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A DISCRETE-TO-CONTINUUM APPROACH TO FREQUENCY BANGAPS IN 1D BIATOMIC METAMATERIALS

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Abstract. *We formulate a discrete-to-continuum approach to the dispersion relation of one-dimensional lattice metamaterials. With reference to a generic lattice structure that can be described as a biatomic mass-spring chain, we formulate a higher order gradient continuum theory of the competent dynamical problem using a homogenization approach. The proposed theory allows us to obtain an analytic description of the bandgap-type response of the homogenized chain, and to estimate the frequency boundary region that is affected by the full transmission of mechanical waves. Numerical applications of the proposed discrete-to-continuum approach are given with reference to tensegrity metamaterials, which exhibit a prestress-tunable bandgap response over a wide range of frequencies.*

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1 Introduction

Recently, researchers have focused on exploring the bandgap-type response of lattice metamaterials, e.g. [1–4]. These materials are artificial periodic structures with potential variations in density and/or geometry (refer, e.g., to [4], [5], [6] and references therein). They are characterized by unusual mechanical properties. It has been shown that such systems support phononic bandgaps, and have frequency ranges with inhibited wave propagation.
This study refers to mass-spring tensegrity lattices for which it has been demonstrated that the dispersion relation can be tuned by varying the applied internal self-stress induced by prestretching the cables forming the tensegrity units, and/or the global prestress induced by the application of compression forces to the terminal bases [7]. This paper addresses the bandgap-type response of the homogenized chain. Previous homogenization methods have been illustrated for both low and high frequency regimes, e.g. in [8], [9], in [10] and [11]. In contrast to these studies, the present work approaches the problem from the point of view of the theory of mixtures. The sense is that, for thermodynamic reason, the existence of bandgap in elastic continuum materials, at least two kinematical descriptors are necessary. The first is that on which it is evident the attenuation in the prohibited frequency band. The second is that on which the energy is stored. Thus, in a bi-atomic chain the kinematical descriptors correspond to the position of those masses and one side of the stiffer spring and on the same side of the softer spring. It is a matter of facts that if the springs have the same rigidity, no band gap is possibile.

In the first part of the paper, we develop a discrete-to-continuum approach to the dispersion relation of 1D periodic arrays, both for monoatomic and biatomic chains. Then, we apply the developed theory to tensegrity metamaterials, and we compare the results with those obtained using the discrete approach [7]. Our key findings are summarized in Sect. 4, where we also propose future research lines for the manufacturing and testing of bandgap metamaterials with tensegrity architecture.

2 Discrete-to-continuum models of bandgap systems

The present section studies continuous mathematical models able to describe the behavior of both monoatomic and biatomic systems. It is a matter of facts that a standard continuous model is not sufficient to represent an intrinsically discrete system. The reason is inside the continuous approximation, that is however attenuated by the use of non standard models. In order to provide a well-posed system of Partial Differential Equations for such models, the use of variational procedure is suggested, see e.g. [12]. Thus, an Action functional $\mathcal{A}$ is defined in terms of kinetic $\mathcal{K}$, internal $\mathcal{U}_{int}$ and external $\mathcal{U}_{ext}$ energies,

$$\mathcal{A} = \int_{t_1}^{t_2} \mathcal{K} - \mathcal{U}_{int} + \mathcal{U}_{ext}, \tag{1}$$

where t_1 and t_2 are two instants of time and where the action $\mathcal{A}$ is a functional of the fundamental kinematical fields of the continuum. We will see in Subsection 2.1 that for the continuous model of monoatomic chain we need only a single placement field χ and in Subsection 2.2 that for the continuous model of biatomic chain we need two placement fields χ_1 and χ_2. For the sake of simplicity, we consider for both cases a one dimensional continuous body of length L and an abscissa X, in the straight reference configuration, such that $X \in [0, L]$.

