

On the Computation of Value Correspondences

Christopher Sleet

*Department of Economics
University of Iowa
Iowa City, IA 52242, USA*
E-mail: christopher-sleet@uiowa.edu

Şevin Yeltekin

*M.E.D.S., Kellogg School of Management
Northwestern University
Evanston, IL 60208, USA*
E-mail: s-yeltekin@northwestern.edu

Recursive game theory provides theoretical procedures for computing the equilibrium payoff sets of repeated games and the equilibrium payoff correspondences of dynamic games. In this paper, we propose and implement outer and inner approximation methods for value correspondences that naturally occur in the analysis of dynamic games. The procedure utilizes set-valued step functions. We provide an application to a bilateral insurance game with storage.

1. INTRODUCTION

In two influential papers, Abreu, Pearce and Stacchetti (APS) (1986, 1990) developed set-valued dynamic programming techniques for solving certain classes of repeated game. They showed that, for these games, the set of sequential equilibrium payoffs can be computed as a fixed point of an operator analogous to the Bellman operator in dynamic programming. These methods can be extended to cover a large class of dynamic games that arise naturally in industrial organization, macroeconomics, and public finance.¹ In the dynamic case, the object of interest is a correspondence that maps a physical state variable to sets of equilibrium payoffs. APS-based methods imply that the equilibrium payoff correspondence is a fixed

¹See Atkeson (1991), Phelan and Stacchetti (2001), Sleet (1996) and Fernandez-Villaverde and Tsyvinski (2002) for applications to international lending and macroeconomic policy.

point of a monotone operator and can be obtained via an iteration of this operator. In principle, this iterative procedure can be implemented numerically to solve for the equilibrium payoff correspondence. However, its practical numerical implementation requires an approximation scheme for correspondences that is efficient and consistent with the underlying structure of the monotone operator. This paper provides such a scheme.

As a first step, following Cronshaw and Luenberger's (1994) and Judd, Yeltekin and Conklin's (JYC)(2002) analyses of repeated games, we convexify the dynamic game by introducing a public randomization device. We then show that convex-valued step correspondences provide an efficient and consistent means of approximating the equilibrium payoff correspondences from such convexified games. Moreover, they enable us to obtain inner and outer approximations to the equilibrium payoff correspondence. Since the latter lies between these approximations, the difference between them provides us with an error bound for gauging the accuracy of the procedure. This approach improves both on the discretization and linear interpolation methods that have been used elsewhere.² The former is non-parsimonious and is vulnerable to the curse of dimensionality, the latter delivers neither an inner nor an outer approximation and lacks error bounds. To illustrate the value of our methods, we briefly describe an application of them to a bilateral insurance game with storage.

2. CORRESPONDENCE-VALUED DYNAMIC PROGRAMMING

Assume that N infinitely lived players play a dynamic game. Let the state of the i -th player be denoted by a variable $k_i \in K_i$, K_i compact. Let $k \in K = \times_{i=1}^N K_i$ denote an 'aggregate' state. Suppose that in every period each agent selects an economic variable from a feasible choice correspondence $A_i : K \rightrightarrows \mathfrak{R}^m$.³ Assume that this correspondence has a compact graph. A profile of player choices will be denoted a , where $a \in A(k) = \times_{i=1}^N A_i(k)$. The aggregate state evolves according to a continuous function $f : \text{Graph}(A) \rightarrow K$. Each player has a continuous per period utility function $u_i : \text{Graph}(A) \rightarrow \mathfrak{R}$. Let $U : K \rightrightarrows \mathfrak{R}^N$ be a correspondence that describes the set of conceivable one period payoffs available at each value of the state variable, i.e. $U(k) = \{w \in \mathfrak{R}^N : w_i = u_i(a_i, k), a_i \in A_i(k), i = 1, \dots, N\}$. The graph of U is clearly compact. Given a fixed value of k , the stage

²See Chang (1998) for an application of discrete methods to a repeated policy game. See Phelan and Stacchetti (2001) for an application of linear interpolation methods to a dynamic policy game.

³These choice correspondences are assumed to be separable across players for notational convenience. If some subset of players moves before another this need not be so. However, the ideas are easily extended to this case.

game, $S(k)$, is described by the triple $(A(k), \{u_i\}_{i=1}^N, f)$. It will be assumed throughout that for all values of $k \in K$, $S(k)$ possesses a pure Nash equilibrium.

