Euler's Pioneering Equation:
'The most beautiful theorem in mathematics'

[Written by Robin Wilson and presented in person by Leonhard Euler]

1. Title page

In this talk I'd like to tell you about my pioneering equation, called 'the most beautiful theorem in mathematics'. But first let me introduce myself, Leonhard Euler.

I was born in Switzerland, but spent many years in the Imperial courts of St Petersburg and Berlin, and have been called 'the most prolific mathematician of all time', having published over 800 books and papers in over 70 volumes. Ranging across almost all branches of mathematics and physics at the time, these amounted to about one-third of all the 18th-century publications in these subjects.

2. 'The most beautiful theorem in mathematics'

Why is it '*the most beautiful theorem in mathematics*'?

This description came from a poll run by the *Mathematical Intelligencer*, an American mathematics magazine, when my equation topped the list.

But such polls aren't restricted to mathematicians. A similar poll, for '*the greatest equation ever*', was taken by *Physics World*, with my equation appearing in the top two, and way ahead of such equations as Einstein's $E = mc^2$ and Newton's laws of motion.

Other people have been equally impressed.

Indeed, when only 14, the future Nobel prize-winning physicist Richard Feynman called my equation '*the most remarkable formula in math*', while Fields-Medal winner Sir Michael Atiyah has described it as '*the mathematical equivalent of Hamlet's 'To be or not to be'*: *very succinct, but at the same time very deep*'. And the mathematical populariser Keith Devlin waxed even more eloquent, saying:

'*Like a Shakespearian sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler's equation reaches down into the very depths of existence*'.

It has even featured in two episodes of *The Simpsons*, and was crucial in a criminal court case when an American physics graduate student was sentenced to eight years in prison after vandalizing 100 luxury sports cars by spray-painting slogans onto them. He was identified after spraying my equation (which had just popped into his head) onto a Mitshubishi Montero. As he announced at his trial,

'*I've known Euler's equation since I was* 5 *– everyone should know Euler's equation*'.

So what *is* this result of mine that '*everyone should know*'?

3. Euler's equation

My equation's important because it combines five of the most important constants in mathematics:

1 (the basis of our counting system),

0 (the number that expresses 'nothingness'),

π (the basis of circle measurement),

e (the number linked to exponential growth),

and i (an 'imaginary' number, the square root of minus 1).

It also involves the fundamental mathematical operations of addition, multiplication, and taking powers.

Now, if we take e, and raise it to the power (i times π), and then add 1, we get 0 – or equivalently $e^{i\pi} = -1$.

As one participant in the *Physics World* poll remarked:

What could be more mystical than an imaginary number interacting with real numbers to produce nothing?

4. Euler's equation (2)

. . . and the numbers have even featured in a nursery rhyme:

Leonhard Euler had a farm, e, i, e, i, 0,

And on that farm he had 1 π-g, e, i, e, i, 0.

5. Euler's identity

As we'll see, my equation is a special case of a more general result that I published in 1748: this beautifully relates the exponential function and the trigonometrical functions cos x and sin x. But why should the exponential function e^x which goes 'shooting off to infinity' as x becomes large, have anything to do with sine and cosine which oscillate forever between the values 1 and −1?

Indeed, there is no *real* reason why there should be such a relationship – no real reason, but there is a *complex* reason! Introducing the complex number i leads to such connections, and realising this was one of my greatest achievements. My result has even appeared on a Swiss postage stamp, where it appears up the left-hand side.

6. Some applications

Although my results may seem rather abstract, they're also of fundamental importance in physics and engineering. This is because exponentials of the form e^{kt} describe things that grow if k is positive, or decay if k is negative, while those of the form e^{ikt} describe circular motion. But, by my identity, e^{ikt} is made up from cos kt and sin kt and so can be used to represent things that oscillate: for example, $e^{i\omega t}$ refers to an alternating electric current with angular frequency ω.

These 'imaginary exponentials' are much easier to deal with mathematically, than sines and cosines – and indeed, for more advanced topics such as quantum mechanics or image processing, many calculations cannot be done without them.

In this talk I'll introduce the five constants one at a time, before showing you how they combine to give what we've called my equation.

7. 1*: the counting number*

We'll start with 1, the basis of our counting system.

It's been said that: *There are three types of people – those that can count and those who can't* – but how *do* we count?

We use a *decimal system*, using only the ten digits 1 to 9 and 0. But it's also a *place-value system*, with the placing of each number determining its value. For example, the number 5157 means 5 thousands, 1 hundred, 5 tens, and 7 ones. Here the number 5 plays two different roles, depending on its position – as 5 thousands, and as 5 tens – and the advantage of such a place-value system is that we can carry out our number calculations column by column.

