

LIE SYMMETRIES, OBSERVABILITY AND MODEL TRANSFORMATION OF NONLINEAR SYSTEMS WITH UNKNOWN INPUTS

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Abstract. *This work investigates the use of Lie symmetries for assessing and improving the observability of dynamical systems under examined sensor setups. The framework of Lie symmetry is extended to account for unmeasured and hence unknown inputs. An efficient algorithm is developed to calculate the translation and scaling symmetries of nonlinear systems with unknown inputs. The use of the algorithm to assess the observability of a given nonlinear system is demonstrated, i.e. in theory whether it would be successful to identify the dynamic states, the parameters and the unknown inputs of the system from given measurements. The work further shows the potential application of the calculated symmetries for transforming the model of an unobservable system to an equivalent model with a minimum number of unobservable states and unknown inputs. The proposed method and algorithm are illustrated on the observability properties of a dynamical system with a Bouc-Wen nonlinearity.*

1 Introduction

During a system identification campaign, a priori observability analysis of a dynamical system [1, 2] is used to predict whether it is feasible to infer the state variables and parameters of the system from a given set of input-output measurements [3, 4, 5]. Recent developments of novel identification methods such as [6, 7, 8, 9] allow for estimating the unmeasured and hence unknown inputs of a system reliably as well as its state variables and parameters. To meet the demand of observability analysis when invoking such identification methods, the Observability Rank Condition (ORC) [3] was extended to alleviate the constraint that all inputs need to be measured [10, 11]. Recently the authors achieved efficient automated observability testing of large linear systems with unknown parameters [12], relaxing the significant computational constraints of the default implementations of the ORC.

It has been found in [4, 5, 13, 14] that the observability properties of a system are closely associated with the existence of Lie symmetries in the system. Symmetry was first introduced by Lie [15, 16] to define a way of variable transformations that leaves differential equations invariant. Sedoglavic [4, 5] used the concept of Lie symmetry for dynamical systems with fully known inputs and output measurements, and demonstrated the connections between symmetry and observability. The computational framework of Lie symmetry was explored in [4, 17, 18, 14], resulting in methods for the efficient calculations of certain types of Lie symmetries for a given system.

In this work, the concept of Lie symmetry and the corresponding computational framework are further used and extended for nonlinear systems with partially measured inputs, i.e. where some or potentially all of the inputs are unmeasured. The occurring method, and consequently algorithm, can be used as an alternative tool to assess the observability of such systems. Furthermore, the work illustrates that, relying on the results of symmetries, an unobservable dynamical model can be transformed to an equivalent model with a minimum number of unobservable states and unknown inputs. Such a model, either through the use of a minimal set of assumptions or through re-defining a new state vector of reduced length, can lead to a fully observable model. The proposed method of model transformation will potentially find useful applications in improving the observability and identifiability of dynamical systems under examined sensor setups. In the end of the paper, the proposed algorithm and the idea of model transformation are illustrated through a carefully chosen example.

2 Lie symmetries of nonlinear systems with unknown inputs

This work focuses on nonlinear systems with unknown inputs that can be generally written in the following state space representation:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})\end{aligned}\tag{1}$$

where $\mathbf{x} = [x_1, x_2, \dots, x_n]$ denotes the augmented state vector [1] containing both the dynamic states and the time-invariant parameters of the underlying system. $\mathbf{w} = [w_1, w_2, \dots, w_m]$ denotes the unmeasured and hence unknown inputs, $\mathbf{u} \in \mathbb{R}^l$ the vector of measured inputs and $\mathbf{y} \in \mathbb{R}^p$ the vector of output measurements. Assume system (1) contains r groups of Lie symmetries. The i^{th} ($1 \leq i \leq r$) group of Lie symmetries of system (1) is a one-parameter, $\epsilon_i \in \mathbb{R}$, group of transformations:

$$\begin{aligned}{}^i\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i) &= [{}^i\phi_{x,1}, {}^i\phi_{x,2}, \dots, {}^i\phi_{x,n}] \\ {}^i\phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, \epsilon_i) &= [{}^i\phi_{w,1}, {}^i\phi_{w,2}, \dots, {}^i\phi_{w,m}]\end{aligned}\tag{2}$$

