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Ramification in higher norm fields

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Abstract

In [5], Fontaine and Wintenberger developed the theory of norm fields to study certain p-adic representations of local fields with perfect residue fields. In [3], Scholl has extended the theory to more general local fields of characteristic zero with imperfect residue fields; however, differing from Fontaine and Wintenberger, no appeal is made to higher ramification theory. In this paper, we use the ramification filtration of Abbès and Saito to initiate a study of Scholl's theory from Fontaine and Wintenberger's viewpoint of "arithmetically profinite" extensions.

Contents

- In Section 1, we give a brief overview of Abbès and Saito's ramification theory for local fields with imperfect residue fields, and of Scholl's construction of the field of norms for strictly deeply ramified towers. With the exception of Example 2.1, this section contains no original material.
- In Section 2, which is entirely original, we determine the ramification properties of strictly deeply ramified extensions, and study how these translate via the field of norms functor. Applications, which are expected to lead towards a generalization of a result of Wintenberger, are given.

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1 Introduction

In [5], Fontaine and Wintenberger developed the theory of norm fields to study certain p-adic representations of local fields with perfect residue fields. In [3], Scholl has extended the theory to more general local fields of characteristic zero with imperfect residue fields; however, differing from Fontaine and Wintenberger, no appeal is made to higher ramification theory. In this paper, we use the ramification theory of Abbès and Saito to initiate a study of Scholl's theory from Fontaine and Wintenberger's viewpoint of "arithmetically profinite" (APF) extensions, characterized in terms of ramification groups.

In the first part, we give an overview of Abbès and Saito's definition of an upper numbering ramification filtration of the absolute Galois group of a local field with imperfect residue field. We illustrate the construction by computing the ramification groups of a natural non-abelian generalization of cyclotomic extensions ("Kummer towers"). We proceed to define the field of norms of so-called "strictly deeply ramified" extensions of a local field of mixed characteristic $(0, p)$, under some mild assumptions on the residue field (namely, that it has a finite p-basis). If L/K is such an extension, the field of norms X of L/K is a complete, discretely valued field of characteristic p, and the study of finite separable extensions of X amounts to the study of finite extensions of L.

In the second part, we determine the ramification properties of strictly deeply ramified extensions, and study how these translate to norm fields. The motivation behind this is to show the following: *M/L is an infinite extension, union of finite extensions of L, then M/K is strictly deeply ramified if an only if the corresponding extension of norm fields Y/X is strictly deeply ramified, in which case the field of norms of M/K is isomorphic to the field of norms of Y/X.* This result is already known in the classical case ([5], Proposition 3.4.1). The starting observation here is that strictly deeply ramified extensions are APF, for the Abbès-Saito ramification filtration, and this allows us to translate the problem into a study of ramification properties of the extensions M/K and M/L. We then define strictly deeply ramified extensions of norm fields (in lack of a satisfactory definition in positive characteristic), and show how the ramification filtrations of the corresponding Galois groups behave via the field of norms functor. This allows us to conclude with a partial generalization of Wintenberger's result.

Notation

If K is a complete, discretely valued field, we denote by v_K its normalized discrete valuation, by $\mathcal{O}_K$ its ring of integers, by π_K a uniformizer, and by k_K its residue field. We fix a separable closure K^s of K, and denote by $\mathcal{O}$ the integral closure of $\mathcal{O}_K$ in K^s. All separable extensions L/K will implicitly be assumed to lie inside K^s, and any such gives rise to a unique valuation extending v_L, abusively denoted v_L, and an absolute value $|x|_L = \theta^{v_L(x)}$, where θ is a real number such that $0 < \theta < 1$, fixed throughout the paper. Let $a \in \mathbb{Q}_{>0}$; we denote by D_K^n, resp. $D_K^{n,(a)}$, the closed n-dimensional polydisc of radius one, resp. a, over K. When there is no possibility for confusion, we drop subscripts and write $\pi = \pi_K, v = v_K, |\cdot| = |\cdot|_K$ instead. Finally, for a separable extension L/K, we denote by $\Omega(L/K)$ the module of relative Kähler differentials of $\mathcal{O}_L/\mathcal{O}_K$.

