

# Transreal Limits and Elementary Functions

Tiago S. dos Reis, James A. D. W. Anderson

**Abstract** We extend all elementary functions from the real to the transreal domain so that they are defined on division by zero. Our method applies to a much wider class of functions so may be of general interest.

**Key words:** transreal continuity, transreal elementary functions, transreal limits, transreal numbers, transreal sequences, transreal series.

## 1 Introduction

The set of real numbers,  $\mathbb{R}$ , is extended to the set of transreal numbers,  $\mathbb{R}^T$ , by the addition of three, definite, non-finite numbers: negative infinity,  $-\infty = -1/0$ ; positive infinity,  $\infty = 1/0$ ; and nullity,  $\Phi = 0/0$ . These three, non-finite numbers are called *strictly transreal* numbers. Nullity was introduced in [1]. Transreal arithmetic is axiomatised and proved consistent by machine proof in [4]. A construction of the transreal numbers from the real numbers is given in [5]. That construction provides a human proof of the consistency of transreal arithmetic. Transreal limits are given in [3]. Transreal derivatives are given in [6].

Here we consider transreal limits and extend elementary functions of real numbers to transreal numbers. In [3] we define a topology on  $\mathbb{R}^T$  which extends the topology of real numbers. Because of this we can define, in a rigorous way, limits of sequences, series and functions and continuity of functions on  $\mathbb{R}^T$  so that wher-

---

Tiago S. dos Reis

Federal Institute of Education, Science and Technology of Rio de Janeiro; 27215-350; Brazil and Program of History of Science, Technique and Epistemology; Federal University of Rio de Janeiro; 21941-916; Brazil; e-mail: tiago.reis@ifrj.edu.br

James A. D. W. Anderson

School of Systems Engineering; University of Reading; Whiteknights; Reading; England; RG6 6AY; e-mail: j.anderson@reading.ac.uk

ever real numbers occur in real limits, they occur identically in transreal limits and wherever infinities occur as symbols in extended real limits, they occur identically in transreal limits but as definite numbers. This means that transreal topology agrees with the usual, real topology.

We extend every elementary function to the transreal domain. Some of these functions, such as the exponential and trigonometric functions, were defined, in [2], at strictly transreal numbers, motivated by power series. However the convergence of series in [2] is taken in an intuitive way, without any rigorous definition of the limit of sequences and series on  $\mathbb{R}^T$ . That development also lacks a topology on  $\mathbb{R}^T$ . Here we also use power series but first we extended some results about series to  $\mathbb{R}^T$  using its well-defined topology [3].

## 2 Topological Foundations

In [3] we define a topology, limit of sequences and limit and continuity of functions on  $\mathbb{R}^T$ . Below we summarise this content.

**Definition 1** *A set is open on  $\mathbb{R}^T$  if and only if it is composed of arbitrarily many unions of finitely many intersections of the following four kinds of interval:*

- i)  $(a, b)$  where  $a, b \in \mathbb{R}$ ,
- ii)  $[-\infty, b)$  where  $b \in \mathbb{R}$ ,
- iii)  $(a, \infty]$  where  $a \in \mathbb{R}$  and
- iv)  $\{\Phi\}$ .

The reader can verify that these open sets do, in fact, make a topology on  $\mathbb{R}^T$ .

**Proposition 1**  *$\mathbb{R}^T$  is a Hausdorff, disconnected, separable, compact and completely metrisable space. For definitions of these terms and proofs see [3] and a forthcoming paper [7].*

Notice that  $\Phi$  is the unique isolated point of  $\mathbb{R}^T$ .

**Proposition 2** *The topology on  $\mathbb{R}$ , induced by the topology of  $\mathbb{R}^T$ , is the usual topology of  $\mathbb{R}$ . That is if  $A \subset \mathbb{R}^T$  is open on  $\mathbb{R}^T$  then  $A \cap \mathbb{R}$  is open (in the usual sense) on  $\mathbb{R}$  and if  $A \subset \mathbb{R}$  is open (in the usual sense) on  $\mathbb{R}$  then  $A$  is open on  $\mathbb{R}^T$ .*

Remember the definition of a sequence. A sequence in  $\mathbb{R}^T$  is a function  $x : \mathbb{N} \rightarrow \mathbb{R}^T$ . We customarily write  $x_n$  in place of  $x(n)$  and write  $(x_n)_{n \in \mathbb{N}}$  in place of  $x : \mathbb{N} \rightarrow \mathbb{R}^T$ . We use the usual definition for the convergence of a sequence in a topological space. That is a sequence,  $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$ , converges to  $x \in \mathbb{R}^T$  if and only if for each neighbourhood,  $V \subset \mathbb{R}^T$  of  $x$ , there is  $n_V \in \mathbb{N}$  such that  $x_n \in V$  for all  $n \geq n_V$ . Notice that since  $\mathbb{R}^T$  is a Hausdorff space, the limit of a sequence, when it exists, is unique.

**Remark 1** *Let  $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  and let  $L \in \mathbb{R}$ . Notice that  $\lim_{n \rightarrow \infty} x_n = L$  in  $\mathbb{R}^T$  if and only if  $\lim_{n \rightarrow \infty} x_n = L$  in the usual sense in  $\mathbb{R}$ . Furthermore,  $(x_n)_{n \in \mathbb{N}}$  diverges, in*

the usual sense, to negative infinity if and only if  $\lim_{n \rightarrow \infty} x_n = -\infty$  in  $\mathbb{R}^T$ . Similarly  $(x_n)_{n \in \mathbb{N}}$  diverges, in the usual sense, to infinity if and only if  $\lim_{n \rightarrow \infty} x_n = \infty$  in  $\mathbb{R}^T$ .

**Remark 2** Let  $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$ . Notice that  $\lim_{n \rightarrow \infty} x_n = \Phi$  if and only if there is  $k \in \mathbb{N}$  such that  $x_n = \Phi$  for all  $n \geq k$ .

**Proposition 3** Every monotone sequence of transreal numbers is convergent.

**Proposition 4 (Transreal version of the Bolzano-Weierstrass Theorem)** Every sequence of transreal numbers has a convergent subsequence.

### 3 Transreal series

In this section we extend some results on series to the transreal domain.

