

THE STONE-ČECH COMPACTIFICATION

by

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ABSTRACT

The Stone-Čech compactification βX has been a topic of increasing study since its introduction in 1937. The algebraic content of this research is collected in the 1960 textbook, Rings of Continuous Functions, by L. Gillman and M. Jerison. Here we take a more purely topological viewpoint of the Stone-Čech compactification and attempt to collect the most important results which have emerged since Rings of Continuous Functions.

The construction of βX is described in an historical perspective. The theory of Boolean algebras is developed and used as a tool, primarily in a detailed investigation of $\beta \mathbb{N}$ and $\beta \mathbb{N} \setminus \mathbb{N}$. The relationships between a space X and its "growth" $\beta X \setminus X$ are examined, including the non-homogeneity of $\beta X \setminus X$, the cellularity of $\beta X \setminus X$, and mappings of βX to $\beta X \setminus X$. The Glicksberg product theorem which characterizes the products such that $\beta(\prod X_\alpha) = \prod (\beta X_\alpha)$ and related results are presented. Finally, the Stone-Čech compactification is studied in a categorical context.

INTRODUCTION

This manuscript had its origins in the author's fascination with L. Gillman and M. Jerison's text, Rings of Continuous Functions, particularly with the Stone-Čech compactification. The number of research papers relating to the Stone-Čech compactification since the publication of Rings of Continuous Functions in 1960 shows that I am not alone in my interest.

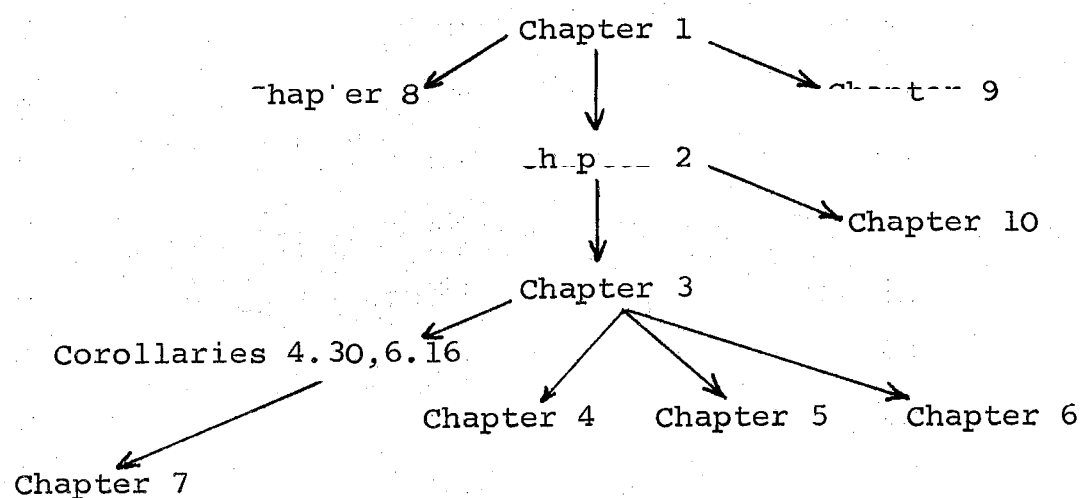
My main objective in this manuscript is to collect these results in a single source in order to make them more accessible to the mathematical community. However, the point of view here is more topological than that of Gillman and Jerison and little of the algebraic interest in the Stone-Čech compactification is apparent beyond the first chapter.

Only a very few of the results here are new. Theorem 10.46 concerning epi-reflective hulls and the results in Sections 10.28-31 concerning pushouts in the category of completely regular spaces were first presented by the author to a categorical topology seminar at Carnegie-Mellon University in 1971. The proof of Theorem 10.22 characterizing the realcompact spaces as closed subspaces of products of real lines appears to be new. Finally, Lemma 2.26 is a Boolean algebra result which is crucial to the investigation of $\beta \mathbb{N} \setminus \mathbb{N}$. The lemma first appeared in I. I. Parovičenko's 1963 paper as a generalization of a result of W. Rudin. However, Parovičenko's proof contains a gap. To

the best of my knowledge, the proof given here is the first complete proof of this result.

This manuscript is intended to be both a reference for research mathematicians and to be a text for second level graduate students in topology. An extensive bibliography has been included as an aid to further reading and the original papers are cited in the text by year and author. A list of exercises follows most chapters to supplement the text and to give the reader an opportunity to become familiar with the concepts.

Chapters 1, 2, and 10 are introductory in nature. Chapter 1 contains the basic material about the Stone-Čech compactification and parts of the chapter will be repetitive for the reader already familiar with the concept. Chapter 2 is an introduction to Boolean algebras. Chapter 10 provides a brief introduction to category theory and discusses the Stone-Čech compactification in the categorical context of reflections. Each of Chapters 3 - 9 develops a particular topic in the theory of Stone-Čech compactifications. Chapters 3 and 7 are devoted to $\beta\mathbb{N}$ and its growth $\beta\mathbb{N}\setminus\mathbb{N}$. For the reader mainly interested in these two spaces, Corollary 4.30 which shows the existence of P-points in $\beta\mathbb{N}\setminus\mathbb{N}$ and the construction used in the proof of Corollary 6.16 comprise the material needed between Chapters 3 and 7. The major pattern of dependence among the chapters is given below.



Because the later chapters are essentially independent, the manuscript can be used as a text in several ways. As a one semester second level graduate course, portions of Chapters 1, 2, 3, and 10 together with one of the more specialized chapters would appear to be a good combination of about the right length. The reader already familiar with Rings of Continuous Functions will need only to read Chapter 2 and part of Chapter 3 in order to have easy access to any of the later chapters. As a two semester course, the student who has completed an introductory topology course should be able to cover the complete manuscript.

ACKNOWLEDGEMENTS

I would like to express my gratitude to Professor Stanley P. Franklin under whose guidance this dissertation has been written. Without his advice and encouragement, the project would have been nearly impossible as well as much less enjoyable. I would also like to thank Greg Naber and Barbara Thomas for their constructive comments and criticism. The author would also like to express his gratitude to Carnegie-Mellon University for financial support during the writing of this manuscript and also to Professors Richard A. Moore and Maury Weir for their encouragement during the first years of graduate study. Finally, I would like to thank Nancy Colmer for her excellent typing of the final manuscript.

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CHAPTER ONE:

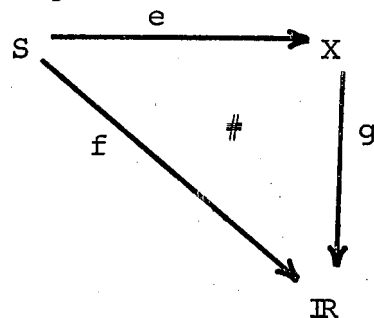
DEVELOPMENT OF THE STONE-ČECH COMPACTIFICATION

1.1. A compactification of a topological space X is a compact space K together with an embedding $e : X \rightarrow K$ with $e[X]$ dense in K . We will usually identify X with $e[X]$ and consider X as a subspace of K . Our main topic is a very special type of compactification -- one in which X is embedded in such a way that every bounded, real-valued continuous function on X will extend continuously to the compactification. Such a compactification of X will be called a Stone-Čech compactification and will be denoted by βX . In this chapter, several constructions of βX will be examined. We will find that βX is a useful device to study relationships between topological characteristics of X and the algebraic structure of the real-valued continuous functions defined on X and that many topological properties of X can be translated into properties of βX . The discussion will proceed in a rough approximation of historical order, although many anachronisms have been included to smooth the way and at the same time to introduce material that will be useful later.

Much of the first chapter will be a review of material developed in the classic L. Gillman and M. Jerison text, Rings of Continuous Functions which will henceforth be referred to by [GJ]. The development will be almost entirely self-contained. Only a very few results which rely on the more algebraic approach

of [GJ] will be stated without proof. The basic reference for results in general topology will be J. Dugundji's text, Topology, which will be referred to by [D].

1.2. For the most part, notation will be as in [GJ]. The ring of all real-valued continuous functions defined on a space X will be denoted by $C(X)$ and the subring consisting of all bounded members of $C(X)$ will be denoted by $C^*(X)$. We will say that a subspace S of X is C -embedded (resp. C^* -embedded) in X if every member f of $C(S)$ (resp. $C^*(S)$) extends to a member g of $C(X)$ (resp. $C^*(X)$). The following diagram illustrates C -embedding:



The symbol $\#$ indicates that the composition of g with the embedding e is equal to the mapping f . In such an instance, we will say that the diagram is commutative.

The set of points of X where a member f of $C(X)$ is equal to zero is called the zero-set of f and will be denoted by $Z(f)$. We will frequently say that f vanishes at x to mean $f(x) = 0$. The complement of the zero-set $Z(f)$ is called a cozero-set and is denoted by $Cz(f)$. The collections of all zero-sets and all cozero-sets of X will be denoted by

$Z[X]$ and $CZ[X]$, respectively. The ring and lattice operations in the rings $C(X)$ and $C^*(X)$ will be defined pointwise and it will be convenient to be familiar with such relationships as $Z(|f| + |g|) = Z(f) \cap Z(g)$ and $\{x \in X : f(x) \geq 1\} = Z((f-1) \wedge 0)$. Note that in the second equation, the symbol " 1 " is used to represent the function which is constantly equal to 1 . The symbol f^{-1} will be reserved to denote the reciprocal of a member f of $C(X)$ if the reciprocal is defined, i.e. if f does not vanish anywhere on X . The term mapping will always refer to a continuous function. The composition of two functions f and g will be denoted by $g \circ f$ and inverse images under a function f will be written $f^{-1}(S)$. The closure of a subspace S of X will be denoted by $cl_X S$ and the subscript will be omitted if no confusion can result. Two subsets A and B of a space X are said to be completely separated in X if there exists a mapping f in $C(X)$ such that $f(a) = 0$ for all a in A and $f(b) = 1$ for all b in B . One can easily see that two sets are completely separated if and only if they are contained in disjoint zero-sets.

The concept of C^* -embedding is important because as we have observed, the Stone-Čech compactification of a space X is a compact Hausdorff space containing X as a dense C^* -embedded subspace. The following theorem is [GJ, 1.17] and will be the main tool used to show that a subspace is C^* -embedded. The argument is a modification of one used by Urysohn in 1925 to show that a closed subset of a normal space is C^* -embedded.

URYSOHN'S EXTENSION THEOREM:

A subspace S of a space X is C^* -embedded in X if and only if any two completely separated sets in S are completely separated in X.

Proof: If S is C^* -embedded in X and A and B are completely separated in S, then the extension to X of the mapping which separates them will separate them in X.

Now assume that f_1 belongs to $C^*(S)$. Then $|f_1| \leq m$ for some integer m. For convenience, put

$$r_n = \left(\frac{m}{2}\right) \left(\frac{2}{3}\right)^n$$

for every n in $\mathbb{N}$. Then $|f_1| \leq 3r_1$. Proceeding inductively, suppose that we have obtained an element f_n of $C^*(S)$ such that $|f_n| \leq 3r_n$. Define

$$A_n = \{s \in S : f_n(s) \leq -r_n\} \text{ and } B_n = \{s \in S : f_n(s) \geq r_n\}.$$

A_n and B_n are completely separated in S and are therefore completely separated in X by hypothesis. Hence, there exists a mapping g_n in $C^*(X)$ such that g_n is constantly equal to $-r_n$ on A_n and to r_n on B_n and $|g_n| \leq r_n$. Now define

$$f_{n+1} = f_n - g_n|_S.$$

Since $|f_n(s) - g_n(s)| \leq 2r_n$ for every s in S, it is clear that $|f_{n+1}| \leq 3r_{n+1}$, and the induction step is complete. The Weierstrass M-test shows that the sequence $\{\sum_{i=1}^n g_i\}$ converges uniformly to a continuous function on X. Since

$$\begin{aligned} g_1 + \dots + g_n|_S &= (f_1 - f_2) + \dots + (f_n - f_{n+1}) \\ &= f_1 - f_{n+1} \end{aligned}$$

and $f_{n+1}(s)$ converges to zero for every s in S , the limit of the sequence extends f_1 . $\blacksquare$

1.3. The next result is [GJ, 1.18] and tells when a C^* -embedded subspace will also be C -embedded.

THEOREM:

A C^* -embedded subspace is C -embedded if and only if it is completely separated from every zero-set disjoint from it.

Proof: Let S be C -embedded in X . If a zero-set $Z(h)$ misses S , define f in $C(S)$ by $f(s) = 1/h(s)$. If g is the extension of f to all of X , then gh completely separates $Z(h)$ from S .

Now assume that S is a C^* -embedded subspace which is completely separated from every zero-set which misses it. Let f belong to $C(S)$. Then the composition $\arctan \circ f$ belongs to $C^*(S)$ and has an extension to a mapping g in $C(X)$. The zero-set

$$Z = \{x \in X : |g(x)| \geq \frac{\pi}{2}\}$$

misses S so that there exists g mapping X into the closed unit interval I such that g is constantly equal to 1 on S and to 0 on Z . Then gh agrees with $\arctan \circ f$ on S and satisfies $|(gh)(x)| < \frac{\pi}{2}$ for every x in X . Hence, $\tan \circ (gh)$ is defined, continuous, and real-valued on X and is an extension of f to all of X . $\blacksquare$

COMPLETELY REGULAR SPACES

1.4. The investigation of properties of a topological space through an embedding into a compact space is clearly limited to subspaces of compact spaces. Thus, it was a major step when Tychonoff characterized this class of spaces in 1929. A space X is completely regular if every closed subspace F of X is completely separated from any point x not in F and if each point is closed. This new class of spaces proved to be precisely the class that can be studied through the method of compactification.

THEOREM: (Tychonoff)

The completely regular spaces are precisely those spaces which can be embedded in a product of copies of the closed unit interval I .

1.5. The proof of the preceding theorem will be immediate from the following more general result and from there it will be an easy step to the simplest construction of the Stone-Čech compactification. A family $\mathfrak{F}$ of functions on X is said to distinguish points if for each pair of distinct points x and y , there exists an f in $\mathfrak{F}$ such that $f(x)$ is not equal to $f(y)$. $\mathfrak{F}$ is said to distinguish points and closed sets if for each closed set F in X and each point x not in F there exists f in $\mathfrak{F}$ such that $f(x)$ misses $\text{cl } f[F]$.

EMBEDDING LEMMA:

Let $\mathfrak{F}$ be a family of mappings such that each member f of $\mathfrak{F}$ maps the space X to a space Y_f . Then:

(a) The evaluation mapping $e : X \rightarrow \times Y_f$ defined by $e(x)_f = f(x)$ for all points x of X is continuous.

(b) The mapping e is an open mapping onto $e[X]$ if $\mathfrak{F}$ distinguishes points and closed sets.

(c) The mapping e is one-to-one if and only if $\mathfrak{F}$ distinguishes points.

(d) The mapping e is an embedding if $\mathfrak{F}$ distinguishes points and $\mathfrak{F}$ distinguishes points and closed sets.

Proof: (a) The function e is continuous since its composition with each projection is continuous ($\pi_f \circ e = f$).

(b) If U is an open set in X and $x \in U$, choose $f \in \mathfrak{F}$ such that $f(x) \notin \text{cl } f[X \setminus U]$. Then the set of all z in $e[X]$ such that $z_f \notin \text{cl } f[X \setminus U]$ is a neighborhood of $e(x)$ and is contained in $f[U]$. Therefore, $f[U]$ is open in $e[X]$.

(c) is clear and (d) follows from (b) and (c). $\blacksquare$

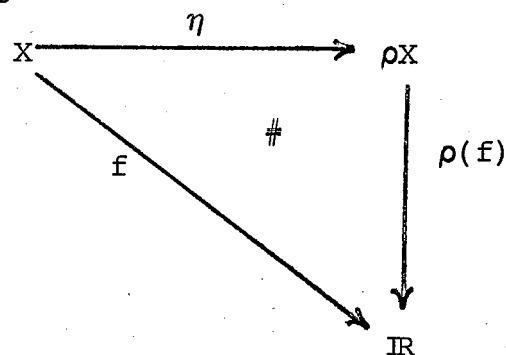
1.6. Tychonoff's characterization of complete regularity shows that no larger class can be studied by the method of compactification. Complete regularity plays a similar role if one attempts to study the topological properties of a space through algebraic or lattice properties of rings of continuous functions. The following theorem was obtained independently by E. Čech and M. H. Stone in 1937 and shows that completely regular spaces are not distinguishable from more general spaces through the

algebraic properties of rings of continuous functions.

THEOREM:

For any topological space X , there exists a completely regular space ρX which is a continuous image of X such that any real-valued mapping from X factors through ρX .

Proof: The situation described in the theorem is illustrated in the following diagram:



Define two points x and y of X to be equivalent if $f(x) = f(y)$ for all f in $C(X)$. This relation partitions X into equivalence classes. Let ρX denote the set of equivalence classes and let $\eta : X \rightarrow \rho X$ assign to each point of X its equivalence class. Since every f in $C(X)$ is constant on each equivalence class, we can define $\rho(f) : \rho X \rightarrow \mathbb{R}$ by $\rho(f)(\eta(x)) = f(x)$. It is immediate that this definition of $\rho(f)$ makes the diagram commute. Now provide ρX with the smallest topology such that each $\rho(f)$ is continuous. Then the closed sets of ρX are of the form

$$F = \bigcap \rho(f_\alpha)^{-1}(F_\alpha)$$

where each F_α is closed in $\mathbb{R}$. With this topology, ρX is

Hausdorff since points of X which are not separated by some member of $C(X)$ are identified in ρX . If F is closed in ρX and y is not in F , then there exists α such that y is not in $\rho(f_\alpha)^{\leftarrow}(F_\alpha)$. The point $\rho(f_\alpha)(y)$ is completely separated from F_α by some real-valued mapping g on $\mathbb{R}$ and $g \circ \rho(f_\alpha)$ completely separates y from F , making ρX completely regular. We will use the form of the closed sets of ρX to show that η is continuous. If F is a closed subset of ρX , then

$$\eta^{\leftarrow}(F) = \eta^{\leftarrow}(\cap \rho(f_\alpha)^{\leftarrow}(F_\alpha)) = \cap f_\alpha^{\leftarrow}(F_\alpha)$$

since the diagram is commutative. Because each f_α is continuous, $\eta^{\leftarrow}(F)$ is the intersection of closed sets and is therefore closed. $\blacksquare$

The importance of this theorem is that the correspondence $f \mapsto \rho(f)$ preserves both the ring and lattice structures of $C(X)$ and is an algebraic and lattice isomorphism between $C(X)$ and $C(\rho X)$. Thus, algebraic and lattice properties of $C(X)$ which are valid for an arbitrary space X also hold for $C(\rho X)$. [GJ, Chapter 3] provides a more detailed discussion of this aspect of ρX .

1.7. The proof of Theorem 1.6 reveals the form of the closed sets in a completely regular space. If f is a mapping of X to Y and g belongs to $C(Y)$, then $Z(g \circ f) = f^{\leftarrow}Z(g)$, so that the inverse image of a zero-set is again a zero-set. Since any closed set in $\mathbb{R}$ is a zero-set, we have shown that

any closed set in ρX is the intersection of zero-sets. We shall say that a family $\mathcal{B}$ of subsets of X is a base for the closed sets of X if any closed set is the intersection of members of $\mathcal{B}$. Hence, the family of zero-sets is a base for the closed sets of ρX . This property can easily be seen to characterize the class of completely regular spaces.

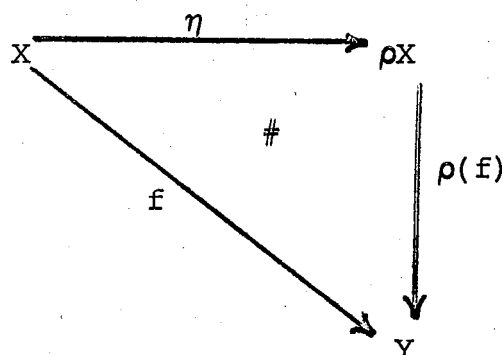
PROPOSITION:

A space is completely regular if and only if the family of zero-sets of the space form a base for the closed sets (or equivalently, the family of cozero-sets forms a base for the open subsets).

1.8. By utilizing Tychonoff's characterization of complete regularity, we can show that any mapping of a space X into a completely regular space will factor through ρX .

COROLLARY:

If f is a mapping of the space X into a completely regular space Y , then there is a mapping $\rho(f)$ of ρX into Y such that the diagram commutes:



Proof: For each g in $C(Y)$, let $\mathbb{R}_g$ be a copy of the real line and let e be the evaluation map embedding Y into their product, $\times \mathbb{R}_g$. Theorem 1.6 gives a mapping $\rho(g \circ f)$ of ρX into $\mathbb{R}_g$ such that $g \circ f = \rho(g \circ f) \circ \eta$ since the composition $g \circ f$ maps X into $\mathbb{R}_g$. To show that f factors through ρX , define a mapping h of ρX to $\times \mathbb{R}_g$ by

$$h(z)_g = \rho(g \circ f)(z)$$

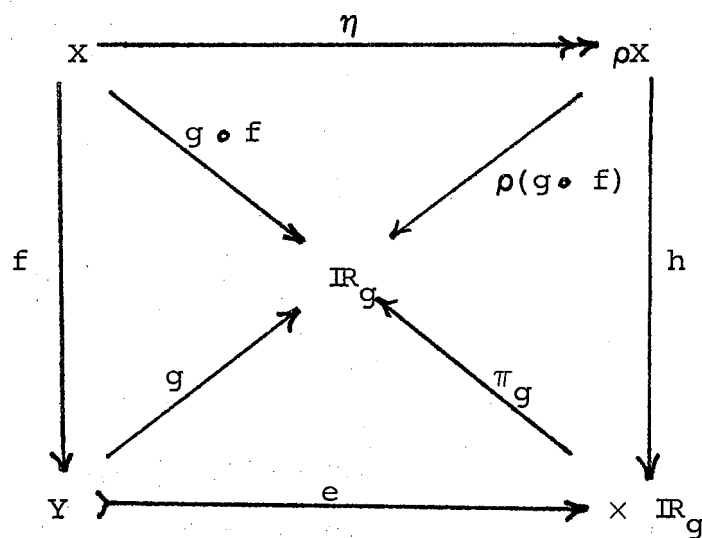
for all z in ρX . Then h is continuous. $h[\rho X]$ is contained in $e[Y]$ since for each projection map π_q ,

$$(\pi_g \bullet h)(z) = \rho(g \bullet f)(z)$$

so that

$$(\pi_g \circ h) [\rho X] \subset g[Y].$$

Since e is an embedding and $h[X]$ is contained in $e[Y]$, putting $\rho(f) = e \circ h$ gives the required factorization of f through ρX . The following diagram describes the situation:



To summarize the importance of the class of completely regular spaces, Theorem 1.4 shows that no larger class can be studied by means of embeddings into compact Hausdorff spaces. Theorem 1.6 establishes that no additional information can be gained by investigating algebraic properties of rings of continuous functions for any larger class of spaces. Finally, Proposition 1.7 exhibits a relationship between the topology of a completely regular space and the real-valued mappings defined on the space which will prove to be very useful. For these reasons, unless otherwise noted,

ALL SPACES MENTIONED WILL BE PRESUMED TO BE COMPLETELY REGULAR.

βX AND THE EXTENSION OF MAPPINGS

1.9. The year 1937 was an important one in the developing theory of topology and its relations to algebra. M. H. Stone and E. Čech each published important papers which provided independent proofs of the existence of the compactification βX . H. Cartan introduced the notions of filter and ultrafilter in a fundamental paper which led to a new theory of convergence by generalizing both sequences and neighborhoods of the diagonal.

Stone's paper treated the relations of algebra and topology through applications of Boolean rings. The most important result in this theory is the representation of Boolean algebras utilizing totally disconnected compact Hausdorff spaces. We will consider this topic in Chapter 2. We will not describe

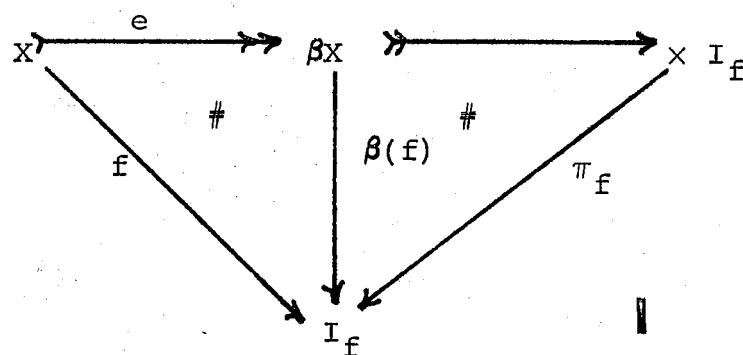
Stone's development of the compactification, but will treat the outgrowths of his work found in the papers of Wallman, Samuel, Frink, Hewitt, Gelfand and Kolmogoroff, Gillman, Jerison, and Henriksen.

Čech demonstrated the existence of the compactification βX in his paper and used it to investigate properties of X by embedding X into βX . We will use modifications of Čech's methods to obtain the results of Stone. The following theorem can be interpreted algebraically as showing that $C^*(X)$ and $C^*(\beta X)$ are isomorphic.

THEOREM: (Stone-Čech)

Every completely regular space X has a Hausdorff compactification βX in which it is C^* -embedded.

Proof: For every f in $C^*(X)$, let I_f denote the range of f . Since f is bounded, $cl I_f$ is compact. By taking $\mathfrak{F} = C^*(X)$ in the Embedding Lemma, 1.5, X can be embedded into $\prod I_f$ by means of the evaluation mapping $e(x)_f = f(x)$. Put $\beta X = cl(e[X])$. Then for f in $C^*(X)$, the extension $\beta(f) : \beta X \rightarrow I_f$ is the restriction to βX of the projection π_f . The situation is described in the following diagram:



A "tail" on an arrow will denote an embedding and a "double tail" will denote a closed embedding. A double headed arrow will indicate a mapping with dense range.

From the above construction and the observation that a mapping from a compact space to a Hausdorff space is closed, we obtain the following result.

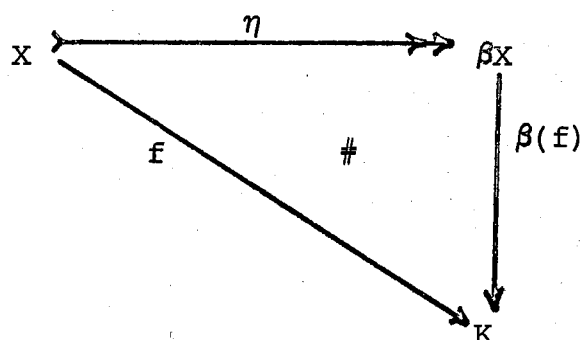
1.10. COROLLARY:

If X is a compact space, βX is homeomorphic to X .

1.11. Stone showed that not only would members of $C^*(X)$ extend to βX , but that any mapping of X into a compact space would extend to βX . We will prove Stone's result by combining the two previous results with Theorem 1.4. The similarity between the following proof and the proof of Corollary 1.8 will be investigated in Chapter 10 when ρX and βX are viewed in a common categorical context.

THEOREM: (M. H. Stone)

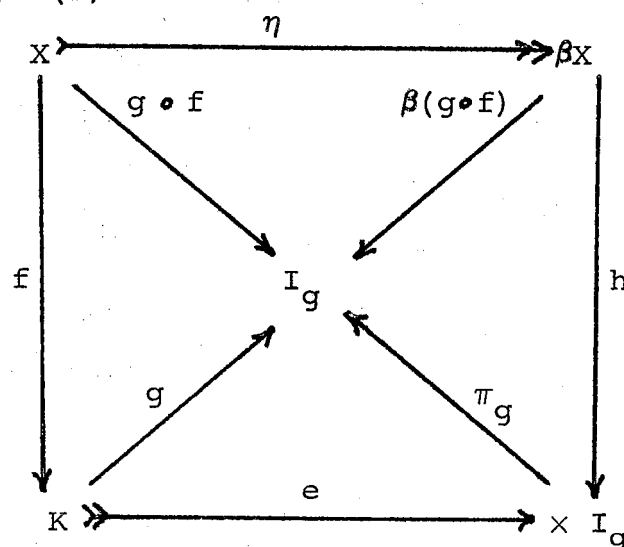
Every completely regular space X has a compactification βX such that any mapping of X to a compact space K will extend uniquely to βX .



Proof: For each map g in $C^*(K)$, let I_g denote the range of g and let e be the evaluation map embedding K into their product, $\times I_g$. Then since $g \circ f$ maps X to I_g , Theorem 1.9 provides an extension $\beta(g \circ f)$ of $g \circ f$ to βX . To show that f extends to βX , define a mapping h of βX to $\times I_g$ by

$$h(p)_g = \beta(g \circ f)(p)$$

for all p in βX . Then h is continuous since its composition with each projection is continuous, i.e. $(\pi_g \circ h)(p) = \beta(g \circ f)(p)$ for all g in $C^*(K)$.



Since e is an embedding, it remains to show that the image of βX under h is contained in the image of K under e . Since for each g in $C^*(K)$, $\beta(g \circ f)[X]$ is contained in $g[K]$,

$$h[\beta X] = h[cl_{\beta X} X] \subset cl[eX] = cl K = K$$

because K is compact.

The uniqueness of the extension h is immediate since

any two extensions of f must agree on the dense subspace X of βX . $\square$

The above proof is a modification of the technique used by Čech to obtain the following special case of the theorem.

1.12. COROLLARY:

Any compactification of X is a continuous image of βX under a mapping which leaves points of X fixed.

A partial ordering can be induced on the set of compactifications of a space X in the following way: If K_1 and K_2 are two compactifications of X , define $K_1 \leq K_2$ whenever there exists a mapping g of K_2 onto K_1 which leaves the points of X fixed. Then Corollary 1.12 shows that βX is a maximal element in the set of compactifications of X . The maximality of βX enables us to show that the Stone-Čech compactification of a space is essentially unique.

1.13. COROLLARY:

Any compactification of X to which every mapping of X to a compact space has an extension is homeomorphic to βX under a homeomorphism which leaves points of X fixed.

Proof: If K is a compactification of X satisfying the stated factorization property, then the embedding of X into βX has an extension f to K and similarly the embedding of X into K has an extension g to βX . Since the restriction of $f \circ g$ to X is the identity on X , $f \circ g$ is the identity on

βX since X is dense in βX . Similarly, $g \circ f$ is the identity on K . Thus, f and g are homeomorphisms leaving points of X fixed and $f = g^{-1}$. $\square$

1.14. Čech gave an additional characterization of βX which is important as a forerunner of the construction of βX via zero-sets as in [GJ, Chapter 6]. We will consider a variant of the [GJ] approach beginning in Section 1.34.

THEOREM: (Čech)

βX is that compactification of a space X in which completely separated subsets of X have disjoint closures.

Proof: If two subsets of X are completely separated by a mapping f in $C^*(X)$, then the extension $\beta(f)$ of f to βX completely separates the closures of the sets. Thus, βX satisfies the condition of the theorem.

Let K be a compactification of X in which completely separated subsets of X have disjoint closures. By Corollary 1.12, there exists a mapping h of βX onto K which leaves points of X fixed. Because βX is compact, h is a closed mapping and it is sufficient to show that h is one-to-one.

Let p and q be distinct points of βX and let f in $C(\beta X)$ be such that $f(p) = 0$ and $f(q) = 1$. The sets $A_0 = \{x \in X : f(x) \leq \frac{1}{3}\}$ and $A_1 = \{x \in X : f(x) \geq \frac{2}{3}\}$ are completely separated in X and thus have disjoint closures in K . Since $h(p)$ is in $cl_K A_0$ and $h(q)$ is in $cl_K A_1$, we must have $h(p) \neq h(q)$. Thus, h is a closed continuous bijection and is therefore a homeomorphism between K and βX . $\square$

1.15. By observing that in a normal space any two disjoint closed sets are completely separated and applying the theorem, Čech obtained the

COROLLARY:

When X is a normal space, βX is that compactification of X in which disjoint closed subsets of X have disjoint closures.

$\mathcal{B}$ -FILTERS AND $\mathcal{B}$ -ULTRAFILTERS

1.16. We now turn to the developments which follow from Cartan's ultrafilter concept and from the set-theoretic content of M. H. Stone's work. The following ideas are adapted from H. Wallman's 1938 paper and from P. Samuel's 1948 paper and are formulated while looking back at M. H. Stone's work and forward in the directions to be taken by E. Hewitt in 1948 and by O. Frink in 1964.

A family $\mathcal{B}$ of subsets of a space X is called a ring of sets if it is closed under finite intersections and unions. A subfamily $\mathcal{F}$ of non-empty members of a ring $\mathcal{B}$ is called a $\mathcal{B}$ -filter if $\mathcal{F}$ satisfies the following conditions:

- (1) $\mathcal{F}$ is closed under finite intersections.
- (2) A member of $\mathcal{B}$ containing a member of $\mathcal{F}$ is in $\mathcal{F}$.

1.17. A $\mathcal{B}$ -filter $\mathcal{F}$ is a $\mathcal{B}$ -ultrafilter if it is not properly contained in any other $\mathcal{B}$ -filter. The following characterization of $\mathcal{B}$ -ultrafilters is easily verified.

PROPOSITION:

A $\mathfrak{g}$ -filter $\mathfrak{u}$ is a $\mathfrak{g}$ -ultrafilter if and only if any member of $\mathfrak{g}$ which meets every member of $\mathfrak{u}$ is in $\mathfrak{u}$.

1.18. In his fundamental 1937 paper, H. Cartan considered $\mathfrak{g}$ -filters and $\mathfrak{g}$ -ultrafilters where $\mathfrak{g}$ is the family of all subsets of a set X . In such a case, the reference to $\mathfrak{g}$ is usually suppressed and the terms filter and ultrafilter are used. An example of such a filter is the collection of all neighborhoods of a point x in X which is denoted by $\mathfrak{B}(x)$ and called the neighborhood filter of x . The filter concept was introduced in order to study convergence. A filter $\mathfrak{F}$ is said to converge to a point x if $\mathfrak{F}$ contains $\mathfrak{B}(x)$. A filter $\mathfrak{F}$ is said to cluster at a point x if every member of $\mathfrak{F}$ meets every member of $\mathfrak{B}(x)$, i.e. if x belongs to $\bigcap \{ \mathfrak{C} \in \mathfrak{F} : x \in \mathfrak{C} \}$.

Observe that the definition of convergence of a filter is valid in an arbitrary topological space. Convergence of $\mathfrak{g}$ -filters for rings other than the power set is usually defined only for classes of spaces in which each point has a neighborhood base contained in the ring. The proper choice of $\mathfrak{g}$ for the class of completely regular spaces will be considered in Section 1.27.

1.19. In addition to describing convergence, collections of $\mathfrak{g}$ -filters, usually $\mathfrak{g}$ -ultrafilters, have been used to construct topological spaces. Let $\omega_X(\mathfrak{g})$ denote the collection of all $\mathfrak{g}$ -ultrafilters on X . When no confusion can result, we shall

simply write $\omega(\mathcal{B})$. For Z in $\mathcal{B}$, let Z^ω denote the members of $\omega(\mathcal{B})$ which contain Z . Taking $\{Z^\omega : Z \in \mathcal{B}\}$ as a base for the closed sets imposes a topology on $\omega(\mathcal{B})$ which is frequently useful in the formation of compactifications.

(a) M. H. Stone introduced this topology with $\mathcal{B}$ the family of all open-and-closed subsets of X , which we shall call clopen sets. He showed that $\omega(\mathcal{B})$ is compact and that if X has a base for the open sets composed of clopen sets, then X can be embedded into $\omega(\mathcal{B})$.

(b) H. Wallman in 1938 considered the case where $\mathcal{B}$ is the family of all closed sets of a T_1 -space X . Wallman showed that under these hypotheses, $\omega(\mathcal{B})$ is compact and contains X as a dense subspace, but that $\omega(\mathcal{B})$ is not necessarily Hausdorff. He showed that $\omega(\mathcal{B})$ would be Hausdorff precisely when X is normal. In fact, we will later show that when X is embedded in $\omega(\mathcal{B})$, Z^ω is the closure of the set Z in $\mathcal{B}$, and we will see that when X is normal, $\omega(\mathcal{B})$ is βX . The definition of Z^ω shows that disjoint closed sets of X have disjoint closures in $\omega(\mathcal{B})$ and that coupled with Cech's characterization of βX for normal X contained in Corollary 1.15 gives $\omega(\mathcal{B}) = \beta X$. It was this result which was to motivate Frink's work as we shall see later. Wallman utilized his compactification to show that it is not possible to distinguish between T_1 -spaces and compact spaces by means of homology theory.

(c) The space $\omega(\mathcal{B})$ is compact if $\mathcal{B}$ is the power set of X , but X cannot be embedded into $\omega(\mathcal{B})$ except when X is discrete.

If X is discrete, the discussion of the Wallman compactification above shows that $\omega(\beta X) = \beta X$. This is the interpretation of βX which will be most useful in the case of discrete spaces in later chapters.

βX AND MAXIMAL IDEAL SPACES

1.20. In his 1937 paper, M. H. Stone showed that if Y is a compact space, then the maximal ideals of $C^*(Y)$ are in one-to-one correspondence with the points of Y and that the maximal ideal corresponding to a point of Y is the set of mappings which vanish at that point. Stone also introduced a topology for the set of maximal ideals of a Boolean ring, i.e. a ring in which $x^2 = x$ for every x in the ring. In this section, we will consider the relationships between a space X and the rings $C^*(X)$ and $C(X)$ which were developed from Stone's work by I. Gelfand and A. Kolmogoroff in their 1939 paper.

We first consider the topology introduced by Stone for the set of maximal ideals of a Boolean ring and show that the topology can be used in a wider class of rings.

If A is a commutative ring with unity, let $\mathfrak{M}(A)$ denote the collection of all maximal ideals of A . For a subset $\mathfrak{H}$ of $\mathfrak{M}(A)$, the kernel of $\mathfrak{H}$ is defined to be $\bigcap \mathfrak{H}$ and for any ideal I of A , define the hull of I to be

$$\{M \in \mathfrak{M}(A) : I \subset M\}.$$

The Stone topology on the set of maximal ideals $\mathfrak{m}(A)$ is then obtained by defining the closure of any subset $\mathfrak{H}$ of $\mathfrak{m}(A)$ to be the hull of the kernel of $\mathfrak{H}$, i.e.

$$\text{cl } \mathfrak{H} = \{M \in \mathfrak{m}(A) : \cap \mathfrak{H} \subset M\}.$$

For this reason, the Stone topology is often called the hull-kernel topology. With this topology, the space $\mathfrak{m}(A)$ is called the structure space of A .

1.21. For each element a of A , let $C(a)$ denote the set of maximal ideals containing a . Each set $C(a)$ is closed since it is the hull of the ideal consisting of all multiples of a . The family $\{C(a) : a \in A\}$ is a base for the closed sets of $\mathfrak{m}(A)$ since

$$\text{cl } \mathfrak{H} = \cap \{C(a) : a \in \cap \mathfrak{H}\}$$

when $\mathfrak{H}$ is a subset of $\mathfrak{m}(A)$. The following basic result follows the outline of [GJ, ex. 7M].

PROPOSITION:

If A is a commutative ring with unity, $\mathfrak{m}(A)$ with the Stone topology is a compact Hausdorff space if and only if whenever M and M' are distinct maximal ideals there exist a not in M and a' not in M' such that aa' belongs to every maximal ideal of A .

Proof: If A satisfies the stated hypothesis and M and M' are distinct maximal ideals, then

$$U = \mathfrak{m}(A) \setminus C(a) \quad \text{and} \quad U' = \mathfrak{m}(A) \setminus C(a')$$

are neighborhoods of M and M' , respectively. To show that U and U' are disjoint, we write:

$$\begin{aligned} U \cap U' &= (\mathfrak{m}(A) \setminus C(a)) \cap (\mathfrak{m}(A) \setminus C(a')) \\ &= \mathfrak{m}(A) \setminus (C(a) \cup C(a')) \\ &= \mathfrak{m}(A) \setminus C(aa'). \end{aligned}$$

The last equality holds because every maximal ideal is prime.

Hence, since aa' belongs to every maximal ideal, $C(aa') = \mathfrak{m}(A)$ implying that $U \cap U' = \emptyset$ and $\mathfrak{m}(A)$ is Hausdorff.

Conversely, if $\mathfrak{m}(A)$ is Hausdorff any pair of distinct maximal ideals M and M' must be separated by disjoint basic open sets U and U' as above. But

$$\emptyset = U \cap U' = \mathfrak{m}(A) \setminus (C(a) \cup C(a'))$$

implies that $C(a) \cup C(a')$ is all of $\mathfrak{m}(A)$ so that aa' belongs to every maximal ideal of A .

