Realising the Cognitive Potential of Children 5 to 7 with a Mathematics focus

Michael Shayer, Kings’ College, University of London
e-mail: m.shayer@ukonline.co.uk
01954-231-814
Mundher Adhami, King’s College, University of London

Abstract

This paper describes the application of theories underlying research-in-progress intended to enhance substantially the cognitive development of children in the first two years of Primary school. The intervention is delivered partly in the context of mathematics, and partly through an existing Y1 intervention focused on Piaget’s concrete operations. The research addresses the findings of the CSMS surveys 30 years ago that barely 20% of the population achieve, by the end of adolescence, the cognitive development which Piaget presented as normal. The research aim is to alter this proportion from the very beginning so that more children may benefit from schooling.
Section 1: Introduction

This paper presents research intended to promote the cognitive development of children in the first two years of Primary school (Y1&2). It belongs within the context of other intervention projects with older children which have demonstrated the validity of the underlying theory. All are based on the findings of the CSMS survey\(^1\) of the cognitive development of 12,000 children between the ages of 9 and 16, using three Piagetian tests (Shayer, Kücheman & Wylam, 1976; Shayer & Wylam, 1978). The findings of the survey are shown below:

*Figure 1: Cognitive development Population curves for England*

At the time (1975/76) the most striking finding was that, even by the end of adolescence at 16, only about 30% of the population had developed even early formal operational thinking (3A). This had obvious implications for the transition between selective education in grammar schools and educating the population in comprehensives.

But as time went on an even more important finding was the immense range of cognitive development that was to be found at any age\(^2\). That the situation has not changed since 1974 can be seen in the Key Stage 3 Mathematics results for the whole English population in 2002, shown in Figure 2. ‘Key Stage 3’ is Government-speak for the age-range 11 to 14 whose achievement is assessed by National tests at the end of the third year in secondary education (at age 14). The National

---


\(^2\) This evidence is as yet ‘unknown’ in both the psychological and educational literature. Examination of the 5-year summaries of the Social Citation Index since 1980 reveal no citations in either article or book other than self-citations or by our Greek colleague Andreas Demetriou with whom we published a monograph of a survey of 5 to 10 year-olds, which likewise is never cited. This approaches the fate of Mendel (published in 1866, taken up by Bateson in 1902).
Curriculum is defined in terms of 8 levels which cover the whole range of children’s learning from entry into Primary school at 5.

This cried out for an interpretation, which initially was hypothetical: even if Piaget had correctly described the genotype for cognitive development (his name for this was ‘epistemic subject’) the environment in general must be very unfavourable for its phenotypic expression. The hypothesis was that the genetic potential for the development that Piaget had described was present in all children. As in medicine the ‘diagnosis’ can at least be defended if some intervention based on its assumption is successful in producing competence which otherwise would not have occurred.

The CASE\textsuperscript{3} project was set up in 1984 to test the hypothesis that intervention might alter what was found in the CSMS survey. The method was, over a two year period in Y7 & Y8 (first two years of secondary schooling), to choose contexts within science featuring the main formal operational

\textsuperscript{3} Cognitive Acceleration through Science Education (1984-87). Research project funded by the Social Science Research Project at Chelsea College.
schemata described by Inhelder and Piaget (1958), and generate lessons intended to promote
thinking. The initial research findings and the whole methodology are discussed in Shayer and Adey
(1981) and Adey and Shayer (1994), but the major long-term effects were obtained and reported in
Shayer (1999) after several years of a professional development programme for teachers conducted
from King’s College. A typical result from the data on 8 schools which were reported was that the
proportion of Y8 students in July showing early formal thinking or above was 65%, compared with
25% the year before. This is the evidence that the genetic programme was present in 40% of the
students that otherwise would not have been expressed. That the effect was permanent was shown
by the fact that three years later the students’ 1997 GCSE results in both science, maths and
English were comparably enhanced in relation to control schools, e.g. the same school, expected to
get 29.6% at grade C or above in Maths in fact obtained 69.2%.

Subsequently the CAME programme was developed using the same principles within the
context of maths in Y7&Y8 (Adhami, Johnson & Shayer, 1998; Shayer& Adami , 2004) and a
related intervention programme pCAME for Y5&Y6 (Adhami, Johnson & Shayer, 2002).

Given the above evidence on the magnitude of cognitive gains that can be obtained at the
transition to adolescence a natural question is to ask whether it could be applied successfully on
younger children so that they would have a longer period in their lives to make good use of
schooling. Medical evidence on brain growth shows that there are two major periods in children’s
life at which there are spurts: one between 10 and 11 years of age, and the other between 5 and 6
(Epstein, 1986). These ages coincide with the times described by Piaget for the transition to formal
operations and concrete operations respectively, and so may be part of the genetic programme for
children to adapt to major changes in their social environment as they grow up. This suggests that
the other most likely time for intervention would be in the first two years of Primary education. This
argument led to the current RCPCM research project5

The RCPCM project was designed to build on the expertise already developed in the
research project CASE@KS1.H&F (1997-2000). Children in Y1 were given, every week,
interactive and collaborative learning focused on the major concrete operational schemata described
by Piaget. Unlike the original CASE project for Y7/8 the activities were not placed within the
context of any particular school subject. The effects of this one year intervention are reported in
Adey , Robertson & Venville (2002). Using Piagetian Pre- and Post-tests, effect-sizes, in relation to
Control schools, of 0.47 and 0.43 standard deviations were obtained. This is the same order of

---

4 General Certificate of Secondary Education: exam taken by all students in English schools at age 16.
5 Realising the Cognitive Potential of Children 5 to 7 with a Mathematics focus (2001/2004). Research project funded by the
Economic and Social Research Council at King’s College.
6 CASE@KS1.H&F. Research project based at King’s College funded as part of a Single Regeneration Budget granted to the
Hammersmith and Fulham LEA by the DfES.
magnitude as obtained as a mid-term test after one year of the original CASE research at the end of Y7. The lessons and materials used have now been published as *Let’s Think* (Adey et al. 2001).

Might it be possible to prolong this intervention to the end of Y2 so as to obtain similarly large and long-term effects as earlier were obtained for adolescents? The method proposed in the RCPCM project was twofold. In Y1 ‘Thinking Maths’ lessons (*TM*) would be designed in such a way that the teachers could be led to manage the children’s collaborative learning in the context of maths using similar skills they were also using in their *Let’s Think* activities. In Y2 the *Let’s Think* work would have ceased but the children, already used to the learning strategies practised in the Y1 *TM* lessons, would now receive further *TM* lessons at a rate of about one every 10 days. In addition the teachers would be encouraged, where possible, to use the same teaching skills within the context of their ordinary Numeracy work, and also to establish a link from the children’s *TM* insights and the National Curriculum learning objectives. The research design is shown in Table 1.

