Maths and Voting

Chris Budd
We live in interesting times!

- General election
- Brexit
- Strictly Come Dancing

All involve major decisions based on voting
Want to turn the opinions of a group into a final decision

BUT …

How can we make sure that this process is fair, representative, and anonymous
A rigorous approach to voting

Kenneth Arrow. 1972 Nobel Prize in Economics
Arrow assumed an election had multiple candidates A, B, C ..., and each voter could express a range of opinions on each one.
Arrow’s Axioms for Fair Voting

1. **(Dictator)** The system should reflect the wishes of more than one voter, so there is no dictator.

2. **(Unanimity)** If all voters prefer candidate A to candidate B then A should come out ahead of B in the final vote.

3. **(Universality)** The voting system should always return one clear result.

4. **(Independence of irrelevant alternatives)** In the final result, the ranking of A above B should only depend on how individual votes ranked A compared to B and not how they ranked them when compared to a third alternative C. No tactical voting
Some more reasonable conditions for fair voting

**Majority condition:** A candidate who is top choice for a majority of the voters should get elected.

**Monotonicity:** It should neither be possible to prevent the election of a candidate by ranking them higher on some of the ballots, nor possible to elect an otherwise unelected candidate by ranking them lower on some of the ballots.

**Anonymity:** it should not be possible to tell from the vote, who voted for which candidate.

**Practicality:** The method should be polynomial in time
Problem!

There’s NO voting system which satisfies them all

All voting systems have flaws and are compromises between different priorities
Example: The Borda voting method

Election with $N$ candidates  eg. Recruiting a lecturer

Purest form:

Each voter gives each candidate a mark from $N-1$ to 0

Votes are added

Highest total vote wins

Strictly Come Dancing:

Each judge gives each dance pair a mark from 10 to 1

Marks added to give a ranking. Combined with audience rank
First Borda example:

A wins with 5 votes

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>V1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>V2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>V3</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>5</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Second Borda example:

Result is a tie

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>V1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>V2</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>V3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>
How well does the Borda method match Arrow’s conditions?

**Dictator** The Borda system is a consensus system electing broadly supported candidates. However, if there are only a few voters it is possible for one to be a dictator by giving an otherwise popular candidate a very low score or vice versa.

**Unanimity** and **Monotonicity** are satisfied. If all voters prefer A to B then A will always get a higher ranking than B. It follows the sum of the rankings of A will be higher than the sum of the rankings of B.

Or mathematically

\[
\text{if } X > Y \text{ and } W > Z \text{ then } X + W > Y + Z.
\]
Universality fails. We have seen this in the election above where A and B tie for first place. Rerunning the vote may well sort this out, but this cannot be guaranteed.

Independence also fails. This is more subtle, and is a weakness of the Borda system which can be exploited in tactical voting.

Linked to this

The Borda method fails the majority condition!!
Third Borda example:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>V1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>V2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>V3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>V4</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>V5</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Total</td>
<td>9</td>
<td>10</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

Candidate B wins the election.

But …

A has won the majority of the voter preferences

B has never come top with any voter

A could have lost due to tactical voting by V4 and V5
Strictly Come Dancing

One pair gets votes: 9 9 9 1 = 28

Another gets votes: 8 8 8 8 8 = 32
Second pair wins despite first pair being preferred by 3 judges

The last judge is a dictator
Commonly used voting methods

First Past the Post (FPTP)
Single constituency or single issue

Voters express a single preference for a candidate or issue

The one with the most votes wins

Eg. Parliamentary votes, Brexit

Credit: ©UK Parliament/Jessica Taylor/Stephen Pike CC BY 3.0
To work the first past the post you need the mathematics of simultaneous linear equations.
Party A: 310 members          Party B: 250 members

All of Party B will vote against a motion. 10 members of Party A will abstain

How many of Party A have to vote for the motion for it to carry?

\[ x : \text{votes for} \quad y : \text{votes against} \]

\[ x + y + 10 = 310 \quad \text{Members of party A} \]

\[ x = 250 + y + 1 \quad \text{Votes to carry the motion} \]

\[ 2x + y + 10 = 560 + y + 1 \]

\[ 2x = 551 \quad \text{so that} \quad x = 275.5 \]

\[ 276 \]
Advantages: Traditional, clear decision

Disadvantage 1: No allowance for a preference

Disadvantage 2: Split votes

1. Increase Secondary School funding,
2. Increase Primary School funding,
3. Give no extra funding to schools.

Voting: 25% for 1, 35% for 2, and 40% for 3.