**Definition 2** Let  $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$ . For each  $n \in \mathbb{N}$ , we define

$$s_n := \sum_{i=1}^n x_i = x_1 + \cdots + x_n.$$

The sequence  $(s_n)_{n \in \mathbb{N}}$  is called a series and is denoted by  $\sum x_n$ , each  $s_n$  is called a partial sum of  $\sum x_n$  and  $x_n$  is called the  $n$ -th term of  $\sum x_n$ . We say that  $\sum x_n$  converges or is convergent if and only if there is the  $\lim_{n \rightarrow \infty} s_n$ . Otherwise,  $\sum x_n$  diverges or is divergent. When  $\sum x_n$  is convergent we denote

$$\sum_{n=1}^{\infty} x_n := \lim_{n \rightarrow \infty} s_n.$$

**Definition 3** Let  $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^T$ . We say that a series  $\sum x_n$  converges absolutely or is absolutely convergent if and only if  $\sum |x_n|$  is convergent.

Customarily, in a calculus course, the first example of a convergent series is the geometric series,  $\sum r^n$ . Recall that  $\sum r^n$  converges in  $\mathbb{R}$  if and only if  $r \in (-1, 1)$  and, in this case,  $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ .

**Example 1** Let  $r \in \mathbb{R}^T$ . The series  $\sum r^n$  converges in  $\mathbb{R}^T$  if and only if  $r \in \{-\infty, \Phi\} \cup (-1, \infty]$ . Indeed, if  $r \in \{-\infty, \Phi\}$  then  $\sum_{n=1}^{\infty} r^n = \Phi$ , if  $r \in (-1, 1)$  then  $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ , if  $r \in [1, \infty]$  then  $\sum_{n=1}^{\infty} r^n = \infty$  and if  $r \in (-\infty, -1]$  then  $s_{2n-1} < 0$  and  $s_{2n} \geq 0$  for all  $n \in \mathbb{N}$  whence  $\sum r^n$  diverges.

**Remark 3** In the real domain we know that if  $\sum x_n$  is convergent then  $\lim_{n \rightarrow \infty} x_n = 0$ . In the transreal domain there is no similar result. That is, we can have a convergent series,  $\sum x_n$ , such that  $(x_n)_{n \in \mathbb{N}}$  converges to anywhere in  $\mathbb{R}^T$ . Indeed, if  $a \in \{\Phi, -\infty, \infty\}$  and  $x_n = a$  for all  $n \in \mathbb{N}$  then  $\sum x_n$  is convergent (to  $a$ ) and  $\lim_{n \rightarrow \infty} x_n = a$ . If  $a \in \mathbb{R}$  is positive, let  $x_n = a - \frac{1}{n^2}$  for each  $n \in \mathbb{N}$ . Remember that

$\sum \frac{1}{n^2}$  converges increasingly to  $\frac{\pi^2}{6}$  whence  $\sum_{i=1}^n \frac{1}{i^2} < \frac{\pi^2}{6}$  for all  $n \in \mathbb{N}$ . Given an arbitrary positive  $M \in \mathbb{R}$ , there is  $n_M \in \mathbb{N}$  such that  $n_M a > M + \frac{\pi^2}{6}$ . Hence if  $n \geq n_M$  then  $na \geq n_M a > M + \frac{\pi^2}{6} > M + \sum_{i=1}^n \frac{1}{i^2}$  whence  $\sum_{i=1}^n x_n = na - \sum_{i=1}^n \frac{1}{i^2} > M$ , that is,  $\sum_{i=1}^n x_n \in (M, \infty]$ . Thus  $\sum x_n$  is convergent (to  $\infty$ ) and  $\lim_{n \rightarrow \infty} x_n = a$ . If  $a \in \mathbb{R}$  is negative, let  $x_n = a + \frac{1}{n^2}$  for each  $n \in \mathbb{N}$  then, by a similar argument, we have that  $\sum x_n$  is convergent (to  $-\infty$ ) and  $\lim_{n \rightarrow \infty} x_n = a$ .

**Remark 4** In the real domain, we know that a series  $\sum x_n$  is convergent if and only if for each positive  $\varepsilon \in \mathbb{R}$  there is  $n_\varepsilon \in \mathbb{N}$  such that

$$\left| \sum_{i=m}^n x_i \right| < \varepsilon \quad (1)$$

whenever  $n \geq m \geq n_\varepsilon$ . This is nothing more than the application of the Cauchy criterion. As  $\mathbb{R}$  is a complete metric space, a sequence is convergent if and only if it is Cauchy. The inequality (1) is, in fact,  $|s_n - s_{m-1}| < \varepsilon$ . We can naturally enunciate that result in the transreal domain by using its complete metric, denoted as  $d$ . A series  $\sum x_n$  is convergent if and only if for each positive  $\varepsilon \in \mathbb{R}$  there is  $n_\varepsilon \in \mathbb{N}$  such that  $d(s_n, s_{m-1}) < \varepsilon$  whenever  $n \geq m \geq n_\varepsilon$ . This is correct but is not as helpful as in the real domain because, unlike in real domain, the fact that  $d(s_n, s_{m-1})$  is less than an arbitrary, positive, real number is not equivalent to the fact that  $|\sum_{i=m}^n x_i|$  is less than some positive, real number.

We can enunciate the Cauchy criterion, in the real domain, in another way. A series  $\sum x_n$  is convergent if and only if for each neighbourhood  $V$  of zero there is  $n_V \in \mathbb{N}$  such that  $\sum_{i=m}^n x_i \in V$  whenever  $n \geq m \geq n_V$ . In this way we might imagine that there is a similar result for the transreal domain. For example, let  $x_n = n^2$  for all  $n \in \mathbb{N}$ . The series  $\sum x_n$  is convergent (to  $\infty$ ) and for each neighbourhood  $V$  of infinity there is  $n_V \in \mathbb{N}$  such that  $\sum_{i=m}^n x_i \in V$  whenever  $n \geq m \geq n_V$ . However this is not always the case. Let  $x_n = 1$  for all  $n \in \mathbb{N}$ . The series  $\sum x_n$  is convergent (to  $\infty$ ), but for all  $n \in \mathbb{N}$ ,  $\sum_{i=n}^n x_i = 1$  does not belong to neighbourhood of infinity  $(M, \infty]$  where  $M > 1$ .

We might wish to enunciate, in the transreal domain, something like: if the series  $\sum x_n$  is convergent then either  $\sum_{i=m}^n x_i$  belongs to a neighbourhood of zero for  $n$  and  $m$  large enough or  $\sum_{i=m}^n x_i$  belongs to a neighbourhood of infinity for  $n$  and  $m$  large enough or else  $(x_n)_{n \in \mathbb{N}}$  is constant. But this is still not true. Let  $x_n = \frac{1}{n}$ . The series  $\sum x_n$  is convergent (to  $\infty$ ) but does not satisfy any of the three conditions above. In fact, let there be arbitrary, positive  $\varepsilon, M \in \mathbb{R}$  such that  $\varepsilon < M$ . There is  $n \in \mathbb{N}$  such that  $\sum_{i=n}^n x_i = \frac{1}{n} \in (-\varepsilon, \varepsilon)$  and there is  $m > n$  such that  $\sum_{i=n}^m x_i \in (M, \infty]$ .