It remains to demonstrate the compactness of $\mathfrak{m}(A)$. Let $\{F_\alpha\}$ be a family of closed sets. We will show that if $\{F_\alpha\}$ has empty intersection, then some finite subfamily has empty intersection. Since each F_α is an intersection of basic closed sets, it is sufficient to assume that each set in the family is a basic set, i.e. that for every α there is some a_α in A such that $F_\alpha = C(a_\alpha)$. The result will follow from determining when the intersection of a basic family $\{C(a_\alpha)\}$ will be empty in terms of the elements $\{a_\alpha\}$. We will show that

$\bigcap C(a_\alpha) = \emptyset$ exactly when the subset $\{a_\alpha\}$ of A generates A . If $\bigcap C(a_\alpha) = \emptyset$, then for every M in $\mathfrak{M}(A)$ there is some a_α not belonging to M . Thus, the only ideal containing $\{a_\alpha\}$ is the ring A itself so that $\{a_\alpha\}$ generates A . The converse is clear since each of the steps is reversible.

Now if $\bigcap C(a_\alpha) = \emptyset$, $\{a_\alpha\}$ generates A and there exist members r_i of A such that the identity element of A can be written $1 = \sum_{1 \leq i \leq n} r_i a_{\alpha_i}$ for some finite family a_{α_i} . But then the ideal generated by $\{a_{\alpha_i}\}$ contains the identity and hence is all of A . Thus, the finite subfamily $\{a_{\alpha_i}\}$ generates A so that $\bigcap C(a_{\alpha_i}) = \emptyset$. Hence, $\mathfrak{M}(A)$ is compact. $\square$

1.22. The next step in relating $C^*(X)$ to X is to characterize the maximal ideals of $C^*(X)$. One need only consider a compact space since $C^*(X)$ and $C^*(\beta X)$ are isomorphic.

PROPOSITION: (M. H. Stone)

If Y is a compact space, the maximal ideals of $C^*(Y)$ are in one-to-one correspondence with the points of Y and are given by

$$M^{*p} = \{f \in C^*(Y) : f(p) = 0\}$$

for p a point of Y .

Proof: Each M^{*p} is clearly an ideal. Since distinct points of Y are separated by a member of $C^*(Y)$, M^{*p} and M^{*q} are distinct whenever p is not equal to q . To complete the proof it is sufficient to show that any proper ideal I is contained

in M^*p for some p . Suppose on the contrary that for every point p of Y , there is a member f_p of I such that $f_p(p) \neq 0$. Then there is some neighborhood $U(p)$ of p on which f_p is never equal to zero. Since Y is compact, the covering $\{U(p)\}$ of Y has a finite subcover, $\{U(p_i)\}$. Then the mapping defined by $g = f_{p_1}^2 + \dots + f_{p_n}^2$ belongs to I and is never zero so that its reciprocal g^{-1} belongs to $C^*(Y)$. Thus, every h in $C^*(Y)$ can be written as $h = g \cdot g^{-1} \cdot h$ and therefore belongs to I so that I is not a proper ideal. This is a contradiction. $\blacksquare$

1.23. In order to obtain the following main theorem, it is sufficient to show that the correspondence between the points of a compact space Y and the maximal ideals of $C^*(Y)$ is a homeomorphism. Let $m^*(Y)$ denote $m(C^*(Y))$ and $m(Y)$ denote $m(C(Y))$.

THEOREM:

A compact space Y is homeomorphic with the maximal ideal space $m^*(Y)$.

Proof: If p and q are distinct points of Y , there exist f and g in $C^*(Y)$ such that $f(p) = g(q) = 1$ and $fg = 0$. Hence, f does not belong to M^*p and g does not belong to M^*q although their product fg belongs to every maximal ideal of $C^*(Y)$. Proposition 1.21 thus shows that $m^*(Y)$ is a compact Hausdorff space.

Denote the bijection of Y with $m^*(Y)$ which sends a point p of Y to the ideal M^*p by τ^* . To show that τ^*

is a homeomorphism it is sufficient to show that for S a subset of Y , $\text{cl}(\tau^*[S]) = \tau^*[\text{cl } S]$. If p is in $\text{cl } S$, then every member of $C^*(Y)$ which vanishes on all of S also vanishes at p . But then by the definition of closure in a space of maximal ideals,

$$\text{cl}(\tau^*[S]) = \cap \{ \mathcal{C}(f) : f[S] = \{0\} \},$$

and this shows that $\tau^*(p) = M^p$ belongs to $\text{cl}(\tau^*[S])$. On the other hand, if p is not in $\text{cl } S$, then there is a member of $C^*(Y)$ which vanishes on all of S but not at p . But this shows that $\tau^*(p) = M^p$ fails to belong to $\text{cl}(\tau^*[S])$. $\square$

1.24. The following corollary is immediate from the observation that $C^*(X)$ is isomorphic to $C^*(\beta X)$.

COROLLARY:

βX is homeomorphic with the maximal ideal space $\mathfrak{M}^*(X)$.

1.25. If $C^*(X)$ and $C^*(Y)$ are isomorphic for compact spaces X and Y , then $\mathfrak{M}^*(X)$ and $\mathfrak{M}^*(Y)$ are homeomorphic. The following result expresses the fact that the ring of bounded real-valued mappings of a compact space determines the space to within homeomorphism.

COROLLARY: (M. H. Stone)

If X and Y are compact spaces, then X and Y are homeomorphic if and only if $C^*(X)$ and $C^*(Y)$ are isomorphic.

1.26. Gelfand and Kolmogoroff also showed that βX is homeomorphic to $m(X)$. The proof is more complex than that just given in the case of the maximal ideal space $m^*(X)$. The difficulty involved is to characterize the maximal ideals of $C(X)$ in terms of the points of βX . Once this has been achieved, it will remain to exhibit the homeomorphism of βX with $m(X)$, and this step will be similar to that for $m^*(X)$. In the case of $C^*(X)$, we can utilize the ring isomorphism $f \mapsto \beta(f)$ of $C^*(X)$ and $C^*(\beta X)$ to characterize the maximal ideals of $C^*(X)$ in terms of βX .

PROPOSITION:

The maximal ideals of $C^*(X)$ are in one-to-one correspondence with the points of βX and are given by

$$M^*p = \{f \in C^*(X) : \beta(f)(p) = 0\}$$

where p is a point of βX .

In terms of zero-sets, this tells us that M^*p is the collection of all members of $C^*(X)$ such that p belongs to $Z(\beta(f))$. The maximal ideals of $C(X)$ can also be characterized in terms of zero-sets, but the process is more complicated and will require the development of the notion of $\mathcal{B}$ -filters in such a way as to relate the topological structure of X to the algebraic structure of $C(X)$.

1.27. We first consider a description of convergence in completely regular spaces. In a completely regular space X , if U is an open set containing the point x , there exists a map f in $C(X)$ such that $f(x) = 1$ and $f[X \setminus U] = \{0\}$. Then we have that the set Z defined by

$$Z \equiv \{x : f(x) \geq \frac{1}{2}\} = Z((f - \frac{1}{2}) \wedge 0)$$

is contained in U and is both a zero-set and a neighborhood of x . Thus, in order to describe convergence in a completely regular space, it is sufficient to consider zero-set neighborhoods.

Further, if $Z(f_1)$ and $Z(f_2)$ are zero-sets, then $Z(f_1 f_2) = Z(f_1) \cup Z(f_2)$ and $Z(f_1^2 + f_2^2) = Z(f_1) \cap Z(f_2)$ so that $Z[X]$ is a ring of sets and we can consider $\mathfrak{z}$ -filters and $\mathfrak{z}$ -ultrafilters with $\mathfrak{z} = Z[X]$. We shall refer to the ring of zero-sets often and $\mathfrak{z}$ -filters where $Z[X]$ is the ring will be called z-filters. For each point x of X , let O^x denote the neighborhoods of x which are also zero-sets. Then if $\mathfrak{B}(x)$ is the neighborhood filter of x , $O^x = \mathfrak{B}(x) \cap Z[X]$. We have shown above that O^x contains enough neighborhoods of x to describe convergence. O^x is clearly a z -filter. A z -filter $\mathfrak{F}$ converges to x if O^x is contained in $\mathfrak{F}$ and $\mathfrak{F}$ clusters at x if x belongs to $\bigcap \mathfrak{F}$. The following property makes it convenient to discuss convergence in terms of z -ultrafilters as opposed to z -filters.

1.28. PROPOSITION:

A z-ultrafilter converges to any cluster point.

Proof: If $\mathcal{U}$ is a z-ultrafilter and a point x belongs to $\bigcap \mathcal{U}$, then every member of O^x meets every member of $\mathcal{U}$. But then O^x is contained in $\mathcal{U}$ Proposition 1.17. $\square$

1.29. We now relate z-filters and z-ultrafilters to the ring $C(X)$. The relationships between z-filters and ideals of $C(X)$ were first explored by E. Hewitt in his fundamental 1948 paper on rings of continuous functions. Consider the function

$$Z : C(X) \rightarrow Z[X]$$

which sends each mapping in $C(X)$ to its zero-set. The following result is [GJ, 2.3] and shows that the image of an ideal under Z is a z-filter and that the pre-image of a z-filter is an ideal.

PROPOSITION:

(a) If I is a proper ideal in $C(X)$, then $Z[I] = \{Z(f) : f \in I\}$ is a z-filter on X .

(b) If $\mathcal{F}$ is a z-filter on X , then $Z^{-1}[\mathcal{F}] = \{f \in C(X) : Z(f) \in \mathcal{F}\}$ is an ideal in $C(X)$.

Proof: (a) Since a proper ideal can contain no unit and the units of $C(X)$ are those maps which have void zero-sets, all members of $Z[I]$ are non-empty. If f_1 and f_2 are in I , then $f_1^2 + f_2^2$ is in I and since $Z(f_1) \cap Z(f_2) = Z(f_1^2 + f_2^2)$, $Z[I]$ is closed under finite intersections. Let $Z(f)$ be in $Z[I]$ for f in I . If $Z(g) \supset Z(f)$, then

$$Z(g) = Z(f) \cup Z(g) = Z(fg)$$

is in $Z[I]$. Thus, $Z[I]$ is a z -filter.

(b) Let $J = Z^{\leftarrow}[\mathfrak{F}]$. Since the empty set is not in $\mathfrak{F}$, J does not contain a unit. If f and g are in J ,

$$Z(f-g) \supset Z(f) \cap Z(g)$$

and $f - g$ is in J since $\mathfrak{F}$ is closed under supersets in $Z[X]$ and finite intersections. Thus, J is an additive subgroup. If f is in J and g is in $C(X)$, then $Z(fg) \supset Z(f)$ and fg is in J since $\mathfrak{F}$ is closed under supersets in $Z[X]$. $\square$

Since Z preserves containment, it is clear that the proposition yields a one-to-one correspondence between the z -ultrafilters on X and the maximal ideals of $C(X)$. The previous proposition together with the characterization of $\mathfrak{z}$ -ultrafilters given in Proposition 1.17 allows the identification of those members of $C(X)$ which belong to a maximal ideal M : f is in M if $z(f)$ meets the zero-set of every member of M . We can now establish the main result of Gelfand and Kolmogoroff's 1939 paper.

1.30. THEOREM: (Gelfand and Kolmogoroff)

The maximal ideals of $C(X)$ are in one-to-one correspondence with the points of βX and are given by

$$M^p = \{f \in C(X) : p \in \text{cl}_{\beta X} Z(f)\}$$

for p in βX .

Proof: We show first that each M^p is a maximal ideal by showing that $Z[M^p]$ is a z -ultrafilter. It is clear that $Z[M^p]$ is closed under supersets in $Z[X]$ and that $Z[M^p]$ does not contain the empty set. Since disjoint zero-sets of X are completely separated, they would have disjoint closures in βX . Thus, since p is in the closure of every Z in $Z[M^p]$, no two members of $Z[M^p]$ can be disjoint by Theorem 1.14. To show that $Z[M^p]$ is maximal, suppose that a zero-set Z meets every member of $Z[M^p]$. Then if p is not in $cl_{\beta X} Z$, there exists a zero-set neighborhood Z' of p in βX which misses Z . But then $Z' \cap X$ is in $Z[M^p]$ and misses Z , which is a contradiction. Thus, M^p is a maximal ideal.

It remains to show that every maximal ideal is of the form M^p for some p in βX . If M is maximal, then $\{cl_{\beta X} Z : Z \in Z[M]\}$ is a family of closed sets with the finite intersection property in a compact space. Thus, there exists p in $\bigcap \{cl_{\beta X} Z : Z \in Z[M]\}$ so that $Z[M]$ clusters at p . But then $Z[M]$ converges to p and must converge to p alone since βX is Hausdorff. Thus, $M = M^p$. $\blacksquare$

1.31. COROLLARY:

βX is homeomorphic with the maximal ideal space $\mathfrak{M}(X)$.

Proof: The theorem establishes a one-to-one correspondence τ between βX and $\mathfrak{M}(X)$. The proof that τ is a homeomorphism is similar to the proof that τ^* is a homeomorphism in Corollary 1.23. $\blacksquare$

The proofs here are based on a 1954 paper of Gillman, Henriksen, and Jerison in which Theorem 1.30 is discussed with its applications. The characterization of maximal ideals is also treated in [GJ, Chapter 7].

1.32. The proof of Theorem 1.30 exhibits a one-to-one correspondence between the points of βX and the maximal ideals of $C(X)$. We also have a one-to-one correspondence between the maximal ideals of $C(X)$ and the z -ultrafilters on X . Following [GJ], if p belongs to βX , we will denote the z -ultrafilter corresponding with the maximal ideal M^p by A^p . Then Proposition 1.29 shows that

$$A^p = Z^{\leftarrow}[M^p] = \{Z(f) : f \in M^p\}.$$

From Theorem 1.30, it is evident that if p belongs to X , then $\bigcap A^p = \{p\}$, and that if p belongs to $\beta X \setminus X$, then $\bigcap A^p = \emptyset$. In the first case, we shall say that A^p is fixed and in the second, that A^p is free. The same terms will also be applied to the ideals M^p as well as to any z -filter or ideal. Thus we can view the construction of βX as adding a point p to X for each free maximal ideal M^p in such a way that the resulting space is compact and that A^p converges to p .

1.33. Gelfand and Kolmogoroff observed that if $\mathbb{N}$ is the countable discrete space, then $C^*(\mathbb{N})$ and $C^*(\beta \mathbb{N})$ are isomorphic, but that $C(\mathbb{N})$ and $C(\beta \mathbb{N})$ are not isomorphic. Thus, $C(X)$ is a more sensitive invariant than $C^*(X)$ for distinguishing

between topological spaces. We shall see in Theorem 1.56 that E. Hewitt developed this idea by introducing the class of real-compact spaces which play a role with respect to $C(X)$ analogous to that played by the class of compact spaces with respect to $C^*(X)$ in Corollary 1.25.

SPACES OF $\mathfrak{B}$ -ULTRAFILTERS

1.34. We have seen two different ways to represent the Stone-Čech compactification. We first showed that βX could be obtained as the closure of a copy of X embedded in a product of intervals and we later saw that βX is homeomorphic to the spaces of maximal ideals of the rings $C^*(X)$ and $C(X)$. In the next several sections we will consider the Stone-Čech compactification in the context of spaces of $\mathfrak{B}$ -ultrafilters.

In 1.19b, it was mentioned that H. Wallman showed that for a normal space X , βX is homeomorphic to the space of $\mathfrak{B}$ -ultrafilters on X when $\mathfrak{B}$ is the ring of closed subsets of X . In Chapter 6 of [GJ], βX is described as the space of all z -ultrafilters on X by considering the ring of zero-sets of X .

1.35. Motivated largely by these two familiar examples of rings of sets -- the zero-sets of a completely regular space and the closed sets of a normal space, in 1964 O. Frink introduced the concept of a normal base. A collection of closed subsets of a T_1 -space X is called a normal base for X if $\mathfrak{B}$

satisfies the following conditions:

- (1) $\mathcal{B}$ is a ring of sets.
- (2) $\mathcal{B}$ is disjunctive, i.e. if a closed subset of X does not contain a point of X , then there exists a member of $\mathcal{B}$ containing the point and missing the closed set.
- (3) $\mathcal{B}$ is a base for the closed sets of X , i.e. every closed set is an intersection of members of $\mathcal{B}$.
- (4) $\mathcal{B}$ is normal, i.e. disjoint members of $\mathcal{B}$ are contained in disjoint complements of members of $\mathcal{B}$.

After we obtain the necessary preliminary results, we will impose the topology used by Wallman on the space of $\mathcal{B}$ -ultrafilters when $\mathcal{B}$ is a normal base and show that the resulting space is a Hausdorff compactification.

1.36. The following result can be obtained by a straightforward application of Zorn's Lemma and will be used to show that $\omega(\mathcal{B})$ is compact.

PROPOSITION:

Every $\mathcal{B}$ -filter is contained in a $\mathcal{B}$ -ultrafilter if $\mathcal{B}$ is any ring of sets.

1.37. The next result will allow X to be embedded into $\omega(\mathcal{B})$. If x is a point in X , let $\phi(x) = \{Z \in \mathcal{B} : x \in Z\}$.

PROPOSITION:

Each $\phi(x)$ is a $\mathcal{B}$ -ultrafilter if $\mathcal{B}$ is a disjunctive ring of sets.

Proof: Suppose that Z belongs to $\mathcal{B}$ and that x is not in Z . Since Z is closed and $\mathcal{B}$ is disjunctive, there exists Z' containing x and missing Z . Then Z does not belong to $\varphi(x)$ and every element of $\mathcal{B}$ which meets every member of $\varphi(x)$ contains x . Hence, Proposition 1.17 shows that $\varphi(x)$ is a $\mathcal{B}$ -ultrafilter. $\blacksquare$

Thus, we have obtained a function

$$\varphi : X \rightarrow \omega(\mathcal{B})$$

by associating to each point x of X the $\mathcal{B}$ -ultrafilter $\varphi(x)$ consisting of all members of $\mathcal{B}$ containing x . Because X is a T_1 -space and $\mathcal{B}$ is a base for the closed sets of X , the function φ is one-to-one. We will identify X with its image $\varphi[X]$ and regard X as a subset of $\omega(\mathcal{B})$. We now show that $\omega(\mathcal{B})$ with the topology defined in Section 1.19 is a compact Hausdorff space and that the function φ is an embedding. Recall that a base for the closed sets of $\omega(\mathcal{B})$ is given by

$$\{Z^\omega : Z \in \mathcal{B}\}$$

where $Z^\omega = \{u \in \omega(\mathcal{B}) : Z \in u\}$. Since $\mathcal{B}$ is a base for the closed sets of X and $Z^\omega \cap X = Z$, φ is an embedding.

1.38. The closure of any subset of $\omega(\mathcal{B})$ is the intersection of all the basic closed sets containing the given set. The description of the closures of sets in $\mathcal{B}$ is useful in relating X to $\omega(\mathcal{B})$.

PROPOSITION:

If Z is in $\mathcal{B}$, then Z^ω is the closure of Z in $\omega(\mathcal{B})$.

Proof: Since $Z \subset Z^\omega$, it is clear that $cl Z \subset Z^\omega$. Now suppose that Z_0^ω is a basic set containing Z . Then $Z_0 = Z_0^\omega \cap X \supset Z$ so that $Z^\omega \subset Z_0^\omega$. Thus, $Z^\omega \subset cl Z$. $\square$

Since two members of $\mathcal{B}$ both belong to a $\mathcal{B}$ -ultrafilter exactly when their intersection belongs to the $\mathcal{B}$ -ultrafilter, it is clear that if Z_1 and Z_2 are in $\mathcal{B}$,

$$(a) \quad (Z_1 \cap Z_2)^\omega = Z_1^\omega \cap Z_2^\omega.$$

Similarly, we have

$$(b) \quad (Z_1 \cup Z_2)^\omega = Z_1^\omega \cup Z_2^\omega.$$

The basic open sets of $\omega(\mathcal{B})$ can be identified by taking complements of the basic closed sets:

$$\begin{aligned} \omega(\mathcal{B}) \setminus Z^\omega &= \{u \in \omega(\mathcal{B}) : Z \notin u\} \\ &= \{u \in \omega(\mathcal{B}) : Z^\dagger \subset X \setminus Z \text{ for some } Z^\dagger \in u\}. \end{aligned}$$

For $U = X \setminus Z$, denote the basic open set obtained from U by

$${}^\omega U = \{u \in \omega(\mathcal{B}) : Z^\dagger \subset U \text{ for some } Z^\dagger \in u\}.$$

1.39. We can now establish Frink's basic result.

THEOREM:

If $\mathcal{B}$ is a normal base for a T_1 -space X , then $\omega(\mathcal{B})$ is a compact Hausdorff space and ϕ is a dense embedding of X

into $\omega(\mathcal{B})$.

Proof: (a) $\omega(\mathcal{B})$ is Hausdorff:

For u and v distinct points of $\omega(\mathcal{B})$, Proposition 1.17 shows that there exists Z_1 in u and Z_2 in v such that $Z_1 \cap Z_2 = \emptyset$. Now since $\mathcal{B}$ is normal, there exist A_1 and A_2 in $\mathcal{B}$ such that $Z_i \subset X \setminus A_i$ for both values of i and $(X \setminus A_1) \cap (X \setminus A_2) = \emptyset$. But then ${}^\omega(X \setminus A_1)$ and ${}^\omega(X \setminus A_2)$ are disjoint basic neighborhoods of u and v , respectively.

(b) $\omega(\mathcal{B})$ is compact:

It is sufficient to show that any family $\mathcal{G}^\omega$ of basic closed sets with the finite intersection property has non-empty intersection. Let $\mathcal{G} = \{Z \in \mathcal{B} : Z^\omega \in \mathcal{G}^\omega\}$. Then $\mathcal{G}$ has the finite intersection property and is contained in a $\mathcal{B}$ -ultrafilter u . But then since $u \supset \mathcal{G}$, u is in Z^ω for each Z in $\mathcal{G}$, and u belongs to $\bigcap \mathcal{G}^\omega$.

Since we have seen earlier that ϕ is an embedding, all that remains is to show that:

(c) $\phi[X]$ is dense in $\omega(\mathcal{B})$:

Let ${}^\omega U$ be a non-empty basic open set of $\omega(\mathcal{B})$. Then there exists u in ${}^\omega U$. If Z is a member of u such that Z is a subset of U , then the image under ϕ of a point of Z belongs to ${}^\omega U \cap \phi[X]$. $\blacksquare$

With X regarded as a subspace of $\omega(\mathcal{B})$, it is clear from the definition of closure that a $\mathcal{B}$ -ultrafilter u belongs to $Z^\omega = \text{cl } Z$ for any Z in u . Further, since distinct

$\mathcal{B}$ -ultrafilters must contain disjoint $\mathcal{B}$ -sets and $Z_1^\omega \cap Z_2^\omega = (Z_1 \cap Z_2)^\omega$, $\mathcal{U}$ is actually the intersection of such closures, i.e.

$$\{\mathcal{U}\} = \bigcap \{Z^\omega : Z \in \mathcal{U}\}.$$

Hence, the point $\mathcal{U}$ is the limit of the $\mathcal{B}$ -ultrafilter $\mathcal{U}$, and $\mathcal{U}$ is the only $\mathcal{B}$ -ultrafilter on X which can have the point $\mathcal{U}$ as its limit.

The ring of closed subsets of a regular T_1 -space which fails to be normal will satisfy all the requirements of a normal base except normality. In such a case, $\omega(\mathcal{B})$ will not be Hausdorff. Thus, in order to guarantee that the space of $\mathcal{B}$ -ultrafilters will be Hausdorff, it is necessary to add the additional condition that the ring $\mathcal{B}$ be a normal family, a condition which abstracts a crucial property of the family of closed subsets of a normal space. Also note the similarity between the normality condition and the condition of Proposition 1.21 for maximal ideal spaces, particularly as it is applied in the proof of Theorem 1.23.

1.40. Observe that when X is a normal space and $\mathcal{B}$ is the normal base of all closed subsets of X , the definition of closure of members of $\mathcal{B}$ and 1.38(a) make the following corollary immediate from Corollary 1.15.

COROLLARY:

If $\mathcal{B}$ is the family of all closed subsets of a normal space X , then $w(\mathcal{B})$ is βX .

1.41. The Alexandroff one point compactification αX of a locally compact space X can also be obtained in this way by choosing as the normal base the zero-sets of those mappings in $C(X)$ which are constant on the complement of a compact set.

(See Exercise 1F.)

Many compactifications can be obtained by the methods of Wallman and Frink. However, it is not known if every compactification can be achieved in this way. In every case where a compactification has been shown to be a Wallman-type compactification, the normal base has been a subcollection of the family of zero-sets. In 1968B, R. A. Alo and H. L. Shapiro showed that any compact space which has a normal base consisting of regular closed sets will be a Wallman-type compactification of each of its dense subspaces. A set is said to be regular closed if it is the closure of its interior.

1.42. In [GJ, Chapter 6], the Stone-Čech compactification is obtained as a Wallman-type compactification. A similar approach will be taken here. We will first show that the zero-sets of X form a normal base and then show that the resulting compactification is βX .

PROPOSITION:

The collection $Z[X]$ of zero-sets of a completely regular space X is a normal base which is closed under countable intersections, and $\omega(Z[X])$ is βX .

Proof: We have already seen that $Z[X]$ is closed under finite unions and is a base for the closed sets of X . Let $\{Z(f_n) : n \geq 1\}$ be countable family of zero-sets and put $g_n = |f_n| \wedge \frac{1}{2^n}$. The Weierstrass M-test shows that the limit of the sequence $\{\sum_{1 \leq i \leq n} g_i\}$ is a continuous function g and it is evident that the zero-set of g is $\bigcap Z(f_n)$.

It is easy to see that $Z[X]$ is disjunctive by noting that the zero-set neighborhoods form a base for the neighborhoods of a point.

Let Z_1 and Z_2 be disjoint zero-sets. Then there is a real-valued mapping f which completely separates Z_1 and Z_2 . The zero-sets $\{x : f(x) \leq \frac{1}{2}\}$ and $\{x : f(x) \geq \frac{1}{2}\}$ have disjoint complements which show that $Z[X]$ is a normal family.

Any two completely separated subsets of X are contained in disjoint zero-sets and 1.38(a) shows that disjoint zero-sets have disjoint closures in $\omega(Z[X])$. Thus, completely separated sets of X have disjoint closures in $\omega(Z[X])$ and Theorem 1.14 shows that $\omega(Z[X])$ is βX . $\square$

1.43. Since the completely regular spaces are precisely the subspaces of compact Hausdorff spaces by Tychonoff's Theorem 1.4, Proposition 1.42 and Theorem 1.39 combine to yield another characterization of the class of completely regular spaces.

COROLLARY: (Frink)

A T_1 -space is completely regular if and only if it admits a normal base.

Note that this characterization is an internal one in the sense that it contains no reference to the real-valued mappings on the space. Frink's motivation for considering normal bases was to obtain such a characterization.

1.44. We have seen that βX can be obtained as a space of z -ultrafilters in such a way that each point of $\beta X \setminus X$ is the limit of a unique free z -ultrafilter. To relate this property to the characterizations of βX in terms of extensions of mappings, it is necessary to consider the behavior of z -filters under mappings.

If f is a mapping of X to Y and $\mathcal{F}$ a z -filter on X , the family of sets $f[\mathcal{F}] = \{f[Z] : Z \in \mathcal{F}\}$ need not be a z -filter. This can easily be seen by choosing f to be the identity from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$ with the usual topology. However, a subfamily of $f[\mathcal{F}]$ is sufficient to reflect the most interesting properties of $\mathcal{F}$. Define

$$f^{\#}\mathcal{F} = \{Z \in \mathcal{Z}[Y] : f^{-1}(Z) \in \mathcal{F}\}.$$

It is clear that $f^{\#}\mathcal{F}$ is a z -filter. Since the zero-set neighborhoods of a point form a base for the neighborhoods of the point, it can easily be seen that if a z -filter $\mathcal{F}$ converges to x in X , then $f^{\#}\mathcal{F}$ converges to $f(x)$ in Y . Further

discussion of the function $f^\#$ can be found in [GJ, 4.12, 4.13, 10.17].

In Proposition 1.28, we saw that convergence can be described in terms of z -ultrafilters. It would be convenient if $f^\#$ preserved z -ultrafilters but this is not the case. Following [GJ, ex. 4H], let X be $[0,1]$ with the discrete topology and let f from X to the closed unit interval I be the identity mapping. Since $I = [0,1]$ is compact and any z -filter is a family of closed sets with the finite intersection property, every z -filter on I is fixed. Thus, if $\mathcal{U}$ is any free z -ultrafilter on X , $f^\#\mathcal{U}$ is fixed. But any fixed z -ultrafilter on I contains its limit point as a member, and $f^\#\mathcal{U}$ can contain no finite subset set since $\mathcal{U}$ is free. Thus, $f^\#\mathcal{U}$ is not a z -ultrafilter.

However, $f^\#$ does preserve the prime z -filters, and we shall see that this family includes the z -ultrafilters and shares many of the characteristics of z -ultrafilters. A z -filter $\mathfrak{F}$ is prime when the union of two zero-sets belongs to $\mathfrak{F}$ only if one of the zero-sets belongs to $\mathfrak{F}$.

Let $\mathfrak{F}$ be a prime z -filter on X and suppose that $Z_1 \cup Z_2$ is in $f^\#\mathfrak{F}$. Then $f^\leftarrow(Z_1 \cup Z_2) = f^\leftarrow(Z_1) \cup f^\leftarrow(Z_2)$ is in $\mathfrak{F}$ as is either $f^\leftarrow(Z_1)$ or $f^\leftarrow(Z_2)$. But then the corresponding Z_i is in $f^\#\mathfrak{F}$ and we have demonstrated the following useful property of prime z -filters.

PROPOSITION:

If f is a mapping of X to Y and $\mathfrak{F}$ is a prime z -filter on X , then $f^{\#}\mathfrak{F}$ is a prime z -filter on Y .

1.45. Additional properties of the class of prime z -filters are contained in the next proposition.

PROPOSITION:

(a) Any z -ultrafilter is prime.

(b) If $\mathfrak{F}$ is a prime z -filter on X , the following are equivalent for a point x in X :

(1) x is a cluster point of $\mathfrak{F}$.

(2) $\mathfrak{F}$ converges to x .

(3) $\bigcap \mathfrak{F} = \{x\}$.

Proof: (a) Let $Z_1 \cup Z_2$ be in the z -ultrafilter $\mathfrak{u}$. Then if neither Z_1 nor Z_2 is in $\mathfrak{u}$, there exist A_1 and A_2 in $\mathfrak{u}$ such that $A_1 \cap Z_1 = \emptyset$ and $A_2 \cap Z_2 = \emptyset$. But then $(Z_1 \cup Z_2) \cap (A_1 \cap A_2) = \emptyset$, and $Z_1 \cup Z_2$ could not be in $\mathfrak{u}$, which is a contradiction.

(b) Let the prime z -filter $\mathfrak{F}$ cluster at x and let V be a zero-set neighborhood of x . By the complete regularity of X , V contains a neighborhood of x of the form $X \setminus Z$ for Z a zero-set. Then since $V \cup Z = X$ is in $\mathfrak{F}$, either V or Z is in $\mathfrak{F}$. But Z misses x and therefore cannot be in $\mathfrak{F}$. Thus, V is in $\mathfrak{F}$ and $\mathfrak{F}$ converges to x . This shows that (1) implies (2) and the other implications are clear. $\square$

Prime z -filters are discussed in [GJ, Chapter 2 and Section 3.17] which is the source of the previous Proposition.

CHARACTERIZATIONS OF βX

1.46. We now summarize the characteristic properties of the Stone-Čech compactification by displaying their equivalence in a somewhat encyclopedic theorem similar to Theorem 6.5 of [GJ].

THEOREM:

Every completely regular space X has a unique compactification βX which has the following equivalent properties:

- (1) X is C^* -embedded in βX .
- (2) Every mapping of X into a compact space extends uniquely to βX .
- (3) Every point of βX is the limit of a unique z -ultrafilter on X .
- (4) If Z_1 and Z_2 are zero-sets in X , then

$$cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2 = cl_{\beta X} (Z_1 \cap Z_2).$$
- (5) Disjoint zero-sets in X have disjoint closures in βX .
- (6) Completely separated sets in X have disjoint closures in βX .
- (7) βX is maximal in the partially ordered set of compactifications of X .

Proof: The existence of a compactification satisfying one or more of the listed properties has already been demonstrated. That βX satisfies (1), (2), (6), and (7) seems to be clearest by considering βX through embedding X into a product of intervals as in Theorem 1.9. Condition (3) seems to stand out most clearly through the isomorphism of βX with the space of maximal ideals $\mathfrak{M}(X)$ established in Theorem 1.31. Conditions (4) and (5) are most transparent by applying Theorem 1.39 to the normal base $Z[X]$.

The uniqueness of a compactification satisfying (2) was shown in Corollary 1.13. It remains to show that the seven properties are equivalent.

(3) $\Rightarrow$ (2):

Let A^p be the unique z -ultrafilter converging to the point p of βX . If f is a mapping of X into a compact Hausdorff space Y , $f^\# A^p$ is a family of closed sets with the finite intersection property and therefore $\bigcap f^\# A^p$ contains a point y of Y . By Propositions 1.44 and 1.45, $f^\# A^p$ is a prime z -filter converging to y and $\bigcap f^\# A^p = \{y\}$. This defines a function $\beta(f)$ from βX to Y .

If p is in X , p belongs to $\bigcap A^p$ and $y = \beta(f)(p)$ is in $\bigcap f^\# A^p$ so that $\beta(f)$ is an extension of f .

To show that $\beta(f)$ is continuous, let F be a zero-set neighborhood of $\beta(f)(p)$ and let F' be a zero-set in Y such that $Y \setminus F'$ is contained in F and is a neighborhood of $\beta(f)(p)$. Then $F \cup F' = Y$ and if Z and Z' are the inverse images of

F and F' under f , we have that $Z \cup Z' = X$. Thus, $cl_{\beta X} Z \cup cl_{\beta X} Z' = \beta X$. Since $\beta(f)(p)$ is not in F' , p is not in $cl_{\beta X} Z'$ so that $\beta X \setminus cl_{\beta X} Z'$ is a neighborhood of p . Every point in this neighborhood belongs to $cl_{\beta X} Z$, so that by the definition of $\beta(f)$, $\beta(f)[\beta X \setminus cl_{\beta X} Z'] \subset F$, and $\beta(f)$ is continuous. The uniqueness of the extension is immediate since any two extensions must agree on the dense subspace X .

(2) $\Rightarrow$ (1) is clear since I is compact.

(1) $\Rightarrow$ (6) follows from Urysohn's Extension Theorem, Theorem 1.2 which actually shows that the two are equivalent.

(6) $\Rightarrow$ (5) is immediate since disjoint zero-sets are completely separated.

(5) $\Rightarrow$ (4):

It is clear that $cl_{\beta X}(Z_1 \cap Z_2) \subset cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2$. On the other hand, if $p \in cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2$, then for every zero-set neighborhood V of p in βX , $p \in cl_{\beta X}(Z_1 \cap V)$ and $p \in cl_{\beta X}(Z_2 \cap V)$. Then (5) implies that $V \cap Z_1$ and $V \cap Z_2$ cannot be disjoint so that $V \cap (Z_1 \cap Z_2) \neq \emptyset$ and $p \in cl_{\beta X}(Z_1 \cap Z_2)$.

(4) $\Rightarrow$ (3):

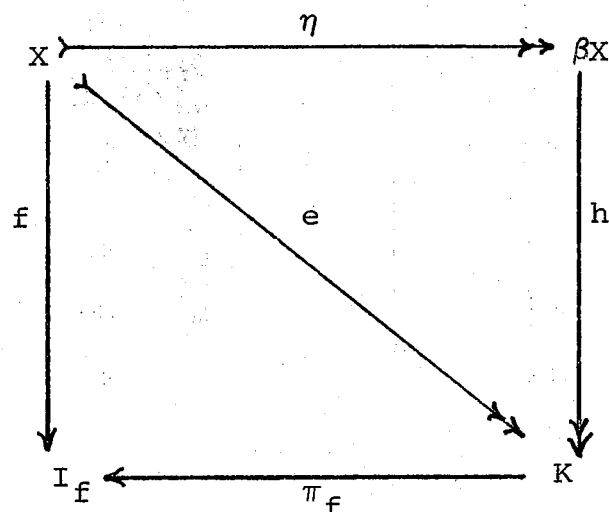
Since X is dense in βX , the trace on X of the zero-set neighborhoods of a point p in βX is a z -filter $\mathfrak{F}$ on X . By Proposition 1.36, $\mathfrak{F}$ is contained in a z -ultrafilter $\mathfrak{u}$ and $\mathfrak{u}$ converges to p . But distinct z -ultrafilters must contain disjoint zero-sets by Proposition 1.17, and (4) shows that any pair of disjoint zero-sets must have disjoint closures

in βX . Hence, exactly one z -ultrafilter converges to p .

(2) $\implies$ (7) is shown in Corollary 1.12.

(7) $\implies$ (1):

As in Theorem 1.9, use the evaluation mapping e to embed X into a product of unit intervals indexed by $C^*(X)$. Then if K is the closure of X in the product, K is a compactification of X and thus is a continuous image of the maximal compactification βX under a mapping h which leaves points of X fixed. A member f of $C^*(X)$ extends to K by composing e with the projection π_f , and $\pi_f \circ h$ extends f to βX .



Note that in the proof, the compactness of βX has been used only once and that was in condition (7). Thus, if we replace βX with any space T which contains X as a dense subspace, conditions (1)-(6) remain equivalent.

The main steps in the development of the Stone-Čech compactification which led up to the preceding theorem are outlined below. The arrows indicate major sources of influence.

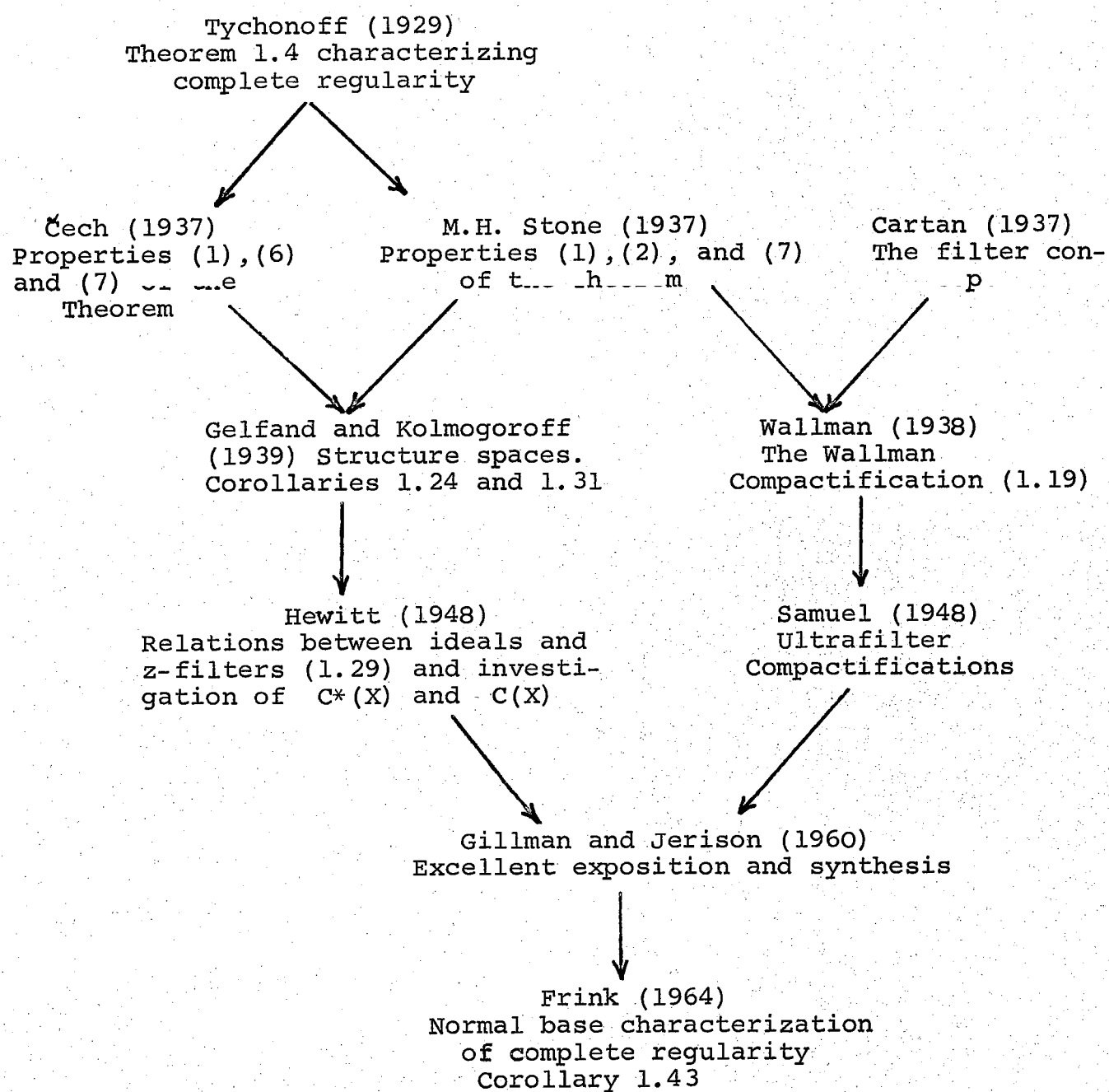


Figure 1.1

1.47. We now consider a few of the most useful properties of the Stone-Čech compactification. First we have a technical result which is taken from [GJ, 3.11].

