SUPPORTING INFORMATION

Analytic expression for electrophoretic mobility of soft particles with hydrophobic inner core at different electrostatic conditions

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A-Coefficients appearing in mobility expression

The coefficients $C_i (i = 1, 2, 3, 4)$ and $D_i (i = 1, 2, 3)$ appearing in equation (16) for $h(r)$ are

$$
\begin{align*}
C_1 &= \frac{1}{3} \int_a^b G(r) dr - \frac{a}{3(a + 2\beta)} \lambda \int_a^b \left\{ (M_3 + M_4) - 3(a + 2\beta) \left( \frac{r}{a} - \frac{1}{\lambda^2 a^2} \right) \right\} \\
\sinh[\lambda(r - a)] - \left\{ \lambda r M_3 + \frac{M_4}{\lambda r} \right\} + 3(a + 2\beta) \left( \frac{r}{\lambda^2 a} - \frac{1}{\lambda^2 a^2} \right) \\
\cosh[\lambda(r - a)] \right\} G(r) dr - \frac{a}{3(a + 2\beta)} C_3 \left[ M_5 \cosh(\lambda d) + M_6 \sinh(\lambda d) \right] + \frac{a}{3(a + 2\beta)} C_4 \\
\left[ M_5 \sinh(\lambda d) + M_6 \cosh(\lambda d) \right] \\
C_2 &= -\frac{1}{3} \int_a^b r^3 G(r) dr + \frac{a^2}{3(a + 2\beta)} \lambda \int_a^b \left\{ (M_3 + M_4) \sinh[\lambda(r - a)] - \left( \lambda r M_3 + \frac{M_4}{\lambda r} \right) \\
\cosh[\lambda(r - a)] \right\} G(r) dr + \frac{a^2}{3(a + 2\beta)} C_3 \left[ M_1 \cosh(\lambda d) + M_2 \sinh(\lambda d) \right] - \frac{a^2}{3(a + 2\beta)} C_4 \\
C_3 &= \frac{1}{M_6} C_4 + \frac{\beta^2}{3\lambda^2} \int_a^b G(r) dr - \frac{1}{3\lambda M_6} \int_a^b r^3 G(r) dr \\
C_4 &= -\left( \frac{a + 2\beta}{\lambda M_6} \right) \int_a^b G(r) dr + \frac{1}{\lambda M_6} \int_a^b \left\{ (M_3 + M_4) - 3(a + 2\beta) \left( \frac{r}{a} - \frac{1}{\lambda^2 a^2} \right) \right\} \\
\sinh[\lambda(r - a)] - \left\{ \lambda r M_3 + \frac{M_4}{\lambda r} \right\} + 3(a + 2\beta) \left( \frac{r}{\lambda^2 a} - \frac{1}{\lambda^2 a^2} \right) \right\} \cosh[\lambda(r - a)] G(r) dr \\
- \frac{3\beta^2(a + 2\beta)}{2a M_6} + \frac{\lambda^2(a + 2\beta)}{2a M_6} \int_a^b r^2 G(r) dr + \frac{\beta^2 M_7}{3\lambda^3 M_6} \int_a^b G(r) dr \\
- \left( \frac{\lambda^2(a + 2\beta)}{2a M_6} + \frac{M_7}{3\lambda^3 M_6} \right) \int_a^b \lambda^3 G(r) dr \\
\end{align*}
$$

and

$$
\begin{align*}
D_1 &= \frac{b_5}{30} \int_\infty^b G(r) dr + \frac{1}{30} \int_\infty^b r^5 G(r) dr - C_2 + C_3 \left( \lambda b + \frac{\lambda^3 b^3}{6} \right) - C_4 \left( \frac{\lambda^2 b^2}{2} + 1 \right) \\
D_2 &= 0, \quad D_3 = 0 \\
\end{align*}
$$
The coefficients $C_1, C_2, C_3, C_4$ involves $M_i (i = 1, 2, ..., 12)$ are given by