2 The main constructions

2.1 The ramification filtration of Abbès and Saito

Let L/K be a finite separable extension of complete, discretely valued fields, and let $Z = (z_1, \ldots, z_n)$ be a system of generators of the $\mathcal{O}_K$-algebra $\mathcal{O}_L$. The kernel of the natural surjection $\mathcal{O}_K[X_1, \ldots, X_n] \twoheadrightarrow \mathcal{O}_L$ is finitely generated, say by the polynomials $f_1, \ldots, f_m$. For a rational number $a > 0$, the set

$$X_Z^a = \{x \in D_K^n \mid |f_i(x)|_K \leq |\pi_K|_K^a \ (i = 1, \ldots, m)\}$$

is an affinoid subdomain of D_K^n, and the set of connected components $\pi_0(X_Z^a)$ of its geometric points, $X_Z^a(K^s)$, with respect to the weak or strong G-topology, is finite and independent of the choice of Z ([1], Lemma 3.1).

If L/K is Galois, with Galois group G, then G acts on $\pi_0(X_Z^a)$ via the natural surjection $G \twoheadrightarrow \pi_0(X_Z^a)$ obtained by sending $\sigma \in G$ to the connected component of the point $\sigma(Z) = (\sigma(z_1), \ldots, \sigma(z_n)) \in X_Z^a(K^s)$. Let G^a be the subgroup of elements of G acting trivially on $\pi_0(X_Z^a)$. Clearly this is well-defined, i.e. independent of the choice of Z. If b is a rational number $\geq a$, then the inclusion $X_Z^b \subseteq X_Z^a$ induces an inclusion $G^b \subseteq G^a$, and the filtration $(G^a)_{a>0}$, extended by $G^0 = G$, is called the *ramification filtration of* G. It satisfies the "Herbrand property": if H is a subgroup of G, then $(G/H)^a = G^a H/H$ (*loc.cit.*, Proposition 2.1). Hence, we can define the ramification filtration of infinite Galois extensions M/K by setting $\mathrm{Gal}(M/K)^a = \varprojlim \mathrm{Gal}(L/K)^a$, the limit being taken over the finite Galois extensions L of K contained in M. For G finite and infinite, the ramification filtration of G enjoys the following properties:

1. It is left continuous, with rational jumps: if we set $G^{a+} = \overline{\bigcup_{b \in \mathbb{Q}_{>a}} G^b}$, and $G^{a-} = \overline{\bigcap_{b \in \mathbb{Q}_{<a}} G^b}$, then $G^{a-} = G^a$ if a is rational, and $G^{a-} = G^{a+}$ otherwise (*loc.cit* Theorem 3.8).
2. For $0 < a \leq 1$, G^a is the inertia subgroup of G, and G^{1+} is the wild inertia subgroup of G (*loc.cit*, Proposition 3.7).
3. It is exhaustive: $\bigcup_{a \geq 0} G^a = \{0\}$ (*loc.cit.*, Theorem 3.3).
4. It coincides, in the case where K has perfect residue field, with the classical upper numbering ramification filtration ([4], Chapter 4), shifted by one ([1], Proposition 3.7).

Example 2.1. Let K be a local field of mixed characteristic $(0, p)$, whose residue field k_K has a p-basis of cardinality $d \geq 1$ (cf. Section 2.2); for instance, we could take $K = \mathrm{Frac}(\mathbb{Z}_p[x_1, \ldots, x_d]^\wedge_{(p)})$. Let $\{t_1, \ldots, t_d\} \subset \mathcal{O}_K^\times$ be a lift of such a basis. Let $(\varepsilon_n)_{n \geq 0}$ be a compatible system of primitive p^n'th roots of unity, and, for each $\alpha = 1, \ldots, d$, let $(t_{\alpha,n})_{n\geq 0}$, be a compatible system of p^n'th roots of t_α. Assume for simplicity that $\varepsilon_1 \notin K$. Let $Z_n = (\varepsilon_n, t_{1,n}, \ldots, t_{d,n}) \in \mathcal{O}^n$, and set $K_n = K(Z_n)$, and $K_\infty = \bigcup K_n$. We have $G_n = \mathrm{Gal}(K_n/K) = (\mathbb{Z}/p^n\mathbb{Z})^\times \ltimes (Z/p^n\mathbb{Z})^d$ and $G = \mathrm{Gal}(K_\infty/K) = \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^d$. A presentation for $\mathcal{O}_{K_n}$ is given by $\mathcal{O}_{K_n} = \mathcal{O}_K[X, Y_1, \ldots, Y_d]/(f_n(X), g_{n,1}(Y_1), \ldots, g_{n,d}(Y_d))$, where $f_n(X)$ denotes the p^n'th cyclotomic polynomial, and where $g_{n,\alpha}(Y_\alpha) = Y_\alpha^p - t_{n,\alpha}$. Thus, for $a \in \mathbb{Q}_{>0}$, the affinoid variety $X^a_{Z_n}$ decomposes as a product $U^a_n \times V^a_{1,n} \times \ldots \times V^a_{d,n}$, where

$$\begin{aligned} U^a_n(K^s) &= \{x \in \mathcal{O} \mid |f_n(x)| \leq |\pi|^a\} \\ V^a_{\alpha,n}(K^s) &= \{x \in \mathcal{O} \mid |g_{n,\alpha}(x)| \leq |\pi|^a\}, \quad (1 \leq \alpha \leq d). \end{aligned}$$