where the j^{th} ($1 \leq j \leq n$) component of ${}^i\phi_{\mathbf{x}}$, i.e. ${}^i\phi_{x,j}$, is a Lie symmetry of the j^{th} component of $\mathbf{x}$, i.e. x_j , and it is an analytic function of $\mathbf{x}$, $\mathbf{w}$ and the real constant parameter ϵ_i . Similarly, ${}^i\phi_{w,j}$ ($1 \leq j \leq m$) is a Lie symmetry of w_j and it is also an analytic function of $\mathbf{x}$, $\mathbf{w}$ and ϵ_i . A fundamental property of Lie symmetries is that such group of transformations of $\mathbf{x}$ and $\mathbf{w}$ fulfills the equations of system (1) leaving the measured inputs and the output measurements unchanged, i.e.:

$$\begin{aligned} {}^i\dot{\phi}_{\mathbf{x}} &= \mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}}) \\ \mathbf{y} &= \mathbf{h}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}}) \end{aligned} \quad (3)$$

where it should be noted that the functions $\mathbf{f}$ and $\mathbf{h}$ between equations (1) and (3) remain the same. Further based on Lie's First Fundamental Theorem [15, 16], the following differential equations hold with known initial conditions of ${}^i\phi_{\mathbf{x}}$ and ${}^i\phi_{\mathbf{w}}$:

$$\begin{aligned} \frac{\partial {}^i\phi_{\mathbf{x}}}{\partial \epsilon_i} &= {}^i\xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}), \quad {}^i\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, 0) = \mathbf{x} \\ \frac{\partial {}^i\phi_{\mathbf{w}}}{\partial \epsilon_i} &= {}^i\xi_{\mathbf{w}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}), \quad {}^i\phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, 0) = \mathbf{w} \end{aligned} \quad (4)$$

where ${}^i\xi_{\mathbf{x}}$ and ${}^i\xi_{\mathbf{w}}$ are analytic functions. $\frac{\partial {}^i\phi_{\mathbf{x}}}{\partial \epsilon_i} \big|_{\epsilon_i=0} = {}^i\xi_{\mathbf{x}}(\mathbf{x}, \mathbf{w})$ and $\frac{\partial {}^i\phi_{\mathbf{w}}}{\partial \epsilon_i} \big|_{\epsilon_i=0} = {}^i\xi_{\mathbf{w}}(\mathbf{x}, \mathbf{w})$ are called the infinitesimals of the Lie group of transformations.

3 Computations of translation and scaling Lie symmetries

In this section, the computational framework of Lie symmetries of system (1) is derived by extending the works in [18, 13] to account for the existence of the unmeasured inputs. The symmetry computation relies on setting up a system of differential equations whose solution provides the information of ${}^i\xi_{\mathbf{x}}$ and ${}^i\xi_{\mathbf{w}}$ appearing in equations (4). The first differential equation of the system is derived starting from the equation:

$$\frac{\partial \left(\frac{d {}^i\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i)}{dt} \right)}{\partial \epsilon_i} = \frac{d \left(\frac{\partial {}^i\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i)}{\partial \epsilon_i} \right)}{dt} \quad (5)$$

Equation (5) is proved to hold for all realizations of $\mathbf{x}$, $\mathbf{w}$ and ϵ_i in Appendix A. The second differential equation is derived based on the property that the measurements remain unchanged with respect to any realization of the value of ϵ_i , and therefore:

$$\frac{d\mathbf{y}}{d\epsilon_i} = \mathbf{0} \quad (6)$$

Applying the chain rule to equations (5) and (6) yields:

$$\begin{cases} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} {}^i\xi_{\mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{w}} {}^i\xi_{\mathbf{w}} = \frac{\partial {}^i\xi_{\mathbf{x}}}{\partial \mathbf{x}} \mathbf{f} + \frac{\partial {}^i\xi_{\mathbf{x}}}{\partial \mathbf{w}} \frac{d\mathbf{w}}{dt} \\ \frac{\partial \mathbf{h}}{\partial \mathbf{x}} {}^i\xi_{\mathbf{x}} + \frac{\partial \mathbf{h}}{\partial \mathbf{w}} {}^i\xi_{\mathbf{w}} = \mathbf{0} \end{cases} \quad (7)$$