PROPOSITION:

Any two disjoint closed sets, one of which is compact, are completely separated. Hence, every compact subspace is C^* -embedded.

Proof: Let K and F be disjoint closed subsets of X with K compact. For each x in K , choose a zero-set neighborhood Z_x of x and a zero-set Z'_x containing F and missing Z_x . The cover $\{Z_x\}$ of K has a finite subcover $\{Z_{x_i}\}$. Then $\bigcup Z_{x_i}$ and $\bigcap Z'_{x_i}$ are disjoint zero-sets containing K and F respectively. Hence, K and F are completely separated.

To prove the second statement, we show that a compact subspace S of X is C^* -embedded by applying Theorem 1.2. Any two subsets of S which are completely separated in S have disjoint compact closures and thus are completely separated in X by the first statement. $\square$

1.48. PROPOSITION:

A subspace S of X is C^* -embedded in X if and only if $\beta S = cl_{\beta X} S$.

Proof: If S is C^* -embedded in X , S is clearly C^* -embedded in βX and therefore $cl_{\beta X} S$ is a compactification of S in which S is C^* -embedded. Conversely, $cl_{\beta X} S$ is compact and therefore is C^* -embedded in βX . It follows that S is C^* -embedded in X since S is C^* -embedded in $cl_{\beta X} S$ by assumption. $\square$

The Proposition will be most useful when S is a closed, C^* -embedded copy of the countable discrete space $\mathbb{N}$.

1.49. The following proposition will be useful in the creation of Examples. A subspace T of βX which contains X is clearly dense in βX . Further, such a subspace T is C^* -embedded in βX since a mapping in $C^*(T)$ can first be restricted to X and then extended to βX . Thus, we have verified the

PROPOSITION:

If T is a subspace of βX containing X , then βT is βX .

1.50. The work of Wallman and Frink make it clear that the process of forming βX is one of "fixing" the free z -ultrafilters of X by attaching a point p for each free z -ultrafilter A^p in such a way that A^p converges to p . Because of this interpretation of βX , we will often identify the points of $\beta X \setminus X$ with the free z -ultrafilters of X . Since any z -filter is contained in a z -ultrafilter and it is clear that a free z -filter cannot be contained in a fixed z -ultrafilter, we have verified the following characterizations of compactness:

PROPOSITION:

The following are equivalent:

- (1) X is compact.
- (2) Every z -ultrafilter on X converges to a point of X .
- (3) Every z -filter on X is fixed.
- (4) Every ideal in $C(X)$ is fixed.

We see then that an ideal M^p or a z -ultrafilter A^p is fixed for p in X and free for p in $\beta X \setminus X$. The subspace $\beta X \setminus X$ will be referred to as the growth of X and will be denoted by X^* .

Note that if Z belongs to a z -filter $\mathfrak{F}$, then the trace of the elements of $\mathfrak{F}$ on Z is a family of closed sets of Z having the finite intersection property. Hence, if Z is compact, the trace must have non-empty intersection and $\mathfrak{F}$ must then be fixed. Thus, we have shown that compact zero-sets can belong only to fixed z -ultrafilters and that no free z -filter can contain a compact zero-set.

1.51. Since a space X is C^* -embedded in βX , one is led to expect that there is a simple and perhaps useful relationship between the zero-sets of X and those of βX . It would be tempting to conjecture that a zero-set of βX is just the closure of a zero-set of X . However, this is not the case. Consider the mapping f on $\mathbb{R}$ defined by $f(x) = 1/(1 + |x|)$. It is clear that f never vanishes on $\mathbb{R}$, but since f approaches zero on every non-compact subset of $\mathbb{R}$, $\beta(f)(p) = 0$ for all p in $\beta \mathbb{R} \setminus \mathbb{R}$. Thus, $\beta \mathbb{R} \setminus \mathbb{R}$ is the zero-set of $\beta(f)$ but is not the closure of any zero-set of $\mathbb{R}$.

The actual relationship between the two families of zero-sets is somewhat more complicated, but will still prove to be useful. For the proof of the following proposition, it is helpful to recall that the z -ultrafilters on X are of the form

$$A^p = \{Z \in Z[X] : p \in \text{cl}_{\beta X} Z\}$$

for p in βX . Thus, the closure of a zero-set is given by

$$\text{cl}_{\beta X} Z = \{p \in \beta X : Z \in A^p\}.$$

PROPOSITION:

The zero-sets of βX are countable intersections of closures in βX of zero-sets of X .

Proof: If Z is in $Z(\beta X)$, then $Z = Z(\beta(f))$ for some f in $C^*(X)$. Thus,

$$Z(\beta(f)) = \bigcap_{n=1}^{\infty} \{p \in \beta X : |\beta(f)(p)| \leq \frac{1}{n}\}.$$

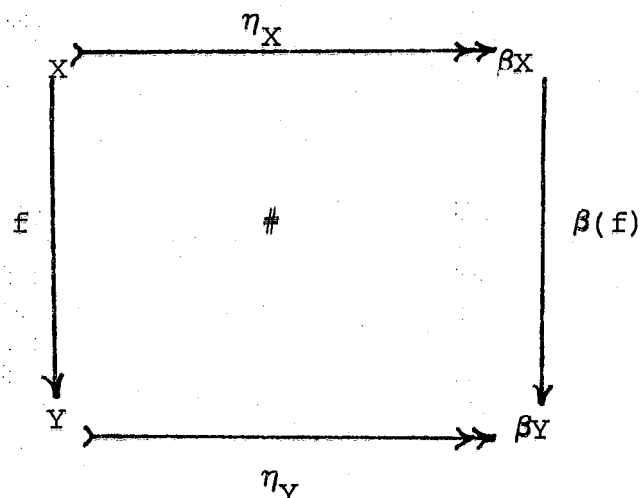
But then we can write that

$$Z(\beta(f)) = \bigcap_{n=1}^{\infty} \text{cl}_{\beta X} \{x \in X : |f(x)| \leq \frac{1}{n}\}$$

and the result follows since $\{x \in X : |f(x)| \leq \frac{1}{n}\}$ is a zero-set for each n . $\square$

Observe that from the definition of closure, the proof also shows that the zero-sets of X involved in computing $Z(\beta(f))$ belong to A^p if p is in $Z(\beta(f))$.

1.52. If f is a mapping of X to Y , then the composition of f and the embedding of Y into βY extends to a mapping $\beta(f)$ of βX into βY . Thus, we have the following commutative diagram:

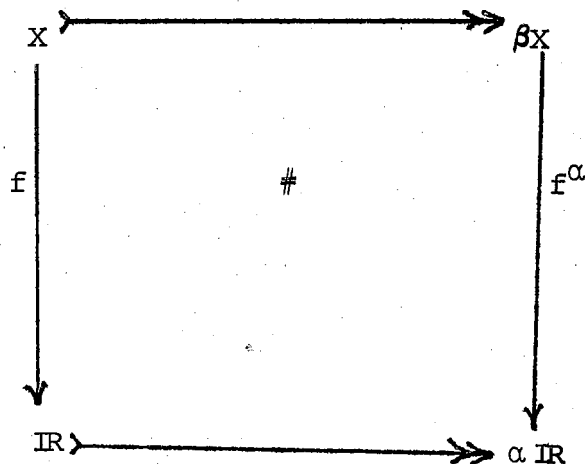


We shall see in Chapter 10 that the existence of this diagram is a major step in placing the Stone-Čech compactification in its categorical context.

GENERALIZATIONS OF COMPACTNESS

1.53. The classes of realcompact spaces, pseudocompact spaces, and locally compact spaces each contain the class of compact spaces and will be defined later. The main application of these classes of spaces will be to study the interactions between a space and its "growth" $\beta X \setminus X$. The subspace $\beta X \setminus X$ will be denoted by X^* . We will give a criterion to recognize a space of each class in terms of the growth of its Stone-Čech compactification.

Any member f of $C(X)$ is a mapping of X into $\alpha \mathbb{R} = \mathbb{R} \cup \{\infty\}$, the one point compactification of $\mathbb{R}$, and thus has an extension f^α which maps $\beta \mathbb{R}$ into $\alpha \mathbb{R}$.



If f is unbounded, there will be a point in the growth of X at which f^α will take the value ∞ . For each map f in $C(X)$, define

$$\cup_f X \equiv \beta X \setminus \{p \in \beta X : f^\alpha(p) = \infty\}.$$

Thus, $\cup_f X$ is the set of points of βX at which f is finite, and we will call $\cup_f X$ the set of real points of f . Let $\cup X$ be the subspace of βX consisting of points which are real points for every f in $C(X)$, i.e.

$$\cup X = \bigcap \{\cup_f X : f \in C(X)\}.$$

A space X is said to be realcompact if $X = \cup X$, i.e. if the only points which are real points for every f in $C(X)$ are the points of X itself. It is immediate that any compact space is realcompact and that the subspace $\cup X$ of βX is the largest subspace of βX to which every member of $C(X)$ can be extended without any extension taking on the value ∞ . If the extension of f to $\cup X$ is denoted by $\cup(f)$, then the correspondence $f \mapsto \cup(f)$ is an isomorphism of $C(X)$ with $C(\cup X)$.

Our main use of realcompactness will be to recognize the subspace $\mathcal{O}X$ of βX as a "dividing line" between two different kinds of points of βX . The distinction between points of $\mathcal{O}X$ and points of $\beta X \setminus \mathcal{O}X$ can be recognized through the corresponding z -ultrafilters. A z -ultrafilter $\mathcal{A}^p$ is called a real z -ultrafilter if the intersection of any countable subfamily of $\mathcal{A}^p$ is non-empty. The ideal $\mathcal{M}^p$ will be called real if $\mathcal{A}^p$ is a real z -ultrafilter. We can now use the representation of the zero-sets of βX obtained in Proposition 1.51 to characterize realcompactness in terms of zero-sets and also in terms of z -ultrafilters.

THEOREM: (Hewitt)

The following are equivalent for any space X :

- (1) X is realcompact.
- (2) Every point of $\beta X \setminus X$ is contained in a zero-set of βX which misses X .
- (3) Every real z -ultrafilter on X is fixed.

Proof:

(1) $\Leftrightarrow$ (2):

If $X = \mathcal{O}X$, then for every point p in $\beta X \setminus X$ there is a mapping f in $C(X)$ such that $f > 0$ and $f^\alpha(p) = \infty$. But then if g is the reciprocal of f , $p \in Z(\beta(g)) \subset \beta X \setminus X$. The converse is obtained by reversing the steps.

(2) $\Rightarrow$ (3):

Suppose that p is in $\beta X \setminus X$. We will show that the corresponding free z -ultrafilter $\mathcal{A}^p$ cannot be real. By (2),

there is a zero-set Z such that $p \in Z \subset \beta X \setminus X$. Proposition 1.51 implies that $Z = \bigcap \text{cl}_{\beta X} Z_n$. But since this intersection is contained in the growth, $\bigcap Z_n = \emptyset$ and A^p is not a real z -ultrafilter.

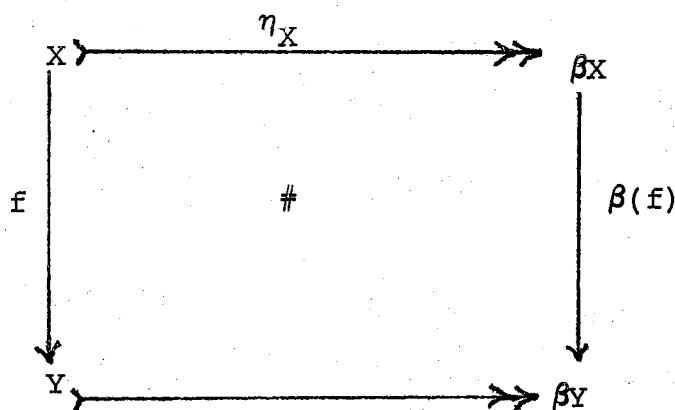
(3) $\Rightarrow$ (2):

For p in $\beta X \setminus X$, A^p is not real. Hence, there exists a countable family $\{Z(f_n)\}$ contained in A^p such that $\bigcap Z(f_n) = \emptyset$ and f_n is in $C^*(X)$ for every n . Then the zero-set $Z(\beta(f_n))$ of the extension of f_n contains p . Then $\bigcap Z(\beta(f_n))$ is a zero-set containing p and missing X . $\blacksquare$

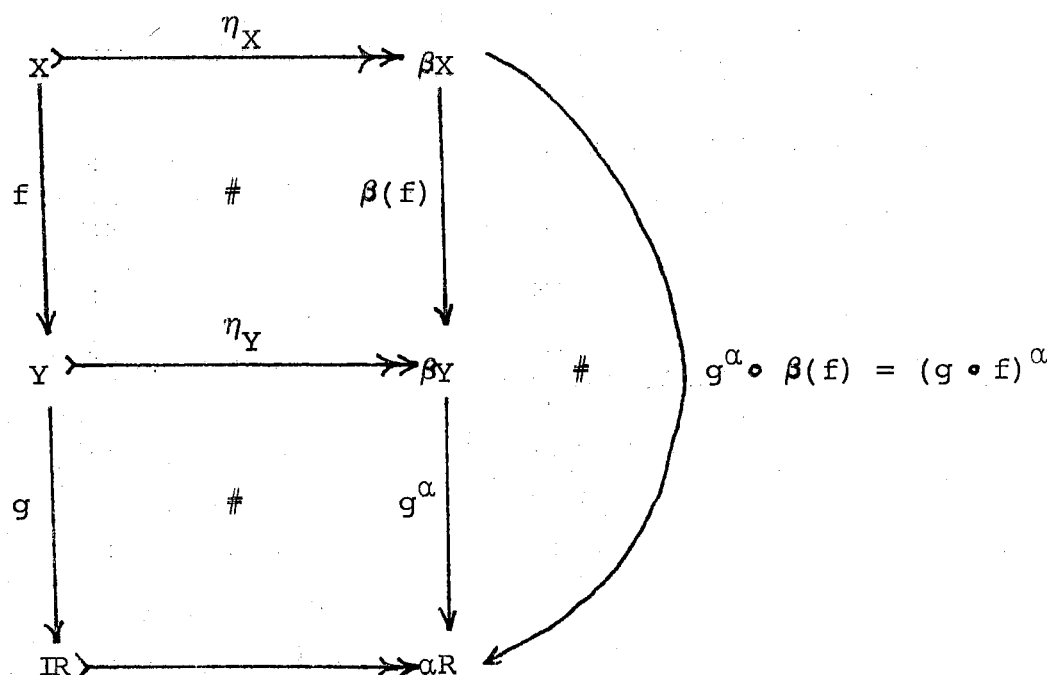
1.54. EXAMPLES:

The argument given in Section 1.51 to show that $\beta \mathbb{R} \setminus \mathbb{R}$ is a zero-set of $\beta \mathbb{R}$ also shows that $\mathbb{R}$ is realcompact. A similar argument will show that the discrete space $\mathbb{N}$ of natural numbers is realcompact. The space of rationals, $\mathbb{Q}$, is also realcompact as will be shown in Exercise 1D.

1.55. Let f be a mapping of X to Y . We saw in Section 1.52 that there is an extension $\beta(f)$ of f such that the following diagram commutes:



We will establish an analogous result for $\cup X$ by showing that the image of the restriction $\beta(f)|_{\cup X}$ is a subspace of $\cup Y$. The result will follow from the definition of $\cup X$. If g belongs to $C(Y)$, then there is an extension g^α of g which maps βY to $\alpha \mathbb{R}$. Similarly, $g \circ f$ belongs to $C(X)$ and therefore has an extension $(g \circ f)^\alpha$ of βX into $\alpha \mathbb{R}$. Since the two mappings $(g \circ f)^\alpha$ and $g^\alpha \circ \beta(f)$ both agree with f on X , they are equal and we have the following commutative diagram:

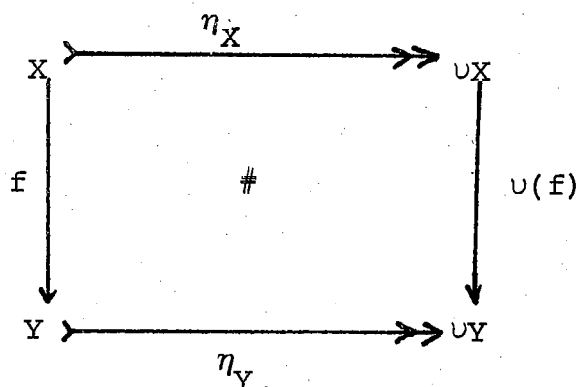


Now if q belongs to $\cup X$, we must show that $\beta(f)(q)$ is in $\cup Y$. From the definition of $\cup X$, we see that q is a real point of $g \circ f$ for every g in $C(Y)$. Thus, for every g in $C(Y)$, $(g^\alpha \circ \beta(f))(q) = (g \circ f)^\alpha(q) \neq \infty$. Hence, we have shown that

$$\beta(f)(q) \in \cup Y = \bigcap \{ \cup_g Y : g \in C(Y) \}.$$

By defining $\cup(f)$ to be the restriction $\beta(f)|_{\cup X}$, we have

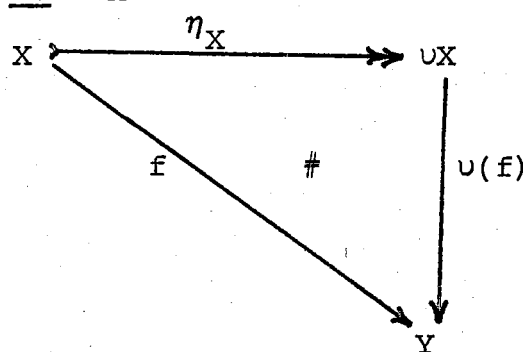
obtained the following commutative diagram:



If we consider the special case where the space Y in the preceding diagram is realcompact, then we have verified the following result:

PROPOSITION:

Every mapping of the space X into a realcompact space Y will extend uniquely to υX .



Note that this result is analogous to Theorem 1.11 for βX and to Corollary 1.8 for ρX .

The space υX is called the Hewitt-Nachbin realcompactification of X since E. Hewitt and L. Nachbin gave independent constructions of the extension υX . Hewitt's discussion of the class of realcompact spaces appears in his 1948 paper. Nachbin's

construction of $\mathcal{U}X$ is based on uniformities and was not published. [GJ, Chapter 15] discusses Nachbin's work and Shirota's important contribution which came in 1952 and showed that barring measurable cardinals, the class of realcompact spaces is the same as the class of spaces which admit a complete uniform structure.

1.56. In Corollary 1.25, we saw that $C^*(X)$ is an algebraic invariant for the class of compact spaces. Since $\mathcal{U}X$ is a subspace of βX and hence of the space of ideals $\mathfrak{m}(X)$, we have the analogous theorem for $C(X)$.

THEOREM: (Hewitt)

If X and Y are realcompact spaces, X is homeomorphic to Y if and only if $C(X)$ and $C(Y)$ are isomorphic.

The theorem follows from the fact that the property of being a real maximal ideal is preserved under the ring isomorphism of $C(X)$ with $C(Y)$.

The brief treatment of realcompactness given here is a very narrow one designed only to facilitate the study of βX by utilizing the subspace $\mathcal{U}X$. The material included is based on [GJ, Chapter 8] which offers a much more complete discussion of realcompactness. The most complete treatment of realcompactness is given in the thesis of M. Weir.

1.57. A space X is pseudocompact if every real-valued mapping on X is bounded, i.e. if $C^*(X)$ is equal to $C(X)$.

THEOREM: (Hewitt)

The following are equivalent for any space X :

- (1) X is pseudocompact.
- (2) Every non-empty zero-set of βX meets X .
- (3) Every z -ultrafilter on X is real.
- (4) Every z -filter on X has the countable intersection property.

Proof:

(1) $\Rightarrow$ (2):

If $Z(\beta(f))$ is contained in $\beta X \setminus X$, f does not vanish on X and is not bounded away from zero on X . Thus, f^{-1} is an unbounded member of $C(X)$.

(2) $\Rightarrow$ (1):

If there is an unbounded member of $C(X)$, then there is an unbounded member f of $C(X)$ which does not vanish on X , and the zero-set of the extension of f^{-1} is non-empty and misses X .

(2) $\Leftrightarrow$ (3):

The zero-sets containing a point p of the growth of X are of the form $\bigcap \text{cl}_{\beta X} Z_n$ with $\{Z_n\} \subset A^p$. Since such a zero-set can miss X exactly when $\bigcap Z_n = \emptyset$, it is clear that A^p is a real z -ultrafilter precisely when every zero-set containing p meets X .

(3) $\Leftrightarrow$ (4) is clear. $\blacksquare$

1.58. Since every z -ultrafilter is real when X is pseudo-compact and every real z -ultrafilter is fixed for X real-compact, we have verified the following characterization of compactness. The second statement in the corollary follows since νX is the subspace of βX corresponding to the real z -ultrafilters on X .

COROLLARY:

A space is compact if and only if it is realcompact and pseudocompact. Further, $\nu X = \beta X$ if and only if X is pseudo-compact.

1.59. A space is locally compact if each point of the space has a basis of compact neighborhoods. The locally compact spaces are precisely those spaces which can be compactified by the addition of a single point. (See Exercise 1F.) Since such a compactification αX is the image of βX , $\beta X \setminus X$ is a closed subspace of βX since it is the inverse image of the added point. We have shown the necessity in the following characterization.

THEOREM:

A space is locally compact if and only if it is an open subspace of its Stone-Ćech compactification.

Sufficiency follows from the fact that an open subspace of a compact space is locally compact.

F-SPACES AND P-SPACES

1.60. In addition to the generalizations of compactness just discussed, the class of F-spaces will be of frequent interest in the investigation of Stone-Čech compactifications. This class of spaces was introduced in 1956B by L. Gillman and M. Henriksen as the class of spaces for which $C(X)$ is a ring in which every finitely generated ideal is a principal ideal. This is also the definition which is used in the discussion of F-spaces contained in [GJ, Chapter 14] from which the material in this section is drawn. Since the algebraic content is not emphasized here, we will adopt one of the characterizations of the class of F-spaces given by Gillman and Henriksen as our definition: An F-space is a space in which every cozero-set is C^* -embedded. The next theorem offers the characterizations of the class which will be most useful in the present context. If f belongs to $C(X)$, let $\text{pos}(f)$ and $\text{neg}(f)$ denote the set of points on which f is positive and negative, respectively.

THEOREM:

The following are equivalent for any space X :

- (1) X is an F-space.
- (2) Disjoint cozero-sets of X are completely separated.
- (3) For f in $C(X)$, $\text{pos}(f)$ and $\text{neg}(f)$ are completely separated.
- (4) βX is an F-space.

Proof:

(1) $\Rightarrow$ (2):

Let U and V be disjoint cozero-sets of X . Then the function which is constantly equal to 0 on U and to 1 on V is continuous on the cozero-set $U \cup V$ and its extension separates U from V .

(2) $\Rightarrow$ (3):

This is immediate since $\text{pos}(f) = \text{Cz}(f \vee 0)$ and $\text{neg}(f) = \text{Cz}(f \wedge 0)$.

(3) $\Rightarrow$ (4):

Let $\text{Cz}(h)$ be a cozero-set in βX and let A and B be completely separated subsets of $\text{Cz}(h)$. Then there exists k in $C^*(\text{Cz}(h))$ which is positive on A and negative on B . Define f on X by

$$f(x) = \begin{cases} 0 & \text{on } X \setminus \text{Cz}(h) \\ k(x) |h(x)| & \text{on } X \cap \text{Cz}(h) \end{cases}.$$

Since k is bounded, f is continuous on X and $A \cap X$ is contained in $\text{pos}(f)$ and $B \cap X$ is contained in $\text{neg}(f)$. Since $\text{pos}(f)$ and $\text{neg}(f)$ are completely separated in X , their closures are disjoint in βX and contain A and B , respectively. Thus, A and B are completely separated in βX showing that $\text{Cz}(h)$ is C^* -embedded in βX by Theorem 1.2 and therefore βX is an F -space.

(4) $\Rightarrow$ (1):

This is immediate since any cozero-set of X is the inter-

section of X with a cozero-set of βX . $\square$

1.61. COROLLARY:

Any cozero-set or C^* -embedded subspace of an F-space is also an F-space.

1.62. Examples of F-spaces include the growths of certain Stone-Ćech compactifications. A σ -compact space is one which is the union of countably many compact subspaces.

PROPOSITION:

The growth of a σ -compact locally compact space is an F-space.

The proof of the preceding result is particularly suited to the more algebraic techniques developed in detail in [GJ] and will not be included here. The proof given in [GJ, 14.27] shows that X^* satisfies (3) of Theorem 1.60 and is based on an algebraic treatment of the extension of a member of $C(X)$ to a mapping of βX into $\alpha \mathbb{R}$, the one-point compactification of $\mathbb{R}$.

1.63. Since a zero-set $Z(f)$ can be written

$$Z(f) = \bigcap \{x \in X : |f(x)| < 1/n\},$$

any zero-set is a G_δ , i.e. an intersection of countably many open sets. The complement of a G_δ is a union of countably many closed sets and is called an F_σ .

If Z is a zero-set of βX contained in $X^* = \beta X \setminus X$,

then $\beta X \setminus Z$ is an open F_σ in the compact space βX and is therefore locally compact and σ -compact. By Proposition 1.48, $\beta(\beta X \setminus Z) = \beta X$, and we have verified the

COROLLARY:

Any zero-set of βX which does not meet X is an F -space.

1.64. Another relationship between Stone-Ćech compactifications and the class of F -spaces is that every infinite compact F -space contains a copy of $\beta \mathbb{N}$.

PROPOSITION:

Every countable subspace of an F -space is C^* -embedded and hence, every infinite compact F -space contains a copy of $\beta \mathbb{N}$.

Proof: Let X be an F -space and let N be a countable subspace of X . Let $\{x_n\}$ and $\{y_n\}$ be subsets of N which are completely separated in N . We will show that $\{x_n\}$ and $\{y_n\}$ are completely separated in X in order to apply Urysohn's Extension Theorem, 1.2. Since neither sequence meets the closure of the other, $\{x_n\}$ and $\{y_n\}$ are contained in disjoint cozero-sets, U and V , of X : Choose closed neighborhoods of each x_n and y_n by alternating the choices between the two sequences and each time choosing the neighborhood to miss the union of the closure of the other sequence with the neighborhoods previously chosen for the other sequence. Since each neighborhood contains a cozero-set neighborhood, U and V can each be chosen to be the union of countably many cozero-sets. Since the cozero-sets U and V are completely separated in X , $\{x_n\}$ and $\{y_n\}$

are completely separated in X and N is C^* -embedded in X by Theorem 1.2.

Since any infinite Hausdorff space contains a countable discrete subspace IN and the closure of such a subspace in a compact F -space X is a compactification in which IN is C^* -embedded, an infinite compact F -space contains a copy of βIN . |

1.65. Gillman and Henriksen also introduced the class of P -spaces. The class was first defined in their 1954 paper and was also discussed in 1956B.

A point of a topological space is called a P -point if every G_δ containing the point is a neighborhood of the point. Recall that every zero-set is a G_δ . Further, it is easy to see that in a completely regular space, every G_δ containing a point contains a zero-set which also contains the point. Hence, a point is a P -point if and only if every zero-set containing the point is a neighborhood of the point. A space is called a P -space if every point of the space is a P -point. Thus, a P -space is one in which every G_δ , or equivalently every zero-set, is open. Since every cozero-set in a P -space is clopen, it is immediate that every P -space is an F -space. But while Proposition 1.62 indicates that there are many compact F -spaces, the additional open sets in a P -space make it hard for a P -space to be compact. In fact,

PROPOSITION:

Every pseudocompact P -space is finite.

Proof: Since the complement of a countable set is a G_δ , every sequence in a P-space is closed, discrete, and is also C^* -embedded by the preceding proposition since a P-space is an F-space. Since a zero-set is clopen, every sequence in a P-space is completely separated from any disjoint zero-set and thus is C-embedded by Theorem 1.3. Hence, an infinite P-space contains a C-embedded copy of $\mathbb{N}$ and fails to be pseudo-compact. $\blacksquare$

1.66. Just as a P-space can rarely be compact, the product of two F-spaces cannot often be an F-space. The following result is contained in the 1960 paper of D. C. Curtis, Jr. and several related results are contained in the thesis of N. Hindman and the 1969 paper of Comfort, Hindman, and Negrepontis.

PROPOSITION:

If $X \times Y$ is an F-space, then either X or Y is a P-space.

Proof: If X and Y both fail to be P-spaces, then there exist points x_0 in X and y_0 in Y together with non-negative mappings f in $C(X)$ and g in $C(Y)$ such that $f(x_0) = g(y_0) = 0$ and x_0 belongs to $\text{cl pos}(f)$ and y_0 to $\text{cl pos}(g)$. Define h in $C(X \times Y)$ by $h(x, y) = f(x) - g(y)$. Then the point (x_0, y_0) belongs to the closures of both $\text{pos}(h)$ and $\text{neg}(h)$ which contradicts (3) of Theorem 1.60. $\blacksquare$

1.67. As an application of the preceding proposition, consider the following example which lays to rest the tempting, but false, conjecture that $\beta X \times \beta Y$ is $\beta(X \times Y)$. (In Chapter 8 we will investigate the problems inherent in describing the Stone-Cech compactifications of product spaces.)

EXAMPLE:

Proposition 1.62 shows that the growth of $\mathbb{R} \times \mathbb{R}$ is an F-space. In $\beta \mathbb{R} \times \beta \mathbb{R}$, the complement of $\mathbb{R} \times \mathbb{R}$ is $(\beta \mathbb{R} \times \mathbb{R}^*) \cup (\mathbb{R}^* \times \beta \mathbb{R})$. If this subspace were an F-space, then the C^* -embedded subspace $\beta \mathbb{R} \times \mathbb{R}^*$ would also be an F-space. But neither $\beta \mathbb{R}$ nor $\mathbb{R}^*$ is a P-space by Proposition 1.65.

P-points and P-spaces are discussed in [GJ, ex. 4 JKL, 14.29] and the class of P-spaces is investigated in detail in the thesis and paper of A. K. Misra.

OTHER APPROACHES TO βX

1.68. We have seen three approaches to the construction of βX : as an embedding of X into a product of intervals indexed by $C^*(X)$, as the structure space of $C^*(X)$ or $C(X)$, and as the space of z -ultrafilters on X . A fourth approach is through uniformities and had its beginning in 1937 when A. Weil obtained βX via a construction based on uniform structures. The following use of uniformities is discussed in [GJ, Chapter 15]. If for each f in $C^*(X)$ we define $\psi_f : X \times X \rightarrow \mathbb{R}$ by

$$\psi_f(x, y) = |f(x) - f(y)|,$$

then ψ_f is a pseudometric on X . Every member of $C^*(X)$ is uniformly continuous with respect to the uniformity generated by all such pseudometrics as f runs through $C^*(X)$ and the uniformity generated is precompact. Thus, the completion of X with respect to this uniformity is a compact space containing X as a dense, C^* -embedded subspace and hence is βX . The realcompactification νX is obtained in the same manner by allowing f to run through all of $C(X)$.

Another characterization of βX in terms of uniformities was given by R. Alo in 1968. Call a uniformity on X admissible if it generates the topology of X and continuous if it generates a topology weaker than that of X . Then βX is that compactification of X to which every admissible precompact uniformity has a continuous extension.

M. H. Stone's original construction in 1937 was based on a combination of algebraic properties of $C^*(X)$ with a ring of subsets of X . His construction was simplified by I. Gelfand and G. Shilov in 1941. In 1941, S. Kakutani gave a construction of βX based on Banach lattices.

P. Alexandroff, in 1939, modified the Wallman technique of compactification described in Section 1.19 to obtain a compactification (not Hausdorff, of course) of an arbitrary regular space. J. Flachsmeier, in 1961, obtained βX and other compactifications by using inverse limit spaces. P. Zenor, in 1970, constructed both βX and νX as an inverse

limit of systems of metric spaces.

In the first volume of their treatise, Linear Operators, N. Dunford and J. Schwartz demonstrate the existence of the Stone-Čech compactification from the Stone-Weierstrass Theorem for algebras of complex-valued functions.

Speaking very loosely, a proximity on a set is a relation on the power set of the set in which related sets are thought of as being "close". Each proximity will induce a topology on the set by defining the closure of a set to be the set of singletons which are "close" to the set. Proximities were introduced by V. A. Efremovic in his 1951 and 1952 papers. The class of completely regular spaces can be characterized as that class of spaces in which the topology is induced by a proximity. There is a one-to-one correspondence between the proximities which induce the topology of a space and the set of compactifications of a space. The relationship between proximities and compactifications was considered by Yu M. Smirnov in his 1952 and 1967 papers. Basic discussions of proximities are contained in the texts of R. Engelking and S. Willard.

Finally, in 1971, R. Alo and L. Sennott characterized βX as that compactification of X to which every mapping of X into a Fréchet space such that the image of X is totally bounded will extend. Here the term Fréchet space is not used in the usual topological sense, but refers to a topological vector space in which the topology is induced by a complete metric.

EXERCISES

1A. TOPOLOGICALLY COMPLETE SPACES

In 1937, Čech termed a space X topologically complete if X is a G_δ in some compactification. His motivation was the fact that a metrizable space is completely metrizable if and only if it is topologically complete.

1. A space is topologically complete if and only if it is a G_δ in its Stone-Čech compactification.
2. Let X be a topologically complete space contained in a space T . Then X is a G_δ in its closure in T . [$\text{cl}_{\beta T} X$ is a compactification of X and βX is maximal.]

References: Čech, 1937. For a proof of the stated characterization of complete metric spaces, see Nagata, 1968.

1B. σ -COMPACT AND LOCALLY COMPACT SPACES

1. A space X is σ -compact if it is the union of at most countably many compact subspaces. A space X is σ -compact if and only if X^* is a G_δ in βX .
2. Every G_δ containing a compact subspace contains a zero-set containing the compact subspace.
3. Every compact G_δ is a zero-set.
4. A space X is σ -compact and locally compact if and only if X^* is a zero-set in βX .

5. A σ -compact and locally compact space is realcompact.

1C. REALCOMPACT SPACES

A subspace F of X is real-closed in X if every point not in F is contained in a G_δ which misses F . The real-closure of a subset S of X is the smallest real-closed subset containing S .

1. The real-closure of a subspace is contained in its closure.
2. $\cup X$ is the real-closure of X in βX .
3. X is realcompact if and only if it is real-closed in some compactification.
4. A real-closed subspace of a realcompact space is realcompact.
5. Every subspace of a first countable realcompact space is realcompact, hence every subspace of $\mathbb{R}$ is realcompact.

Reference: Real-closure has also been called Q -closure.
Mrówka, 1957.

1D. MORE REALCOMPACT SPACES

A Lindelöf space is one in which every open cover has a countable subcover. A second countable space is one which has a countable base for the open sets.

1. Every second countable space is Lindelöf.
2. A space is Lindelöf if and only if every family of closed sets with the countable intersection property has non-empty

intersection.

3. Every Lindelöf space is realcompact.
4. Every subspace of $\mathbb{R}^n$ is realcompact if n is a positive integer.
5. The rationals, $\mathbb{Q}$, and the irrationals, $\mathbb{P}$, are realcompact.
6. Every countable space is realcompact.
7. A separable metric space is second countable. Hence, every separable metric space is realcompact.

Reference: [GJ, 8.2]. The various spaces were originally shown to be realcompact in Hewitt, 1948.

1E. THE MAXIMAL COMPACTIFICATION

1. Any space X admits $2^{2^{|X|}}$ filters.
2. Any space X admits a set $\{K_\alpha\}$ of compactifications.
3. Any space X admits a maximal compactification. [Embed X as the diagonal in $\prod_\alpha K_\alpha$.]

Reference: This construction of βX was suggested to the author by B. Banaschewski.

1F. ONE-POINT COMPACTIFICATION

1. An open subspace of a compact space is locally compact.
2. A space X has a Hausdorff compactification αX whose growth is a single point if and only if X is locally compact. [Let neighborhoods of the added point be the point together with the complement of a compact subspace of X .]

3. If X is locally compact and Z is a compact subspace contained in an open set U , then there exists an open set V such that $Z \subset V \subset \text{cl } V \subset U$ and $\text{cl } V$ is compact.

In the remainder of the problem, let X be a non-compact, locally compact space. Let $\mathcal{B}$ be the zero-sets of those mappings in $C^*(X)$ which are constant on the complement of a compact set.

4. $\mathcal{B}$ is a disjunctive ring and is a base for the closed subsets of X .
5. If $Z(f)$ belongs to $\mathcal{B}$, then either $Z(f)$ or $\text{cl}(Cz(f))$ is compact.
6. If Z_1 and Z_2 in $\mathcal{B}$ are both non-compact, then $Z_1 \cap Z_2 \neq \emptyset$.
7. $\mathcal{B}$ is a normal base. [(3) and Proposition 1.47.]
8. $\omega(\mathcal{B})$ is αX , the one-point compactification of X .

References: The one-point compactification appears in Alexandroff and Urysohn, 1929. αX was shown to be a Wallman-type compactification by Brooks, 1967. In 1952, Fan and Gottesman obtained a similar result using the family of open sets such that either the closure of the set or the complement of the set is compact.

1G. COMPLETELY REGULAR ULTRAFILTERS

A collection of sets $\mathcal{B}$ contained in a filter $\mathcal{F}$ is a base for $\mathcal{F}$ if the intersection of two sets in $\mathcal{F}$ contains a member of $\mathcal{B}$. A filter $\mathcal{F}$ on X is said to be a completely regular filter if $\mathcal{F}$ has a base $\mathcal{B}$ of open sets such that

for each set A in $\mathcal{B}$, there exists B in $\mathcal{B}$ such that B is contained in A and B is completely separated from $X \setminus A$.

1. Any completely regular filter is contained in a maximal completely regular filter. [Zorn's Lemma.]
2. A completely regular filter $\mathcal{F}$ is maximal if and only if for each pair of open sets A and B with B contained in A and B completely separated from $X \setminus A$, either A belongs to $\mathcal{F}$ or A does not belong to $\mathcal{F}$ and B misses some set of $\mathcal{F}$. [If $f : X \rightarrow I$ completely separates B from $X \setminus A$, consider the filter generated by $\mathcal{F}$ and $\{f^{-1}([0, a)) : 0 < a < 1\}$.]
3. If p belongs to $\beta X \setminus X$, then the trace of the neighborhoods of p on X is a maximal completely regular filter.
4. For x in X , $\mathcal{B}(x)$, the neighborhood filter of x is a maximal completely regular filter.
5. Distinct maximal completely regular ultrafilters on X cluster at distinct points of βX . [Distinct maximal completely regular filters contain completely separated sets.]
6. There is a one-to-one correspondence between maximal completely regular filters on X and the points of βX .

References: Adapted from N. Bourbaki, 1966, Exercise IX 1.8.

This approach to the Stone-Čech compactification appears in Alexandroff, 1939, and is also discussed in Banachewski, 1964.

CHAPTER TWO:
BOOLEAN ALGEBRAS

2.1. This chapter is intended mainly as an introduction for the reader who is not already familiar with Boolean algebras. The central result is Stone's Representation Theorem which shows that any Boolean algebra can be identified with the family of clopen subsets of some totally disconnected compact space. The reader who is familiar with the relationships between Boolean algebras and totally disconnected compact spaces will probably wish to omit this chapter and perhaps return to specific topics within the chapter as they are used later. The material on separability will be used in connection with $\beta\mathbb{N}\setminus\mathbb{N}$ in Chapters 3 and 7. The homomorphism described in Proposition 2.16 will be used in Chapter 3 to relate $\mathbb{N}$ to $\beta\mathbb{N}\setminus\mathbb{N}$. The only occasion in this chapter where the Stone-Čech compactification plays an important part is in Examples 2.14 and 2.15 which are based on properties of $\beta\mathbb{Q}$. Propositions 2.3 and 2.5 together with the Stone Representation Theorem will be needed in Chapter 10 in the context of projectives.

Many of the routine verifications are left to the exercises following the chapter. The reader who would like a more detailed account of Boolean algebras should consult the books of Dwinger, Halmos, and Sikorski.