The action-state space for the dynamic game is contained within $\text{Graph}(A)^\infty$. A t -period history, h_t , is a sequence $\{\{a_s, k_s\}_{s=0}^{t-1}, k_t\}$ with $k_s = f(k_{s-1}, a_{s-1})$ and $a_s \in A(k_s)$. The set of feasible t -period histories will be denoted H_t . It will be assumed that the i -th player values sequences of points in $\text{Graph}(A)^\infty$ according to the function:

$$w_i(a^\infty, k^\infty) = (1 - \beta) \sum_{s=0}^{\infty} \beta^s u_i(a_{is}, k_s), \quad (1)$$

where β is the common discount factor amongst players. A strategy for player i is a sequence of functions $\{\sigma_{it}\}_{t=0}^{\infty}$ with $\sigma_{it} : H_t \rightarrow \{a_i \in A_i(k)\}$, some k , and $\sigma_{it}(h_{t-1}, a_{t-1}, k_t) \in A_i(k_t)$. Let V^* denote the subgame perfect equilibrium payoff correspondence and \mathcal{V} the set of all correspondences $V : K \rightrightarrows \mathfrak{R}^N$ such that i) the graph of V is compact and ii) the graph of V is contained within the graph of U . V^* may be shown to be an element of \mathcal{V} .

We follow the recursive approach of APS. In the recursive formulation each subgame perfect equilibrium payoff vector is supported by a profile of actions consistent with Nash play in the current period and a vector of continuation payoffs that are themselves payoffs in some subgame perfect equilibrium. The key to finding V^* is the construction of self-generating correspondences. The concept may be formalized by introducing the operator $B : \mathcal{V} \rightarrow \mathcal{V}$:

$$B(V)(k) = \{v \mid \quad (2)$$

$$v' \in V(f(k, a)), a \in A(k), \text{ and for each } i \in 1, \dots, N$$

$$v_i = (1 - \beta)u_i(a_i, k) + \beta v'_i \quad (3)$$

$$v_i \geq \max_{a' \in A_i(k)} \min_{\bar{v} \in V(f(k, a-i, a'))} \in (1 - \beta)u_i(k, a-i, a') + \beta \bar{v}_i \quad (4)$$

A value v is in $B(V)(k)$ if two conditions are satisfied. Firstly, (3), there is some action profile a and continuation payoff $v' \in V(f(k, a))$ such that the i -th player receives v_i when a is chosen in the current period and the player's continuation payoff is given by v'_i . Secondly, (4), each player is better off adhering to a_i in the current period and receiving the continuation payoff v'_i than deviating and receiving the worst feasible continuation payoff consistent with V . (4) will be referred to as the incentive compatibility constraint. A correspondence V is self-generating if $\text{Graph}V \subseteq \text{Graph}B(V)$. An extension of arguments in APS establishes that the graph of any self-generating correspondence is contained within the graph of the equilibrium

payoff correspondence and that the equilibrium payoff correspondence is itself self-generating.⁴ It follows that the equilibrium payoff is a fixed point of the operator B . In fact, it is the largest fixed point in \mathcal{V} .

The operator B is monotone in the set inclusion ordering, so that if $\text{Graph}V_1 \subseteq \text{Graph}V_2$, then $\text{Graph}B(V_1) \subseteq \text{Graph}B(V_2)$. Additionally, it preserves compactness. It can be readily shown that these facts ensure that $B^n(U)$ converges to V^* pointwise in the Hausdorff metric and this convergence is uniform on the domain of V^* when V^* is continuous and the domain is compact.⁵

3. APPROXIMATING VALUE CORRESPONDENCES

The numerical implementation of the APS procedure requires that the graphs of candidate value correspondences be efficiently represented on a computer and that the monotonicity property of the B-operator is preserved. We proceed in two steps. First, we convexify the underlying game and its equilibrium value correspondence. Then we develop methods for the approximation of convex-valued correspondences.