Table 1: RCPCM Main Study schedule

<table>
<thead>
<tr>
<th>Year</th>
<th>Main Study Experimental Schools</th>
<th>Main Study Control Schools</th>
</tr>
</thead>
</table>
| Sept. 2002-July 2003 | Pre-tests: NFER Maths 5 & Piagetian Spatial test  
Y1 teachers use Let’s Think each week during the year  
Y1 use 10 TM lessons during the year |
| Sept. 2003-July 2004 | Y2 teachers use 15 TM lessons during the year and also ‘bridge’ to their Numeracy work  
Post-tests: NFER Maths 7, Piagetian Spatial test and KS1 SATs in Maths |
| July 2004     | Pre-tests: NFER Maths 5 & Piagetian Spatial test                                             | Post-tests: NFER Maths 7, Piagetian Spatial test and KS1 SATs in Maths |

This applied research was never going to be as simple (or simple-minded) as the original CASE (1984-7) work. There the original research on the reasoning patterns underlying learning in science (formal operational schemata) had already been done for us by Inhelder and Piaget (1958). All we had to do was to find relevant contexts within biology, physics and chemistry and set the activities up so that the students were able to look below the surface and construct the schema which underlay many concepts in that area of science. But in mathematics the relevant Piagetian research (Piaget & Szeminska, 1941, & Piaget, 1952) only begins at age 5. There is a curious lacuna in Piaget’s description of cognitive development. Although in Piaget (1953) what his own children could actually do between birth and their second birthday, is described in his usual meticulous detail, the development of children from 2 to 5 years is absent from his publications. All we find are a few 4 year-olds in.Number (Piaget, 1952) quoted for what they were unable to do. In the next
section the other work needed to fill out the developmental picture for mathematics will be
described. *Why* it is necessary to know what is going on up to age of 5, rather than after may be
shown in Figure 3.

![Figure 3: Pre-test result for two schools in RCPCM Project](image)

The distribution of cognitive development in the Control school shown (with an intake
somewhat above the National average) is very similar to that reported for children aged 5 to 6
illustrated in Piaget (1952), with most at Early or Middle Concrete. Our experimental schools were
from neighbourhoods for which the Government *Department for Education and Skills* (DfES)
considered it appropriate to provide the Local Education Authority with substantial funds in
support. The ‘1B’ level indicates the 3 children from the experimental school shown probably had
just 1:1 correspondence but no other concrete operational schemas. Comparison with Figure 1
shows that, in mental age terms, the Y1 experimental class contained 54% of children with
competencies anywhere between the average 3 and 5 year-old. The wide spread of development
already well known in older children is already there at entry to Primary school in Y1: hence the
need for intervention.

In order to show and explain the methodology used in the RCPCM project, first the sources
in the research literature drawn on will be described.

**Section 2: Background literature**

Somewhere in the course of Y1 and Y2 children are expected to use each of the four
operations of addition, subtraction, multiplication and division on relatively simple numbers. They
are also expected to represent relations between numbers and quantities in data-handling, and to look at aspects of shape and measure in the context of space.

Our approach to this is to ask the question, What, below the surface, do children have to attend to and process, to make sense of what they are asked to do with numbers and space? There would be answers to this question coming both from maths and psychology.

**Aspects of Measure**

From maths comes the work of Cable (1997). Although Cable begins his study both with a discussion of the philosophical development of maths and number and also Piaget’s experiments on children’s concepts of conservation, his main argument is from measurement theory. He asserts that all measurements are essentially *ratios*: that is ratios of one quantity to that of the measurement unit (degrees of quantity). They are to do with the order properties of quantities being made precise: ‘greater than’ improved to ‘5 times as large as (the unit)’. Hence all measurement implies multiplicative relations between quantities, rather than additive, even if they can be counted. This then leads to intensive quantities such as density being regarded as ratios of degrees of quantity. The educational implications of this are then that, not only do children need ‘to use the language and symbolism of number in appropriate contexts’, they need from the outset, more than at present, concrete experience of the quantity and measurement aspects of number as these relative aspects of numbers give children access to multiplicative relations.

This suggestion, coming from an analysis of purely mathematical properties, is paralleled in the work of Vergnaud, from a psychology of maths perspective. In Vergnaud (1983) he gives an analysis of the conceptual field of Multiplicative Structures, as, from a psychological point of view, ‘…multiplication, division, fraction, ratio, rational number, linear and n-linear function, dimensional analysis and vector space…are not mathematically independent of each other, and they are all present simultaneously in the very first problems students meet.’. This analysis goes deeper than is needed here, but, as with Cable, he treats multiplicative relations as essentially to do with measures, of which the first is ‘Isomorphism of Measures’ (the higher ones are Product of Measures and Multiple Proportion other than Product).

Here we find the familiar problems, e.g.:

*Richard buys 4 cakes priced at 15 cents each, How much does he pay?* and *Connie wants to share her sweets with Jan and Susan. Her mother gives her 12 sweets. How many sweets will each receive?*

each of which can be analysed in terms of the structure of direct proportion between the measure spaces $M_1$ and $M_2$.

\[
\begin{array}{c|c|c}
M_1 & M_2 \\
\end{array}
\]
\[
\begin{array}{cc}
\text{l} & a \\
b & x
\end{array}
\]

So in the first example \(M_1 = \text{[numbers of cakes]}\) and \(M_2 = \text{[costs]}\), and \(a = 15\) and \(b = 4\). In Cable’s terms ‘1 cake’ is the measurement unit for food, and the quantity is 4 of these, and ‘15 cents’ is the costs measurement unit corresponding to one cake, and the child typically uses the scalar relation of ‘4’ on both measures to arrive at the answer. The second problem is then solved in a parallel way by division as the inverse of multiplication, here with a scalar relation of ‘3’.

These are both, for children, the simplest type of ratio relation that they can process. Cable and Vergnaud concur in seeing measurement as underlying multiplicative relations between quantities.

**Brain function, Number and Quantity**

Experiments done with infants, observing their interest or boredom when they are presented with various visual displays, shows that, even at 8 to 10 months, they can distinguish the numerosity of a set of objects up to 3, above which they become ‘many’ (Butterworth, 1999, p112-120). This subitizing ability is clearly not counting. Butterworth describes this as the foundation upon which a Number Module is developed in the brain, located in the parietal zone of the cerebral cortex. But, right from the outset of development, there are two aspects of numerosity in operation, the one to do with Cardinalities and the other to do with Quantities. Dehaene (1997, p28-40) presents an ‘Accumulator Metaphor’ to account for the ability of infants (and animals like chimpanzees, rats and pigeons) to subitize. They possess a processor for magnitudes which is ‘fuzzy’—that is, having a relatively large amount of error, but in which the error is small enough to allow ‘1’, ‘2’ and ‘3’ to be differentiated, but becoming increasing unreliable for differentiating between higher numbers of objects.

While there is no doubt that it is the left parietal lobe that is responsible for processing most of the conscious and learned aspects of number (Butterworth, 1999, p214,215; Dehaene, 1997, p195), Dehaene (1998) goes on to argue that:

‘…number processing is associated with a specific cerebral network located in the inferior intra-parietal area of both hemispheres.’

Some of the evidence for this comes from well-established research on reaction times. Given two numbers, the largest of which is 9, subjects take systematically longer to decide that 9 is larger than 8, than if they are asked about 9 and 5, and the reaction time gets less the greater is the distance between the two numbers. This means that, in addition to processing the cardinal aspect of the numbers, the brain is simultaneously processing the magnitude aspect in an analogical mode. Two lines of evidence bear on the location of both these aspects of processing (Deheane, 1998, p360). Event-related potential effects ‘…although always bi-lateral, were stronger over the left parietal
areas during multiplication and over the right parietal area during number comparison.’ This was replicated with fMRI recordings:

‘Relative to letter reading, digit comparison yielded greater activity in the right parietal area, multiplication yielded greater activity in the left parietal area, and subtraction yielded a bilateral increase.’