2.2. A partially ordered set is a set X together with a binary relation $\leq$ defined on X which satisfies the following conditions for arbitrary elements x, y , and z of X :

- (1) $\leq$ is reflexive: $x \leq x$ for all x
- (2) $\leq$ is transitive: $x \leq y$ and $y \leq z$ implies that $x \leq z$
- (3) $\leq$ is antisymmetric: $x \leq y$ and $y \leq x$ implies that $x = y$.

An upper bound for a subset A of X is an element x such that $a \leq x$ for all a in A , and a lower bound for A is an element z such that $z \leq a$ for all a in A . A least upper bound for the set A is an upper bound x such that for any other upper bound y , $x \leq y$. A greatest lower bound is defined similarly. The least upper bound of a family $\{x_\alpha\}$ is written $\bigvee_\alpha x_\alpha$ and the greatest lower bound is written $\bigwedge_\alpha x_\alpha$. The terms join and supremum (resp. meet and infimum) are often used instead of least upper bound (resp. greatest lower bound).

A lattice is a partially ordered set in which every pair of elements has a supremum and an infimum. A lattice is said to be complete if every set has a supremum and an infimum.

EXAMPLES:

The rings $C^*(X)$ and $C(X)$ for any space X are lattices with the partial order defined by $f \leq g$ if $f(x) \leq g(x)$ for every point x of X .

A normal base as defined in Section 1.35 is a lattice where the partial order is defined by set inclusion and meets and joins

are given by intersections and unions, respectively. The normal base of zero-sets of a space X is a lattice in which every countable set has a greatest lower bound.

The family of all subsets of a set is a lattice with the relation of containment and the operations of union and intersection.

A lattice is called distributive if the operations meet and join satisfy the two identities:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

The two identities can be shown to be equivalent, which is a useful fact when working with examples since it simplifies verification of the properties of a distributive lattice.

A lattice is called complemented if it contains distinct elements 0 and 1 such that $0 \leq x \leq 1$ for all elements x and to each element x is assigned an element x' such that $x \vee x' = 1$ and $x \wedge x' = 0$. The element x' is called the complement of x . The elements 0 and 1 are often referred to as the zero element and the unit, respectively. A Boolean algebra is a complemented distributive lattice. A Boolean algebra is said to be complete if it is complete as a lattice. Observe that in order to show that a Boolean algebra is complete, it suffices to show only that every family has a supremum because the existence of infima follows from the existence of suprema by taking complements.

2.3. Of the examples listed in the preceding section, only the family of all subsets of a set is a Boolean algebra, and that example is complete. We now consider a second example of a complete Boolean algebra -- the family of all regular closed subsets of a topological space. A subset F of a space X is called a regular closed set if $F = \text{cl}(\text{int } F)$. Similarly, a subset G of X is said to be a regular open set if $G = \text{int}(\text{cl } G)$.

PROPOSITION:

The family $R(X)$ of regular closed subsets of a space X is a complete Boolean algebra with the following operations:

- (1) $A \leq B$ if and only if $A \subseteq B$.
- (2) $\bigvee_{\alpha} A_{\alpha} = \text{cl}(\bigcup_{\alpha} \text{int } A_{\alpha})$
- (3) $\bigwedge_{\alpha} A_{\alpha} = \text{cl}(\text{int}(\bigcap_{\alpha} A_{\alpha}))$
- (4) $A^{\ast} = \text{cl}(X \setminus A)$.

Proof: Let $\{A_{\alpha}\}$ be any collection of regular closed subsets of X .

Set $B = \text{cl}(\bigcup_{\alpha} \text{int } A_{\alpha})$. Then B is the closure of an open set and is clearly in $R(X)$. Since B contains $\text{cl}(\text{int } A_{\alpha}) = A_{\alpha}$ for each α , B is an upper bound of $\{A_{\alpha}\}$. Suppose that E is another upper bound. Then $\text{int } A_{\alpha}$ is contained in $\text{int } E$ for every α and therefore

$$B = \text{cl}(\bigcup_{\alpha} \text{int } A_{\alpha}) \subseteq \text{cl}(\text{int } E) = E.$$

Hence, B is the least upper bound of $\{A_{\alpha}\}$.

Set $F = \text{cl}(\text{int}(\bigcap_{\alpha} A_{\alpha}))$. Since $\text{int}(\bigcap_{\alpha} A_{\alpha})$ is contained in $\text{int } A_{\alpha}$ for all α , F is contained in A_{α} for every α and F

is a lower bound for $\{A_\alpha\}$. Now suppose that E is another lower bound. Then E is contained in $\bigcap A_\alpha$ so that $\text{int } E$ is contained in $\text{int}(\bigcap A_\alpha)$. Therefore,

$$E = \text{cl}(\text{int } E) \subseteq \text{cl}(\text{int}(\bigcap A_\alpha)) = F$$

and F is the least upper bound.

Take the empty set and X to be the zero element and the unit element, respectively. For any A in $R(X)$, put $A^* = \text{cl}(X \setminus A)$. Then $A \wedge A^*$ is the closure of the interior of the boundary of A and is therefore empty. Similarly, $A \vee A^*$ is the closure of a dense set and is therefore all of X .

It remains to show that the distributive law holds. Let A , B , and C belong to $R(X)$. Note first that

$$\text{cl}(\text{int } B \cup \text{int } C) = B \vee C = \text{cl}(\text{int}(B \vee C)).$$

Now recall that in any topological space, if two sets S and T have the same closure and G is an open set, then

$$\text{cl}(G \cap S) = \text{cl}(G \cap T).$$

Applying these two formulas, we obtain the distributive law:

$$\begin{aligned} A \wedge (B \vee C) &= \text{cl}(\text{int } A \cap \text{int}(B \vee C)) \\ &= \text{cl}(\text{int } A \cap (\text{int } B \cup \text{int } C)) \\ &= \text{cl}((\text{int } A \cap \text{int } B) \cup (\text{int } A \cap \text{int } C)) \\ &= \text{cl}(\text{int } A \cap \text{int } B) \cup \text{cl}(\text{int } A \cap \text{int } C) \\ &= (A \wedge B) \cup (A \wedge C) \\ &= (A \wedge B) \vee (A \wedge C). \end{aligned}$$

By interchanging the roles of the closure and interior operators, one can show that the Boolean algebra of regular open sets of a space is also a complete Boolean algebra.

2.4. Denote the family of clopen subsets of a space X by $CO(X)$. Since the complement of a clopen set is clopen, it is easy to see that the clopen subsets form a Boolean algebra with the operations of set-theoretic complementation, union, and intersection. However, if X is connected, the only clopen sets are X itself and the empty set. We now consider three classes of highly non-connected spaces which are important because a space X belonging to any one of the classes possesses a sufficiently rich supply of clopen subsets to make $CO(X)$ an interesting Boolean algebra.

A space is said to be totally disconnected if the only connected subsets are the points. In his 1937 paper, M. H. Stone showed that any distributive lattice can be represented as the family of clopen subsets of some totally disconnected space. A space is said to be zero-dimensional if it has a base of clopen sets. From the definition, we can expect to be able to relate $CO(X)$ to the space X if X is zero dimensional. Note that this is not the same definition adopted by [GJ], who term a space zero-dimensional if every finite cover of the space by basic open sets may be refined by a partition. Their definition implies that the space has a basis of clopen sets, but not conversely. [GJ, 16.17] shows that the two definitions are equivalent for Lindelöf spaces.

It is easy to see that a zero-dimensional space is totally

disconnected. If a space is also compact, the converse is true.

PROPOSITION:

A compact space is zero-dimensional if and only if it is

totally disconnected.

Proof: Let X be a compact, totally disconnected space and

let U be an open neighborhood of a point x of X . The component

of a point in a compact space is the intersection of the clopen

sets containing the point [GJ, 16.17]. Now because X is

totally disconnected, we can write $\{x\} = \bigcap \{U_\alpha : \alpha \in A\}$ where

$\{U_\alpha : \alpha \in A\}$ is the family of clopen sets containing x . For

every point y not equal to x , there is some U_α which misses

y . Hence, $\{U_\alpha : \alpha \in A\}$ is an open cover of X . Because

X is compact, there is a finite subcover $\{U_1 \cup \dots \cup U_n\}$.

But then $X \setminus U$ is contained in $U(X \setminus U)^c$ so that $\bigcap U_i$ is a

clopen neighborhood of x contained in U . Thus, X has a

base of clopen sets and is zero-dimensional. $\square$

2.5. A space is said to be extremely disconnected if the closure

of every open subspace is open. The importance of this class

of spaces in the context of Boolean algebras is described in the

following

PROPOSITION:

The Boolean algebra of clopen subsets of a zero-dimensional

space is complete if and only if the space is extremely dis-

connected.

Proof: Let X be a zero-dimensional space. We first show that $CO(X)$ is complete if X is extremely disconnected. Let $\{A_\alpha\}$ be a family of clopen subsets of X . Then the set $B = cl(\bigcup A_\alpha)$ is an upper bound for $\{A_\alpha\}$ since B is open if X is extremely disconnected. Since any clopen set containing every A_α also contains $\bigcup A_\alpha$, B is the least upper bound for $\{A_\alpha\}$.

Now assume that $CO(X)$ is complete and let U be an open subset of X . Let $\{A_\alpha\}$ be the family of clopen subsets of X contained in U and let V be the least upper bound of $\{A_\alpha\}$. The proof is completed by showing that $cl U = V$ and is therefore open. Since X is zero-dimensional, $U = \bigcup A_\alpha$ and therefore

$$cl U = cl(\bigcup A_\alpha) \subseteq cl V = V.$$

To show that V is contained in $cl U$, consider the set $V \setminus cl U$. This is an open set and if it is non-empty it contains a non-empty clopen set S since X is zero-dimensional. Then $V \setminus S$ is clopen and contains U and is therefore an upper bound for $\{A_\alpha\}$. But since V is the least upper bound, S must be empty and V is contained in $cl U$. $\blacksquare$

THE STONE REPRESENTATION THEOREM

2.6. An ideal in a Boolean algebra L is a subset $\mathcal{I}$ of L satisfying:

- (1) $a \vee b$ belongs to $\mathcal{I}$ whenever both a and b belong to $\mathcal{I}$.
- (2) If b belongs to $\mathcal{I}$ and $a \leq b$, then a belongs to $\mathcal{I}$.

It is clear that an ideal is not all of L only if the unit element is not in the ideal. Such an ideal is called a proper ideal. For any subset A of L , there is a smallest ideal containing A , called the ideal generated by A . The members of the ideal generated by A are all members of L which are dominated by the supremum of some finite subset of A . A principal ideal is an ideal generated by a single element of L , and the principal ideal generated by a is denoted by L_a .

A filter in L is a subset $\mathfrak{F}$ of L satisfying:

- (1) $a \wedge b$ belongs to $\mathfrak{F}$ whenever both a and b belong to $\mathfrak{F}$.
- (2) If b belongs to $\mathfrak{F}$ and $b \leq a$, then a belongs to $\mathfrak{F}$.

A filter is proper, i.e. not all of L , exactly when it does not contain the zero element.

The relationship between ideals and filters in a Boolean algebra is apparent by comparing their definitions and observing that a Boolean algebra satisfies the de Morgan Laws:

$$(1) \quad a \vee b = (a^* \wedge b^*)^*$$

and

$$(2) \quad a \wedge b = (a^* \vee b^*)^*.$$

Thus, the complements of members of an ideal form a filter and conversely. For this reason, a filter is frequently called a dual ideal and a principal dual ideal is the set of all elements greater than or equal to a single element. The principal dual ideal generated by an element a is denoted by L^a .

A filter or ideal is called maximal if the only filter or ideal properly containing it is L itself.

PROPOSITION:

A proper filter (resp. ideal) is maximal if and only if for every element a of L , either a or a' belongs to the filter (resp. ideal).

Proof: If a proper ideal $\mathcal{J}$ satisfying the stated condition is properly contained in an ideal $\mathcal{J}'$, then there exists an element a in $\mathcal{J}'$ but not in $\mathcal{J}$. Then a' belongs to $\mathcal{J}$ so that a and a' belong to $\mathcal{J}'$. Thus, $a \vee a' = 1$ belongs to $\mathcal{J}'$ and $\mathcal{J}'$ is all of L .

Now suppose that $\mathcal{J}$ is maximal and that a is not in $\mathcal{J}$. Then the ideal generated by $\mathcal{J}$ and $\{a\}$ contains $\mathcal{J}$ properly and is therefore all of L . Hence, 1 is the supremum of a and finitely many elements of $\mathcal{J}$. But this implies that a' is dominated by the supremum of finitely many elements of $\mathcal{J}$ and therefore belongs to $\mathcal{J}$. $\blacksquare$

2.7. A field of sets is a family of subsets of a set X which is closed under finite unions, finite intersections, and complementation. An example is the family of clopen subsets of any topological space. It is clear that any field is a Boolean algebra. The converse of this statement is known as the Stone Representation Theorem which states that every Boolean algebra can be represented as the field of clopen subsets of some totally disconnected compact space. A field of subsets of X is called a reduced field if for every pair of distinct points of X , there

is a member of the field containing one of the points but not the other. The field of clopen sets of a zero-dimensional space is easily seen to be reduced.

A filter (resp. ideal) in a field is said to be determined by a point if it is the set of all members of the field containing the (resp. not containing) the point. It is clear that such a filter or ideal is necessarily maximal. A perfect field is a field in which every maximal ideal, or equivalently every maximal filter, is determined by a point. Because any filter of clopen subsets of a compact space has non-empty intersection, the clopen sets of a compact space form a perfect field. Every maximal filter is determined by each point belonging to the intersection of the filter. Because an infinite discrete space has maximal filters of clopen sets with empty intersection, such a field is not perfect. It is, however, reduced since a discrete space is zero-dimensional. On the other hand, the field of clopen sets of a compact connected space having more than two points is perfect, but not reduced.

The following proposition shows that every field which is both perfect and reduced is the family of clopen subsets of a totally disconnected compact space.

PROPOSITION:

If $\mathcal{C}$ is a perfect reduced field of subsets of a set X , then a topology can be defined on X such that the space X is compact and totally disconnected and $\mathcal{C}$ is the Boolean algebra of clopen subsets of X .

Proof: Take $\mathcal{E}$ to be the basis for a topology on X , i.e. a set G contained in X is open if and only if G is a union of members of $\mathcal{E}$. Since every set of $\mathcal{E}$ is open in this topology and $\mathcal{E}$ is a field, the sets of $\mathcal{E}$ are also closed. Since $\mathcal{E}$ is a reduced field, for any two points of X , there is a clopen set containing one and missing the other so that X is totally disconnected.

To show that X is compact, we show that any open covering $\mathcal{U}$ of X has a finite subcover. We can assume that $\mathcal{U}$ is made up of members of $\mathcal{E}$. Let $\mathcal{J}$ be the ideal of $\mathcal{E}$ generated by $\mathcal{U}$. If X is not the union of finitely many elements of $\mathcal{U}$, then X does not belong to $\mathcal{J}$ and $\mathcal{J}$ is proper. Choose a member E of $\mathcal{E}$ which is not contained in any of the sets of $\mathcal{J}$. Then the union of any chain of ideals containing $\mathcal{J}$ but not containing E is a proper ideal. Hence, Zorn's Lemma shows that $\mathcal{J}$ is contained in a maximal ideal. Because $\mathcal{E}$ is perfect, there is some point of X which does not belong to any member of the maximal ideal, and hence, does not belong to any member of $\mathcal{U}$. Thus, $\mathcal{U}$ cannot be a covering of X and the assumption that $\mathcal{U}$ does not admit a finite subcover leads to a contradiction.

It remains to prove that every clopen subset U of X belongs to $\mathcal{E}$. The clopen set U is the union of a family $\{V_\alpha\}$ of members of $\mathcal{E}$. But then $\{V_\alpha\} \cup \{X \setminus U\}$ is an open cover of X and therefore has a finite subcover. Hence, U is the union of finitely many sets belonging to $\mathcal{E}$ so that U belongs to $\mathcal{E}$. $\blacksquare$

2.8. A Boolean algebra homomorphism h from a Boolean algebra L to a Boolean algebra M is a function which preserves the Boolean operations, i.e.

$$(1) \quad h(a \wedge b) = h(a) \wedge h(b)$$

$$(2) \quad h(a \vee b) = h(a) \vee h(b)$$

$$(3) \quad h(a^*) = (h(a))^*.$$

Note that in the presence of (3), de Morgan's Laws show that (1) and (2) are equivalent. Thus, to verify that a function is a Boolean algebra homomorphism, it is sufficient to demonstrate that it satisfies (3) and either (1) or (2). A Boolean algebra homomorphism is called a monomorphism if it is one-to-one. An isomorphism is a monomorphism which is onto.

PROPOSITION:

A Boolean algebra homomorphism h is a monomorphism if and only if $h(a) = 0$ implies that a is 0 .

Proof: Necessity is clear. To show sufficiency, assume that $h(a) = h(b)$. Then $a \wedge b^* = 0$ since $h(a \wedge b^*) = h(a) \wedge (h(b))^* = 0$. Similarly, $a^* \wedge b = 0$. But this implies that $a \leq b$ and $b \leq a$, and therefore $a = b$. $\square$

2.9. The following proposition shows that the set of all maximal filters of a Boolean algebra admits a reduced field of subsets which is a homomorphic image of the Boolean algebra. This fact combined with Proposition 2.7 will yield a proof of the Stone Representation Theorem. The proofs are based on those given in Sikorski's book.

PROPOSITION:

If $S(L)$ is the set of maximal filters of a Boolean algebra L and if for every element a of L , $h(a)$ denotes the set of maximal filters containing a , then the family $\mathcal{C} = \{h(a) : a \in L\}$ is a reduced field of subsets of $S(L)$ and h is a homomorphism onto $\mathcal{C}$. If in addition, every non-zero element of L belongs to some maximal filter, then h is an isomorphism.

Proof: Let $\mathfrak{F}$ be a maximal filter of L . Then by definition of h ,

$$a \in \mathfrak{F} \text{ if and only if } \mathfrak{F} \in h(a).$$

Since $a \wedge b$ is in $\mathfrak{F}$ if and only if a is in $\mathfrak{F}$ and b is in $\mathfrak{F}$,

$$h(a \wedge b) = h(a) \cap h(b).$$

Because $S(L)$ is made up of maximal filters,

$$h(a') = S(L) \setminus h(a).$$

Thus, h is a homomorphism and $\mathcal{C} = \{h(a) : a \in L\}$ is a field of subsets of $S(L)$ since it is the image of L under h .

If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are distinct maximal filters of L , then there is an element a of L belonging to $\mathfrak{F}_1$ but not $\mathfrak{F}_2$. Hence, $\mathfrak{F}_1$ belongs to $h(a)$ and $\mathfrak{F}_2$ does not, thus showing that $\mathcal{C}$ is a reduced field.

Finally, if every non-zero element of L belongs to some maximal filter, $h(a) = \emptyset$ implies that $a = 0$, and h is therefore an isomorphism. **|**

2.10. THE STONE REPRESENTATION THEOREM:

Every Boolean algebra is isomorphic to the Boolean algebra of clopen sets of a compact totally disconnected space.

Proof: Let L be a Boolean algebra and let $S(L)$, $\mathcal{C}$, and h be as in the preceding proposition. If a in L is not 0, then the principal filter generated by a is proper, and therefore can be extended to a maximal filter. It follows from the preceding proposition that h is an isomorphism of L onto the reduced field $\mathcal{C}$.

We show that $\mathcal{C}$ is a perfect field so that we can apply Proposition 2.7. Let $\mathcal{Q}$ be a maximal filter of $\mathcal{C}$. Then the set $\mathfrak{F}$ of all elements a of L such that $h(a)$ belongs to $\mathcal{Q}$ is a maximal filter of L by isomorphism, so that $\mathfrak{F}$ is also a point of $S(L)$. If b belongs to L , we have

$$h(b) \in \mathcal{Q} \text{ if and only if } b \in \mathfrak{F}$$

and as in the proof of the preceding proposition,

$$h(b) \in \mathcal{Q} \text{ if and only if } \mathfrak{F} \in h(b).$$

Hence, the filter $\mathcal{Q}$ is determined by the point $\mathfrak{F}$ of $S(L)$.

The proof is completed by appealing to Proposition 2.7 to show that $\mathcal{C}$ is the Boolean algebra of clopen subsets of $S(L)$ for the topology generated by $\mathcal{C}$ and that this topology is compact and totally disconnected. $\blacksquare$

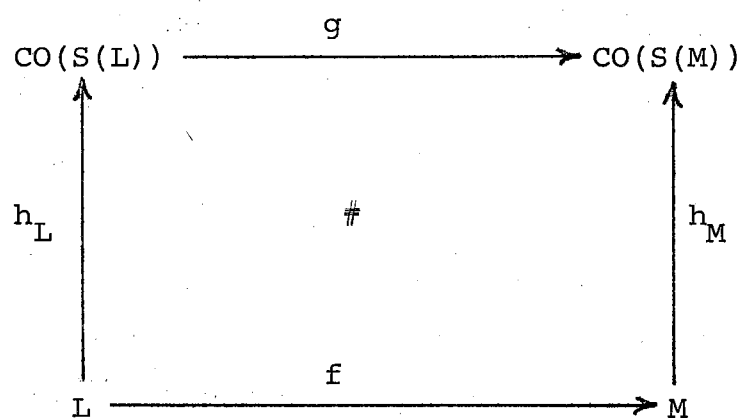
The set $S(L)$ with the topology described in Proposition 2.9 is called the Stone space of L . This topology was introduced

in 1937 by M. H. Stone in his development of the Representation Theorem. As we saw in Chapter 1, the same topology has been used to construct a variety of compactifications of topological spaces. Note that if D is an infinite discrete space, the clopen sets and the zero-sets of D coincide and the Stone space of the algebra of clopen sets of D is βD .

If L and M are Boolean algebras, there is a natural one-to-one correspondence between homomorphisms of L into M and mappings of $S(M)$ into $S(L)$. We will first show that a homomorphism f of L into M induces a homomorphism g of $CO(S(L))$ into $CO(S(M))$. Let $h_L : L \rightarrow CO(S(L))$ and $h_M : M \rightarrow CO(S(M))$ be the isomorphisms provided by the Stone Representation Theorem and for a in L , define

$$g(a) = h_M \circ f \circ h_L^{-1}(a).$$

Then g is a homomorphism and we have the following diagram:



Now we will use g to define a mapping $\varphi : S(M) \rightarrow S(L)$. For every x in $S(M)$, let $\mathfrak{F}(x)$ denote the maximal filter of

$\text{CO}(S(M))$ determined by x . Then it is easily seen that the family

$$g^{\leftarrow} \mathfrak{F}(x) \equiv \{U \in \text{CO}(S(L)) : g[U] \in \mathfrak{F}(x)\}$$

is a maximal filter in $\text{CO}(S(L))$, and is therefore determined by a point y of $S(L)$. Define $\phi(x) = y$. The function ϕ is well-defined because distinct points of $S(L)$ determine different maximal filters since $S(L)$ is zero-dimensional. If U is a clopen subset of $S(L)$, the definition of ϕ shows that

$$(*) \quad \phi(x) \in U \text{ if and only if } x \in g[U].$$

Thus, $\phi^{\leftarrow}(U) = g[U]$ is clopen because g is a homomorphism. Hence, ϕ is continuous since $S(L)$ has a base of clopen sets. Note that the process of obtaining the mapping ϕ from the original homomorphism f has "reversed the arrows":

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ S(L) & \xleftarrow{\phi} & S(M) \end{array}.$$

The inverse procedure to obtain a homomorphism sending L to M from a mapping of $S(M)$ into $S(L)$ is less complicated. The definition of the homomorphism is actually dictated by $(*)$ above. If $\phi : S(M) \rightarrow S(L)$ is a mapping, then $\phi^{\leftarrow}$ preserves the Boolean operations and sends clopen sets to clopen sets. Thus, the definition

$$g[U] = \phi^{\leftarrow}(U)$$

defines a homomorphism of $CO(S(L))$ into $CO(S(M))$. The homomorphism g then gives a homomorphism f of L into M by defining $f = h_M^{-1} \circ g \circ h_L$. Further, it is evident that if f is an isomorphism, then ϕ is a homeomorphism, and conversely.

TWO EXAMPLES

2.11. By an automorphism of a topological space we shall mean a homeomorphism of the space onto itself. Similarly, an automorphism of a Boolean algebra is an isomorphism of the algebra with itself. A topological space is said to be rigid if the identity map is the only automorphism of the space. In 1948, G. Birkhoff asked if there existed an infinite Boolean algebra whose only automorphism was the identity. The Stone Representation Theorem and the preceding discussion of duality between homomorphisms and mappings show that an affirmative answer to Birkhoff's question is equivalent to demonstrating the existence of a rigid totally disconnected compact space. In 1951, M. Katětov provided an affirmative answer by exhibiting a rigid space whose Stone-Čech compactification is both rigid and totally disconnected. Two preliminary results are required before examining Katětov's example. A point in a topological space is called a κ -point if it is the limit of a sequence of distinct points of the space. It is immediate that any homeomorphism must carry κ -points to κ -points. The next result will be used to study automorphisms of Stone-Čech compactifications.

2.12. PROPOSITION:

No point belonging to a G_δ of βX which misses X can be a κ -point of βX .

Proof: Assume on the contrary that a sequence $\{x_n\}$ of distinct points of βX converges to a point p and that p belongs to a G_δ which misses X . Put $Z = X \cup \{x_n\} \cup \{p\}$. Then there exists a sequence $\{G_i\}$ of open sets of the subspace Z such that $\bigcap G_i = \{p\}$. By perhaps passing to a subsequence and eliminating a point of the sequence from some of the G_i 's, we can assume that for each i , $\{x_j : j > i\}$ is contained in $\bigcap \{G_j : j \leq i\}$ and $\{x_j : j \leq i\}$ misses $\bigcap \{G_j : j \leq i\}$. In other words, we are assuming that for each i , x_i is the only point of the sequence contained in the set $(X \setminus G_i) \cap (\bigcap \{G_j : j < i\})$.

Because any two disjoint closed sets one of which is compact are completely separated, there exists a sequence of mappings $\{h_n\}$ in $C^*(Z)$ such that:

- (a) $0 \leq h_n(z) \leq 2^{-n}$ for each z in Z ,
- (b) $h_n(x_i) = 0$ for $i > n$,
- (c) $h_n(p) = 0$,
- (d) $h_n(z) = 2^{-n}$ for each z in $Z \setminus \bigcap \{G_i : i \leq n\}$.

Put $g_n = \sum_{i \leq n} h_i$ for each n . The Weierstrass M-test shows that the sequence g_n converges to a mapping g in $C^*(Z)$, and the construction of $\{g_n\}$ assures that:

- (e) $0 \leq g(z) \leq 1$ for each z in Z ,
- (f) $g(p) = 0$
- (g) $g(z) > 0$ if $z \neq p$,

$$(h) \quad g(x_n) = \sum_{i \leq n} 2^{-i}.$$

Condition (h) shows that $\{g(x_{2k})\}$ and $\{g(x_{2k+1})\}$ are disjoint closed subspaces of the half-open interval $(0,1]$. Because $(0,1]$ is normal, there exists a mapping ι of $(0,1]$ into $[0,1]$ such that $\iota[\{g(x_{2k})\}] = \{0\}$ and $\iota[\{g(x_{2k+1})\}] = \{1\}$. Put $f = \iota \circ (g|Z \setminus \{p\})$ and consider the extension $\beta(f)$ of f to βX . Then $\{\beta(f)(x_{2k})\}$ converges to 0 and $\{\beta(f)(x_{2k+1})\}$ converges to 1. However, this is a contradiction of the continuity of $\beta(f)$ because both subsequences $\{x_{2k}\}$ and $\{x_{2k+1}\}$ converge to p . Thus, p cannot be a κ -point. $\blacksquare$

2.13. The following result is [GJ, 16.16] and will be employed in Katětov's example to show that if X is a countable zero-dimensional space, then βX is totally disconnected. A topological space is said to be a Lindelöf space if every open covering of the space has a countable subcovering. A disconnection of a space is a covering of the space by two disjoint clopen sets.

PROPOSITION:

Disjoint closed subspaces of a zero-dimensional Lindelöf space are separated by a disconnection.

Proof: Let X be such a space and let F and K be disjoint closed subsets of X . For each point x of X , choose a clopen neighborhood $U(x)$ of x which meets at most one of the sets F and K . Since X is Lindelöf, the covering $\{U(x)\}$ has a countable subcovering $\{U_n\}$. Put $V_n = U_n \setminus \bigcup_{i < n} U_i$. The V_n are thus disjoint clopen sets and cover X . Then setting

$$W = \bigcup \{V_n : V_n \cap F = \emptyset\}$$

yields a disconnection $\{W, X \setminus W\}$ where F is contained in $X \setminus W$ and K is contained in W . $\square$

2.14. We will need one further result before considering the example. In 1937, B. Pospíšil showed that the cardinality of $\beta\mathbb{N}$ is 2^c . [GJ, 3.2] provides one proof of this result and another will be given in Theorem 3.2. Since every non-degenerate interval G of $\mathbb{Q}$, the space of rationals, contains a closed C^* -embedded copy of $\mathbb{N}$, it follows from Proposition 1.48 that $G^* = cl_{\beta\mathbb{Q}} G \setminus \mathbb{Q}$ contains 2^c points. We will require this result in the

EXAMPLE: (M. Katětov)

There exists a countable, rigid, normal space such that every point of the space is a κ -point.

Proof: Consider the space $\mathbb{Q}$ of rationals and write $\mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q}$ and $(\beta\mathbb{Q})^2 = \beta\mathbb{Q} \times \beta\mathbb{Q}$. For every point y of $(\beta\mathbb{Q})^2$, let $\Phi(y)$ denote the set of all points of $(\beta\mathbb{Q})^2$ that are images of y under mappings of the subspace $\mathbb{Q}^2 \cup \{y\}$ into $(\beta\mathbb{Q})^2$ which have $\mathbb{Q}^2$ as the image of itself. Since there are at most c mappings of $\mathbb{Q}^2$ into itself and $\mathbb{Q}^2$ is dense in $(\beta\mathbb{Q})^2$, the cardinality of $\Phi(y)$ is at most c . Let $\{G_n\}$ denote a countable open base for the topology of $\mathbb{Q}$ and put $G_n^* = cl_{\beta\mathbb{Q}} G_n \setminus G_n$ for each n . Then the cardinality of each G_n^* is 2^c .

The required space Y will be constructed by adding countably many points of $(\beta\mathbb{Q})^2$ to $\mathbb{Q}^2$. We will use Proposition 2.12 to

show that any automorphism of Y must send Q^2 into Q^2 .

The fact that each $\Phi(y)$ has only c points while each G_n^* contains 2^c points will allow the added points to be chosen so that they force the automorphism to be the identity on Q^2 . Since Q^2 is dense in Y , the automorphism must therefore also be the identity on Y .

We will first describe the points to be added to Q^2 .

(a) There exists a sequence of points (p_n, q_n) in $(\beta Q)^2$ such that:

- (1) $(p_n, q_n) \in G_n^* \times G_n^*$,
- (2) If (p, q) and $(\tilde{p}, \tilde{q})$ are any points of Q^2 , then for all $m, n = 1, 2, \dots$ with $m < n$, $(p_n, \tilde{q}) \notin \Phi(p_m, q)$, $(p_n, \tilde{q}) \notin \Phi(p, q_m)$, $(\tilde{p}, q_n) \notin \Phi(p, q_m)$, $(\tilde{p}, q_n) \notin \Phi(p_m, q)$, and $(p, q_n) \notin \Phi(p_n, q)$.

Choose p_1 arbitrarily in G_1^* . Then $\bigcup_{q \in Q} \Phi(p_1, q)$ contains at most c points. Since G_1^* contains 2^c points and $Q \times \{r\}$ and $Q \times \{s\}$ are disjoint for distinct points r and s of G_1^* , there must exist a point q_1 in G_1^* such that $Q \times \{q_1\}$ misses $\bigcup_{q \in Q} \Phi(p_1, q)$.

Now suppose that p_m and q_m have been chosen for all $m < n$. Then the set

$$\bigcup_{m < n} \{ (\bigcup_{q \in Q} \Phi(p_m, q)) \cup (\bigcup_{q \in Q} \Phi(p, q_m)) \}$$

can contain at most c points of $G_n^* \times G_n^*$, so that by a similar argument to that for q_1 , there must exist a point p_n in G_n^* such that $\{p_n\} \times Q$ misses the above union. Then the point q_n is chosen so that $Q \times \{q_n\}$ misses the union

$$\bigcup_{m < n} \{ (\bigcup_{q \in Q} \Phi(p, q_m)) \cup (\bigcup_{q \in Q} \Phi(p_m, q)) \} \cup \bigcup_{q \in Q} \Phi(p_n, q).$$

This completes the induction step.

Because no point of Q has a compact neighborhood in Q , Proposition 1.59 shows that every neighborhood in βQ of a point of Q must meet $\beta Q \setminus Q$. In fact, because $\{G_n\}$ is a base for the open sets of Q , every neighborhood in βQ of a point of Q must contain some G_n^* . It therefore follows that:

(b) Each of the sequences $\{p_n\}$ and $\{q_n\}$ is dense in βQ .

Now put $A_n = \{p_n\} \times Q$ and $B_n = Q \times \{q_n\}$ for each n .

Then (a) shows that:

(c) If x belongs to $A_m \cup B_m$ for some $m < n$, then

$(A_n \cup B_n) \cap \Phi(x) = \emptyset$, and

(d) If x belongs to A_n , $B_n \cap \Phi(x) = \emptyset$.

Set $Y = Q^2 \cup (\bigcup_n (A_n \cup B_n))$. We will show that Y is the required space. Any automorphism of Y must send a point which has a countable base to a point which also has a countable base. Thus, to show that any automorphism of Y will send points of Q^2 to Q^2 it will suffice to show:

(e) A point y of Y has a countable neighborhood base if and only if y belongs to Q^2 : It is clear from Proposition 2.12 that no point outside of Q^2 can have a countable neighborhood base. Now let $\{U_n\}$ be a countable neighborhood base at a point y of Q^2 and let V be a neighborhood of y in Y . Choose a neighborhood W of y in Y such that

$$y \in W \subset c\ell_Y W \subset V.$$

Then there is an integer m such that U_m is contained in $W \cap Q^2$. Now we have

$$y \in Y \setminus cl_Y(Y \setminus cl_Y U_m) \subset cl_Y U_m \subset (cl_Y W) \cap Q^2 \subset V,$$

so that the family $\{Y \setminus cl_Y(Y \setminus cl_Y U_n)\}$ is a countable neighborhood base at y in Y .

If h is an automorphism of Y , (e) shows that $h[Q^2] = Q^2$ and therefore that for every y in Y , $h(y)$ is in $\Phi(y)$ and y is in $\Phi(h(y))$. Further, (c) and (d) imply that $h[A_n] = A_n$ and $h[B_n] = B_n$ for every n .

Now assume that some point (p, q) of Q^2 is not mapped to itself by h , i.e. that $h(p, q) = (\tilde{p}, \tilde{q})$ and, without loss of generality, that $q \neq \tilde{q}$. Choose a neighborhood U of q in βQ such that $\tilde{q}$ is not in $cl_{\beta Q} U$. Let B be the union of the sets B_n such that q_n belongs to $cl_{\beta Q} U$. Since $h[B_n] = B_n$, we have $h[B] = B$. Also, (p, q) belongs to $cl_Y B$ since the set of all q_n is dense in βQ and therefore every neighborhood of (p, q) contains points (p, q_n) with q_n in $cl_{\beta Q} U$. Because h is continuous, $(\tilde{p}, \tilde{q}) = h(p, q)$ belongs to $cl_Y B = cl_Y h[B]$. But this is a contradiction since U was chosen so that $\tilde{q}$ was not in $cl_{\beta Q} U$. Hence, $h(p, q) = (p, q)$. Since Q^2 is dense in Y , h must also be the identity on Y and Y is rigid.

It is immediate that every point of Y is a $\mathbb{K}$ -point since if (p_n, q) is in Y , there is a sequence $\{r_i\}$ in Q converging to q and (p_n, r_i) converges to (p_n, q) .

Finally, Y is Lindelöf because it is countable. But a regular Lindelöf space is paracompact, and hence, normal.

[D, pp. 163, 174.] ¶

2.15. The previously advertised rigid totally disconnected compact space is now obtained as βY where Y is the preceding example.

EXAMPLE: (M. Katětov)

There exists a rigid totally disconnected compact space.

Proof: By Proposition 2.13, disjoint closed subsets of Q are separated by a disconnection. Then we can show that:

(a) βQ is totally disconnected: If p and q are distinct points of βQ , choose disjoint closed neighborhoods U and V of p and q , respectively. Then if W is a clopen set of Q containing $V \cap Q$ and missing $U \cap Q$, $cl_{\beta Q} W$ is a clopen set of βQ containing p and missing q . Thus, no connected subspace of βQ contains more than a single point.

Hence, Proposition 2.4 shows that βQ and therefore $(\beta Q)^2$ and its countable subspace Y all have bases consisting of clopen sets. Therefore, by the same argument as in (a),

(b) βY is totally disconnected.

Now let h be an automorphism of βY . Since Y is countable, every point in $\beta Y \setminus Y$ is contained in a G_δ which misses Y . Thus, Proposition 2.12 together with the fact that every point of Y is a $\mathbb{K}$ -point shows that:

(c) $h[Y] = Y$.

But since Y is rigid, the restriction $h|_Y$ is the identity on Y , and it follows that h is the identity on βY . Hence, βY is also rigid. ¶

2.16. The second example is a Boolean algebra homomorphism of $R(X)$ into $R(\beta X \setminus X)$. Recall that in the first chapter, we denoted $\beta X \setminus X$ by X^* . If A is a closed subspace of X , the definition

$$A^* = \text{cl}_{\beta X} A \setminus X$$

is consistent with the notation of X^* for the growth $\beta X \setminus X$ because X is dense in βX . If we restrict our attention to the regular closed sets of X , then the assignment

$$A \mapsto A^*$$

is sometimes a homomorphism of $R(X)$ into $R(X^*)$. We will use this assignment in the next chapter to study IN^* . In 1971B, R. G. Woods has shown that this operation will be a homomorphism for any realcompact, metric, or nowhere locally compact space.

In the Boolean algebra of regular closed sets, complementation is defined by

$$A^{\dagger} = \text{cl}_X(X \setminus A).$$

Thus, the hypothesis in the following proposition is just the requirement that $A \mapsto A^*$ preserve complementation.

PROPOSITION:

The function that sends a subset A of X to A^* is a Boolean algebra homomorphism from $R(X)$ into $R(X^*)$ if and only if for every A in $R(X)$,

$$[\text{cl}_X(X \setminus A)]^* = \text{cl}_{X^*}(X^* \setminus A^*).$$

Proof: To show sufficiency, we must first demonstrate that if A belongs to $R(X)$, then A^* belongs to $R(X^*)$:

$$\begin{aligned}
 cl_{X^*}(int_{X^*} A^*) &= cl_{X^*}[X^* \setminus cl_{X^*}(X^* \setminus A^*)] \\
 &= cl_{X^*}[X^* \setminus [cl_X(X \setminus A)]^*] \\
 &= [cl_X[X \setminus cl_X(X \setminus A)]]^* \\
 &= [cl_X(int_X A)]^* \\
 &= A^*
 \end{aligned}$$

since A is a regular closed set in X . Now since the closure operation distributes over finite unions and the join of two regular closed sets is just their union, we have

$$(A \vee B)^* = (A \cup B)^* = A^* \cup B^* = A^* \vee B^*$$

and using the hypothesis once again,

$$(A^*)^* = [cl_X(X \setminus A)]^* = cl_{X^*}(X^* \setminus A^*) = (A^*)^*$$

and the function is a Boolean algebra homomorphism.

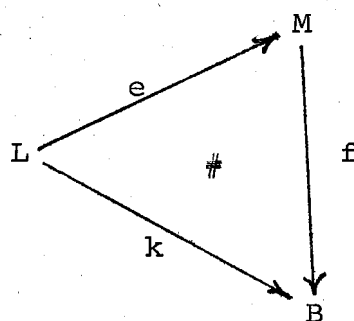
The converse is evident from the last line since a Boolean algebra homomorphism must preserve complementation. $\square$

THE COMPLETION OF A BOOLEAN ALGEBRA

2.17. A Boolean algebra L contained in a Boolean algebra M is said to generate M if every element of M is a supremum of elements of L . A homomorphism of Boolean algebras is said to

be complete if it preserves any suprema which exist. A completion of L is a pair (M, e) where M is a complete Boolean algebra and e is a complete isomorphism of L into M and $e[L]$ generates M . We will usually think of L as a subalgebra of M . The Stone Representation Theorem shows immediately that every Boolean algebra L has a completion since L is isomorphic with the algebra of clopen sets of $S(L)$ which can be seen from Proposition 2.3 to generate the complete Boolean algebra of regular closed subsets of $S(L)$ because $S(L)$ is zero-dimensional. We will see that not only is $R(S(L))$ a completion of L , but that in a sense it is a "best" completion in that it adds the fewest additional points to L . The discussion of completions is based on that given in the book of Halmos.