**Remark 5** In the real domain, we know that if  $\sum x_n$  is absolutely convergent then  $\sum x_n$  is convergent. In the transreal domain this is not true. Indeed, according to Example 1, the series  $\sum (-1)^n$  is absolutely convergent ( $\sum |(-1)^n|$  converges to  $\infty$ ), but it is not convergent.

**Proposition 5** In the transreal domain every series of non-negative terms is convergent. In other words, if  $(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^T$  and  $x_n \not\prec 0$  for all  $n \in \mathbb{N}$  then  $\sum x_n$  is convergent.

*Proof.* As  $x_n \not\leq 0$  for all  $n \in \mathbb{N}$ , for some index  $n_0$ , the sequence of partial sums of  $\sum_{n \in \mathbb{N}} x_{n+n_0}$ ,  $(s_{n+n_0})_{n \in \mathbb{N}}$  is monotone. By Proposition 3,  $\sum_{n \in \mathbb{N}} x_{n+n_0}$  is convergent whence  $\sum_{n \in \mathbb{N}} x_n$  is convergent.

Obviously if  $(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^T$  and  $x_n \not\geq 0$  for all  $n \in \mathbb{N}$  then  $\sum x_n$  is convergent too. That is, every series of non-positive terms is convergent.

**Remark 6** Notice that, since in the transreal domain every series of non-negative terms is convergent, the Comparison Test, the Ratio Test and the Root Test are not appropriate for transreal series.

**Definition 4** Let  $(c_n)_{n \in \mathbb{N}} \in \mathbb{R}^T$ . The series  $c_0 + \sum c_n x^n$  is called a power series. A power series defines a function at the values  $x$  for which it is convergent. That is, if  $A$  is the set of all transreal numbers  $x$  such that  $c_0 + \sum c_n x^n$  converges then

$$\begin{aligned} f : A &\longrightarrow \mathbb{R}^T \\ x &\longmapsto c_0 + \sum_{n=1}^{\infty} c_n x^n \end{aligned}$$

is a well-defined function.

## 4 Elementary functions

Recall that a real, elementary function is defined in the following way. Every polynomial, root, exponential, logarithmic, trigonometric and inverse trigonometric function is an elementary function; any finite composition of elementary functions is an elementary function; and any finite combination, using the four arithmetical operations, between elementary functions is an elementary function. We wish to extend the real elementary functions to the transreal numbers.

There are some choices which need to be made. The power series of some elementary functions have a value, at a strictly transreal number, that is different to the limit of the function when the argument tends to that strictly transreal number. Let us illustrate this with the exponential function. It is known that  $e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$  for all  $x \in \mathbb{R}$ . We have that  $\lim_{x \rightarrow -\infty} e^x = 0$  but  $1 + \sum_{n=1}^{\infty} \frac{(-\infty)^n}{n!} = \Phi$ . How should we define  $e^{-\infty}$ ? By  $e^{-\infty} = 0$  or  $e^{-\infty} = \Phi$ ? Geometrically speaking, it is intuitive to us that, in the graph of the exponential function, when  $x$  ‘‘hits’’ minus infinity,  $e^x$  ‘‘hits’’ zero. In this way we choose to define  $e^{-\infty} = 0$ . This has the advantage that the exponential function becomes continuous at  $-\infty$ . At  $\infty$  there is no trouble because  $\lim_{x \rightarrow \infty} e^x = \infty = 1 + \sum_{n=1}^{\infty} \frac{\infty^n}{n!}$ . Nullity,  $\Phi$ , is an isolated point, whence it is nonsense to speak of the limit at  $\Phi$ ; because of this we choose to define  $e^\Phi$  by way of the power series. We have that  $1 + \sum_{n=1}^{\infty} \frac{\Phi^n}{n!} = \Phi$  whence we define  $e^\Phi = \Phi$ . However we do not always define a function by way of limits or power series. For example the function  $\frac{\sin(x)}{x}$ , calculated exactly at zero is  $\frac{\sin(0)}{0} = \frac{0}{0}$  and  $\frac{0}{0} = \Phi$  but  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$ . Again we have a choice to make. The transreal numbers were defined in order to allow division by zero. It seems to us counter-productive to ignore

the fact that, in  $\mathbb{R}^T$ ,  $\frac{0}{0}$  is well-defined. Hence we choose to define  $\frac{\sin(0)}{0} = \Phi$ . Of course the function  $\frac{\sin(x)}{x}$  becomes discontinuous at 0 but this is the price to pay for having transreal arithmetic.

In summary we adopt the following procedure to extend a function from the real domain to the transreal domain. If the expression of the function is lexically well-defined, at a transreal number, then we define the function by simply applying its expression at that transreal number. If the function  $f$  is not lexically well-defined at a transreal number,  $x_0$ , but there is a limit,  $\lim_{x \rightarrow x_0} f(x)$ , then we choose to define the function at  $x_0$  by  $\lim_{x \rightarrow x_0} f(x)$ . Otherwise we choose to define the function by way of its power series if it converges. And if, nevertheless, its power series does not converge, we keep the function undefined.

Firstly we recall some results about the limit and continuity of functions obtained in [3]. Recall that if  $X$  is a topological space then  $x_0 \in A \subset X$  is a limit point of  $A$  if and only if for every neighbourhood  $V$  of  $x_0$  it follows that  $V \cap (A \setminus \{x_0\}) \neq \emptyset$ . The set of all limit points of  $A$  is denoted as  $A'$ . We use the usual definition of the limit of functions in a topological space. That is if  $A$  is a subset of  $\mathbb{R}^T$ ,  $f : A \rightarrow \mathbb{R}^T$  is a function,  $x_0$  is a limit point of  $A$  and  $L$  is a transreal number, we say that  $\lim_{x \rightarrow x_0} f(x) = L$  if and only if, for each neighbourhood  $V$  of  $L$ , there is a neighbourhood  $U$  of  $x_0$  such that  $f(A \cap U \setminus \{x_0\}) \subset V$ .

