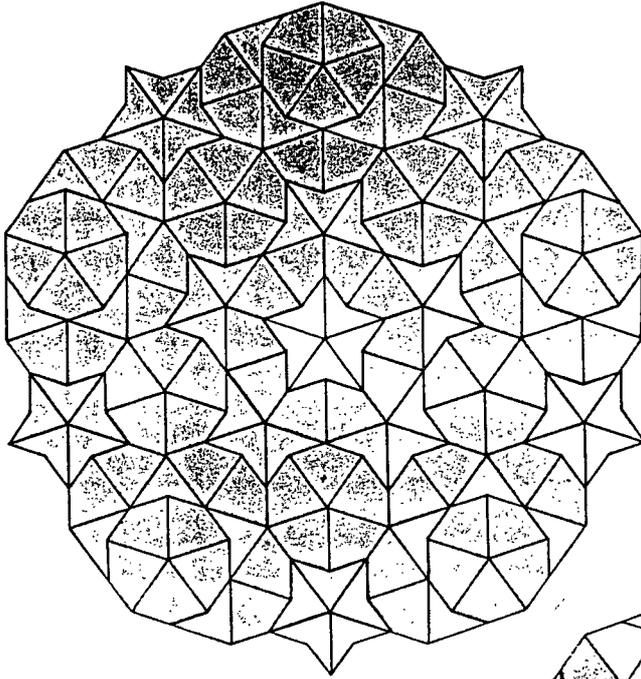
THEORY OF TILES

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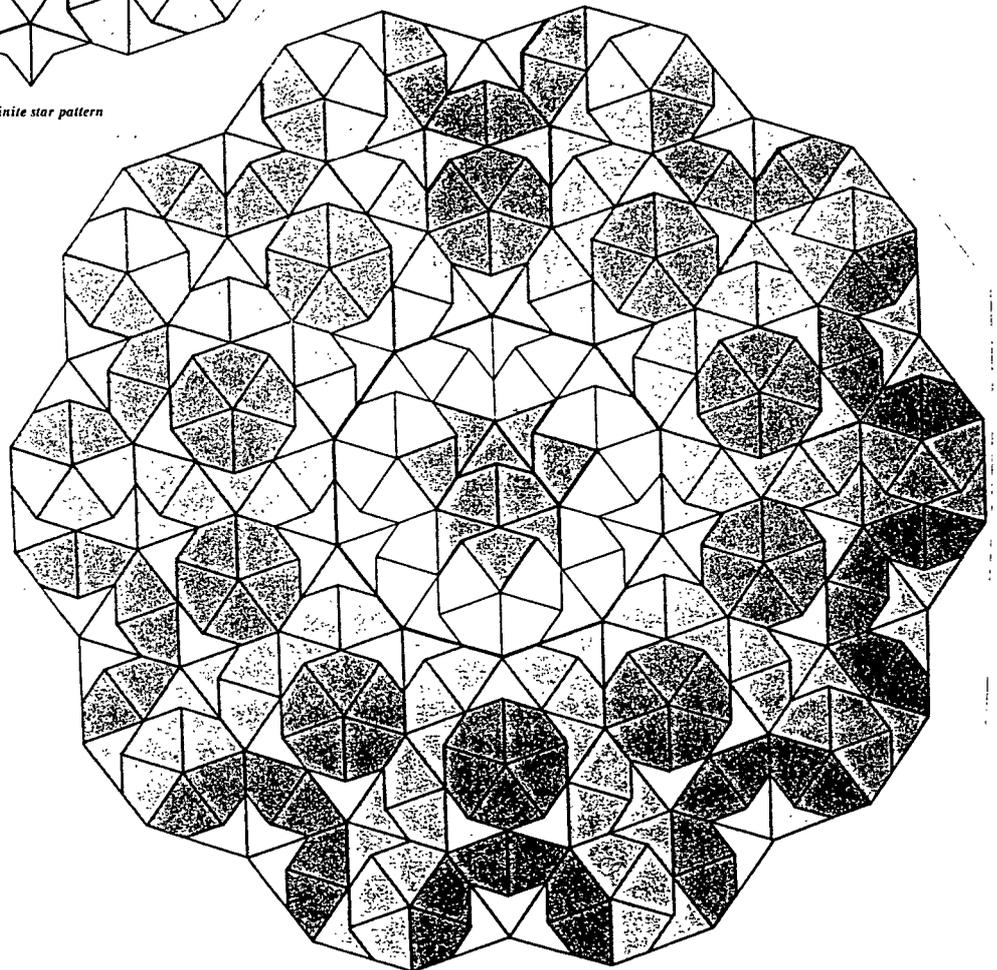
Fig.8 Penrose tiles go public.