Presumably the multiplication activity utilised stored multiplication bonds from the left parietal area, the number comparison necessitated quantity comparisons on the right parietal area before the appropriate left brain computations could be completed, and subtraction needed both.

No simple left-brain/right-brain model can be inferred from this: even when the right parietal area is active the corresponding left parietal area is also involved, and the same goes for visual identification of digits in the occipital area. But it does suggest that the quantity aspect of number—underlying multiplicative relations and measurement— is processed analogically by the two hemispheres in collaboration, whereas the cardinal aspect—underlying additive relations— is processed digitally in the left brain.

In a review paper rich in varied detail Houdé et al. (2003) cites a study done in collaboration with Peter Bryant in which French and English two and three-year-olds are compared on tasks similar to those cited above on infants’ subitizing. Already at the age of two children’s language is beginning to interact with their subitizing ability, with the French word ‘un’ giving ambiguity,. being used both for the English ‘a’ and ‘one’. This problem disappeared with the older children.

The work of Gelman & Gallistel (1978) on counting continues the developmental story for 3 and 4 year-olds. It shows a development from a primitive form of 1:1 correspondence—one for you and one for me—that some two-year-olds can do, to a process of tagging where each member of a collection is counted once and once only and the last object counted gives the cardinal value. But it has to be said that, as with Piaget, the samples were small (20 or less for each year of age) and even with the 5 year-olds only 27% were able to count to 9 with complete success (33% of the 4 year-olds). As with so many generalisations about what e.g. ‘3-year-olds’ can do such statements need to read in relation to the population evidence shown in Figure 1.

It is interesting that Gelman (1978, p208) summarises her work as describing, in Piaget’s sense of the word, a schema for counting, with self-correcting processes equivalent to his ‘accommodation’.

Proto-quantitative schemata

Developmentally, what lies between Gelman’s description of counting in 3 to 5 year-olds and Piaget’s description of the integration of cardinality and ordinality in Number? Resnick et al. (1991) offers just such an intermediate account (without, it has to be granted, her conceding any utility or validity to what Piaget offers). She describes three Protoquantitative schemata by which,
in terms of Steven’s classification, the children are able, to some extent, to use order structures but nothing more powerful, and this needs bearing in mind in addressing their learning in mathematics.

The *Compare* schema makes greater-smaller judgements of amounts of material. Language to do with *more, less or the same* begins to be used, and may be used on numbers of sets also.

The *Increase/Decrease* schema interprets change as increases or decreases in quantities. This covers things getting larger or smaller, and also the inverse, that if there are changes in quantity then something must have been added or taken away.

Lastly, the *Part-Whole* schema underlies additive relations. Resnick states:

‘This protoquantitative knowledge allows them to make judgements about the relations between parts and the wholes, including class inclusion and the effects of changes in the size of parts on the size of the whole.’

This last appears to be claiming much too much for children of pre-school age, and to be directly contradicted by the empirical evidence on 5 to 10 year-olds given in the monograph Shayer, Demetriou and Pervez, (1988) let alone that of Piaget and co-workers. But it may be possible to interpret it in the same order relation terms as the other two schemata, so that closure of the sets is not yet attained.

Yet in Gelman and Gallistel (1978) there is evidence that children as young as 3 or 4 years of age implicitly know the key principles of counting. Resnick argues that:

‘…many children who know how to count sets do not spontaneously count in order to compare sets. This means that counting and protoquantitative schemata exist initially as separate knowledge systems, isolated from each other.’

Hence an important objective for early school work in mathematics is to manage the children’s learning so that they integrate these separate knowledge systems.

**Other developmental evidence and ‘Street Mathematics’**

From both Piaget and Bryant comes evidence that (some!) 5 and 6 year-olds have, at the same time as they have some insight into additive relations between numbers, primitive concepts of the multiplicative relations between numbers. Thus in *Number* Piaget, (1952, p 215-218) shows children developing the relations ‘two-for-one’and ‘three-for-one’ between, e.g. eggs and egg-cups. In Bryant & Pireeni (1996) the two to one and three to one relations were of relative magnitudes rather than cardinals. Yet learning directed toward these number relationships is relatively absent from current Primary school Y1&2 practice. In a paper introducing the strategy of ‘correspondence reasoning’ Nunes (2002) illustrates from her earlier work on street traders (Nunes, Schleumann &
Caraher, 1993) a possible way by which young children may be helped to cope with some aspects of multiplicative relations.

Nunes gives this instance, which shows how the actual capacities of the human brain may be utilised to give people access to mathematics. A nine-year old girl is selling fruit from a street stall. She has both to guard her produce and also quickly to deal with the money side of things without getting distracted. She knows a lemon costs 5 cruzeiros, and a customer asks how much twelve will cost. She takes two in her hand, saying, These cost 10 cruzeiros. Then, still keeping her eye on the customer and her stall, she murmurs, 4 will cost 20, six will cost 30, 8 will cost 40, 10 will cost 50, So, it’s 60 cruzieros.

That is, she knows intuitively that it is a multiplicative relation she has to process, but she chooses the procedure that will make least demand on her working memory, a succession of simple additive steps until she reaches the end. Note that these also serve to show the customer, easily, how she got there. Nunes prefers to express the strategy as ‘replication’ rather than just addition. This is another way (other than using a scalar operation directly) of solving multiplicative tasks modelled by Vergnaud’s analysis of Isomorphism of Measures above.

Nunes quotes evidence (Kornilaki, 1999) that 6 year-olds can use correspondence reasoning (presumably using all their working memory to do it) to solve simple problems involving one-many relations, like 1:2 and 1:3. In addition she argues from Vygotsky’s concept of ‘cultural tools’ that correspondence reasoning gives young children access to multiplicative relations that the cultural tools of formal paper-and-pencil arithmetic do not.

Metacognition and Development

In the Houdé (2003) paper cited above evidence was given on adults working on an implication task such as modelled originally by Peter Wason. This was made slightly harder by being in the form ‘if not $p$, then $q$’—‘if there is not a red square on the left, then there is a yellow circle on the right’— and the failure rate was 90%. That is, on being asked to choose objects which would make the rule false, they put a red square on the left of a yellow circle. Houdé’s argument was that people have two competing strategies in their mental workspace—one perceptual and the other logical—but they have trouble inhibiting the perceptual one. The perceptual strategy leads to their using the shapes that are mentioned in the rule rather than to see whether a blue square on the left gave, say, a green shape on the right.

Giving the subjects logical explanations or just more practice examples made little difference to their failure rate, but when they were given emotional warnings about the error risk if they didn’t specifically inhibit their intuitive perceptual strategy, then their success rate became 90%. Hence it was not their logic that caused their initial error. fMRI imaging showed that it was
the right prefrontal cortex as well as the left (these are the brain areas to do with control strategies) that was involved in the cognitive inhibition, with the implication that emotion probably assists reasoning\textsuperscript{7}.

This suggests that in cognitive development processes very often perceptual inhibition may be a necessary condition. For example, in Piaget’s classical 1:1 correspondence problem children will not succeed unless they inhibit their perception that the length or the space occupied by one of the set of six counters affects their numerosity. Houdé’s comment on this is:

‘Developmental studies also indicate that children often fail to inhibit reasoning biases, especially perceptual ones (or semantic belief biases), and that they are even more receptive to bias-inhibition training than are adults.’

This suggests a precise focus for the mediation of children’s metacognitive activities.