A completion (M, e) is said to be minimal if for any other completion (B, k) , there is a complete monomorphism f from M to B such that $k = f \circ e$.



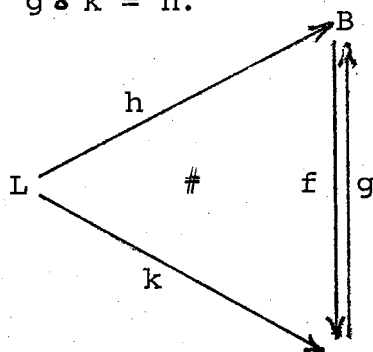
The following proposition shows that any two minimal completions are isomorphic. Observe that the proof is analogous to the proof of Proposition 1.13 concerning uniqueness of a maximal compactification. Here the key will be that complete homomorphisms which agree on a generating subalgebra must be the same, whereas

in Proposition 1.13 we used that mappings which agree on a dense subspace are the same.

PROPOSITION:

Any two minimal completions of a Boolean algebra L are isomorphic by an isomorphism which leaves points of L fixed.

Proof: Let (B, h) and (C, k) be minimal completions of L . Then there exist complete one-to-one homomorphisms f and g such that $f \circ h = k$ and $g \circ k = h$.



Then by substitution, $f \circ g \circ k = h$ and $g \circ f \circ h = k$. But then the two complete homomorphisms $f \circ g$ and $g \circ f$ agree with the identity of L on the generating algebra L and therefore f and g are isomorphisms and are inverses of each other. $\square$

2.18. THEOREM:

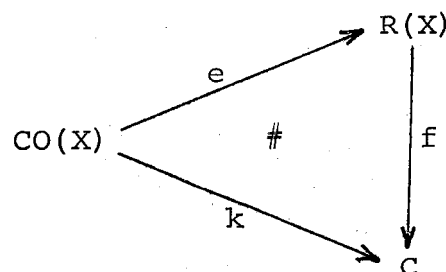
The minimal completion of the Boolean algebra of clopen subsets of a compact totally disconnected space is the Boolean algebra of regular closed subsets of the space.

Proof: Let X be a compact totally disconnected space. Then $R(X)$ is complete by Proposition 2.3 and the injection e of $CO(X)$ into $R(X)$ is a homomorphism since for clopen sets the operations of $R(X)$ reduce to the ordinary set theoretic operations

of $CO(X)$. We must show that if a family $\{A_\alpha\}$ has a supremum in $CO(X)$, then its supremum in $R(X)$ is the same set. If $F = cl(\bigcup A_\alpha)$ is the supremum in $R(X)$, then because every clopen set is regular closed, the supremum B of $\{A_\alpha\}$ in $CO(X)$ contains F . If $B \setminus F$ is non-empty, $B \setminus F$ contains a clopen set E and every A_α is contained in $B \setminus E$, contradicting the assumption that B is the supremum in $CO(X)$. Thus, $B = F$.

To establish that $R(X)$ is a completion of $CO(X)$, it remains to show that $CO(X)$ generates $R(X)$. If F belongs to $R(X)$, let $\{A_\alpha\}$ be the family of clopen subsets of $\text{int } F$. Then $\bigcup A_\alpha = \text{int } F$ and $\bigvee A_\alpha = cl(\bigcup A_\alpha) = F$, and F is a supremum of members of $CO(X)$.

We now must show that $R(X)$ is the minimal completion, i.e. if (C, k) is another completion, then there exists a complete monomorphism f from $R(X)$ to C such that



is a commutative diagram. If F belongs to $R(X)$, write $\text{int } F$ as the union of the clopen sets which it contains, i.e. $\text{int } F = \bigcup A_\alpha$ and define $f(F)$ to be the supremum of $\{k(A_\alpha)\}$. It is then evident that the diagram commutes since, if F is clopen, F is not only the supremum of $\{A_\alpha\}$, but F is also equal to some A_{α_0} so that $f(F) = f(A_{\alpha_0}) = k(A_{\alpha_0})$. If $f(U) = 0$, then $f(V) = 0$ for all clopen subsets V of U and hence $k(V) = 0$.

But k is one-to-one, so that each V must be the empty set. Hence, U is empty, and f is one-to-one.

To show that f preserves complementation, write the interiors of a set F in $R(X)$ and its complement F^i as unions of clopen sets, i.e. $\text{int } F = \bigcup_{\alpha} A_{\alpha}$ and $\text{int } F^i = \bigcup_{\beta} B_{\beta}$. Since any pair A_{α} and B_{β} are disjoint, it follows that $k(A_{\alpha}) \wedge k(B_{\beta}) = 0$. It therefore follows from

$$(\bigvee_{\alpha} A_{\alpha}) \wedge (\bigvee_{\beta} B_{\beta}) = \bigvee_{\alpha, \beta} (A_{\alpha} \wedge B_{\beta})$$

that $k(F) \wedge k(F^i) = 0$. Since $F \vee F^i = 1$ in $R(X)$, there can be no proper clopen set of X containing all the A_{α} 's and B_{β} 's so that $\bigvee_{\alpha, \beta} (A_{\alpha} \vee B_{\beta}) = 1$. Then $\bigvee_{\alpha, \beta} (k(A_{\alpha}) \vee k(B_{\beta})) = 1$ and hence $f(F) \vee f(F^i) = 1$ so that $f(F^i) = (f(F))^i$.

To show that f preserves all suprema, let $\{F_{\alpha}\}$ be a family of members of $R(X)$ and let $F = \bigvee_{\alpha} F_{\alpha}$. Since f is clearly order preserving, $f(\bigvee_{\alpha} F_{\alpha}) \leq f(F)$. To verify the opposite inequality, we show that for any clopen subset A of $\text{int } F$, $f(A) \leq f(\bigvee_{\alpha} F_{\alpha})$. Let $\{A_{\alpha, \beta}\}$ be the family of all clopen subsets of $\text{int } F_{\alpha}$ and write

$$A = A \cap F = \bigvee_{\alpha} (A \cap F_{\alpha}) = \bigvee_{\alpha} (A \cap (\bigvee_{\beta} A_{\alpha, \beta})).$$

Then it follows that

$$A = \bigvee_{\alpha, \beta} (A \cap F_{\alpha, \beta})$$

and hence that

$$k(A) = \bigvee_{\alpha, \beta} (k(A) \wedge k(A_{\alpha, \beta})) = \bigvee_{\alpha} (k(A) \wedge f(F_{\alpha})) = k(A) \wedge (\bigvee_{\alpha} f(F_{\alpha})).$$

Hence, $k(A) \leq \bigvee_{\alpha} f(F_{\alpha})$ and it follows from the definition of f that $f(A) \leq \bigvee_{\alpha} f(F_{\alpha})$. $\blacksquare$

SEPARABILITY IN BOOLEAN ALGEBRAS

2.19. In the remainder of the chapter we consider three conditions on a Boolean algebra which deal with the insertion of its elements into certain strictly increasing sequences. We will use the notation $a < b$ to mean that a is less than but not equal to b .

The first such condition deals with the simplest type of sequence, i.e. one involving only two elements. A Boolean algebra is said to be dense in itself if whenever one element is properly less than another, then a third element can be properly interposed between them, i.e. if $a < b$, then there exists c such that $a < c < b$. The next Proposition shows that a Boolean algebra is dense in itself exactly when its Stone space has no isolated points.

PROPOSITION:

A compact zero-dimensional space Y has no isolated points if and only if the algebra $CO(Y)$ is dense in itself.

Proof: If y is an isolated point of Y , put $A = \emptyset$ and $B = \{y\}$. Then no member of $CO(Y)$ can be interposed between A and B and $CO(Y)$ is not dense in itself.

On the other hand, if $CO(Y)$ fails to be dense in itself,

there exist members A and B of $CO(Y)$ such that A is contained in B and no proper clopen subset of B is a proper superset of A . But since Y is zero-dimensional and $B \setminus A$ is clopen, $B \setminus A$ must be a singleton, and Y contains an isolated point. $\square$

2.20. We will consider two kinds of separability for infinite sequences. A Boolean algebra is said to be Cantor separable if no strictly increasing sequence has a least upper bound, i.e. if whenever

$$a_1 < \dots < a_n < \dots < b,$$

then there exists an element c such that $a_n < c < b$ for every n . A Boolean algebra is said to be DuBois-Reymond separable if a strictly increasing sequence can be separated from a strictly decreasing sequence dominating the increasing one, i.e. if whenever

$$a_1 < \dots < a_n < \dots < b_n < \dots < b_1,$$

then there exists h such that $a_n < h < b_n$ for all n .

Both kinds of separability imply topological properties of the Stone space. The next result shows that if L is Cantor separable, then every non-empty G_δ in $S(L)$ has non-empty interior.

PROPOSITION:

Every non-empty G_δ in a zero-dimensional space Y has non-empty interior if $CO(Y)$ is Cantor separable.

Proof: Let $\cap U_i$ be a non-empty G_δ in Y . Choose a point y in $\cap U_i$ and let V_i be a clopen neighborhood of y contained in U_i . We can assume that $\{V_i\}$ is a decreasing sequence. Hence,

$$Y \setminus V_1 \subset \dots \subset Y \setminus V_n \subset \dots \subset Y$$

is an increasing sequence. Because $CO(Y)$ is Cantor separable, there exists a clopen set C such that $Y \setminus V_n \subset C \subset Y$ for all n . Therefore, $\emptyset \subset Y \setminus C \subset V_n$ for all n . Hence, $Y \setminus C$ is contained in $\cap V_n$, and $\cap U_n$ therefore has non-empty interior. $\square$

2.21. Next we will see that if L is DuBois-Reymond separable, then no sequence of distinct points in $S(L)$ converges, i.e. $S(L)$ contains no κ -points.

PROPOSITION:

A zero-dimensional space Y contains no κ -points if $CO(Y)$ is DuBois-Reymond separable.

Proof: Suppose that $\{y_n\}$ is a sequence of distinct points converging to y and that $y_n \neq y$ for any n . We will construct a neighborhood of y which misses infinitely many points of the sequence, thus contradicting convergence and showing that no point of Y can be a κ -point.

Choose U_1 to be a clopen set containing y_1 and missing y and the other points of the sequence. Choose U_2 to be a clopen set containing y_2 and missing U_1 , y , and all other points of the sequence. Continuing by induction, we obtain a sequence $\{U_i\}$ of disjoint clopen sets such that y_i is in U_i

for each i and y belongs to none of the sets.

For each $n \geq 1$, put $A_n = \bigcup \{U_{2i-1} : i \leq n\}$ and $B_n = Y \setminus (\bigcup \{U_{2i} : i \leq n\})$. Observe that A_n contains the first n points having odd indices and B_n excludes the first n points having even indices. Further, the sequences $\{A_n\}$ and $\{B_n\}$ satisfy:

$$A_1 \subset \dots \subset A_n \subset \dots \subset B_n \subset \dots \subset B_1.$$

DuBois-Reymond separability implies that there exists a clopen set H such that $A_n \subset H \subset B_n$ for every n . Thus, H contains all points having even indices and excludes all points having odd indices. Then y belongs to either H or $Y \setminus H$. Either set fails to contain infinitely many points of the sequence, which contradicts the assumption that it converged to y . $\blacksquare$

2.22. We now consider sufficient conditions for the algebra of clopen sets to be Cantor or DuBois-Reymond separable. In the next chapter, we will see that $\mathbb{N}^*$ satisfies the hypotheses of the following two propositions and therefore is both Cantor and DuBois-Reymond separable. The next two results are taken from R. G. Wood's forthcoming paper.

PROPOSITION:

The Boolean algebra of clopen subsets of a totally disconnected compact space without isolated points and in which every zero-set is regular closed is Cantor separable.

Proof: Let $\{A_n\}$ be a strictly increasing sequence of clopen subsets of a space X satisfying the stated hypotheses and let B be a clopen subset properly containing each A_n . Because B is compact, $\bigcup A_n$ is properly contained in B since $\{A_n\}$ is an open covering of its union $\bigcup A_n$ which has no finite sub-covering. The set $\bigcup A_n$ is a cozero-set since it is a countable union of cozero-sets. Therefore, $B \setminus \bigcup A_n = \bigcap (B \setminus A_n)$ is a non-empty G_δ which must contain a non-empty zero-set. Since the zero-sets are regular closed, $B \setminus \bigcup A_n$ contains a non-void open set and therefore $\text{cl}(\bigcup A_n)$ is a proper subset of B . Since X has no isolated points, there exist distinct points x and y contained in $B \setminus \text{cl}(\bigcup A_n)$. But then $\text{cl}(\bigcup A_n) \cup \{x\}$ and $(X \setminus B) \cup \{y\}$ are disjoint compact subsets of the compact totally disconnected space X . Since X has a base of clopen sets by Proposition 2.4, there is a clopen subset C of X containing the first set and missing the second. But then

$$\begin{array}{c} A_n \subset C \subset B \\ \quad \nearrow \quad \searrow \end{array}$$

and the Boolean algebra of clopen sets is Cantor separable. |

2.23. Recall that one of several equivalent defining properties of the class of F-spaces which was demonstrated in Theorem 1.60 is that disjoint cozero-sets are completely separated.

PROPOSITION:

The Boolean algebra of clopen subsets of a totally disconnected compact F-space is DuBois-Reymond separable.

Proof: Let X be such a space and let $\{A_n\}$ and $\{B_n\}$ be sequences of clopen subsets of X such that

$$A_1 \subset \dots \subset A_n \subset \dots \subset B_n \subset \dots \subset B_1.$$

Then $\bigcup A_n$ and $X \setminus \bigcap B_n$ are disjoint cozero-sets and therefore have disjoint closures since X is an F -space. Hence, there exists a clopen subset H of X containing $\text{cl}(\bigcup A_n)$ and missing $\text{cl}(X \setminus \bigcap B_n)$. Thus, $A_n \subset H \subset B_n$ for each n , and the Boolean algebra of clopen subsets of X is DuBois-Reymond separable. $\square$

2.24. The following sequence of three lemmas concerning Boolean algebras satisfying separability conditions will demonstrate the utility of the Stone Representation Theorem and will be used in Chapters 3 and 7 to investigate the space $\mathbb{N}^*$. The results are highly technical in nature and the reader may wish to omit them until they are actually applied. The first application occurs in Theorem 3.31.

LEMMA:

In a Boolean algebra which is Cantor separable, each ideal containing a cofinal sequence contains any lower bound of the set of upper bounds of the ideal.

Proof: Let A be such an ideal in the Boolean algebra L .

Without loss of generality, we can assume that the cofinal

sequence of A is an increasing sequence $a_1 < \dots < a_n < \dots$

Further, consider L to be the Boolean algebra of clopen sets

of a compact totally disconnected space Y . Suppose that an element a of L satisfies $a \leq b$ for any b containing $\bigcup a_n$. If a is not in A , then a is strictly greater than every a_n . Because the intersection of all members b containing $\bigcup a_n$ is $\text{cl}(\bigcup a_n)$, we have that a is contained in $\text{cl}(\bigcup a_n)$. Since a is open, $a \cap (\bigcup a_n)$ contains a clopen set and there must exist an integer n_0 such that $a \cap a_n > 0$ for all $n \geq n_0$ because $\{a_n\}$ is an increasing sequence. If the decreasing sequence $\{a \setminus a_n\}$ is eventually constant, say from n_1 on, then $(a \setminus a_{n_1}) \cap (\bigcup a_n) = \emptyset$ and this implies that $a \leq a_{n_1}$. Then a must belong to A . In this case, Cantor separability is not required.

If the sequence $a \setminus a_n$ is not eventually constant, then we have

$$a \setminus a_1 \supset \dots \supset a \setminus a_n \supset \dots \supset 0$$

which implies that

$$a \cap a_1 \subset \dots \subset a \cap a_n \subset \dots \subset a.$$

By Cantor separability, there exists c in L such that $a \cap a_n \subset c \subset a$. Then for every n , $0 \subset a \setminus c \subset a \setminus a_n$. Thus, $a \setminus c$ is a non-empty clopen set such that $0 \subset a \setminus c \subset \text{cl}(\bigcup a_n)$ and $(a \setminus c) \cap (\bigcup a_n) = \emptyset$. But this is a contradiction. Thus, $a \leq a_n$ for some n , and a is therefore in A . $\blacksquare$

2.25. LEMMA:

In a Boolean algebra L which is Cantor and DuBois-Reymond separable, if an ideal A contains a cofinal sequence and C is a countable subset of $L \setminus A$, then there exists a principal ideal L_h containing A and missing C .

Proof: Let $a_1 < \dots < a_n < \dots$ be the sequence cofinal with the ideal A and let $C = \{c_i\}$. Because C misses A and L is Cantor separable, the preceding lemma shows that each element of C is not less than or equal to every upper bound of A . In other words, for each c_i there exists e_i in L such that

$$a_1 < \dots < a_n < \dots < e_i$$

and c_i is not less than or equal to e_i . For each $j \geq 1$, put

$$b_j = \bigwedge_{i=1}^j e_i.$$

Then $\{b_j\}$ is a decreasing sequence such that c_i is not less than or equal to b_j for all $j \geq i$. Further, $a_i < b_j$ for all i and j . If the sequence $\{b_j\}$ is eventually constant, say from some point b_{j_0} on, then $L_{b_{j_0}}$ is the required principal ideal. Otherwise, we have

$$a_1 < \dots < a_n < \dots < b_n < \dots < b_1$$

and DuBois-Reymond separability yields h such that $a_n < h < b_m$ for all n and m and c_i is not less than h for all i . Thus, L_h is the required principal ideal. $\blacksquare$

Although the definition of Cantor separability deals with increasing sequences, complementation shows that it is actually equivalent to the statement that no strictly decreasing sequence has a greatest lower bound. Because the complement of an ideal is a dual ideal or a filter, an equivalent form of the lemma is obtained by interchanging meet and join and reversing the order in the definitions.

DUAL LEMMA:

In a Boolean algebra L which is Cantor and DuBois-Reymond separable, if a dual ideal F contains a cointial sequence and C is a countable subset of $L \setminus F$, then there exists a principal dual ideal L^h containing F and missing C .

2.26. The final lemma is a Boolean algebra formulation of a lemma proved in 1956 by W. Rudin and applied by him to construct an automorphism of $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. We will see Rudin's original use of the lemma in Chapter 7. The present form of the lemma is drawn from the 1963 paper (in which the proof of the lemma contains a gap) of I. I. Parovičenko who used it to obtain a characterization of $\mathbb{N}^*$. His result will be Theorem 3.31. In both instances, the lemma is used at the crucial stage in the construction of an isomorphism between the algebras of clopen sets of two compact totally disconnected spaces. The duality between Boolean algebra isomorphisms and the homeomorphisms of the respective Stone spaces is then used to obtain a homeomorphism.

The lemma can be viewed as an extended version of DuBois-Reymond separability. It not only asserts that an element of the Boolean algebra can be interposed between an increasing sequence and a decreasing sequence dominating it, but that the element can also be chosen so that the principal ideal and principal dual ideal which it generates both exclude a given countable set of elements.

LEMMA:

Let a Boolean algebra L be dense in itself and be both Cantor and DuBois-Reymond separable. For all l, m , and n , let the countable (perhaps finite) sets $\{a_l\}$, $\{b_m\}$ and $\{c_n\}$ of L satisfy:

- (a) $a_1 < \dots < a_l < \dots < b_m < \dots < b_1$,
- (b) $c_n \not\leq a_l$,
- (c) $b_m \not\leq c_n$.

Then there exists d in L such that:

- (d) $a_l < d < b_m$,
- (e) e is not comparable to any c_n , i.e. $\{c_i\}$ is contained in $L \setminus (L_d \cup L^d)$.

Note that the statement of the lemma allows the sequences to be finite. In such instances, we can use the density of the algebra in place of the separability conditions. The proof will also make use of the density to show that the Stone space has no isolated points and of Cantor separability to show that every non-empty G_δ in the Stone space has non-empty interior.

Proof: First we use DuBois-Reymond and Cantor separability to find g_0 and g_1 in L such that

$$a_1 < \dots < a_n < \dots < g_0 < g_1 < \dots < b_m < \dots < b_1.$$

We will now find a d between g_0 and g_1 which is not comparable to any of the c_i . First we must construct an ideal and a dual ideal to which we can apply the previous lemma and its dual. Let $A = \bigcup_{i \geq 1} L_{a_i}$ and $F = \bigcup_{m \geq 1} L_{b_m}^{b_m}$. Then A is an ideal and F a dual ideal cofinal and coinitial, respectively, with a countable set. The lemma and its dual show that there exist h_0 and h_1 in L such that:

$$A \subseteq L_{h_0}, \quad F \subseteq L_{h_1}^{h_1}, \quad \text{and } \{c_i\} \subseteq L \setminus (L_{h_0} \cup L_{h_1}^{h_1}).$$

Let $t_0 = g_0 \wedge h_0$ and $t_1 = g_1 \vee h_1$. Then we have

$$A \subseteq L_{t_0} \quad \text{and} \quad F \subseteq L_{t_1}^{t_1}.$$

Further, because $t_0 \leq h_0$ and $h_1 \leq t_1$, we also have

$$\{c_i\} \subseteq L \setminus (L_{t_0} \cup L_{t_1}^{t_1}).$$

It remains to choose a d between t_0 and t_1 such that d is not comparable to any of the c_i 's, i.e. such that

$$\{c_i\} \subseteq L \setminus (L_d \cup L_d^d).$$

For the remainder of the proof, it is convenient to consider L to be a Boolean algebra of clopen sets of a compact, totally disconnected space. Further, because L is dense in itself and

Cantor separable, the space has no isolated points (Proposition 2.19) and every non-empty G_δ has non-empty interior (Proposition 2.20).

The set d which we will construct will be properly contained between t_0 and t_1 . Because of the conditions that are already satisfied by the c_n 's, i.e. $c_n \not\subseteq t_0$ and $t_1 \not\subseteq c_n$, we will first show that some c_n 's may not be comparable to any set between t_0 and t_1 and therefore can be disregarded. We will see that the only c_n 's which we must consider are those such that both c_n and its complement meet $t_1 \setminus t_0$.

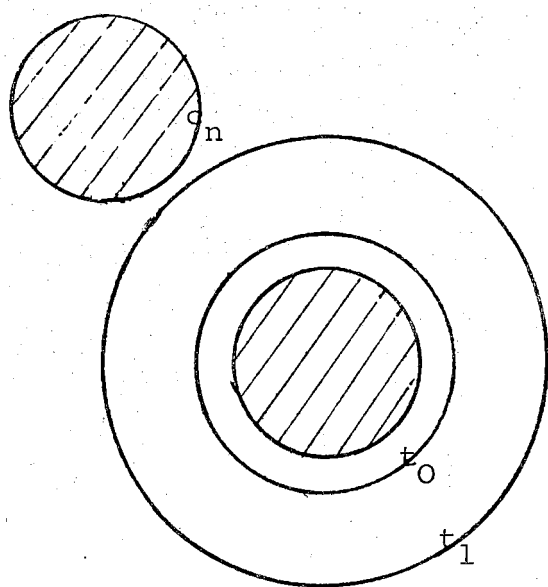
Some of the possible relationships between a c_n and $t_1 \setminus t_0$ are illustrated in Figure 2.1 below.

If c_n misses $t_1 \setminus t_0$ (Case (a)), then c_n must contain points outside of t_1 since c_n cannot be contained in t_0 . Since d will be contained in t_1 , c_n will not be contained in d . Because d will contain points of $t_1 \setminus t_0$, d will not be contained in c_n . Hence, such a c_n can be disregarded.

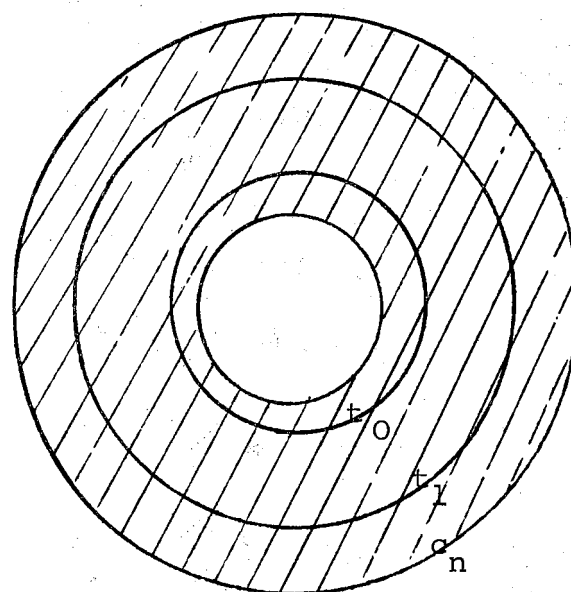
Now suppose that c_n hits $t_1 \setminus t_0$:

Then c_n either contains $t_1 \setminus t_0$ or it doesn't. If it does contain $t_1 \setminus t_0$ (Case (b)), then it cannot contain all of t_0 since then it would contain t_1 , which is impossible. Then because d will contain t_0 , c_n will not contain d . Because d will not contain all of $t_1 \setminus t_0$, d will not contain c_n . Hence, such a c_n can be disregarded.

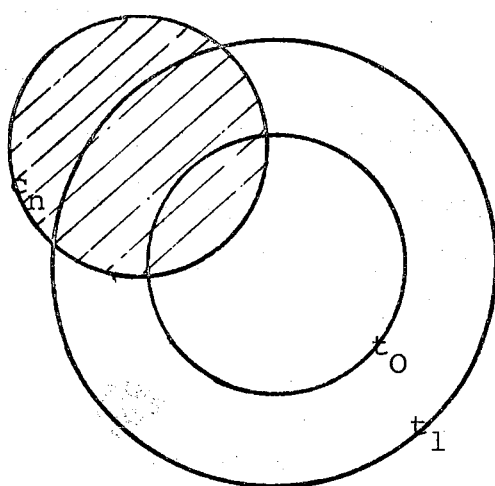
If c_n doesn't contain $t_1 \setminus t_0$ (Cases (c) and (d)), then c_n' must also hit $t_1 \setminus t_0$.



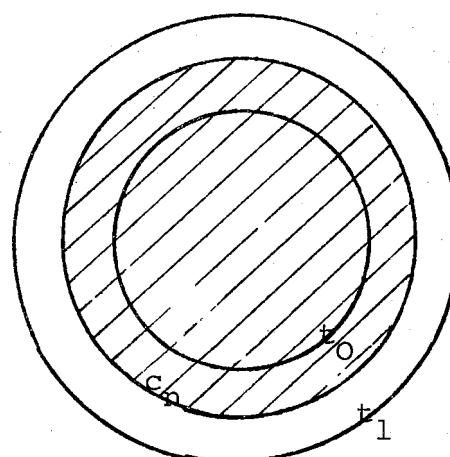
(a)



(b)



(c)



(d)

The shaded region is c_n in each of the cases.

Figure 2.1

Thus, we need only carry out the construction to show that d is not comparable to any c_n such that both c_n and c'_n meet $t_1 \setminus t_0$. So we will now assume that all the c_n 's have this property. We will actually construct a set d which will not be comparable to any c_n or c'_n . Thus d must satisfy the following four conditions: $d \not\subseteq c_n$, $c_n \not\subseteq d$, $d \not\subseteq c'_n$, $c'_n \not\subseteq d$.

Because both c_n and c'_n meet $t_1 \setminus t_0$, we will be able to make one further assumption about the family $\{c_n\}$. Choose a point x_0 in $t_1 \setminus t_0$. Then for each n , either c_n or c'_n contains x_0 . By relabeling if necessary, we can assume that x_0 belongs to c_n for every n and therefore that $\bigcap c_n \neq \emptyset$.

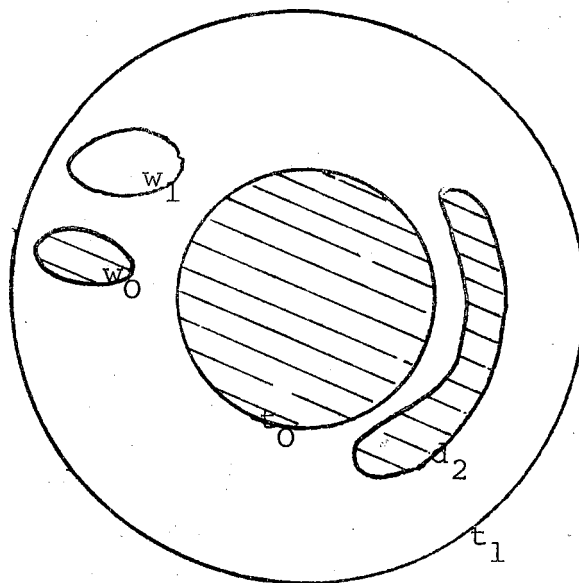
We can now begin the construction of the required set d . We have just seen that the set $(\bigcap c_n) \cap (t_1 \setminus t_0)$ is a non-empty G_δ . Since every non-empty G_δ has non-empty interior and there are no isolated points, $(\bigcap c_n) \cap (t_1 \setminus t_0)$ contains a disjoint pair w_0 and w_1 of non-empty clopen sets. Define $d_1 = t_0 \vee w_0$. The set d_1 is not the required set, but an examination of its properties will show that it satisfies three of the four conditions which d must satisfy and will help to indicate what must be done to complete the proof. The set d_1 has the following properties for each n :

- (i) $d_1 \not\subseteq c'_n$: This is immediate because w_0 is contained in c_n for each n .
- (ii) $c_n \not\subseteq d_1$: This follows because w_1 is contained in every c_n but misses d_1 .
- (iii) $c'_n \not\subseteq d_1$: Each c'_n meets $t_1 \setminus t_0$, but the only points of $t_1 \setminus t_0$ which belong to d_1 belong to every c_n . Thus, there

is a point of c'_n outside of d_1 .

Unfortunately, it is quite possible that d_1 is contained in some c_n . For instance, this will be the case for any c_n which is contained between t_0 and t_1 (Case (d) in Figure 2.1). In order to eliminate such a possibility, we will add to d_1 points from each c'_n while at the same time being careful not to add all of the points of c'_n for any n . The procedure which we will use will be somewhat reminiscent of the proof of Proposition 2.21 where we used DuBois-Reymond separability to obtain a clopen set which contained all points of odd index and excluded all points of even index in a particular sequence.

Here we will use DuBois-Reymond separability to construct a set d_2 which will contain points of c'_n for every n and at the same time exclude points of each c'_n . We then add d_2 to d_1 and show that this yields the required set. The situation is illustrated in Figure 2.2 below:



The shaded regions will make up d .

Figure 2.2

Put $r = t_1 \setminus (t_0 \vee w_0 \vee w_1)$. Then r is a clopen subset of $t_1 \setminus t_0$ and the definition of d_1 shows that r contains $c'_n \cap (t_1 \setminus t_0)$ for every n . We will define sequences $\{p_n\}$ and $\{q_n\}$ of clopen subsets of r such that

$$p_1 < \dots < p_n < \dots < q_n < \dots < q_1.$$

The sequence $\{p_n\}$ will be designed so that each c'_n will have points included in $\cup p_n$ and the sequence $\{q_n\}$ will be designed so that each c'_n has points excluded from $\cap q_n$. DuBois-Reymond separability will provide the set d_2 such that $p_n < d_2 < q_n$ for every n . The two sequences will be constructed by induction.

Choose a point y_1 in $c'_1 \cap r$. Then for all n , y_1 belongs to either c_n or c'_n . Let $u'_1 = \{c'_n : y_1 \in c'_n\}$ and $u_1 = \{c_n : y_1 \in c_n\}$. The set $(\cap u'_1) \cap (\cap u_1) \cap (t_1 \setminus t_0)$ contains y_1 and therefore is a non-empty G_δ . Hence, it contains a pair e_1 and f_1 of non-empty, disjoint clopen sets since it must have non-empty interior and the space contains no isolated points. Put $p_1 = f_1$ and $q_1 = r \setminus e_1$. Then we have $p_1 < q_1$ because e_1 and f_1 are disjoint. Note that p_1 contains points of every c'_n belonging to u'_1 and that q_1 excludes points of every c'_n belonging to u'_1 .

Now let j_2 be the least index such that y_1 is not in c'_{j_2} . Choose y_2 in $c'_{j_2} \cap r$ and let $u'_2 = \{c'_n : y_2 \in c'_n\}$ and $u_2 = \{c_n : y_2 \in c_n\}$. Because $e_1 \cup f_1$ is contained in c'_{j_2} , y_2 also belongs to $(e_1 \cup f_1)'$. Thus, the following set

$$(\cap u_2^i) \cap (\cap u_2) \cap ((e_1 \cup f_1)^c) \cap (t_1 \setminus t_0)$$

is a non-empty G_δ and therefore contains a disjoint pair e_2 and f_2 of non-empty clopen subsets. Observe that it is important to include $(e_1 \cup f_1)^c$ in the intersection so that no points of $e_2 \cup f_2$ belong to $e_1 \cup f_1$. Put $p_2 = p_1 \vee f_2$ and $q_2 = q_1 \setminus e_2$. Then we have

$$p_1 < p_2 < q_2 < q_1.$$

We will continue the construction by induction yielding the situation described in Figure 2.3 below.

Now suppose that for each $i < m$, we have chosen a point y_i in r and that $u_i^i = \{c_n^i : y_i \in c_n^i\}$ and $u_i = \{c_n : y_i \in c_n\}$. Suppose further that we have defined sequences of non-empty clopen sets $\{e_i : i < m\}$ and $\{f_i : i < m\}$ such that:

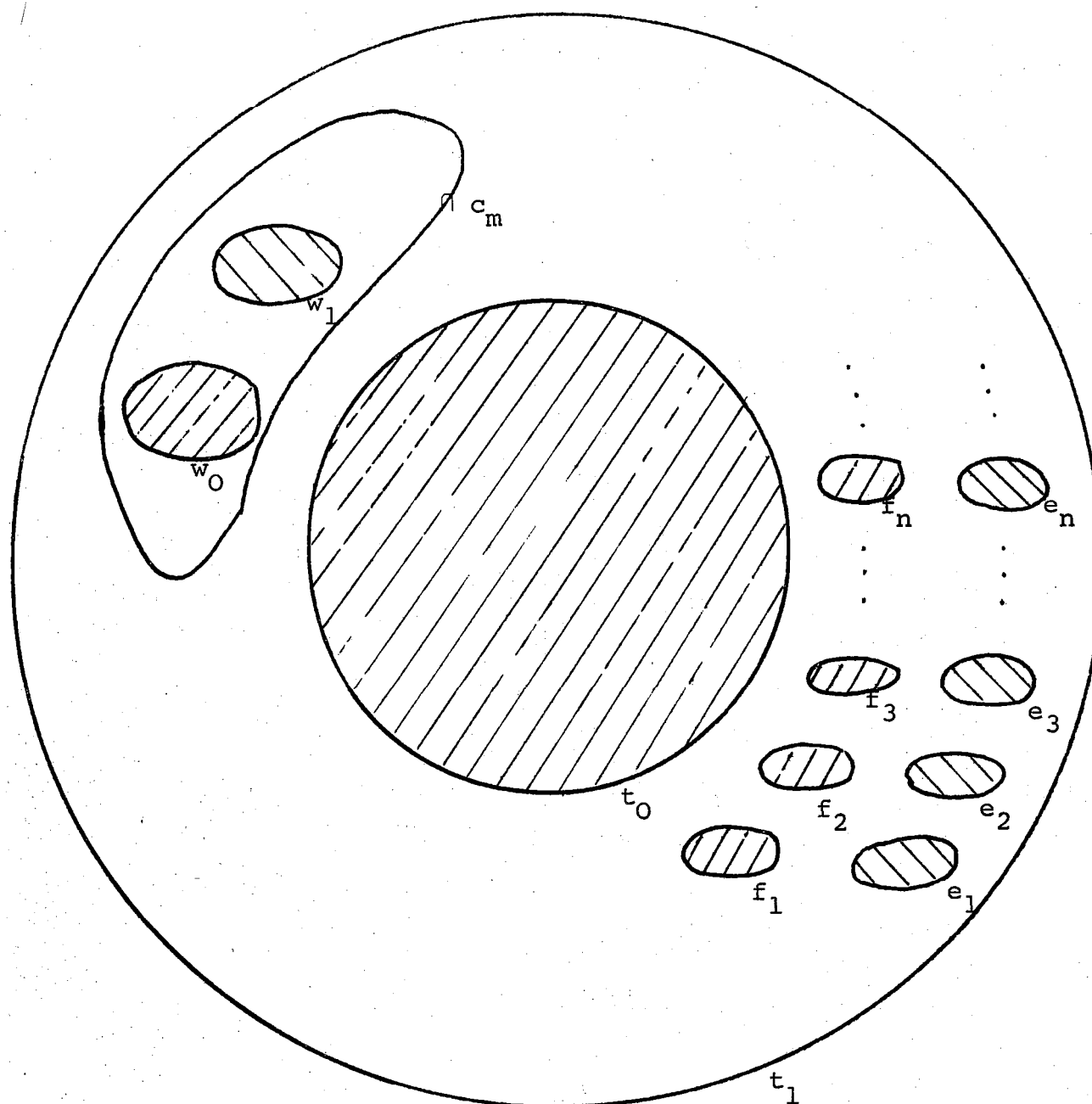
- (1) $\{e_i\} \cup \{f_i\}$ is a pairwise disjoint family, and
- (2) $e_i \cup f_i \subset (\cap u_1^i) \cap (\cap u_i) \cap ((\cup_{j < i} (e_j \cup f_j))^c) \cap (t_1 \setminus t_0)$ for each i .

It follows that if we put $p_i = \vee \{f_j : 1 \leq j \leq i\}$ and $q_i = r \setminus (\vee \{e_j : 1 \leq j \leq i\})$ for each $i < m$, then

$$p_1 < \dots < p_{m-1} < q_{m-1} < \dots < q_1.$$

Further, each p_i contains points belonging to $\cap u_1^i$ and each q_i excludes points belonging to $\cap u_1^i$.

Now let j_m be the least index (if it exists, since if it does not, we are done), such that $c_{j_m}^i$ is not included in $\cup \{u_1^i : i < m\}$. Choose a point y_m in $c_{j_m}^i \cap r$ and let



For every m , points of c_m^i belong to at least one of the e_n 's and are excluded, and points of c_m^i belong to at least one of the f_n 's and are included. For every m , points of c_m belong to w_1 and are excluded, while points of c_m belong to w_0 and are included.

Figure 2.3

$w_m^i = \{c_n^i : y_m \in c_n^i\}$ and $u_m = \{c_n : y_m \in c_n\}$. Because c_{j_m} belongs to u_i for all $i < m$, y_m is also contained in the complement of $U\{e_i \cup f_i : i < m\}$. Hence, the set

$$(\cap w_m^i) \cap (\cap u_m) \cap ((U\{e_i \cup f_i : i < m\})^c) \cap (t_1 \setminus t_0)$$

is a non-empty G_δ and therefore contains a disjoint pair e_m and f_m of non-empty clopen sets. Define $p_m = p_{m-1} \vee f_m$ and $q_m = q_{m-1} \setminus e_m$. Then since $e_m \cup f_m$ misses $U\{e_i \cup f_i : i < m\}$ and $e_m \cap f_m = \emptyset$, we have

$$p_1 < \dots < p_{m-1} < p_m < q_m < q_{m-1} < \dots < q_1.$$

The induction is now complete and yields an increasing sequence dominated by a decreasing sequence:

$$p_1 < \dots < p_m < \dots < q_m < \dots < q_1.$$

By DuBois-Reymond separability if the sequence is infinite or the density of the algebra for the finite case, there exists a clopen set d_2 such that

$$p_m < d_2 < q_m$$

for all m . Put $d = d_1 \vee d_2$. Because d_2 misses each e_m , d fails to contain all points of c_m^i for each m and we have:

(iv) $c_m^i \not\subseteq d$ for every m .

However, since d_2 contains each f_m , d contains some points of every c_m^i . Thus,

(v) $d \not\subseteq c_m^i$ for every m .

Since d_2 misses w_0 and w_1 , we see that d retains the

properties (i) and (ii) of d_1 above:

(vi) $d \not\subseteq c_m^t$ for all m .

(vii) $c_m \not\subseteq d$ for all m .

Therefore, d is the required set. $\blacksquare$

An investigation of the argument given in Parovičenko's paper will show that if there exists a c_n such that

$$a_1 < \dots < a_l < \dots < c_n < \dots < b_m < \dots < b,$$

and if $c_i = c_n^t$ for some i , then his argument will fail. Such a c_n may well exist since L is DuBois-Reymond separable.

