

Proof of Church's Thesis

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We prove that if our calculating capability is limited to that of a universal Turing machine with a finite tape, then Church's thesis is true. This way we accomplish Post (1936) program.

Keywords: Church's thesis, Post program, universal Turing machine.

§1 Church's Thesis

¶1 · Church's thesis, also known as Church-Turing thesis, says, see [Gandy \(1980\)](#):

◇ Thesis 1 *What is effectively calculable is computable.*

¶2 · What is *computable* is anything that any Turing machine can compute, where the Turing machine was defined by [Turing \(1936\)](#). There are other definitions of computable, using for example Church's λ -calculus, but all of them are mathematically equivalent. In any case, the term computable is defined with mathematical rigor.

¶3 · But, because what can be calculated was considered a vague notion, it was assumed that Church's thesis could not be formally proven. In what follows, we will make some definitions and assumptions in order to overcome the ambiguity. Then we will deduce Church's thesis from our calculating limitations, which we will assume are those of a universal Turing machine with a finite tape. In doing so, we are realizing the old program of [Post \(1936\)](#), for whom Church's thesis is not a definition nor an axiom, but a *natural law* stating the limitations of the mathematicizing power of our species *Homo sapiens*.

This is DOI: [10.6084/m9.figshare.4955501](https://doi.org/10.6084/m9.figshare.4955501) ([arXiv:1209.5036](https://arxiv.org/abs/1209.5036)), version 20170430.

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§2 Syntax Engine

¶1 · *Syntax*, as opposed to Semantics, is concerned with transformations of strings of symbols, irrespective of the symbols meanings, but according to a precise and finite set of well-defined rules. We call this set of rules *algorithm*.

¶2 · The previous definition of Syntax generalizes the linguistic one by Chomsky (1957): “Syntax is the study of the principles and processes by which sentences are constructed in particular languages” (page 11). A sentence is a string of words, and a word is a particular case of symbol.

¶3 · We will assume here that persons, that is, the members of our own species *Homo sapiens*, have a syntactic capability. If a person speaks a particular language, then she has a syntactic capability according to the linguistic definition. Also, most of mathematics is pure Syntax according to the general definition, see Hilbert (1922), and mathematics are produced and consumed by persons.

¶4 · We will call whatever that implements a syntactic capability a *syntax engine*. So, if persons have a syntactic capability, then each person has a syntax engine. In the case of a general-purpose computer, the central processing unit (CPU) is its syntax engine.

§3 Finite Turing Machine

¶1 · A Turing machine can then be seen as a mathematical model for syntax engines, see Chomsky (1959). The only part of a Turing machine that cannot be physically built is its infinite tape. So we will define a *finite Turing machine* as a Turing machine with a finite tape instead of the infinite tape. The tape is just read and write memory.

¶2 · For each finite Turing machine there is one *corresponding Turing machine*, which is the Turing machine that is identical to the finite one, except for the tape. Defining *processor* as the whole Turing machine except its tape, then any finite Turing machine and its corresponding Turing machine have the same processor.

¶3 · If two or more finite Turing machines have the same corresponding Turing machine, then we say that they are *equivalent*. Equivalent finite Turing machines only difference is the length of their tapes; they have all the same processor.

¶4 · The processor has some finite memory itself to store what Turing (1936) called the “*m*-configurations”. From the processor point of view, this *m*-configurations memory is internal memory while the tape is external memory. Thus, basically, a processor is a finite-state automaton, a Turing machine is a finite-state automaton attached to an infinite external memory, and a finite Turing machine is a finite-state automaton attached to a finite external memory.

¶5 · Because a universal Turing machine is a Turing machine, see Turing (1936), we can define finite universal Turing machines the same way. A *finite universal Turing machine* is a universal Turing machine, but with a finite tape, instead of the infinite tape. The corresponding Turing machine of a finite universal Turing machine is a universal Turing machine; both have the same processor. But, as noted below, a finite universal Turing machine is not universal, because it is finite.

¶6 · With these definitions and considerations, we can say that the general-purpose computer CPU is a finite universal Turing machine, by design.

§4 Effective Computation

¶1 · We will call any computation done by a finite Turing machine in a finite time an *effective computation*. Effective computations are physically achievable, because finite Turing machines can be physically built, as the general-purpose computer CPU proves.

¶2 · Finite Turing machines will fail on those computations that require more tape than they have available. Also, in practical terms, and particularly if the tape is long, time available could be exhausted before reaching a tape end or a HALT instruction, and then the computation would have to be aborted, failing. These computations that fail in the finite Turing machine because of a lack of tape (external memory) or a lack of time would continue in its corresponding Turing machine, and they will eventually succeed, or not. Summarizing: computing is more successful than effectively computing.

¶3 · But, how much successful? A computation is *successful* when the Turing machine has reached a HALT instruction in a finite time. And, whenever a computation has reached a HALT instruction in a finite time, the Turing machine has only had time to inspect a finite number of tape cells. This means that any successful computation could be done in some finite Turing machine in a finite time. So the answer is: not too much.

¶4 · In addition, each finite Turing machine computation that HALTS will be run identically by its corresponding Turing machine, because no limitations of tape nor time were found, and there are not any other differences between the two machines. So any computation done by a finite Turing machine will be identically computed by its corresponding Turing machine. Summarizing: an effective computation is a computation.

¶5 · A finite universal Turing machine is not a universal finite Turing machine, that is, finite universal Turing machines are not universal, because, taking any finite universal Turing machine, there will always be some computations that fail in it, but that do not fail in other equivalent finite Turing machines that have more tape or more time.

¶6 · On the other hand, any finite universal Turing machine computation that HALTS will be run identically by its corresponding Turing machine, which is a universal Turing machine. And this means that, except for tape or time limitations, a finite universal Turing machine computes exactly as its corresponding universal Turing machine.