where $\mathbf{f}$ and $\mathbf{h}$ correspond to the functions in (3). It is noted that equations (7) hold for any realization of ϵ_i including $\epsilon_i = 0$. Evaluating equations (7) at $\epsilon_i = 0$ leads to:

$$\begin{cases} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} {}^i\xi_{\mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{w}} {}^i\xi_{\mathbf{w}} = \frac{\partial {}^i\xi_{\mathbf{x}}}{\partial \mathbf{x}} \mathbf{f} + \frac{\partial {}^i\xi_{\mathbf{x}}}{\partial \mathbf{w}} \frac{d\mathbf{w}}{dt} \\ \frac{\partial \mathbf{h}}{\partial \mathbf{x}} {}^i\xi_{\mathbf{x}} + \frac{\partial \mathbf{h}}{\partial \mathbf{w}} {}^i\xi_{\mathbf{w}} = \mathbf{0} \end{cases} \quad (8)$$

where $\mathbf{f}$ and $\mathbf{h}$ correspond to the functions in (1). Analytically solving the system of equations (8) yields ${}^i\xi_{\mathbf{x}}(\mathbf{x}, \mathbf{w})$ and ${}^i\xi_{\mathbf{w}}(\mathbf{x}, \mathbf{w})$, and the Lie's First Fundamental Theorem ensures ${}^i\xi_{\mathbf{x}}(\mathbf{x}, \mathbf{w})$ and ${}^i\xi_{\mathbf{w}}(\mathbf{x}, \mathbf{w})$ contain the essential information for characterizing ${}^i\xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}})$ and ${}^i\xi_{\mathbf{w}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}})$. Subsequently solving equations (4) allow for obtaining the group of Lie symmetries ${}^i\phi_{\mathbf{x}}$ and ${}^i\phi_{\mathbf{w}}$.

In general, obtaining the analytical solution of the above system of differential equations is challenging if no assumptions are made for the symmetries [18]. However, if ${}^i\phi_{\mathbf{x}}$ and ${}^i\phi_{\mathbf{w}}$ are assumed to be certain types of symmetries, such as translations and scalings, the system of differential equations can be solved automatically and efficiently. Section 3.1 and 3.2 discuss in detail the efficient computations of one-parameter translation and scaling types of symmetries of system (1). It is further assumed in the following sections that all the symmetries occurring in the system are related to translation and scaling symmetries and their combinations. This simplifying assumption is often satisfied for a wide range of real world engineering systems. Other types of Lie transformations not studied in this work include affine, quadratic, Mobius and some more general higher-order polynomial symmetries investigated in [18, 14] for systems with fully measured inputs. Efficient computations of those types of symmetries are potentially also available for the system described in (1) and will be the focus of future extensions of this work.

3.1 Translation symmetries

If the i^{th} group of Lie symmetries are translation symmetries then:

$$\begin{aligned} {}^i\phi_{x,1} &= x_1 + \alpha_{i,1}\epsilon_i \\ &\vdots \\ {}^i\phi_{x,n} &= x_n + \alpha_{i,n}\epsilon_i \\ {}^i\phi_{w,1} &= w_1 + \alpha_{i,n+1}\epsilon_i \\ &\vdots \\ {}^i\phi_{w,m} &= w_m + \alpha_{i,n+m}\epsilon_i \end{aligned} \quad (9)$$

where $\alpha_{i,1}, \dots, \alpha_{i,n+m}$ are constant coefficients to be determined. If the expressions of ${}^i\phi_{\mathbf{x}}$ and ${}^i\phi_{\mathbf{w}}$ from equations (9) are used in equations (7), consequently equations (8) are simplified to:

$$\begin{cases} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1} \\ \vdots \\ \alpha_{i,n} \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \dots & \frac{\partial f_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial w_1} & \dots & \frac{\partial f_n}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1} \\ \vdots \\ \alpha_{i,n+m} \end{bmatrix} = \mathbf{0} \\ \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \dots & \frac{\partial h_p}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1} \\ \vdots \\ \alpha_{i,n} \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial w_1} & \dots & \frac{\partial h_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial w_1} & \dots & \frac{\partial h_p}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1} \\ \vdots \\ \alpha_{i,n+m} \end{bmatrix} = \mathbf{0} \end{cases} \quad (10)$$

