

Definite Integration

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Introduction

Mathematica's capability for definite integration gained substantial power in Version 3.0. Comprehensiveness and accuracy were two major trends that were given strong consideration and have been successfully accomplished in the new development. Definite integration procedures were tested against all major handbooks of integrals. *Mathematica* now is able to calculate about half of the definite and almost all indefinite integrals from the well-known collection of integrals compiled by Gradshteyn and Ryzhik. Moreover, Version 3.0 makes it possible to calculate thousands of new integrals not included in any published handbooks. Other essential features of **Integrate** are convergence tests, principal-value integrals, and the assumptions mechanism. In the next sections, we will discuss each of these features.

New Classes of Integrals

Integrate was significantly extended to handle new classes of integrals. Here are some of them:

■ integrals of rational functions

$$\int_1^2 \frac{x^3 + 1}{x^7 + x^3 + 1} dx$$
$$-\text{RootSum}[\#1^7 + \#1^3 + 1 \ \&, \frac{\log(1 - \#1) \#1^3 + \log(1 - \#1)}{7 \#1^6 + 3 \#1^2} \ \&] +$$
$$\text{RootSum}[\#1^7 + \#1^3 + 1 \ \&, \frac{\log(2 - \#1) \#1^3 + \log(2 - \#1)}{7 \#1^6 + 3 \#1^2} \ \&]$$

■ logarithmic and polylogarithmic integrals

$$\int_0^1 \frac{\text{Li}_2(x)}{(x+1)^2} dx$$

$$\frac{\log^2(2)}{2}$$

$$\int_0^1 \frac{\log(1-x) \log(1-x^2)}{x^2} dx$$

$$\frac{1}{4} (\pi^2 - 4 \log^2(2))$$

$$\int_0^\infty \frac{\eta e^{-x} - \frac{1-e^{-\eta x}}{x}}{x} dx$$

$$\text{If } [\text{Re}(\eta) > 0, \eta (\log(\eta) - 1), \int_0^\infty \frac{e^{-x} \eta - \frac{1-e^{-x\eta}}{x}}{x} dx]$$

■ elliptic integrals

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\sqrt{\cos(x)}} dx$$

$$4 F\left(\frac{\pi}{8} \mid 2\right)$$

$$\int_0^{\frac{\pi}{2}} \frac{x}{\sqrt{\sin(x)}} dx$$

$${}_3F_2\left(\frac{1}{2}, \frac{3}{4}, 1; \frac{3}{2}, \frac{3}{2}; 1\right)$$

■ integrals of Airy functions

$$\int_0^\infty x \text{Ai}(x) \text{Ai}(-x) dx$$

$$\frac{1}{2\sqrt{3}\pi}$$

$$\int_0^\infty \text{Ai}(x)^2 dx$$

$$-\frac{\Gamma(-\frac{1}{6})}{12\sqrt[3]{2}\sqrt[6]{3}\pi^{3/2}}$$

■ integrals involving Bessel functions

$$\int_0^{\infty} t e^{-t^2} I_2(t) K_2(t) dt$$

$$-3 + \frac{1}{4} \sqrt{e} K_2\left(\frac{1}{2}\right)$$

$$\int_0^{\infty} \frac{J_1(x)}{x} e^{-\frac{2}{x}} dx$$

$$2 J_1(2) K_1(2)$$

■ integrals involving non-analytic functions

$$\int_{-1-i}^{2+2i} x |x| dx$$

$$\frac{14 i \sqrt{2}}{3}$$

$$\int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} \text{Min}[\sin(x), \cos(x)] dx$$

$$-\frac{1}{\sqrt{2}} - 2\sqrt{2} + \frac{1}{2}(-2 - \sqrt{2})$$

RootReduce [%]

$$-1 - 3\sqrt{2}$$

Conditions

The major feature of the new integration code is that **Integrate** is now "conditional." In most cases, if the integrand or limits of integration contain symbolic parameters, **Integrate** returns an **If** statement of the form

`If[conditions, answer, held integral]`

which gives necessary conditions for the existence of the integral. For example:

$$\text{In[2]:= } \int_0^{\infty} \frac{x^{\lambda-1}}{x+1} dx$$

$$\text{If}[\text{Re}(\lambda) > 0 \wedge \text{Re}(\lambda) < 1, \pi \csc(\pi \lambda), \int_0^{\infty} \frac{x^{\lambda-1}}{x+1} dx]$$