January 1977

Penrose tiles can cover the plane in many different ways, none of them with lattice periodicity. Here are two examples: one has perfect fivefold symmetry about *one* point, and the other 'nearly' has:



*The infinite star pattern*



*The cartwheel pattern*

**Fig.9** The star and the cartwheel.

You can play Escherlike games too:

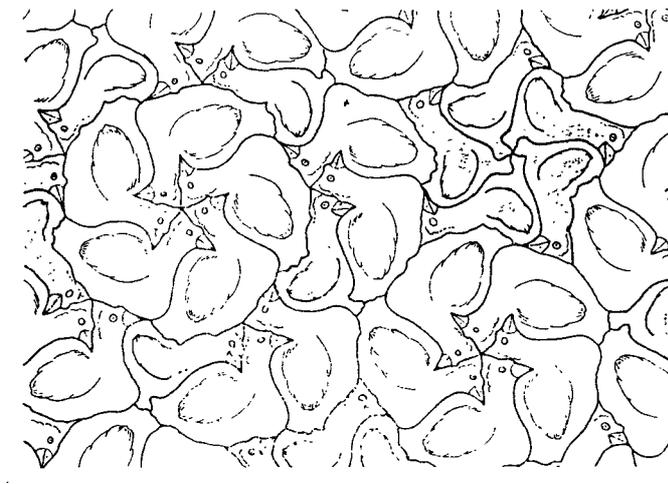


Fig.10 Penrose's aperiodic chickens.

All *Penrose patterns* have 'almost' 5-fold symmetry around infinitely many distinct points. Indeed all Penrose patterns are 'locally isomorphic': given two of them, each contains arbitrarily large patches that are identical to patches of the other. Nonetheless there are many globally different patterns.

If atoms are arranged in Penrose-like quasilattices, you get quasicrystals, and the 5-fold almost-symmetries of the quasilattices give rise to macroscopic structures that look 5-fold symmetric.

### ***Fullerenes***

What's football got to do with it?

Be patient — all will be revealed shortly...

There is another novel source of 5-fold symmetry in nature, the remarkable molecule  $C_{60}$ , or *buckminsterfullerene* ('buckyball'). Buckminster Fuller was an architect renowned for his *geodesic domes*, quasiperiodic polyhedral structures.



Fig.11 Buckminster Fuller.