Once the set of kinematical fields and their admissible variation have been defined, the equation of motion are achieved by imposing that the variation of the action, among all the admissible variation of the kinematical fields, is null, i.e., $\delta\mathcal{A} = 0$

2.1 Monoatomic systems

Kinetic energy of the microstructural continuum representing the monoatomic chain contains not only the standard contribution in terms of the mass density ρ but also a non standard one in terms of the microinertia η,

$$\mathcal{K} = \int_0^L \frac{1}{2}\rho\dot{\chi}^2 + \frac{1}{2}\eta\dot{\chi'}^2, \tag{2}$$

where dot $\dot{a}$ and prime a' represents, respectively, the derivatives with respect to time t and position X.

Analogously, the internal energy contains not only the standard contribution in terms of the strain $\epsilon = u'$ but also in terms of the strain gradient $\epsilon' = u''$, where $u = \chi - X$ is the displacement field,

$$\mathcal{U}_{int} = \int_0^L \frac{1}{2}k_{el1}u'^2 + \frac{1}{2}k_{el2}u''^2, \tag{3}$$

and where k_{el1} is the standard axial stiffness and k_{el2} is the non-standard strain gradient axial stiffness.

The definition of an external energy should provide the necessary ingredients for solving boundary value problems. For the sake of simplicity, here we assume no external energy, i.e. $\mathcal{U}_{ext} = 0$.

Both standard arguments for the derivation of the variation of the action and imposing that its variation among all the admissible variation $\delta\chi$ of the fundamental kinematical field χ is null, i.e. $\delta\mathcal{A} = 0$, yields the Partial Differential Equation of the problem,

$$\rho\ddot{u} - \eta\ddot{u}'' = k_{el1}u'' - k_{el2}u''''. \tag{4}$$

By assuming a plane wave solution for the displacement field u in terms of the real part (i.e. Re) of complex exponential form,

$$u = Re\left(u_0 \exp i\left(\omega t - kX\right)\right), \tag{5}$$

where u_0, ω and k are, respectively, the complex amplitude, the angular frequency and the wave number, we obtain the following dispersion relation

$$\omega = \omega_c\left(k\right) = k\sqrt{\frac{k_{el1} + k_{el2}k^2}{\rho + \eta k^2}}, \tag{6}$$

where the function $\omega_c\left(k\right)$ has been introduced for the present continuous system. A comparison between the dispersion relations for discrete [7] and and for continuous (18) systems can be done only for large wave length $\lambda = 2\pi/k$, that means for small wave number k. Thus, it is reasonable to perform a Taylor's series expansion of both functions $\omega_d\left(k\right)$ and $\omega_c\left(k\right)$,

$$\omega_d\left(k\right) = ka\sqrt{\frac{K_{el}}{M}}\left(1 - \frac{\left(ka\right)^2}{24}\right) + o\left(k^3\right). \tag{7}$$

$$\omega_c\left(k\right) = k\sqrt{\frac{k_{el1}}{\rho}} + k^3\sqrt{\frac{k_{el1}}{\rho}}\frac{k_{el1}\eta - k_{el2}\rho}{2k_{el1}\rho} + o\left(k^3\right), \tag{8}$$

An identification of the two dispersion relations (7) and (8) yields

$$\rho = \frac{M}{a}, \quad k_{el1} = aK_{el}, \quad \eta = \rho\left(\frac{a^2}{12} + \frac{k_{el2}}{k_{el1}}\right) = \frac{M}{a}\left(\frac{a^2}{12} + \frac{k_{el2}}{aK_{el}}\right) \tag{9}$$

Eqns. $(27)_1$ and $(27)_2$ are the standard and intuitive identifications of both continuous mass density ρ and standard axial rigidity k_{el1} in term of the constitutive parameters of the discrete model M, K_{el} and a. Eq. $(27)_3$ gives a relation between the two strain gradient parameters η and k_{el2}. It is worth to be noted that the identification of null microinertia $\eta = 0$ is forbidden by the thermodynamic assumption $k_{el2} \geq 0$, that is implied by positive definiteness of the internal energy (3).

2.2 Biatomic systems

We now examine a continuous model for biatomic chains. As we have already pointed out, the fundamental kinematical fields of the continuum model of biatomic systems are the two placements χ_1 and χ_2.

