

On Compactness and Projections

by

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0. It is a well known and frequently useful fact that whenever a topological space X is compact, the projection π of $X \times Y$ on any space Y is a closed mapping ([Du], p.227). (The converse, although not so well known is also true [M].) Analogues of this theorem concerning countable compactness, sequential compactness, m -compactness, etc., fairly abound in the literature ([PL], [H₂], [I], [N], etc.) In this note, using convergence as our basic concept, we propose to derive improvements and extensions of many of these results, as well as a number of related facts, in a somewhat uniform manner.

Whenever X is countably compact and Y is a sequential space, π is closed. Following Isiwata [I] take (\S) to be the class of space Y such that π is closed for each countably compact X . (3) As Isiwata points out, each subspace and each closed continuous image of a space in (\S) is again in (\S) . Hence every subspace of a sequential space is in (\S) . (We are unable to decide whether every space in (\S) is a subspace of a sequential space.) As an example of a space not in (\S) , take a point $p \in \beta\mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Cech compactification of the natural numbers, and consider $\mathbb{N} \cup \{p\}$ as a subspace of $\beta\mathbb{N}$. Then $\mathbb{N} \setminus \{p\}$ is countably compact, but the projection of $(\mathbb{N} \setminus \{p\}) \times (\mathbb{N} \cup \{p\})$ on $\mathbb{N} \cup \{p\}$ is not closed. Thus $\mathbb{N} \cup \{p\}$ does not belong to (\S) . It follows immediately that no superset of $\mathbb{N} \cup \{p\}$ belongs to (\S) and, in particular $\beta\mathbb{N} \notin (\S)$ (see [I]2.3).*

Letting S_n be a space consisting of a convergent sequence and its limit, i.e., homeomorphic to the subset $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ of the real line, the following is a strong converse to the first assertion of the previous paragraph: if the projection of $X \times S_1$ is closed, then X is countably compact. Let us denote

MAR 21 1968

by (f) the class of spaces Y such that X is countably compact whenever the projection of $X \times Y$ on Y is closed. Hanai has

$$(5) \quad \neg \text{P-space} \rightarrow \text{(T)}$$

shown that every space which is not a P-space belongs to (T) ([H,] ; Theorem 4). It follows immediately that $\text{P-space} \rightarrow \text{(T)}$. Conversely, no P-space belongs to (T) , so that (T) is precisely the complement of the class of P-spaces. Also, no non-discrete P-space belongs to (T) , from which it follows that the non-discrete spaces in (N) also belong to (T) . However, $\mathbb{N} \cup \{p\}$, since it is not a P-space belongs to (T) but not to (T) . Every non-discrete P-space belongs to neither (T) nor (T) .

Since sequential compactness implies countable compactness and every Frechet space is sequential, it follows that whenever X is sequentially compact and Y is a Frechet space, XY is closed. Conversely if the projection of $X \times S_1$ on S_1 is closed, then X is sequentially compact. (7) These results suggest that classes (g_s) and (T_s) be defined bearing the same relation to sequential compactness as (N) and (j) do to countable compactness. Since there are compact spaces which are not sequentially compact (for example \mathbb{N}), (5) is empty. Since sequential compactness implies countable compactness (S) contains (S) . We conjecture that the inclusion is proper.

It is easily seen that the corresponding classes, (S_c) and (T_c) , for compactness are respectively all spaces and empty. Given Y , let X be the set of ordinals less than the initial ordinal of cardinality $2^{|X|}$. X is not compact but the projection of $X \times Y$ on Y is closed.

The unproved assertions of the preceding paragraphs follow readily from the more general considerations which are sketched out in the next section.

1. X and Y are topological spaces; π is the projection of $X \times Y$ on Y ; D and E are classes of nets into X , Y or $X \times Y$ closed under composition with and cancellation of maps between these spaces.

The following are equivalent: (i) For every $S \subseteq X \times Y$, Y is a limit of an E -net in $\text{TT}(S)$ only if $\text{TT}^{-1}(y)$ contains a limit of a D -net in S . (ii) For every (not necessarily continuous) function from Y to X , the range of every convergent E -net in Y contains a D -net converging to the same limit whose image under f converges in X .

Remarks: (i) implies (iii): If S contains all limits of its D -nets, $\text{TT}(S)$ will contain all limits of its E -nets. Conversely, if the adjunction of all D -limits to sets is an idempotent closure operator in $X \times Y$ (for example, if D is closed under iteration in the sense of the construction of [K] p.69), then (iii) implies (i).