**Remark 7** Notice that given  $x_0, L \in \mathbb{R}$ ,  $\lim_{x \rightarrow x_0} f(x) = L$ , in the transreal sense, if and only if  $\lim_{x \rightarrow x_0} f(x) = L$ , in the real sense. The same can be said about  $\lim_{x \rightarrow x_0} f(x) = -\infty$ ,  $\lim_{x \rightarrow x_0} f(x) = \infty$ ,  $\lim_{x \rightarrow -\infty} f(x) = L$ ,  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ ,  $\lim_{x \rightarrow -\infty} f(x) = \infty$ ,  $\lim_{x \rightarrow \infty} f(x) = L$ ,  $\lim_{x \rightarrow \infty} f(x) = -\infty$  and  $\lim_{x \rightarrow \infty} f(x) = \infty$ .

**Remark 8** Let  $x_0 \in \mathbb{R}^T$ , notice that  $\lim_{x \rightarrow x_0} f(x) = \Phi$  if and only if there is a neighbourhood  $U$  of  $x_0$  such that  $f(x) = \Phi$  for all  $x \in U \setminus \{x_0\}$ .

We use the usual definition of continuity in a topological space. That is if  $A \subset \mathbb{R}^T$ ,  $f : A \rightarrow \mathbb{R}^T$  is a function and  $x_0 \in A$ , we say that  $f$  is continuous in  $x_0$  if and only if, for each neighbourhood  $V$  of  $f(x_0)$ , there is a neighbourhood  $U$  of  $x_0$  such that  $f(A \cap U) \subset V$ . We say that  $f$  is continuous in  $A$  if and only if  $f$  is continuous in  $x$  for all  $x \in A$ .

**Remark 9** Of course if  $x_0$  is a limit point of  $A$  then  $f$  is continuous in  $x_0$  if and only if  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .

**Remark 10** Notice that given  $x_0 \in \mathbb{R}$ ,  $f$  is continuous in  $x_0$ , in the transreal sense, if and only if  $f$  is continuous in  $x_0$ , in the real sense.

**Remark 11** Notice that if  $\Phi$  belongs to the domain of  $f$  then  $f$  is continuous in  $\Phi$ .

## 4.1 Polynomial Functions

A function,  $f$ , is a real, polynomial function if and only if there is  $n \in \mathbb{N}$  and  $a_0, \dots, a_n \in \mathbb{R}$  such that  $f(x) = a_n x^n + \dots + a_1 x + a_0$  for all  $x \in \mathbb{R}$ . Of course,  $f$

can be written  $f(x) = a_mx^m + \dots + a_1x + a_0$  for any  $m > n$  provided that  $a_{n+1} = \dots = a_m = 0$ . And if  $a_k = 0$  for some  $k \in \{1, \dots, n\}$  then  $f$  can also be written  $f(x) = a_nx^n + \dots + a_{k+1}x^{k+1} + a_{k-1}x^{k-1} + \dots + a_1x + a_0$ . This multiplicity, in the representation of  $f$ , is a problem in the transreal domain. In the real domain,  $0 \times x^k = 0$  for all real  $x$  but  $0 \times x^k = 0$  does not hold for all transreal  $x$ . We have  $0 \times (-\infty)^k = 0 \times \infty^k = 0 \times \Phi^k = \Phi \neq 0$ . In order to avoid this problem we define that a function  $f$  is a real, polynomial function if and only if there is  $n, k \in \mathbb{N}$ ,  $n_1, \dots, n_k \in \{1, \dots, n-1\}$  and  $a_0, a_{n_1}, \dots, a_{n_k}, a_n \in \mathbb{R}$  such that  $a_{n_1}, \dots, a_{n_k}, a_n$  are different from zero and

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto a_nx^n + a_{n_k}x^{n_k} + \dots + a_{n_1}x^{n_1} + a_0 \end{aligned} \quad (2)$$

By following our procedure, as every arithmetical operation is well-defined in transreal numbers, we extend the function  $f$  to  $\mathbb{R}^T$  naturally in the following way.

**Definition 5** A function  $f$  is a transreal, polynomial function if and only if there is  $n, k \in \mathbb{N}$ ,  $n_1, \dots, n_k \in \{1, \dots, n-1\}$  and  $a_0, a_{n_1}, \dots, a_{n_k}, a_n \in \mathbb{R}$  such that  $a_{n_1}, \dots, a_{n_k}, a_n$  are different from zero and

$$\begin{aligned} f : \mathbb{R}^T &\longrightarrow \mathbb{R}^T \\ x &\longmapsto a_nx^n + a_{n_k}x^{n_k} + \dots + a_{n_1}x^{n_1} + a_0 \end{aligned}$$

**Remark 12** For every non-constant, transreal, polynomial function,  $f$ , we have that  $f(\Phi) = \Phi$ .

Notice that, in the transreal domain, some polynomial functions are not continuous.

**Example 2** Let  $n \in \mathbb{N}$  such that  $n \geq 2$  and let non-zero numbers  $a_0, \dots, a_n \in \mathbb{R}$  such that  $a_n a_{n-1} < 0$  and  $f(x) = a_nx^n + \dots + a_1x + a_0$  then  $f(\infty) = \Phi$  but  $\lim_{x \rightarrow \infty} f(x) = \text{sgn}(a_n) \times \infty$ , whence  $f$  is not continuous at  $\infty$ .

**Example 3** Let  $n \in \mathbb{N}$  such that  $n \geq 2$  and let non-zero numbers  $a_0, \dots, a_n \in \mathbb{R}$  such that  $a_n a_{n-1} > 0$  and  $f(x) = a_nx^n + \dots + a_1x + a_0$  then  $f(-\infty) = \Phi$  but  $\lim_{x \rightarrow -\infty} f(x) = \text{sgn}(a_n) \times (-\infty)^n$ , whence  $f$  is not continuous at  $-\infty$ .

If the reader wishes to extend the function in (2) continuously to  $\mathbb{R}^T$  then boundary conditions are needed at  $-\infty$  and  $\infty$ :

$$\begin{aligned} f : \mathbb{R}^T &\longrightarrow \mathbb{R}^T \\ x &\longmapsto \begin{cases} a_nx^n & , \text{ if } x \in \{-\infty, \infty\} \\ a_nx^n + a_{n_k}x^{n_k} + \dots + a_{n_1}x^{n_1} + a_0 & , \text{ otherwise} \end{cases} \end{aligned}$$

Notice that just by using Definition 5, some particular, transreal, polynomial functions become continuous – as exemplified next.

**Example 4** Let  $a, b \in \mathbb{R}$  such that  $a \neq 0$  and  $f(x) = ax + b$  for all  $x \in \mathbb{R}^T$ . Notice that  $\lim_{x \rightarrow -\infty} f(x) = f(-\infty)$  and  $\lim_{x \rightarrow \infty} f(x) = f(\infty)$  whence  $f$  is continuous in  $\mathbb{R}^T$ .