Kinetic energy of such a continuum is not only the standard contribution in terms of a quadratic form of the two velocity fields $\dot{\chi}_1$ and $\dot{\chi}_2$ but also a non standard one in terms of a quadratic form of the two velocity gradient fields $\dot{\chi}'_1$ and $\dot{\chi}'_2$,

$$\mathcal{K} = \int_0^L \frac{1}{2} \begin{pmatrix} \dot{\chi}_1 & \dot{\chi}_2 \end{pmatrix} A \begin{pmatrix} \dot{\chi}_1 \\ \dot{\chi}_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \dot{\chi}'_1 & \dot{\chi}'_2 \end{pmatrix} B \begin{pmatrix} \dot{\chi}'_1 \\ \dot{\chi}'_2 \end{pmatrix}, \tag{10}$$

where the two matrixes A and B,

$$A = \begin{pmatrix} \rho_1 & \rho_{12} \\ \rho_{12} & \rho_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \eta_1 & \eta_{12} \\ \eta_{12} & \eta_2 \end{pmatrix}, \tag{11}$$

are positive defined.

Analogously, the internal energy contains not only the standard contribution in terms of a quadratic form of the two axial strains $\epsilon_1 = u'_1$ and $\epsilon_2 = u'_2$ but also in terms of the two axial strain gradient $\epsilon'_1 = u''_1$ and $\epsilon'_2 = u''_2$, where the two displacements $u_1 = \chi_1 - X$ and $u_2 = \chi_2 - X$ are defined. Besides, a contribution in terms of the square of the difference between the two displacements u_1 and u_2 is objective and is conceived as follows,

$$\mathcal{U}_{int} = \int_0^L \frac{1}{2} k_{el0} (u_1 - u_2)^2 + \frac{1}{2} \begin{pmatrix} u'_1 & u'_2 \end{pmatrix} C \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} u''_1 & u''_2 \end{pmatrix} D \begin{pmatrix} u''_1 \\ u''_2 \end{pmatrix}, \tag{12}$$

where the two matrixes C and D,

$$C = \begin{pmatrix} k_{el1} & k_{el12} \\ k_{el12} & k_{el2} \end{pmatrix}, \qquad D = \begin{pmatrix} k_{2el1} & k_{2el12} \\ k_{2el12} & k_{2el2} \end{pmatrix}, \tag{13}$$

are positive defined. For the sake of simplicity we assume $D = 0$. Besides, we assume no external energy, i.e. $\mathcal{U}_{ext} = 0$.

Both standard arguments for the derivation of the variation of the action and imposing that its variation among all the admissible variation $\delta\chi_1$ and $\delta\chi_2$ of the fundamental kinematical fields χ_1 and χ_2 are null, i.e. $\delta\mathcal{A} = 0$, yields the following system of Partial Differential Equations of the problem,

$$\rho_1 \ddot{u}_1 + \rho_{12} \ddot{u}_2 - \eta_1 \ddot{u}''_1 - \eta_{12} \ddot{u}''_2 = k_{el1} u''_1 + k_{el12} u''_2 - k_{el0}(u_1 - u_2) = 0 \tag{14}$$

$$\rho_2 \ddot{u}_2 + \rho_{12} \ddot{u}_1 - \eta_2 \ddot{u}''_2 - \eta_{12} \ddot{u}''_1 = k_{el2} u''_2 + k_{el12} u''_1 + k_{el0}(u_1 - u_2) = 0 \tag{15}$$