Another example is the *binary system* used in computing, which is based on 2, rather than 10. It's been said that: *There are 10 types of people – those that can count in binary and those that can't*. So a binary number such as 1101 means

1 lot of 2^3 + 1 lot of 2^2 + no lots of 2^1 + 1 unit, corresponding to our decimal number 13. In fact, it's as easy as 1, 10, 11.

8. Egyptian counting

So how did our counting systems arise? How did early civilizations count? Let's look at some of them.

Around 1800 BC the Egyptians (who wrote on papyrus) used a decimal system – but it wasn't a place-value system because they used different symbols for 1, 10, 100, and so on, repeating them as often as necessary. So the number below (reading from right to left), is 2 lotus flowers, 6 coiled ropes, 5 heel bones, and 8 rods, or 2658.

9. Mesopotamian counting

Around the same time, the Mesopotamians (or Babylonians) were imprinting their numbers on clay tablets. They *did* use a place-value system, but it was based on 60, not on 10 – a method of counting we still use when we measure time: 60 seconds in a minute, 60 minutes in an hour.

Using a vertical symbol for 1 and a horizontal one for 10, the number 1, 12, 37 shown here means

(one lot of 60^2) + (twelve lots of 60) + (thirty-seven units),

which add up to our decimal number 4357.

10. The classical world

Moving forward by over a thousand years takes us to classical Greece and Rome.

We're all familiar with Roman numerals – a decimal system with letters representing numbers. But it's not a place-value system, because different letters are used for 1, 10, 100 and 1000, and for 5, 50 and 500. Because calculating with these letters isn't easy, they used a counting-board or abacus for their everyday calculations.

The Greek system seems even more confusing. It's also a decimal system, but again it's not a place-value system since different Greek letters represent the units from 1 to 9, the tens from 10 to 90, and the hundreds from 100 to 900. So a number like 888 would be written as 800 + 80 + 8, or $\omega\pi\eta$.

11. Chinese counting

In China they used counting-boards for their arithmetic, placing small bamboo rods into separate compartments for units, tens, hundreds, and so on. This was a decimal place-value system – one of the first.

Here each number came in two forms – vertical and horizontal – which alternate. So 1713 is a horizontal 1, a vertical 7, another horizontal 1, and a vertical 3. Notice that, for the number 6036, the zero gives us an empty box, and the two forms of 6 are different.

12. Mayan counting

A different method of counting was used for the calendar calculations of the Mayans of Mexico and Central America. These survive in a small number of codices drawn on tree-bark and then folded.

Here, counting was based mainly on 20, combining dots and lines to give all the numbers from 1 to 19 (on the left). For larger numbers they piled these numbers on top of each other, so here you can see twelve 20s plus 13, corresponding to our decimal number 273.

An attractive feature of Mayan counting was that each number also had a pictorial *head-form*, like the ones below. They also had a symbol for 0 – the shell-like symbol below.

13. Indian counting

So this leads us to our second number, 0. In India around 250 BC, the edicts of Ashoka, the first Buddhist monarch, were carved on pillars around the kingdom. Some of these contained early examples of Indian base-10 numerals, as a decimal place-value system began to emerge, using only the digits 1 to 9, and (later) 0.

14. 0 *– the nothingness number*

So how did 0 arise? We've seen how the Chinese left spaces in their counting boards, while other civilizations left spaces in the sand to distinguish numbers like 35 and 305. But gradually, special symbols began to emerge. Here's a cave in Gwalior in India, where 270 is clearly seen on the wall.

But there was great excitement last autumn when some birch-bark that had lain undiscovered for 100 years was found in the Bodleian library in Oxford. It had hundreds of blobs on it, each representing 0, and it pre-dated all other known appearances of 0 by some 400 years.

15. Brahmagupta

Note that 0 can play two roles – as a place-holder, as we've seen, but also as a number to calculate with. Positive and negative numbers were already used in money markets for profits and debts, and around the year 600 rules for calculating with them, and with 0, were given by the Indian mathematician Brahmagupta.

Here are some examples: *adding zero and a negative number gives a negative number*; *a negative number taken from zero becomes positive*, and so on. The only meaningless one was the last one, relating to division by 0. But this is forbidden, because if we cancel the 0s in an equation like $4 \times 0 = 9 \times 0$, we get $4 = 9$, which is nonsense.

16. Development of number systems

This picture shows how our number systems developed over the centuries, leading to the numerals at the bottom which we can clearly recognize. Also developing were the Arabic numerals on the right, which are still used in the Middle East.