## 4.2 Exponential Functions

A function,  $f$ , is a real, exponential function if and only if there is a positive  $a \in \mathbb{R}$  such that  $a \neq 1$  and

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto a^x. \end{aligned} \quad (3)$$

The natural, exponential function is  $f(x) = e^x$  where  $e$  is Euler's number. We know that every exponential function  $f(x) = a^x$  can be written  $f(x) = e^{\ln(a)x}$  where  $\ln a$  is the natural logarithm of  $a$ .

Let

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto e^x. \end{aligned}$$

By following our procedure, as the function  $f$  is not defined by way of arithmetical operations, we define  $f(-\infty) := \lim_{x \rightarrow -\infty} e^x$  and  $f(\infty) := \lim_{x \rightarrow \infty} f(x)$  whence  $f(-\infty) = 0$  and  $f(\infty) = \infty$ . Furthermore, we know that  $e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$  for all  $x \in \mathbb{R}$ . Following our method, in the absence of limit, we define  $f(\Phi) = 1 + \sum_{n=1}^{\infty} \frac{\Phi^n}{n!}$ , whence  $f(\Phi) = \Phi$ . Thus we extend  $f$  to  $\mathbb{R}^T$  by defining

$$\begin{aligned} f : \mathbb{R}^T &\longrightarrow \mathbb{R}^T \\ x &\longmapsto e^x = \begin{cases} 0 & , \text{ if } x = -\infty . \\ 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} & , \text{ otherwise} \end{cases} \end{aligned}$$

Now we extend the function in (3) to  $\mathbb{R}^T$  in the following way.

**Definition 6** A function,  $f$ , is a transreal, exponential function if and only if there is a positive  $a \in \mathbb{R}$  such that  $a \neq 1$  and

$$\begin{aligned} f : \mathbb{R}^T &\longrightarrow \mathbb{R}^T \\ x &\longmapsto a^x = e^{\ln(a)x}. \end{aligned}$$

**Proposition 6 (Properties of exponential functions)** Let  $f$  be the transreal, exponential function,  $f(x) = a^x$ . We have that, for all  $x, y \in \mathbb{R}^T$ :

1.  $f$  is continuous.
2. The image of  $f$  is  $[0, \infty] \cup \{\Phi\}$ .
3.  $f$  is strictly monotone.
4.  $f$  is injective.
5.  $a^{x+y} = a^x a^y$ .
6.  $a^{x-y} = \frac{a^x}{a^y}$ .

**Remark 13** In [6] we define the transreal derivative in the following way. Let  $A \subset \mathbb{R}^T$ ,  $f : A \rightarrow \mathbb{R}^T$  and  $x_0 \in A$ . If  $x_0 \in \mathbb{R} \cap A'$  and  $f$  is differentiable at  $x_0$ , in the real sense, then the transreal derivative of  $f$  at  $x_0$  is the real derivative  $f'(x_0)$ . If  $x_0 \in \{-\infty, \infty\} \cap D'$  (where  $D$  denotes the set of points in  $A$  at which  $f$  is differentiable, in the real sense) then the transreal derivative of  $f$  at  $x_0$  is  $f'(x_0) = \lim_{x \rightarrow x_0} f'(x)$  if this limit exist. And if  $x_0 \notin A'$  then the transreal derivative of  $f$  at  $x_0$  is  $f'(x_0) := \Phi$ .



**Remark 14** According to Remark 13, we have that  $\frac{d}{dx}a^x = \ln(a)a^x$  for all  $x \in \mathbb{R}^T$ . In particular,  $\frac{d}{dx}e^x = e^x$  for all  $x \in \mathbb{R}^T$ .

### 4.3 Logarithmic Functions

A function,  $f$ , is a real, logarithmic function if and only if there is a positive  $a \in \mathbb{R}$  such that  $a \neq 1$  and

$$\begin{aligned} f : (0, \infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto \log_a(x) \end{aligned} \quad (4)$$

where  $\log_a(x)$  denotes the unique, real number  $y$ , for which  $a^y = x$ .

By following our procedure, as the transreal exponential is injective and has image  $[0, \infty] \cup \{\Phi\}$ , the function  $f$  in (4) is lexically well-defined in  $[0, \infty] \cup \{\Phi\}$ . Because of this we extend the function  $f$  to  $\mathbb{R}^T$  in the following way.

**Definition 7** A function,  $f$ , is a transreal, logarithmic function if and only if there is a positive  $a \in \mathbb{R}$  such that  $a \neq 1$  and

$$\begin{aligned} f : [0, \infty] \cup \{\Phi\} &\longrightarrow \mathbb{R}^T \\ x &\longmapsto \log_a(x) \end{aligned}$$

where  $\log_a(x)$  denotes the unique, transreal number  $y$ , for which  $a^y = x$ .

Like in the real domain, we call  $\log_e(x)$  the transreal, natural logarithm of  $x$  and denote it as  $\ln(x)$ . In particular we have that  $\ln(0) = -\infty$ ,  $\ln(\infty) = \infty$  and  $\ln(\Phi) = \Phi$ .

**Proposition 7 (Properties of logarithmic functions)** Let  $f$  be the transreal, logarithmic function,  $f(x) = \log_a(x)$ . We have that for all  $x, y \in \mathbb{R}^T$ :

1.  $f$  is continuous.
2. The image of  $f$  is  $\mathbb{R}^T$ .
3.  $f$  is strictly monotone.
4.  $f$  is injective.
5.  $\log_a(x \times y) = \log_a(x) + \log_a(y)$ .
6.  $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ .

**Remark 15** We have already said that the derivative of a function  $f$  at  $\infty$ , if  $\infty \in D'$ , is  $f'(\infty) := \lim_{x \rightarrow \infty} f'(x)$  if this limit exists. Fortunately this definition has sense for all  $x_0 \in D'$  so we define, for all  $x_0 \in D'$ ,  $f'(x_0) := \lim_{x \rightarrow x_0} f'(x)$ , if this limit exists.

**Remark 16** According to Remark 15,  $\frac{d}{dx} \log_a(x)|_0 = \lim_{x \rightarrow 0} \frac{1}{\ln(a)} \times \frac{1}{x} = \lim_{x \rightarrow 0^+} \frac{1}{\ln(a)} \times \frac{1}{x} = \frac{1}{\ln(a)} \times \infty = \begin{cases} -\infty, & \text{if } 0 < a < 1 \\ \infty, & \text{if } a > 1 \end{cases}$ . Therefore we have that  $\frac{d}{dx} \log_a(x) = \frac{1}{\ln(a)} \times \frac{1}{x}$  for all  $x \in \mathbb{R}^T$ . In particular,  $\frac{d}{dx} \ln(x) = \frac{1}{x}$  for all  $x \in \mathbb{R}^T$ .

