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HEARING THE SHAPE OF A DRUM

A Lecture by

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Hearing the Shape of a Drum

Ian Stewart

Can you hear the shape of a drum?

Mathematicians can find deep and fundamental problems where nobody else would think to look; and this strange question, posed by the late Mark Kac in 1966, is much more important than its quirky formulation might suggest. The frequency of a sound is the number of vibrations per second; the spectrum of an object is the list of basic frequencies at which it can vibrate. In that language, a more impressive-sounding version would be this: what information about a shape can you infer from its vibrational spectrum? When an earthquake hits, the entire Earth rings like a bell, and seismologists deduce a great deal about the internal structure of our planet from the 'sound' that it produces and the way those sounds echo around, bouncing off different layers of rock. Kac's celebrated question is the simplest and tidiest one that we can ask about such techniques: reconstructing information about an object from the range of vibrations that it can undergo.

Kac showed that some features of a drum are determined by its sound: for example its area and its perimeter. "Personally, I believe that one cannot 'hear' the shape... but I may well be wrong and I am not prepared to bet large sums either way," he wrote. It has taken over a quarter of a century to prove that Kac's instincts were correct: you can't, in general, infer the shape of a vibrating membrane from its spectrum. Carolyn Gordon and David Webb at Washington University in St. Louis, and Scott Wolpert at the University of Maryland, have constructed two distinct mathematical drumskins that produce the identical range of sounds (**Fig.1**). The curious shapes to which their analysis leads show that the problem is decidedly weird, and justifies Kac's diffidence about the correctness of his guess.

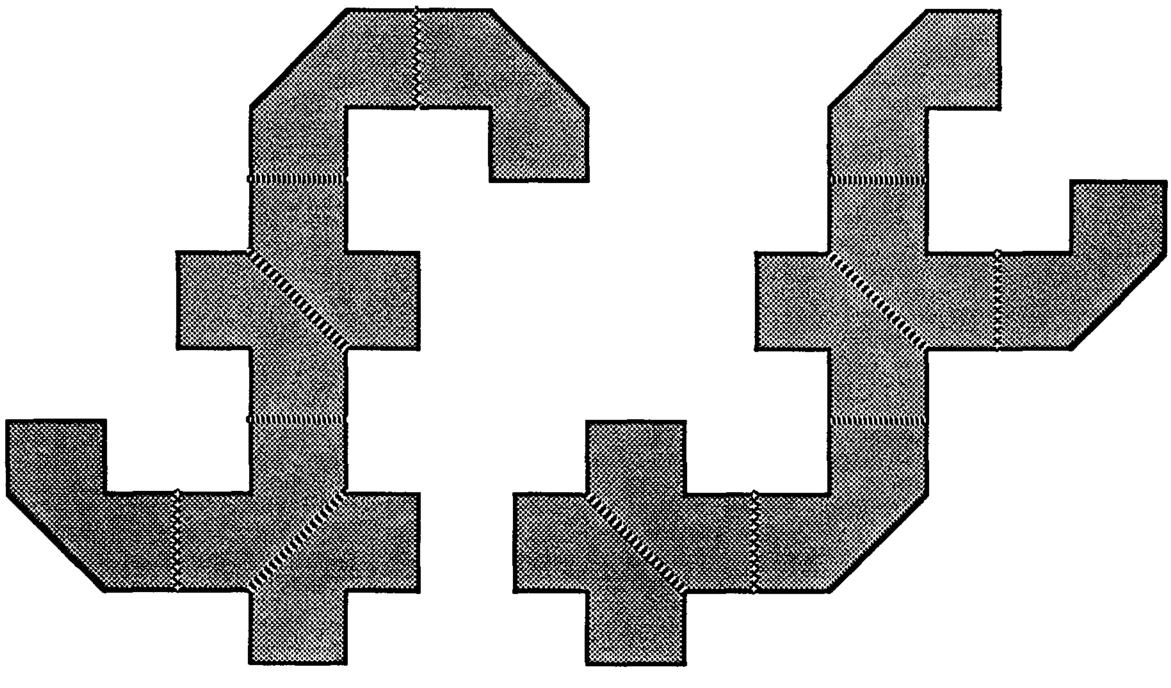


Fig.1 Different drums with the same sound, each assembled from seven halves of a Maltese cross.