17. The Hindu–Arabic numerals

But it took many centuries for what we now call the *Hindu–Arabic numerals* to become fully established. Here's a 16th-century picture representing *Arithmetic* and contrasting the modern 'algorist' with his Hindu–Arabic numerals and the old-fashioned 'abacist' with his counting board. Meanwhile, arithmetic books promoting the Hindu–Arabic numerals began to appear – by Fibonacci in 1202, Luca Pacioli in 1494, and Robert Recorde in 1543. The drawing on the right from Pacioli's *Summa* shows you how to calculate on your fingers.

18. π: the 'circle number'

We now turn to our third number, π, which arises in two ways – as the ratio of the circumference of a circle to its diameter: $\pi = C/d$, *so* $C = \pi d$, *or* $2\pi r$, *where r is the radius*; this ratio is the same for circles of any size, from a pizza to the Moon.

And π is also the ratio of the area of a circle to the square of its radius: $\pi = A/r^2$, *so* $A = \pi r^2$. This ratio is also the same for all circles, as proved by Euclid in the 3rd century BC.

19. π to 500 decimal places

We can never write down π exactly – its decimal expansion goes on for ever – but if my six figures weren't enough for you, here are a few more . . .

20. The Vienna metro (Karlsplatz)

. . . but if you've forgotten any of these and happen to live in the Karlsplatz area of Vienna, don't worry – you'll find them at your local metro stop.

21. π to 2000 decimal places

. . . and if they're not enough for you, here are some more.

But the point is, we can never write out π in full. Although π has been memorised to 100,000 decimal places (what a way to spend a life!), and calculated to 20 trillion – even they're only a beginning, and there's still a long long way to go!

22. Mnemonics for π

But you can easily remember the first few digits from these mnemonics. In the question: '*May I have a large container of coffee?*', the number of letters in each word spell out the first 8 digits: 3 1 4 1 5 9 2 6. And from the second sentence we get 14 decimal places: *How I need a drink, alcoholic of course, after all these lectures informing quantum mechanics.* And there's even one in Greek which gives us 22 decimal places.

23. Mesopotamian value of π

When did people start to measure circles? Several early civilizations needed to estimate the circumference or area, and although they had no conception of π as a number, their results yield approximations to it.

A Mesopotamian clay tablet relates the perimeter of a regular hexagon to the circumference of the surrounding circle as the sexagesimal number 0;57,36. If the radius is r, then each side of the hexagon is also r (these are equilateral triangles), and so this ratio of $6r / 2\pi r$ (or $3/\pi$) is ${}^{57}/_{60} + {}^{36}/_{3600}$. This gives $\pi = 3^1/_8$, or 3.125 in decimals – a lower estimate that's within one per cent of the true value.

24. An Egyptian problem in geometry

Around the same time, an Egyptian papyrus asked the following question: *given a round field of diameter 9 khet, what is its area?* The answer is given in steps: *Take away* ${}^1/_9$ *of the diameter, which is* 1, *leaving* 8. *Multiply* 8 *times* 8, to *give* 64 setat *of land*.

So to find the area they reduced the diameter by one-ninth and squared the result: this method was probably found by experience. In terms of the radius, the area turns out to be ${}^{256}/_{81}\, r^2$, so that π is about 3.160, an upper estimate that's also within one per cent of the true value.

25. The Biblical value

A very convenient (but much less accurate) value appeared about 1000 years later in the Old Testament. From I Kings and II Corinthians we learn that a worker in bronze named Hiram made a molten sea with diameter 10 cubits and circumference 30 cubits, giving $\pi = 3$.

26. Using polygons

A better method for finding π was introduced by the Greeks, and would be used for almost 2000 years. Often credited to Archimedes, it dates back to the 5th century BC when the Greek sophists Antiphon and Bryson approximated a circle by regular polygons, and then tried to obtain better and better estimates by repeatedly doubling the number of sides until the polygons eventually 'became' the circle.

Antiphon first took a square inside a unit circle and found its area to be 2. He then doubled the number of sides to an octagon, with area $2\sqrt{2}$, or 2.828. Bryson's approach was similar, except that he also considered polygons *outside* the circle, getting upper bounds of 4 for the square and 3.32 for the octagon.

Two hundred years later Archimedes adopted the same idea, but worked with perimeters rather than areas. Starting with hexagons inside and outside the circle, he doubled the number of sides to 12, 24, 48 and 96, obtaining bounds for π of $3^{10}/_{71}$ and $3^1/_7$. This gives π to two decimal places.

27. Chinese values for π

What was happening elsewhere? In China, around the year 263, Liu Hui also used polygons to approximate π. Starting with hexagons and dodecagons, he found simple methods for calculating the successive areas and perimeters when he doubled the number of sides, and for polygons with 192 sides he obtained bounds of about 3.14. Four more doublings led to polygons with 3072 sides and to $\pi = 3.14159$.