\[
\begin{align*}
M_1 &= \frac{6\lambda^2 \beta}{a^4} + \frac{\lambda^2 \beta}{a^3} + \frac{3\lambda^3}{a^2} \\
M_2 &= \frac{3\lambda^4 \beta}{a^4} + \frac{6\lambda^3 \beta}{a^3} + \frac{\lambda^4}{a^2} + \frac{3\lambda^3}{a} \\
M_3 &= \beta \left( \frac{3}{a^2} - \frac{6}{a^2 + a^2} \right) - \left( \frac{\beta}{a} + \frac{3}{a^2 + a^2} \right) \\
M_4 &= \beta \left( \frac{6r}{a^2} + \frac{\lambda^2 r}{a} \right) + \frac{3r}{a^2} \\
M_5 &= \frac{\lambda^2 \beta}{a^2} \\
M_6 &= \frac{3\lambda^4 \beta}{a^2} + \frac{\lambda^4}{a} \\
M_7 &= \frac{\lambda^2 \beta}{a} \cosh(\lambda d) + \left( \frac{3\lambda^4 \beta}{a^2} + \frac{\lambda^4}{a} \right) \sinh(\lambda d) - \frac{\lambda^3 (a + 2\beta)}{2a} \\
M_8 &= \frac{\lambda^2 \beta}{a} \sinh(\lambda d) + \left( \frac{3\lambda^4 \beta}{a^2} + \frac{\lambda^4}{a} \right) \cosh(\lambda d) - \frac{3\lambda^3 (a + 2\beta)}{2ab} \\
M_9 &= \left( \frac{\lambda^2 \beta}{a^2} - \frac{3\lambda^4 \beta}{a^2 b} - \frac{\lambda^4}{a b} \right) \sinh(\lambda d) + \left( \frac{3\lambda^4 \beta}{a^2} + \frac{\lambda^4}{a} - \frac{\lambda^3 \beta}{a b} \right) \cosh(\lambda d) - \frac{\lambda^4 (a + 2\beta)}{ab} \\
M_{10} &= b^2 \lambda^4 + \frac{\lambda^4}{a^2} + \alpha^2 \lambda^4 + a^2 \lambda^4 \cosh(\lambda d) + \lambda^2 \sinh(\lambda d) \\
M_{11} &= b^2 \lambda^4 + \frac{\lambda^4}{a^2} + a^2 \lambda^4 \sinh(\lambda d) + \lambda^2 \cosh(\lambda d) \\
M_{12} &= \left[ \frac{\lambda^2 b}{b} - \frac{\lambda^2}{a^2 + a^2} \right] \cosh(\lambda d) + \left( \frac{3\lambda^4 \beta}{a^2 b} + \frac{\lambda^4}{a b} \right) \sinh(\lambda d) + \left[ \frac{\lambda^2 b}{b} - \frac{\lambda^2}{a^2 + a^2} \right] \cosh(\lambda d) + \left( \frac{3\lambda^4 \beta}{a^2 b} + \frac{\lambda^4}{a b} \right) \sinh(\lambda d)
\end{align*}
\]

The constant $L_i (i = 1, 2, 3, 4, 5)$ in general mobility expression (18) are given by

\[
\begin{align*}
L_1' &= \left[ a^2 \lambda^2 + \frac{a^4}{3(a + 2\beta)} - \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \sinh(\lambda d) - \left[ a^2 \lambda^2 + \frac{a^4}{3(a + 2\beta)} - \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \cosh(\lambda d) \\
L_2' &= \left[ a^2 \lambda^2 - \frac{a^4}{3(a + 2\beta)} + \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \sinh(\lambda d) - \left[ a^2 \lambda^2 - \frac{a^4}{3(a + 2\beta)} + \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \cosh(\lambda d) \\
L_3' &= \left[ a^2 \lambda^2 - \frac{a^4}{3(a + 2\beta)} + \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \sinh(\lambda d) - \left[ a^2 \lambda^2 - \frac{a^4}{3(a + 2\beta)} + \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \cosh(\lambda d) \\
L_4' &= \left[ a^2 \lambda^2 - \frac{a^4}{3(a + 2\beta)} + \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \sinh(\lambda d) - \left[ a^2 \lambda^2 - \frac{a^4}{3(a + 2\beta)} + \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \cosh(\lambda d) \\
L_5' &= \left[ a^2 \lambda^2 - \frac{a^4}{3(a + 2\beta)} + \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \sinh(\lambda d) - \left[ a^2 \lambda^2 - \frac{a^4}{3(a + 2\beta)} + \frac{a^2}{2b^2} + \frac{a^4}{2b^2} + \frac{ab}{b(a + 2\beta)} + \frac{ab}{b(a + 2\beta)} \right] \cosh(\lambda d)
\end{align*}
\]
The addition term $\mu_{add}$ involved in mobility expression (22) is given below