3.2 Scaling symmetries

If the i^{th} group of Lie symmetries are scaling symmetries then:

$$\begin{aligned} {}^i\phi_{x,1} &= e^{\alpha_{i,1}\epsilon_i} x_1 \\ &\vdots \\ {}^i\phi_{x,n} &= e^{\alpha_{i,n}\epsilon_i} x_n \\ {}^i\phi_{w,1} &= e^{\alpha_{i,n+1}\epsilon_i} w_1 \\ &\vdots \\ {}^i\phi_{w,m} &= e^{\alpha_{i,n+m}\epsilon_i} w_m \end{aligned} \quad (11)$$

where $\alpha_{i,1}, \dots, \alpha_{i,n+m}$ are constant coefficients to be determined. If the expressions of ${}^i\phi_x$ and ${}^i\phi_w$ from equations (11) are used in equations (7), consequently equations (8) are simplified to:

$$\begin{aligned} &\left\{ \begin{aligned} &\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1}x_1 \\ \vdots \\ \alpha_{i,n}x_n \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \dots & \frac{\partial f_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial w_1} & \dots & \frac{\partial f_n}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1}w_1 \\ \vdots \\ \alpha_{i,n+m}w_m \end{bmatrix} \\ &- \begin{bmatrix} \alpha_{i,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_{i,n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \end{aligned} \right\} = \mathbf{0} \\ &\left\{ \begin{aligned} &\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \dots & \frac{\partial h_p}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1}x_1 \\ \vdots \\ \alpha_{i,n}x_n \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial w_1} & \dots & \frac{\partial h_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial w_1} & \dots & \frac{\partial h_p}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1}w_1 \\ \vdots \\ \alpha_{i,n+m}w_m \end{bmatrix} \end{aligned} \right\} = 0 \end{aligned} \quad (12)$$

Both systems of equations (10) and (12) can be converted to a linear in the coefficients system:

$$\mathbf{M}\alpha_i = \mathbf{0} \quad (13)$$

where $\alpha_i = [\alpha_{i,1}, \dots, \alpha_{i,n+m}]$ and $\mathbf{M}$ is a matrix of functions of $\mathbf{x}$, $\mathbf{w}$ and $\mathbf{u}$. $\mathbf{M}$ can be calculated symbolically by:

$$\mathbf{M} = \frac{\partial \mathbf{P}}{\partial \alpha_i} \quad (14)$$

where $\mathbf{P}$ is a vector of the left hand side of equations (10) and (12) that can be commonly expressed in terms of equations (8):

$$\mathbf{P} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} {}^i\boldsymbol{\xi}_{\mathbf{x}} + \frac{\partial \mathbf{f}}{\partial \mathbf{w}} {}^i\boldsymbol{\xi}_{\mathbf{w}} - \frac{\partial {}^i\boldsymbol{\xi}_{\mathbf{x}}}{\partial \mathbf{x}} \mathbf{f} \\ \frac{\partial \mathbf{h}}{\partial \mathbf{x}} {}^i\boldsymbol{\xi}_{\mathbf{x}} + \frac{\partial \mathbf{h}}{\partial \mathbf{w}} {}^i\boldsymbol{\xi}_{\mathbf{w}} \end{bmatrix} \quad (15)$$

The vector of coefficients α_i can then be determined by calculating the kernel of $\mathbf{M}$ symbolically, i.e. $\alpha_i = \ker(\mathbf{M})$ giving rise to different bases as groups of realizations of α_i . Each basis corresponds to a different parameter ϵ_i . An algorithm was presented in [18] to calculate α_i efficiently for rational nonlinear systems by specializing symbolic variables in $\mathbf{M}$ to random values and performing computations over a finite field.

4 r -parameter group of Lie symmetries, observability and model transformation

4.1 Combination of translation and scaling symmetries

It is often the case that a dynamical system contains multiple one-parameter groups of Lie symmetries. Multiple translation and scaling symmetries of system (1) can be obtained by calculating the kernel of M for multiple solutions of bases with each basis corresponding to the coefficients of an independent group of translation or scaling symmetries. These obtained groups of translations and scalings can be treated separately, or alternatively they can be combined into a single multi-parameter group of Lie symmetries. The principle behind the way of combination is based on the property that Lie transformations of ${}^i\phi_x$ and ${}^i\phi_w$ are themselves Lie transformations of x and w that satisfy the equations of the original system [16]. Assume there exists r one-parameter groups of symmetries including r_t groups of translations and r_s groups of scalings. If one successively applies the translations and scalings in spite of their sequence to transform x and w iteratively, the result gives an r -parameter group of Lie symmetries of system (1).