By assuming plane wave solutions for the displacement fields u_1 and u_2 in terms of the real part of complex exponential form,

$$u_1 = Re \left(u_{10} \exp i \left(\omega t - kX \right) \right), \quad u_2 = Re \left(u_{20} \exp i \left(\omega t - kX \right) \right), \tag{16}$$

where u_{10} and u_{20} are the complex amplitudes, we obtain the following dispersion relation

$$\det\begin{pmatrix} \rho_1\omega^2+\eta_1 k^2\omega^2-k_{el1}k^2-k_{el0} & \rho_{12}\omega^2+\eta_{12}k^2\omega^2-k_{el12}k^2+k_{el0} \\ \rho_{12}\omega^2+\eta_{12}k^2\omega^2-k_{el12}k^2+k_{el0} & \rho_2\omega^2+\eta_2 k^2\omega^2-k_{el2}k^2-k_{el0} \end{pmatrix}=0, \tag{17}$$

that in the explicit form is

$$\omega=\omega_{cc}(k)=\frac{E(k)\pm\sqrt{E(k)^2-4F(k)}}{2G(k)}, \tag{18}$$

where

$$\begin{aligned} E(k) &= k^4\left(k_{el1}\eta_2+k_{el2}\eta_1-2k_{el12}\eta_{12}\right)+ \\ &\quad k^2\left(k_{el1}\rho_2+k_{el2}\rho_1-2k_{el12}\rho_{12}+k_{el0}\left(\eta_1+\eta_2+2\eta_{12}\right)\right)+ \\ &\quad k_{el0}\left(\rho_1+\rho_2+2\rho_{12}\right) \end{aligned} \tag{19}$$

$$F(k) = k^2\left(k_{el0}\left(k_{el1}+k_{el2}+2k_{el12}\right)+k^2\left(k_{el1}k_{el2}-k_{el12}^2\right)\right)G(k) \tag{20}$$

$$G(k) = k^4\left(\eta_1\eta_2-\eta_{12}^2\right)+k^2\left(\rho_1\eta_2+\rho_2\eta_1-2\eta_{12}\rho_{12}\right)+\rho_1\rho_2-\rho_{12}^2 \tag{21}$$

and the function $\omega_{cc}(k)$ has been introduced for the present continuous system. A comparison between the dispersion relations for discrete and for continuous systems can be done only for large wave length $\lambda=2\pi/k$, that means for small wave number k. Thus, it is reasonable to perform a Taylor's series expansion of both functions $\omega_{dd}(k)$ and $\omega_{cc}(k)$, that we perform with the assumption

$$\rho_1=\rho, \quad \rho_2=\rho, \quad \eta_1=\eta, \quad \eta_2=\eta, \tag{22}$$

that yields,

$$\omega_{dd1}(k)^2 = (ka)^2\frac{K_1K_2}{2M\left(K_1+K_2\right)} \tag{23}$$

$$\omega_{dd2}(k)^2 = 2\frac{K_1+K_2}{M}-(ka)^2\frac{K_1K_2}{2M\left(K_1+K_2\right)} \tag{24}$$

$$\omega_{cc1}(k)^2 = k^2\frac{k_{el1}+k_{el2}+2k_{el12}}{2\left(\rho+\rho_{12}\right)}, \tag{25}$$

$$\omega_{cc2}(k)^2 = 2\frac{k_{el0}}{\rho-\rho_{12}}+k^2\left[\frac{k_{el1}+k_{el2}-2k_{el12}}{2\left(\rho-\rho_{12}\right)}-2\frac{k_{el0}\left(\eta-\eta_{12}\right)}{\left(\rho-\rho_{12}\right)^2}\right] \tag{26}$$

An identification of the four dispersion relations (23), (24), (25) and (26) yields

$$\rho=\frac{M}{a}, \quad k_{el0}=\frac{K_1+K_2}{a}, \quad k_{el1}+k_{el2}=a\frac{K_1K_2}{K_1+K_2}, \quad \eta=\frac{MaK_1K_2}{2\left(K_1+K_2\right)^2} \tag{27}$$

with the assumption

$$\rho_{12}=\eta_{12}=k_{el12}=0 \tag{28}$$

3 Numerical results

We shall now apply the theory outlined above to a biatomic tensegrity chain modeled as a mass-spring chain and featuring linear springs with two alternating stiffness constants K_1 and K_2 (where $K_1<K_2$) [7]. This section includes a numerical validation of the approach

