Escher and Coxeter

A Mathematical Conversation

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M. C. Escher (1898 – 1972)

Hand with Reflecting Sphere
(Lithograph, 1935)
Education

- Early education in Arnhem.
- Admitted 1919 to School for Architecture and Decorative Arts in Haarlem.

*White Cat* (Woodcut, 1919)
Travels in Italy and Spain

San Gimignano
(Woodcut, 1923)

Atranti, Coast of Amalfi
(Lithograph, 1931)
A new direction

*Metamorphosis I* (Woodcut, 1937)
Regular Tilings of the plane

Sarah Hart

Escher and Coxeter
A regular tiling uses one shape of tile, which is a regular polygon, and the same number of tiles meet at each point.
A regular tiling uses one shape of tile, which is a regular polygon, and the same number of tiles meet at each point.
What is a regular polygon?

A regular polygon is a shape made with straight edges, and all edges are equal length, and all internal angles are equal.
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What is a regular polygon?

A regular polygon is a planar shape made with straight edges, and all edges are equal length, and all internal angles are equal.
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What is a regular polygon?

A regular polygon is a planar shape

- made with straight edges,
What is a regular polygon?

A regular polygon is a planar shape

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A regular polygon is a planar shape

- made with straight edges,
- and all edges are equal length,
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A regular polygon is a **convex** planar shape
- made with straight edges,
- and all edges are equal length,
- and all internal angles are equal.
What is a regular polygon?

A regular polygon is a convex planar shape
▶ made with straight edges,
▶ and all edges are equal length,
▶ and all internal angles are equal.
Regular Tilings of the plane

A regular polygon is a convex planar shape made from straight edges where all edges are equal length and all internal angles are equal size.
Regular Tilings of the plane

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A regular tiling uses one shape of tile, which is a regular polygon, and the same number of tiles meet at each point.
A \( \{k, n\} \)-tiling has \( k \) tiles (\( n \)-gons) at each point.

The regular tilings of the plane are \( \{4, 4\} \), \( \{6, 3\} \) and \( \{3, 6\} \).
What about the 17 wallpaper patterns?
We can tile the sphere with regular polygons (we have to curve them to lie properly on the surface).
Regular Tilings of the Sphere

- We can tile the sphere with regular polygons (we have to curve them to lie properly on the surface).
- Start with flat tiles making a 3-d shape and ‘inflate’ to make a sphere.

Five possibilities:
- \{3, 3\}, \{4, 3\}, \{5, 3\}, \{3, 4\}, \{3, 5\}
We can tile the sphere with regular polygons (we have to curve them to lie properly on the surface).

Start with flat tiles making a 3-d shape and ‘inflate’ to make a sphere.

Remember we need the same number of the same tile meeting at each point.
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Angels and Devils on a Sphere (1942)
Donald Coxeter (1907 – 2003)

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Sarah Hart, Birkbeck, University of London

Escher and Coxeter
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International Congress of Mathematicians, 1954

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Escher and Coxeter
Coxeter’s Diagram
Escher’s New Tiling

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Escher and Coxeter
Three Geometries
Three Geometries
The Poincaré Disc
Hyperbolic Tilings
I’ve been killing myself, [...] for four days with clenched teeth, to make another nine good prints of that highly painstaking circle-boundary-in-colour. Each print requires twenty impressions: five blocks, each block printing four times. (Escher to his son Arthur, March 1960)
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Influence on Coxeter
Circle Limit IV (woodcut, 1960)
Classification of Regular Tilings

- A \( \{k, n\} \) regular tiling is a tiling by regular \( n \)-gons, with \( k \) meeting at each vertex.

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- We must have \( n \geq 3 \) as the smallest number of sides a polygon can have is 3.
**Theorem**

For any $k$ and $n$ with $k \geq 3$ and $n \geq 3$, there exists a regular tiling in exactly one of plane, spherical and hyperbolic geometry.

- $\frac{1}{k} + \frac{1}{n} > \frac{1}{2} \rightarrow$ regular spherical tiling.
- $\frac{1}{k} + \frac{1}{n} = \frac{1}{2} \rightarrow$ regular plane tiling.
- $\frac{1}{k} + \frac{1}{n} < \frac{1}{2} \rightarrow$ regular hyperbolic tiling.

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Example

For a tiling with equilateral triangles ($n = 3$) we have $\frac{1}{2} - \frac{1}{n} = \frac{1}{6}$.

So if $k$ is 3, 4 or 5 we get a spherical tiling; if $k = 6$ it's a plane tiling; if $k$ is 7 or higher it's a hyperbolic tiling.
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Thank you!