Decision: Give no extra funding!
Disadvantage 3: Possibility of error

What happens if not everyone votes?

2016 Brexit result was 52%-48% in favour of Leave

The turnout rate was 72%

37% of the British electorate actually chose to vote Leave.

Q. Is a 52% majority on a 72% sample strong enough evidence for a greater than 50% majority from the whole sample?

Subtle question in statistics which is a matter of hot debate!
Uncertainty is a good reason for insisting on a margin of error in referendums such as 60:40
Multiple constituency voting

How to make a bad thing worse!
UK/US General Elections

FPTP run in many constituencies

The party with the most elected members wins

Gives very distorted results

Eg. Recent US election

Donald Trump and the Republicans got 304 electoral votes and 46.1% of the vote

Hilary Clinton and the Democrats got 227 electoral votes and 48.2% of the vote
Example: Opposite to Borda voting

A wins the election

B comes second in all constituencies and wins no seats

B gets the most votes
Gerrymandering

Governor of Massachusetts, Elbridge Gerry approved of an oddly-shaped voting district with the explicit outcome of trying to rig the election

It worked!!

Practice still goes on today!
Proportional Representation

Avoids many of these problems

IDEA: Divide N seats in proportion to the votes received from the voters

Widely used around the world eg. EU Parliament
Main Problem 1

You end up with a fractional number of seats

Eg. 11 seats, with votes in proportion

\[
\frac{1}{2} : \frac{1}{3} : \frac{1}{6}
\]

Gives seats allocated as:

5.5 3.66666 1.83333333
Main Problem 2

Parties/countries with a small number of votes may get no seats at all in a strict proportional election.

Addressed in the European elections where following 2009, each country has to have between 6 and 96 seats.

Smaller countries over-represented degressive representation.
The d’Hondt method for seat allocation in PR

Parties ‘buy seats’ with their votes until no seats are left
Seat price starts high, and is reduced as seats are allocated.

Operation:

• The party with the most votes wins one seat
• The votes $V$ for each party are divided by the number of seats $s$ it has plus one to give

$$ N = V/(s + 1) $$

• Second seat is given to the party with the largest value of $N$
• Process continues until all of the seats have been allocated
Used in the European elections on May 22, 2014

Example: 7 seats to allocate to 4 parties A, B, C, D with votes

A = 100 000, B = 80 000, C = 30 000, D = 20 000

1: A gets one seat
2: N(A) = 50 000, N(B) = 80 000, N(C) = 30 000, N(D) = 20 000        B gets one seat
3: N(A) = 50 000, N(B) = 40 000, N(C) = 30 000, N(D) = 20 000        A gets one seat
4: N(A) = 33 333, N(B) = 40 000, N(C) = 30 000, N(D) = 20 000        B gets one seat
5: N(A) = 33 333, N(B) = 26 666, N(C) = 30 000, N(D) = 20 000        A gets one seat
6: N(A) = 25 000, N(B) = 26 666, N(C) = 30 000, N(D) = 20 000        C gets one seat
7: N(A) = 25 000, N(B) = 26 666, N(C) = 15 000, N(D) = 20 000        B gets one seat

The seat allocation is          A:3,        B:3,        C:1,        D:0

The method tends to favour larger parties
Mathematically sophisticated voting

Many mathematicians have considered how to produce ‘optimal’ voting strategies.

18th-century French mathematician Marie Jean Antoine Nicolas Caritat, the Marquis de Condorcet (1743-1794) ‘Gold standard for voting’
In a pure Condorcet method the choices of each voter are compared against everyone else in a series of tournaments.

If one candidate wins all of the tournaments then they win overall. They become the Condorcet Winner.

Condorcet winners don’t always exist, but if they do we would want our voting method to select them.
Example 1: A clear winner

<table>
<thead>
<tr>
<th>Number of voters</th>
<th>Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>A &gt; B &gt; C</td>
</tr>
<tr>
<td>1</td>
<td>A &gt; C &gt; B</td>
</tr>
<tr>
<td>5</td>
<td>C &gt; A &gt; B</td>
</tr>
<tr>
<td>0</td>
<td>C &gt; B &gt; A</td>
</tr>
<tr>
<td>9</td>
<td>B &gt; C &gt; A</td>
</tr>
<tr>
<td>5</td>
<td>B &gt; A &gt; C</td>
</tr>
</tbody>
</table>