Without loss of generality, it is assumed that all the translation symmetries are combined first. Let $\alpha_1, \dots, \alpha_{r_t}$ be the coefficients of the translations, and the j^{th} component of the combination is given by:

$$x_j + \alpha_{1,j}\epsilon_1 + \dots + \alpha_{r_t,j}\epsilon_{r_t} \quad (16)$$

Next all the scaling symmetries are combined. Let $\alpha_{r_t+1}, \dots, \alpha_r$ be the coefficients of the scalings, and the j^{th} component of the combination is given by:

$$e^{\alpha_{r_t+1,j}\epsilon_{r_t+1} + \dots + \alpha_{r,j}\epsilon_r} x_j \quad (17)$$

Combining all the one-parameter groups of symmetries using equations (16) and (17) gives:

$$\phi_{x,j} = e^{\alpha_{r_t+1,j}\epsilon_{r_t+1} + \dots + \alpha_{r,j}\epsilon_r} (x_j + \alpha_{1,j}\epsilon_1 + \dots + \alpha_{r_t,j}\epsilon_{r_t}) \quad (18)$$

where $\phi_{x,j}$ is the j^{th} component of ϕ_x . $\phi_x(x, w, \epsilon)$ is used to denote the r -parameter group of Lie symmetries with $\epsilon = [\epsilon_1, \dots, \epsilon_r]$. $\phi_w(x, w, \epsilon)$ can be obtained in a similar fashion.

4.2 From Lie symmetries to observability

Same as their one-parameter sub-groups of Lie symmetries, ϕ_x and ϕ_w as the transformations of x and w satisfy the equations of system (1) leaving u and y unchanged, i.e.:

$$\begin{aligned} \dot{\phi}_x &= f(\phi_x, u, \phi_w) \\ y &= h(\phi_x, u, \phi_w) \end{aligned} \quad (19)$$

This fundamental property builds the relationship between the r -parameter group of Lie symmetries and the observability of system (1). If $\phi_{x,j} \neq x_j$, then x_j is unobservable given the measurements of u and y . Similarly, if $\phi_{w,j} \neq w_j$, then w_j is unobservable given the measurements of u and y . On the contrary, if $[\phi_x, \phi_w] \equiv [x, w]$, then all the states and unmeasured inputs are observable resulting in a fully observable underlying system.

4.3 Model transformation

The model of an unobservable system can be transformed to an equivalent model with a minimum number of unobservable states and unmeasured inputs utilizing the results of Lie

symmetries. If system (1) was found to contain an r -parameter group of Lie symmetries, then at least a total of r states and unmeasured inputs are unobservable whose symmetries are functions of ϵ . There exist however a set of transformations of the states and unmeasured inputs, $\mathbf{x}_T$, with respect to which the system model can be re-written or transformed such that the transformed model contains up to $n+m-r$ observable states and unmeasured inputs and a minimum number of unobservable variables. In the case where all the unobservable variables could vanish, the model transformation would lead to a fully observable model.

To obtain $\mathbf{x}_T$ in the case of transforming an unobservable model to be observable, if possible, a transformation $\mathbf{S}$ is first sought by combining the components of ϕ_x and ϕ_w , i.e. $\mathbf{S}(\phi_x, \phi_w)$, where the goal is to eliminate all the parameters ϵ . $\mathbf{x}_T$ is then introduced to be identically equal to $\mathbf{S}(\phi_x, \phi_w)$. This process will be demonstrated through the example where some added properties of this transformation allow for obtaining an observable reduced model.

5 Algorithm

An algorithm is presented as follows to summarize the procedure of computing translation and scaling symmetries of system (1). All the computations involved are symbolic, and the resulting multi-parameter group of Lie symmetries from the algorithm is a combination of all the translations and scalings computed using the methods described in Section 3.1, 3.2 and 4.1.