At a theoretical level, convexification requires modifying the underlying game to incorporate lotteries. Thus, at the beginning of each period a lottery is held over a state space Ω . Strategies are then assumed to condition player choices on histories of lottery outcomes as well as histories of actions and states. The lottery probabilities are determined by a profile of functions that map histories into probability distributions on Ω . These and the lottery outcomes are common knowledge. It is readily shown that this formulation's value correspondence, V^{L*} , is convex-valued as desired. Analogues of the results described in the previous section go through with only minor modifications for this case. In particular, V^{L*} can be shown to be a fixed point of an operator B^L , where $B^L(V)(k) = coB(V)(k)$ and coX denotes the convex hull of the set X . Furthermore, repeatedly applying the operator B^L to the correspondence U^L , where $U^L(k) = coU(k)$ induces a sequence of convex-valued correspondences that converge to V^{L*} .

Before turning to this in detail some definitions are introduced for future use.

DEFINITION 3.1. *Approximation Scheme*

Let \mathcal{W} be a collection of compact, convex-valued correspondences with domain a compact subset K of \mathfrak{R}^m . Let $\hat{\mathcal{W}} \subset \mathcal{W}$ be a collection of finitely parameterized elements of \mathcal{W} . Let $F : \mathcal{W} \rightarrow \hat{\mathcal{W}}$ be a rule for approximating an element of \mathcal{W} with an element of $\hat{\mathcal{W}}$. The three-tuple $(\mathcal{W}, F, \hat{\mathcal{W}})$ will be

⁴Atkeson provides these extensions in a related environment.

⁵See Sleet and Yeltekin (2001).

called an approximation. A sequence of approximations, $\{\mathcal{W}, F_n, \hat{\mathcal{W}}_n\}_{n=0}^\infty$, will be called an approximation scheme.

1. An approximation scheme $\{\mathcal{W}, F_n, \hat{\mathcal{W}}_n\}_{n=0}^\infty$ will be called pointwise (resp. uniformly) convergent if, $\forall W \in \mathcal{W}$, the sequence $F_n(W)$ converges to W pointwise (resp. uniformly) in the Hausdorff metric.

2. An approximation $(\mathcal{W}, F, \hat{\mathcal{W}})$ is monotone if whenever $\text{Graph}W_1 \subseteq \text{Graph}W_2$, $W_1, W_2 \in \mathcal{W}$, then $\text{Graph}F(W_1) \subseteq \text{Graph}F(W_2)$. A monotone approximation scheme is an approximation scheme composed of monotone approximations.

3. An approximation $(\mathcal{W}, F, \hat{\mathcal{W}})$ is said to be outer (inner) if $\text{Graph}W \subseteq \text{Graph}F(W)$ ($\text{Graph}W \supseteq \text{Graph}F(W)$), $W \in \mathcal{W}$. An outer (inner) approximation scheme is an approximation scheme composed of outer (inner) approximations.

Outer and inner approximations are useful as they give upper and lower bounds to a correspondence. The difference between them (in the Hausdorff metric) provides an error bound - if they are close, then each is close to the true correspondence. Approximation of the (convexified) equilibrium value correspondence requires using an approximation scheme in conjunction with the B^L operator. Thus, in practice, it is necessary to iterate with an operator $\hat{B} = F \circ B^L$, on $\hat{\mathcal{V}}$, a space of finitely parameterized elements of \mathcal{V} . If F is monotone, then \hat{B} will also be monotone and if $\hat{B}(U^L) \subseteq U^L$, then $\{\hat{B}^n(U^L)\}$ will be a decreasing convergent sequence of correspondences. However, in general, $V^{L*} \neq \hat{B}(V^{L*})$, and for given F , the fixed points of \hat{B} and the limit of the sequence of $\{\hat{B}^n(U^L)\}$, may not be close to V^{L*} . On the other hand, if F is a monotone outer approximation, $W_0 \in \mathcal{V}$, $\text{Graph}\hat{B}(W_0) \subseteq \text{Graph}W_0$ and $\text{Graph}V^{L*} \subseteq \text{Graph}W_0$ then $\text{Graph}V^{L*} \subseteq \text{Graph}F(V^{L*}) = \text{Graph}\hat{B}(V^{L*}) \subseteq \text{Graph}\hat{B}^n(W_0)$, so the limit of an iteration of \hat{B} from W_0 is an outer approximation of V^{L*} . Similarly, if $\text{Graph}W_0 \subseteq \text{Graph}V^{L*}$, $\text{Graph}W_0 \subseteq \text{Graph}\hat{B}(W_0)$ and F is a monotone, inner approximation, the limit of $\{\hat{B}^n(W_0)\}$ is an inner approximation to V^{L*} . Thus, it is desirable to find monotone outer and inner approximation schemes for correspondences since these can be combined with B^L to yield outer and inner approximations of V^{L*} . We now turn to such schemes.