A is the Condorcet winner  (B is the Borda winner)
Example 2: A clear tie
Example 3: Not quite a winner

- No Condorcet winner
- FPTP: A
- Borda: B

Who has won?
Copeland’s method

Elect the candidate who wins the most head to head contests

In the example, A and B tie for first place in Copeland’s method

This is not unusual and means that the method is not widely used in practice

A version is used in Premier League football
Shulze’s method

A complicated but very fair method. Now used for internet voting eg. Debian and Wikimedia

Draw the same graph as before but show how much each candidate beats the other by
Construct paths from candidate X to candidate Y

Strength $P(X,Y)$ of path is value of the ‘weakest link’

<table>
<thead>
<tr>
<th></th>
<th>Path to A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Path from A</td>
<td>x</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>x</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>4</td>
<td>x</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>x</td>
</tr>
</tbody>
</table>

The winner of a Schulze election is the candidate X so that $P(X,Y) > P(Y,X)$ for all possible Y. In this case it is B i.e. it takes more voters to fancy X over Y than Y over X

Fair and fast. Widely used on the internet
Dodgson Method
Dodgson winner:

The candidate for whom the smallest number of (adjacent) changes are needed for them to be the Condorcet winner.
<table>
<thead>
<tr>
<th>Number of votes</th>
<th>Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>A &gt; B &gt; C &gt; D</td>
</tr>
<tr>
<td>1</td>
<td>D &gt; B &gt; A &gt; C</td>
</tr>
<tr>
<td>1</td>
<td>D &gt; C &gt; A &gt; B</td>
</tr>
<tr>
<td>1</td>
<td>C &gt; D &gt; B &gt; A</td>
</tr>
<tr>
<td>1</td>
<td>B &gt; D &gt; C &gt; A</td>
</tr>
</tbody>
</table>

Only one of the first three voters has to change their mind for B to be the Condorcet winner

**B is the Dodgson winner**

Fair method, but **VERY** hard to use in practice (NP Hard)
Reasonable practical compromises

IRV/AV, STV Voting Methods

Instant Run-Off Voting

Used in Australia to elect a single candidate

Voters express their preferences for each candidate
The ballots are initially counted for each voter's top choice. If a candidate has more than half of the vote based on first-choices, that candidate wins.

If not, the candidate with the fewest votes is eliminated.

The voters who selected the defeated candidate as a first choice have their votes for that candidate added to the totals of their next choice.

Process continues until a candidate has more than half of the votes.
Candidates A, B, C  
Voters a, b, c, d, e

Round 1

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>b</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>c</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>e</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

B > A  
A > C  
C > B  
No Condorcet winner

C is eliminated on the first round
**Round 2**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>b</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>e</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

B is the winner of the IRV vote

Note: A is the Borda and the Shulze winner
Single Transferable Vote (STV)

Invented in 1819 by Thomas Hill

Similar to the IRV method, used if there are multiple candidates elected to N posts

Used, for elections to the London Mathematical Society and many different countries

In each round the last candidate is removed and their votes reallocated. Process continues until only N candidates remain

STV is easy to use and approximately proportional, but does not necessarily deliver the Condorcet winner
<table>
<thead>
<tr>
<th>Rank</th>
<th>Country</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Belarus</td>
<td>48</td>
</tr>
<tr>
<td>2</td>
<td>Denmark</td>
<td>281</td>
</tr>
<tr>
<td>3</td>
<td>Estonia</td>
<td>19</td>
</tr>
<tr>
<td>4</td>
<td>The Netherlands</td>
<td>114</td>
</tr>
<tr>
<td>5</td>
<td>Hungary</td>
<td>84</td>
</tr>
<tr>
<td>6</td>
<td>Norway</td>
<td>214</td>
</tr>
<tr>
<td>7</td>
<td>Russia</td>
<td>174</td>
</tr>
<tr>
<td>8</td>
<td>Georgia</td>
<td>50</td>
</tr>
<tr>
<td>9</td>
<td>United Kingdom</td>
<td>23</td>
</tr>
<tr>
<td>10</td>
<td>Ukraine</td>
<td>214</td>
</tr>
<tr>
<td>11</td>
<td>Greece</td>
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<tr>
<td>12</td>
<td>Azerbaijan</td>
<td>234</td>
</tr>
<tr>
<td>13</td>
<td>Italy</td>
<td>126</td>
</tr>
<tr>
<td>14</td>
<td>Germany</td>
<td>18</td>
</tr>
<tr>
<td>15</td>
<td>Malta</td>
<td>120</td>
</tr>
<tr>
<td>16</td>
<td>France</td>
<td>14</td>
</tr>
<tr>
<td>17</td>
<td>Lithuania</td>
<td>17</td>
</tr>
<tr>
<td>18</td>
<td>Moldova</td>
<td>71</td>
</tr>
<tr>
<td>19</td>
<td>Finland</td>
<td>13</td>
</tr>
<tr>
<td>20</td>
<td>Belgium</td>
<td>71</td>
</tr>
<tr>
<td>21</td>
<td>Spain</td>
<td>8</td>
</tr>
<tr>
<td>22</td>
<td>Romania</td>
<td>65</td>
</tr>
<tr>
<td>23</td>
<td>Ireland</td>
<td>5</td>
</tr>
</tbody>
</table>

