

# Approximating planets' orbits in closed-form

<b>Description</b>	A sufficiently accurate and efficient model for planetary orbits
<b>Period</b>	Fall 2014
<b>Languages &amp; Libs</b>	Matlab
<b>Tags</b>	Astronomy Applied mathematics



I wanted to find or create a formula which would accept an epoch timestamp, latitude and longitude and it would produce the Sun's observed azimuth and altitude in radians. It needs to take into account details earth's axial tilt and its position on its orbit around the sun. To my surprise I wasn't able to find such formula, so I had to develop it from scratch. Luckily earth's orbit (and orbits in general) is a well studied and documented problem, so I could take some shortcuts.

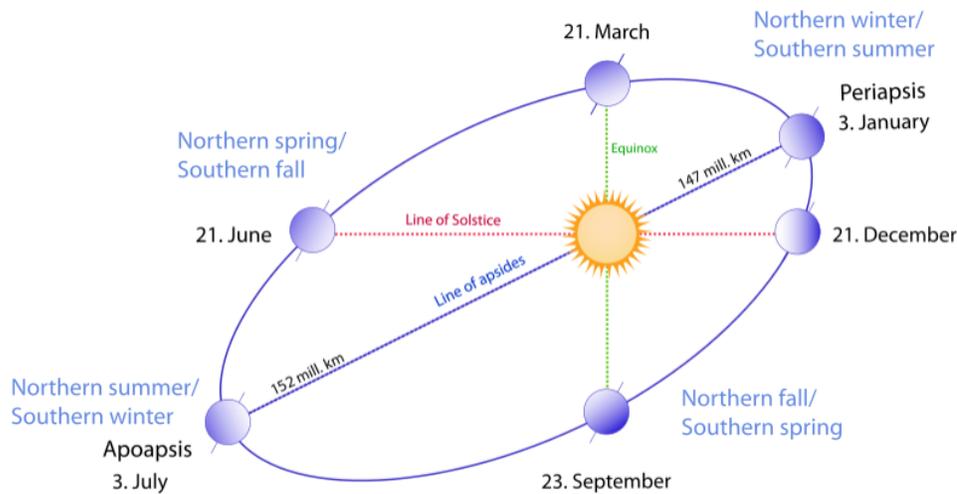


Figure 1: An illustration of earth's orbit and its relation to calendar day (from [Wikimedia commons](#)).

[Kepler's equation](#) does not have an analytical solution, but it is easy apply in the "opposite" direction. Here an approximate solution is created. First an even sampling of true anomalies is generated, covering the whole  $360^\circ$  orbit rotation:

$$\theta = 0 \dots 2\pi \tag{1}$$

Next corresponding eccentric anomaly  $E$  is calculated from  $\theta$  and the orbit's eccentricity  $e$ :

$$E = \text{atan2} \left( \sqrt{1-e} \cdot \sin(\theta/2), \sqrt{1+e} \cdot \cos(\theta/2) \right) \tag{2}$$

Mean anomaly  $M$  is calculated from  $E$  by using Kepler's equation:

$$M = E - e \sin E \tag{3}$$

The  $M$  is trivial to calculate for any desired calendar date, so we want to have an approximation formula  $\theta = f_e(M)$  so that previous equation hold reasonably well. I noticed experimentally that it would be a good approach to approximate

$$f'_e(M) = (M - \theta) / e \tag{4}$$

for  $e > 0$ , and has a somewhat constant shape for small values of  $e$ . Then the following holds:

$$\theta = f_e(M) = M - e \cdot f'_e(M) \tag{5}$$

The effect of  $e$  on  $f'_e(M)$  shape can be seen in Figure 2. Most importantly its magnitude is around  $\pm 2$ , and has values of exactly zero at  $M = 0$ ,  $M = \pi$  and  $M = 2\pi$ . Due to these properties, it is approximated by the polynomial

$$f'_e(M) \approx a(e) \cdot (M - \pi) \cdot (M \cdot (2\pi - M))^{-b(e)} \tag{6}$$

The shape of  $f'_e(M)$  for various values of  $e$  can be seen in Figure 2. It is notable that for  $M \approx \pi/2$  it has a value of about -2, which indicates that  $\theta > M$  around that region. This occurs because when  $M \approx 0$  the planet is closest to the Sun, which means it has a higher angular velocity. Thus  $\theta$  grows initially faster than the mean anomaly  $M$ .