Algorithm I

Input: state-space and measurement equations of system (1)

Output: an r -parameter group of translation and scaling Lie symmetries

1. Compute $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$, $\frac{\partial \mathbf{f}}{\partial \mathbf{w}}$, $\frac{\partial \mathbf{h}}{\partial \mathbf{x}}$ and $\frac{\partial \mathbf{h}}{\partial \mathbf{w}}$
2. First compute the translation symmetries. Let ${}^1\alpha = [\alpha_1, \dots, \alpha_n]$ and ${}^2\alpha = [\alpha_{n+1}, \dots, \alpha_{n+m}]$. Compute $\mathbf{P}_1 = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}({}^1\alpha)^T + \frac{\partial \mathbf{f}}{\partial \mathbf{w}}({}^2\alpha)^T$ and $\mathbf{P}_2 = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}({}^1\alpha)^T + \frac{\partial \mathbf{h}}{\partial \mathbf{w}}({}^2\alpha)^T$
3. Let $\mathbf{P} = [\mathbf{P}_1^T, \mathbf{P}_2^T]^T$, and compute $\mathbf{M} = \frac{\partial \mathbf{P}}{\partial \alpha}$
4. Compute $\alpha = \ker(\mathbf{M})$ and obtain r_t bases of α
5. Substitute the coefficients to equations (9) to obtain the translations $[{}^1\phi_x, {}^1\phi_w], \dots, [{}^{r_t}\phi_x, {}^{r_t}\phi_w]$
6. Now compute the scaling symmetries. Compute $\mathbf{P}_1 = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \text{diag}(\mathbf{x})({}^1\alpha)^T + \frac{\partial \mathbf{f}}{\partial \mathbf{w}} \text{diag}(\mathbf{w})({}^2\alpha)^T - \text{diag}({}^1\alpha)\mathbf{f}$ and $\mathbf{P}_2 = \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \text{diag}(\mathbf{x})({}^1\alpha)^T + \frac{\partial \mathbf{h}}{\partial \mathbf{w}} \text{diag}(\mathbf{w})({}^2\alpha)^T - \text{diag}({}^1\alpha)\mathbf{h}$
7. Let $\mathbf{P} = [\mathbf{P}_1^T, \mathbf{P}_2^T]^T$, and compute $\mathbf{M} = \frac{\partial \mathbf{P}}{\partial \alpha}$
8. Compute $\alpha = \ker(\mathbf{M})$ and obtain r_s bases of α
9. Substitute the coefficients to equations (11) to obtain the scalings $[{}^{r_t+1}\phi_x, {}^{r_t+1}\phi_w], \dots, [{}^r\phi_x, {}^r\phi_w]$
10. Combine $[{}^1\phi_x, {}^1\phi_w], \dots, [{}^r\phi_x, {}^r\phi_w]$ to obtain ϕ_x and ϕ_w following equations (16), (17) and (18)

6 Example: a 2 degrees of freedom (DOFs) mass-spring system with a Bouc-Wen element

Consider a 2 DOFs mass-spring system as shown in Figure 1. The displacements of the two masses m are denoted as x_1 and x_2 respectively, and the corresponding velocities are denoted as v_1 and v_2 . k_1 and k_2 are the effective stiffness of the springs. The first spring is assumed to be a Bouc-Wen element with the elastic displacement r . The 2 DOFs system is driven by an unmeasured force $F(t)$ applied at the second mass. The state-space equations of the underlying

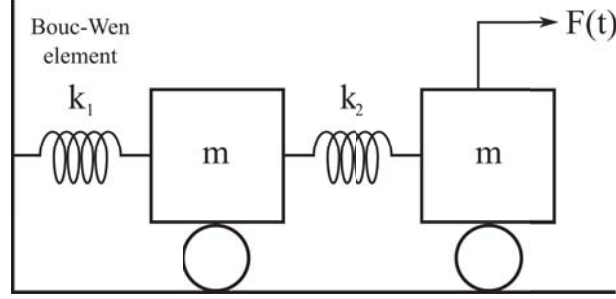


Figure 1: a 2 DOFs mass-spring system with a Bouc-Wen element

system are given by:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ r \\ k_1 \\ k_2 \\ m \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ (-k_1 r + k_2(x_2 - x_1))/m \\ (k_2(x_1 - x_2) + F)/m \\ v_1 - \beta|v_1||r|r - \gamma v_1|r|^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (20)$$