Even more impressively, around the year 500 Zu Chongzhi and his son doubled the number of sides three more times to over 24,000, and obtained π to six decimal places. They also improved Archimedes' fractional value of $^{22}/_{7}$ to $^{355}/_{113}$, which also gives π to six decimal places. This latter approximation wasn't rediscovered in Europe for another thousand years.

28. Ludolph van Ceulen (Dutch)

After this, everyone got in on the game, as the number of sides continued to double, with corresponding increases in accuracy – leading eventually to the remarkable Dutchman Ludolph van Ceulen, with polygons of over 500 billion sides, giving π to 20 decimal places: his upper and lower estimates appear here, below his portrait.

Not content with this, he then used polygons with 2^{62} sides to find π to 35 decimal places. He asked for this latter value to appear on his tombstone in Leiden, and for many years π was known in Germany as the *Ludolphian number*.

29. The $\tan^{-1}$ (or arctan) function

A new and highly productive method for estimating π, which was used extensively in the 18th and 19th centuries, involved the *inverse tangent function*, usually written as $\tan^{-1}$ or arctan. [This next bit gets a little technical, but hang in there – it won't last long! In particular, we'll use radian measure for angles, where π means 180°.]

If $\tan \theta = a/b$, then $\theta = \tan^{-1}(a/b)$, so $\tan^{-1}$ simply undoes whatever tan does. For example,

$\tan \pi/4$ (that's tan 45°) is 1, so $\tan^{-1} 1$ is $\pi/4$,

and $\tan \pi/6$ (that's tan 30°) is $1/\sqrt{3}$, so $\tan^{-1} 1/\sqrt{3}$ is $\pi/6$.

We can combine different values of $\tan^{-1}$ – for example, when we add $\tan^{-1}(^1/_2)$ and $\tan^{-1}(^1/_3)$ we get $\pi/4$. We can see this from the picture on the left, and we can also prove it by simple geometry. And in general we can combine any two inverse tangents by using the formula below.

30. The series for $\tan^{-1} x$

Many functions can be written as infinite series. For example, $\tan^{-1} x$ is the infinite series shown here, with only odd powers of x, and with odd numbers as denominators: this result was already known in 15th-century India, but is usually named after the Scotsman James Gregory (shown here), who rediscovered it 300 years later.

If we now let $x = 1$, we get a series expression for $\pi/4$, a result also found in India but usually credited to Leibniz. This is surely one of the most amazing results in the whole of mathematics, since by just adding and subtracting numbers of the form $1/n$ we get a result involving the circle number π.

Unfortunately, the Leibniz series converges so slowly that we cannot use it to find π in practice; for example, the first 300 terms of the series give π to only two decimal places, while the first half-a-million terms give us only five places.

But we *can* use Gregory's series to estimate π if we substitute values other than 1. Remember that $\tan^{-1}(^1/_2)$ and $\tan^{-1}(^1/_3)$ add up to $\pi/4$, so we can substitute $x = {}^1/_2$ and $x = {}^1/_3$ into the series for $\tan^{-1} x$, giving the two series here.

And because of the increasing powers of 2 and 3 in the denominators, these converge much faster, yielding good estimates for π. Indeed, in 1861 a gentleman from Potsdam used these very series to find π to 261 decimal places.

31. Machin's $\tan^{-1}$ formula (1706)

Are there other series for π which converge even faster?

In 1706 the Englishman John Machin repeatedly used the addition formula to show that π is $16 \tan^{-1} (^1/_5) - 4 \tan^{-1} (^1/_{239})$, and then wrote out the two $\tan^{-1} x$ series shown here.

Now these series converge rapidly because of the powers of 5 and 239 in the denominators – for example, taking just three terms already gives the value 3.14. Also, 5 is an easy number to divide by, and Machin was able to calculate π by hand to 100 decimal places – a great improvement on anything that went before.

32. William Jones introduces π (1706)

1706 was a good year for π. As well as Machin's result, a Welsh maths teacher called William Jones wrote *A New Introduction to the Mathematicks*, in which he introduced, for the first time, the symbol π for the circle number.

Here are two extracts from his book. Above you can see Machin's series, followed by the first ever appearance of π. And below is Machin's value for π in full, described as: *True to above a 100 places*; *as Computed by the Accurate and Ready Pen of the Truly Ingenious Mr. John Machin*.

33. William Shanks's value of π

Such results can be used to obtain improved values for π, and most notorious of all was one obtained by William Shanks, who in 1873 used Machin's formula to calculate π to an impressive 707 decimal places. These were later inscribed in a ceiling frieze in the π-room of the Palace of Discovery in Paris, where they can still be seen. Unfortunately for him, and for the Palace, it was later found that only the first 527 of these decimal places are correct.