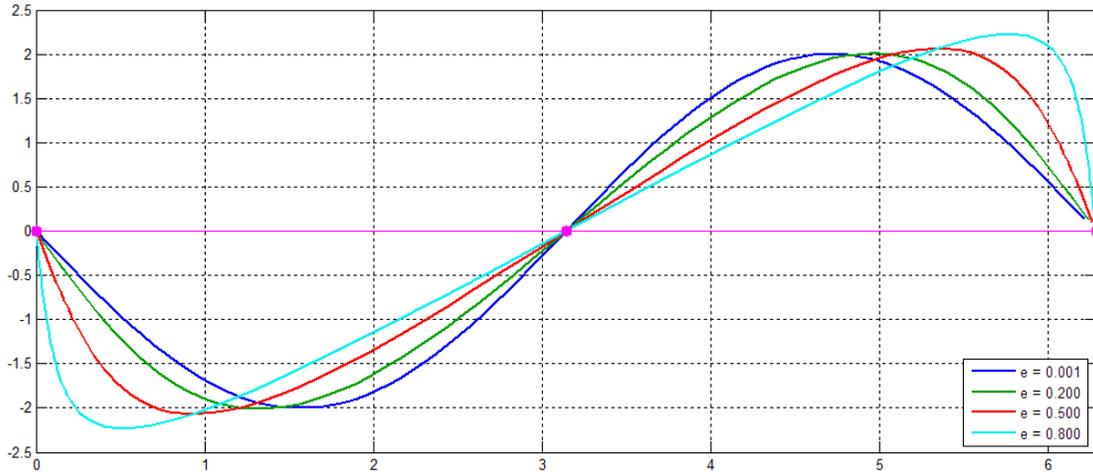


Figure 2: A plot of  $f'_e(M)$  for various values of  $e$ . X-axis has the value of  $M$  from 0 to  $2\pi$ .

The eccentricity  $e$ 's role on  $a(e)$  and  $b(e)$  is solved by finding optimal  $a$  and  $b$  for various different values of  $e$  and fitting a model to those findings. Parameters are estimated by minimizing the standard sum of squared errors, which is achieved by Matlab's `lsqnonlin` function. These results are shown in Figure 3. Luckily both parameters seem to follow the same formula  $y(e) = y_0 + \exp(k_1 \cdot \log(e) + k_2)$ , and  $k_1$  and  $k_2$  can be reliably estimated by fitting a linear trend to the log-log data points. The constant  $y_0$  (which corresponds to  $a_0$  and  $b_0$ , one for both parameters) is estimated so that the best log-linear model can be obtained. If that value is off then the log-log plot is no longer linear, and a more complex model is needed.

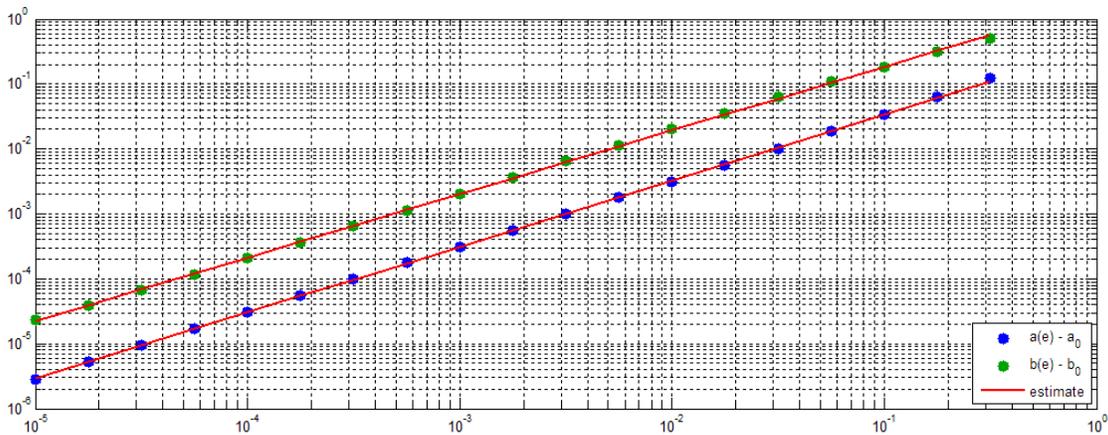


Figure 3: Estimating  $a(e)$  and  $b(e)$  from sampled models.