Setting the option **GenerateConditions** to **False** prevents **Integrate** from returning conditional results (as in Version 2.2):

$$\text{Integrate}\left[\frac{x^{\lambda-1}}{x+1}, \{x, 0, \infty\}, \text{GenerateConditions} \rightarrow \text{False}\right]$$

$$\pi \csc(\pi \lambda)$$

One might ask why conditions are so necessary. The answer is because we are certain that all development should progress toward to complete rigor, or to the greatest rigor it is possible to achieve. If a given definite integral has symbolic parameters then the result of integration essentially always depends on certain specific conditions on those parameters. In the above example the restrictions $\text{Re}(\lambda) > 0$ and $\text{Re}(\lambda) < 1$ came from conditions of the convergence. Even when a definite integral is convergent, some other conditions on parameters might appear. For instance, the presence of singularities on the integration path could lead to essential changes when the parameters vary. The next section is devoted to the convergence of definite integrals.

Convergence

The new **Integrate** contains criteria for the convergence of the integral. Every time you call **Integrate**, it will examine the integrand for convergence:

$$\int_0^1 \frac{\cos(x)}{x} dx$$

Integrate::idiv : Integral of $\frac{\cos(x)}{x}$ does not converge on {0, 1}.

$$\int_0^1 \frac{\cos(x)}{x} dx$$

This integral has a non-integrable singularity at $x = 0$. Thus, **Integrate** generates a warning message and returns unevaluated. Consider another integral with a symbolic parameter α :

$$\int_0^1 x^\rho \tan^{-1}(x) dx$$

$$\text{If}[\text{Re}(\rho) > -2, \frac{\psi^{(0)}\left(\frac{\rho+2}{4}\right) - \psi^{(0)}\left(\frac{\rho+4}{4}\right) + \pi}{4(\rho+1)}, \int_0^1 x^\rho \tan^{-1}(x) dx]$$

The integral has one singular point at $x = 0$, which is integrable only if $\text{Re}(\rho) > -2$.

If you are sure that a particular integral is convergent or you don't care about the convergence, you can avoid testing the convergence by setting the option **GenerateConditions** to **False**. It will make **Integrate** return an answer a bit faster. In this example, almost 40% of the time is taken by convergence tests.

```
Timing[Integrate[(Sin[x]^20*Cos[x])/x^20,
  {x, 0, Infinity}];][[1]] /
Timing[Integrate[(Sin[x]^20*Cos[x])/x^20,
  {x, 0, Infinity}, GenerateConditions -> False];][[1]]
1.47624
```

Setting **GenerateConditions** to **False** also lets you evaluate integrals in the Hadamard sense (that is, the finite part of divergent integrals):

```
Integrate[1/x, {x, 0, 2}, GenerateConditions -> False]
log(2)
```

Principal-Value Integrals

In Version 3.0, definite integrals in the Riemann sense and principal-value integrals are distinguished by the new option **PrincipalValue**. If you want to evaluate an integral in the Cauchy sense, set the option **PrincipalValue** to **True** (the default setting is **False**). For example:

$$\int_{-1}^2 \frac{1}{x} dx$$

Integrate::idiv : Integral of $\frac{1}{x}$ does not converge on $\{-1, 2\}$.

$$\int_{-1}^2 \frac{1}{x} dx$$

Here, **Integrate** detects that the integral does not exist in the Riemann sense, generates a warning message, and returns unchanged. However, setting the option **PrincipalValue** to **True**, we get

```
Integrate[1/x, {x, -1, 2}, PrincipalValue -> True]
log(2)
```

Here are two more advanced examples:

$$\text{Integrate}\left[\frac{1}{4 - \tan^2(x)}, \{x, 0, \frac{\pi}{2}\}, \text{PrincipalValue} \rightarrow \text{True}\right]$$

$$\frac{\pi}{10}$$

$$\text{Integrate}\left[\frac{x \csc(x)}{\cos(x) - \sin(x)}, \{x, 0, \frac{\pi}{2}\}, \text{PrincipalValue} \rightarrow \text{True}\right]$$

$$-C + \frac{1}{4} \pi \log(2)$$

Assumptions

The new option **Assumption** is used to specify particular assumptions on parameters in definite integrals. Consider this integral with the arbitrary parameter y :

$$\int_0^{\infty} \frac{e^{-x^2 - xy}}{\sqrt{x}} dx$$

Setting the option **Assumptions** to $\text{Re}(y) > 0$, we obtain

$$\text{Integrate}\left[\frac{e^{-x^2 - xy}}{\sqrt{x}}, \{x, 0, \infty\}, \text{Assumptions} \rightarrow \text{Re}(y) > 0\right]$$

$$\frac{1}{2} e^{\frac{y^2}{8}} \sqrt{y} K_{\frac{1}{4}}\left(\frac{y^2}{8}\right)$$