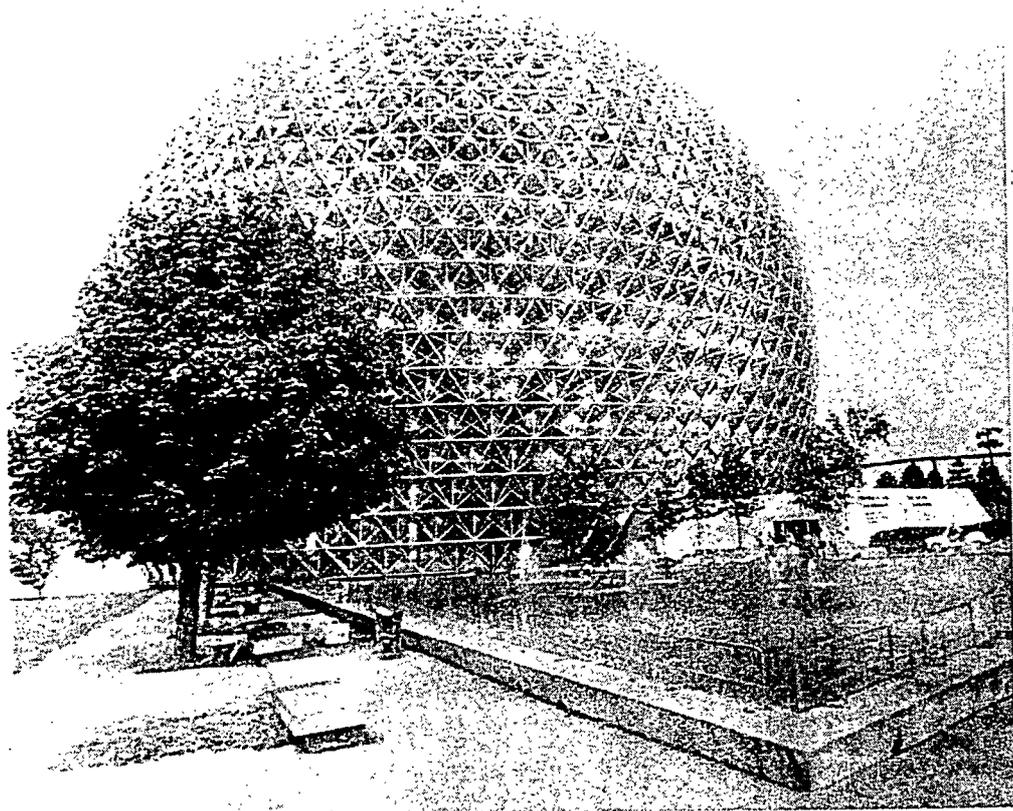


Fig.12 Buckminster Fuller's geodesic dome at the Montreal Expo 1967.

Geodesic domes are polyhedra. There are five more familiar *regular polyhedra*: here they are drawn in Johannes Kepler's *Harmonices Mundi* of 1619.