34. Buffon's needle experiment (1777)

Let's now look at a very different way to find π. In 1777 the Comte de Buffon described an *experiment* for estimating it.

Suppose you throw a large number of needles (or matchsticks) of length L onto a grid of parallel lines at a distance D apart, and record the proportion of needles that cross a line. It's not difficult to show that this proportion is $2/\pi \times L/D$, from which we can calculate a value for π. Here L/D is $^4/_5$, and exactly five of the ten needles cross lines; this gives us $\pi = 3.2$, which isn't too bad for just ten needles.

Incidentally, in 1901 an Italian mathematician called Mario Lazzerini carried out such an experiment in which L/D was $^5/_6$, performing 3408 trials and claiming 1808 crossings. This gave $\pi = {}^{355}/_{113}$ which (as we saw), gives six decimal places. He was lucky. If just one needle had landed differently, his result would have been correct to only two decimal places.

35. Legislating for π

In 1897 a bizarre event took place in the American State of Indiana, where the House of Representatives unanimously passed 'A bill introducing a new Mathematical Truth'. This attempted to legislate an incorrect value for π proposed by a local physician, who'd then allow the State to use his value freely, but who'd expect royalties from anyone out-of-state who used it:

A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only in the State of Indiana free of cost by paying any royalties whatever on the same . . .

According to the proposer, *the ratio of the diameter and circumference is as five-fourths to four*, which gives $\pi = 3.2$.

For some reason the bill was then passed on to the House Committee on Swamp Lands, who in turn passed it on to the Committee on Education:

It has been found that a circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side . . .

This makes no sense, but even so it proceeded to the Committee on Temperance, who recommended its passage. Fortunately, a mathematician from Purdue University happened to visit the statehouse when the bill was about to be finally ratified, and he persuaded the senators to stop it just in time. As far as we know, it's still with the Committee on Temperance . . .

36. Some weird results

The 20th century saw several new discoveries – many of them completely bizarre. Here are three of them.

In 1914 the Indian mathematician Ramanujan found some remarkable exact formulas for $1/\pi$, including this one – an infinite series in which strange numbers, such as 1103 and 26,390, seem to appear from nowhere. Such series converge extremely rapidly and form the basis of some of today's fastest algorithms for calculating π. Many years later, in 1989, David and Gregory Chudnovsky of New York produced a similar result with even larger numbers, as you can see.

My third example is simpler, but caused much surprise at the time. Its importance is that, if we work in base 16 rather than base 10, we can calculate each digit of π one at a time without having to re-calculate any preceding digits first.

37/38. Circling the earth

To end my discussion of π, here's a simple puzzle that appeared in 1702, in a book on Euclid's *Elements* by the Cambridge mathematician William Whiston. If you haven't seen it before, you may find the answer surprising.

The circumference of the Earth is about 25,000 miles, or 132 million feet. Assuming the Earth to be a perfect sphere, suppose we tie a long piece of string tightly around it. We then extend this string by 2π (or just over 6) feet, and prop it up equally all around the Earth. How high above the ground is the string? Most people think that the resulting gap must be extremely small – perhaps a tiny fraction of an inch – but the answer is *one foot*!

In fact, we get the same answer whether we tie the string around the Earth, a tennis ball, or any other sphere. For, if the sphere has radius r feet, then the original string has length $2\pi r$. When we extend it by 2π feet, the new circumference is $2\pi r + 2\pi$, which is $2\pi \times (r + 1)$.

So the new radius is $r + 1$: one foot more than before.

39. The exponential number' e?

Let's now move on to our next number, the 'exponential number' $e = 2.71828$, etc., another number whose decimal expansion goes on for ever. Here, we're concerned with how quickly things grow. We often use the phrase 'exponential growth' to indicate something that grows very fast, but how fast is this?

The letter e first appeared in print in 1736 here, in my *Mechanica*, a book on the mathematics of motion. It's in the penultimate line: *where e denotes the number whose hyperbolic logarithm is* 1.

40/41. A chessboard problem

To show what we mean by 'exponential growth', here's a story about the origins of the game of chess.

A wealthy king was so impressed by this new game that he offered the wise man who invented it any reward he wished – to which the wise man replied:

My prize is for you to give me 1 grain of wheat for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, and so on, doubling the number of grains on each successive square until the chessboard is filled . . .

The king was amazed to be asked for such a tiny reward (or so he believed), until his treasurers calculated the total number of grains of wheat. This works out at $2^{64} - 1$ grains, enough wheat to form a pile the size of Mount Everest. Placed end to end they'd reach to the nearest star, *Alpha Centauri*, and back again!

42. Exponential growth

Let's see how quickly other sequences can grow.