Let $P = \times_{i=1}^m [a_i, b_i] \subseteq \mathfrak{R}^m$. Suppose that each interval $[a_i, b_i]$ is subdivided into k_i sub-intervals. The Cartesian product of these sub-intervals will be referred to as a partition of P . The j -th element of a partition will be called a block and will be denoted Q_j .

DEFINITION 3.2. Block specific and step correspondences Let $P = \times_{i=1}^m [a_i, b_i]$ have a partition P_n . Let Λ be a correspondence with domain P . Λ is said to be block specific if it is empty-valued outside of a specific

block in P_n . Λ is said to be a step correspondence if it is constant on the interior of each block in P_n .

The following approximation methods rely on correspondences constructed from finite numbers of block specific or step correspondences.

3.1. An outer approximation method

This method approximates the underlying correspondence with a collection of convex valued step correspondences. The procedure is as follows. Let $W : P \rightrightarrows \mathfrak{R}^n$ be the underlying correspondence. Assume that W is uhc, compact and convex valued. Fix a given partition of P and define the correspondence ω_j , where

$$\omega_j(k) = \begin{cases} \text{co } \cup_{k' \in Q_j} W(k') & \text{if } k \in Q_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, ω_j is a convex-valued step correspondence. Its value on any Q_j can be approximated using the outer piecewise linear procedure for approximating convex sets developed by JYC and discussed in the appendix. Let ω'_j denote an approximation to ω_j obtained in this way. As shown in Sleet and Yeltekin (2002), ω'_j can be made arbitrarily close to ω_j by selecting a fine enough piecewise linear procedure. Next, define the approximation of W , W' , pointwise as $W'(k) = \cup_j \omega'_j(k)$ each k .

Let $\{W'_n\}_{n=0}^\infty$ be a sequence of such approximations and let P_n denote the partition associated with the n -th approximation. Denote by $\omega'_{j,n}$ the j -th block specific correspondence of the n -th approximation. Assume that P_n is constructed from 2^n uniformly distributed blocks. Additionally, for a fixed decreasing sequence of positive numbers $\{\epsilon_n\}$ with limit 0, assume that the JYC outer procedure is implemented so as to ensure $\rho(\omega'_{j,n}, \omega_{j,n}) < \epsilon_n$ for $j = 1, \dots, 2^n$.

THEOREM 3.1. *The sequence $\{W'_n\}_{n=0}^\infty$ is a pointwise convergent, monotone outer approximation scheme. Each W'_n is uhc and compact valued.*

Proof. That each $\omega'_{j,n}$ is uhc is immediate. The uhc of W'_n then follows since W'_n is equal to the union of a finite number of uhc correspondences. Each $W'(k)$ is bounded by construction. For the closed valuedness see Beer (1980). Fix a positive number ϵ and a value $k \in P$. Let $\mathcal{O}_\epsilon(k)$ denote a ball of radius ϵ around the point k . By the uhc of W , it follows that there exists some $\lambda > 0$ such that: $\cup\{W(k') : \|k - k'\| < \lambda\} \subset \mathcal{O}_\epsilon(W(k))$. Now, there exists an $n_1(k)$ such that for all $m > n_1(k)$, the diameter of each block in the m -th partition is less than λ . Additionally, by construction there is some n_2 such that for all $m > n_2$, $\rho(\omega'_{j,m}, \omega_j) < \epsilon$, $\forall j = 1, \dots, 2^m$.

Fix $m > \max(n_1(k), n_2)$ and let Q be a block from the m -th partition that contains k . It follows that for all $k' \in Q$, $\cup\{W(k') : k' \in Q\} \subset \mathcal{O}_\epsilon(W(k))$. Since W is convex valued, it follows that $\mathcal{O}_\epsilon(W(k))$ is convex. Hence,

$$co \cup \{W(k') : k' \in Q\} \subset \mathcal{O}_\epsilon(W(k))$$

Denote the step correspondence associated with block Q by ω_Q and its approximation by ω'_Q . Since $m > n_2$, $\rho(\omega'_Q, \omega_Q) < \epsilon$. Hence, $\omega'_Q(k) \subset \mathcal{O}_{2\epsilon}(W(k))$ and $W'_n(k) \subseteq \mathcal{O}_{2\epsilon}(W(k))$. ■

In conjunction with the previous result, the following theorem shows that a uniformly convergent approximation scheme can be obtained if the correspondence to be approximated is continuous and compact valued.