An easy computation shows that two primitive p^n'th roots of unity, u_1 and u_2, belong to the same connected component of $U^a_n(K^s)$ if and only if $u_1^{p^{n-m+1}} = u_2^{p^{n-m+1}}$, where m is the smallest integer $\geq a$. Similarly, for $\alpha = 1, \ldots, d$, two p^n'th roots of $t_{\alpha,n}$, v_1 and v_2, belong to the same connected component of $V^a_{\alpha,n}(K^s)$ if and only if $v_1^{p^{n-m+1}} = v_2^{p^{n-m+1}}$, where m is the integer such that $m + 1/(p-1) < a \leq m + p/(p-1)$. Hence, if we let $G_n(i,j) = \mathrm{Gal}(K_n/K(\varepsilon_i)) \ltimes \mathrm{Gal}(K_n/K(\{t_{\alpha,j}\}))$, and $G(i,j) = \varprojlim_n G_n(i,j) \simeq (1 + p^i\mathbb{Z}_p) \ltimes (p^j\mathbb{Z}_p)$, we have $G^a_n = G_n$ and $G^a = G$, for $0 \leq a \leq 1$; $G^a_n = G_n(m, m-1)$ and $G^a = G(m, m-1)$ for $m < a \leq m + 1/(p-1)$ and $m = 1, \ldots, n$, resp. $m \geq 1$; $G^a_n = G_n(m,m)$ and $G^a = G(m,m)$ for $m + 1/(p-1) < a \leq m+1$ and $m = 1, \ldots, n$, resp. $m \geq 1$; and $G^a_n = \{1\}$ for $a \geq n+1$.

2.2 The field of norms functor of Scholl

Fix a prime p and an integer $d \geq 0$. Let K be a d-big local field of mixed characteristic $(0,p)$. Recall that this means that K is complete with respect to a discrete valuation, and that the residue field k_K has a p-basis of cardinality d. Let $K_\bullet = (K \subseteq K_1 \subseteq \ldots \subset K_\infty = \bigcup K_n)$ be a tower of finite extensions of d-big local fields satisfying the following condition: there exists an integer $n_0 \geq 0$ and an ideal $\xi \subset \mathcal{O}_{K_{n_0}}$ containing p such that

$$\begin{gathered}\text{For all } n \geq n_0,\ [K_{n+1} : K_n] = p^{d+1}, \text{ and there exists a} \\ \text{surjection } \Omega(K_{n+1}/K_n) \twoheadrightarrow (\mathcal{O}_{K_{n+1}}/\xi)^{d+1}.\end{gathered} \tag{$*$}$$

Such a tower is said to be *strictly deeply ramified*. [In order to specify an integer n_0 and an ideal ξ satisfying $(*)$, we refer to the triple $(K_\bullet, n_0, \xi)$ as being strictly deeply ramified.] For $n \geq n_0$, we have $e(K_{n+1}/K_n) = p$, $k^p_{K_{n+1}} = k_{K_n}$, and the Frobenius endomorphism of $\mathcal{O}_{K_{n+1}}/\xi$ induces a surjection $f : \mathcal{O}_{K_{n+1}}/\xi \to \mathcal{O}_{K_n}/\xi$ ([3], Proposition 1.2.1). The inverse limit

$$X^+_{K_\bullet} = \varprojlim_{n \geq n_0} (\mathcal{O}_{K_n}/\xi, f),$$

is a complete discrete valuation ring of characteristic p, with residue field $k' = \varprojlim_{n\geq n_0}(k_{K_n}, f)$ and uniformizer $\Pi = (\pi_{K_n} \mod \xi)_{n\geq n_0}$, for a suitable choice of uniformizers π_{K_n} of the fields K_n (*loc.cit.*, Theorem 1.3.2). It does not depend on the choice of pair (ξ, n_0) satisfying $(*)$, and if $K'_\bullet$ is a tower equivalent to $K_\bullet$ (i.e. if there exists $r \in \mathbb{Z}$ such that $K_n = K'_{n+r}$ for all n sufficiently large), then $K'_\bullet$ is also strictly deeply ramified, and $X^+_{K_\bullet} \simeq X^+_{K'_\bullet}$. Thus, denoting by $\mathcal{K}$ the equivalence class of $K_\bullet$, we put $X^+_{\mathcal{K}} = \varinjlim X^+_{K_\bullet}$, the limit being taken over all equivalent towers $K_\bullet \in \mathcal{K}$, and all pairs (ξ, n_0) satisfying $(*)$, and the transition maps being isomorphisms $X^+_{K_\bullet} \simeq X^+_{K'_\bullet}$, $K_\bullet, K'_\bullet \in \mathcal{K}$ (for an explicit description of these, see the proof of *loc.cit.*). The fraction field