**Remark 17** The Definition 6 defines powers  $x^y$  when  $x \in \mathbb{R}$  is positive and  $x \neq 1$ . The transreal logarithm allows us to define  $x^y$  for all non-negative transreal  $x$ . Let  $x \in [0, \infty] \cup \{\Phi\}$  and  $y \in \mathbb{R}^T$ , we define  $x^y := e^{\ln(x)y}$ . In particular,

- a)  $0^y = \infty$  for all  $y < 0$ .
- b)  $0^0 = \Phi$ .
- c)  $0^y = 0$  for all  $y > 0$ .
- d)  $0^\Phi = \Phi$ .
- e)  $1^y = 1$  for all  $y \in \mathbb{R}$ .
- f)  $1^y = \Phi$  for all  $y \in \{-\infty, \infty, \Phi\}$ .
- g)  $\infty^y = 0$  for all  $y < 0$ .
- h)  $\infty^0 = \Phi$ .
- i)  $\infty^y = \infty$  for all  $y > 0$ .
- j)  $\infty^\Phi = \Phi$ .
- k)  $\Phi^\Phi = \Phi$ .

#### 4.4 Trigonometric functions

We know that the real, trigonometric functions can be defined as follows.

- a)  $\sin : \mathbb{R} \rightarrow \mathbb{R}$   
 $x \mapsto \sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1}$ ,
- b)  $\cos : \mathbb{R} \rightarrow \mathbb{R}$   
 $x \mapsto \cos(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ ,
- c)  $\tan : \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi; k \in \mathbb{Z}\} \rightarrow \mathbb{R}$   
 $x \mapsto \tan(x) = \frac{\sin(x)}{\cos(x)}$ ,
- d)  $\csc : \mathbb{R} \setminus \{k\pi; k \in \mathbb{Z}\} \rightarrow \mathbb{R}$   
 $x \mapsto \csc(x) = \frac{1}{\sin(x)}$ ,
- e)  $\sec : \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi; k \in \mathbb{Z}\} \rightarrow \mathbb{R}$   
 $x \mapsto \sec(x) = \frac{1}{\cos(x)}$  and
- f)  $\cot : \mathbb{R} \setminus \{k\pi; k \in \mathbb{Z}\} \rightarrow \mathbb{R}$   
 $x \mapsto \cot(x) = \frac{\cos(x)}{\sin(x)}$ .

If  $f$  is a trigonometric function, that is, if  $f$  is one of the six functions above then there are no limits  $\lim_{x \rightarrow -\infty} f(x)$  and  $\lim_{x \rightarrow \infty} f(x)$ . So by following our procedure, we define  $\sin(-\infty) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} (-\infty)^{2n-1}$ ,  $\cos(-\infty) := 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (-\infty)^{2n}$ ,  $\tan(-\infty) := \frac{\sin(-\infty)}{\cos(-\infty)}$ ,  $\csc(-\infty) := \frac{1}{\sin(-\infty)}$ ,  $\sec(-\infty) := \frac{1}{\cos(-\infty)}$  and  $\cot(-\infty) := \frac{\cos(-\infty)}{\sin(-\infty)}$ . And similarly for  $\infty$  and  $\Phi$ . Hence  $\sin(-\infty) = \cos(-\infty) = \tan(-\infty) = \csc(-\infty) = \sec(-\infty) = \cot(-\infty) = \sin(\infty) = \cos(\infty) = \tan(\infty) = \csc(\infty) = \sec(\infty) = \cot(\infty) = \sin(\Phi) = \cos(\Phi) = \tan(\Phi) = \csc(\Phi) = \sec(\Phi) = \cot(\Phi) = \Phi$ .

Notice that  $\tan$ ,  $\csc$ ,  $\sec$  and  $\cot$  are lexically well-defined at  $\frac{\pi}{2} + k\pi$  and  $k\pi$  for all  $k \in \mathbb{Z}$ . In this way we extend the trigonometric functions to  $\mathbb{R}^T$  in the following way.

**Definition 8** *A function is a transreal, trigonometric function if and only if it is one of the functions below.*

$$\begin{aligned}
 & \sin : \mathbb{R}^T \longrightarrow \mathbb{R}^T \\
 a) \quad & x \longmapsto \sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1}, \\
 & \cos : \mathbb{R}^T \longrightarrow \mathbb{R}^T \\
 b) \quad & x \longmapsto \cos(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \\
 & \tan : \mathbb{R}^T \longrightarrow \mathbb{R}^T \\
 c) \quad & x \longmapsto \tan(x) = \frac{\sin(x)}{\cos(x)}, \\
 & \csc : \mathbb{R}^T \longrightarrow \mathbb{R}^T \\
 d) \quad & x \longmapsto \csc(x) = \frac{1}{\sin(x)}, \\
 & \sec : \mathbb{R}^T \longrightarrow \mathbb{R}^T \\
 e) \quad & x \longmapsto \sec(x) = \frac{1}{\cos(x)} \quad \text{and} \\
 & \cot : \mathbb{R}^T \longrightarrow \mathbb{R}^T \\
 f) \quad & x \longmapsto \cot(x) = \frac{\cos(x)}{\sin(x)}.
 \end{aligned}$$

**Proposition 8 (Properties of trigonometric functions)** 1. *Every transreal, trigonometric function is discontinuous.*

2. *The image of  $\sin$  and  $\cos$  is  $[-1, 1] \cup \{\Phi\}$ .*
3. *The image of  $\tan$  and  $\cot$  is  $\mathbb{R}^T$ .*
4. *The image of  $\csc$  and  $\sec$  is  $(-\infty, -1] \cup [1, \infty) \cup \{\Phi\}$ .*
5. *Every transreal, trigonometric function has period  $2\pi$ .*
6.  *$\sin$  restricted to  $[-\pi, \pi] \cup \{\Phi\}$  is injective.*
7.  *$\cos$  restricted to  $[0, 2\pi] \cup \{\Phi\}$  is injective.*
8.  *$\tan$  restricted to  $[-\frac{\pi}{2}, \frac{\pi}{2}] \cup \{\Phi\}$  is injective.*
9.  *$\csc$  restricted to  $[-\frac{\pi}{2}, \frac{\pi}{2}] \cup \{\Phi\}$  is injective.*
10.  *$\sec$  restricted to  $[0, \pi] \cup \{\Phi\}$  is injective.*
11.  *$\cot$  restricted to  $[0, \pi] \cup \{\Phi\}$  is injective.*
12.  *$\sin^2(x) + \cos^2(x) = 1^x$  for all  $x \in \mathbb{R}^T$ .*
13.  *$\tan^2(x) + 1^x = \sec^2(x)$  for all  $x \in \mathbb{R}^T$ .*
14.  *$\cot^2(x) + 1^x = \csc^2(x)$  for all  $x \in \mathbb{R}^T$ .*

