Introduction to Fixed Point Iteration Method and its application

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Introduction

- What is fixed point?

Does anybody knows, What is it?

Introduction

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  $\rightarrow$If a function $\phi(x)$ intersected by $y = x$ then, the x-co-ordinate of an intersection is known as fixed point.

Introduction

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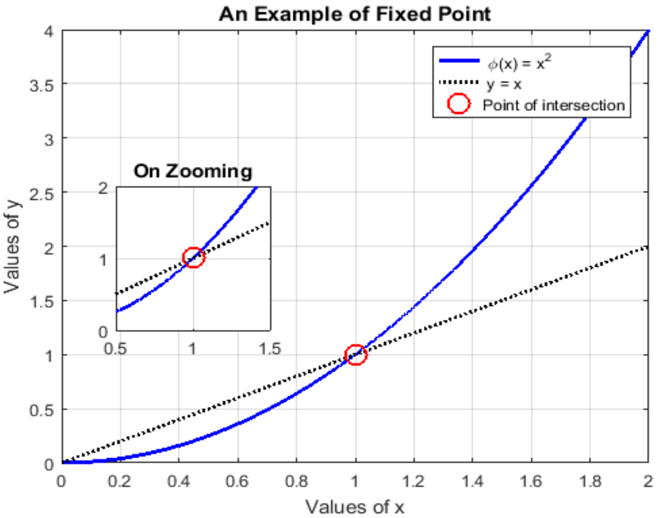
  →If a function $\phi(x)$ intersected by $y = x$ then, the x-co-ordinate of an intersection is known as fixed point.

Formal definition

"A fixed point of a function is an element of function's domain that is mapped to itself by the function."

Lets see an example 1 [See its matlab code in Appendix Section];



- In the previous figure, what are the fixed points?

Fixed Point's response on your answer:

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"Finally, you knew me!"

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  You'll get roots for your desire equation.

What is the reason behind changing my equation into $x = \phi(x)$ form?

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$\rightarrow$ You're in the right place! This presentation is for you!

Hehe, Thank Me!

Fixed Point Iteration Method

Let me answer your question first!

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$$f(x) = 0$$

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In the previous example 1, $y = x$ is a line. Lets say, $y = \phi(x)$ then, the function $\phi(x) = x$ becomes $y = x$. Is this an equation of a line that will intersect the equation $\phi(x)$?

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ANSWER: No! It is a fixed point that will satisfy $f(x) = 0$.

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1. First, Convert your function $f(x)$ into $x = \phi(x)$ form.

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1. First, Convert your function $f(x)$ into $x = \phi(x)$ form.
2. Iterate the value of x to the function $\phi(x)$ until you get the desire precision such that the precision value should be repeated after you again iterate it.

Can you show me in geometrical perspective?

Can you show me in geometrical perspective?

CLICK THE **"ICON"** TO GLORIFY YOURSELF!

Condition for Convergence

We can make lots of functions in the form of $x = \phi(x)$ by using the $f(x)$. Then, Can every functions will give me the root of the $f(x)$?

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$\rightarrow$ May be! But, the rate of convergence would be slow.

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Lets talk about it!

Condition for Convergence

"If α be the root of the $f(x)$ which is equivalent to $x = \phi(x)$ and α is contained in the interval I of $\phi(x)$ and $|\phi'(x)| < 1 \quad \forall \quad x \in I$. Then, if x_0 is any point in I so that, the sequence defined by