Indeed Nunes (1999) speculates that a refinement of correspondence reasoning may lead children to being able to process functional relations between the elements of two sets (an important step up the hierarchy of the multiplicative field as described by Vergnaud). She quotes Vygotsky as suggesting that the teacher’s mediating role in this may sometimes be to ask the child to invert the figure-ground relation in a task. This would also be a metacognitive inhibition step, assisted by every kind of alternative representations (Tables, bar-charts, graphs, etc.) that can be devised.

\textsuperscript{7} Hume: ‘Reason is, and ought only to be, the slave of the passions’
Section 3: The Y1 Bridging to Maths lessons (BM)

The collaborative learning aspect

In Resnick, Bill & Lesgold (1992) one of the largest educational intervention effect-sizes (2 standard deviations) ever achieved is reported. The study describes work done in a first year Primary class in arithmetic (6 to 7 year-olds):

‘The children with whom we have worked come disproportionally from among the least favoured of American families. Many of them are considered to be educationally at risk…Yet these children learned effectively in a type of programme that, if present in schools at all, has been reserved for children judged able and talented–most often those from favoured social groups’.

Underlying the teaching is the same emphasis on a ‘context-delivered’ model of teaching-for-thinking as was used in the CASE project discussed in Adey & Shayer (1994). In order to allow mathematics to become a context for developing thinking:

‘…we needed to revise mathematics teaching in the direction of treating mathematics as if it were an ill-structured discipline…we needed to take seriously, with and for young learners, the propositions that mathematical statements can have more than one interpretation, that interpretation is the responsibility of every individual using mathematical expressions, and that argument and debate about interpretation and their implications are a normal part of mathematical activity.’

- So the first task of maths schooling is to provide contextualised stories within which children can apply their counting skills within the relations specified by the protoquantitative schemata. The way in which this is actualised is governed by six principles:

1. Draw children’s informal knowledge, developed outside school, into the classroom.
2. Develop children’s trust in their own knowledge.
3. Use formal notations (identify sentences and equations) as a public record of discussion and conclusions.
4. Introduce key mathematical structures as quickly as possible
5. Encourage everyday problem finding
6. Talk about mathematics, don’t just do arithmetic

This involves rejecting the behaviourist notion that ‘...it is necessary to learners to master simpler components before they try to learn complex skills.’

To see the qualitative flavour of how this actualises in the classroom it would be necessary to consult Bill et al. (1992) and Resnick et al.(1991). Bill’s classroom practice is splendidly exemplary. But an essential principle is that an effort model of development and learning is being substituted for the usual ability model. If teacher and pupil alike can assert that differences in ability
are a myth, and all that is necessary for learning is to apply enough effort, then this has interesting implications for classroom management. The ‘I’m smart so I don’t need to work much, plus, I’m not smart so there’s no point in my trying’ syndrome goes with individualism and competitive behaviour. So you hide your work from others for fear of being accused of cheating. But the effort model implies—at least in Resnick’s treatment of it—that cognition is a social practice rather than ‘a set of competencies-in-the head’. She asserts ‘a shift in focus that denies a fundamental distinction between competence and performance’. Strangely, Vygotsky is not cited as a source for this view, but what actually happens in these classrooms is that all the learning becomes collaborative, and the ‘public record’ (in 3. above) is a translation (by the teacher, with the children contributing) of the ideas the whole class have come up with into mathematical language that is common to all.

Video-recordings of these lessons, plus the lesson transcripts in the two papers cited in the previous paragraph, show a surprising similarity to the lesson transcripts of the lessons taught by the teachers in the RCPCM project, with much the same emphasis on collaborative learning. How could this be, given that the theoretical basis of the two projects appears to be mutually exclusive?

Our view is that the individual differences which are manifestly already there at 5 years of age are real, as can be seen in Figure 3, and need to be taken into account. Hence the need for an intervention theory and programme aimed at realising the genetic programme in all. To us, a view that cognition is a social practice rather than an individual competence appears to be the cognitive equivalent of ‘colour-blindness’ in social behaviour. Yet we do not deny that Resnick’s approach—particularly in the American school context—may be politically the most powerful tactic possible, and even concede that its power might be vitiated, at this time, by allowing Piaget into the picture.

Our interventions specifically invoke Vygotsky because his account of cognitive development possesses a dynamic aspect which Piaget’s does not (Shayer, 2003). Much cognitive development is social in origin because it is promoted by collaborative learning and mediated by others—mainly his/her peers—as well as created by a child’s own efforts. Hence if this mediation process is promoted in the classroom all the children should benefit. It can therefore be seen that if just the social processes in the classroom are considered application of Resnick’s six principles should result in some similarities to the control of children’s maths learning by the teacher as in the RCPCM project.

Our own practice differs from hers in that peer-peer mediation is regarded as the ‘driver’ rather than teacher-pupil mediation (Shayer, 2003). The RCPCM teachers mediate the peer-peer mediation process.

The cognitive aspect

And the objection to that is: ‘You’re not relating to me, my style, my problems’
The essential difference between the practice of Resnick and her teachers and RCPCM lies in the treatment of the cognitive aspect of children’s learning. How this relates to the specifics of lesson design and conduct will now be discussed.

As can be seen in Figure 3, in any one Year 1 Primary classroom one can expect the children to be at three different Piagetian sub-levels, whether the intake be an advantaged or a disadvantaged one. We believe that this needs to be born in mind in the design of a maths activity intended as part of an intervention to promote cognitive development. Such a consideration is in effect vetoed by Resnick’s insistence that cognition is a social practice rather than ‘a set of competencies-in-the head’. But here it leads to the requirement that a learning context be chosen that contains genuine opportunities for thinking and fresh insights at least at three different levels. The lesson is then conducted in such a way that each child gets the chance to make what Vygotsky called ‘revelutionary jumps’ from where they presently are (Shayer, 2003). This leads to episodes of 3-Act sequences, usually two per lesson. In Act 1 the context is introduced to the children—often in the form of the teacher casting it in the form of a story the children can enter into—at a level that all the children can process. Then by a series of questions and requests to the children to suggest their solutions the teacher prepares the children for Act 2 where they will work in pairs or in small groups of three or four at a problem. The children know that the purpose of Act 2 is not to carry out ‘neat work’ but is to produce ideas they can soon show to and share with the rest of the class. The teacher encourages the children to talk to each other while they are working. In Vygotsky’s terms the ‘Zone of Proximal Development’ (ZPD) has both a social and an individual meaning. When children are working actively together in a small group they can thought to be sharing a communal ZPD to which all are contributing. If the frontier of the shared ZPD moves forward as a result of their collaboration then more than one may benefit. The individual, and the original Vygotskian meaning of ZPD is that each child, before they are able to show an assured competence, has in their head some partially developed concepts which are often completed when the child witnesses a successful show of that competence in a peer similar to themselves. Much of cognitive development is like this, and appears ‘revelutionary’ because the child’s new insight appears suddenly as though out of nowhere.

The teacher’s behaviour in Act 2 is different from much Primary school practice. She inhibits her impulse to help the children in their groups in favour of watching and noting what is happening in each of the groups, and who is involved. This is in order that Act 3 may be managed as beneficially as possible. The teacher needs to be aware of the different demand levels of each of the lesson’s concepts, and needs also to be able to see at a glance (or hearing) the level at which each child is operating. In Act 3 each group then quickly shows the rest of the class (either from the tables where they were, or back on the carpet at the front of the room) where they have got to in
working on the problem. So the teacher will have noticed in Act 2 some of the least able children getting somewhere on some aspect of the problem, and asks them to report first. This helps their self-esteem and also gives the other less able children the chance to complete their ZPD at this level. Then she asks the groups who have made further progress to report, until all the ideas she has noticed in Act 2 are made public for all the class. In this way, if the ‘successful performance’ a girl or boy needed to complete their individual ZPD happened to have occurred in a different group, all now get the opportunity to benefit.