where β and γ are the Bouc-Wen hysteretic parameters. The parameters k_1 , k_2 , m , β and γ are unknown and thus are to be identified given measurements. It should be noted that the exponent of the Bouc-Wen model is assumed to be known and equal to 2. This is not due to the limitations of the method suggested in this work which would allow for studying non-rational systems, but for presenting the results within a more concise way. The displacements x_1 and x_2 are measured, and therefore the measurement equations of the system are given by:

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (21)$$

The proposed method in this work cannot be applied directly to the underlying system because the system is not smooth due to the existence of the absolute value operators. However, as discussed in [1], it can be divided into several smooth branches under different conditions of the states, and these smooth branches are then allowed to be examined separately. The function of $\dot{r}$ for each branch is formulated in the following while the equations of the other states remain the same.

$$\begin{aligned} A : \dot{r} &= v_1 - \beta v_1 r^2 - \gamma v_1 r^2, & \text{when } v_1 > 0 & \text{ and } r > 0 \\ B : \dot{r} &= v_1 + \beta v_1 r^2 - \gamma v_1 r^2, & \text{when } v_1 < 0 & \text{ and } r > 0 \\ C : \dot{r} &= v_1 + \beta v_1 r^2 - \gamma v_1 r^2, & \text{when } v_1 > 0 & \text{ and } r < 0 \\ D : \dot{r} &= v_1 - \beta v_1 r^2 - \gamma v_1 r^2, & \text{when } v_1 < 0 & \text{ and } r < 0 \end{aligned} \quad (22)$$

For the sake of brevity, only Lie symmetries of branch A are studied, while the detailed discussions on the observability properties of the complete non-smooth system can be referred

in the works [1, 11]. For branch A , Algorithm I gives a 2-parameter group of Lie symmetries that is a combination of a group of translations and a group of scalings:

$$[\phi_x, \phi_w] = [x_1, x_2, v_1, v_2, r, e^{\epsilon_1} k_1, e^{\epsilon_1} k_2, e^{\epsilon_1} m, \beta + \epsilon_2, \gamma - \epsilon_2, e^{\epsilon_1} F] \quad (23)$$

As can be seen from the results of symmetries, the symmetries of x_1 , x_2 , v_1 , v_2 and r are identically equal to themselves, which indicates that those states are observable within branch A . On the contrary, the parameters k_1 , k_2 , m , β and γ are unidentifiable and the unmeasured excitation F is unobservable. The observability results suggested by the symmetries are in agreement with the results output from the observability algorithm ORC-DF [11].

In the following, a transformation S is determined for combining the resulting Lie symmetries so as to eliminate all ϵ_1 and ϵ_2 :

$$S(\phi_x, \phi_w) = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ r \\ \frac{e^{\epsilon_1} k_1}{e^{\epsilon_1} m} \\ \frac{e^{\epsilon_1} k_2}{e^{\epsilon_1} m} \\ \beta + \epsilon_2 + \gamma - \epsilon_2 \\ \frac{e^{\epsilon_1} F}{e^{\epsilon_1} m} \end{bmatrix}^T = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ r \\ \frac{k_1}{m} \\ \frac{k_2}{m} \\ \beta + \gamma \\ \frac{F}{m} \end{bmatrix}^T \quad (24)$$

A set of new variables, f_1 , f_2 , δ and F_m , are then introduced such that:

$$\mathbf{x}_T = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ r \\ f_1 \\ f_2 \\ \delta \\ F_m \end{bmatrix}^T \equiv \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ r \\ \frac{k_1}{m} \\ \frac{k_2}{m} \\ \beta + \gamma \\ \frac{F}{m} \end{bmatrix}^T \quad (25)$$

and the model of branch A can be re-written with respect to $\mathbf{x}_T$ as follows:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ r \\ f_1 \\ f_2 \\ \delta \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ -f_1 r + f_2 (x_2 - x_1) \\ f_2 (x_1 - x_2) + F_m \\ v_1 - \delta v_1 r^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (26)$$

Note in the transformed model (26), $f_1 = \frac{k_1}{m}$ and $f_2 = \frac{k_2}{m}$ are related to the natural frequencies of the system, δ is the new Bouc-Wen parameter and $F_m = \frac{F}{m}$ is a new unmeasured input to the system. Given the same measurements model (26) is observable with all its state variables and parameters as well as the unmeasured input observable, which can be verified by the observability algorithm ORC-DF.