A very simple form of growth is *linear growth*, as in the counting numbers $n = 1, 2, 3, 4, 5$, etc. Somewhat quicker is the way the perfect squares $n^2 = 1, 4, 9$, etc. grow, and even faster is that of the cubes n^3. These are all examples of *polynomial growth*, since they involve powers of n.

Alternatively, we could look at powers of 2, or of any other number. As we saw in the chessboard story, the numbers 2^n (the powers of 2) start slowly, but soon gather pace because each successive number is twice the previous one – and the powers of 3 grow even more quickly. These are examples of *exponential growth*, where n appears as the exponent.

To compare these types of growth, let's see the running times of some polynomials and exponentials when $n = 10$, 30, and 50, for a computer performing a million operations per second. For polynomial growth, such as n^3, such a computer would take about one-eighth of a second when n is 50. But exponential growth, such as 2^n, is much greater, as we've seen: when n is 50, the computer would take over 35 years, and would be vastly greater than this for 3^n.

So, in the long run, exponential growth tends to exceed polynomial growth, often by a huge margin. Algorithms that run in polynomial time are generally thought of as 'efficient', while those that run in exponential time normally take much longer to implement, and are regarded as 'inefficient'.

43. An interest-ing problem

Returning to e, what exactly is this number, and how did it arise?

In 1683 the Swiss mathematician Jakob Bernoulli was calculating compound interest. Given a sum of money to invest at a given rate of interest, how does it grow? The answer depends on how often we calculate the interest. How much is earned if we calculate it yearly? or twice a year? or every month? every week? every day? continuously?

As an example, to keep the calculations simple, let's see what happens if we invest £1 at the unlikely annual rate of 100 per cent. After one year our pound has doubled to £2. But if we calculate the interest twice a year (50% each time), then after six months our £1 is multiplied by $1^1/_2$ to give £1.50, and after another months *that* amount is multiplied by $1^1/_2$ to give £2.25, which is more than before.

Now let's calculate the interest every three months. Then there are four periods, and after each one the amount's multiplied by $1^1/_4$ – first to £1.25, then to about £1.56, then to £1.95, and by the end of the year to £2.44, which is £1 × ($1^1/_4$ to the power 4). So the final amount is larger still!

As the periods get shorter, what happens? Do the final amounts increase without bound, or do they settle down to a limiting value? The results are shown in this table, to five decimal places. To find them, note that if the year's divided into n periods, then after each period the amount's multiplied by $1 + {}^1/_n$, so that the final amount is $(1 + {}^1/_n)^n$.

We also see from this table that, as n increases, these final amounts approach a limiting value that corresponds to when we calculate the interest continuously. This limiting amount of about 2.81828 is the exponential number e.

44. Leonhard Euler

The greatest advances in understanding exponentials were made in the early 18th century. After Bernoulli, the main figure in this story was myself, who investigated the number e and the related exponential function e^x. In 1748 my *Introduction to the Analysis of Infinites*, one of the most important mathematics books ever written, brought together many of my results from earlier works.

45. Some properties of e

Here are some of my main findings.

We've just seen that e is the limit of the numbers $(1 + 1/n)^n$ as n increases indefinitely, and similarly we can show that e^x is the limit of $(1 + x/n)^n$ for any number x.

But, as Isaac Newton had already discovered, e is also the sum of the infinite series shown here: the denominators are the factorials: 1, 1 × 2, 1 × 2 × 3, and so on – and more generally, there's a similar series for e^x which converges for all values of x. In fact, all these series converge extremely fast because the factorials increase so rapidly; for example, just the first ten terms of the series for e already give the correct value to five decimal places.

On the right is the graph of $y = e^x$, where an important feature is that, at each point x, the slope of the graph is also e^x – that is, the slope at any point is the y-value – so the curve becomes steeper and steeper as x increases.

46. Exponential growth

We'll end our discussion of exponentials by returning to exponential growth. In 1798 Thomas Malthus wrote his *Essay on Population*, where he contrasted the steady *linear growth* of food supplies with the *exponential growth* in population. He concluded that, however one may cope in the short term, exponential growth would win in the long term and there'd be severe food shortages – a conclusion that was borne out in practice.

How fast does a population grow? If $N(t)$ is the size of a population at time t, and if the population grows at a fixed rate k proportional to its size, then we have the differential equation $dN/dt = kN$. This can be solved to give $N(t) = N_0e^{kt}$, where N_0 is the initial population – an example of exponential growth. In the same way we can model *exponential decay* as, for example, in the decay of radium, or in the cooling of a cup of tea.

47. Girolamo Cardano (1545)

We come now to the last of our constants – the 'imaginary' square root of minus 1 – which can be traced back to the 16th century when Cardano, one of the Italian mathematicians who first wrote about cubic equations, was trying to solve a number puzzle: *divide* 10 *into two parts whose product is* 40.