Model accuracy is determined by calculating the maximum error from the true location within the whole  $360^\circ$  orbit. If  $e = 0$  then the orbit would be exactly circular and  $\theta = M$ . Interesting cases are small values of  $e$ , for example on earth's orbit  $e \approx 0.0167$ . Two different models are compared in figures 5 and 6: a simpler model which only uses constants  $a_0$  and  $b_0$  regardless of the value of  $e$ , and the more accurate model with  $e$ -dependent corrections.

The biggest benefit of these corrections occur when  $0.05 < e < 0.3$ . When  $e \approx 0$  these corrections are small anyway, and on bigger values of  $e$  the model loses its accuracy even with these corrections. The ratio of simple model's maximum error and better model's error is show in Figure 4. When  $e = 0.1$  the maximum ratio of 4 is reached, which means that the simpler model has 300% higher maximum error than the  $e$ -corrected version.

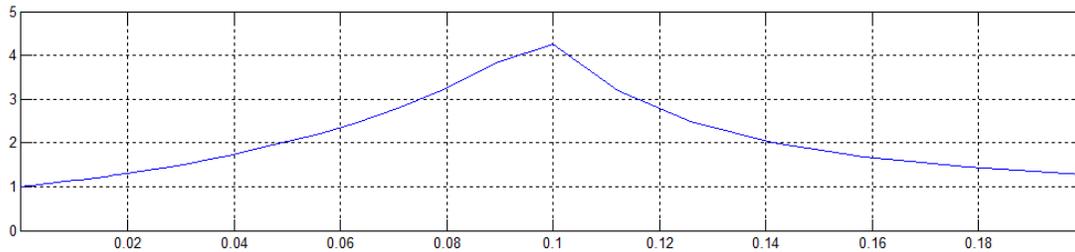


Figure 4: The ratio of simpler model's error to  $e$ -corrected version's error for varying values of  $e$ . Biggest benefit of the more advanced model is gained when  $e \approx 0.1$ .

Error rates (in degrees) and the fitted trend curve for  $0 < e < 0.1$  are shown in Figure 5. Within this region the error seems to grow approximately linearly in respect to  $e$ , and for Earth's  $e \approx 0.0167$  the maximum error is less than  $0.1^\circ$ . It should be sufficient for most applications.

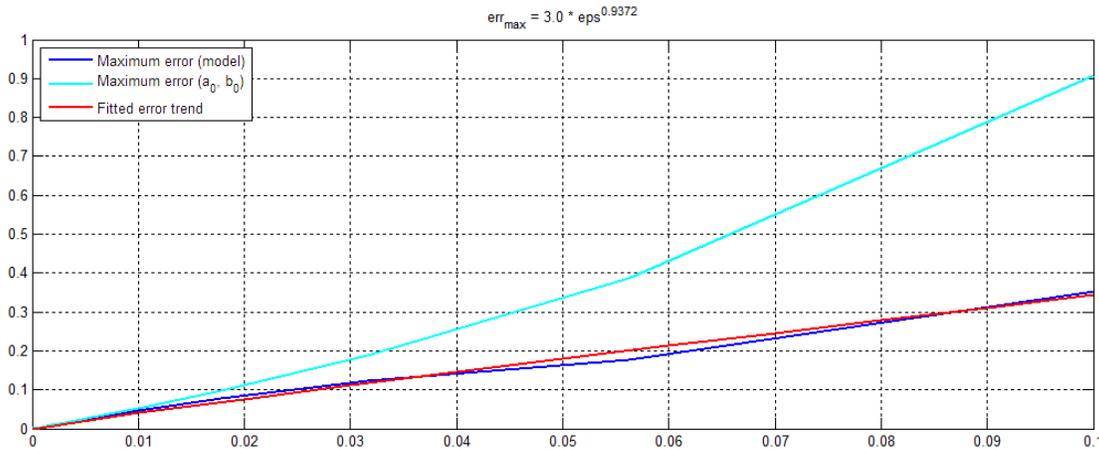


Figure 5: A trend of the maximum error (in degrees) of models for different small values of  $e$ .