THEOREM 3.2 (Beer, 1980).

Let K be a compact topological space and let Y be a metric space. Suppose that $\Gamma : K \rightrightarrows Y$ is a continuous, compact, non-empty valued correspondence. Let $\{\Gamma_n\}_{n=1}^\infty$ be a sequence of upper hemicontinuous, compact valued correspondences such that for each $k \in K$ and n , $\Gamma(k) \subseteq \Gamma_{n+1}(k) \subseteq \Gamma_n(k)$. If $\bigcap_{n=1}^\infty \Gamma_n(k) = \Gamma(k)$ for each k , then Γ_n converges to Γ uniformly in the Hausdorff metric.

Under the assumptions made in Section 2 the B -operator preserves continuity. Thus, the equilibrium value correspondence can be obtained as the limit of a sequence of continuous correspondences. It is the elements of this sequence that must be approximated. Consequently, the previous theorem ensures that there are uniformly convergent approximation schemes available for this purpose.

3.2. The inner approximation step correspondence method

This section describes a method for approximating a lhc correspondence with collections of step correspondences whose graphs are contained by the graph of the approximant. Let W be a lhc correspondence whose images are compact and convex sets. Fix a given partition of P and define the correspondence ω_j , where

$$\omega_j(k) = \begin{cases} \bigcap_{k' \in Q_j} W(k') & \text{if } k \in Q_j, \\ \mathfrak{R}_n & \text{otherwise.} \end{cases}$$

Thus, ω_j is a convex-valued step correspondence. It has a convex graph on Q_j and this convex set can be approximated using the JYC inner approximation method described in the appendix. Let ω'_j denote an approximation to ω_j obtained in this way. Next, define the approximation W' to W pointwise by $W'(k) = \bigcap_j \omega'_j(k)$ each k .

Now, let $\{W'_n\}_{n=0}^\infty$ be a sequence of such approximations. Let P_n denote the partition associated with the n -th approximation and define $\omega'_{j,n}$ and $\omega_{j,n}$ defined analogously to ω'_j and ω_j above (i.e. $\omega'_{j,n}$ is the j -th block specific correspondence of the n -th approximation). Assume that P_n is constructed from 2^n uniformly distributed blocks. Additionally, for a fixed decreasing sequence of positive numbers $\{\epsilon_n\}$ with limit 0, assume that the JYC inner approximation procedure is implemented so as to ensure that $\rho(\omega_{j,n}, \omega'_{j,n}) < \epsilon_n$.

THEOREM 3.3. *The sequence $\{W'_n\}_{n=0}^\infty$ is a pointwise convergent, monotone inner approximation scheme. Each W'_n is lhc, compact and convex valued.*

The proof is an adaptation of arguments in Beer (1980).

4. APPLICATION: A BILATERAL INSURANCE GAME

This section considers a problem in which two risk averse agents mutually insure one another against fluctuations in their endowments and neither can commit. The problem extends the physical environment of Kocherlakota (1996) by permitting the agents to store. Thus, the pair of agents can smooth their consumption by a combination of two methods: intra-temporal risk sharing and inter-temporal storage. The former is achieved by transfers from a joint storage fund. Agents face a short run temptation to refuse to make their prescribed payments into the fund (negative transfers) and to raid the fund itself. However, the numerical results below indicate that the threat of dissolution of the insurance arrangement, is sufficient to support mutually beneficial risk sharing at low enough capital levels. The details of the model now follow.

Consider an environment inhabited by two infinitely lived agents with identical preferences. The i -th agent receives a random endowment, $s_t^i \in \mathbf{S}$, of a consumption good in each period t . Assume that $\mathbf{S} = \{s_1, \dots, s_P\}$, $P < \infty$ and $s_p < s_{p+1}$. Assume that the s_t^i are i.i.d. over dates t and agents i and let Π denote the distribution of s_t^i with $\text{Prob}(s_t^i = s_p) = \Pi_p$, $p = 1, \dots, P$. The history of endowments received by each agent is common knowledge. In each period, after observing the period's endowment realization, the agents make a transfer to a joint fund. If agent i receives endowment s_p in period t and the size of the fund is k_t then the size of that agent's transfer is restricted to lie in $\Gamma(k_t, s_p) = [-k_t, s_p]$. Let τ_t^i denote agent i 's t -th period transfer and c_t^i that agent's consumption. Any amount that is not consumed by the two agents is stored in the joint fund. Thus this fund evolves according to $k_{t+1} = f(k_t + \tau_t^1 + \tau_t^2)$, where f describes the storage technology. f is assumed to be linear in the remainder of the paper. At any point in time, an agent is free to keep her endowment,

grab fraction $\gamma \in [0, 1]$ from storage. Subsequently, she can always live in autarky, making and receiving no transfers from the other agent.