To do so, he let the two numbers be x and $10 - x$, so that x times $(10 - x) = 40$. Solving this quadratic equation he found the solutions to be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$, which seemed meaningless. Commenting that: '*Nevertheless we will operate, putting aside the mental tortures involved* ', he checked that the answers worked, but complained that : '*So progresses arithmetic subtlety, the end of which is as refined as it is useless*.'

48. The imaginary number $\sqrt{-1}$

Trying to take the square root of a negative number doesn't seem to make sense – after all, 1 × 1 is 1 and −1 × −1 is also 1. As the Victorian Augustus De Morgan remarked, 300 years later: '*We have shown the symbol $\sqrt{-a}$ to be void of meaning, or rather self-contradictory and absurd*', while his contemporary, the astronomer George Airy commented:

'I have not the smallest confidence in any result which is essentially obtained by the use of imaginary symbols'. However, Leibniz was more encouraging, claiming that:

'The imaginary numbers are a wonderful flight of God's spirit: they are almost an amphibian between being and not being.' To my shame, even I, who used them so effectively, criticised them.

49. Complex numbers

For many purposes our ordinary real numbers are enough. But suppose we now agree to allow this mysterious object called '$\sqrt{-1}$', or i, as I named it. We can then form many more numbers, such as $1 + 3i$ and $2 + i$. Ignoring for the moment what these actually mean, we can then carry out some simple calculations. Adding is straightforward – we just add the bits without i and the bits with i separately, so $(1 + 3i) + (2 + i) = 3 + 4i$. And so is multiplying, as long as we remember to replace i^2 (wherever it appears) by –1.

50. The complex plane

We can also represent these numbers geometrically, as was first done by Caspar Wessel of Norway, and later by Gauss and by Jean-Robert Argand. It's often called the *Gaussian plane* or *Argand diagram* – but neither name is historically correct, so it's much better to call it the *complex plane*.

We represent each complex number $a + bi$ by the point with coordinates (a, b); the first picture shows four points (such as $1 + 2i$ and $3 + i$) so represented. We can then add two complex numbers by using the *parallelogram rule*, as shown on the right: so as before,

$(1 + 3i) + (2 + i) = 3 + 4i$.

To multiply by i we simply rotate through 90° – as a telephone operator said to me the other day, "The number you have dialled is purely imaginary – please rotate your phone through 90° and try again". Doing this again, we're multiplying by i^2 or –1, which is a rotation through 180°.

51. William Rowan Hamilton

As we've seen, there was much suspicion in Victorian times about these 'imaginary' numbers. The Irish mathematician and astronomer William Rowan Hamilton largely ended this suspicion, by defining the complex numbers as *pairs* (a, b) of real numbers, which combine according to the particular rules shown here.

So, since complex numbers can be represented in the plane, a natural question is: Can we extend this idea to three dimensions, with numbers of the form $a + bi + cj$, where i^2 and j^2 are both –1? Addition is still OK, but multiplication is not, because of the product $i \times j$ that arises when you try.

Hamilton tried many ways to deal with this term, but all failed.

As he later wrote to his son Archibald: *Every morning on my coming down to breakfast, your brother William Edwin and yourself used to ask me. 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply with a sad shake of the head, 'No, I can only add and subtract them.'*

52. Hamilton's quaternions

After struggling with these troublesome triplets for many years, Hamilton had his moment of glory on 16 October 1843, while walking with his wife along Dublin's Royal Canal, when

an electric current seemed to close, and a spark splashed forth . . . I pulled out on the spot a pocket book . . . and made an entry there and then. Nor could I resist the impulse . . . to cut with a knife on a stone on Brougham Bridge, as we passed it, the fundamental formula with the symbols i, j, and k – namely, $i^2 = j^2 = k^2 = ijk = -1$.

These were the *quaternions*, with *four* terms $a + bi + cj + dk$. Here i^2, j^2, and k^2 are all –1 but, unlike our ordinary multiplication which is commutative ($3 \times 4 = 4 \times 3$), this one isn't: here ij is not ji, but $-ji$, jk is $-kj$, and ki is $-ik$. You can then multiply any two quaternions, as long as you stick to these rules.

53. Euler's identity

Now that we've got our five constants, let's return to my equation and my identity. Recall that my identity connects the exponential function which goes shooting off to infinity, with the functions cosine and sine which oscillate between 1 and –1.

To show this connection, recall that these functions can all be expanded as infinite series, valid for all values of x. What now happens if we allow ourselves to introduce the complex number i, the square root of -1, as I did in 1737?