$$X_{\mathcal{K}} = \mathrm{Frac}\ X^+_{\mathcal{K}}$$

is the *field of norms of* $\mathcal{K}$. In case $d = 0$, i.e. when K has perfect residue field, this coincides with the field of norms of the extension K_∞/K, as defined by Fontaine and Wintenberger ([5], Remark 2.2.3.3).

If L_∞ is a finite extension of K_∞, let L_0 be a finite extension of K such that $L_\infty = K_\infty L_0$. Let $L_n = K_n L_0$, and let $\mathcal{L}$ denote the equivalence class of the tower $L_\bullet$ (this class depends only on L_∞). If $\mathcal{K}$ is strictly deeply ramified, then so is $\mathcal{L}$ ([3], Theorem 1.3.3), and $X_{\mathcal{L}}$ is a finite separable extension of $X_{\mathcal{K}}$ (*loc.cit.*, Theorem 1.3.4). In this case, if L'_∞ is another finite extension of K_∞, and if $\mathcal{L}'$ denotes the corresponding equivalence class of towers, then a K_∞-homomorphism $\tau : L_\infty \to L'_\infty$ induces an injection $X^+_{\mathcal{L}} \hookrightarrow X^+_{\mathcal{L}'}$, and $X_{\mathcal{L}'}/X_{\mathcal{L}}$ is a separable extension of degree $[L'_\infty : \tau L_\infty]$ (*loc.cit.*, Theorem 1.3.4). The corresponding functor

$$L_\infty \mapsto X_{\mathcal{L}},$$

denoted $X_{\mathcal{K}}(-)$, in fact defines an equivalence between the category of finite extensions of K_∞ and the category of finite separable extensions of $X_{\mathcal{K}}$ (*loc.cit.*, Theorem 1.3.5). In particular, if L_∞/K_∞ is Galois, then so is $X_{\mathcal{L}}/X_{\mathcal{K}}$, and we have an isomorphism $\mathrm{Gal}(L_\infty/K_\infty) \simeq \mathrm{Gal}(X_{\mathcal{L}}/X_{\mathcal{K}})$.

3 Ramification theory for higher norm fields

3.1 Strictly deeply ramified extensions

Let K be a complete, discretely valued field. Following Fontaine and Wintenberger [5], we say that a separable extension L/K is *arithmetically profinite*, for short *APF*, if, for any rational $a \geq 0$, the group $G^a_K G_L$ is open in G_K, where $(G^a_K)_{a\geq 0}$ denotes the ramification filtration of G_K defined in Section 2.1. If K is a d-big local field of characteristic zero, for some integer $d \geq 0$, then a separable extension L/K is said to be *strictly deeply ramified* if it has a refinement by a tower $K \subseteq K_1 \subseteq \ldots \subset K_\infty = \bigcup K_n$ which is strictly deeply ramified (cf. Section 2.2).

For a separable extension L of K, let $c(L/K)$ denote the conductor of L/K, i.e. the rational $c \geq 0$ such that $G^c_K G_L/G_L \neq G^{c+}_K G_L/G_L = \{1\}$.

Lemma 3.1. *Suppose that K is a d-big local field of characteristic zero, for some integer $d \geq 0$. If L/K is strictly deeply ramified, then it is APF.*

Proof. In view of the Herbrand property for $(G^a)_{a\geq 0}$, and since $L=\bigcup K_n$, it suffices to show that, for any $a\geq 0$, there exists an integer $N\geq 0$ such that $G_K^a G_{K_N}/G_{K_N}$ is non-trivial. Thus it must be shown that the conductor $c(K_n/K)$ becomes arbitrarily large as $n\to\infty$. Possibly after replacing K by one of the finite extensions K_{n_0}, we may assume that the tower $K_\bullet$ satisfies $(*)$ for $n_0=0$. By hypothesis, and by the sequence below, the $\mathcal{O}_{K_n}$-module $\Omega(K_n/K)$, $n\geq 1$, is then generated by $d+1$ elements; write $\Omega(K_n/K)=\bigoplus_{i=0}^d \mathcal{O}_{K_n}/\pi_{K_n}^{\alpha_{n,i}}\mathcal{O}_{K_n}$. By [1], Proposition 7.3, we have $c(K_n/K)>\alpha_{n,i}/e(K_n/K)=\alpha_{n,i}/p^n$ for all i, and by induction on n using the conormal sequence for differentials