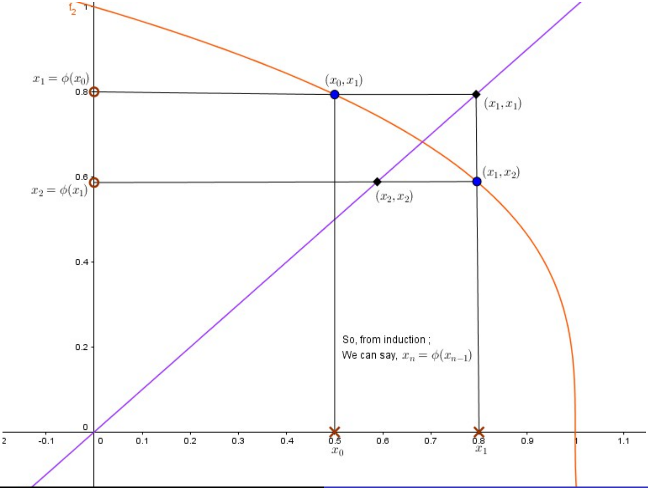
$$x_n = \phi(x_{n-1}) \quad | \quad n \geq 1$$

will converges to the unique fixed point x in I. Thus, that unique fixed point x of $\phi(x)$ will be the root of $f(x)$."

Basic Preliminaries

1. Visualization of $x_n = \phi(x_{n-1})$
2. Mean Value Theorem

Visualization of $x_n = \phi(x_{n-1})$



Mean Value Theorem

Lagrange's MVT

If $\phi(x)$ is a real valued function continuous on the closed interval $[x, x_{n-1}]$, and differentiable in the open interval (x, x_{n-1}) then, there exists a point $\epsilon \in (x, x_{n-1})$ such that

$$\phi(x_{n-1}) - \phi(x) = \phi'(\epsilon)(x_{n-1} - x)$$

Proof

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Now, lets write as ;
$|x_n - x| = |\phi(x_{n-1}) - \phi(x)|$
$or, |x_n - x| = |\phi'(\epsilon)(x_{n-1} - x)|$ Applying Lagrange MVT in right part.

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Since, $\epsilon \in (x, x_{n-1})$ so, assume $|\phi'(\epsilon)| = k$.
$\therefore |x_n - x| \leq k|x_{n-1} - x|$

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$\ldots$
$\therefore |x_n - x| \leq k^n|x_{n-n} - x| = k^n|x_0 - x|$

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Suppose, $k < 1$ and taking $\lim\limits_{n\to\infty}$ in both part.

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$$\lim_{n \to \infty} |x_n - x| \leq \lim_{n \to \infty} k^n |x_0 - x|$$

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Suppose, $k < 1$ and taking $\lim_{n\to\infty}$ in both part.

$\lim_{n\to\infty} |x_n - x| \leq \lim_{n\to\infty} k^n |x_0 - x|$

$\Rightarrow \lim_{n\to\infty} |x_n - x| < 0$ Since, $\lim_{n\to\infty} k^n \to 0$

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$\Rightarrow \lim\limits_{n\to\infty} |x_n - x| < 0$ Since, $\lim\limits_{n\to\infty} k^n \to 0$

Proof.

So, $\{x\}_{n=0}^{\infty}$ converges to x. $\square$

Application

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   $\rightarrow$ Given, equation $f(x) = x^3 + x - 1$. To find the root of this equation, we say $f(x) = 0$ i.e.
   $$x^3 + x - 1 = 0$$
   By testing on $f(a).f(b) < 0$. We estimated, the root should lies on $[0, 1]$.

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There are many ways to change the equation to the fixed point form $x = \phi(x)$ using simple algebraic manipulation.

My possible choices for $\phi(x)$ are;

- $x = \phi_1(x) = 1 - x^3$
- $x = \phi_2(x) = (1 - x)^{1/3}$

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Previous theorem will suggests us, which form of $\phi(x)$ will converges reliably and rapidly! So that, we can be able to reject the one which converges slowly.

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Taking the derivative of above forms with respect to x. We get
$\phi_1'(x) = -3x^2$
$\phi_2'(x) = \frac{-1}{3(1-x)^{\frac{2}{3}}}$

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$|\phi_1'(0.8)| = |-1.92| > 1$
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Application

Test on the value of $x = 0.8$. On putting in above equation and taking absolute value of it. We get
$|\phi_1'(0.8)| = |-1.92| > 1$
$|\phi_2'(0.8)| = |-0.97| < 1$

Since,$|\phi_1'(0.8)| > 1$ So,we reject $\phi_1(x)$ and we choose $\phi_2(x)$ for iteration.