39 of 39 countries voting
Eurovision Facts

Run every year since the 1950s

26 entrants in the final

Ridiculous staging, hilarious costumes, cringe worthy announcers, sarcastic commentators

*Waterloo* from Abba in 1974.

1994 winner was the interval music

1978 Norway Null Points
1968

Cliff Richard Congratulations

beaten by La La La La

by one point!!

Widespread accusations of vote rigging
The voting at the end is by far the best part!

A true conflict between fair assessment of each song, and outrageous tactical and political voting

**Double Borda Method is used for the voting**

**Juries rank songs:** Give 12,10,8,7,6,5,4,3,2,1,0 points

**Countries rank songs by tele-voting:** Give points as above

Jury votes accumulate. Tele-votes added at the end
<table>
<thead>
<tr>
<th>Country</th>
<th>Jury score</th>
<th>Jury Rank</th>
<th>Tele-vote score</th>
<th>Tele-vote rank</th>
<th>Total score</th>
<th>Total rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sweden</td>
<td>365</td>
<td>1</td>
<td>272</td>
<td>3</td>
<td>627</td>
<td>1</td>
</tr>
<tr>
<td>Italy</td>
<td>171</td>
<td>6</td>
<td>356</td>
<td>1</td>
<td>527</td>
<td>2</td>
</tr>
<tr>
<td>Russia</td>
<td>234</td>
<td>3</td>
<td>286</td>
<td>2</td>
<td>520</td>
<td>3</td>
</tr>
<tr>
<td>Belgium</td>
<td>186</td>
<td>5</td>
<td>190</td>
<td>4</td>
<td>376</td>
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</tr>
<tr>
<td>Australia</td>
<td>224</td>
<td>4</td>
<td>124</td>
<td>6</td>
<td>348</td>
<td>5</td>
</tr>
<tr>
<td>Latvia</td>
<td>249</td>
<td>2</td>
<td>88</td>
<td>8</td>
<td>337</td>
<td>6</td>
</tr>
<tr>
<td>Norway</td>
<td>163</td>
<td>7</td>
<td>37</td>
<td>10</td>
<td>200</td>
<td>7</td>
</tr>
<tr>
<td>Estonia</td>
<td>53</td>
<td>11</td>
<td>144</td>
<td>5</td>
<td>197</td>
<td>8</td>
</tr>
<tr>
<td>Israel</td>
<td>77</td>
<td>8</td>
<td>102</td>
<td>7</td>
<td>179</td>
<td>9</td>
</tr>
<tr>
<td>Georgia</td>
<td>62</td>
<td>10</td>
<td>51</td>
<td>9</td>
<td>113</td>
<td>10</td>
</tr>
</tbody>
</table>
Advantages of the method:

Real time scoring

Build up of tension as the results are announced

Disadvantages:

Not a Condorcet method,

Doesn’t necessarily elect the song favoured by the majority

Extremely vulnerable to tactical voting!!!!!!
The Illustrated Beginner's Guide to Eurovision Voting

Nordic Clique
- Oh Norway! I can't believe we're alone!
- Take your 8 points and get yourself embarrassing again!
- Oh Sweden!
- That was your friend?

Baltic Circle Jerk

Soviet Block
- Oh, another Russia? We have traveled from here and back to here and back.
- Oh, can I still own a crowd?

Ankgo-Irish Charity

Yugoslavian Back-Pattery
- Hey guys, I have an idea. Why don't we get together and form our own independent voting bloc. Who knows what we'll come up with?

German Mini-Block
- Oh man, I'm so high.

Iberian Mini-Block

LyricsTranslations.Com
Have fun on December 12th