$$\mathcal{O}_{K_{n+1}}\otimes_{\mathcal{O}_{K_n}}\Omega_(K_n/K)\to\Omega(K_{n+1}/K)\to\Omega(K_{n+1}/K_n)\to 0,$$

we get $\alpha_{n,i}\geq np^n$, and hence $c(K_n/K)>n$. □

A separable extension M/N is said to be *elementary* if $G_N^{c(M/N)}G_M/G_N=G_N$, i.e. if $(G_N^aG_M/G_N)_{a\geq 0}$ has a single jump. In the theory of Fontaine-Wintenberger (perfect residue field), such extensions arise as the fixed fields of two successive ramification groups: if L/K is APF, if $(b_n)_{n\geq 0}$ denotes the ordered set of jumps of $(G_K^aG_L/G_L)_{a\geq 0}$ (i.e. $G_K^{b_n-}G_L/G_L=G_K^{b_n}G_L/G_L\neq G_K^{b_n+}G_L/G_L=G_K^{b_{n+1}}G_L/G_L$), and if κ_n denotes the fixed field of $G_K^{b_n}G_L/G_L$, then κ_{n+1}/κ_n is elementary, for all $n\geq 0$. This follows directly from the fact that κ_n/K is finite, and that, for a fixed n, the ramification filtration $(G_{\kappa_n}^aG_L/G_L)_{a\geq 0}$ is induced by the ramification filtration $(G_K^aG_L/G_L)_{a\geq 0}$ (although after renumbering). In the general case, the Abbès-Saito ramification filtration does not behave well with respect to subgroups, and the latter fact cannot be expected to be true; for a counterexample, see [2], Example p. 24. However, if L/K is strictly deeply ramified, we have the following:

Proposition 3.2. *Let K be as in Lemma 3.1, and let $K_\bullet$ be a strictly deeply ramified tower satisfying condition $(*)$ for $n_0=0$. Suppose furthermore that K_∞ is Galois over K, and let $G=\mathrm{Gal}(K_\infty/K)$. Let H be an open subgroup of G, and let $L=K_\infty^H$; then for each $a\in\mathbb{Q}_{\geq 0}$, $G^a\cap H=H^{ae}$, where e denotes the ramification index of L/K.*

Proof. Since H is open in G, there exists an integer $m\geq 0$ such that $K_m\supseteq L$; we may assume that K_m/K is Galois, with Galois group G_m; let $H_m=\mathrm{Gal}(K_m/L)$. By the Herbrand property, L is the fixed field of G_m^a. Since $k_{K_{n+1}}^p=k_{K_n}$, we fix a compatible system of lifts $(\{t_{\alpha,n}\}_{1\leq\alpha\leq d})_n$ of p-bases of the fields k_{K_n}, i.e. satisfying $t_{\alpha,n+1}\equiv t_{\alpha,n} \pmod{\pi_{K_n}}$, for $\alpha=1,\ldots,d$. We have $\mathcal{O}_{K_m}=\mathcal{O}_K[\pi_{K_m},\{t_{\alpha,m}\}_{1\leq\alpha\leq d}]$, for some uniformizer π_{K_m} of K_m ([3], (1.2.3)); furthermore, $\mathcal{O}_L=\mathcal{O}_K[\pi_{K_i},\{t_{\alpha,j}\}_{\alpha\in I,j\in J}]$, with $0\leq i\leq m$, $I\subseteq\{1,\ldots,d\}$, $J\subset\{0,\ldots,m\}$. For each $\alpha=1,\ldots,d$, $t_{\alpha,m}$ satisfies a fake Eisenstein polynomial over $\mathcal{O}_K$, i.e. a polynomial of the form $t^{p^m}+\sum_{i=1}^{p^{m-1}}a_it^i+t_{\alpha,0}$, with $\pi_K|a_i$. If L contains the element $t_{\alpha,j}$, for some $j=0,\ldots,m$, and no element $t_{\alpha,j'}$, for $j'>j$, then $t_{\alpha,m}$ satisfies the fake Eisenstein polynomial $t^{p^m}+\sum_{i=1}^{p^{m-1}}a_it^i+t_{\alpha,0}$ over $\mathcal{O}_L$.