Often the first 3-Act cycle addresses the lower conceptual levels the context affords, and the second cycle gives more opportunities to the more able children to develop their understanding. But the other children appear to benefit from hearing and seeing the more able children’s explanations—perhaps extending their ZPDs so that some weeks or months later they can go further.

**The collaborative and cognitive aspect together**

The application of these design principles can be seen in *BM4: Streets*, a Y1 lesson on Data-Handling intended to be used soon before or after Christmas. In terms of the maths the top level aim is to realise that if they are to compare the sizes of different sets, as many-many relationships, then the units they use on a bar-chart need to have equal sizes. The first 3-Act episode has the teacher showing the children on the carpet a series of pictures (horses, sheep, people) and asking them to name and sort them. Then she offers them multilinks (2cm. cubes in different colours each able to slot into the next) to achieve 1:1 correspondence with the numerosity of each group. Their Act 2 is to go away with a collection representing a street (houses, pets, people and cars) and find some way of showing their collection using the multilinks or by sorting the pictures. The Act 3 for this episode has the children discussing issues like ‘having bigger horses doesn’t mean there are more of them’ etc. and seeing different ways of representation.

The second 3-Act Episode begins by the teacher, at the whiteboard, showing children how to draw round their multilinks in pencil on an A3 sheet so as to make a bar-chart of their collections. She actually shows a small but equal space between each multilink, but does not ‘teach’ it explicitly. The groups of 3 or 4 then, in Act 2, go away to make their bar-chart of their street collection. In Act 3 the teacher then quickly shows, bluetacked on the whiteboard, each group’s bar-chart, and asks the others to comment. Obviously she begins with the ones showing different faults and problems. Finally, with a good one on show, the class are then asked to interpret the many-many relationships (e.g. ‘Twice as many pets as houses, 6 and 3—what might that mean?’; ‘How about the cars and people?’—‘2 cars for 4 people’, ‘maybe a man and wife—or Daddy and Mummy—share each car’ &).

(Figure 4: Diagram of Episodes in BM4: Data-Handling) *about here*
On the Piagetian diagram—Figure 4—for the lesson (mapped also onto National Curriculum levels, with Y1 pupils working toward level 2) the cognitive demand of each phase of the lesson is shown, showing the different conceptual achievements afforded by the lesson context. It can be seen that the teachers with responsibilities for each of the two school classes shown in Figure 3 could use the diagram to plan how far they could go in each of the two episodes so that, for each class, it would be a satisfying experience.

**BM4: Data Handling**

<table>
<thead>
<tr>
<th>Piagetian Levels</th>
<th>NC Levels</th>
<th>Attainment Points for different pupils</th>
</tr>
</thead>
<tbody>
<tr>
<td>1B (Pre-Operational)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2A (Early Concrete)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2A/2B* (Middle Concrete)</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2B (Mature Concrete)</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

The planning of the maths lessons for the Y1 children had to bear in mind the concurrent use, as referred to earlier, of the other intervention lessons, *Let’s Think*. As they were presented in terms of Piaget’s major concrete operational schemata we had to plan the maths in such a way that those schemata were used in the development of mathematical understanding: in doing this we used Resnick’s description of Proto-quantitative schemata as well as Piaget’s. It was also necessary to bear in mind the National Curriculum for Mathematics and its prescribed content for Y1&2 (Key Stage 1). Our view of our task was that the National Curriculum is presented largely in terms of surface competencies, whereas the *Let’s Think* agenda addresses the logic underlying number. Resnick’s protoquantitative schemata were viewed as describing the link between Piaget’s descriptions and children’s elementary number work. As with the work of Resnick’s team we viewed our task as to go beneath the surface of the children’s procedural maths learning so as to get
children to think about the number concepts which often are treated as unproblematic in procedural learning. For example in BM4: Streets children are asked to construct, collaboratively, from first principles, how a bar-chart works, what the rules-of-the-game are, and how meaning can be extracted from it.

Figure 5 about here

(Figure 5: Mathematical Strands in Y1 Bridging to Maths activities)

In Figure 5 it can be seen that we used four main strands in describing the maths: Number (additive compositions), Measures (leading to multiplicative relations), Data-Handling, and Space and Shape. As with our preceding intervention projects the intention behind these lessons was twofold. Of course they are intended as strenuous and effective thinking exercises for the children (they often tell us they are). But this, on its own, would not be expected to produce the cognitive gains that are intended. They are also intended to show the teachers, implicitly, how to work for thinking in their ordinary Numeracy lessons. This might in part involve the teachers linking the ideas children have developed in their Bridging to Maths lessons to the ordinary Numeracy lessons. But, perhaps more importantly, if they can conduct their Numeracy lessons (themselves having an implicit 3-Act structure) using some of the same teaching skills they use in the TM and the Let’s Think lessons, then there would be a large multiplier effect on their children’s understanding of maths.

In the Number sequence BM1, BM3 and BM5 it can be seen that, in addition to encouraging finger work, as recommended by Butterworth (1999), there is a deliberate attempt to extend children’s subitizing abilities to higher numbers by using images to which number can be attached and recognised. Eventually (BM5) these patterns are used as an auxiliary to developing a count-on strategy when they are given an unseen collection of which the only clue is the digit ‘7’.

In the Measure sequence (BM2, BM4 and BM8) the underlying strategy is derived from Cable (1997). First children are presented with a story about trying to buy shoes for a sibling. What are these shoe sizes? They are presented with three different size footprints said to have been made by drawing round stockinged feet. They are asked to use their multilinks to arrive at their lengths (3, 6 & 10 links), which typically they do by making three separate ‘sticks’ of multilinks. The question, Do you need all three of your sticks? can provoke further ideas about measuring. Then they are asked to quickly ‘measure’ three different dowel rods, and then, when they present their answers, the implicit zero comes up, where some say ‘7’ (correct) and others say ‘8’. So maybe it’s only ‘1’ when they come to the end of their first multilink. Finally, Cable suggests that children should have their early experiences of fractions as numbers. At the whiteboard the teacher shows them a rod (9cm) and asks one group to bring up 4 multilinks—not enough. Another child brings up 5 multilinks—too much. So How long is my rod? ‘It’s half-way between the two’. So in this activity
the equal interval quality of measurement is ‘given’: only the zero is slightly problematised. In BM4, in the context of data-handling, the zero is ‘given’, but now the equal interval property is problematised and they have to construct it. Finally in Footprints both the zero and the equal interval property become problematised and they need to construct both on the basis of their initial errors. Similar developmental strategies underly the Shape and Space activities.