Assume that the agents' preferences are described by the utility functions $u^i(c)$, $i = 1, 2$, where u^i is a continuous, strictly concave, strictly increasing function. The payoff to individual i is given by $(1 - \beta)u^i(c^i) + \beta \sum_{s^{i'} \in \mathbf{S}} \sum_{s^{j'} \in \mathbf{S}} v^{i'}(s^{i'}, s^{j'}) \Pi_{s^{i'}} \Pi_{s^{j'}}$ where $v^{i'}(s^{i'}, s^{j'})$ represents the continuation value for agent i in future endowment state $(s^{i'}, s^{j'})$. Assume that all variables belong to compact intervals. Under these assumptions it is straightforward to verify that the lowest equilibrium payoff to player i when the joint fund contains k is given by:

$$\max_{(\hat{\tau}^i, \hat{m}^i) \in \Gamma(\gamma k, s^i)} (1 - \beta)u^i(s^i - \hat{\tau}^i - \hat{m}^i) + \beta v_{aut}^i(\hat{m}^i).$$

Here $v_{aut}^i(\hat{m}^i)$ gives the autarkic payoff to player i with initial wealth \hat{m}^i . Thus, the B operator associated with the subgame perfect equilibria of this game is of the form:

$$\begin{aligned} B(V)(k, s^1, s^2) = \{ & v = (v^1, v^2) \mid \text{for } i = 1, 2, j \neq i \\ & v^i = \{(1 - \beta)u^i(s^i - \tau^i) + \beta \sum_{s^{i'} \in \mathbf{S}} \sum_{s^{j'} \in \mathbf{S}} v^{i'}(s^{i'}, s^{j'}) \Pi_{s^{i'}} \Pi_{s^{j'}}, \\ & v^j \in V(f(k + s^1 + s^2 - \tau^1 - \tau^2), s^{1'}, s^{2'}) \\ & (1 - \beta)u^j(s^j - \tau^j) + \beta \sum_{s^{i'} \in \mathbf{S}} \sum_{s^{j'} \in \mathbf{S}} v^{i'}(s^{i'}, s^{j'}) \Pi_{s^{i'}} \Pi_{s^{j'}} \geq \\ & \left. \max_{(\hat{\tau}^i, \hat{m}^i) \in \Gamma(\gamma k, s^i)} (1 - \beta)u^i(s^i - \hat{\tau}^i - \hat{m}^i) + \beta v_{aut}^i(\hat{m}^i) \right\} \end{aligned}$$

Notice that here the two shocks are explicitly treated as state variables.

4.0.1. Calculations

The following parametric forms are assumed for our numerical example. $u^i(c) = \frac{c}{1-\sigma} 1-\sigma$, $f(k) = \min(\bar{k}, \max(k, \underline{k}))$. Additionally, the preference and technology parameter are set to: $\beta = 0.8$, $\sigma = 1$, and $k \in [\underline{k}, \bar{k}] = [0, 4]$ and $\mathbf{S} = \{0.1, 0.5\}$, $\Pi = \{0.5, 0.5\}$, $\gamma = 0.8$. Figure 1 below illustrates the approximated value correspondence for the case of high endowment shocks for both agents. Figure 2 displays the cross-section of the correspondence in Figure 1 for a sample of k levels. The axes $P1$ and $P2$ represent the payoffs for Agent 1 and 2 respectively. The worst equilibrium payoff function for each agent (labeled "autarky" in Figure 1), is concave in capital and is independent of the payoff received by the other agent. This functional form is, of course, inherited from the autarkic value function. As the capital level increases, autarky becomes more attractive and the set of