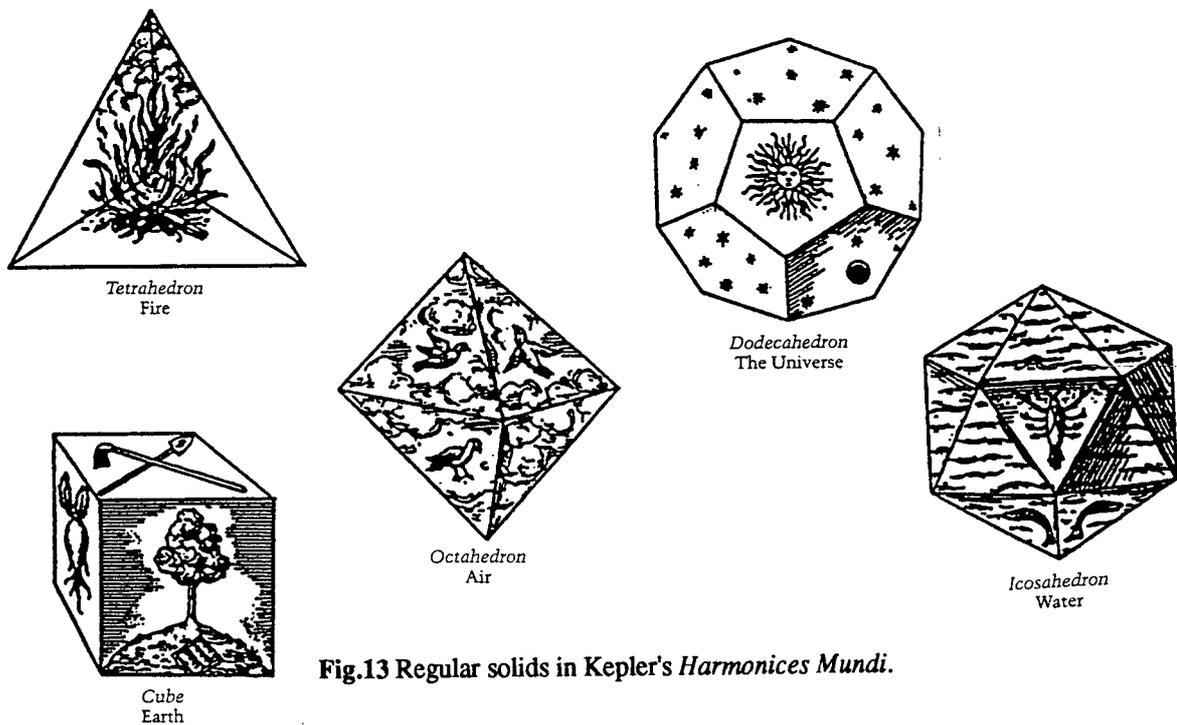
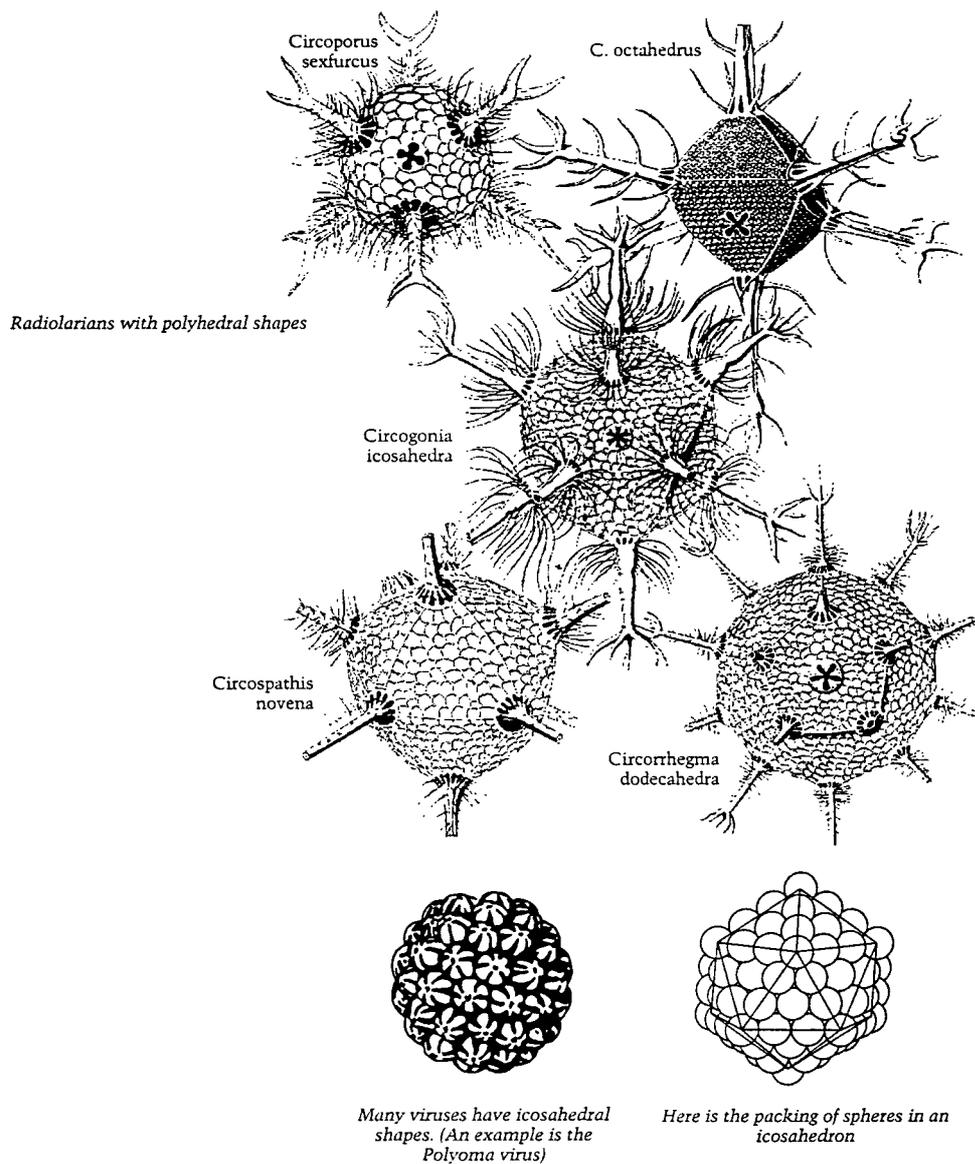


Fig.13 Regular solids in Kepler's *Harmonices Mundi*.

They occur in various forms in nature: radiolarians and viruses are examples.



**Fig.14** Polyhedra in radiolarians and viruses.

More complex polyhedra can be formed by 'truncating' the regular solids.

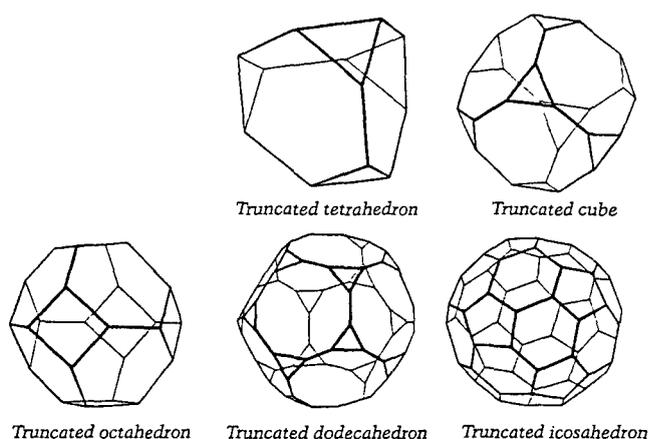


Fig.15 Truncated regular solids.