(ii) follows for every Y if every E -net in X has a convergent D -subnet. Conversely, this holds for X if it satisfies (ii) with any class of Y containing for each member of E a point whose neighborhood filter induces on the range of some one-to-one E -net with the same domain exactly the image of the filter of final subsets. If the ranges of these one-to-one nets in the discrete topology (or more generally, in any for which only the eventually constant D -nets converge) along with their limits actually appear among the Y , then (ii) may be weakened to (iii). (For $X = T_1$, take for S the range of the product E -net; in general, take the union of its point-closures.)

We have use for two special cases: D the class of nets and

E the class of sequences; and $D = E$ the class of sequences.

We obtain respectively:

For every function from Y to X , every convergent sequence in Y has a subsequence whose image clusters in X iff ir sends sequential cluster points onto sequential limits iff ir sends sets containing their sequential cluster points onto sets containing their sequential limits. If X is countably compact this holds for every Y ; conversely if it holds for some Y containing S_{\perp} , X is countably compact.

For every function from Y to X , every convergent sequence in Y has a subsequence whose image converges in X iff ir sends sequential limits onto sequential limits. If X is sequentially compact this holds for every Y ; conversely if it holds for some Y containing S_{\perp} , X is sequentially compact; if, moreover, it holds for $Y = S_{\perp}$, then X is sequentially compact if ir only sends sets containing their sequential limits onto sets containing their sequential limits. (8)

For the remainder of the paper we deal only with $D = E =$ all directed sets with maps. (i) is equivalent to (iii) which says that TT is closed; they hold for all Y iff X is compact. We shall, however, want to take X countably compact and inquire about the corresponding Y .

One result follows from what we have shown so far: If ir is closed, it preserves closures of countable sets, hence sends sets containing sequential cluster points on sets containing sequential limits. Thus if Y contains S_{\perp} , ir closed implies X countably compact. The implication already holds if Y only contains a non-closed F_0 (Hanai [H^{\wedge}]); and for no other Y . We shall show this in a somewhat more general setting: If Y contains a non-closed

union of \mathcal{J}^c closed sets, then if closed implies that X contains no locally finite collection of \mathcal{J}^c closed sets (else the union of the products of paired closed sets in X and Y under any one-to-one pairing would be a closed set with a non-closed projection); conversely, if every \mathcal{J}^c closed subsets of Y have a closed union, then with X any T_1 space of cardinality at most $|\hat{V}|$, XY is closed. (9) Similarly, if Y contains an ascending well-ordered chain of closed subsets, with each set indexed by a limit ordinal the union of those preceding, which is not closed, and if X contains a corresponding descending chain of closed sets with empty intersection, then the obvious union in the product is a closed set whose projection is not closed; in particular, if Y is a non-discrete P -space and X a suitable transfinite ordinal, XY is not closed.

FOOTNOTES

- (1) Call a set sequentially open if no sequence outside the set converges to a point of the set, A space is sequential if every sequentially open subset is open $[F, \underline{1}]$.
- (2) This result improves Lemma 1.1 of [I] and one direction of each of Corollary 1.7 of $[K\underline{1}]$, Corollary 2 of $[H_2]$, and Theorem 1 of [N].
- (3) Let us note here that our (\tilde{N}) is apparently slightly different from Isiwata's since he deals only with completely regular spaces. The difference is real, since countably compact subsets of spaces in his (S) are closed ([I] Lemma 1.2) while countably compact subsets of a sequential space are closed iff sequential limits are unique ($[F_2\underline{J}]$ Prop. 5.4).
- (4) Lynn Imler points out that this improves an unpublished result of E. Michael which asserts that $\underline{N} \cup \{p\}$ cannot be embedded in any Hausdorff sequential space. Since it is not in (\tilde{N}) our result implies it is not a subspace of any sequential space.
- (5) A P-space is one in which every F_σ is closed ([G-J] p. 62).
- (6) A space is a Fréchet space if the closure of each of its subsets may be obtained by taking the limits of the sequences in the subset (c.f. $[F, \underline{1}]$). This is precisely the condition imposed by Isiwata in Lemma 1.1 of [I].
- (7) This strengthens the other direction of Theorem 1 of [N] (see (2)).
- (8) One might in the interests of symmetry, wish to consider also *ir* sending sequential cluster points onto sequential cluster points. This leads to a new kind of compactness implied by ordinary and implying countable; and properly as shown by $ox\underline{1}$ and $\#N \setminus \{p\}$; not implying sequential (else ordinary compactness would) and, we suppose, not implied by it, although we have no counterexample.
- (9) The authors wish to express their thanks to Renababy for the delectable Japanese meal which inspired this observation.

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