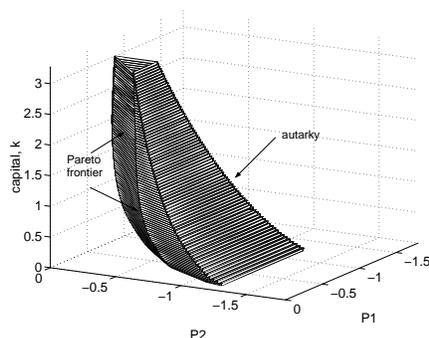


FIG. 1. Value Correspondence

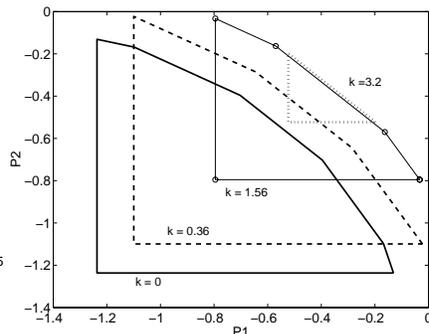


FIG. 2. Cross-section

equilibrium values shift upwards and becomes smaller. Higher capital levels have two effects. On the one hand, they provide for higher consumption levels and better insurance against aggregate shocks; on the other hand, they raise the autarky payoff and increase the incentive for one agent to grab resources from the joint fund. These effects can be clearly seen in Figure 2. In this example, for capital levels above 3.2, bilateral insurance is no longer possible. For $k \in (3.2, 4]$, the autarky values are too high, thus the incentives to deviate are too strong to enable risk-sharing. There is, consequently, a limit to the amount of capital that can be accumulated in this insurance arrangement. This limit depends on the parameters of the model, in particular on the amount of capital that can be grabbed by one agent.

APPENDIX

The following algorithm provides the details of the computational procedure for the outer monotone approximation $B(W)(k)$ using hyperplanes. The inner monotone approximation procedure is similar; the differences are discussed below, along with the numerical procedure for computing error bounds. The correspondence $W(k)$ is a candidate for the equilibrium value correspondence.

Outer Monotone Approximation of $B(W)(k)$

1. **Parameters:** Search subgradients: L subgradients $H = \{h_1, \dots, h_L\} \subset \mathbb{R}^{LN}$, each $h_i \in \mathbb{R}^N$, and partition P_n .

2. Input: Description of W : Approximation subgradients, $H = \{h_1, \dots, h_L\} \subset \mathbb{R}^{LN}$, and levels, $C(j) = \{c_\ell(j) | \ell = 1, \dots, L\}, j = 1, \dots, n$ such that

$$\omega_j(k) = \begin{cases} \bigcap_{\ell=1}^L \{z \mid h_\ell \cdot z \leq c_\ell(j)\}, & \text{if } k \in Q_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus $W(k)$ can be described as $\cup_j \omega_j(k)$.

3. Optimization: Find extremal points of $B(W)$.

For each $h_\ell \in H$ and for each partition element Q_j ,

$$c_\ell^+(j) = \max_{a, w', k \in Q_j} h_\ell \cdot [(1 - \beta)u(k, a) + \beta w'] \quad (\text{A.1})$$

(i) $w' \in W(f(k, a))$

(ii) for $i=1, \dots, N$, $(1 - \beta)u(a_i, k) + \beta w'_i \geq$

$$\max_{a' \in A_i(k)} \min_{\tilde{w} \in W(f(k, a_{-i}, a'))} (1 - \beta)u_i(k, a_{-i}, a') + \beta \tilde{w}_i$$

4. Output: $W^+(k) = B(W)(k) = \cup_j \omega_j^+(k)$, where

$$\omega_j^+(k) = \begin{cases} \bigcap_{\ell=1}^L \{z \mid g_\ell \cdot z \leq c_\ell^+(j)\} & \text{if } k \in Q_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

The main computational work is done in the optimization step. In this step, for each search subgradient h_ℓ and element of the partition P_n , we find an action profile a , continuation values $w' \in W(f(k, a))$ and current state k in Q_j which makes a incentive compatible and maximizes a weighted sum of player payoffs where the weights are given by h_ℓ .¹

In the outer approximation case, the search subgradients and the approximation subgradients coincide. Correspondences are represented by sets of bounding hyperplanes, one set for each block in the partition. In this way, an approximated correspondence can be reduced to a collection $\{c_\ell(j)\}$ in \mathbb{R}^{nL} , with each $c_\ell(j)$ determining the position of the ℓ -th hyperplane on the j -th block. Inner approximations are more complicated. In this case, the search and approximation subgradients need not coincide. The optimizations in (A.1) are modified: it is necessary to maximize over (a, w')