His problem is just the tip of a mathematical iceberg, with far-reaching ramifications and generalizations, and more unsolved problems than answers. At a meeting of the American Mathematical Society at Alabama a few years back, Dennis DeTurck (University of Pennsylvania) used a computer to play the Alabama Jubilee as it would sound on a so-called flat torus, and a quartet played upon four different projective spaces (real, complex, quaternionic, and Cayley) — or equivalently on spheres of dimensions 1, 2, 6, and 12. Harmony of the spheres? Not entirely. Writing in the *Mathematical Intelligencer*, Gordon remarked that "The audience would perhaps be happy to learn that flat tori and low-dimensional projective spaces are uniquely determined by their spectra. No two of them produce the same terrible sound." Subscribers to the *Intelligencer* were treated to a free record of similar music by DeTurck, including the Romanza movement from Beethoven's *Sonatina in G* on a 6-dimensional sphere.

Kac's question is an 'inverse problem': it runs the opposite way compared to what is most natural. The sensible, far easier question is: given the shape of an object, how does it vibrate? As the seismic example shows, however, inverse problems have considerable practical importance. Even the direct problem took several centuries to solve. Probably the earliest major result was obtained in 1714 by Brook Taylor, who calculated the fundamental vibrational frequency of a violin string in terms of its length, tension, and density. The ancient Greeks knew that a vibrating string can produce many different musical notes, depending on the position of the 'nodes', or rest-points (**Fig.2**). For the fundamental frequency, only the end points are at rest. If the string has a node at its centre, then it produces a note one octave higher; and the more nodes there are, the higher the frequency of the note will be. In modern language the Greeks discovered that the vibrational spectrum of the string consists of all whole number multiples of the fundamental frequency. The higher vibrations are called *overtones*. Taylor's work shows that the length of the string can be deduced from the fundamental frequency, the smallest term of the vibrational spectrum — provided the tensions and density are already known. In short, you *can* hear the length of a violin string — a one-dimensional drum.

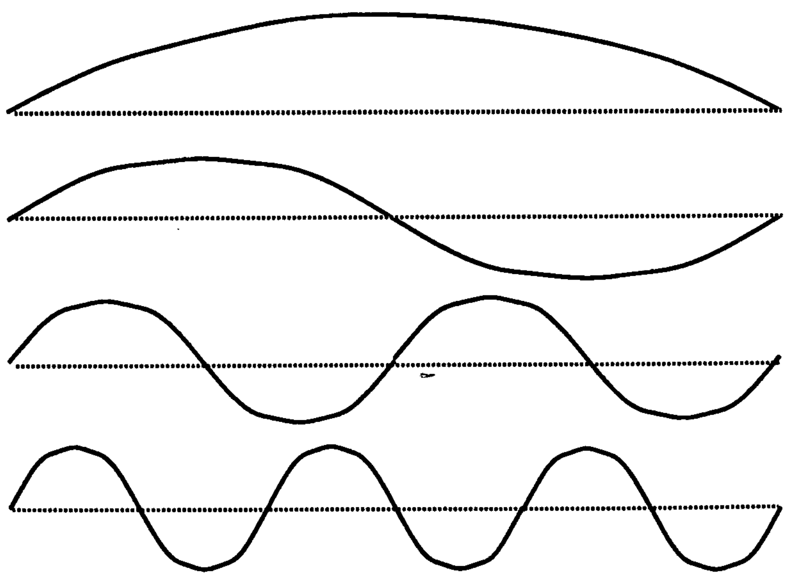


Fig.2 Normal modes of a violin string: fundamental (top) and overtones (below).

The vibrations shown in Fig.2 are standing waves — the shape of the string at any instant is the same, except that it is stretched or compressed in the direction at right angles to its length. The maximum amount of stretching is the amplitude of the wave, which physically determines how loud the note sounds. The waveforms shown are sinusoidal in shape; and their amplitudes vary sinusoidally with time. 'Pure' standing waves of this type are called normal modes.

In 1746 Jean le Rond d'Alembert showed that the full story isn't quite that simple. There are many vibrations of a violin string that are not normal modes, not sinusoidal standing waves. In fact, he proved that the shape of the wave can start out being anything you like. In response to d'Alembert's work, Leonhard Euler took the question much further, and by 1748 had worked out, and solved, the 'wave equation' for a string. These discoveries started a century-long controversy, whose end result was that you get all possible vibrations of the string by *superimposing* normal modes in suitable proportions. The normal modes, the pure sinusoidal standing waves, are the basic components; the vibrations that can occur are all possible sums of constant multiples of finitely or infinitely many normal modes. As Daniel Bernoulli expressed it in 1753: "all new curves given by d'Alembert and Euler are only combinations of the Taylor vibrations".