Note that $K(t_{\alpha,j})/K$ is a totally fiercely ramified extension ([3], Section 1.1), so in particular the ramification index is one. Using a similar argument for π_{K_m} (which satisfies an Eisenstein polynomial over $\mathcal{O}_K$ and over $\mathcal{O}_L$), we deduce that, if Z_m, resp. Z_L, is a system of generators of $\mathcal{O}_{K_m}$ over $\mathcal{O}_K$, resp. $\mathcal{O}_L$, then $\pi_0(X^a_{Z_m}) = \pi_0(X^{ae}_{Z_L})$. Hence $G^a_m \cap H_m = H^{ae}_m$. The claim follows by passing to the limit. $\square$

Corollary 3.3. *Let K be as in Lemma 3.1, and let $K_\bullet$ be a strictly deeply ramified tower. Set $K_\infty = \bigcup K_n$, and denote by $(b_n)_{n\geq 0}$ the ordered set of jumps of the ramification filtration $(G^a_K G_{K_\infty}/G_{K_\infty})_{a\geq 0}$, and, for each n, let κ_n be the fixed field of $G^{b_n}_K G_{K_\infty}/G_{K_\infty}$. Then, for n sufficiently large, κ_{n+1}/κ_n is elementary.*

Proof. Since we are only concerned with the asymptotic behaviour of κ_n, and since κ_n/K is finite by Lemma 3.1, we may assume, possibly after truncating $K_\bullet$, that $K_\bullet$ satisfies $(*)$ for $n_0 = 0$. The claim now follows from Proposition 3.2. $\square$

Remark 3.4. If $K_\bullet$ is strictly deeply ramified, it is convenient to define a function $\psi_{K_\infty/K} : \mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ by

$$\psi_{K_\infty/K}(a) = e(L_a/K)a,$$

where L_a denotes the fixed field of $G^a_K G_{K_\infty}/G_{K_\infty}$ in K_∞; it is well-defined by Lemma 3.1. In view of Proposition 3.2, it generalizes the Herbrand ψ-function (as defined in [4], Chapter 4) for strictly deeply ramified towers.

Suppose that L/K is APF, and let $(\kappa_n)_{n\geq 0}$ be the tower of fixed fields of the ramification groups $G^a_K G_L/G_L$, $a \in \mathbb{Q}_{\geq 0}$. We say that L/K is *strictly APF* if there exists a real number $c > 0$ such that, for all n,

$$\frac{c(\kappa_{n+1}/\kappa_n)}{e(\kappa_{n+1}/K)} > c.$$

This generalizes the definition of [5], Section 1.4.1, since in the perfect residue field case, the extensions κ_{n+1}/κ_n, $n \geq 2$, are totally ramified.

Corollary 3.5. *If L/K is strictly deeply ramified, then it is strictly APF.*

Proof. Since the filtration $(G^a_K G_L/G_L)_{a\geq 0}$ is exhaustive, for each integer $n \geq 0$ there exists an integer $m \geq 0$ such that $\kappa_n \supset K_m$. The claim follows from Proposition 3.2 together with the lower bound of $c(K_m/K) > n$ determined in the proof of Lemma 3.1. $\square$

Remark 3.6. The group $G^{1+}_K G_L/G_L$ is pro-p, and hence if L/K is APF, the extensions κ_{n+1}/κ_n, $n \geq 2$, are p-extensions. We therefore expect a converse of Corollary 3.5 to be true as well, under some assumptions (for instance, k_L must be the perfect closure of k_{κ_n}, for n sufficiently large). If k_K is perfect, this is already the case: assuming κ_{n+1}/κ_n is Galois, we have $\mathrm{Gal}(\kappa_{n+1}/\kappa_n) \simeq (\mathbb{Z}/p\mathbb{Z})^{d_n}$, for some $d_n \geq 1$ ([4], Chapter IV, Corollary 3 to Proposition 7). For

$i = 0, \ldots, d_n$, letting $K_{n,i}$ denote the subfield of κ_{n+1} fixed by $(\mathbb{Z}/p\mathbb{Z})^{d_n - i}$, the Hilbert formula (*loc.cit.*, Proposition 4) gives the equality $v_p(\mathfrak{D}_{K_{n,i+1}/K_{n,i}}) = c(K_{n,i+1}/K_{n,i}) \cdot (p-1)/e_K \cdot [K_{n,i+1} : K]$, where $\mathfrak{D}_{K_{n,i+1}/K_{n,i}}$ denotes the different of the extension $K_{n,i+1}/K_{n,i}$ (the annihilator of $\Omega(K_{n,i+1}/K_{n,i})$). Using the strict APF condition, it follows that L/K is strictly deeply ramified.