Section 4: The Y2 TM lessons and the overall strategic planning

The principle, enunciated in the previous section, that one needs to bear in mind the different cognitive levels at which the children may be has an important corollary. In order to plan either a plausible curriculum for children or, in this case, a mathematics-contextualised intervention programme, one needs to be able to estimate the cognitive demand of each aspect of maths the children are going to be asked to process. Go too far, and, as Vygotsky originally pointed out (Shayer, 2003), the children will merely be mystified. Go not far enough, and there is no stimulus for children to think more deeply. In the monograph Shayer, Demetriou & Pervez (1988) it was shown that among children from Pakistan, Greece, Australia and England, already by the age of 7 20% of the populations were at Piaget’s mature concrete (2B) level of thinking. Inspection of Figure 1 indicates that the RCPCM aim for all children at the end of Y2 should be this level. If the curves for the upper 20% in Figure 1 are followed up the diagram the full genetic programme as described by Inhelder and Piaget (1958) can be seen, leading to Mature Formal (3A/3B) by the age of 16. The 2B level, then, is the maximum for any of the Y2 lessons (see Figure 4, where 2A/2B is the maximum for Y1). It is also the level aimed at for the children at the end of Y2.

At this point, in thinking of intervention strategies, one needs to think ahead on behalf of the children, using the well-established knowledge about 10 to 14 year-olds’ difficulties in handling ratios, fractions and proportions (Hart, 1981). How can children be given a foundation for their number work equally sure both for additive compositions and multiplication relations, given that they actually have the ability, at ‘some intellectually honest level’ (Bruner, 1968, p44). A good answer to this question should make children’s work in Y3 onwards much more confident.

One line of answer lies in the design of activities intending to develop children’s involvement with measures and measurement, as suggested in Section 2. If fractions are introduced to children as measures rather than parts of wholes, then from the word Go their concept will be of them as numbers. At present this interpretation is offered to them typically in Y6, where they find it mystifying after having got used to ‘1/4’ as one cell in a four-cell box. In terms of our cultural history—and this ties in with Nunes’ work on street traders—joiners and cabinet-makers used the measure meaning easily and intuitively. If the joiner’s work on a house only required minimal
accuracy then *quarters* of an inch might be close enough. The cabinet maker may have needed to go three more steps down this progression to *thirty-seconds*. So ‘three quarters’ is easily interpreted by the child as half-way between a half and one, and ‘one and a quarter’ readily visualised on a ruler.

Likewise data-handling exercises involving bar-chart representation can be angled toward *relative* numbers of different variables as well as their number *differences*. All of these are intended to provide children with a foundation for their thinking in multiplicative terms equal to that which they presently get for additive compositions.

The 1995 version of the English National Curriculum cites, for 8 to 11 year-olds, as a Key Stage 2 aim:

‘*understand multiplication as repeated addition, and division as sharing and repeated subtraction*’,

but the Key Stage 1 aim (for 5 to 7 year-olds) has:

‘*understand the operations of multiplication, and division as sharing and repeated subtraction*’

a curious inversion of relative difficulty, indicating some confusion of purpose. It is important not to confuse computing algorithms for multiplication and division with the inherent meaning of the concepts.

Our interpretation makes use of various research cited by Nunes & Bryant (1996, chapter 7). Multiplicative relations between numbers are essentially to do with *relative* quantities, leading eventually with exact work to ratio relations between measures and numbers. The key meaning of division is as the inverse of multiplication, one step on the road to which is seeing that if one quantity is twice as large as the other, then the second is half as big as the first. Early work by Bryant (1974) suggested that ‘young children are much better at working with relations than at solving problems that involve considering absolute values’ (presumably using mainly right-brain numerosity notions). This suggests that intervention work should focus initially at connecting the inverse relation between multiplication and division with what children *can* do.

One general principle with all these intervention lessons, which apparently are to do with ‘difficult’ concepts, is always to embed them, as indeed is nearly all the research cited in Nunes & Bryant, (1996), in specific contexts which can be made meaningful to children. This was also one of the strong points in Dienes’ original work with young children—today rarely mentioned—together with the principle of employing multiple representations of any abstract concept. Thus our first Y2 activity in this line starts with twins, having a birthday party. They have a picture of a bag of 12 marbles, with the marbles drawn sufficiently separated so they can split them in various ways with their pencils. This was Granny’s present. How can the twins share the present so it is fair? Different groups show and explain their solutions, and the teacher records or draw their solutions on the board.
as a record. But she also keeps a running record on half her whiteboard of the major solution as a bar-chart, with one stick-child underneath twelve units and two stick-children under six units to show the sharing.

Then their older brother and sister come in from the park and ask if there is a present for them, but Granny says it is one present for all. What can the children do now? This time, after the groups have presented their solutions the teacher records the sharing out both in bar-chart form and in array form (four children, 3 marbles under each) so as to link the children’s work with mathematical formalisms they may have met already, perhaps without understanding. The activity continues with two more steps, until on the board there is both a 6x2 and a 2x6 array, and a 4by3 and a 3by4 array showing. Finally the children are asked what relations they can see between the different solutions (commutativity, and division the inverse of multiplication)?

This suggests, as an intervention strategy, that first we design activities both in the data-handling strand and the spatial where the stress is on processing relative amounts, initially in a proto-quantitative way, as in *Slugs and Flowers*, where they first count the worms, slugs and flowers in three parts of the garden, then represent the three areas in terms of bar-charts of each of the ‘inhabitants’, and finally see if there is any relation between the amounts of flowers and the amounts of slugs in the different sectors. From this, then, as a step in making the relations more quantitative, another activity is designed, *Jelly Babies*, that utilises the simplest of multiplicative relations of quantities, that of two to one. First they are shown an image in which the head depth has the same depth as the body, which they recognise and name. Then they are shown JellyDaddies, where the body length is twice that of the head, and are asked to talk about the difference. So words get used like ‘twice as much’, ‘compared with a baby, smaller head’. How much smaller? ‘half as big’.

Finally, contexts were invented which allow children to see that various additive (correspondence) strategies such as described in Nunes (2002), and mentioned in section 2, would get them to solutions of problems involving multiplication which otherwise would be quite difficult, and require memorising of all the multiplication tables. Note that the emphasis here would be on the children becoming conscious of the difference between multiplicative relations and additive compositions, but being aware (as indeed an adult using a computer and EXCEL may be) of various economical strategies they might use to get their solutions to problems,

Note that we do not call these street traders’ and workers’ strategies Nunes describes as ‘Alternative Conceptions’, as though they are some inherent human weakness our children are to be persuaded to overcome. On the contrary, they are robust and reliable strategies (as are some uses of fingers for additive work) which we believe can lead children to better understanding of mathematics than the current emphasis on procedures and algorithms. In fact, in contradiction to the
National Curriculum of ‘multiplication as successive addition’ as a *definition*, we expect our children to realise that multiplicative relations like ‘three times as big’ are *different* from additive ones, to do with comparing quantities in a mathematical way, but sometimes procedures based on additions will give the answer.

Work on additive compositions is not neglected in the planning for Y2, but is here focused on the well-known problems all children have when working on meant-to-be ‘simple’ Word Problems. For analysis of the maths involved the work of Vergnaud (Nunes & Bryant, 1997, chapter 1) is relied on. Nunes & Bryant, (1996, chapter 6) provides empirical evidence on the relative difficulty of the different additive structures Vergnaud describes. Here it is necessary to think of the neuro-psychological evidence summarised in section 2. In *BM4 Streets* (Figure 4) perhaps the strategy is that the children are being asked to use and relate left-brain operations to the mainly right-brain estimates of quantity they have before their eyes. In *BM1, Number Patterns*, children are challenged to assist their left-brain counting abilities with both mainly right-brain subitizing and the use of imagery. Our preliminary work (which will surprise no-one who has worked in this field) with children indicates that their processing in words bears only a tenuous relation to the additive algorithms which most can already do. So their problem (and ours) is to recognize that, as the task is one of gradual integration of two different kinds of brain activity, involving different sites, it cannot be taught directly as a particular skill. Rather, successive familiarisation with all aspects of the words/numbers/operations relations over a period of time may be needed.

