Quantitative Analysis of Attachment Time of Air Bubbles to Solid Surfaces in Water

Seongsoo Han\textsuperscript{1,2}, Anh V. Nguyen\textsuperscript{3*}, Kwanho Kim\textsuperscript{1}, Jaikoo Park\textsuperscript{2}, Kwangsuk You\textsuperscript{1*}

\textsuperscript{1} Convergence Research Center for Development of Mineral Resources (DMR), Korea Institute of Geoscience and Mineral Resources (KIGAM), 124 Gwahak-ro, Yuseong-gu, Daejeon 34132, Republic of Korea

\textsuperscript{2} Department of Earth Resources and Environmental Engineering, Hanyang University, 222, Wangsimni-ro, Seongdong-gu, Seoul 04763, Republic of Korea

\textsuperscript{3} School of Chemical Engineering, The University of Queensland, Brisbane, Queensland 4072, Australia

*Corresponding Author Information

Kwang suk You (youks@kigam.re.kr)
Anh.V. Nguyen (Anh.Nguyen@eng.uq.edu.au)

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Figure S1. Images of the water droplet static contact angles (a) 0° (pristine glass slide), (b) 55° (0.01 mM OTS in toluene for 10 min), (c) 75° (0.1 mM OTS in toluene for 10 min), and (d) 105° (1 mM OTS in toluene for 10 min)
Figure S2. Images of the hysteresis of the water contact angles (a) 0° (pristine glass slide) (b) 55 ± 10° (0.01 mM OTS in toluene for 10 min), (c) 75 ± 5° (0.1 mM OTS in toluene for 10 min), and (d) 105 ± 5° (1 mM OTS in toluene for 10 min)
Figure S3. Corrected force curves incorporating the capillary effect of the needle produced by the water in the cell (solid line: after correction, dotted line: before correction)
Conversion to dimensionless forms

Capillary number: \( Ca = \mu V / \sigma \)

\[
x = \frac{r}{R} Ca^{-1/4}; \quad y = \frac{h}{R} Ca^{-1/2}; \quad Y = \frac{H}{R} Ca^{-1/2}; \quad p = \frac{R}{\sigma} P; \quad \tau = \frac{\sigma \sqrt{Ca}}{R \mu} t; \quad \tau_0 = \frac{\sigma \sqrt{Ca}}{R \mu} t_0;
\]

\[
\varphi = \frac{R}{\sigma} \Pi; \quad \Phi = \frac{F}{2\pi R \sigma Ca^{1/2}};
\]

\[
\frac{\partial y}{\partial \tau} = \frac{1}{4} \frac{\partial}{x} \left\{ x^2 \frac{\partial p}{\partial x} \right\} + \frac{d^2}{x} \frac{\partial}{\partial x} \left\{ x^2 \frac{\partial p}{\partial x} \right\} = \frac{1}{4} \left\{ \frac{y^3}{x} \frac{\partial p}{\partial x} + 3y^2 \frac{\partial y}{\partial x} \frac{\partial p}{\partial x} + y^3 \frac{\partial^2 p}{\partial x^2} \right\} + \ldots
\]

\[
p(y, x) = 2 - \frac{1}{x} \frac{\partial}{\partial x} \left\{ x \frac{\partial y}{\partial x} \right\} - \varphi(y) = 2 - \left\{ \frac{\partial^2 y}{\partial x^2} + \frac{1}{x} \frac{\partial y}{\partial x} \right\} - \varphi(y)
\]

\[
y(x, \tau = 0) = Y + x^2 / 2
\]

\[
p(x, y, \tau = 0) = 0
\]

\[
\frac{\partial y_N}{\partial \tau} = -1 - \frac{\partial \Phi}{\partial \tau} \left\{ \log \frac{x_y Ca^{1/4}}{2} + B(\theta_o) \right\}
\]

\[
p(y, x \to \infty) = 0
\]

\[
\Phi = \int_0^{y_o} \left\{ p(x, \tau) + \varphi(x, \tau) \right\} dx
\]

At \( x = 0 \), we obtain the following limits:

\[
\left( \frac{\partial Y}{\partial x} \right)_{x=0} = 0; \quad \left( \frac{\partial p}{\partial x} \right)_{x=0} = 0
\]

\[
p(Y, 0) = 2 - 2 \left\{ \frac{\partial^2 Y}{\partial x^2} \right\}_{x=0} - \varphi(Y)
\]

\[
\frac{\partial Y}{\partial \tau} = \left\{ \frac{3}{2} y^3 + d^2 y^2 \right\} \left[ \frac{\partial^2 p}{\partial x^2} \right]_{x=0}
\]
Numerical Solution by the three-point Finite Difference Method

We divide the film radius into N-1 sections (stages) and consider N points on the 1D mesh.

Applying the three-point FDM, we obtain the following ordinary differential equations (ODEs):

**Section 1 (around the film symmetry):**  \( Y = y_1; \Delta = \text{film radius} / (N-1) \)

\[
\frac{\partial y_1}{\partial \tau} = \left[ \frac{3}{2} y_1^3 + d y_1^2 \right] - \frac{2 p_1 + 2 p_2}{\Delta^2}; \quad p_1 = 2 - \frac{4 y_1 + 4 y_2}{\Delta^2} - \varphi(y_1)
\]

**Section k = 2 to N-1 (intermediate sections):**

\[
\frac{\partial y_k}{\partial \tau} = \frac{m}{2} \left\{ \frac{y_k^3}{x_k} \frac{p_{k+1} - p_{k-1}}{2\Delta} + 3 y_k^2 \frac{y_{k+1} - y_{k-1}}{2\Delta} + \frac{y_k^3}{\Delta^2} \left( \frac{p_{k+1} - 2 p_k + p_{k-1}}{2\Delta} \right) \right\} + ...
\]

\[
p_k = 2 - \left\{ \frac{y_{k+1} - 2 y_k + y_{k-1}}{\Delta^2} + \frac{y_{k+1} - y_{k-1}}{2 x_k \Delta} \right\} - \varphi(y_k)
\]

**Section N (at the film boundary):**

\[
\frac{\partial y_N}{\partial \tau} = -1 \frac{\partial \Phi}{\partial \tau} \left\{ \log \frac{x_c C_a^{1/4}}{2} + B(\theta_c) \right\}; \quad p_N = 0
\]

This system of ODEs can be solved using the Matlab ODE solver “ODE15S.”
Figure S4. Force with respect to the drainage time calculated by Eq. 15 (before attachment)

The y-axis in Figure S3 is the force calculated by Eq. (15) in the manuscript:

$$F(t) = 2\pi \int_{0}^{\infty} \left\{ P(x,t) + \Pi(x,t) \right\} x \, dx$$  \hspace{1cm} (15)
Figure S5. Hamaker function of silica showing increment with the IGE thickness.