The first work on drums was also Euler's, in 1759. Again he derived a wave equation, describing how the displacement of the drumskin in the vertical direction varies over time. Its physical interpretation is that the acceleration of a small piece of the drumskin is proportional to the average tension exerted on it by all nearby parts of the drumskin. Drums differ from violin strings not only in their dimensionality — a drum is a flat two-dimensional membrane — but in having a much more interesting *boundary*. In this whole subject, boundaries are absolutely crucial. The boundary of a drum can be any closed curve — usually a smooth one, but nowadays it may well be a fractal. The key condition is that the boundary of the drum is fixed. The rest of the drumhead can move, but its rim is firmly strapped down. This 'boundary condition' greatly restricts the possible motions of the drum. There are boundary conditions on violin strings too: the ends must be fixed. Among other things, those boundary conditions prevent the occurrence of travelling waves, moving sideways along the string.

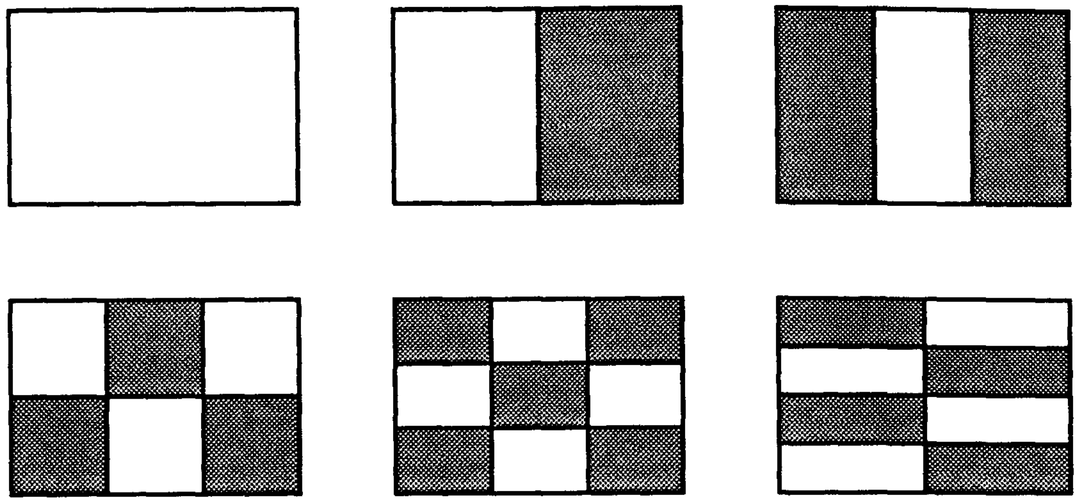
The mathematicians of the eighteenth century were able to solve the equations for the motion of drums of various shapes. Again they found that all vibrations can be built up from simpler ones, the normal modes, and that those yield a specific list of frequencies. The simplest case is the rectangular drum, whose normal modes are combinations of sinusoidal ripples in the two perpendicular directions (**Fig.3a**). A more difficult case is the circular drum, whose normal modes involve more complicated expressions called Bessel functions (**Fig.3b**). The amplitudes of these normal modes still vary sinusoidally with time; but their spatial structure is more complicated.

The wave equation is exceedingly important. Waves arise not only in musical instruments, but in the physics of light and sound. Euler found a three-dimensional version of the wave equation, which he applied to sound waves. Roughly a century later, James Clerk Maxwell extracted the same mathematical expression from his equations for electromagnetism, and predicted the existence of radio waves. Without the early mathematicians' work on musical instruments, we would not today have television.

We can now explain, with greater precision, what Kac's question was. Choose a closed curve, defining the boundary of the drum, and imagine a flat membrane stretched between it, of constant density and tension. The possible vibrations of such a membrane are determined by the two-dimensional wave equation, with the condition that the boundary of the drum, the original curve, remains fixed for all time. The solutions of the wave equation are combinations of normal modes, standing waves whose amplitude varies sinusoidally over time. The set of frequencies of the normal modes is the drum's *spectrum*. In general it consists of an infinite sequence of numbers $\mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots$, the smallest frequency μ_1 being the fundamental. Unlike violin strings, the other frequencies need *not* be integer multiples of the fundamental (which is why drums and bells have distinctive sounds, not entirely consistent with the usual rules of musical harmony).

Clearly the drum's shape determines the spectrum: you just have to solve the wave equation and see what the normal mode frequencies are. But is the converse true? Does the spectrum determine the shape? That was what Kac asked.