$$
\mu_{add} = -\frac{(\kappa a)^2(L_3' + L_4')}{2(\lambda b)^3L_1'} \left[ \left\{ \frac{1}{\lambda b} \left( 1 + \frac{a^3}{2b^3} \right) + \frac{\kappa}{\kappa + \lambda} \right\} e^{-(\kappa + \lambda)(b-a)} - \frac{3}{2\lambda a} - \frac{\kappa}{\kappa + \lambda} \right]
- \frac{(\kappa a)^2(L_3' - L_4')}{2(\lambda b)^3L_1'} \left[ \left\{ \frac{1}{\lambda b} \left( 1 + \frac{a^3}{2b^3} \right) - \frac{\kappa}{\kappa - \lambda} \right\} e^{-(\kappa - \lambda)(b-a)} - \frac{3}{2\lambda a} + \frac{\kappa}{\kappa - \lambda} \right]
+ \frac{\kappa^2}{\lambda^2} \frac{L_3'}{L_1'} - 1 + \frac{a^3L_3'}{2b^3L_1'} \left[ \left( 1 + \frac{2}{\kappa a} + \frac{2}{\kappa^2 a^2} \right) \right] - \frac{2ka}{3} \left[ 1 - \frac{L_2'}{L_1'} (1 + \frac{1}{\kappa b}) \right]
- \frac{\kappa}{\lambda^2 b} \left( 1 + \frac{a^3}{2b^3} \right) \left( e^{-(\kappa - \lambda)(b-a)} - 1 \right) + \frac{\beta (kb)^2 L_5'}{3\lambda^2 b^2(a + 2b)L_1'} \left( \frac{3}{\lambda^2 b^2} - \frac{a}{\lambda^2 b^2} \right) \left( \frac{\kappa}{\kappa + \lambda} \right)
\left[ e^{-(\kappa + \lambda)(b-a)} - 1 \right] - \frac{\beta (kb)^2 L_5'}{3\lambda^2 b^2(a + 2b)L_1'} \left( \frac{3}{\lambda^2 b^2} + \frac{a}{\lambda^2 b^2} \right) \left( 1 + \frac{a^3}{2b^3} \right) e^{(\kappa - \lambda)(a-b)}
- \frac{\beta (kb)^2 L_5'}{3\lambda^2 b^2(a + 2b)L_1'} \left( \frac{a}{\lambda^2 b^2} - 3 \frac{\kappa}{\lambda^2 b^2} \right) \left( 1 + \frac{a^3}{2b^3} \right) e^{(\kappa + \lambda)(a-b)} + \frac{\beta (kb)^2 L_5'}{\lambda^2 b^4(a + 2b)L_1'}
$$

(A.5)

The coefficient $f_i$ ($i = 1, 2, 3, 4$) appears in mobility expression (38) are given below

$$
\begin{align*}
 f_1 &= 1 - \frac{1}{\kappa b} + \frac{(1-\kappa a)(1+kb)}{(1+\kappa a)\kappa b} e^{-2\kappa(b-a)} \\
 f_2(\kappa, \lambda) &= \frac{\kappa a}{2} e^{(\kappa + \lambda)a} \left[ E_3((\kappa + \lambda)a) - (\frac{\kappa}{2})^2 E_3((\kappa + \lambda)b) \right] - \frac{3}{2\lambda a} e^{(\kappa + \lambda)a} \left[ E_5((\kappa + \lambda)a) \right] \\
 - (\frac{\kappa}{b})^3 E_5((\kappa + \lambda)b) + \frac{3}{2\lambda a} + \frac{\kappa}{\kappa + \lambda} - \left\{ \frac{1}{\lambda b} \left( 1 + \frac{a^3}{2b^3} \right) + \frac{\kappa}{\kappa + \lambda} \right\} e^{-(\kappa + \lambda)(b-a)} \\
 f_3(\kappa) &= \frac{3}{2} e^{\kappa a} \left[ E_5(\kappa a) - (\frac{\kappa}{b})^3 E_5(kb) \right] + \frac{a^3}{2b^3} e^{-(\kappa(b-a))(1 + \frac{a^3}{2b^3})} - \frac{3}{2} \\
 f_4 &= \frac{a^3}{2b^3} e^{-(\kappa(b-a))(1 + \frac{a^3}{2b^3} + \frac{3}{\kappa a})} - \frac{3a^3}{4b^3} \left( 1 + \frac{2}{\kappa a} + \frac{2}{\kappa^2 a^2} \right)
\end{align*}
$$

(A.6)

B-Evaluation of drag on a composite soft particle with hydrophobic inner core

The hydrodynamic drag $F_D$ can be obtained as

$$
F_D = 2\pi b^2 \int_0^\pi \left[ \tau_{rr} \cos \theta - \tau_{r\theta} \sin \theta \right] \sin \theta d\theta \quad (B.1)
$$

where $\tau_{rr}$ and $\tau_{r\theta}$ are normal and shear stress, respectively. Equation (B.1) involves the velocity field and pressure term which may be obtained by solving the fluid flow equations

$$
\begin{align*}
\eta \nabla^2 \mathbf{u} - \nabla p - \eta \mathbf{u} \cdot \nabla \mathbf{u} &= 0, \quad a \leq r < b \\
\eta \nabla^2 \mathbf{u} - \nabla p &= 0, \quad r > b
\end{align*}
$$