Application

Taking initial approximation of x as 0.5. We can see in the attached video, how it converges!

CLICK THE "**ICON**" TO GLORIFY YOURSELF!

Application

From the "Fixed Point Iteration" program in Matlab[See the code of it in Appendix Section], we can get the result as,

```
Command Window
 ----------------------------------------------------------
 |WELCOME TO THE PROGRAM OF "FIXED POINT ITERATION METHOD"|
 ----------------------------------------------------------
The equivalent equation is: (1-x).^(1/3)
"The initial approximation should lies on the root's interval"
The initial approximation is: 0.5
No. of iterations: 20
Decimal upto: 2
The approximate root of an equation  is 0.68.
fx >>
```

Hence, the root of the given equation corrected upto 2 decimal places is 0.68

Appendix

Matlab Code for the figure(example 1)

Matlab Code for the figure(example 1)

Script File:

```
function [d,px,py] = variable(x,y,z)
%Find the point of intersection
    d = find(y==z & x >0);
    px = x(d);
    py = y(d);
    plot(x,y,'-b','linewidth',2)
    hold on
    plot(x,z,':k','linewidth',2)
    hold on
    plot(px,py,'ro', 'linewidth',1.5,'MarkerSize',12)
end
```

Matlab Code for the figure(example 1)

Script File:

```
%Script file: fixed.m
%TOPIC: INTRODUCING DEFINITION OF FIXED POINT!
%AIM: In plot, we want to merge the two functions' graphs into one.
%DEMO: For plotting, we used the functions; y = x^2 and z = x.
%AUTHOR: Damodar Rajbhandari
clc
x = 0:0.0001:2;
y = x.^2;
z = x;
%Calling the function defined in file variable.m
[~,~,~] = variable(x,y,z);
%After calling [d,px,py] = variable(x,y,z),
%d got the index number 10001
%In x(d) gives the value of x from the index number 10001
%In y(d)gives the value of y from the index number 10001
```

Matlab Code for the figure(example 1)

```
title('An Example of Fixed Point','Fontsize',12)
legend(' \phi(x) = x^2',' y = x','Point of intersection')
xlabel('Values of x');
ylabel('Values of y');
hold off
grid on
axes('units','centimeters','position',[3 4 3 3])
[~,~,~] = variable(x,y,z);
title('On Zooming')
axis([.5 1.5 0 2])
box on
hold off
grid on
```

Appendix

Matlab Code for "Fixed Point Iteration Method"

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```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%SCRIPT FILE: algorithm_code.m                  %
%AUTHOR: Damodar Rajbhandari                    %
%PROGRAM: Fixed Point Iteration Method          %
%AIM: To find the root by numerical approximation%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc; clear; format long ;
fprintf(['',...
' -----------------------------------------------------------\n ',...
'|WELCOME TO THE PROGRAM OF \"FIXED POINT ITERATION METHOD\"|\n', ...
' -----------------------------------------------------------\n'])
a = input('The equivalent equation is: ','s');
disp('"The initial approximation should lies on the root''s interval"')
x0 = input('The initial approximation is: ');
n = input('No. of iterations: ');
c = input('Decimal upto: ');
```

Matlab Code for "Fixed Point Iteration Method"

```
syms x
y(x) = eval(a);
df = diff(y);
z = eval(subs(df,x0));
if abs(z)<1
   x = x0;
   for i = 1:n
       x = y(x);
    end
    fprintf('The approximate root of an equation  is %.*f.\n',c,x)
else
    disp('Choose another equivalent equation!')
end
```

You can find the MATLAB files of;

1. Plotting code at Click Me!
2. Program code at Click Me!