(a)



(b)

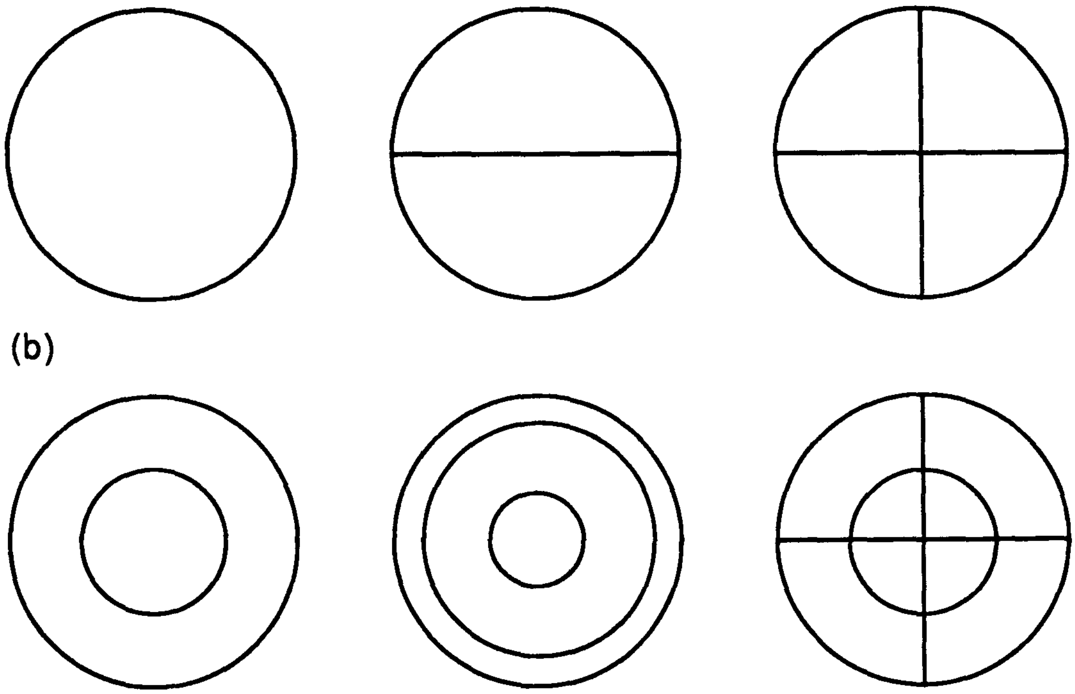


Fig.3 Normal modes of (a) rectangular drum and (b) circular drum. Shaded region starts above the plane of the paper, white region below; amplitude varies sinusoidally.

As he remarked in his original paper, the answer is 'no' in higher dimensions. In 1964 John Milnor (now at SUNY, Stonybrook) wrote a one-page paper in which he exhibited two distinct sixteen-dimensional tori (generalized doughnuts) with identical vibrational spectra. The idea is quite ingenious. One way to get an ordinary torus is to take a unit square in the plane and glue its opposite edges together. Equivalently, draw the integer lattice in the plane and pretend that points whose coordinates differ by those of a lattice point are 'the same'. This effectively rolls the plane up in circles along both coordinate axes and yields the torus. The same construction applies in higher dimensions: just use a higher-dimensional lattice. Milnor proved that the vibrational spectrum of such a torus is determined by the 'length spectrum' of the lattice: the ordered list of distances of lattice points from the origin. Then he observed that the algebraist E.Witt had already found two (rather famous) distinct lattices in 16-dimensional space with the same length spectrum, and everything else followed at once.

Sixteen dimensions are all very well, but what of a drum in the plane? That proved much less tractable. The first results were positive: various features of the shape can indeed be 'heard', that is, deduced from the spectrum. The first was the area. One of the great mathematical centres around the turn of the century was Göttingen. P.Wolfskehl had endowed a prize for a proof of Fermat's Last Theorem, but in the absence of any solution (the problem is still open but the prize has inflated to nothing) the interest was to be used to pay for a series of lectures. In October 1910 the Dutch physicist Hendrik Lorentz, of Lorentz-Fitzgerald contraction fame, gave the Wolfskehl lectures on 'Old and New Problems of Physics'. His lecture included the following passage:

> "There is a mathematical problem which perhaps will arouse the interest of mathematicians who are present. It originates in the radiation theory of Jeans. In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to tones of an organ pipe... Jeans asks for the energy in the frequency interval $d\mu$. To this end he calculates the number of overtones which lie between the frequencies μ and $\mu+d\mu$... It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lies between μ and $\mu+d\mu$ is independent of the shape of the enclosure and is simply proportional to its volume."