(Figure 6: Mathematical Strands in Y2 *Thinking Maths* activities)

The complete set of Y2 TM lessons can be seen in Figure 6, again grouped in terms of the different ‘strands’ of mathematical activity. But perhaps our applied and applicable research strategy is worth describing. Each activity is devised initially, using the principles mentioned above, on the understanding that it is just a *hypothesis* about what might be a fruitful intervention for the children. As it is trialled it has necessarily to be actualised through the mind of one of our (gifted) Hammersmith and Fulham teacher-researchers. Detailed notes, with a time-line, are taken of what goes on in the lesson, what the children and the teacher do and say, and what problems are observed. Since a reality-test of a hypothesis is not worth making unless the researchers are prepared to be shown (possibly many) new aspects of the world focused upon, the team invariably finds more aspects of the lesson that need to be considered than they had anticipated. The authors and the teacher-researchers then consider the evidence at a team meeting, and this then leads usually to a further version to be trialled. This is shown in Table 2 in the case of the lesson *Intensive Relations*, which is based on research presented in Nunes’ paper in this current issue of IJER.

*Table 2 about here*
As an early indication of the relative effects of the RCPCM intervention, Figure 6 shows the extent to which the intent of increasing the proportion of 7 year-old children with mature concrete ability has been realised (the full results will be reported elsewhere when the data is all in).

Figure 6: Proportion of children per class at the Mature Concrete level at Post-test on Spatial Relations

It will be recalled that Shayer, Demetriou & Pervez (1988) showed that about 20% of the population of 7 year-olds were at the 2B level or above. For the 5 year-olds 3.6 (Early Concrete) at Pre-test is about the National average, and it can be seen that the Control school classes developed over two years in line with this earlier evidence. The regression line for the RCPCM classes shows 40% 2B (Mature Concrete) for the same Pre-test level. Likewise those classes matching the Control classes who had 10% at the 2B level average 30% 2B, with two at 55%.

Conclusion

The authors are well aware that this applied and applicable research (*applied* because it necessitates extracting, on a hypothetical basis, fundamental research on children’s learning, and *applicable* because it aims to change what goes on in the classroom) even if successful, would be only the first word. Already, in order to prepare our teachers in the experimental schools, we have realised that their professional development (PD) requires to be conducted so that it parallels the social processes that we wish them to engender among their pupils. For them to gain possession of
the underlying theory—both social and cognitive—they need the same mediation of their own collaborative learning as we are asking them to use with their children. ‘Telling’ won’t do it, either way. So who is to mediate the PD to teachers in other LEAs or schools, and how would they get to the point where they sufficiently own the theory and practice themselves? These are not trivial issues for those with a serious interest in applicable research. There is little evidence that Government thinking has arrived anywhere near to thinking of PD—and the need for new practice to be based properly on evidence-based research—in these terms.

A second issue is the relation of this cognitive intervention research, embedded in the context of mathematics, with the theory and practice of maths teaching itself. Looking a long way ahead, one can imagine, in the spectrum of primary teachers’ skills a seamless integration of instructional teaching—aimed at increasing children’s competence in what they already understand—and interventionist teaching, aimed at enhancing children’s cognitive development. How this might be achieved seems problematical.
References


Table 2: Initial trial of TM lesson on Intensive Properties

**Intensive Properties: Y2 class: June 22, 04**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Children</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.06  ‘This lesson is about Decisions!— during your work I am going to ask you to choose and show your A and B cards’ Shows:</td>
<td>Children are all on the carpet</td>
<td>So at least 80% choose for the salient variable</td>
</tr>
</tbody>
</table>

‘Derek and Alex are working in their gardens. What do you think these grey squares are?’

They are their gardens’

‘When they plant out their roses in their gardens which garden would be the most crowded?’

‘What are your reasons for choosing A?’

‘Next….’

Shows:

<table>
<thead>
<tr>
<th>Derek</th>
<th>Alex</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="Derek.png" alt="Image" /></td>
<td><img src="Alex.png" alt="Image" /></td>
</tr>
</tbody>
</table>

‘Which garden do you think is less crowded?’

‘Turn to your neighbour and discuss it’

**9.14: after a minute or so**

‘Why did you choose A?’

‘Why did you choose B?’

‘Why did you choose same?’

‘Why did you choose A?’

‘Remember when I asked the question I said, less crowded!’

‘Who has changed their mind?’

**9.21: Episode 2**

<table>
<thead>
<tr>
<th>EMMA</th>
<th>MARY</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="EMMA.png" alt="Image" /></td>
<td><img src="MARY.png" alt="Image" /></td>
</tr>
</tbody>
</table>

‘Which garden do you think is less crowded?’

| ? | ? | ? |

Nearly all children hold up their A cards (for Derek’s)

‘lots of flowers’ ‘use up more room’

‘more flowers in A than B’ ‘more plants’ ‘A has got three layers’

| ? | ? | ? |

15 choose B (Mary’s) and 8 choose A (Emma’s)

‘Emma’s garden is bigger—less crowded’

‘Mary has less flowers than Emma’

‘same amount of flowers’

‘A;’s garden was bigger, but there were the same number of flowers’

3 Bs said they had changed to A

‘Emma’s garden is the bigger’ ‘she gets more space’ ‘….more room’

Chooses the other variable

Doubtful reason

Looking to the salient variable only

Processes both variables

Teacher was not overtly indicating that one solution was better

Now only 2 still choose B, and one still chooses ‘same’ The rest now choose A
Shows a bottle of lemon juice.
‘What do you think this is?’
‘What else does it contain?’
‘Why do you think it contains sugar?’

‘lemon juice’
‘water’ ‘sugar’
‘lemons are sour!’

‘Which do you think is the sweeter?—you have a minute or two to talk’
‘Why did you choose A?’
‘Why did you choose B?’
‘Why did you choose A?’

Which was less sweet?
‘No—the sugar lumps are the same size’

‘So how do you all choose now?’

5 pairs choose B; 9 choose A, 2 choose same
‘it has more lemon juice’
‘more sugar (compared to)… the amount of lemon juice—so sweeter’
‘more lemon juice—you have to put more sugar in it’
‘B—cubes are smaller’
‘sugar is tasting sweeter than the lemon’
‘A has more juice: B has less juice, so more for the sugar to do’

4 still choose A; 1 chooses same; the rest now all choose B

5.28:

9.28:
‘Now you can work with your partners’
Gives out:

9.35: ‘Who has finished?’
‘Very few—you can have 3 more minutes; think what you will say to the class’

9.38:
‘Now, what are your decisions?’—show your cards’

‘Why did you choose B?’

‘How about you?’

‘So how could you make A the same sweetness as B?’

‘Why did you choose A?’

7 pairs choose A
6 pairs choose B (with 2 sugar lumps)
‘In B there are 2 sugars with two lemon juices, but in A if I give one sugar to each lemon juice, then there is one lemon juice left over’
‘in B 2 cups to two sugars’ but in A only 3 sugars for 4 cups’

‘Take one cup of juice away’

9.38: ‘Now, what are your decisions?’—show your cards’

‘Why did you choose B?’