3.2 Extensions of norm fields

In this section, we assume that K is a d-big local field of characteristic zero, for some integer $d \geq 0$. Let $K_\bullet = (K \subseteq K_1 \subseteq \ldots \subset K_\infty = \bigcup K_n)$ be a strictly deeply ramified tower, and denote by $\mathcal{K}$ its equivalence class (cf. Section 2.2). Let $X = X_{\mathcal{K}}$ be the field of norms of $\mathcal{K}$. Recall that to a finite separable extension Y/X there is a unique finite extension L_∞ of K_∞ with $X_{\mathcal{K}}(L_\infty) = Y$. Let $X_\bullet = (X \subseteq X_1 \subseteq \ldots \subset X_\infty = \bigcup X_n)$ be a tower of finite separable extensions of X. For each $m \geq 0$, let $L_{\infty,m}$ be the separable extension of K_∞ such that $X_{\mathcal{K}}(L_{\infty,m}) = X_m$, and, for some finite extension $L_{0,m}$ of K, let $L_{\bullet,m} = (K_n L_{0,m})_{n\geq 0}$ be the corresponding strictly deeply ramified tower. Let $M_\infty = \bigcup_m L_{\infty,m}$, and note that if M_∞/K_∞ is Galois, we have an isomorphism $\mathrm{Gal}(M_\infty/K_\infty) \simeq \mathrm{Gal}(X_\infty/X)$. We say that $X_\bullet$ is *strictly deeply ramified* if there exists integers $n_0, m_0 \geq 0$ and ideals $\xi \subset \mathcal{O}_{K_{n_0}}$ and $\xi' \subset \mathcal{O}_{L_{m_0}}$ such that $(K_\bullet, n_0, \xi)$ and $(L_{n_0,\bullet}, m_0, \xi')$ are strictly deeply ramified. This is well-defined, since the property holds independently of the choice of $L_{0,i}$, and since for any $n \geq n_0$, $(L_{n,\bullet}, m_0, \xi')$ is also strictly deeply ramified. [We refer to the tuple $(X_\bullet, n_0, m_0, \xi, \xi')$ as being strictly deeply ramified.] Clearly, for M sufficiently large, the truncated and renumbered tower $((X_{m+M})_{m\geq 0}, n_0, m'_0, \xi, \xi')$ is strictly deeply ramified with $m'_0 = 0$, so that in fact ξ' is an ideal $\mathcal{O}_{K_{n_0}}$.

Theorem 3.7. *Suppose that M_∞/K is Galois, and that $(X_\bullet, n_0, m_0, \xi, \xi')$ is strictly deeply ramified, with $n_0 = m_0 = 0$ and $\xi = \xi'$. Set $G = \mathrm{Gal}(M_\infty/K)$, $H = \mathrm{Gal}(M_\infty/K_\infty)$, and $\mathcal{H} = \mathrm{Gal}(X_\infty/X)$. Equip H with the filtration induced by the the ramification filtration of $\mathcal{H}$ via the isomorphism $H \simeq \mathcal{H}$. Then, for all $a \in \mathbb{Q}_{\geq 0}$, $G^a \cap H = H^{\psi_{K_\infty/K}(a)}$.*

Proof. Let $b = \psi_{K_\infty/K}(a)$, and let $\sigma \in H$. We must show that $\sigma \in H^b$ if and only if $\sigma \in G^a$. Since X_∞/X is Galois, we may assume that X_i/X is Galois, for all $i \geq 0$; set $H_{n,i} = \mathrm{Gal}(L_{n,i}/K_n)$, $H_n = \mathrm{Gal}(L_{n,\infty}/K_n)$, $G_n = \mathrm{Gal}(L_{n,\infty}/K)$, and $\mathcal{H}_i = \mathrm{Gal}(X_i/X)$. By Proposition 3.2, and by the Herbrand property, we have $G_n^a \cap H_n = H_n^b$. Since $G^a = \varprojlim G_n^a$, and $H_n \simeq H$, the claim follows from the following lemma.