The wave equation for electromagnetism is the same as that for a vibrating solid of the same shape as the enclosure. Lorentz was talking about 'asymptotic' properties of the spectrum, depending only upon very high frequencies; and he was asking whether you could hear the volume of the enclosure if only high frequencies were taken into account. Allegedly David Hilbert, the Grand Old Man of Göttingen mathematics, predicted that Lorentz's question would not be answered within his lifetime. For once he was wrong: less than two years later, Hermann Weyl proved the theorem — for waves in any number of dimensions — using the theory of integral equations (much of which had been developed by Hilbert).

Kac himself proved that for an ordinary two-dimensional drum, the spectrum determines the perimeter. One curious consequence is that you can hear whether or not a drum is circular. A circle has the smallest perimeter for given area. If you know the area A and the perimeter p, and it so happens that $p^2 = 4\pi A$ (as it is for a circle), then the drum *is* a circle. Kac also conjectured a formula implying that, should a drum have finitely many holes (whose edges are considered to be part of the boundary and hence also kept fixed) then you can hear how many holes there are.

No serious progress was made for fifteen years after Kac asked his question, but then the problem came off the back boiler. In 1980 Marie-France Vignéras found new high-dimensional spaces with the same spectra but different topology, proving that a topological invariant called the 'fundamental group' cannot be heard. Other examples were found by A.Ikeda in the same year. In 1985 Toshikazu Sunada (Nagoya University) found a general criterion for two distinct shapes to have the same spectrum. Using it, Peter Buser (École Polytechnique, Lausanne), Robert Brooks (University of Southern California), and Richard Tse found distinct curved surfaces with the same spectrum. Surfaces are classified topologically by the number of holes they have, and these examples could have any number of holes greater than or equal to four. You can hear the number of holes, but not the actual shape.

Gordon was describing one of Buser's examples at a geometry conference in the spring of 1991, and Wolpert, in the audience, noticed that this curved surface possesses a particular symmetry allowing it to be 'flattened' in a natural way. He asked whether the result would answer Kac's original question in the negative. Webb reports the suggestion as being "like a cold shower", forcing him and Gordon to think the whole problem through again. They became convinced — wrongly, it later turned out — that Wolpert's idea wouldn't work, but that something more complicated might. Eventually, having filled their offices with huge paper constructions that wouldn't flatten, they got back on the right track, and came up with two drums, each made from seven bisected Maltese crosses, already shown in Fig.1. One drum resembles a pound sign, the other (with artistic licence) a yen sign. The pound and yen have identical spectra, but different overall shapes.

The two drums also have a clear 'family resemblance': they're assembled from identical pieces, and that's important in the clever proof that the spectra are identical. It involves taking any possible vibration of one drum, cutting along the dotted lines between the seven pieces, and showing that the result can be reassembled to give a valid vibration of the other drum — which perforce has the same frequency. This technique is based on Sunada's work, and was pioneered by Pierre Bérard (University of Grenoble). Solutions of the wave equation, in short, can be 'cut-and-pasted' between different drums. The method has now been made more elegant, and many other examples of sound-alike drums are now known.

Despite this cunning answer to Kac's question, most of the subject is still a mystery. What *can* you hear? Volume or area, dimension, a few curvature properties. The topology, for two-dimensional surfaces. Some information about lengths of geodesics, shortest curves. According to a 1988 result by Michel Lapidus (University of Georgia) and Jacqueline Fleckinger-Pellé (Université Paul Sabatier, Toulouse) — confirming a conjecture by the physicist Michael Berry (University of Bristol) that generalizes Weyl's theorem — you can hear the fractal dimension of the drum's boundary. Not the usual Hausdorff-Besicovotch dimension, but a variant called the Minkowski dimension. What *can't* you hear? The fundamental group (and hence the topology). Esoteric items such as patterns of criss-crossing of closed geodesics. Not much else for sure. "The detective work of deciphering just what geometrical information the spectrum holds," says Gordon, "has only just begun."

Further Reading

M Kac, Can one hear the shape of a drum? *American Mathematical Monthly* **73** (1966) 1–23.

Carolyn Gordon, When you can't hear the shape of a manifold, *Mathematical Intelligencer* **11** (1989) 39-47.

Carolyn Gordon, David L Webb, and Scott Wolpert, One cannot hear the shape of a drum, *Bulletin of the American Mathematical Society* **27** (1992) 134-138.

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