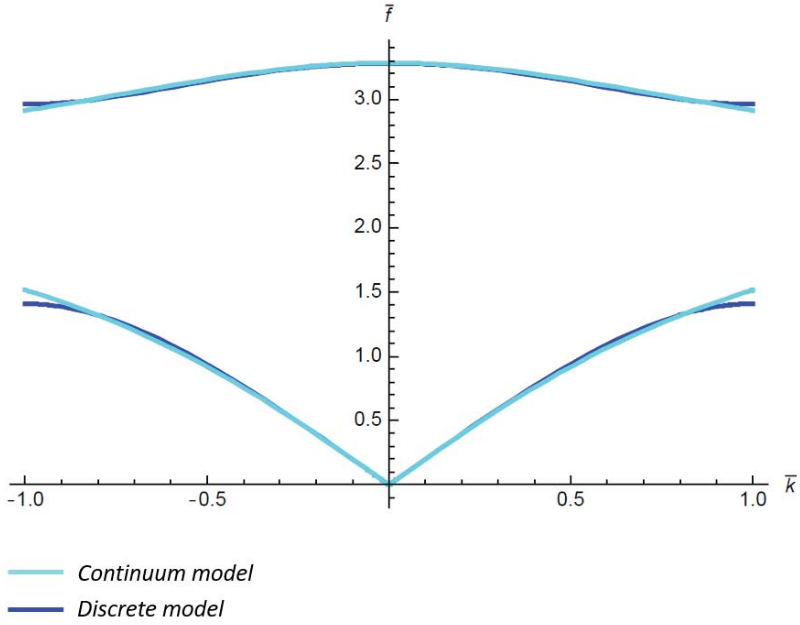

Figure 1: Dispersion relation in the first Brillouin zone for a biatomic chain under $F_0 = 4.21$ N and $\bar{p} = 0.00$, obtained by applying the discrete and continuous models.

presented in Section 2. Our results are compared with those obtained by applying Bloch-Floquet theory to a biatomic tensegrity chain, as illustrated in [7], Subsection 2.2.
We focus on a chain that is uniformly axially loaded with an initial static precompression force $F_0 = 4.21$ N applied to the terminal bases of the structure. Each unit presents an initial self-stress (internal prestress) $\bar{p} = 0.00$.

Fig. 1 compares the dispersion relation of a biatomic tensegrity chain obtained via the discrete model [7] and the continuous model (Section 2), highlighting a good overall agreement between the two approaches.
Here, $\bar{f}$ denotes the normalized frequency $\bar{f} = f\sqrt{\frac{M}{K^*}}$ where K^* is the tangent stiffness of Unit 1 (taken as a reference) in the globally prestressed configuration and M is the total mass of a unit cell. Meanwhile, $\bar{\kappa}$ denotes the normalized wavenumber $\kappa\,\pi/a$ where a is the unit cell size.

It is worth noticing that the identification of the constitutive paramenters of the continuum model in (27-28) has been done assuming small values of the wave number κ and therefore for large wave length $\lambda = 2\pi/\kappa$. Thus, from one hand this is the range in which the continuum approximation is possibile and the comparison that is done in Fig. 1 is relaiable. From the other hand, an investigation in other ranges of wave length for the continuum model still need to be done.

4 Concluding remarks

In this paper, we have analyzed the frequency band structure of a 1D monoatomic and biatomic mass-spring chain. The study led us to formulate a continuum model for these structures based on a higher order gradient continuum theory. In contrast to standard continuous models, this theory allows us to efficiently describe intrinsic discrete systems.

A numerical application of the proposed discrete-to-continuum approach has been presented with reference to tensegrity metamaterials, which exhibit a prestress-tunable bandgap response over a wide range of frequencies while the material properties of the unit cells remain unchanged. A comparison was made between the dispersion relations for discrete and continuous systems in order to validate the mathematical homogenization theory presented.

Possible engineering applications of tensegrity metamaterials may include bandgap systems [13, 14], waveguiding [15, 16], impact protection gear [6, 17], and/or acoustic lenses [18, 19].

This study paves the way for a number of relevant extensions and generalizations that we will address in future work. These may include, for example, the 3D printing [20] and experimental dynamic analysis of tensegrity metamaterial structures at different scales.

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