(B.2)
along with continuity equation as given in equation (2), subject to the following boundary conditions

\[
\begin{align*}
    u_r(r, \theta) &= 0, \quad 0 \leq \theta < 2\pi \\
    u_r(a, \theta) &= \beta r \frac{d}{dr} \left( \frac{u_a}{r} \right) \quad \text{on} \quad r = a, \quad 0 \leq \theta < 2\pi \\
    u_r(b^-, \theta) &= u_r(b^+, \theta), \quad 0 \leq \theta < 2\pi \\
    u_\theta(b^-, \theta) &= u_\theta(b^+, \theta), \quad 0 \leq \theta < 2\pi \\
    \tau_{rr}(b^-, \theta) &= \tau_{rr}(b^+, \theta), \quad 0 \leq \theta < 2\pi \\
    \tau_{r\theta}(b^-, \theta) &= \tau_{r\theta}(b^+, \theta), \quad 0 \leq \theta < 2\pi \\
    u_r(r, \theta) &= -U \cos \theta \quad \text{as} \quad r \to \infty, \quad 0 \leq \theta < 2\pi \\
    u_\theta(r, \theta) &= U \sin \theta \quad \text{as} \quad r \to \infty, \quad 0 \leq \theta < 2\pi
\end{align*}
\]  

(B.3)

Here \( u_r \) and \( u_\theta \) are radial and cross-radial velocity components, respectively. We follow the similar procedure as adopted by Masliyah et al.\(^1 \) to solve the velocity equations evaluate the stress tensors. From equation (B.1) after a lengthy algebra we obtain the drag force as

\[
F_D = -6\pi \eta Ub \Omega' \quad (B.4)
\]

Here the correction factor \( \Omega' \) is due to the presence of surface PEL and it may be derived as follows

\[
\Omega' = \frac{2B'}{3\lambda b} \quad (B.5)
\]

where

\[
\begin{align*}
    B' &= \left( \frac{\lambda b}{2} \sinh(\lambda b) - \frac{1}{2} \cosh(\lambda b) \right) G' + \left( \frac{\lambda b}{2} \cosh(\lambda b) - \frac{1}{2} \sinh(\lambda b) \right) H' \\
    G' &= \frac{1}{N_1} \left( \frac{2\lambda^2 b^3}{a} + \lambda^2 a^2 \right) - \frac{N_2}{N_1} H' \\
    H' &= \frac{N_3}{N_5} \left( \frac{2\lambda^2 b^3}{a} + \lambda^2 a^2 \right) + 2(\lambda b)^3 \left( \frac{1}{\lambda^2 a^2} + \frac{2\beta}{\lambda^2 a^2(\lambda a + \lambda b)} \right) \frac{N_3}{N_5} \\
    &\quad - \left( \frac{2\lambda a}{a + 2\beta} \right) \frac{N_3}{N_5} \\
    N_1 &= \left( \frac{2\lambda^2 b^3}{3a} + \frac{b}{a} + \frac{\lambda^2 a^2}{3} \right) \sinh(\lambda b) - \frac{1}{\lambda a} \cosh(\lambda b) + \frac{1}{\lambda a} \cosh(\lambda a) - \sinh(\lambda a) \\
    N_2 &= \left( \frac{2\lambda^2 b^3}{3a} + \frac{b}{a} + \frac{\lambda^2 a^2}{3} \right) \cosh(\lambda b) - \frac{1}{\lambda a} \sinh(\lambda b) + \frac{1}{\lambda a} \sinh(\lambda a) - \cosh(\lambda a) \\
    N_3 &= \left\{ \frac{\lambda}{\lambda^2 a^2} - \frac{2\beta}{\lambda^2 a^2(\lambda a + \lambda b)} \right\} \sinh(\lambda b) + \left\{ \frac{-\lambda}{\lambda^2 a^2} - \frac{2\lambda a - 2\lambda a^2}{a + 2b} \right\} \cosh(\lambda b) \\
    &\quad + \left\{ \lambda a \right\} + \frac{\beta}{a + 2\beta} \left( \frac{2}{\lambda a} + \lambda a \right) \sinh(\lambda a) \\
    N_4 &= \left\{ \frac{-\lambda}{\lambda^2 a^2} - \frac{2\beta}{\lambda^2 a^2(\lambda a + \lambda b)} \right\} \frac{2\lambda b}{3} \cosh(\lambda b) + \frac{1}{3} \left( \frac{2\lambda a - 2\lambda a^2}{a + 2b} \right) \cosh(\lambda b) \\
    &\quad + \left\{ \frac{\lambda}{\lambda^2 a^2} + \frac{2}{\lambda^2 a^2(\lambda a + \lambda b)} \right\} \sinh(\lambda b) + \left\{ \frac{-\lambda}{\lambda^2 a^2} - \frac{2\lambda a - 2\lambda a^2}{a + 2b} \right\} \sinh(\lambda a) \\
    &\quad + \left\{ \frac{\lambda}{\lambda^2 a^2} + \frac{2}{\lambda^2 a^2(\lambda a + \lambda b)} \right\} \cosh(\lambda a) \\
    N_5 &= N_2N_3 - N_1N_4
\end{align*}
\]  