To do so, take the series for e^x, and replace x everywhere by ix, as I've done here. This gives:

$$e^{ix} = 1 + ix/1! + (ix)^2/2! + (ix)^3/3!, \text{ and so on.}$$

But $i^2 = -1$, so $i^3 = -i$, $i^4 = 1$, etc., and we can simplify and collect terms. This gives the series for cosine x plus i times the series for sine x: that is: $e^{ix} = \cos x + i \sin x$, which is my identity – as I said before, one of the most remarkable equations in the whole of mathematics.

54. Euler's identity: $e^{iv} = \cos v + i \sin v$

In fact, I gave more than one proof of my identity. Here's part of a different approach in which I made use of so-called 'infinitesimals'. This is the proof that appeared in my *Introductio*. The first-ever appearance of my identity is in the penultimate line, as $\mathrm{e}^{+v \times \sqrt{-1}} = \cos v + \sqrt{-1} \times \sin v$.

As I myself commented at the time:

From these equations we can understand how complex exponentials can be expressed by real sines and cosines.

55. Euler's equation

Moreover, from my identity we can deduce my equation. We just let $x = \pi$ (the radian form of 180°). Then, since $\cos \pi = -1$ and $\sin \pi = 0$, we get $e^{i\pi} = -1 + 0i = -1$, so $e^{i\pi} + 1 = 0$.

Although I certainly made this deduction, it doesn't actually appear explicitly in any of my published works.

Can we illustrate my equation pictorially? In 1959 an English school-teacher, L. W. H. Hull, showed how to do so. He took the power series for e^x and put $x = i\pi$. This gives 1, plus $i\pi/1!$, plus $(i\pi)2/2!$, and so on, which simplifies to

$1 + i\pi - \pi^2/2 - i\pi^3/6$, etc. He then successively traced these terms on the complex plane, starting at the point 1, adding $i\pi$, subtracting $\frac{1}{2}\pi^2$ and $\frac{i}{6}\pi^3$, and so on. This produces a spiral path, which converges to the sum of the series which is -1, as expected

56. A near-miss: Johann Bernoulli

We've almost finished, but first I'd like to mention briefly a couple of earlier 'near-misses' to discovering my equation.

In 1702 Johann Bernoulli was investigating the area A of a sector of a circle of radius a – the shaded area on the right above the x-axis and below the line from the origin to the point (x, y). He found it to be the expression given here, involving the logarithm of a complex number. Leaving aside what this means, I later observed that when $x = 0$ this formula simplifies to $(a^2/4i) \times \log(-1)$. But because the sector is now a quarter-circle with area $\pi a^2/4$, we have

$$(a^2/4i) \times \log(-1) = \pi a^2/4, \text{ giving us } \log(-1) = i\pi.$$

Although I wrote down this last result explicitly, I didn't take exponentials to deduce my equation in the form $e^{i\pi} = -1$. Indeed, I often credited Bernoulli with discovering this value for $\log(-1)$.

57. Another near-miss: Roger Cotes

The other near-miss arose from the work of the Cambridge mathematician Roger Cotes, who worked closely with Isaac Newton on the second edition of the *Principia Mathematica*, and who's been credited with introducing radian measure for angles.

Around 1712 he was investigating the surface area of an ellipsoid, obtained by rotating an ellipse around an axis. The details are rather complicated, but he managed to find two different expressions for the area – one involving logarithms, the other involving trigonometry, and both involving an angle φ.

He first proved that the surface area is a certain multiple of log ($\cos \varphi + i \sin \varphi$), and then proved it to be the same multiple of $i\varphi$. Equating these results gave the identity that's shown here, which connects logarithms with the trig functions. If he'd then taken exponentials, he'd have found my identity in the form $e^{i\varphi} = \cos \varphi + i \sin \varphi$ – but he didn't. Another near miss!

58/59. Who discovered 'Euler's equation'?

So to end with, what should we call the equation $e^{i\pi} + 1 = 0$?

We've just seen how it follows easily from results of Johann Bernoulli and Roger Cotes, but that neither of them seems to have done so. Even I never wrote it down explicitly, though I certainly realised that it follows immediately from my identity, $e^{ix} = \cos x + i \sin x$. In fact, we don't know who first published the equation explicitly, though there's an early appearance in a French journal around 1813.

But almost everybody nowadays attributes the result to me, so we're surely justified in naming it 'Euler's equation', to honour the achievements of myself, who's been called a truly great mathematical *pioneer* . . .

. . . a word that (I say modestly) describes me so well, and which appropriately includes among its letters our five constants *pi*, *i*, 0, *one*, and *e*.

60. Euler's pioneering equation

Reference

Robin Wilson, *Euler's Pioneering Equation*, Oxford, 2018.