‘How about you?’

‘So how could you make A the same sweetness as B?’

‘Why did you choose A?’

B is the smaller amount of juice
Child has great difficulty in expressing this, but is considering both variables

Communal ZPD seems to be working without teacher telling them ‘right answer’

This problem was chosen deliberately to start using numbers as well as ordered variables only

Various strategies being attempted
‘So what are your decisions now?’

9.45:

*Shows:*

<table>
<thead>
<tr>
<th>Shop A</th>
<th>Shop B</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="https://via.placeholder.com/150" alt="Image" /></td>
<td><img src="https://via.placeholder.com/150" alt="Image" /></td>
</tr>
</tbody>
</table>

‘This is about cakes and Value-for-money—these are cherry buns. In the two shops they both cost the same, 55p’

‘Which would you choose?’

‘Why did you choose B?’

‘What does Value for money mean?’

‘…and how much you get’

‘Why did you choose A?’

‘Who disagrees?’

‘I would choose A because you get more for the same price’

9.49:

‘Now some work for you with partners—remember it is which is the better bargain’

18 hands go for A; only 4 for B.
‘because in shop B it is smaller’

‘How much money…’

‘the biggest cake’

‘…more cherries on it’

? ? ?

‘…you could give some of it to someone else’

This was their worksheet

<table>
<thead>
<tr>
<th>Shop A</th>
<th>Shop B</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="https://via.placeholder.com/150" alt="Image" /></td>
<td><img src="https://via.placeholder.com/150" alt="Image" /></td>
</tr>
</tbody>
</table>

4 sweets 7 sweets

Points to the group of 6p in A, asks
‘How do six and 12 compare?’

7 pairs choose A; 6 pairs choose B

‘if you gave one cup of A to B it would be the same’

‘…but what about the sugar?!’

3 pairs still choose A, but 8 choose B

Again, communal ZPD seems to do it

Here they have gone spontaneously for the non-salient variable

I think T expressing an opinion is OK for it explains the concrete preparation meaning of value-for-money.

Now they have to use their own thinking on the problem with numbers

Again, they have to think both about the two variables AND about simple numbers, so the comparison is between 6 and 12 and 4 and 7

P is using correspondence reasoning

Correspondence reasoning again, but I did mediate with this girl earlier, pointing to the 12 and the 6 as a question

Reasoning obscure—maybe they were saying that in B it should have been 4 if the bargain was the same
‘So who has changed their mind now?’

time running out here, but we did want
to try Speed as well!

10.04:

Shows:

<table>
<thead>
<tr>
<th>Track A</th>
<th>Track B</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Track A diagram" /></td>
<td><img src="image2" alt="Track B diagram" /></td>
</tr>
</tbody>
</table>

‘See, they both start and finish at the same place, and they both take one minute—which was the faster?’

‘Explain to your partner’

lets them have a minute

‘Now what are your decisions?’

‘So…?’

‘Can you explain that?’

‘What do you all now decide?’

10.11:

Gives out worksheet

10.15:

‘Stop!’

‘What are you decisions?’

‘Explain’

‘2 sweets left in A, but 5 pennies left in B’

11 pairs now choose A and only 2 choose B

We failed to offer them the third alternative of ‘same’

Forced choice!

‘B’ ‘B’ ‘B’

‘the same’

7 choose A; 7 pairs chose B

k ‘B has the longer track

‘……because it is bigger it was faster
to get round in one minute’

h ‘both the same—A is smaller, but B
has more room—both one minute’

h ‘B has the longer track—both took
one minute, so ….’

h ‘A is faster, because it has a smaller
track.’ ‘…because less track to go
round’

12 pairs choose B, and only 2 pairs
choose A

Right sort of reason to give a false
answer

In spite of their hearing misleading
arguments the peer/peer mediation
has somehow worked

Again, I asked one or two to look at
the two distances first

(I only recorded the sexes because in
this class the girls are usually
brighter)

I didn’t mediate this boy

Again correspondence reasoning

Salient variable only used for
comparison

Discussion

Episode 1 Crowds: Concrete Preparation

and Classwork: 15 min

Episode 2 Lemon juice: Concrete Preparation

Pairwork: 10 min

15 mins
Perhaps we should have offered them in each case the third alternative of ‘same’. In general the activity seemed very productive of thinking by the children. A very rough indication of the changes during the lesson is:

<table>
<thead>
<tr>
<th>Episode 3 Value for money: Concrete preparation and classwork</th>
<th>4 min</th>
<th>24 mins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pairwork</td>
<td>4 min</td>
<td></td>
</tr>
<tr>
<td>Class discussion</td>
<td>11 min</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Episode 4 Speed: Concrete Preparation</th>
<th>7 min</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pairwork</td>
<td>4 min</td>
</tr>
<tr>
<td>Class discussion</td>
<td>4 min</td>
</tr>
</tbody>
</table>

. It seems in every case once the children have heard everyone’s discussion and reasons their solutions using both variables rise to around 80%. In addition there is some indication that their opinions when first shown the problems does rise from about the 30% level to the 50% level. Obviously many of them ‘want’ to consider one variable only—yet they do seem to be able to see the force of reasoning that considers both variables.
Figure 5

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.
Word Problems 2:
Clues
Sorting clues about numbers and their placement of in boxes; reasons for sorting; irrelevant clues.

Word problems 1:
True and untrue stories
Checking correctness of word problems; correcting numbers, finding missing numbers in each part of sentence.
1 had 3 marbles at first
Then I found 4 more marbles
Now I have 8 marbles altogether

Number system
How many each?

How many pencils in each box?
24 pencils in all

Six pencils in a box
How many boxes?

Intensive Properties
Which is sweeter?
Shop A Shop B
6p 12p
4 sweets 7 sweets

Chocolate
Which group would get the most chocolate each?
How did you decide?

Jelly babies
What is the difference?
(Varied in size)-
Which is which?

Multlicative Relations
How much Bigger?
Fractions as measures

Measurement
On the Market
Correspondence Reason
	O½ need 12 lemons
	O2 lemons cost 10p
	O20p
	OSo how much for 12?

Broken Rulers
Equal interval scales; Zero Scale-concept

Using intervals to co
Constructing equal interval sc

Crossing the Bridge
Given 10, 5 and 2 cm cylinders
Find how many ways you can make a length of 20 cm

Sharing presents
12 items shared in various ways.

Tile Patterns
2 to 3 relations
O2 more of the set of 3 tiles: How many of the sets of two?

Word Problems

Word Problems 1:
True and untrue stories
Checking correctness of word problems; correcting numbers, finding missing numbers in each part of sentence.
I had 3 marbles at first
Then I found 4 more marbles
Now I have 8 marbles altogether

Tile Patterns
2 to 3 relations
O2 more of the set of 3 tiles: How many of the sets of two?

Crossing the Bridge
Given 10, 5 and 2 cm cylinders
Find how many ways you can make a length of 20 cm

Sharing presents
12 items shared in various ways.

Table: 24 pencils in all
Six pencils in a box

Table: 12 lemons

Table: 20p

Table: 20p

Table: 12 lemons cost 10p

Table: O½ need 12 lemons

Table: O2 lemons cost 10p

Table: O20p

Table: OSo how much for 12?

Table: O½ need 12 lemons

Table: O2 lemons cost 10p

Table: O20p

Table: OSo how much for 12?