(B.6)

It may be noted that the soft particle with hydrophilic inner core (i.e., \( \beta \to 0 \)), the correction factor \( \Omega' \) approaches to \( \Omega \), as given in equation (32b) in Masliyah et al.\(^1 \). After doing lengthy
algebra we may further simplify the equation (B.4) as

\[ F_D = -6\pi \eta b \left[ \frac{a}{b} \left( \frac{L'_2}{L'_1} + \frac{3L'_3}{2\lambda^2 b^2 L'_1} \right) \right]^{-1} U \]  

(B.7)

where the coefficient \( L'_1, L'_2 \) and \( L'_3 \) are defined in equation (A.4).

C- Soft particle with charged superhydrophobic inner core and uncharged PEL

Here we have considered the soft particle with superhydrophobic inner core, i.e., \( \beta/a >> 1 \) and \( \beta/b >> 1 \). Under these assumption the mobility expression given in equation (22)
reduces to

\[
\mu_E = \frac{\epsilon_r\epsilon_0}{\eta} \left[ \frac{(ka)^2}{6} \left( \frac{a}{b} \right)^2 e^{\kappa a} \left[ E_3(\kappa b) - 3 \left( 1 - \frac{2L''_2}{3L''_1} + \frac{2L''_3}{\lambda^2b^2 L''_1} \right) E_5(\kappa b) \right] \right] \\
+ \frac{3(ka)^2(L''_3 + L''_4)}{4(\lambda b)^3 L''_1} e^{(\kappa + \lambda)a} \left[ \frac{\kappa a}{3} \left\{ E_3((\kappa + \lambda)a) - \left( \frac{a}{b} \right)^2 E_3((\kappa + \lambda)b) \right\} \right] \\
- \frac{1}{\lambda a} \left\{ E_5((\kappa + \lambda)a) - \left( \frac{a}{b} \right)^4 E_5((\kappa + \lambda)b) \right\} - \frac{3(ka)^2(L''_3 - L''_4)}{4(\lambda b)^3 L''_1} e^{(\kappa - \lambda)a} \\
\left[ \frac{\kappa a}{3} \left\{ E_3((\kappa - \lambda)a) - \left( \frac{a}{b} \right)^2 E_3((\kappa - \lambda)b) \right\} + \frac{1}{\lambda a} \left\{ E_5((\kappa - \lambda)a) - \left( \frac{a}{b} \right)^4 E_5((\kappa - \lambda)b) \right\} \right] \\
- \left( \frac{a}{b} \right)^2 E_5((\kappa - \lambda)b) \right] \right] + \left( \frac{\kappa}{\lambda} \right)^2 \left( 1 - \frac{L''_2}{L''_1} \right) e^{\kappa a} \left[ E_5(\kappa a) - \left( \frac{a}{b} \right)^4 E_5(\kappa b) \right] \\
- \frac{(ka)^2(L''_3 + L''_4)}{2(\lambda b)^3 L''_1} \left[ \left\{ \frac{1}{\lambda b} \left( 1 + \frac{a^3}{2b^3} \right) + \frac{\kappa}{\kappa + \lambda} \right\} e^{-(\kappa + \lambda)(b-a)} - \frac{3}{2\lambda a} - \frac{\kappa}{\kappa + \lambda} \right] \\
- \frac{(ka)^2(L''_3 - L''_4)}{2(\lambda b)^3 L''_1} \left[ \left\{ \frac{1}{\lambda b} \left( 1 + \frac{a^3}{2b^3} \right) - \frac{\kappa}{\kappa - \lambda} \right\} e^{-(\kappa - \lambda)(b-a)} - \frac{3}{2\lambda a} + \frac{\kappa}{\kappa - \lambda} \right] \\
+ \frac{\kappa}{\lambda^2 b} \left[ \frac{L''_3}{L''_1} - 1 + \frac{a^3 L''_3}{2b^3 L''_1} \left( 1 + \frac{2}{\kappa a} + \frac{2}{\kappa a^2} \right) \right] - \frac{2\kappa a}{3} \left[ 1 - \frac{L''_2}{L''_1} \left( 1 + \frac{1}{\kappa b} \right) \right] \\