It is the *truncated icosahedron* that turned up as an entirely new form of carbon. It was first synthesised in 1985, in a collaboration between Harry Kroto (University of Sussex) and Richard Smalley (Rice University, Houston). On 1 September of that year they vaporised carbon in an atmosphere of hydrogen, nitrogen, and various other elements to simulate the conditions near red giant stars (where this form of carbon was thought perhaps to exist). On 4 September they detected the presence of carbon molecules with molecular weight 720. Carbon's molecular weight is 12, so this corresponded to exactly 60 carbon atoms. There was another trace of a 70-atom form too.

What was their structure? The two scientists tried all sorts of ideas. Their graduate students discovered that the molecule was so stable that it could not have any 'dangling bonds'. This reinforced a growing feeling that it was some kind of polyhedral 'cage'. Smalley recalls sitting up all night on 9 September with scissors and paper, and finding a possible structure. Kroto, a lifelong fan of Buckminster Fuller, says that he was aware of the possibility all along. At any rate, despite this falling-out between the co-discoverers, the structure was like this:

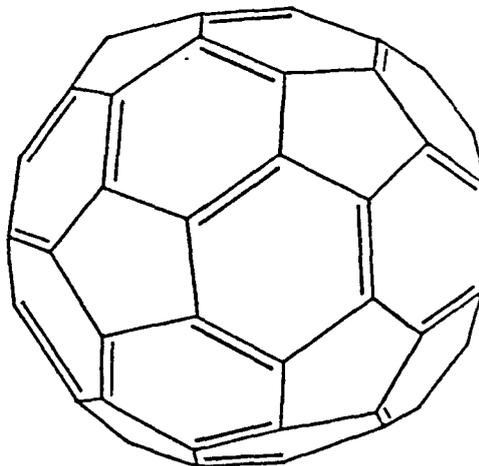


Fig.16 Truncated icosahedral structure of Buckminsterfullerene.

The 70-atom form has a very similar structure:

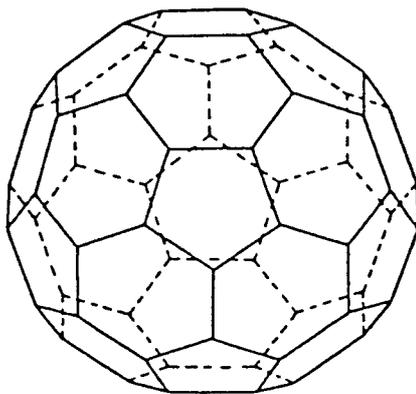


Fig.17 Polyhedral cage for  $C_{70}$ .

In retrospect it is difficult to understand why the two scientists didn't just go and grab a random mathematician, who would (I hope!) have told them about the truncated icosahedron.

Failing that, they might have asked Leonardo da Vinci, who knew the shape several centuries ago:

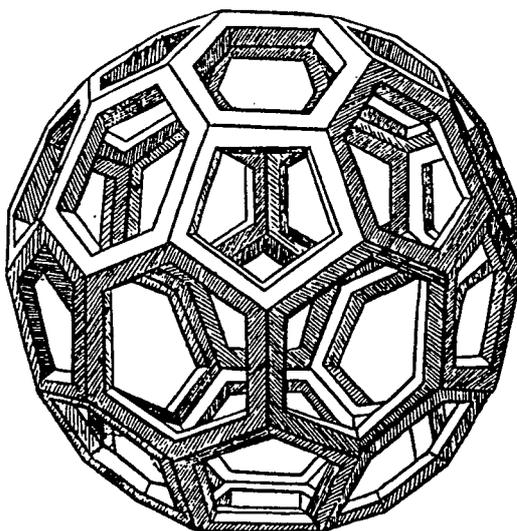


Fig.18 Da Vinci's drawing of the truncated icosahedron.

or gone to any football match:



Fig.19 That's what football's got to do with it!

## FURTHER READING

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