The argument given here follows that of Parovičenko only as far as the construction of t_0 and t_1 .

EXERCISES

2A. LATTICE IDENTITIES

In any partially ordered set X , the operations of meet and join satisfy the following laws whenever the specified expressions exist:

1. $x \wedge x = x, \quad x \vee x = x.$
2. $x \wedge y = y \wedge x, \quad x \vee y = y \vee x.$
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z.$
4. $x \wedge (x \vee y) = x \vee (x \wedge y) = x.$

2B. LATTICE INEQUALITIES

1. In any partially ordered set X , the following are equivalent:

- (a) $x \leq y$
- (b) $x \wedge y = x$
- (c) $x \vee y = y.$

Thus, the partial order could be suppressed by replacing it with a statement involving meet or join.

2. In any lattice, if $x \leq y$, then $x \wedge z \leq y \wedge z$ and $x \vee z \leq y \vee z.$
3. In any lattice, the distributive inequalities are satisfied:

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$$

$$(x \vee y) \wedge (x \vee z) \geq x \vee (y \wedge z).$$

2C. DISTRIBUTIVE LATTICES

1. In any lattice, the following are equivalent for all elements x , y , and z :

$$(a) \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$(b) \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

[Use 2, 3, and 4 of 2A.]

2. In a distributive lattice, if $z \wedge x = z \wedge y$ and $z \vee x = z \vee y$, then $x = y$. [Use 2A.2, 2A.4 and part 1 above.]
3. In a distributive lattice, if an element has a complement, then the complement is unique.

2D. DEMORGAN'S LAWS

Let L be a Boolean algebra and let x , y , and z be arbitrary elements of L .

1. $(x')' = x$.
2. $x \wedge y = 0$ if and only if $x \leq y'$. [2B.2, 2B.3.]
3. The bijection $x \mapsto x'$ inverts order, i.e. $x \leq y$ implies $y' \leq x'$. [2B.2.]
4. DeMorgan's Laws are satisfied:

$$(x \vee y)' = x' \wedge y' \quad \text{and} \quad (x \wedge y)' = x' \vee y'.$$

2E. SUPREMA OF CLOPEN SETS

1. Let $\{U_\alpha\}$ be a family of clopen sets of a zero-dimensional space X . Then the supremum of $\{U_\alpha\}$ exists in $\text{CO}(X)$ and

is equal to $cl(U \cup U_\alpha)$ if and only if $cl(U \cup U_\alpha)$ is open.

[Proposition 2.5.]

2. State and prove the analogous result for infima in $CO(X)$.

2F. COUNTABLE NEIGHBORHOOD BASES

1. A space T is regular if every point has a basis of closed neighborhoods. If T is regular and X is dense in T , then a point of X has a countable neighborhood base in X if and only if it has a countable neighborhood base in T .
[2.14(e).]
2. If a point of x has a countable neighborhood base in X , it also has a countable neighborhood base in βX , and conversely.

2G. $A \mapsto A^*$

Let X be a space such that $[cl_X(X \setminus A)]^* = cl_{X^*}(X^* \setminus A^*)$, for every regular closed subset A of X . Then Proposition 2.16 shows that $A \mapsto A^*$ is a homomorphism of $R(X)$ to $R(X^*)$. The homomorphism is a monomorphism if and only if X admits no non-empty compact regular closed subset.

2H. MORE ONE POINT COMPACTIFICATION

Let D be an infinite discrete space and let L be the Boolean algebra consisting of the finite subsets of D and their complements.

1. Only one maximal filter in L is not determined by a point.
2. $S(L)$ is αD , the one-point compactification of D .
3. L is not dense in itself, Cantor separable, or DuBois-Reymond separable.

2I. BOOLEAN RINGS

A Boolean ring is a ring with identity in which $x^2 = x$ for every x .

1. Every Boolean ring is commutative. $[x + y = (x + y)^2.]$
2. A Boolean algebra L can be made into a Boolean ring by defining addition and multiplication as follows:

$$x + y = (x \wedge y') \vee (x' \wedge y)$$

$$xy = x \wedge y.$$

3. A Boolean ring R can be made into a Boolean algebra by defining meet and join as follows:

$$x \wedge y = xy \quad \text{and} \quad x \vee y = x + y + xy.$$

References: The relationships between Boolean algebras and Boolean rings were first explored by M. H. Stone, 1936, 1937. The Appendix of G. Simmons' book, 1963, contains a discussion of these relationships together with a proof of the Stone Representation Theorem which utilizes Boolean rings.

CHAPTER THREE:
ON $\beta\mathbb{N}$ AND $\mathbb{N}^*$

3.1. The Stone-Čech compactification of the discrete space $\mathbb{N}$ of natural numbers has become one of the most widely studied topological spaces for a number of reasons. First, $\beta\mathbb{N}$ and the growth $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ admit a seemingly endless flow of interesting subspaces. For example, $\beta\mathbb{N}$ contains a countably compact subspace whose square fails to be pseudocompact. On the other hand, there are 2^c subspaces of $\beta\mathbb{N}$ such that every finite power of each of the subspaces is pseudocompact and no two of the subspaces are homeomorphic. $\mathbb{N}^*$ has a basis of c clopen subspaces each of which is homeomorphic with $\mathbb{N}^*$ itself and yet $\mathbb{N}^*$ manages to contain 2^c disjoint copies of $\mathbb{N}^*$.

Despite the apparent pathologies of the spaces $\beta\mathbb{N}$ and $\mathbb{N}^*$, copies of these two spaces occur frequently in the Stone-Čech compactifications of other spaces and their presence greatly facilitates certain investigations of these compactifications as we shall see in Chapter 4.

Finally, despite the apparent complexity of $\beta\mathbb{N}$, some of its properties are easily derived from elementary facts about the natural numbers. The effect of this behavior has been to make $\beta\mathbb{N}$ something of a prototype, in that problems involving Stone-Čech compactifications are first examined and solved in this simple case, and then the arguments are adjusted to achieve more general solutions. One particularly evident example where

this has been done will be seen in Theorem 5.8 which can be viewed as a modification of the proof (in Theorem 3.22) that $\mathbb{N}^*$ admits a family of c disjoint open sets.

THE CARDINALITY OF $\beta\mathbb{N}$

3.2. In his fundamental 1937 paper, E. Čech used the cardinality of $\beta\mathbb{N}$ to estimate the cardinalities of various subspaces of Stone-Čech compactifications. However, at the time he knew only that the cardinality of $\beta\mathbb{N}$ was at least c and at most 2^c and he emphasized the importance of determining the exact cardinality. In the same year, B. Pospíšil answered the question in more general terms by showing that if D is the infinite discrete space of cardinality m , then the cardinality of βD is 2^{2^m} .

THEOREM: (B. Pospíšil)

If D is the infinite discrete space of cardinality m , then the cardinality of βD is 2^{2^m} .

Proof: Since the product of no more than 2^m spaces each admitting a dense subset of cardinality m or less also has a dense subset of the same cardinality [Hewitt, 1946, or Pondiczery, 1944, contain proofs, and D, p. 175 establishes the special case $m = \aleph_0$], D can be mapped onto a dense subspace of the product I^{2^m} . Such a mapping then extends to βD and the image of βD under the extension is compact and contains a dense subspace, and thus is all of I^{2^m} . Hence,

$$|\beta D| \geq |I^{2^m}| = c^{2^m} = 2^{2^m}.$$

On the other hand, D admits at most 2^{2^m} ultrafilters, each of which must converge to a point of βD . Hence, the cardinality of βD is at most 2^{2^m} . $\square$

3.3. COROLLARY:

The cardinality of $\beta \mathbb{N}$ is $2^{2^{\aleph_0}} = 2^c$.

The proof of Pospíšil's theorem given here is based on the proof given for $\beta \mathbb{N}$ in 1959 by S. Mrówka.

3.4. The first application which we will make of our knowledge of the cardinality of $\beta \mathbb{N}$ will be to show that every infinite closed subspace of $\beta \mathbb{N}$ contains a copy of $\beta \mathbb{N}$ and therefore has cardinality 2^c . This result is usually attributed to J. Novák in 1953B, but in that source, Novák remarks that the result was known to Čech as early as 1939. Again the proof is that of S. Mrówka.

THEOREM:

The cardinality of each infinite closed subset of $\beta \mathbb{N}$ is 2^c .

Proof: Let F be an infinite closed subspace of $\beta \mathbb{N}$. Then F contains a countable, discrete subspace E since F is an infinite Hausdorff space. $\mathbb{N} \cup E$ is a countable completely regular space and hence is regular and Lindelöf. But a regular Lindelöf space is paracompact and therefore is normal [D, pp. 163, 174]. Since every point of $\mathbb{N}$ is isolated, E is closed in $\mathbb{N} \cup E$,

and thus is C^* -embedded in the normal space $IN \cup E$. Any bounded continuous real-valued function on E can therefore be extended to $IN \cup E$, and from there will extend to βIN . Consequently, E is dense and C^* -embedded in $cl_{\beta IN} E$ which is contained in F so that $cl_{\beta IN} E$ is βE . Then since βE is homeomorphic to βIN , F contains a copy of βIN and must have cardinality 2^C . $\square$

3.5. Since every uncountable open subset of βIN must contain an infinite closed subset, we have the immediate

COROLLARY:

Every uncountable open subset of βIN has cardinality 2^C .

3.6. The proof of Theorem 3.4 contains the proof of the following interesting and important property of βIN

PROPOSITION:

Every countable subspace of βIN is C^* -embedded.

Note that the proposition implies that the closure in βIN of any discrete, countably infinite subspace of βIN is homeomorphic with βIN .

3.7. The cardinality of βIN can also be used to obtain a lower bound for the cardinality of any zero-set contained in the growth of any Stone-Ćech compactification.

THEOREM:

Every zero-set of βX which misses X contains a copy of $\mathbb{N}^*$ and therefore its cardinal is at least 2^c .

Proof: Let $Z(f)$ be a zero-set contained in $\beta X \setminus X$ and put $Y = \beta X \setminus Z(f)$. Then $h = (f|_Y)^{-1}$ is well-defined and continuous because f does not vanish on Y . Further, h is unbounded. Therefore, the range of h contains an unbounded (and hence closed) copy M of $\mathbb{N}$ and Y therefore contains a copy N of $\mathbb{N}$ which is mapped by h homeomorphically onto M .

(a) N is C -embedded in Y : Let g belong to $C(N)$. Since $\theta = (h|_N)$ is a homeomorphism, $g \circ \theta^{\leftarrow}$ belongs to $C(M)$. Since M is C -embedded in $\mathbb{R}$, there exists a mapping ι in $C(\mathbb{R})$ which extends $g \circ \theta^{\leftarrow}$. Then $\iota \circ h$ belongs to $C(Y)$ and for y in N ,

$$(\iota \circ h)(y) = (g \circ \theta^{\leftarrow} \circ \theta)(y) = g(y)$$

so that $\iota \circ h$ extends g to all of Y .

It now follows from Proposition 1.48 that $cl_{\beta Y} N = \beta N$.

(b) $\beta N \setminus N$ is contained in $Z(f)$: Since f approaches zero along N , $f(p) = 0$ for any p in $\beta N \setminus N$. But such a point p belongs to $Z(f)$.

Finally, since $\beta N \setminus N$ has the same cardinality as $\mathbb{N}^*$, the cardinality of $Z(f)$ is at least 2^c . $\square$

3.8. Since every point contained in a G_δ belongs to a zero-set which is also contained in the G_δ , a G_δ point is a zero-set. Hence the following corollary is immediate.

COROLLARY: (Čech)

No point of $\beta X \setminus X$ is a G_δ in βX .

The preceding two results are contained in [GJ, Chapter 9] together with a proof of Theorem 3.2 obtained by showing that an infinite discrete space of cardinality m admits 2^{2^m} ultrafilters and also a more comprehensive examination of cardinals of closed sets in βX . Čech showed in 1937 that a zero-set contained in a growth must contain a copy of $\mathbb{N}^*$ and derived Corollary 3.8, however, he did not yet know the cardinality of $\beta \mathbb{N}$.

THE CLOPEN SETS OF $\beta \mathbb{N}$ AND $\mathbb{N}^*$

3.9. Proposition 1.42 shows that there is a one-to-one correspondence between the free ultrafilters on $\mathbb{N}$ and the points of $\mathbb{N}^*$. Under this correspondence, a point p of $\mathbb{N}^*$ is identified with the unique free ultrafilter A^p on $\mathbb{N}$ which converges to p in $\beta \mathbb{N}$. A^p consists of precisely those subsets of $\mathbb{N}$ which have p in their closure in $\beta \mathbb{N}$. Thus, every neighborhood of p meets $\mathbb{N}$ in a set belonging to A^p . The points of $\mathbb{N}$ are isolated in $\beta \mathbb{N}$ and are the only isolated points of $\beta \mathbb{N}$ since $\mathbb{N}$ is dense in $\beta \mathbb{N}$.

If X is any space, the closure in βX of any clopen subspace S of X is easily seen to be clopen by observing that $cl_{\beta X} S$ is the set of points where the extension of the characteristic function of S is equal to 1. By applying this result to $\beta \mathbb{N}$,

we will see in the next proposition that the closures in $\beta\mathbb{N}$ of clopen sets of $\mathbb{N}$ separate points of $\beta\mathbb{N}$ so that $\beta\mathbb{N}$ is totally disconnected.

PROPOSITION:

$\beta\mathbb{N}$ is totally disconnected and therefore is also zero-dimensional.

Proof: Let p and q be distinct points of $\beta\mathbb{N}$ and choose a subset Z of $\mathbb{N}$ belonging to $\mathcal{A}^p$ but not to $\mathcal{A}^q$. Then $\text{cl}_{\beta\mathbb{N}} Z$ is a clopen neighborhood of p which misses q . $\beta\mathbb{N}$ is therefore totally disconnected since the only connected subspaces of $\beta\mathbb{N}$ are the singletons and is zero-dimensional since every compact totally disconnected space is zero-dimensional (Proposition 2.4). $\blacksquare$

3.10. Since every subset of $\mathbb{N}$ is closed in $\mathbb{N}$ and also C^* -embedded in $\mathbb{N}$, Proposition 1.48 shows that if A is an infinite subset of $\mathbb{N}$, then $\text{cl}_{\beta\mathbb{N}} A$ is homeomorphic with $\beta\mathbb{N}$ and $A^* = \text{cl}_{\beta\mathbb{N}} A \setminus A$ is homeomorphic with $\mathbb{N}^*$. Further, $\text{cl}_{\beta\mathbb{N}} A$ is clopen in $\beta\mathbb{N}$. The next proposition shows that every infinite clopen subspace of $\beta\mathbb{N}$ is of this form and hence is a copy of $\beta\mathbb{N}$.

PROPOSITION:

Every clopen subspace of $\beta\mathbb{N}$ is of the form $\text{cl}_{\beta\mathbb{N}} A$ for some subset A of $\mathbb{N}$.

Proof: Let U be a clopen subset of $\beta\mathbb{N}$. Then U is compact, and hence if U is contained in $\mathbb{N}$, U is finite and $U = \text{cl}_{\beta\mathbb{N}} U$.

If U meets $\mathbb{N}^*$, then $U \cap \mathbb{N}$ is non-empty since $\mathbb{N}$ is dense in $\beta\mathbb{N}$. Because U is closed, $\text{cl}_{\beta\mathbb{N}}(U \cap \mathbb{N})$ is contained in U . Since U is open, $U \setminus \text{cl}_{\beta\mathbb{N}}(U \cap \mathbb{N})$ is open in $\beta\mathbb{N}$ and misses $\mathbb{N}$. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, this difference must be empty and $U = \text{cl}_{\beta\mathbb{N}}(U \cap \mathbb{N})$. $\blacksquare$

3.11. Since $\beta\mathbb{N}$ has a base of clopen subsets, we are now able to state the following improvement of Corollary 3.5:

COROLLARY:

Every open set of $\beta\mathbb{N}$ which meets $\mathbb{N}^*$ has cardinality 2^c .

3.12. Subspaces of $\beta\mathbb{N}$ have yielded a wealth of interesting examples. In their 1929 memoir on compact spaces, P. Alexandroff and P. Urysohn posed the following question: Does there exist a compact Hausdorff space with no isolated points and such that no point is a limit of a sequence of distinct points? Čech showed in 1937 that the space $\mathbb{N}^*$ is such a space. Recall from 2.11 that a point which is a limit point of a sequence of distinct points is called a κ -point.

PROPOSITION:

$\mathbb{N}^*$ is a compact Hausdorff space containing no isolated points and no κ -points.

Proof: $\mathbb{N}^*$ is clearly a compact Hausdorff space since it is a closed subspace of $\beta\mathbb{N}$. $\mathbb{N}^*$ cannot contain an isolated point since an open subset of $\beta\mathbb{N}$ which meets $\mathbb{N}^*$ contains 2^c points. If a point p of $\mathbb{N}^*$ is a limit of the sequence $\{x_n\}$,

then $\{x_n\} \cup \{p\}$ is a countably infinite closed subspace of $\beta\mathbb{N}$ which is impossible by Theorem 3.4. $\square$

3.13. Since $\mathbb{N}$ is normal, any two disjoint subsets of $\mathbb{N}$ have disjoint closures in $\beta\mathbb{N}$ by Corollary 1.15. Thus, if A is an infinite subset of $\mathbb{N}$, we have that

$$[cl_{\mathbb{N}}(\mathbb{N} \setminus A)]^* = cl_{\mathbb{N}^*}(\mathbb{N}^* \setminus A^*).$$

By Proposition 2.16, this equation shows that the assignment

$$A \mapsto A^*$$

is a homomorphism of the Boolean algebra of regular closed subsets of $\mathbb{N}$ to the Boolean algebra of regular closed subsets of $\mathbb{N}^*$. We now describe this homomorphism in detail and show that its image is the Boolean algebra of clopen subsets of $\mathbb{N}^*$.

3.14. PROPOSITION:

If A and B are infinite subsets of $\mathbb{N}$, then B^* is contained in A^* if and only if $B \setminus A$ is finite.

Proof: To show sufficiency, write $B = (B \cap A) \cup (B \setminus A)$ and assume that $B \setminus A$ is finite. Then if $B \cap A$ is finite, B is finite and B^* is empty. Thus, we can assume that $B \cap A$ is infinite.

Then:

$$B^* = (cl_{\beta\mathbb{N}}(B \cap A) \cup cl_{\beta\mathbb{N}}(B \setminus A)) \setminus B = cl_{\beta\mathbb{N}}(B \cap A) \setminus B$$

since $B \setminus A$ is its own closure. Hence, B^* is contained in A^* .

To establish necessity, assume that $B \setminus A$ is infinite and

exhibit a point in $B^* \setminus A^*$. Since $B \setminus A$ is infinite and misses A , it belongs to some free ultrafilter $\mathcal{A}^p$ which does not contain A . (See 3.9.) Then p belongs to B^* but not to A^* . $\square$

3.15. COROLLARY:

If A and B are infinite subsets of $\mathbb{N}$, then $A^* = B^*$ if and only if $(A \setminus B) \cup (B \setminus A)$ is finite. B^* is properly contained in A^* if and only if $A \setminus B$ is infinite and $B \setminus A$ is finite. A^* meets B^* if and only if $A \cap B$ is infinite.

3.16. We will now see that every clopen set of $\mathbb{N}^*$ is an image under the homomorphism described in Section 3.13 and that sets of the form A^* constitute a base for both the open and the closed sets of $\mathbb{N}^*$.

PROPOSITION:

Every clopen subspace of $\mathbb{N}^*$ is of the form A^* for some infinite subset A of $\mathbb{N}$. The sets A^* form a base for both the open and the closed sets of $\mathbb{N}^*$.

Proof: Let U be a clopen subspace of $\mathbb{N}^*$. Since $\mathbb{N}^*$ is C^* -embedded in $\beta\mathbb{N}$, the characteristic function of U as a subspace of $\mathbb{N}^*$ is the restriction of some mapping f in $C^*(\beta\mathbb{N})$. Put $A = \{n \in \mathbb{N} : f(n) \geq \frac{1}{2}\}$. Then U is contained in A^* and $\mathbb{N}^* \setminus U$ is contained in $(\mathbb{N} \setminus A)^*$. Since $\mathbb{N}^*$ is the disjoint union of A^* and $(\mathbb{N} \setminus A)^*$, U is equal to A^* .

$\mathbb{N}^*$ is zero-dimensional since $\beta\mathbb{N}$ is zero-dimensional. Thus, every open set of $\mathbb{N}^*$ is a union of sets of the form A^* by the first statement of the proposition. By taking complements,

every closed set of $\mathbb{N}^*$ is seen to be the intersection of sets of the same form. $\square$

3.17. Since $\mathbb{N}$ has c infinite subsets and only c subsets in all, the above propositions establish the cardinalities of bases for both $\beta\mathbb{N}$ and $\mathbb{N}^*$.

COROLLARY:

$\beta\mathbb{N}$ and $\mathbb{N}^*$ each have a base consisting of c clopen sets.

3.18. The set of automorphisms of $\beta\mathbb{N}$ clearly forms a group under composition. The orbit of a point p of $\beta\mathbb{N}$ is the set of points of $\beta\mathbb{N}$ which are images of p under the automorphisms of $\beta\mathbb{N}$. Knowing the form of the clopen sets of $\beta\mathbb{N}$ and $\mathbb{N}^*$ allows a partial description of the orbit of a point of $\mathbb{N}^*$. Since any automorphism must exchange isolated points of the space, it is clear that all automorphisms of $\beta\mathbb{N}$ are extensions of permutations of $\mathbb{N}$. Thus, since there are c permutations of $\mathbb{N}$, there are just c automorphisms of $\beta\mathbb{N}$. A space is called homogeneous if for every pair of points of the space, there is an automorphism of the space which exchanges the pair of points. Since $\beta\mathbb{N}$ contains both isolated and non-isolated points, it is immediate that $\beta\mathbb{N}$ is not homogeneous.

As we have also seen that $\beta\mathbb{N}$ admits just c automorphisms but contains 2^c points, the non-homogeneity of $\beta\mathbb{N}$ is also clear from cardinality considerations. Observe that this does not imply that $\mathbb{N}^*$ is not homogeneous since there may be automorphisms of $\mathbb{N}^*$ which are not obtained from permutations

of $\mathbb{N}$. The question of the non-homogeneity of $\mathbb{N}^*$ is quite complex and will be explored from a cardinality viewpoint later in the chapter.

3.19. Although the orbit of any point in $\mathbb{N}^*$ under auto-homeomorphisms of $\beta\mathbb{N}$ contains at most c points, we can show that the orbit is dense in $\mathbb{N}^*$.

PROPOSITION:

If A^* and B^* are proper clopen subspaces of $\mathbb{N}^*$, there exists an automorphism of $\beta\mathbb{N}$ and hence of $\mathbb{N}^*$ carrying A^* onto B^* .

Proof: Since A^* and B^* are both proper subsets of $\mathbb{N}^*$, the complements in $\mathbb{N}$ of the associated infinite sets A and B are both infinite. Let σ be a permutation of $\mathbb{N}$ which carries A onto B and $\mathbb{N}\setminus A$ onto $\mathbb{N}\setminus B$. Then the extension $\beta(\sigma)$ of σ to $\beta\mathbb{N}$ is an automorphism and carries A^* onto B^* by continuity. $\square$

3.20. Since the sets A^* form a base for the open sets of $\mathbb{N}^*$, the following is immediate:

COROLLARY:

The orbit of any point of $\mathbb{N}^*$ is a dense subspace of $\mathbb{N}^*$.

The preceding sequence of results is based on [GJ, 6.10(a) and ex. 6S]. The cited exercise in [GJ] has its origin in the 1956 paper of W. Rudin and the 1953A paper of J. Novák.

3.21. The orbit of a point in $\mathbb{N}^*$ under automorphisms of $\beta\mathbb{N}$ is a dense subset of $\mathbb{N}^*$ of cardinality c . As an application of properties of the basic sets A^* of $\mathbb{N}^*$, we will see that the next result concerning subsets of $\mathbb{N}$ implies that every dense subset of $\mathbb{N}^*$ must contain at least c points. The proof is from [GJ, ex. 6Q]. A family of sets is said to be almost disjoint if the intersection of any two of the sets is finite.

PROPOSITION:

$\mathbb{N}$ admits a family of c almost disjoint infinite subsets.

Proof: Consider a one-to-one mapping ϕ of $\mathbb{N}$ onto $\mathbb{Q}$.

Select an increasing sequence $\{q_n\}$ of rationals converging to each irrational. For each such sequence, define

$$E = \{\phi^{-1}(q_n)\}$$

and let $\mathcal{E}$ be the collection of all such sets. Then it is clear that $\mathcal{E}$ has c members and that the intersection of any two members is finite. $\square$

3.22. The cellularity of a topological space Y is the smallest cardinal number m for which each pairwise disjoint family of open sets of Y has m or fewer members. The family of c almost disjoint infinite subsets of $\mathbb{N}$ which exist by the preceding proposition will yield a family of c disjoint open subsets of $\mathbb{N}^*$ and thus provide a lower bound for the cellularity of $\mathbb{N}^*$. Actually, this is the largest possible such family.

THEOREM:

The cellularity of $\mathbb{N}^*$ is c .

Proof: Consider the family $\mathcal{E}$ of c almost disjoint infinite subsets of $\mathbb{N}$ provided by the preceding proposition. For distinct members E and F of $\mathcal{E}$, $cl_{\beta\mathbb{N}} E \cap cl_{\beta\mathbb{N}} F \cap \mathbb{N}^* = \emptyset$ since any point in this intersection would belong to the closure of $E \cap F$ which is just $E \cap F$ since $E \cap F$ is finite. Thus, $\{E^* : E \in \mathcal{E}\}$ is a family of c pairwise disjoint non-void open subsets of $\mathbb{N}^*$.

On the other hand, it is clear that there can be no such family of larger cardinality since the power set of $\mathbb{N}$ has cardinality c . $\square$

Thus, any dense subset of $\mathbb{N}^*$ must contain at least c points.

3.23. The previous theorem shows that $\mathbb{N}^*$ contains a family of c disjoint clopen subsets. Thus, a copy of $\mathbb{N}$ may be obtained in $\mathbb{N}^*$ by choosing a point from each of countably many members of such a family of clopen sets. By Proposition 3.6, the copy of $\mathbb{N}$ just obtained is C^* -embedded in $\mathbb{N}^*$ and its closure is homeomorphic with $\beta\mathbb{N}$. Thus, we have shown again a result which was stated earlier in the proof of Theorem 3.4.

COROLLARY:

$\mathbb{N}^*$ contains a copy of $\beta\mathbb{N}$.

3.24. Since $\mathbb{N}$ is both σ -compact and locally compact, Proposition 1.62 implies that $\mathbb{N}^*$ is an F -space. However,

$\mathbb{N}^*$ actually satisfies a stronger condition in that disjoint cozero-sets are separated by a partition of $\mathbb{N}^*$, i.e. they are contained in disjoint clopen sets whose union is all of $\mathbb{N}^*$. Theorem 1.60 requires only that disjoint cozero-sets be completely separated. We will shortly see an example which will show that the same statement fails for arbitrary open sets by showing that disjoint open sets of $\mathbb{N}^*$ need not have disjoint closures.

PROPOSITION:

Two disjoint co-zero sets of $\mathbb{N}^*$ are separated by a partition and hence $\mathbb{N}^*$ is an F-space.

Proof: Since $\mathbb{N}^*$ is C^* -embedded in $\beta\mathbb{N}$, disjoint co-zero sets G and H of $\mathbb{N}^*$ are traces on $\mathbb{N}^*$ of co-zero sets $Cz(g)$ and $Cz(h)$ of $\beta\mathbb{N}$, respectively. Since G and H are disjoint, either $Cz(g) \cap Cz(h)$ is finite or both g and h take on arbitrarily small values on the intersection. Choose $U = Cz(g) \setminus Cz(h)$ and $V = \mathbb{N} \setminus U$. Then G is contained in U^* , H is contained in V^* , and $U^* \cup V^* = \mathbb{N}^*$ and $\mathbb{N}^*$ is partitioned as required.

Since sets which are separated by a partition are completely separated, disjoint cozero-sets of $\mathbb{N}^*$ are completely separated and $\mathbb{N}^*$ is therefore an F-space. $\blacksquare$

3.25. Since $\mathbb{N}^*$ is also totally disconnected and compact, the following corollary is immediate from Proposition 2.23.

COROLLARY:

The Boolean algebra of clopen subsets of $\mathbb{N}^*$ is DuBois-Reymond separable.

3.26. The next result will ultimately lead to a proof that the Boolean algebra of clopen sets of $\mathbb{N}^*$ also is Cantor separable.

PROPOSITION:

If a sequence of clopen subsets of $\mathbb{N}^*$ has the finite intersection property, then the intersection of the sequence of sets contains a non-empty clopen set.

Proof: Let $\{A_n^*\}$ be a sequence of clopen sets of $\mathbb{N}^*$ having the finite intersection property. Then we may assume that the sequence is nested, i.e., that

$$A_1^* \supset A_2^* \supset \dots \supset A_n^* \supset \dots$$

and therefore that the cardinality of $A_{n+1}^* \setminus A_n^*$ is finite for all n . Choose a sequence of distinct points $A = \{x_n\}$ such that x_n belongs to $\bigcap_{i=1}^n A_i^*$. Then $|A \setminus A_n^*| \leq n-1$ for each n so that A^* is contained in A_n^* and A^* is contained in $\bigcap A_n^*$. $\parallel$

3.27. COROLLARY:

Every non-empty G_δ in $\mathbb{N}^*$ has non-empty interior.

3.28. COROLLARY:

The zero-sets of $\mathbb{N}^*$ are regular closed sets.

Proof: Since a zero-set is a G_δ , the preceding corollary shows that a non-empty zero-set of $\mathbb{N}^*$ has non-empty interior. Now

assume that a zero-set Z_1 of $\mathbb{N}^*$ is not regular closed, i.e. that there is a point p in $Z_1 \setminus \text{cl}_{\mathbb{N}^*} \text{int}_{\mathbb{N}^*} Z_1$. Then there exists a zero-set neighborhood Z_2 of p such that Z_2 misses $\text{int}_{\mathbb{N}^*} Z_1$. Thus, $Z_1 \cap Z_2$ is a zero-set of $\mathbb{N}^*$ which is contained in the boundary of Z_1 and hence has empty interior. But this is a contradiction of the assumption that Z_1 is not a regular closed set. $\blacksquare$

3.29. All of the hypotheses of Proposition 2.22 have now been verified, and we can make the following observation:

COROLLARY:

The Boolean algebra of clopen subsets of $\mathbb{N}^*$ is Cantor separable.

3.30. We have seen in Proposition 3.6 that every countable subspace of $\mathbb{N}^*$ is C^* -embedded in $\mathbb{N}^*$. Further, $\mathbb{N}^*$ contains many copies of $\mathbb{N}^*$ and $\beta\mathbb{N}$, each of which is C^* -embedded because each is compact. (Proposition 1.47.) However, we now see that no dense subspace of $\mathbb{N}^*$ is C^* -embedded and therefore, $\mathbb{N}^*$ is not the Stone-Čech compactification of any of its dense subspaces. The following proposition appears in L. Gillman's 1967 paper in which he discusses results about the spaces $\mathbb{N}^*$ and $\mathbb{R}^*$ which have been obtained with the aid of the Continuum Hypothesis. It is a good example of the way in which the Continuum Hypothesis arises naturally in considerations involving $\mathbb{N}^*$. (Results which use the Continuum Hypothesis will be indicated by [CH].)

PROPOSITION [CH]:

Dense subsets of $\mathbb{N}^*$ are not C^* -embedded.

Proof: By Proposition 1.49, it is sufficient to show that $\mathbb{N}^* \setminus \{p\}$ is not C^* -embedded in $\mathbb{N}^*$. We will show that $\mathbb{N}^* \setminus \{p\}$ is the union of disjoint open sets A and B each of which contains p in its closure. Thus, there exists a two-valued mapping on $\mathbb{N}^* \setminus \{p\}$ which will not extend continuously to $\mathbb{N}^*$.

Assuming the Continuum Hypothesis, the basis of c zero-set neighborhoods of p can be indexed by ω_1 and written $\{Z_\alpha\}_{\alpha < \omega_1}$. Proceeding by transfinite induction, assume for a given $\alpha < \omega_1$ that cozero sets A_σ and B_σ have been defined for all $\sigma < \alpha$ such that

$$p \notin A_\gamma \cup B_\tau \quad \text{and} \quad A_\gamma \cap B_\tau = \emptyset$$

for all $\gamma, \tau < \alpha$. Since a countable union of cozero sets is a cozero set, Proposition 3.24 shows that there exist complementary clopen sets A'_α and B'_α such that

$$\bigcup_{\sigma < \alpha} A_\sigma \subset A'_\alpha \quad \text{and} \quad \bigcup_{\sigma < \alpha} B_\sigma \subset B'_\alpha.$$

The set $Z_\alpha \cap (\bigcap \{\mathbb{N}^* \setminus (A_\sigma \cup B_\sigma) : \sigma < \alpha\})$ contains p and thus is a non-void zero-set of $\mathbb{N}^*$ and has a non-void interior by Corollary 3.28. Since $\mathbb{N}^*$ contains no isolated points, the set

$$Z_\alpha \cap (\bigcap \{\mathbb{N}^* \setminus (A_\sigma \cup B_\sigma) : \sigma < \alpha\}) \setminus \{p\}$$

contains disjoint non-void cozero-sets A''_α and B''_α . Now define

$$A_\alpha = (A'_\alpha \setminus Z_\alpha) \cup A''_\alpha \quad \text{and} \quad B_\alpha = (B'_\alpha \setminus Z_\alpha) \cup B''_\alpha.$$

Then A_α and B_α are disjoint cozero sets both of which fail to contain p and the induction hypothesis is satisfied for all $\sigma, \tau \leq \alpha$.

Now define

$$A = \bigcup_{\alpha < \omega_1} A_\alpha \quad \text{and} \quad B = \bigcup_{\alpha < \omega_1} B_\alpha.$$

A and B are disjoint open sets, and neither contains p . If q is a point of $\mathbb{N}^*$ other than p , then some neighborhood Z_α of p misses q so that q is in $A_\alpha \cup B_\alpha$ by construction. Hence, $A \cup B = \mathbb{N}^* \setminus \{p\}$. Since each basic neighborhood Z_α of p contains $A_\alpha \cup B_\alpha$, every neighborhood of p meets both A and B so that p is in the closures of both A and B . $\square$

In Proposition 3.24 we saw that disjoint cozero-sets of $\mathbb{N}^*$ are separated by a partition. The preceding proof shows that it was necessary to assume in Proposition 3.24 that the sets were cozero-sets since A and B above are disjoint open sets of $\mathbb{N}^*$ which not only are not separated by a partition but also fail to have disjoint closures.

A CHARACTERIZATION OF $\mathbb{N}^*$

3.31. Among the properties satisfied by $\mathbb{N}^*$ and its Boolean algebra of clopen sets are six which characterize $\mathbb{N}^*$ as a topological space. $\mathbb{N}^*$ is clearly compact and in Proposition 3.9 we saw that it is totally disconnected. Corollaries 3.25 and 3.29 establish that the Boolean algebra of clopen sets of $\mathbb{N}^*$ is both DuBois-Reymond and Cantor separable. The weight of a space

is the least cardinal number of a basis for the space, and thus Corollary 3.17 shows that $\mathbb{N}^*$ has weight c . Corollary 3.11 established that $\mathbb{N}^*$ has no isolated points and this result together with Proposition 2.19 shows that the Boolean algebra of clopen sets is dense in itself. In 1963, I. I. Parovičenko showed that in the presence of the Continuum Hypothesis, these six properties characterize the space $\mathbb{N}^*$. The Continuum Hypothesis is used in the proof to index the c clopen sets of $\mathbb{N}^*$ by the countable ordinals. The proof is a modification of an argument used in 1956 by W. Rudin and we will see the argument in its original context later in Theorem 7.6.

In addition, without using the Continuum Hypothesis, Parovičenko showed that $\mathbb{N}^*$ maps continuously onto every compact space having weight at most $\aleph_1$. We will see in Chapter 6 that this result will aid in the description of the growths of other compactifications of $\mathbb{N}$.

THEOREM [CH]: (Parovičenko)

A totally disconnected compact space Y without isolated points and such that $CO(Y)$ is both Cantor and DuBois-Reymond separable is homeomorphic to $\mathbb{N}^*$. Further, $\mathbb{N}^*$ maps continuously onto any compact space having weight at most $\aleph_1$.

Proof: The first statement will be proven first. We have just seen that $\mathbb{N}^*$ satisfies all of the conditions stated in the hypothesis. The proof will be accomplished by constructing a Boolean algebra isomorphism σ of $CO(Y)$ onto $CO(\mathbb{N}^*)$. Then our remarks following the Stone Representation Theorem, 2.10,

show that σ induces a homeomorphism of the respective Stone spaces. But because both $\mathbb{N}^*$ and Y are compact and totally disconnected, they are easily seen to be homeomorphic to the Stone spaces of $\text{CO}(\mathbb{N}^*)$ and $\text{CO}(Y)$, respectively, so that $\mathbb{N}^*$ and Y are therefore homeomorphic.

It remains to construct the isomorphism σ . We will accomplish the construction by transfinite induction. The induction will be carried out by showing that if σ has been defined for a countable field of clopen subsets of Y which does not include a clopen set U , then because σ preserves the Boolean operations, σ can be extended to a countable field containing U . We need only consider countable fields because of the assumption of the Continuum Hypothesis.

Using the Continuum Hypothesis, we can index the c members of the respective Boolean algebras by the countable ordinals, i.e. we can write $\text{CO}(Y) = \{U_\alpha : \alpha < \omega_1\}$ and $\text{CO}(\mathbb{N}^*) = \{V_\alpha : \alpha < \omega_1\}$. We can also accomplish the indexing so that $U_0 = Y$ and $V_0 = \mathbb{N}^*$. Now define $\sigma(U_0) = V_0$ and $\sigma(\emptyset) = \emptyset$ so that the family for which σ is initially defined forms a field. Now assume that σ has been defined for a countable field C contained in $\text{CO}(Y)$ and that α is the least ordinal for which U_α has not been defined. Write C as the union of the families $\{F_i\}$, $\{G_i\}$, and $\{C_i\}$ such that $F_i \subset U_\alpha$, $U_\alpha \subset G_i$, and no inclusion relation holds between C_i and U_α for all values of i . Put $A_n = F_1 \cup \dots \cup F_n$ and $B_n = G_1 \cap \dots \cap G_n$ so that

$$A_1 \subset \dots \subset A_n \subset \dots \subset U_\alpha \subset \dots \subset B_n \subset \dots \subset B_1,$$

and no C_i is contained in any A_n and no C_i contains any B_n . Because σ preserves containment, we have

$$\sigma(A_1) \subset \dots \subset \sigma(A_n) \subset \dots \subset \sigma(B_n) \subset \dots \subset \sigma(B_1)$$

and no $\sigma(C_i)$ is contained in any $\sigma(A_n)$ and no $\sigma(C_i)$ contains any $\sigma(B_n)$. Now by Lemma 2.26, there exists V_β in $CO(\mathbb{N}^*)$ such that no inclusion relation holds between any $\sigma(C_i)$ and V_β and

$$\sigma(A_1) \subset \dots \subset \sigma(A_n) \subset \dots \subset V_\beta \subset \dots \subset \sigma(B_n) \subset \dots \subset \sigma(B_1).$$

Define $\sigma(U_\alpha) = V_\beta$ and $\sigma(Y \setminus U_\alpha) = \mathbb{N}^* \setminus V_\beta$, and let C' be the field generated by C and U_α . The requirement that σ preserve the Boolean algebra operations dictates a unique extension of σ to C' . This accomplished, $\sigma^{\leftarrow}$ satisfies the same induction hypothesis and we can define $\sigma^{\leftarrow}(V_\delta)$ where δ is the least ordinal such that V_δ is not in the range of σ . The process is continued to yield the required Boolean algebra isomorphism.

We now establish the second statement of the theorem. Observe that in the construction of the isomorphism σ above, we required all of the hypothesis on both $\mathbb{N}^*$ and Y in order to apply Lemma 2.26. The hypotheses on $\mathbb{N}^*$ were required to construct σ and on Y to construct $\sigma^{\leftarrow}$. If we did not wish to construct an inverse, then $CO(Y)$ could be replaced with any Boolean algebra L having cardinality at most $\aleph_1$ and the construction would yield a one-to-one homomorphism of L into $CO(\mathbb{N}^*)$. Thus, if X is a compact space having weight at most $\aleph_1$, since X

has a base of regular closed sets there is a one-to-one homomorphism σ of $R(X)$ into $CO(\mathbb{N}^*)$. We will use this homomorphism to construct a mapping of $\mathbb{N}^*$ onto X .