- \frac{\kappa}{\lambda^2 b} \left( 1 + \frac{a^3}{2b^3} \right) e^{-\kappa(b-a)} - \frac{(kb)^2 L''_5}{4\lambda^2 b^2 L''_1} e^{(\kappa - \lambda)a} \left( \frac{3}{\lambda^2 b^2} + \frac{a}{\lambda b^2} \right) \left[ \frac{\kappa a}{3} \right] \\
\left\{ E_3((\kappa - \lambda)a) - \left( \frac{a}{b} \right)^2 E_3((\kappa - \lambda)b) \right\} + \frac{1}{\lambda a} \left\{ E_5((\kappa - \lambda)a) - \left( \frac{a}{b} \right)^4 E_5((\kappa - \lambda)b) \right\} \right] \\
- \frac{(kb)^2 L''_5}{4\lambda^2 b^2 L''_1} e^{(\kappa + \lambda)a} \left( \frac{3}{\lambda^2 b^2} - \frac{a}{\lambda b^2} \right) \left[ \frac{\kappa a}{3} \right] \left\{ E_3((\kappa + \lambda)a) \right\} \\
- \left( \frac{a}{b} \right)^2 E_3((\kappa + \lambda)b) \right\} - \frac{1}{\lambda a} \left\{ E_5((\kappa + \lambda)a) - \left( \frac{a}{b} \right)^4 E_5((\kappa + \lambda)b) \right\} \right] \\
+ \frac{(kb)^2 L''_5}{6\lambda^2 b^2 L''_1} \left( \frac{3}{\lambda^2 b^2} + \frac{a}{\lambda b^2} \right) \left( \frac{\kappa}{\kappa - \lambda} \right) e^{-(\kappa - \lambda)(b-a)} - 1 \right] \\
+ \frac{(kb)^2 L''_5}{6\lambda^2 b^2 L''_1} \left( \frac{3}{\lambda^2 b^2} - \frac{a}{\lambda b^2} \right) \left( \frac{\kappa}{\kappa + \lambda} \right) e^{-(\kappa + \lambda)(b-a)} - 1 \right] \\
- \frac{(kb)^2 L''_5}{6\lambda^2 b^2 L''_1} \left( \frac{3}{\lambda^3 b^3} + \frac{a}{\lambda^2 b^3} \right) \left( 1 + \frac{a^3}{2b^3} \right) e^{(\kappa - \lambda)(a-b)} \\
- \frac{(kb)^2 L''_5}{6\lambda^2 b^2 L''_1} \left( \frac{a}{\lambda^2 b^3} - \frac{3}{\lambda^3 b^3} \right) \left( 1 + \frac{a^3}{2b^3} \right) e^{(\kappa + \lambda)(a-b)} + \frac{(kb)^2 L''_5}{2\lambda^4 b^4 L''_1} \right]
\]

(C.1)
where

\[
\begin{align*}
L_1'' &= \left[ \frac{3a}{2\lambda^2b^2} - \frac{3a^2}{2\lambda^2b^3} - \frac{a^4}{4b^4} - \frac{a}{2b} + \frac{3a^3}{4b^3} + \frac{3}{2} \right] \cosh[\lambda(b - a)] \\
&\quad - \left[ \frac{3a^2}{2\lambda^2b^3} - \frac{3a^3}{2\lambda^2b^4} - \frac{a^4\lambda^2}{4b^5} - \frac{a\lambda^2}{2} + \frac{3a^3\lambda^2}{4b^4} + \frac{3}{2} \right] \frac{\sinh[\lambda(b-a)]}{\lambda b} \\
L_2'' &= \left[ \frac{3a}{2\lambda^2b^2} + \frac{3a^3}{4b^3} + \frac{3}{2} \right] \cosh(\lambda d) + \left[ \frac{3a^2}{2b^2} + \frac{\lambda^2ab}{2} + \frac{\lambda^2a^2}{4b^2} \right] \frac{\sinh(\lambda d)}{\lambda b} - \frac{3a}{2\lambda^2b^2} \\
L_3'' &= \left[ \frac{3a}{2\lambda^2b^2} - \frac{a}{2b} \right] \cosh(\lambda d) - \left[ \frac{3a}{2\lambda b} - \frac{1}{2\lambda} \right] \cosh(\lambda d) + \frac{1}{\lambda b} \\
L_4'' &= \left( \frac{3a^2}{2b^2} - \frac{3a}{2b} \right) \cosh(\lambda d) + \left( \frac{3a^2}{2b^2} - \frac{3a^2\lambda}{2b} \right) \sinh(\lambda d) - ab\lambda^2 - \frac{a^2\lambda^2}{2b^2}
\end{align*}
\] (C.2)