7 Conclusions

This work derives the computational framework of Lie symmetries of nonlinear systems with unmeasured inputs, and proposes an efficient algorithm used to calculate the translation and scaling symmetries of such systems. The computed Lie symmetries find applications in predicting and improving the observability of dynamical systems under examined sensor setups. Future works will be devoted to extending the algorithm to compute more general forms of Lie symmetries as well as to introducing a systematic methodology of model transformation. Other applications of Lie symmetries in improving models and identification results of engineering systems will be explored.

8 Appendix A: a proof of equation (5)

Applying the chain rule to the left hand side of equation (5), one obtains:

$$\frac{\partial(\frac{d^i \phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i)}{dt})}{\partial \epsilon_i} = \frac{\partial(\frac{\partial^i \phi_{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} + \frac{\partial^i \phi_{\mathbf{x}}}{\partial \mathbf{w}} \frac{d\mathbf{w}}{dt})}{\partial \epsilon_i} \quad (27)$$

Using the product rule equation (27) is equivalent to:

$$\frac{\partial(\frac{d^i \phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i)}{dt})}{\partial \epsilon_i} = \frac{\partial^2 i \phi_{\mathbf{x}}}{\partial \mathbf{x} \partial \epsilon_i} \frac{d\mathbf{x}}{dt} + \frac{\partial^i \phi_{\mathbf{x}}}{\partial \mathbf{x}} \frac{\partial(\frac{d\mathbf{x}}{dt})}{\partial \epsilon_i} + \frac{\partial^2 i \phi_{\mathbf{x}}}{\partial \mathbf{w} \partial \epsilon_i} \frac{d\mathbf{w}}{dt} + \frac{\partial^i \phi_{\mathbf{x}}}{\partial \mathbf{w}} \frac{\partial(\frac{d\mathbf{w}}{dt})}{\partial \epsilon_i} \quad (28)$$

Since $\dot{\mathbf{x}}$ and $\dot{\mathbf{w}}$ are independent of ϵ_i , $\frac{\partial(\frac{d\mathbf{x}}{dt})}{\partial \epsilon_i}$ and $\frac{\partial(\frac{d\mathbf{w}}{dt})}{\partial \epsilon_i}$ vanish and equation (28) simplifies to:

$$\frac{\partial(\frac{d^i \phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i)}{dt})}{\partial \epsilon_i} = \frac{\partial^2 i \phi_{\mathbf{x}}}{\partial \mathbf{x} \partial \epsilon_i} \frac{d\mathbf{x}}{dt} + \frac{\partial^2 i \phi_{\mathbf{x}}}{\partial \mathbf{w} \partial \epsilon_i} \frac{d\mathbf{w}}{dt} \quad (29)$$

Similarly applying the chain rule to the right hand side of equation (5), one obtains:

$$\frac{d(\frac{\partial^i \phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i)}{\partial \epsilon_i})}{dt} = \frac{\partial(\frac{\partial^i \phi_{\mathbf{x}}}{\partial \epsilon_i})}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} + \frac{\partial(\frac{\partial^i \phi_{\mathbf{x}}}{\partial \epsilon_i})}{\partial \mathbf{w}} \frac{d\mathbf{w}}{dt} \quad (30)$$

Equation (30) simplifies to:

$$\frac{d(\frac{\partial^i \phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i)}{\partial \epsilon_i})}{dt} = \frac{\partial^2 i \phi_{\mathbf{x}}}{\partial \epsilon_i \partial \mathbf{x}} \frac{d\mathbf{x}}{dt} + \frac{\partial^2 i \phi_{\mathbf{x}}}{\partial \epsilon_i \partial \mathbf{w}} \frac{d\mathbf{w}}{dt} \quad (31)$$

Comparing equations (29) and (31), the left and right hand sides of equation (5) coincide and therefore equation (5) is proved to hold.

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