Lemma 3.8. *For all $n \geq 0$, the isomorphisms $X_{\mathcal{K}}(-) : \mathcal{H}_{n,i} \to H_i$ preserve the ramification filtration, i.e. $X_{\mathcal{K}}(\mathcal{H}_{n,i}^a) = H_i^a$, for all $a \in \mathbb{Q}_{\geq 0}$.*

Proof. Fix a set of generators $Z_{n,i}$ for the $\mathcal{O}_{K_n}$-algebra $\mathcal{O}_{L_{n,i}}$, and, for a fixed a, let $X_{n,i}^a = X_{Z_{n,i}}^a$ be the corresponding affinoid subdomain of $D_{K_n}^{d+1}$, as defined in Section 2.1. Let $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ be a system of points $x_n, y_n \in \mathcal{O}^{d+1}$ such that, for all n, x_n and y_n belong to $X_{n,i}^a$, and such that $x_{n+1}^p \equiv x_n \pmod{\xi}$, and $y_{n+1}^p \equiv y_n \pmod{\xi}$.

Claim. *If x_0 and y_0 belong to the same connected component of $X^a_{0,i}$, then x_m and y_m belong to the same connected of $X^a_{m,i}$, for all $m \geq 0$.*

Proof. Let I_n denote the kernel of the natural surjection $\mathcal{O}_{K_n}[X_1, \dots, X_{d+1}] \to \mathcal{O}_{L_n,i}$ defined by $Z_{n,i}$; in fact we can assume that $I_n = (f_{n,1}, \dots, f_{n,d+1})$, since $\mathcal{O}_{L_n,i}$ is a complete intersection over $\mathcal{O}_{K_n}$ ([1], Lemma 7.1). Let $f^a_n : D^{d+1}_{K_n} \to D^{d+1}_{K_n}$ be the map defined by the polynomials $f_{n,1}, \dots, f_{n,d+1}$, and let $\varphi^a_n : X^a_{Z_{n,i}} \to D^{d+1,(a)}_{K_n}$ be the induced map. Since $e(K_{n+1}/K_n) = p$, Frobenius induces a surjection $D^{d+1,(a)}_{K_{n+1}} \to D^{d+1,(a)}_{K_n} = D^{d+1,(pa)}_{K_{n+1}}$, and the result follows since $\varphi^a_n(x^p) \equiv \varphi^a_{n+1}(x)^p \pmod{\xi}$. □

The Lemma now follows by taking $\sigma \in H^a_i$, $x_0 = Z_{0,i}$ and $y_0 = \sigma(x_0)$; this also completes the proof of the Theorem. □

Corollary 3.9. *Let M_∞/K_∞ be an infinite extension, union of finite extensions $L_{\infty,n}$ of K_∞, and let $X_\mathcal{M}/X$ be the corresponding extension of norm fields. If M_∞/K is strictly deeply ramified, then $X_\mathcal{M}/X$ is strictly APF. Conversely, if $X_\mathcal{M}/X$ is strictly deeply ramified, then M_∞/K is strictly APF.*

Proof. Possibly after truncating towers, we may throughout assume that strictly deeply ramified conditions are satisfied at $n_0 = 0$. Let M'_∞ be a Galois extension of K such that M'_∞/M_∞ is finite, and set $G = \mathrm{Gal}(M'_\infty/K)$, $H = \mathrm{Gal}(M'_\infty/K_\infty)$, $H' = \mathrm{Gal}(M'_\infty/M_\infty)$. By Theorem 3.7, if $(H^a)_{a\geq 0}$ is the filtration induced by the ramification filtration of $\mathrm{Gal}(X_\mathcal{M}/X)$, then, for all $a \in \mathbb{Q}_{\geq 0}$, $G^a \cap H = H^{\psi_{K_\infty/K}(a)}$, and hence $[G : H'G^a] = [G : HG^a][H : H^{\psi_{K_\infty/K}(a)}H']$. The first part follows from Lemma 3.1, since K_∞/K and M_∞/K are strictly deeply ramified, hence APF. For the second part, it suffices to show that $X_\mathcal{M}/X$ is APF, which is immediate from the isomorphism $\mathrm{Gal}(X_\mathcal{M}/X) \simeq \mathrm{Gal}(L^\infty_0/K)$ preserving the ramification filtration (cf. the proof of Theorem 3.7) and the fact that M^∞_0/K is strictly deeply ramified, hence strictly APF. □

Remark 3.10. Since k_{K_∞} is perfect, we may assume, possibly after truncation of the tower $X_\bullet$, that $k_{M_\infty} = k_{K_\infty}$, the perfect closure of k_{K_n}, for n sufficiently large. Hence, as remarked in 3.6, we could expect that M_∞/K is strictly deeply ramified if and only if $X_\mathcal{M}/X$ is strictly deeply ramified, and that the field of norms of M_∞/K is identified with the field of norms of $X_\mathcal{M}/X$, for a suitable definition of the latter, an obvious candidate for this being the projective limit of the field of norms of the towers $L_{n,\bullet}$.

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