Various limiting situations for composite soft particle with superhydrophobic inner core are given below. The modified Huc'c and Smoluchowski formula for electrophoretic mobility of soft particle with superhydrophobic inner core may be derived as follows

\[
\mu_E = \frac{\epsilon_r \epsilon_0 \zeta}{\eta} \frac{a}{b} \left( \frac{L_2''}{L_1''} + \frac{3L_3''}{2\lambda^2b^2L_1''} \right)
\] (C.3)

and

\[
\mu_E = \frac{\epsilon_r \epsilon_0 \zeta}{\eta} \left[ - \frac{5}{\lambda^2a^2} + \frac{3L_4''}{2L_1''} \left( \frac{a}{b} \right)^3 \left( \frac{1}{\lambda a} + \frac{5}{\lambda^2a^2} \right) + \frac{L_3''}{L_1''} \left( \frac{5}{\lambda^2a^2} + \left( \frac{15}{2\lambda^2a^2} + \frac{4}{\lambda^2a^2} \right) \left( \frac{a}{b} \right)^3 \right) \right] \\
\quad - \frac{\beta L_5''}{(a + 2\beta)L_1''} \left( \frac{10}{\lambda^4a^2b^2} + \frac{1}{\lambda^2b^2} \right)
\] (C.4)

For a rigid superhydrophobic colloid, the corresponding electrophoretic mobility is given as

\[
\mu_E = \frac{\sigma a}{(1 + \kappa a)\eta} \left[ 1 + 2e^{\kappa a}E_5(\kappa a) - 5e^{\kappa a}E_7(\kappa a) \right] + \frac{1}{3} \frac{\sigma a}{\eta} \left[ 1 + \frac{(\kappa a)^2}{2(1 + \kappa a)}e^{\kappa a}E_5(\kappa a) \right]
\] (C.5)

For plate-like soft particle comprised of superhydrophobic inner core, we may derive the simple formula for electrophoretic mobility as follows

\[
\mu_E = \frac{\epsilon_r \epsilon_0 \zeta}{\eta} \left[ \frac{\kappa}{\lambda \sinh(\lambda d)} \frac{\exp(-kd)}{\sinh(\lambda d)(\kappa^2 - \lambda^2)} \left( (\lambda^2 \sinh(\lambda d) - \kappa \lambda \sinh(\kappa d)) + \kappa \lambda (\cosh(\lambda d) - \cosh(\kappa d)) \right) \right]
\] (C.6)

which further reduces to the simple expression \( \kappa >> \lambda \) as follows

\[
\mu_E = \frac{\epsilon_r \epsilon_0 \zeta}{\eta} \frac{\kappa}{\lambda \sinh(\lambda d)}
\] (C.7)
D- Soft particle with uncharged superhydrophobic inner core and charged PEL

For a soft particle with superhydrophobic inner core (i.e., \( \beta/a >> 1 \) and \( \beta/b >> 1 \)), the mobility expression (38) reduces to