¹The surface of the step correspondence contains discontinuities at the joins of the partition elements. Application of a non-linear optimizer requires either that the correspondence is smoothed across these joins, or that the optimizations are split into a collection of sub-optimizations in each of which the continuation state variable is restricted to an element of the partition.

given k and minimize over k . After completing the set of optimizations, the convex hull of the resulting extreme points is constructed on each partition block. These convex hulls define a new set of approximation subgradients $\{g_m(j)\}$ that, along with a corresponding set of constants $\{c_m(j)\}$, describe the inner approximation.

For an approximation of the equilibrium value correspondence V , the procedure described above is executed repeatedly until $d(W(k), W^+(k))$ is small where $d(W_i, W(k)^+)$ is defined as

$$d(W(k), W^+(k)) = \max_j \max_\ell |c_\ell(j) - c_\ell^+(j)|$$

for an outer approximation and as

$$d(W(k), W^+(k)) = \max_j \max_\ell \|z_\ell(j) - z_\ell^+(j)\|$$

for an inner approximation where z_ℓ represents the extreme point in the ℓ -th direction on the j -th partition block.

Once the inner ($W^I(k)$) and outer ($W^O(k)$) approximations of the equilibrium value correspondence have been constructed, an error bound can be computed. The Hausdorff distance between $W^I(k)$ and $W^O(k)$ is defined as

$$\rho(W^O(k), W^I(k)) = \max_j \rho(\omega_j^O, \omega_j^I)$$

where $\rho(\omega_j^O, \omega_j^I) = \max_{v^O \in \omega_j^O} \min_{v^I \in \omega_j^I} \|v^O - v^I\|$. Since ω_j^I and ω_j^O are step correspondences whose graphs are polytopes, the above expression can be replaced by

$$\rho(\omega_j^O, \omega_j^I) = \max_{z^O \in Z_j^O} \min_{v^I \in \omega_j^I} \|z^O - v^I\|$$

where Z_j^O is the set of vertices of ω_j^O . Thus, $\rho(W^O(k), W^I(k))$ can be computed by solving $L \times n$ (number of search gradients (or vertices) times number of k in partition P_n) concave minimization problems with linear constraints.

REFERENCES

1. Abreu, Dilip, David Pearce and Ennio Stachetti. 1986. Optimal cartel equilibria with imperfect monitoring. *Journal of Economic Theory* 39:251-269.
2. Abreu, Dilip, David Pearce and Ennio Stacchetti. 1990. Towards a theory of discounted repeated games with imperfect monitoring. *Econometrica* 56:383-396.

3. Atkeson, Andrew. 1991. International lending with moral hazard and risk of repudiation. *Econometrica* 59:1069-1089.
4. Beer, Gerald. 1980. The approximation of upper semicontinuous multifunctions by step multifunctions. *Pacific Journal of Mathematics* 87:11-19.
5. Chang, Roberto. 1998. Credible monetary policy in an infinite horizon model: recursive approaches. *Journal of Economic Theory* 81:431-461.
6. Cronshaw, Mark and David Luenberger. 1994. Strongly symmetric subgame perfect equilibrium in infinitely repeated games with perfect monitoring and discounting. *Games and Economic Behavior* 6:220-237.
7. Fernandez-Villaverde, Jesus and Aleh Tsyvinski. 2002. Optimal fiscal policy in a business cycle model without commitment. Mimeo. University of Pennsylvania.
8. Kenneth Judd, Şevin Yeltekin and James Conklin. 2002. Computing supergame equilibria. Unpublished paper, Stanford University.
9. Kocherlakota, Narayana. 1996. Efficient bilateral risk sharing without commitment. *Review of Economic Studies* 43: 595-609.
10. Phelan, Christopher and Ennio Stacchetti. 2001. Subgame perfect equilibria in a Ramsey taxes model. *Econometrica* 69:1491-1518.
11. Sleet, Christopher. 1996. New recursive methods for macroeconomic policy games. Mimeo. Stanford University.
12. Sleet, Christopher and Şevin Yeltekin. 2001. Notes on the approximation of value correspondences. Mimeo. University of Iowa.