**Remark 18** *We have already defined the derivative of a function  $f$  at  $x_0$ , if  $x_0 \in D'$ , by  $f'(x_0) := \lim_{x \rightarrow x_0} f'(x)$  if this limit exists. Regrettably if  $f$  is a trigonometric function and  $x_0 \in \{-\infty, \infty\}$  then there is no limit  $\lim_{x \rightarrow x_0} f(x)$ . If we appeal to lexical application of the derivative, we can adopt the following. If  $x_0 \in D'$  and there is no limit  $\lim_{x \rightarrow x_0} f'(x)$  and the expression of  $f'$  is lexically well-defined at  $x_0$  then we define the derivative of  $f$  at  $x_0$  as  $f'(x_0)$ .*

**Remark 19** *For example  $\frac{d}{dx} \sin(x) = \cos(x)$  for all  $x \in \mathbb{R}$ ,  $-\infty$  and  $\infty$  are limit points of the set where  $\sin$  is differentiable, in the real sense, and there are no*

$\lim_{x \rightarrow -\infty} \cos(x)$  and  $\lim_{x \rightarrow \infty} \cos(x)$ . As  $\cos(x)$  is lexically well-defined for  $x = -\infty$  and  $x = \infty$ , according to Remark 18,  $\frac{d}{dx} \sin(x)|_{-\infty} = \cos(-\infty)$  and  $\frac{d}{dx} \sin(x)|_{\infty} = \cos(\infty)$ . And since  $\Phi$  is not a limit point of the domain of  $\sin$ , according to Remark 13,  $\frac{d}{dx} \sin(x)|_{\Phi} = \Phi = \cos(\Phi)$ . Because of this,  $\frac{d}{dx} \sin(x) = \cos(x)$  for all  $x \in \mathbb{R}^T$ .

As another example  $\frac{d}{dx} \tan x = \sec^2(x)$  for all  $x \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi; k \in \mathbb{Z}\}$ . As, for every  $k \in \mathbb{Z}$ ,  $\frac{\pi}{2} + k\pi$  is a limit point of the set where  $\tan$  is differentiable, in the real sense, and there is  $\lim_{x \rightarrow \frac{\pi}{2} + k\pi} \sec^2(x)$ , according to Remark 15, we define  $\frac{d}{dx} \tan(x)|_{\frac{\pi}{2} + k\pi} = \lim_{x \rightarrow \frac{\pi}{2} + k\pi} \sec^2(x) = \infty = \sec^2(\frac{\pi}{2} + k\pi)$ . Since  $-\infty$  and  $\infty$  are limit points of the set where  $\tan$  is differentiable, in the real sense, and there are no limits  $\lim_{x \rightarrow -\infty} \sec^2(x)$  and  $\lim_{x \rightarrow \infty} \sec^2(x)$  but  $\sec^2(x)$  is lexically well-defined for  $x = -\infty$  and  $x = \infty$  so, according to Remark 18,  $\frac{d}{dx} \tan(x)|_{-\infty} = \sec^2(-\infty)$  and  $\frac{d}{dx} \tan(x)|_{\infty} = \sec^2(\infty)$ . And since  $\Phi$  is not a limit point of the domain of  $\tan$ , according to Remark 13,  $\frac{d}{dx} \tan(x)|_{\Phi} = \Phi = \sec^2(\Phi)$ . Because of this,  $\frac{d}{dx} \tan(x) = \sec^2(x)$  for all  $x \in \mathbb{R}^T$ .

Generally, we have, for all  $x \in \mathbb{R}^T$ :

1.  $\frac{d}{dx} \sin(x) = \cos(x)$ .
2.  $\frac{d}{dx} \cos(x) = -\sin(x)$ .
3.  $\frac{d}{dx} \tan(x) = \sec^2(x)$ .
4.  $\frac{d}{dx} \csc(x) = -\csc(x) \cot(x)$ .
5.  $\frac{d}{dx} \sec(x) = \sec(x) \tan(x)$ .
6.  $\frac{d}{dx} \cot(x) = -\csc^2(x)$ .

At this point the reader can already define the inverse trigonometric functions and deduce their properties.

## 5 The limit and value of a function at a point

We tabulate both the values and the limits of the above functions so that we can see whether a function is extended to  $\mathbb{R}^T$  continuously or not.

$a_2x^2 + a_1x$ where $a_2 > 0$ and $a_1 > 0$			
$x_0$	$a_2x^2 + a_1x$	$\lim_{x \rightarrow x_0} (a_2x^2 + a_1x)$	Continuity at $x_0$
$-\infty$	$\Phi$	$\infty$	not continuous
$\infty$	$\infty$	$\infty$	continuous
$\Phi$	$\Phi$	not applicable	continuous

$a_2x^2 + a_1x$ where $a_2 > 0$ and $a_1 < 0$			
$x_0$	$a_2x^2 + a_1x$	$\lim_{x \rightarrow x_0} (a_2x^2 + a_1x)$	Continuity at $x_0$
$-\infty$	$\infty$	$\infty$	continuous
$\infty$	$\Phi$	$\infty$	not continuous
$\Phi$	$\Phi$	not applicable	continuous

$e^x$			
$x_0$	$e^{x_0}$	$\lim_{x \rightarrow x_0} e^x$	Continuity at $x_0$
$-\infty$	0	0	continuous
$\infty$	$\infty$	$\infty$	continuous
$\Phi$	$\Phi$	not applicable	continuous

$\ln(x)$			
$x_0$	$\ln(x_0)$	$\lim_{x \rightarrow x_0} \ln(x)$	Continuity at $x_0$
0	$-\infty$	$-\infty$	continuous
$\infty$	$\infty$	$\infty$	continuous
$\Phi$	$\Phi$	not applicable	continuous

$\sin(x)$			
$x_0$	$\sin(x_0)$	$\lim_{x \rightarrow x_0} \sin(x)$	Continuity at $x_0$
$-\infty$	$\Phi$	none	not continuous
$\infty$	$\Phi$	none	not continuous
$\Phi$	$\Phi$	not applicable	continuous