For each point p of $\mathbb{N}^*$, the members of $CO(\mathbb{N}^*)$ containing p form a maximal filter. Let $\mathfrak{F} = \{\sigma^{\leftarrow}(U) : p \in U\}$. Since σ is a monomorphism, Proposition 2.8 implies that $\sigma^{\leftarrow}(U) \wedge \sigma^{\leftarrow}(V) \neq 0$ for any two members of $\mathfrak{F}$. Since $\mathfrak{F}$ is clearly closed under supersets in $R(X)$, $\mathfrak{F}$ is a filter. Further, if F in $R(X)$ is such that $F \wedge \sigma^{\leftarrow}(U) \neq 0$ for all members $\sigma^{\leftarrow}(U)$ of $\mathfrak{F}$, then $\sigma(F) \wedge U \neq 0$ for all U containing p . Hence, $\sigma(F)$ contains p and F belongs to $\mathfrak{F}$. Thus, $\mathfrak{F}$ is a maximal filter. Since X has a base of regular closed sets and is compact, $\bigcap \mathfrak{F}$ is a singleton. Hence, we can define a function f from $\mathbb{N}^*$ to X by putting $f(p) = \bigcap \mathfrak{F}$. The function f is onto since the image of every maximal filter in $R(X)$ has non-empty intersection in $\mathbb{N}^*$.

We now show that f is continuous. Let U be a neighborhood of x in X . Then there exists an open neighborhood V of x such that

$$x \in V \subset cl V \subset U.$$

Since $cl V$ is regular closed, $\sigma(cl V) = W$ with W in $CO(\mathbb{N}^*)$. For q in W , $f(q)$ belongs to $\sigma^{\leftarrow}(W) = cl V$. Hence, $f[W]$ is contained in U , and f is continuous. $\blacksquare$

3.32. By Proposition 2.22, the condition of Cantor separability in the theorem can be replaced by requiring that the zero-sets be regular closed sets. Proposition 2.23 shows that the requirement that the space be an F -space can replace DuBois-Reymond separability. Thus, we can restate Parovičenko's characterization without involving the separability conditions. This restatement appears in R. G. Woods 1971B paper.

THEOREM [CH]:

A totally disconnected compact F -space without isolated points and having weight c and such that every zero-set is a regular closed set is homeomorphic with $\mathbb{N}^*$.

3.33. EXAMPLES:

Each of the four conditions in addition to total disconnectivity and compactness is necessary in the above characterization. There exists a totally disconnected compact space which satisfies any three of the conditions but not the fourth.

(a) Consider the disjoint union $\mathbb{N}^* \cup \{x\}$ of $\mathbb{N}^*$ and an isolated point. $\mathbb{N}^* \cup \{x\}$ is easily seen to satisfy all the requirements except the non-existence of isolated points.

(b) Let D be the discrete space of cardinality 2^c and consider $D^* = \beta D \setminus D$. D^* is clearly compact and is easily seen to be totally disconnected by a proof similar to that of Proposition 3.9. Since D^* is C^* -embedded in the F -space βD , it is itself an F -space. By Lemma 4.21 the zero-sets of D^* are regular closed sets. However, an argument similar to that of Theorem 3.22 will show in Chapter 5 that the weight

of D^* is strictly greater than c .

(c) Consider a strictly increasing sequence $\{A_n^*\}$ of clopen subsets of $\mathbb{N}^*$ and put $X = \text{cl}_{\mathbb{N}^*}(\bigcup A_n^*)$. X is clearly totally disconnected, compact, and without isolated points. Since X is C^* -embedded in the F -space $\mathbb{N}^*$, X is an F -space and the weight of X is easily seen to be c . Since X is the supremum of $\{A_n^*\}$ in the algebra of clopen sets of $\mathbb{N}^*$, the clopen sets of X cannot have Cantor separability and hence the zero-sets of X are not all regular closed sets.

(d) Finally, $\mathbb{N}^* \times \mathbb{N}^*$ fails to satisfy only the F -space condition. $\mathbb{N}^* \times \mathbb{N}^*$ is clearly totally disconnected, compact, and without isolated points and it is easy to see that $\mathbb{N}^* \times \mathbb{N}^*$ has weight c . By Proposition 1.66, if the product of two F -spaces is an F -space, one of the spaces must be a P -space. Since a compact P -space must be finite, $\mathbb{N}^* \times \mathbb{N}^*$ fails to be an F -space. It remains to show either that the zero-sets of $\mathbb{N}^* \times \mathbb{N}^*$ are regular closed or, from the point of view of Theorem 3.31, that the clopen sets of $\mathbb{N}^* \times \mathbb{N}^*$ have Cantor separability. We will show the latter. Let $\{A_n\}$ be a strictly increasing sequence of clopen subsets of $\mathbb{N}^*$ which is dominated by a clopen subset B , i.e.

$$A_1 \subset \dots \subset A_n \subset \dots \subset B.$$

A basic open set of $\mathbb{N}^* \times \mathbb{N}^*$ is a rectangle $R = U \times V$ formed by the product of two basic clopen sets of $\mathbb{N}^*$. Since B is compact, B is the union of finitely many basic clopen rectangles,

$$B = \bigcup_{1 \leq i \leq n} R_i,$$

and we can assume that the R_i are mutually disjoint. Then each member of the sequence $\{A_n\}$ can be written

$$A_n = \bigcup_{1 \leq i \leq n} (A_n \cap R_i).$$

Since a projection map is always open and the projection parallel to a compact factor is also closed [D, p. 227], for $j = 1, 2$ and $i = 1, \dots, n$, we have that

$$\pi_j[A_1 \cap R_i] \subseteq \dots \subseteq \pi_j[A_n \cap R_i] \subseteq \dots \subseteq \pi_j[R_i],$$

is a sequence of clopen sets of $\mathbb{N}^*$. The properties of the clopen sets of $\mathbb{N}^*$ can now be applied to each of these sequences. There are three cases. First, if infinitely many of the containments of the sequence are proper, then the Cantor separability of the clopen sets of $\mathbb{N}^*$ yields a clopen subset $C_{j,i}$ of $\mathbb{N}^*$ such that

$$\pi_j[A_n \cap R_i] \subset C_{j,i} \subset \pi_j[R_i]$$

for every $n = 1, 2, \dots$. Second, if the sequence has only finitely many distinct elements but $\pi_j[A_n \cap R_i]$ is never equal to $\pi_j[R_i]$ for any n , then the density of the algebra of clopen sets provides a clopen set $C_{j,i}$ such that

$$\pi_j[A_n \cap R_i] \subset C_{j,i} \subset \pi_j[R_i]$$

for every $n = 1, 2, \dots$. Finally, if $\pi_j[A_n \cap R_i] = \pi_j[R_i]$ for some n , then put $C_{j,i} = \pi_j[R_i]$.

The hypothesis on the original sequence in $\mathbb{N}^* \times \mathbb{N}^*$ make it impossible that the final case can occur for every choice of j and i . Thus, we have that

$$A_1 \subset \dots \subset A_n \subset \dots \subset \bigcup_{1 \leq i \leq n} (C_{1,i} \times C_{2,i}) \subset \bigcup_{1 \leq i \leq n} R_i = B$$

and the clopen sets of $\mathbb{N}^* \times \mathbb{N}^*$ have Cantor separability.

(e) Note that (a), (c), and (d) show that no two of the conditions dense in itself, Cantor separable, and DuBois-Reymond separable imply the third. Also note that (d) shows that the converse to Proposition 2.21 is false. No sequence in $\mathbb{N}^* \times \mathbb{N}^*$ can converge, although $\mathbb{N}^* \times \mathbb{N}^*$ fails to be DuBois-Reymond separable.

3.34. The following proposition characterizes the class of spaces which have zero-dimensional Stone-Čech compactifications, and, together with Parovičenko's characterization of $\mathbb{N}^*$, it will allow us to recognize certain homeomorphs of $\mathbb{N}^*$. A space is called strongly zero-dimensional if for every pair of disjoint zero-sets of the space, there is a clopen set containing one zero-set and missing the other. Any discrete space is clearly strongly zero-dimensional.

PROPOSITION:

A space X is strongly zero-dimensional if and only if βX is zero-dimensional (or equivalently, βX is totally disconnected).

Proof: The equivalence of the two conditions on βX is

Proposition 2.4. Let X be strongly zero-dimensional. If p and q are distinct points of βX , then we can choose disjoint zero-set neighborhoods Z_1 and Z_2 of p and q , respectively, so that $Z_1 \cap X$ and $Z_2 \cap X$ are disjoint zero-sets of X . Then there exists a clopen set U containing $Z_1 \cap X$ and missing $Z_2 \cap X$. Thus, $\text{cl}_{\beta X} U$ is a clopen set containing p but not q so that βX is totally disconnected.

Conversely, assume that Z_1 and Z_2 are disjoint zero-sets of X and that βX is zero-dimensional. Then $\text{cl}_{\beta X} Z_1$ and $\text{cl}_{\beta X} Z_2$ are disjoint closed sets of βX and Proposition 2.13 shows that X is strongly zero-dimensional since disjoint closed sets in a zero-dimensional Lindelöf space are separated by a partition. $\blacksquare$

3.35. Since any point has a base of cozero-set neighborhoods and every neighborhood contains a zero-set containing the point, it is easy to see that a strongly zero-dimensional space is zero-dimensional. Proposition 2.13 shows that the converse is true if the space is also Lindelöf. Thus, we have proven the

PROPOSITION:

Zero dimensionality and strong zero-dimensionality are equivalent in Lindelöf spaces.

3.36. In metric spaces, Lindelöf and second countability are equivalent [D, p. 187]. Hence, every subspace of the line $\mathbb{R}$ is Lindelöf since second countability is an hereditary property. It is easy to see that a subspace of $\mathbb{R}$ has a base of clopen

sets if it contains no intervals. Since no interval can be zero-dimensional, we have characterized the strongly zero-dimensional subspaces of $\mathbb{R}$:

PROPOSITION:

A subspace of $\mathbb{R}$ is strongly zero-dimensional if and only if it contains no intervals.

3.37. Parovičenko's characterization of $\mathbb{N}^*$ now enables us to show that a zero-set in the growth of a strongly zero-dimensional space which admits only continuously many real-valued mappings is a copy of $\mathbb{N}^*$. The next result appears in the 1968 paper of W. W. Comfort and S. Negrepontis.

PROPOSITION [CH]:

If X is a strongly zero-dimensional space for which the cardinality of $C^*(X)$ is c , then any zero-set of βX which misses X is homeomorphic with $\mathbb{N}^*$.

Proof: Any such zero-set Z is clearly compact and it is immediate from Proposition 3.34 that it is also totally disconnected. Z has weight c since the cozero-sets of the C^* -embedded space Z all arise from members of $C^*(X)$ and form a base for Z . Corollary 1.63 shows that Z is an F -space. If p is an isolated point of Z , then there exists a zero-set neighborhood Z_1 of p in βX such that $Z_1 \cap Z = \{p\}$. But then $\{p\}$ is a zero-set and Theorem 3.7 shows that a zero-set contained in X^* must contain 2^c points, which is a contradiction. Finally, consider the space $\beta X \setminus Z$. It is clear from

Propositions 1.59 and Theorem 1.53 that $\beta X \setminus Z$ is locally compact and realcompact. In Lemma 4.21, to which the reader may refer directly, we will see that a zero-set in the growth of a locally compact and realcompact space is a regular closed set.

Hence, we have shown that Z satisfies all the conditions of Theorem 3.32, and Z is necessarily homeomorphic with $\mathbb{N}^*$. |

3.38. EXAMPLES:

Since Q contains no intervals, Q is strongly zero-dimensional. Note that we have also verified and used this fact in Example 2.15 and that we have shown there that the space Y of Example 2.14 is strongly zero-dimensional. Therefore, any zero-set of βQ or βY which is contained in Q^* or Y^* , respectively, is a copy of $\mathbb{N}^*$.

3.39. Proposition 3.34 indicates that the conjecture that βX has a base of clopen sets whenever X does is false. The following example was introduced in 1955 by C. H. Dowker and shows that a zero-dimensional space need not be strongly zero-dimensional.

EXAMPLE:

Consider the ordered space $\omega_1 + 1$ of all ordinals not greater than the first uncountable ordinal. Since for any $\alpha < \omega_1$ the set $\{\gamma : \gamma < \alpha\}$ is clopen in the order topology, $\omega_1 + 1$ is zero-dimensional. Dowker's example is a zero-dimensional subspace M of $(\omega_1 + 1) \times I$ such that the growth M^* of M contains a copy of I , therefore showing that βM

is not zero-dimensional.

Begin the construction of M by defining two points of I to be equivalent if their difference is rational. Each equivalence class is dense and countable and so there are c equivalence classes. Choose a collection $\{P_\gamma\}_{\gamma < \omega_1}$ of distinct equivalence classes other than the rationals and put $S_\alpha = I \setminus \bigcup_{\gamma \geq \alpha} P_\gamma$. Then each S_α is zero-dimensional since each fails to contain an interval. Define the following subspaces of $(\omega_1 + 1) \times I$:

$$M_\alpha = \bigcup_{\gamma \leq \alpha} (\{\gamma\} \times S_\gamma), \quad M = \bigcup_{\alpha < \omega_1} M_\alpha, \quad \text{and} \quad M^+ = M \cup (\{\omega_1\} \times I).$$

(a) M is zero-dimensional: Each M_α is a clopen subspace of M and is zero-dimensional since it is a subspace of the zero-dimensional space $\omega_1 \times S_\alpha$. Thus, M is zero-dimensional since it is the union of clopen zero-dimensional subspaces.

(b) M^+ is normal: First note that since each M_α is regular and second countable, the Urysohn Metrization Theorem [D, p. 195] shows that M_α is metrizable and hence, normal. Now let F_1 and F_2 be disjoint closed subsets of M^+ . Then $F_1 \cap (\{\omega_1\} \times I)$ and $F_2 \cap (\{\omega_1\} \times I)$ are disjoint compact sets and therefore are contained in disjoint open sets U_1 and U_2 of M^+ , respectively. Since F_1 and F_2 are closed and neighborhoods of ω_1 in $\omega_1 + 1$ are "tails", there is some $\alpha < \omega_1$ such that

$$F_i \cap (M^+ \setminus M_\alpha) \subset U_i$$

for each i . It follows from the fact that M_α is normal and open in M^+ that the sets F_1 and F_2 are contained in disjoint

open sets of M^+ .

(c) Disjoint closed sets of M have disjoint closures in M^+ : Let F_1 and F_2 be disjoint closed sets in M . If a point (ω_1, r) belongs to the closures in M^+ of both sets, then there exist two sequences $\{(\alpha_i, x_i)\}$ and $\{(\beta_i, y_i)\}$ contained in F_1 and F_2 , respectively, such that each sequence converges to (ω_1, r) and we can assume that $\alpha_i < \beta_i < \alpha_{i+1}$ for each i . But then $\sup\{\alpha_i\} = \sup\{\beta_i\} = \gamma < \omega_1$ and (γ, r) belongs to $A \cap B$, which is impossible.

As an immediate consequence of (b) and (c), we have that M is normal. Further,

(d) M is C^* -embedded in M^+ : By Theorem 1.2, we need only show that completely separated sets of M are also completely separated in M^+ . But completely separated sets in M have disjoint closures in M and also in M^+ by (c). But then their closures in M^+ are completely separated since M^+ is normal.

Finally, since M is dense and C^* -embedded in M^+ , we see that $\beta M = \beta(M^+)$. Then $\{\omega_1\} \times I$ is a copy of I contained in βM so that βM cannot be zero-dimensional.

SUMS OF ULTRAFILTERS AND THE NON-HOMOGENEITY OF IN^*

3.40. If a space X is homogeneous, does it necessarily follow that the growth $X^* = \beta X \setminus X$ is also homogeneous? This question was raised during a seminar at the University of Wisconsin in 1955 and in 1956 W. Rudin showed that the space IN^* provides a

negative answer in the presence of the Continuum Hypothesis. Rudin's result sparked a sequence of results in the area of non-homogeneity of growths which will be discussed in the next chapter.

In the present chapter, we consider Z. Frolik's proof from 1967B that $\mathbb{N}^*$ is not homogeneous. Frolik's method does not require the Continuum Hypothesis and can be modified to show that X^* fails to be homogeneous whenever X is not pseudocompact. This modification will be treated in the next chapter.

3.41. Every permutation σ of $\mathbb{N}$ extends to a homeomorphism

$$\beta(\sigma) : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$$

and the restriction of $\beta(\sigma)$ to $\mathbb{N}^*$, denoted by σ^* , is a homeomorphism of $\mathbb{N}^*$. For a pair of points p and q of $\mathbb{N}^*$, define $p \sim q$ if $\sigma^*(p) = q$ for some permutation σ . Clearly, $\sim$ is an equivalence relation. Let T be the set of equivalence classes and let

$$\mathfrak{J} : \mathbb{N}^* \longrightarrow T$$

be the function which assigns to each free ultrafilter its equivalence class. The elements of T are called types of ultrafilters. If $t = \mathfrak{J}(p)$, t is called the type of p and p is said to be of type t . The following result was obtained by W. Rudin in 1956 by considering ultrafilters as partially ordered sets.

THEOREM:

There are 2^c types of ultrafilters in $\mathbb{N}^*$ and there is a dense set of c ultrafilters of each type.

Proof: Corollary 3.20 established that the images of any point in $\mathbb{N}^*$ under homeomorphisms of the form σ^* comprise a dense subspace of $\mathbb{N}^*$. Since $\mathbb{N}^*$ contains c disjoint open sets by Theorem 3.22, there must be at least c ultrafilters of each type. On the other hand, there are only c permutations of $\mathbb{N}$, so there must be exactly c . Since there are 2^c ultrafilters in $\mathbb{N}^*$, it is clear that $\mathbb{N}^*$ must contain 2^c types of ultrafilters. $\square$

3.42. We saw earlier that any countably infinite discrete subspace of $\beta\mathbb{N}$ is C^* -embedded in $\beta\mathbb{N}$. Therefore, for such a subspace X , $cl_{\beta\mathbb{N}} X \cong \beta\mathbb{N}$ and if we put $X^* = cl_{\beta\mathbb{N}} X \setminus X$, $X^* \cong \mathbb{N}^*$. If z is in X^* , then the trace of the neighborhoods of z on X is an ultrafilter. The type of this ultrafilter will be denoted by $\mathfrak{J}_X(z)$ and is called the type of z relative to X .

If p is in $\mathbb{N}^*$ and Z belongs to A^p , then p is in $cl_{\beta\mathbb{N}} Z$ which is homeomorphic to $\beta\mathbb{N}$. Thus, $\mathfrak{J}_Z(p)$ is defined. In this case, $\mathfrak{J}_Z(p)$ can be determined by showing that $\mathfrak{J}_Z$ is a restriction of $\mathfrak{J}$. Since any homeomorphism of $cl_{\beta\mathbb{N}} Z$ is the extension of a permutation of Z , a homeomorphism of $cl_{\beta\mathbb{N}} Z$ will extend to a homeomorphism of $\beta\mathbb{N}$ by extending the permutation of Z to a permutation of $\mathbb{N}$. We have verified the

PROPOSITION:

If p is a free ultrafilter on $\mathbb{N}$, the type of p relative to a member of $\mathcal{A}^p$ is equal to the type of p .

In the remainder of the chapter, it will be helpful to identify a point p of $\mathbb{N}^*$ with the unique ultrafilter $\mathcal{A}^p$ on $\mathbb{N}$ which converges to p . Thus, we will think of p both as a point of $\mathbb{N}^*$ and as a collection of subsets of $\mathbb{N}$, i.e. as the ultrafilter $\mathcal{A}^p$.

3.43. In order to become more familiar with the notion of the type of an ultrafilter and with the meaning of relative types, consider the following example of a mapping of $\beta\mathbb{N}$ into $\mathbb{N}^*$. The example will depend on the observation that if t is any type and $X = \{x_n\}$ is a discrete sequence contained in $\beta\mathbb{N}$, then there exists p in X^* such that $\mathfrak{J}_X(p) = t$ since $\text{cl } X$ is a copy of $\beta\mathbb{N}$.

EXAMPLE:

There exists a mapping f of $\beta\mathbb{N}$ into $\mathbb{N}^*$ such that $f[\mathbb{N}]$ is discrete and $f^2[\mathbb{N}]$ is a singleton which does not belong to $f[\mathbb{N}]$. Choose a type t and a decomposition $\{N_n\}$ of $\mathbb{N}$ with each N_n an infinite set. Choose an x_n in N_n^* such that each x_n is of type t and choose a point p in $\mathbb{N}^*$ whose type relative to $X = \{x_n\}$ is t . Then there exists a bijection of each N_n onto X which sends the neighborhood traces of x_n onto the neighborhood traces of p on X . If f is the Stone-Ćech extension of the mapping of $\mathbb{N}$ defined in

this way, then f sends each x_n to p , and sends $\mathbb{N}$ to X . Hence, $f^2[\mathbb{N}] = \{p\}$.

3.44. One proof that $\beta\mathbb{N}$ is not homogeneous (Section 3.18) was based on cardinality considerations and consisted of showing that a point of $\mathbb{N}^*$ cannot be mapped to every other point of $\mathbb{N}^*$ under homeomorphisms of $\beta\mathbb{N}$ simply because there are too few such homeomorphisms. However, this does not eliminate the possibility that there might be enough homeomorphisms of $\mathbb{N}^*$ to map every point to every other point. To show that this is not the case, we will ultimately introduce an invariant for points of $\mathbb{N}^*$ which will be preserved under all homeomorphisms of $\mathbb{N}^*$. The description of the invariant will involve the somewhat unwieldy technique of summing ultrafilters. The following example is included to help motivate the definition of a sum of ultrafilters which is to follow. The example was introduced in the 1950 paper of R. Arens.

EXAMPLE:

Let $\{x_n\}$ be a sequence of isolated points converging to a point x_0 . Call the point x_0 the level 0 point and the points of the sequence level 1 points. Denote this convergent sequence by S_1 . To each level 1 point x_n attach a sequence $\{y_m^n\}$ of isolated points converging to x_n and call the points y_m^n level 2 points. Let S_2 be the union of the three levels of points. The level 2 points are isolated and a basic neighborhood of any level 1 point x_n is x_n plus a residual

set (or tail) of the sequence $\{y_m^n\}$. A neighborhood of the level 0 point x_0 is formed by first choosing a tail $\{x_n : n \geq r\}$ of level 1 points. Then since each open neighborhood of x_0 must contain a neighborhood of each x_n that it contains, a tail $\{y_m^n : m \geq s_n\}$ of level 2 points must be included for each $n \geq r$. Thus, a neighborhood of x_0 is a union

$$\{x_n : n \geq r\} \cup \left(\bigcup_{n \geq r} \{y_m^n : m \geq s_n\} \right) \cup \{x_0\}.$$

Note that many different neighborhoods can be formed for the same choice of a tail of level 1 points by choosing different tails of level 2 points.

This description of the basic neighborhoods of x_0 is similar to the operation of summing ultrafilters to be described below. More precisely, summing ultrafilters will correspond closely with the formation of the traces of the neighborhoods on the set of level 2 points, i.e. the sets

$$\bigcup_{n \geq r} \{y_m^n : m \geq s_n\}.$$

In terms of neighborhood filters, the formation of the trace of a neighborhood of x_0 on the set of level 2 points can be described as a three step operation which is very similar to the operation of summing ultrafilters. First, choose an element P of the filter of neighborhoods of x_0 in the subspace S_1 . Second, for each level 1 point x_n belonging to P , choose the trace M_{x_n} of a neighborhood of x_n on the sequence $\{y_m^n : m \geq 1\}$ of level 2 points converging to x_n . Third, form

the union

$$U\{M_{x_n} : x_n \in P\}$$

to obtain the trace of a neighborhood of x_0 on the set of level 2 points.

3.45. We now define the operation of summing ultrafilters. Let X be a collection of ultrafilters on $\mathbb{N}$ and let $\mathcal{U} = \{P_\alpha\}$ be an ultrafilter on X . The collection of all sets of the form

$$U\{M_x : x \in P_\alpha\}$$

where P_α is a set of $\mathcal{U}$ and M_x belongs to the ultrafilter x for each x in P_α is called the sum of X with respect to $\mathcal{U}$ and is denoted by $\Sigma_{\mathcal{U}}X$. Thus, $\Sigma_{\mathcal{U}}X$ is a collection of subsets of $\mathbb{N}$ where each subset is obtained by first choosing a member P_α of $\mathcal{U}$, then choosing a member M_x of x for each ultrafilter x in P_α , and finally taking the union of the chosen sets M_x . Note that as in the example of the neighborhoods of x_0 in S_2 , different members of $\Sigma_{\mathcal{U}}X$ may be obtained for the same choice of P_α by choosing different elements of x for some x in P_α .

We can easily see that $\Sigma_{\mathcal{U}}X$ is a filter. Put $\mathfrak{F} = \Sigma_{\mathcal{U}}X$. First we show that $\mathfrak{F}$ is closed under supersets. If R contains $U\{M_x : x \in P_\alpha\}$, then R is in x for each ultrafilter x in P_α since R contains M_x . Thus, R can be shown to belong to $\mathfrak{F}$ by choosing $M_x = R$ for every x in P_α .

$\mathfrak{F}$ is easily seen to be closed under finite intersections since

$$(U\{N_x : x \in P_\alpha\}) \cap (U\{M_x : x \in P_\beta\}) \supset U\{N_x \cap M_x : x \in P_\alpha \cap P_\beta\}$$

and we have already shown that $\mathfrak{F}$ is closed under supersets.

Hence we have partially verified the following

PROPOSITION:

$\mathfrak{F} = \Sigma_{\mathcal{U}} X$ is an ultrafilter on $\mathbb{N}$.

Proof: We have just seen that $\mathfrak{F}$ is a filter. We will show that $\mathfrak{F}$ is maximal by showing that a subset Z of $\mathbb{N}$ which meets every member of $\mathfrak{F}$ must belong to $\mathfrak{F}$. We first will show that if Z is such a subset, then for every P_α in $\mathcal{U}$, there is some ultrafilter x in P_α such that Z belongs to x . If this is not so for some P_{α_0} , then for every x in P_{α_0} , there is an M_x in x which misses Z . Thus, Z misses $U\{M_x : x \in P_{\alpha_0}\}$, which is a contradiction.

Now let $S = \{x \in X : Z \in x\}$. Since $\mathcal{U}$ is an ultrafilter on X , either S or $X \setminus S$ belongs to $\mathcal{U}$. For each x in $X \setminus S$, choose N_x in x such that $Z \cap N_x = \emptyset$. Then $Z \cap (U\{N_x : x \in X \setminus S\}) = \emptyset$. But since Z must meet every member of $\mathfrak{F}$, $X \setminus S$ is not in $\mathcal{U}$. Thus, S is in $\mathcal{U}$ and $Z = U\{Z : x \in S\}$ by choosing $M_x = Z$ for all x in S . Thus, $\mathfrak{F}$ is an ultrafilter. $\blacksquare$

3.46. If $X = \{x_n\}$ is a sequence in $\mathbb{N}^*$ and p is any ultrafilter on $\mathbb{N}$, then p can be considered to be an ultrafilter on X by identifying each point x_n of X with its subscript

n in $\mathbb{N}$. Thus $\sum_p X$, the sum of X with respect to p , is meaningful since we have identified the points of $\mathbb{N}^*$ with the free ultrafilters on $\mathbb{N}$. It will be necessary to consider only countable discrete subspaces of $\mathbb{N}^*$ and for the remainder of the chapter, X and Y will be used to denote countable, discrete subspaces of $\mathbb{N}^*$.

We now explore the supply of countable discrete subspaces X which might be summed in a fruitful manner. A collection X of ultrafilters on $\mathbb{N}$ is called discrete if M_x can be chosen in x for each x in X such that

$$\{M_x : x \in X\}$$

is a disjoint family of subsets of $\mathbb{N}$. The following proposition justifies the use of the term discrete for such a family.

PROPOSITION:

X is discrete if and only if X is a discrete subspace of $\beta\mathbb{N}$.

Proof: Disjoint basic neighborhoods of the points of X must arise from a family of disjoint members of the ultrafilters x . |

3.47. Note that for any sequence $\{t_n\}$ of types, there is a discrete sequence $X = \{x_n\}$ contained in $\mathbb{N}^*$ such that $\mathcal{I}(x_n) = t_n$. This follows from the observation that there is a dense set of ultrafilters of each type and that $\mathbb{N}^*$ contains c disjoint open sets. The existence of such discrete sequences will allow us to sum types as well as to sum ultrafilters. If

$\{t_n\}$ is any sequence of types and p is any ultrafilter on $\mathbb{N}$, we can define $\Sigma_p\{t_n\}$ to be the type of $\Sigma_p\{x_n\}$ where $\{x_n\}$ is any discrete sequence of $\mathbb{N}^*$ with $\mathcal{J}(x_n) = t_n$. This definition is well-defined as we now see.

PROPOSITION:

If discrete sequences $X = \{x_n\}$ and $Y = \{y_n\}$ of $\mathbb{N}^*$ are such that x_n and y_n are of the same type for each n , then $\Sigma_p X$ and $\Sigma_p Y$ are of the same type for any ultrafilter p on $\mathbb{N}$.

Proof: For each n , let σ_n be a permutation of $\mathbb{N}$ such that $\sigma_n^*(x_n) = y_n$. Choose disjoint families $\{R_n : R_n \in x_n\}$ and $\{S_n : S_n \in y_n\}$. Since $\sigma_n(R_n)$ is in y_n , we have that $\sigma_n(R_n) \cap S_n$ is infinite. Define

$$T_n = \sigma_n^{\leftarrow} (\sigma_n(R_n) \cap S_n) \setminus \{c_n\}$$

for some c_n in $\sigma_n^{\leftarrow} (\sigma_n(R_n) \cap S_n)$ in order to be certain that $\mathbb{N} \setminus \bigcup T_n$ is infinite. Since $\sigma_n^*(x_n) = y_n$, T_n is in x_n and $\sigma_n(T_n)$ is in y_n . Now define a permutation σ of $\mathbb{N}$ by setting $\sigma(m) = \sigma_n(m)$ if m is in T_n and for m in $\mathbb{N} \setminus \bigcup T_n$, let σ be any bijection of $\mathbb{N} \setminus \bigcup T_n$ with $\mathbb{N} \setminus \bigcup \sigma_n(T_n)$. Now suppose that $Z = \bigcup \{M_{x_n} : x_n \in p\}$ is a member of $\Sigma_p X$. Then:

$$\begin{aligned} \sigma(Z) &= \sigma(\bigcup \{M_{x_n} : x_n \in p\}) \\ &\supset \sigma(\bigcup \{M_{x_n} \cap T_n : x_n \in p\}) \\ &= \bigcup \{\sigma(M_{x_n} \cap T_n) : x_n \in p\}. \end{aligned}$$

But this last set belongs to $\Sigma_p Y$ so that $\sigma(Z)$ belongs to $\Sigma_p Y$ and $\sigma^*(\Sigma_p X) = \Sigma_p Y$. $\square$

3.48. The main tool which will be used to relate types of ultrafilters to the topology of $\mathbb{N}^*$ is called the producing relation. The producing relation ϕ on the set T of types is the set of all pairs of types (u, v) such that

$$v = \Sigma_p \{t_n\}$$

for some ultrafilter p of type u and some sequence $\{t_n\}$ in T . The statement $v = \phi(u)$ is read " u produced v " or " v is produced by u ". Thus for u in T , $\phi(u)$ is the set of all types which can be obtained by summing sequences in T over ultrafilters of type u .

The producing relation is interpreted topologically in terms of closures of copies of $\mathbb{N}$ in $\mathbb{N}^*$. Let X be a copy of $\mathbb{N}$ contained in $\mathbb{N}^*$ and for p in X^* , let p_X denote the trace of the neighborhoods of p on X .

PROPOSITION:

If p is in X^* , then $p = \Sigma_{p_X} X$ and $\mathcal{I}_X(p)$ produces $\mathcal{I}(p)$.
Conversely, if y is any free ultrafilter on X , then the sum
 $q = \Sigma_y X$ is in X^* and $y = q_X$.

Proof: If p is in X^* , then $\text{cl}_{\beta \mathbb{N}} Z \cap X \neq \emptyset$ for all Z in p . Thus, Z belongs to x for all x in $\text{cl}_{\beta \mathbb{N}} Z \cap X$. Then for all Z in p , in forming $\Sigma_{p_X} X$ we can choose $M_x = Z$ for all x in $\text{cl}_{\beta \mathbb{N}} Z \cap X$. Thus, Z is in $\Sigma_{p_X} X$ for all Z in p , so that $p = \Sigma_{p_X} X$. Then by definition of the producing relation,

we have $\mathcal{J}(p)$ in $\varphi(\mathcal{J}_X(p))$.

For the converse, let Z be in $q = \sum_Y X$. Then there exists Y in y such that $Z = \bigcup \{M_x : x \in Y\}$. Hence, Z contains M_x so that x is in $cl_{\beta \mathbb{N}} Z$ for each x in Y . Thus, every neighborhood of q meets X and q belongs to X^* . Now let $cl_{\beta \mathbb{N}} Z \cap X$ be in q_X and let Z' be in y . To show that $y = q_X$ it is sufficient to show that $(cl_{\beta \mathbb{N}} Z \cap X) \cap Z' \neq \emptyset$: $Z = \bigcup \{M_x : x \in Y\}$ for some Y in y , so that $Z \supset \bigcup \{M_x : x \in Y \cap Z'\}$. But then Z is in x for x in $Y \cap Z'$. Thus, x is in $cl_{\beta \mathbb{N}} Z$ and $x \in (cl_{\beta \mathbb{N}} Z \cap X) \cap Z'$. $\parallel$

3.49. The definition of the producing relation shows that the preceding proposition can be restated in the following way:

PROPOSITION:

The set of relative types of a point p of $\mathbb{N}^*$ is the set of types which produce the type of the ultrafilter p .

3.50. The central result in Frolik's theory of types is the calculation of the cardinalities of $\varphi(t)$ and $\varphi^{\leftarrow}(t)$ for any type t . The proof is accomplished through an examination of the possible number of relative types of a point p in $\mathbb{N}^*$.

THEOREM: (Frolik)

Any type is produced by at most c types and any type produces 2^c types.

Proof: The second statement amounts to showing that for any type t , the cardinality of $\varphi(t)$ is 2^c .

(a) Let $\{M_n\}$ be any countable partition of $\mathbb{N}$ into infinite sets and let x_n and y_n be in M_n^* for each n . If $x_n \neq y_n$ for each n , then $cl_{\beta\mathbb{N}}\{x_n\} \cap cl_{\beta\mathbb{N}}\{y_n\} = \emptyset$: To show this, write $M_n = R_n \cup S_n$, a disjoint union with R_n in x_n and S_n in y_n for each n . For any p in $\mathbb{N}^*$ and Z in p , write $Z = (Z \cap (\cup R_n)) \cup (Z \cap (\cup S_n))$. Exactly one of these sets is in p . Suppose that p belongs to $cl_{\beta\mathbb{N}}\{x_n\}$. Then we have

$$p \in cl_{\beta\mathbb{N}}\{x_n\} \subset cl_{\beta\mathbb{N}}(\cup R_n).$$

Hence, $\cup R_n$ belongs to p and therefore $\cup S_n$ is not in p so that p cannot belong to $cl_{\beta\mathbb{N}}\{y_n\}$.

(b) Since the cardinality of M_n^* is 2^c for each n , (a) shows that $\mathbb{N}^*$ contains 2^c disjoint sets of the form $cl_{\beta\mathbb{N}}\{x_n\}$. Each is homeomorphic to $\beta\mathbb{N}$ and so for any type t , each contains a point p such that $\mathcal{I}_{\{x_n\}}(p) = t$. The point p is then of type $\phi(t)$ by the preceding proposition. Since only c of these 2^c points can be of any given type, t must produce 2^c types.

The first statement of the theorem will follow after completing two more partial steps.

(c) For any point p in $\mathbb{N}^*$, there is a set $\mathcal{X}$ of at most c discrete, countable subsets of $\mathbb{N}^*$ such that

- (i) Each member of $\mathcal{X}$ contains p in its closure, and
- (ii) If Y is any discrete countable set with p in Y^* , then Y contains some member of $\mathcal{X}$:

For each partition $\{M_n\}$ of $\mathbb{N}$, choose y_n in M_n^* to form Y with p in Y^* , if possible. The same partition may yield many such Y 's. Now for each Y that can be formed in this way when all partitions are considered, take all sets X contained in Y such that p belongs to $cl_{\beta_{\mathbb{N}}} X \setminus Y$. By (a), any two choices of Y from the same partition meet in an infinite set containing p in its closure, so we need consider only one choice of Y from each partition. Since each Y contains only c infinite subsets and since there are only c partitions, we have $|\mathcal{X}| \leq c \cdot c = c$.

By Proposition 3.49, a type t is produced by a type r exactly when there is an ultrafilter p of type t and a discrete sequence X in $\mathbb{N}^*$ with p in X^* and the type of p relative to X is r . We now see that any point can have at most c distinct relative types.

(d) If X is contained in a discrete sequence Y and p belongs to $cl_{\beta_{\mathbb{N}}} X \setminus Y$, then $\mathfrak{J}_X(p) = \mathfrak{J}_Y(p)$. This follows immediately exactly as Proposition 3.42.

Thus, (d) shows that in computing the relative types of p , we need only consider the members of $\mathcal{X}$ and $\mathcal{X}$ contains at most c sets. Since there are only c ultrafilters of each type (Theorem 3.41), the set of ultrafilters of a given type can have a total of at most c relative types. Hence, each type can be produced by at most c types. $\parallel$

The first estimate in Frolik's Theorem cannot be improved. In 1971, the Steiners introduced a method for constructing types and showed the existence of a point in $\mathbb{N}^*$ having c relative types.

3.51. We can now show that $\mathbb{N}^*$ is not homogeneous as an application of the producing relation.

COROLLARY:

$\mathbb{N}^*$ is not homogeneous.

Proof: Let T_p denote the set of all relative types of p for each p in $\mathbb{N}^*$, i.e. $T_p = \phi^{\leftarrow}(\mathcal{J}(p))$. If h is an automorphism of $\mathbb{N}^*$ and $h(p) = q$, then $T_p = T_q$, since the trace of the neighborhoods of p on $\{x_n\}$ is of the same type as the trace of the neighborhoods of q on $\{h(x_n)\}$ for all sequences $\{x_n\}$ having p as a cluster point. Now the cardinality of T_p is at most c since each type is produced by at most c types. Since the sets $\{T_p\}$ cover T and the cardinality of T is 2^c , there must exist points r and s of $\mathbb{N}^*$ for which T_r and T_s are distinct. But then r cannot be mapped to s by a homeomorphism of $\mathbb{N}^*$. $\square$

3.52. The orbits of two points p and q of $\mathbb{N}^*$ under automorphisms of $\mathbb{N}^*$ are disjoint exactly when no automorphism carries p to q . Thus, the set of all such orbits decomposes $\mathbb{N}^*$ into a union of disjoint sets. Since any two points belonging to the same orbit have the same set of at most c relative types, there must be 2^c distinct orbits. We have verified the

COROLLARY:

For any point p of $\mathbb{N}^*$, there are 2^c points of $\mathbb{N}^*$
which cannot be mapped to p by automorphisms of $\mathbb{N}^*$.

EXERCISES

3A. CARDINALITY OF STONE-ČECH COMPACTIFICATIONS

1. If X is separable, $|\beta X| \leq 2^{\mathfrak{c}}$.
2. If X contains a closed C^* -embedded copy of $\mathbb{N}$, then $|\beta X| \geq 2^{\mathfrak{c}}$.
3. The cardinalities of $\beta\mathbb{R}$, $\beta\mathbb{Q}$, and $\beta\mathbb{P}$ are all $2^{\mathfrak{c}}$.

3B. S_2 AS GROWTH

Consider the space S_2 described in Example 3.44.

1. If two points of S_2 are on the same level, there is an automorphism of S_2 which will send one of the points to the other. However, no automorphism will exchange two points of different levels.
2. There is a subspace Y of $\beta\mathbb{N}$ such that Y^* is homeomorphic to S_2 . [3.50(a).]
3. Y can be chosen so that no pair of level 2 points of Y^* can be mapped to each other by an automorphism of $\beta\mathbb{N}$.
4. Y can be chosen so that for any pair of level 2 points of Y^* , there is an automorphism of $\beta\mathbb{N}$ which will exchange the pair of points.

CHAPTER FOUR:
NON-HOMOGENEITY OF GROWTHS

4.1. This chapter will be the first of three to be devoted to the relationships between a space X and its growth $X^* = \beta X \setminus X$. In order to be certain that in considering growths of Stone-Ćech compactifications we will not be discussing a restricted class of spaces, we will first follow the outline of [GJ, ex. 9K] to show that any completely regular space can be expressed in the form $\beta S \setminus S$ for a suitable choice of S .