§5 Calculability

¶1 · We will assume the following thesis:

◇ Thesis 2 *What is effectively calculable is what a person can calculate.*

¶2 · This could be denied from a Platonist view of mathematics, because a super-person, or a super-machine, provided with a syntax super-engine, could calculate what a plain person cannot. But, firstly, we should say that this would be “super-calculable”, not just “calculable”, and even less “effectively calculable”.

¶3 · And, secondly, if those super-calculations are outside our plain syntactic capabilities, then we could never identify those super-calculations as calculations, nor we could follow nor understand them. For seeing this, just imagine any finished calculation. If each and every step of the calculation obeys the finite set of rules for the calculation, that is, if each and every step obeys the algorithm, then we have a plain calculation that we can follow and understand. Or else, the steps that do not obey the algorithm are errors, and then

the calculation is wrong, even if its final result is right. So, for us, a super-calculation can only be a plain calculation, or a wrong calculation.

¶4 · Another objection could be that different persons can have different calculating capabilities. This is true, but to investigate this issue, let us present our own thesis:

◇ Thesis 3 *Persons' syntax engine is a finite universal Turing machine.*

¶5 · If this is the case, then, for normal persons, that is, persons without mental disabilities, the difference can only be the amount of memory or time.

§6 Proof

¶1 · We will now show that if thesis 3 is true, then Church's thesis (thesis 1) is also true.

¶2 · First, we take thesis 1 in §1 and, using thesis 2 in §5, we replace 'what is effectively calculable' with 'what a person can calculate'. Second, using the concept of syntax engine, seen in §2, what a person can calculate is what her syntax engine can compute. Now, if thesis 3 in §5 is true, that is, if persons' syntax engine is a finite universal Turing machine, then what a person's syntax engine can compute is what a finite universal Turing machine can compute. A finite universal Turing machine is a finite Turing machine, as seen in §3, and anything that a finite Turing machine can compute can also be computed by its corresponding Turing machine, as shown in §4. To close this proof just remember the first definition in §1: anything that a Turing machine can compute is computable. QED

§7 Discussion

¶1 · From the proof it is easy to see that it is also possible to imply Church's thesis from a weaker assumption:

◇ Thesis 4 *Persons' syntax engine is a finite Turing machine.*

¶2 · But this thesis 4 does not fit some empirical facts. Because, if thesis 4 were true, but thesis 3 were false, then some computations would be outside the syntactic capability of persons. This would mean that persons could not follow some Turing machine computations. In fact, we could only calculate one algorithm.

¶3 · On the other hand, the stronger thesis 3 implies thesis 4, and it is very close to the converse of Church's thesis. The converse of Church's thesis would be:

◇ Thesis 5 *What is computable is effectively calculable.*

¶4 · For this thesis 5 to be true, person's syntax engine should have to be a universal Turing machine, but then persons could calculate any computation, that is, any algorithm with any data. Sadly, we persons are finite, and therefore neither our syntax engines have an infinite memory, nor ourselves have an infinite time to do calculations.

¶5 · Luckily, if thesis 3 is true, and persons' syntax engines are finite universal Turing machines, then, except for limitations of memory or time, persons can calculate any computation. So, taking a small enough part of any computation, persons can calculate if it is computed rightly or wrongly.

§8 Conclusions

¶1 · We can summarize thesis 2 and thesis 3 in a refined Church's thesis:

◇ Thesis 6 *What is effectively calculable is effectively computable.*

¶2 · Turing could not have designed his universal machine if his own syntax engine were not a finite universal Turing machine.

¶3 · We have shown that if our calculating capability is that of a finite universal Turing machine, then Church's thesis is true. This way Post (1936) program is accomplished.

§9 Epilogue on Post (1936) Program

¶1 · Church (1937) reviewed Post (1936) program unfavorably. For Church, his thesis gives a definition of effective calculability as computability that seems "to be an adequate representation of the ordinary notion", and that is all.

¶2 · But, even if that were the case, considering Church's thesis a definition does not settle the question, but just sweep it under the carpet. For suppose two possibilities: A) Tomorrow someone devises a procedure to perform calculations that are beyond the capability of any Turing machine. B) Tomorrow some machine is found that performs calculations that are beyond the capability of any Turing machine. In either case, the current definition would be wrong.

¶3 · So, it is not that effective calculability does not have an exact definition, and giving it one solves the problem, as Church (1937) argues. What happens is that effective calculability is uncertain because it depends on what we do not currently know.

¶4 · For Post (1936) program, effective calculability is defined exactly (by thesis 2, see §5), and then Church's thesis is not a definition, but a *natural law* that states a limitation of the calculating capability of our species *Homo sapiens* (our thesis 3, see §5, for which thesis 1, see §1, is a consequence, see §6). Therefore, Post law is our thesis 3 and, as any *natural law*, it is refutable and then it is in need of continual verification. It is refutable because it predicts both the impossibility of case A, and that we are blind to case B.

¶5 · As in our argument in favor of thesis 2, see §5, we are blind to case B if our calculating capability is limited to computing, because then we cannot see the super-calculation as a calculation. This is possibly the case of quantum systems. Take for example a Schrödinger's cat in its box. If we suppose that the box is a physical device that performs a calculation the result of which is a dead or an alive cat, then we have to conclude that no algorithm is obeyed, but that some random activity happens.

¶6 · Post (1936) program is deep. As he concludes: "Only so, [that is, only if Church's thesis is a *natural law* stating the limitations of the mathematicizing power of our species *Homo sapiens*], can Gödel's theorem concerning the incompleteness of symbolic logics of a certain general type and Church's results on the recursive unsolvability of certain problems be transformed into conclusions concerning all symbolic logics and all methods of solvability."

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