$\cos(x)$			
$x_0$	$\cos(x_0)$	$\lim_{x \rightarrow x_0} \cos(x)$	Continuity at $x_0$
$-\infty$	$\Phi$	none	not continuous
$\infty$	$\Phi$	none	not continuous
$\Phi$	$\Phi$	not applicable	continuous

$\tan(x)$			
$x_0$	$\tan(x_0)$	$\lim_{x \rightarrow x_0} \tan(x)$	Continuity at $x_0$
$-\infty$	$\Phi$	none	not continuous
$\infty$	$\Phi$	none	not continuous
$\Phi$	$\Phi$	not applicable	continuous
$\frac{\pi}{2} + (2k+1)\pi$	$-\infty$	none	not continuous
$\frac{\pi}{2} + 2k\pi$	$\infty$	none	not continuous

$\tan(x)$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$			
$x_0$	$\tan(x_0)$	$\lim_{x \rightarrow x_0} \tan(x)$	Continuity at $x_0$
$-\frac{\pi}{2}$	$-\infty$	$-\infty$	continuous
$\frac{\pi}{2}$	$\infty$	$\infty$	continuous

$\csc(x)$			
$x_0$	$\csc(x_0)$	$\lim_{x \rightarrow x_0} \csc(x)$	Continuity at $x_0$
$-\infty$	$\Phi$	none	not continuous
$\infty$	$\Phi$	none	not continuous
$\Phi$	$\Phi$	not applicable	continuous
$k\pi$	$\infty$	none	not continuous

$\csc(x)$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$			
$x_0$	$\csc(x_0)$	$\lim_{x \rightarrow x_0} \csc(x)$	Continuity at $x_0$
0	$\infty$	none	not continuous

$\sec(x)$			
$x_0$	$\sec(x_0)$	$\lim_{x \rightarrow x_0} \sec(x)$	Continuity at $x_0$
$-\infty$	$\Phi$	none	not continuous
$\infty$	$\Phi$	none	not continuous
$\Phi$	$\Phi$	not applicable	continuous
$\frac{\pi}{2} + k\pi$	$\infty$	none	not continuous

$\sec(x)$ restricted to $[0, \pi]$			
$x_0$	$\sec(x_0)$	$\lim_{x \rightarrow x_0} \sec(x)$	Continuity at $x_0$
$\frac{\pi}{2}$	$\infty$	none	not continuous

$\cot(x)$			
$x_0$	$\cot(x_0)$	$\lim_{x \rightarrow x_0} \cot(x)$	Continuity at $x_0$
$-\infty$	$\Phi$	none	not continuous
$\infty$	$\Phi$	none	not continuous
$\Phi$	$\Phi$	not applicable	continuous
$2k\pi$	$\infty$	none	not continuous
$(2k+1)\pi$	$-\infty$	none	not continuous

$\cot(x)$ restricted to $[0, \pi]$			
$x_0$	$\cot(x_0)$	$\lim_{x \rightarrow x_0} \cot(x)$	Continuity at $x_0$
0	$\infty$	$\infty$	continuous
$\pi$	$-\infty$	$-\infty$	continuous

$\frac{\sin(x)}{x}$			
$x_0$	$\frac{\sin(x_0)}{x_0}$	$\lim_{x \rightarrow x_0} \frac{\sin(x)}{x}$	Continuity at $x_0$
0	$\Phi$	1	not continuous
$-\infty$	$\Phi$	0	not continuous
$\infty$	$\Phi$	0	not continuous
$\Phi$	$\Phi$	not applicable	continuous

## 6 Conclision

We extend all elementary functions from the real domain to the transreal domain so that they are defined on division by zero. If the expression of the function is lexically well-defined, at a transreal number, then we define the function by simply applying its expression at that transreal number. If the function  $f$  is not lexically well-defined at a transreal number,  $x_0$ , but there is a limit,  $\lim_{x \rightarrow x_0} f(x)$ , then we choose to define the function at  $x_0$  by  $\lim_{x \rightarrow x_0} f(x)$ . Otherwise we choose to define the function by way of its power series if it converges. And if, nevertheless, its power series does not converge, we keep the function undefined. This method for extending functions from the real domain to the transreal domain clearly works for a much wider class of functions so it may be of general interest.

**Acknowledgements** The authors would like to thank the members of Transmathematica for many helpful discussions. The first author's research was financially supported, in part, by the Federal Institute of Education, Science and Technology of Rio de Janeiro, campus Volta Redonda and by the Program of History of Science, Technique and Epistemology, Federal University of Rio de Janeiro. The second author's research was financially supported, in part, by the School of Systems Engineering at the University of Reading and by a Research Travel Grant from the University of Reading Research Endowment Trust Fund (RETF).

## References

1. J. A. D. W. Anderson. Representing geometrical knowledge. *Phil. Trans. Roy. Soc. Lond. Series B.*, 352(1358):1129–1139, 1997.
2. James A. D. W. Anderson. Perspex machine ix: Transreal analysis. In Longin Jan Lateki, David M. Mount, and Angela Y. Wu, editors, *Vision Geometry XV*, volume 6499 of *Proceedings of SPIE*, pages J1–J12, 2007.
3. James A. D. W. Anderson and Tiago S. dos Reis. Transreal limits expose category errors in ieee 754 floating-point arithmetic and in mathematics. In *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2014, WCECS 2014, 22-24 October, 2014, San Francisco, USA.*, volume 1, pages 86–91, 2014.
4. James A. D. W. Anderson, Norbert Völker, and Andrew A. Adams. Perspex machine viii: Axioms of transreal arithmetic. In Longin Jan Lateki, David M. Mount, and Angela Y. Wu, editors, *Vision Geometry XV*, volume 6499 of *Proceedings of SPIE*, pages 2.1–2.12, 2007.
5. Tiago S. dos Reis and James A. D. W. Anderson. Construction of the transcomplex numbers from the complex numbers. In *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2014, WCECS 2014, 22-24 October, 2014, San Francisco, USA.*, volume 1, pages 97–102, 2014.
6. Tiago S. dos Reis and James A. D. W. Anderson. Transdifferential and transintegral calculus. In *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2014, WCECS 2014, 22-24 October, 2014, San Francisco, USA.*, volume 1, pages 92–96, 2014.
7. Tiago S. dos Reis and James A. D. W. Anderson. Transreal calculus. *IAENG International Journal of Applied Mathematics*, 45(1):51–63, 2015.