Error rates for  $e$ -values of up-to  $10^{-0.5} \approx 0.3$  are shown in Figure 6. In this context the error grows in proportion to  $e^{3.09}$ , reaching a maximum error of  $6^\circ$  at  $e = 0.3$ . It might be too high for most applications, but this elliptic orbits are not common in at least our solar system. Our top-3 planets with highest eccentricities are Mercury ( $e = 0.21$ ), Mars ( $e = 0.09$ ) and Saturn ( $e = 0.05$ ). Pluto has a remarkably high eccentricity of 0.25.

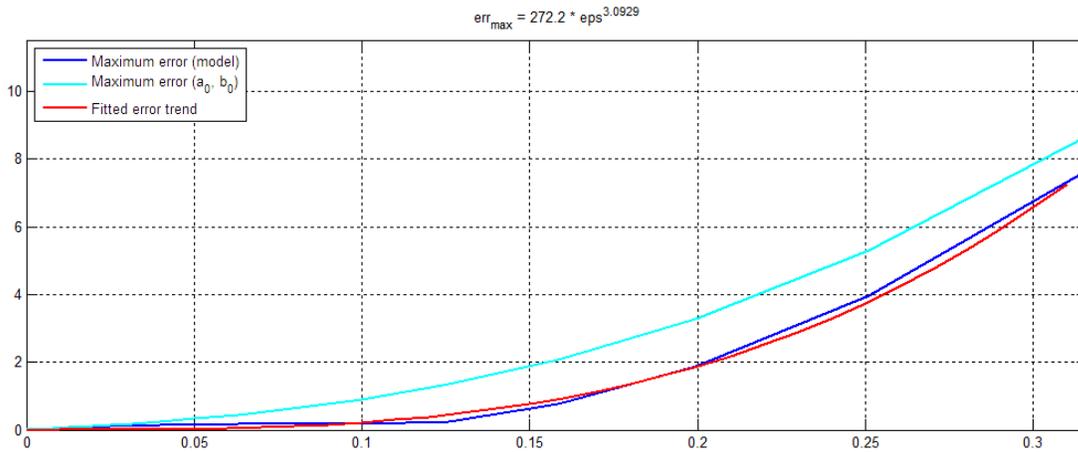


Figure 6: A trend of the maximum error (in degrees) of models for different large values of  $e$ .

After this model is in place, it enables us to calculate Earth's location in respect to Sun in polar coordinates for a given timestamp. It just needs to be converted to "seconds since January 1 of this year", which is further converted to the mean anomaly  $M$  (in radians). True anomaly  $\theta$  is calculated as

$$\begin{aligned}\theta_e(M) &= M - e \cdot a(e) \cdot (M - \pi) \cdot (M \cdot (2\pi - M))^{-b(e)} \\ a(e) &= a_0 + \exp(a_1 \cdot \log(e) + a_2) \\ b(e) &= b_0 + \exp(b_1 \cdot \log(e) + b_2)\end{aligned}\quad (7)$$

Estimated parameters are:

$$\begin{aligned}a_0 &= 0.0804 \\ a_1 &= 0.9869 \\ a_2 &= 1.0123 \\ b_0 &= -1.3848 \\ b_1 &= 0.6143 \\ b_2 &= -1.0715\end{aligned}\quad (8)$$

The next step would be to estimate the local coordinate system from latitude, longitude and time, and to use polar coordinates to calculate the person's orientation with respect to the heliocentric coordinate system where periapsis (around 3.1. each year) is the "zero" direction. Then the Sun's apparent altitude and azimuth can be easily calculated.