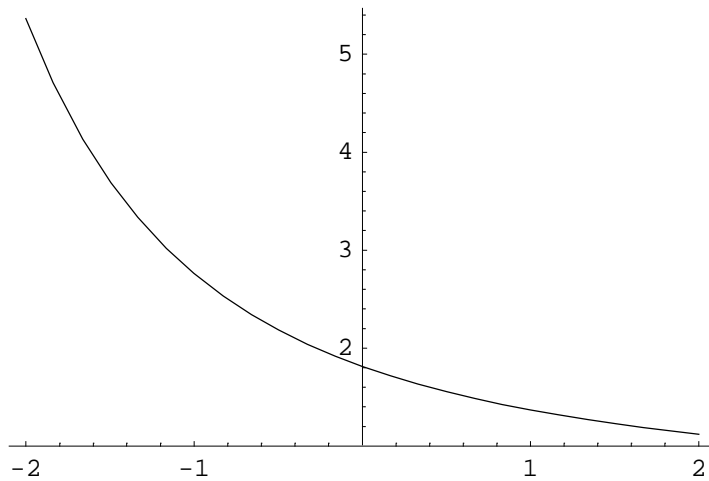
Setting **Assumptions** to $\text{Re}(y) < 0$, we get a different form of the answer:

$$\text{Integrate}\left[\frac{e^{-x^2 - xy}}{\sqrt{x}}, \{x, 0, \infty\}, \text{Assumptions} \rightarrow \text{Re}(y) < 0\right]$$

$$\frac{e^{\frac{y^2}{8}} \pi \sqrt{-y} \left(I_{-\frac{1}{4}}\left(\frac{y^2}{8}\right) + I_{\frac{1}{4}}\left(\frac{y^2}{8}\right)\right)}{2 \sqrt{2}}$$

though the integral is a continuous function with respect to the parameter y :

```
Plot[
  NIntegrate[Exp[-x^2 - x y] / Sqrt[x], {x, 0, Infinity}], {y, -2, 2}]
```



- Graphics -

The next integral is discontinuous with respect to the parameter γ :

```
Integrate[ $\frac{\cos(x) (1 - \cos(\gamma x))}{x^2}$ , {x, 0,  $\infty$ }, Assumptions  $\rightarrow \gamma > 1$ ]
```

$$\frac{1}{2} \pi (\gamma - 1)$$

```
Integrate[ $\frac{\cos(x) (1 - \cos(\gamma x))}{x^2}$ , {x, 0,  $\infty$ }, Assumptions  $\rightarrow 0 < \gamma < 1$ ]
```

0

The syntax of the option **Assumptions** lets you specify assumptions either as a logical **And** function or as a list:

```
Integrate[ $x^{\nu-1} (1-x)^{\mu-1}$ , {x, 0, 1}, Assumptions  $\rightarrow \text{Re}(\mu) > 0 \wedge \text{Re}(\nu) > 0$ ]
```

$$\frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)}$$

```
Integrate[ $x^{\nu-1} (1-x)^{\mu-1}$ , {x, 0, 1}, Assumptions  $\rightarrow \{\text{Re}(\mu) > 0, \text{Re}(\nu) > 0\}$ ]
```

$$\frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu + \nu)}$$

You might already have observed that if the given assumptions exactly match the generated assumptions, then the latter don't show up in the output; otherwise, **Integrate** produces complementary assumptions to given ones:

Integrate $[e^{-\mu x} x^{\nu-1} (\psi(\nu) - \log(x)), \{x, 0, \infty\}, \text{Assumptions} \rightarrow \text{Re}(\mu) > 0]$

If $[\text{Re}(\nu) > 0, \mu^{-\nu} \Gamma(\nu) \log(\mu), \int_0^{\infty} e^{-x\mu} x^{\nu-1} (\psi^{(0)}(\nu) - \log(x)) dx]$