\[
\mu_E = \frac{\rho_{fix} b^2 f_1}{2\eta} \left[ \frac{a^3}{6b^3} e^{kb} E_3(kb) - \frac{a^3}{2b^3} \left( 1 - \frac{2L''}{3L''_1} + \frac{2L'''_3}{\lambda^2 b^2 L''_1} \right) e^{kb} E_5(kb) - \frac{2}{3kb} \right] \\
+ \frac{L'''_2}{\lambda^2 b^2 L''_1} \left( 1 + \frac{a^3}{2b^3} + \frac{1}{kb} \right) + \frac{2L'''_2}{3kb L''_1} \left( 1 + \frac{1}{kb} \right) \right] - \frac{\rho_{fix} a^2 (1 + \kappa b)}{4\eta \lambda^3 b^3 (1 + \kappa a)} e^{-\kappa (b-a)} \\
\left[ \left( \frac{L''_3 + L''_4}{L''_1} \right) \left\{ \left( 1 + \frac{1}{\kappa a} \right) f_2(\kappa, \lambda) + \left( 1 + \frac{1}{\kappa a} \right) f_2(-\kappa, \lambda) \right\} - \frac{\rho_{fix} (1 + \kappa b)}{3\eta \lambda^2 (1 + \kappa a)} e^{-\kappa (b-a)} \left[ \left( 1 - \frac{L''_3}{L''_1} \right) \left\{ \left( 1 + \frac{1}{\kappa a} \right) f_3(\kappa) \right. \right. \\
+ \left( 1 + \frac{1}{\kappa a} \right) f_3(-\kappa) \right\} - \frac{L''_3}{L''_1} \left\{ \left( 1 - \frac{1}{\kappa a} \right) f_4(\kappa) + \left( 1 + \frac{1}{\kappa a} \right) f_4(-\kappa) \right\} \right] \\
- \frac{\rho_{fix} L''_5 (1 + \kappa b)}{12\eta \lambda^3 L''_1 (1 + \kappa a)} e^{-\kappa (b-a)} \left[ \left( \frac{a}{\lambda b^2} - \frac{3}{\lambda^2 b^2} \right) \left\{ \left( 1 - \frac{1}{\kappa a} \right) f_2(\kappa, \lambda) + \left( 1 + \frac{1}{\kappa a} \right) f_2(-\kappa, \lambda) \right\} \\
- \left( \frac{a}{\lambda b^2} + \frac{3}{\lambda^2 b^2} \right) \left\{ \left( 1 - \frac{1}{\kappa a} \right) f_2(\kappa, -\lambda) + \left( 1 + \frac{1}{\kappa a} \right) f_2(-\kappa, -\lambda) \right\} \right] \tag{D.1}
\]

where the constants \( L''_i (i = 1, 2, 3, 4, 5) \) are already given in equation (C.2) We have further illustrated various limiting cases for the undertaken soft particle with superhydrophobic inner core.

The modified Huckel and Smoluchowski formula for electrophoretic mobility of soft particle with superhydrophobic inner core may be derived as follows

\[
\mu_E = \frac{2(b^3 - a^3) \rho_{fix}}{9\eta b} \left( \frac{L''_2}{L''_1} + \frac{3L'''_3}{2\lambda^2 b^2 L''_1} \right) \tag{D.2}
\]

and

\[
\mu_E = \frac{\rho_{fix}}{\eta \lambda^2} \left( 1 + \frac{a^3}{b^3} \right) \left\{ \frac{L''_3}{L''_1} - \frac{1}{3} + \frac{a(1 - \lambda b)}{4\lambda^2 b^3} \left( \frac{L''_3 - L'''_4}{L''_1} \right) e^{\lambda(b-a)} + \frac{a(1 + \lambda b)}{4\lambda^2 b^3} \left( \frac{L''_3 + L'''_4}{L''_1} \right) e^{-\lambda(b-a)} \right\} \\
+ \frac{\rho_{fix} L''_5}{12\eta \lambda^3 L''_1} \left( 1 + \frac{a^3}{b^3} \right) \left\{ \left( \frac{1}{\lambda a} + \frac{b}{a} \right) \left( \frac{a}{\lambda b^2} - \frac{3}{\lambda^2 b^2} \right) e^{-\lambda(b-a)} + \left( \frac{1}{\lambda a} - \frac{b}{a} \right) \left( \frac{a}{\lambda^2 b^2} + \frac{3}{\lambda^2 b^2} \right) e^{\lambda(b-a)} \right\} \tag{D.3}
\]

where \( L''_i (i = 1, 2, 3, 4, \text{ and } 5) \) are defined in equation (C.2).

For the semi-soft particle comprised of superhydrophobic inner core the mobility expression
may be derived as follows

\[ \mu_E = \frac{\rho_{fix}}{2\eta \kappa^2} \left[ \frac{2}{3} \left( 1 + \frac{a^3}{2b^3} \right) - \left( \frac{a}{b} \right)^3 e^{\kappa b} \left\{ 5E_7(\kappa b) - 2E_5(\kappa b) \right\} \right] \]  

(D.4)

For plate-like soft particle comprised of superhydrophobic inner core, we may derive the simple formula for electrophoretic mobility given as

\[ \mu_E = \frac{\rho_{fix}}{\eta \lambda^2} \left[ \frac{1}{\left( \frac{\kappa^2}{\lambda^2} - 1 \right)} \left\{ 1 - e^{-\kappa d} \left\{ \frac{\lambda \sinh(\kappa d)}{\kappa} \left( \frac{\lambda}{\kappa} + \frac{1}{\tanh(\lambda d)} \right) \right\} \right\} \right] + \frac{\rho_{fix}}{\eta \lambda^2} \]  

(D.5)

which further reduces to the simple expression \( \kappa >> \lambda \) as follows

\[ \mu_E = \frac{\rho_{fix}}{\eta \lambda^2} \]  

(D.6)

**Bibliography**