In order to derive this result, we will need to consider spaces of ordinal numbers. The basic material on ordinal numbers can be found in [D]. Recall that every ordinal number is the set of all its predecessors. If α is any ordinal number, the symbol α will also be used to denote the topological space obtained by imposing the interval topology on α . If α is a non-limit ordinal, i.e. if α has an immediate predecessor, the space α is compact. [GJ, 5.11]. With this notation, the celebrated counterexample known as the Deleted Tychonoff Plank is written

$$T = (\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}$$

where ω_0 is the first infinite ordinal and ω_1 is the first uncountable ordinal. The space T was introduced by Tychonoff in 1929 as an example of a completely regular space which is not normal. In his 1949 paper, H. Tong established that the Stone-Ćech compactification of T is obtained by adding the

point (ω_1, ω_0) so that

$$\beta T = (\omega_1 + 1) \times (\omega_0 + 1).$$

This result is also discussed in [GJ, 8.20] and will be obtained below as Proposition 4.4.

The critical properties of ω_1 used in the proof of this fact are that no countable subset of ω_1 is cofinal and that every real-valued continuous function on ω_1 is constant on a tail of ω_1 . By similar properties of larger ordinals, we will show that if Y is a compact space, an ordinal ω_α can be chosen so that

$$\beta(\omega_\alpha \times Y) = (\omega_\alpha + 1) \times Y.$$

This result can then be used to show that any space can be expressed as a growth of some Stone-Čech compactification.

4.2. If Y is a compact space, let $\aleph_\alpha$ be a cardinal number greater than the cardinality of Y and such that α is a non-zero non-limit ordinal. Using this notation, we have the following

PROPOSITION:

$$\beta(\omega_\alpha \times Y) = (\omega_\alpha + 1) \times Y.$$

Proof: It is clear that $(\omega_\alpha + 1) \times Y$ is compact and contains $\omega_\alpha \times Y$ densely. It remains to show that $\omega_\alpha \times Y$ is C^* -embedded in $(\omega_\alpha + 1) \times Y$. Let f be a bounded, real-valued mapping on $\omega_\alpha \times Y$. Then for each point y in Y , the restriction

$f|_{\omega_\alpha \times \{y\}}$ is constantly equal to some value r_y on a tail $[\sigma_y, \omega_\alpha) \times \{y\}$ of the "row" $\omega_\alpha \times \{y\}$. Because the cardinality of Y is less than $\aleph_\alpha$, $\sigma = \sup\{\sigma_y : y \in Y\}$ is less than ω_α since α is a non-limit ordinal. Since $\sigma \geq \sigma_y$, each restriction $f|_{[\sigma, \omega_\alpha) \times \{y\}}$ is constant. Define $\beta(f)$ to be f on $\omega_\alpha \times Y$ and to be r_y at (ω_α, y) . Then $\beta(f)$ is continuous and extends f . $\square$

4.3. We can now show that every space is the growth of some Stone-Ćech compactification.

PROPOSITION:

For any completely regular space X there exists a space S such that X is homeomorphic to $\beta S \setminus S$.

Proof: Let Y be a compactification of X and choose ω_α as above. Now define S to be the following subspace of $(\omega_\alpha + 1) \times Y$:

$$S = (\omega_\alpha + 1) \times Y \setminus \{(\omega_\alpha, x) : x \in X\}.$$

Since S contains $\omega_\alpha \times Y$, the previous result and Proposition 1.49 imply that $\beta S = (\omega_\alpha + 1) \times Y$. The result now follows since $S^* = \{(\omega_\alpha, x) : x \in X\}$ is a copy of X . $\square$

4.4. By a technique similar to that just used we will now show that $(\omega_1 + 1) \times (\omega_0 + 1)$ is the Stone-Ćech compactification of the Deleted Tychonoff Plank.

PROPOSITION:

$$\beta((\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}) = (\omega_1 + 1) \times (\omega_0 + 1).$$

Proof: Let $Y = \omega_0 + 1$. Then we have seen that $\beta(\omega_1 \times (\omega_0 + 1)) = (\omega_1 + 1) \times (\omega_0 + 1)$. Since $T = (\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}$ contains $\omega_1 \times (\omega_0 + 1)$, the result follows. $\blacksquare$

4.5. In Corollary 3.51 we saw that the growth IN^* is not homogeneous. In this chapter we will build upon our previous results for IN^* to show in two distinct ways that X^* will fail to be homogeneous whenever X fails to be pseudocompact, i.e. whenever X admits an unbounded, real-valued mapping. The class of pseudocompact spaces was first investigated by E. Hewitt in 1948 and the next two results appear in [GJ, 1.20, 1.21].

LEMMA:

Any subspace of a space X on which a real-valued mapping of X is unbounded contains a C-embedded copy of IN which is closed in X .

Proof: Observe that the main portion of the proof is given in part (a) of the proof of Theorem 3.7. It remains only to show that the C-embedded copy of IN is closed. For any cluster point of a C-embedded copy N of IN , there is a real-valued mapping on N which is unbounded on every neighborhood of the cluster point. Since such a mapping can have no real-valued extension to the cluster point, N must be closed. $\blacksquare$

4.6. Now the corollary characterizing the class of pseudocompact spaces is immediate.

COROLLARY: (Hewitt)

The pseudocompact spaces are those spaces which contain no C-embedded copy of $\mathbb{N}$.

Taking the closure in βX of a C-embedded copy of $\mathbb{N}$ contained in a space X produces a copy of $\mathbb{N}^*$ contained in X^* . The existence of such a copy of $\mathbb{N}^*$ is the key to showing that the growth of a non-pseudocompact space fails to be homogeneous. We will frequently identify a C-embedded copy of $\mathbb{N}$ with $\mathbb{N}$ and refer to the points of the copy by n .

We will first consider Z. Frolik's adaptation of his argument for $\mathbb{N}^*$ to show that the growth of any non-pseudocompact space fails to be homogeneous. While Frolik's method will demonstrate the non-homogeneity of such growths, no specific kinds of points will be exhibited through this approach. In the remainder of the chapter, we will consider the classification of several kinds of points which can be identified in growths of Stone-Čech compactifications. In particular, we will investigate the existence of P-points and non-P-points in growths.

TYPES OF POINTS IN X^*

4.7. We first establish a notion of types for an arbitrary space Y . For a point y in Y , let m_y denote the set of

all C^* -embedded copies of $\mathbb{N}$ in Y with y in $cl_Y \mathbb{N} \setminus \mathbb{N}$. For such a copy of $\mathbb{N}$, $cl_{\beta Y} \mathbb{N}$ is homeomorphic to $\beta \mathbb{N}$ so that $cl_Y \mathbb{N}$ is a subspace of $\beta \mathbb{N}$. Thus, if y is in $cl_Y \mathbb{N} \setminus \mathbb{N}$, the trace of the neighborhoods of y on $\mathbb{N}$ forms an ultrafilter on $\mathbb{N}$. Hence, we can speak of the type of y with respect to $\mathbb{N}$ as discussed in Chapter 3. If S is a subspace of Y , types of a point y in Y with respect to sets of $\mathfrak{m}_Y$ whose closures are contained in S are called types in S . In the special case that Y is βX and S is X^* , types in S are called ideal types.

4.8. The heart of Frolik's argument lies in showing that if X is a non-pseudocompact space, then certain points of X^* have "too few" ideal types to allow X^* to be homogeneous. The following preliminary results are needed to relate countable sets of X^* to C -embedded copies of $\mathbb{N}$ in X .

LEMMA:

Let $\mathbb{N}$ be C -embedded in X and let Y be a countable subspace of βX . Then if neither subspace meets the βX -closure of the other, the two subspaces have disjoint closures in βX .

Proof: As in the proof of Proposition 1.64, there exist disjoint cozero-sets U and V of βX which contain $\mathbb{N}$ and Y , respectively. Thus, $X \setminus U$ is a zero-set of X missing $\mathbb{N}$.

Since $\mathbb{N}$ is C -embedded in X , Theorem 1.3 implies that $\mathbb{N}$ and $X \setminus U$ are completely separated. Then $cl_{\beta X}(X \setminus U) \cap cl_{\beta X} \mathbb{N} = \emptyset$.

Since $cl_{\beta X} Y$ is contained in $cl_{\beta X}(X \setminus U)$, $cl_{\beta X} Y$ must miss $cl_{\beta X} \mathbb{N}$. **I**

4.9. COROLLARY:

Let $\mathbb{I}\mathbb{N}$ be C-embedded in X and let Y be a countable subspace of βX whose βX -closure is contained in X^* . Then

$$cl_{\beta X} \mathbb{I}\mathbb{N} \cap cl_{\beta X} Y \subset cl_{\beta X} (Y \cap cl_{\beta X} \mathbb{I}\mathbb{N}).$$

Proof: Suppose that p is in $cl_{\beta X} \mathbb{I}\mathbb{N} \cap cl_{\beta X} Y$ but not in $cl_{\beta X} (Y \cap cl_{\beta X} \mathbb{I}\mathbb{N})$. Then there is a neighborhood U of p such that $Y \cap U \cap cl_{\beta X} \mathbb{I}\mathbb{N} = \emptyset$. Since $cl_{\beta X} (Y \cap U) \cap \mathbb{I}\mathbb{N} = \emptyset$, the lemma gives $cl_{\beta X} (Y \cap U) \cap cl_{\beta X} \mathbb{I}\mathbb{N} = \emptyset$. But this is a contradiction since p is in both these closures. Hence, p belongs to $cl_{\beta X} (Y \cap cl_{\beta X} \mathbb{I}\mathbb{N})$. $\square$

4.10. The following special case of the preceding corollary shows that if $\mathbb{I}\mathbb{N}$ is C-embedded in X , then every ideal type of a point in $\mathbb{I}\mathbb{N}^*$ is also a type in $\mathbb{I}\mathbb{N}^*$.

COROLLARY:

Let $\mathbb{I}\mathbb{N}$ be C-embedded in X and let Y be a countable discrete subspace of βX whose βX -closure is contained in X^* . Then a point in $Y^* \cap \mathbb{I}\mathbb{N}^*$ is in Z^* for some Z contained in $Y \cap cl_{\beta X} \mathbb{I}\mathbb{N}$.

4.11. An upper bound for the cardinality of the ideal types of a point of X^* belonging to a copy of $\mathbb{I}\mathbb{N}^*$ can now be obtained by applying Theorem 3.50.

PROPOSITION:

If $\mathbb{N}$ is C-embedded in X , the cardinal of the ideal types of a point in $\mathbb{N}^*$ is at most c .

Proof: Let t be the ideal type of a point p of $\mathbb{N}^*$ with respect to some Y contained in X^* . By the preceding corollary, there exists $Z \subset Y \cap \text{cl}_{\beta X} \mathbb{N}$ such that p is in $\text{cl}_{\beta X} Z$. From Proposition 3.42 involving $\beta \mathbb{N}$, it is clear that the types of p with respect to Y and Z coincide. Thus, since Z is contained in $\mathbb{N}^*$, every ideal type of p is an ideal type of p considered as a point of $\mathbb{N}^* = \beta \mathbb{N} \setminus \mathbb{N}$. But now the result follows from the fact that any type of ultrafilter on $\mathbb{N}$ is produced by at most c types. $\blacksquare$

4.12. The upper bound on the cardinality of ideal types can now be used to show that a growth can only be homogeneous when the space is pseudocompact. The proof is very similar to the proof that $\mathbb{N}^*$ is not homogeneous in Corollary 3.51.

THEOREM: (Frolík)

The growth of a non-pseudocompact space fails to be homogeneous.

Proof: Let X be a non-pseudocompact space. For each point p in X^* , let T_p be the set of all ideal types of p . It is clear that if h is any automorphism of X^* such that $h(p) = q$, then $T_p = T_q$. Now let $\mathbb{N}$ be C-embedded in X and choose p in $\mathbb{N}^*$. It is now sufficient to show that there exists q in $\mathbb{N}^*$ such that $T_p \neq T_q$. Since there are 2^c types of points in $\mathbb{N}^*$, by the previous proposition it is possible to choose a type t not in T_p . Then any q in $\mathbb{N}^*$

having type t with respect to $\mathbb{N}^*$ will do. $\blacksquare$

This approach to non-homogeneity appears in Frolik's 1967C paper.

4.13. The preceding theorem shows that a sufficient condition for X^* to be non-homogeneous is that X is not pseudocompact. However, this condition is not necessary.

EXAMPLE:

Let N contained in $\mathbb{N}^*$ be a copy of $\mathbb{N}$ and choose a point p in $\mathbb{N}^*$. Let $X = \beta\mathbb{N} \setminus (N \cup \{p\})$. Then X^* contains both isolated points and a cluster point and therefore is not homogeneous. X is countably compact, and therefore pseudocompact, since any countable set of X has 2^c cluster points in $\beta\mathbb{N}$ and only countably many points have been removed.

4.14. We now consider another approach to the question of the non-homogeneity of growths. This approach differs from the one just considered in that the Continuum Hypothesis is required. Despite this shortcoming, the technique has the added advantage of singling out two interesting kinds of points which occur in growths, the P -points and the remote points, and showing their existence along parallel lines. P -points were defined in Section 1.65 and will be studied beginning in Section 4.29. Remote points will be defined and discussed beginning in Section 4.37.

The existence of P -points and remote points will first be demonstrated through an algebraic approach developed by D. Plank in 1969. Unfortunately, Plank's method requires restrictive

hypotheses. To show the existence of P-points in X^* for X belonging to a less restricted class of spaces, we will consider the more powerful methods of W. Rudin and Isiwata. To show the existence of remote points under less restrictive conditions, we will utilize a construction developed by Fine and Gillman.

C-POINTS AND C^* -POINTS IN X^*

4.15. In Chapter 1 we saw that βX is homeomorphic to the spaces of maximal ideals $\mathfrak{m}(X)$ of $C(X)$ and $\mathfrak{m}^*(X)$ of $C^*(X)$, each taken with the hull-kernel topology. The respective maximal ideals are of the form

$$M^p = \{f \in C(X) : p \in \text{cl}_{\beta X} Z(f)\}$$

and

$$M^{*p} = \{f \in C^*(X) : \beta(f)(p) = 0\}$$

for p in βX . The homeomorphisms $p \mapsto M^p$ of βX with $\mathfrak{m}(X)$ and $p \mapsto M^{*p}$ of βX with $\mathfrak{m}^*(X)$ are denoted by τ and τ^* , respectively. The above descriptions of the maximal ideals of the two rings enable us to express $\text{cl}_{\beta X} Z(f)$ and $Z(\beta(f))$ in terms of τ and τ^* , respectively: If f is in $C(X)$,

$$\text{cl}_{\beta X} Z(f) = \tau^{\leftarrow}(\{M^p \in \mathfrak{m}(X) : f \in M^p\})$$

and if f is in $C^*(X)$,

$$Z(\beta(f)) = (\tau^*)^{\leftarrow}(\{M^{*p} \in \mathfrak{m}^*(X) : f \in M^{*p}\}).$$

The investigation of the different kinds of points of $\mathcal{I}N^*$ will be carried out by studying subsets of the above form. We will frequently need to consider closures, interiors, and boundaries in the three spaces X , βX , and X^* , and one must be careful to note the subscripts on the operators denoted by cl , int , and ∂ .

4.16. A point p of X^* is called a C-point if

$$p \notin \partial_{X^*}(cl_{\beta X} Z(f) \cap X^*)$$

for all f in $C(X)$. Similarly, a point p of X^* is called a C^* -point if $p \notin \partial_{X^*}(Z(\beta(f)) \cap X^*)$ for all f in $C^*(X)$.

The significance of the C^* -points is easily seen when the definition of C^* -point is compared with that of P-point.

Recall from Section 1.65 that a P-point is one which is in the interior of every zero-set which contains it. Hence, a P-point cannot belong to the boundary of any zero-set. By definition, a C^* -point is a point which cannot lie in the X^* -boundary of the trace on X^* of any zero-set of βX . Thus, it is clear that a P-point of X^* is a C^* -point. The converse is also true.

PROPOSITION:

A point in X^* is a C^* -point if and only if it is a P-point of X^* .

Proof: We have just seen that any P-point of X^* must be a C^* -point.

We will prove the converse by establishing the contra-positive. If a point q of X^* is not a P-point, then there

is a zero-set Z_1 in X^* such that q lies in $\partial_{X^*} Z_1$. Because Z_1 is a G_δ in the subspace X^* , there is a G_δ S of βX such that $Z_1 = X^* \cap S$. But now there also exists a zero-set Z_2 of βX such that

$$q \in Z_2 \subset S.$$

Since q is in the X^* -interior of Z_1 and $Z_2 \cap X^*$ is contained in Z_1 , we must have that q belongs to $\partial_{X^*}(Z_2 \cap X^*)$. Thus, q cannot be a C^* -point. $\square$

4.17. The characterization of C -points is more difficult and cannot be accomplished without making assumptions about X . We will first prove an existence theorem for C^* -points and C -points and then undertake separate examinations of the two kinds of points. The relationship between C -points and remote points will be described in Theorem 4.40.

4.18. The definitions just given indicate that the existence of such points will depend upon the intersection of families of dense open sets being non-void. For example, it is clear that the C^* -points are precisely the points belonging to

$$\bigcap \{X^* \setminus \partial_{X^*}(Z(\beta(f)) \cap X^*) : f \in C^*(X)\}.$$

The following analogue of the Baire Category Theorem is thus important in showing the existence of such points. The proof given is based on a method used by W. Rudin in 1956.

PROPOSITION:

In a locally compact space in which every non-empty G_δ has non-empty interior, the intersection of any family of no more than $\aleph_1$ dense open sets is dense. If the space also has no isolated points, then the intersection contains at least 2^c points.

Proof: Write $\mathfrak{D} = \{U_\alpha : \alpha < \omega_1\}$ for the collection of dense open sets and let G be any non-empty open set in such a space Y . We must show that G meets $\bigcap \mathfrak{D}$. For $\alpha < \omega_1$, suppose that we have defined a collection $\{V_\beta : \beta < \alpha\}$ of open sets of G satisfying:

- (a) $cl_Y V_\beta$ is compact for $\beta < \alpha$,
- (b) $V_\beta \subset U_\beta$ for $\beta < \alpha$,
- (c) $\bigcap_{\beta < \alpha} V_\beta \neq \emptyset$.

Then $\bigcap_{\beta < \alpha} V_\beta$ is a G_δ and thus has a non-void interior which must meet U_α since U_α is dense. Since U_α is also open, there is an open set V_α with compact closure such that

$$cl_Y V_\alpha \subset U_\alpha \cap \left(\bigcap_{\beta < \alpha} V_\beta \right) \subset U_\alpha \cap G.$$

Thus, $\{V_\alpha : \alpha < \omega_1\}$ is defined inductively so that $\{cl_Y V_\alpha : \alpha < \omega_1\}$ is a family of compact sets with the finite intersection property. Therefore,

$$\emptyset \neq \bigcap cl_Y V_\alpha \subset \left(\bigcap \mathfrak{D} \right) \cap G,$$

and $\bigcap \mathfrak{D}$ is dense.

For the second statement, note that if Y has no isolated points, then at each step of the induction, there are two disjoint choices for V_α , so that $|D| \geq 2^{\aleph_1}$. $\square$

4.19. The next sequence of results will conclude with the proof that a non-empty G_δ in the growth of a locally compact real-compact space has non-empty interior. This will be the major step in applying the preceding proposition. The first lemma appears in the 1962 paper of Fine and Gillman and will be employed to show that a C -embedded copy of $\mathbb{N}$ is completely separated from a disjoint closed subset.

LEMMA:

Let f be a non-negative and unbounded member of $C(X)$ and let $\{J_n\}$ be a sequence of disjoint closed intervals in $\mathbb{R}$ such that each interval contains a value of f in its interior. Then the family of sets

$$E_n = f^{\leftarrow}(J_n), \quad n \geq 1,$$

satisfies:

- (a) Each E_n is a zero-set with non-void interior.
- (b) If Z_n is any sequence of zero-sets with Z_n contained in E_n , then $\bigcup Z_n$ is a zero set.

Proof: (a) is clear since each J_n is a zero-set with non-void interior and these properties will be shared by $E_n = f^{\leftarrow}(J_n)$.

To prove (b), we exhibit a map g in $C^*(X)$ such that $Z(g) = \bigcup Z_n$. For each $n \geq 1$, there exists s_n in $C^*(\mathbb{R})$

such that $s_n^{\leftarrow}(1) = J_n$ and $s_n^{\leftarrow}(0) = \bigcup_{m \neq n} J_m$. Now for Z_n given as in (b), write $Z_n = Z(h_n)$, and assume that $0 \leq h_n \leq 2^{-n}$. Consider the sequence of functions defined by

$$g_n(x) = h_1(x) \cdot (s_1 \circ f)(x) + \dots + h_n(x) \cdot (s_n \circ f)(x).$$

The sequence $\{g_n\}$ converges uniformly to a mapping g in $C^*(X)$. It is clear that g is never zero outside of the union $\bigcup E_n$, and that for each integer n g agrees with h_n on E_n , so that $Z(g) = \bigcup Z_n$. $\parallel$

4.20. The next result is [GJ, ex.9M1].

COROLLARY:

A copy of $\mathbb{N}$ which is C-embedded in a space X is completely separated from any closed subset disjoint from it.

Proof: Let $\mathbb{N}$ be C-embedded in X and let the closed set F miss $\mathbb{N}$. For each n in $\mathbb{N}$, there is a zero-set W_n containing F which does not contain n . Then $Z = \bigcap W_n$ is a zero-set containing F and missing $\mathbb{N}$. There exists a non-negative member of $C(X)$ which is unbounded on $\mathbb{N}$ so that by the preceding lemma we can choose zero-sets E_n with n contained in E_n for each n . Further, for each n in $\mathbb{N}$, choose a zero-set Z_n that is contained in E_n , misses Z , and contains n . Then F and $\mathbb{N}$ are completely separated since they are contained in the disjoint zero-sets Z and $\bigcup Z_n$, respectively. $\parallel$

4.21. Note that the next proposition shows that every non-empty G_δ in the growth of a locally compact realcompact space has non-empty interior since every non-empty G_δ contains a non-empty zero-set. The result is taken from the 1960 paper of Fine and Gillman.

PROPOSITION:

The zero-sets in the growth of a locally compact and realcompact space are regular closed sets.

Proof: Let X be a locally compact and realcompact space that is not compact. First observe that the proof of Corollary 3.28 shows that it is sufficient to show that a non-empty zero-set has non-empty interior.

Since X is locally compact, X^* is closed (Theorem 1.59) and therefore is C^* -embedded in βX . Thus, a zero-set Z of X^* is of the form $Z = Z(f) \cap X^*$ for some f in $C(\beta X)$. Let p belong to Z . Since X is realcompact and p lies outside of X , by Theorem 1.53 there is a mapping g in $C(\beta X)$ such that $g(p) = 0$ and g does not vanish on X . Let $h = |f| + |g|$. Then $Z(h)$ is a subset of Z containing p , i.e.

$$p \in Z(h) \subset Z.$$

Let $\{x_n\}$ be a discrete sequence of distinct points of X on which h approaches zero. Since the reciprocal of $h|_X$ is continuous and unbounded on $\{x_n\}$, by Lemma 4.5 we can assume that the sequence is C -embedded in X . Choose a compact

neighborhood V_n of x_n for each n such that the family $\{V_n\}$ is pairwise disjoint and

$$|h(x) - h(x_n)| < \frac{1}{n}$$

for each x in V_n . Then the preceding corollary shows that there exists a mapping ι in $C^*(X)$ such that $\iota(x_n) = 1$ for each n and ι vanishes outside of $\bigcup V_n$. If $\beta(\iota)$ is the extension of ι to βX and q is any point of X^* such that $\beta(\iota)(q) \neq 0$, then every neighborhood of q meets infinitely many of the compact sets V_n and hence $h(q) = 0$. Thus, we have shown that

$$X^* \setminus Z(\beta(\iota)) \subset Z(h) \subset Z,$$

and Z therefore has non-empty interior in X^* . $\blacksquare$

4.22. The local compactness is critical in the preceding proposition as the following example shows.

EXAMPLE:

The space of rationals Q is realcompact but not locally compact. Q^* contains non-empty zero-sets which have empty interior. For example, let h embed Q as a dense subspace of the closed interval I such that $h[Q]$ does not contain 0 . Then the zero-set $Z(\beta(h))$ is non-empty and is contained in Q^* . However, we will show that for any point p in Q^* , $\beta(h)$ is not constant on any Q^* -neighborhood of p so that in particular, $Z(\beta(h))$ has empty interior in Q^* . Let V be any closed

βQ -neighborhood of p . Then $V \cap Q$ contains disjoint closed intervals J and L . Because neither interval is compact, $cl_{\beta Q} J \cap Q^*$ and $cl_{\beta Q} L \cap Q^*$ are disjoint non-empty subsets of $V \cap Q^*$. But because h is a homeomorphism, the values which $\beta(h)$ takes on $cl_{\beta Q} L \cap Q^*$ are distinct from those taken on $cl_{\beta Q} J \cap Q^*$. Hence, $\beta(h)$ is not constant on $V \cap Q^*$. This example was suggested in [GJ, ex. 60].

4.23. Proposition 4.21 shows that the growth of a locally compact and realcompact space will satisfy the hypotheses of Proposition 4.18. Now by assuming the Continuum Hypothesis and restricting the cardinality of the ring of continuous functions, we obtain the central existence theorem for C -points and C^* -points.

THEOREM [CH]: (Plank)

The growth of a locally compact and realcompact but non-compact space will have a dense set of 2^c C -points (resp. C^* -points) if the cardinality of $C(X)$ (resp. $C^*(X)$) is c .

Proof: If X is a non-compact, locally compact and realcompact space, then the space X^* is a non-empty compact space and every non-empty G_δ of X^* has non-empty interior by Proposition 4.21. Further, since X is realcompact, X^* can have no isolated points. If a point p in X^* were isolated, then there would exist a zero-set neighborhood Z_1 of p in βX such that $Z_1 \cap X^* = \{p\}$. Since X is realcompact, there is a zero-set Z_2 containing p but missing X . Thus, $Z_1 \cap Z_2 = \{p\}$. But Theorem 3.7 shows that a zero-set of βX contained in X^* must have cardinality at least 2^c , giving a contradiction.

Using the Continuum Hypothesis, $\mathfrak{D} = \{X^* \setminus \bigcup_{f \in C^*(X)} Z(\beta(f)) : f \in C^*(X)\}$ is a family of $\aleph_1$ dense open subsets of X^* . Since X^* is a compact space in which the non-empty G_δ 's have non-empty interiors, $\bigcap \mathfrak{D}$ is a dense subset of at least 2^c C^* -points. Since $|\beta X| = |m^*(X)| \leq 2^c$, $\bigcap \mathfrak{D}$ must contain exactly 2^c points. The proof for C -points is similar. $\blacksquare$

4.24. The following corollary is immediate from the observation that $C^*(X)$ is contained in $C(X)$.

COROLLARY [CH]:

The growth of a locally compact and realcompact space X which fails to be compact has a dense subset of 2^c points which are both C -points and C^* -points if the cardinality of $C(X)$ is c .

4.25. Note that the hypothesis that the cardinality of $C^*(X)$ or $C(X)$ be no more than c is redundant if X is separable. The following example shows that Theorem 4.23 applies to some non-separable spaces.

EXAMPLE:

Let X be a non-closed cozero-set in $\mathbb{N}^*$. Such a cozero-set must exist in $\mathbb{N}^*$ because $\mathbb{N}^*$ is not a P -space (Proposition 1.65). X is not separable since X contains an open copy of $\mathbb{N}^*$ which in turn contains a family of c disjoint open sets by Theorem 3.22. This family of sets must then be open in X since the copy of $\mathbb{N}^*$ was open in X . The space X is realcompact because a cozero-set in a realcompact space is realcompact

[GJ, 8.14]. Since X is open in the compact space $\mathbb{I}\mathbb{N}^*$, X is locally compact. Proposition 3.24 shows that $\mathbb{I}\mathbb{N}^*$ is an F -space. Therefore the cozero-set X is C^* -embedded in $\mathbb{I}\mathbb{N}^*$ and hence is also C^* -embedded in $\beta\mathbb{I}\mathbb{N}$. Because $\beta\mathbb{I}\mathbb{N}$ is separable, the cardinality of $C(X)$ is just c .

4.26. A sufficient condition for a point p of X^* to be a C -point or a C^* -point can be stated in terms of the corresponding ideals M^p and M^{*p} . Gelfand and Kolmogoroff characterized the maximal ideal M^p of $C(X)$ by showing (Theorem 1.30) that

$$M^p = \{f \in C(X) : p \in \text{cl}_{\beta X} Z(f)\}.$$

A subfamily of M^p is the set O^p of those mappings f in $C(X)$ such that p belongs to the interior of $\text{cl}_{\beta X} Z(f)$, i.e.

$$O^p = \{f \in C(X) : p \in \text{int}_{\beta X} \text{cl}_{\beta X} Z(f)\}.$$

O^p is easily seen to be an ideal contained in M^p . The corresponding ideals of $C^*(X)$ are given (Theorem 1.26) by

$$M^{*p} = \{f \in C^*(X) : p \in Z(\beta(f))\}$$

and

$$O^{*p} = \{f \in C^*(X) : p \in \text{int}_{\beta X} Z(\beta(f))\}.$$

4.27. Now consider the case where O^p is equal to M^p and p is in X^* . For every mapping f in $C(X)$, if p is in $\text{cl}_{\beta X} Z(f)$, then p must be in $\text{int}_{\beta X} \text{cl}_{\beta X} Z(f)$. Thus, p is a C -point. The corresponding argument for O^{*p} is similar and we have verified the following

PROPOSITION:

If p is a point of X^* and $M^p = O^p$ (resp. $M^{*p} = O^{*p}$), then p is a C -point (resp. C^* -point).

4.28. EXAMPLE:

The converse of the proposition is false. We have from Theorem 4.23 that $\mathbb{N}^*$ contains a dense set of C^* -points. The function j in $C^*(\mathbb{N})$ defined by $j(n) = \frac{1}{n}$ for each n in $\mathbb{N}$ belongs to M^{*p} for all p in $\mathbb{N}^*$, but $\text{int}_{\beta X} Z(\beta(j)) = \emptyset$ so that j cannot be in O^{*p} for any p in $\mathbb{N}^*$.

P-POINTS IN X^*

4.29. In Proposition 4.16 we saw that the P -points of X^* are just the C^* -points. Thus, by combining that result with Theorem 4.23, we obtain the following two corollaries.

COROLLARY [CH]:

The growth of a locally compact and realcompact space X which is not compact has a dense set of 2^c P -points if the cardinality of $C^*(X)$ is c .

4.30. The second corollary is the original result in the investigation of the non-homogeneity of growths and was obtained by W. Rudin in 1956.

COROLLARY [CH]:

$\mathbb{N}^*$ has a dense set of 2^c P -points.

4.31. Any homeomorphism must carry P-points to P-points. Thus, any space containing both P-points and non-P-points must fail to be homogeneous. Because a pseudocompact P-space must be finite, $\mathbb{N}^*$ must also contain non-P-points.

PROPOSITION [CH]: (W. Rudin)

$\mathbb{N}^*$ contains a dense set of $2^{\mathfrak{c}}$ non-P-points and fails to be homogeneous if the Continuum Hypothesis is assumed.

Proof: $\mathbb{N}^*$ clearly contains non-P-points as we have just seen. If the Continuum Hypothesis is assumed, it also contains P-points and therefore fails to be homogeneous.

Because every G_δ containing a P-point is a neighborhood of the P-point, no P-point can be a cluster point of a sequence. Every discrete sequence in $\mathbb{N}^*$ has $2^{\mathfrak{c}}$ cluster points, so $\mathbb{N}^*$ therefore contains $2^{\mathfrak{c}}$ non-P-points. Any homeomorphism of $\beta\mathbb{N}$ must send a non-P-point of $\mathbb{N}^*$ to another non-P-point. Because the orbit of a point of $\mathbb{N}^*$ is dense in $\mathbb{N}^*$, the non-P-points form a dense subset. $\blacksquare$

The investigation of P-points and their role in the non-homogeneity of growths of Stone-Ćech compactifications now becomes an exercise in eliminating the restrictive hypotheses of Corollary 4.29.

4.32. We have seen that if X fails to be pseudocompact, then X contains a C-embedded copy of $\mathbb{N}$ (Lemma 4.5), and that X^* therefore contains a copy of $\mathbb{N}^*$. The next result shows that if X is also locally compact, then the P-points of $\mathbb{N}^*$ are

also P-points of X^* .

LEMMA:

If $\mathbb{N}$ is C-embedded in a locally compact space X , then every P-point of $\mathbb{N}^*$ is a P-point of X^* .

Proof: Let p be a P-point of $\mathbb{N}^*$ and suppose that f in $C(X^*)$ vanishes at p . We must show that f vanishes on an X^* neighborhood of p . Since X^* is closed in βX , f can be extended to a mapping g in $C(\beta X)$. Since g vanishes at p , there is an infinite subset E of $\mathbb{N}$ such that p belongs to $\text{cl}_{\beta X} E$ and g approaches zero along E . Now choose a neighborhood V_n of each n in E such that the closures in X of the neighborhoods form a pairwise disjoint family of compact sets and

$$|g(x) - g(n)| < \frac{1}{n}$$

for every x in V_n . Corollary 4.20 shows that there exists a non-negative mapping ι in $C^*(X)$ such that $\iota(n) = 1$ for each n in E and ι vanishes outside of $\bigcup V_n$. Then it is clear that p belongs to the cozero-set $\text{Cz}(\beta(\iota))$. Since ι vanishes outside of $\bigcup V_n$, no point of $X^* \cap \text{Cz}(\beta(\iota))$ is a limit point of $X \setminus \bigcup V_n$. Further, since each V_n has compact closure in X , every neighborhood of a point in $X^* \cap \text{Cz}(\beta(\iota))$ meets infinitely many of the sets $\{V_n\}$. Thus by the choice of the neighborhoods V_n , it is clear that for any point q in $X^* \cap \text{Cz}(\beta(\iota))$, $f(q) = g(q) = 0$. Thus, we have shown that

$$X^* \cap Cz(\beta(\mathcal{U})) \subset Z(f)$$

so that f vanishes on a X^* -neighborhood of p . $\blacksquare$

4.33. The preceding lemma shows that every C -embedded copy of $\mathbb{N}$ in a locally compact space will lead to the existence of P -points in the growth of the space. The next lemma shows that for a realcompact space there will be sufficiently many C -embedded copies of $\mathbb{N}$ to produce a dense set of P -points in the growth. The following result is contained in [GJ, ex. 9D].

LEMMA:

If V is a neighborhood of a point in $\beta X \setminus \alpha X$, then $V \cap \alpha X$ contains a C -embedded copy of $\mathbb{N}$, and hence V contains a copy of $\beta \mathbb{N}$.

Proof: Since βX is completely regular, we can assume that a neighborhood of a point p of X^* is a zero-set neighborhood, Z_p . Since p is in $\beta X \setminus \alpha X$, there is a zero-set $Z(f)$ such that $p \in Z(f) \subset \beta X \setminus \alpha X$. Since $Z_p \cap Z(f) \subset Z_p$, there exists g in $C(X)$ which is unbounded on Z_p and hence, $Z_p \cap \alpha X$ contains a C -embedded copy of $\mathbb{N}$ by Lemma 4.5. Thus, $Z_p \supset c\ell_{\beta X} \mathbb{N}$. $\blacksquare$

4.34. The first statement in the following theorem was established by W. Rudin in 1956 with the additional hypothesis that X be normal. The proofs of the next two theorems follow the outline of [GJ, ex. 9M] as did that of Lemma 4.32.

THEOREM [CH]:

The growth of a non-pseudocompact locally compact space contains both P-points and non-P-points. If the space is also realcompact, then both sets of points are dense.

Proof: Since X is not pseudocompact, X contains a C -embedded copy of $\mathbb{N}$. If the Continuum Hypothesis is assumed, then $\mathbb{N}^*$ contains P-points which are also P-points of X^* by Lemma 4.32. The non-P-points of $\mathbb{N}^*$ will clearly not be P-points of X^* .

If in addition X is realcompact, then every X^* -neighborhood of a point q in X^* contains a copy of $\mathbb{N}^*$ by Lemma 4.33 and thus contains both P-points and non-P-points of X^* . $\blacksquare$

Theorem 4.34 is an improvement over Corollary 4.29 since the restriction on the cardinality of $C^*(X)$ has been dropped. For example, Corollary 4.29 will not apply to an uncountable discrete space.

4.35. Note that the first statement of Theorem 4.34 implies that the growth of a non-pseudocompact, locally compact space fails to be homogeneous. If we drop the local compactness hypothesis, we will still be able to show the non-homogeneity of X^* but will not be able to show the existence of P-points in X^* . The technique will utilize one member of the family of spaces $\cup_f X$ introduced in Section 1.53. Recall that for f in $C(X)$,

$$\cup_f X = \beta X \setminus \{p \in \beta X : f^\alpha(p) = \infty\}$$

where f^α is the extension of f to βX and takes its values in $\alpha\mathbb{R}$, the one point compactification of $\mathbb{R}$. Since $\cup_f X$ is an open subspace of βX , it is clearly locally compact. The family of spaces $\{\cup_f X\}$ was introduced by M. Henriksen and is studied in [GJ, ex. 8B]. The following theorem was proven by T. Isiwata in 1957 and utilizes the subspace $\cup_f X$ for a well-chosen f .

THEOREM [CH]: (Isiwata)

The growth of a non-pseudocompact space fails to be homogeneous.

Proof: If a space X is not pseudocompact, then X contains a C -embedded copy of $\mathbb{N}$. In order to take advantage of the copy of $\mathbb{N}^*$ which $\mathbb{N}$ yields in X^* , we will obtain a convenient locally compact space Y with $X \subset Y \subset \beta X$, to which Lemma 4.32 can be applied. We shall then show that if ϕ is an automorphism of X^* such that $\phi(p)$ is in $\mathbb{N}^*$ for some P -point p of $\mathbb{N}^*$, then $\phi(p)$ is also a P -point of $\mathbb{N}^*$. This will complete the proof since it will show that a P -point of $\mathbb{N}^*$ cannot be mapped to all points of $\mathbb{N}^*$ by homeomorphisms of X^* .

Let f be a mapping in $C(X)$ such that $f(n) = n$ for all n in $\mathbb{N}$. Then consider

$$Y = \cup_f X \setminus \phi^{-1}(\mathbb{N}^*).$$

Since Y is obtained by deleting a compact subspace of βX , Y is locally compact. Since the restriction $f^\alpha|_Y$ is unbounded on $\mathbb{N}$, Lemma 4.5 shows that we can assume that $\mathbb{N}$ is C -embedded in Y . It is clear that $\mathbb{N}^*$ is contained in Y^* .

By Lemma 4.32, every P-point of $\mathbb{N}^*$ is a P-point of Y^* . Further, $\phi|Y^*$ is a homeomorphism onto its range, so that $\phi(p)$ is a P-point of $\phi[Y^*]$. Since $\phi(p) \in \mathbb{N}^* \subset \phi[Y^*]$, $\phi(p)$ is a P-point of $\mathbb{N}^*$. $\square$

4.36. Note that the theorem states only that X^* is not homogeneous. While the proof utilizes P-points, it does not necessarily imply that there exist P-points in X^* . The P-points need only be P-points of a subspace of X^* .

EXAMPLE:

Since Q is clearly not pseudocompact, the theorem shows that Q^* is not homogeneous. However Q^* contains no P-points. We first investigate another characterization of P-points. If f is in $C(X)$ for any X , then $f^{\leftarrow}(r)$ is a zero-set because every point of $\mathbb{R}$ is a zero-set. Thus, if p is a P-point and belongs to $f^{\leftarrow}(r)$, f is constant on a neighborhood of p since every zero-set containing p is a neighborhood of p . Returning to Q^* , in Example 4.22 we exhibited a mapping $\beta(h)$ on Q^* which is not constant on any open set of Q^* . Hence, Q^* can contain no P-point.

REMOTE POINTS IN X^*

4.37. A remote point of βX is a point which does not belong to the closure of any discrete subspace of X . It is clear that any remote point must lie in X^* . Remote points are related to C-points in somewhat the same manner that P-points are

related to C^* -points. However, the existence theorems for remote points require much more stringent hypotheses. The existence of remote points seems to be a phenomenon confined to metric spaces, but by no means common to all metric spaces. For example, if D is an infinite discrete space, it is clear from the definition of remote point that βD can contain no such point. To eliminate this behavior, we shall find it necessary to restrict the existence of isolated points in order to obtain remote points. One property of metric spaces which we shall find useful is that every closed set is a zero-set. The other useful properties will be outlined in the next two lemmas.

4.38. The first lemma uses the following property of realcompact spaces: If X is a realcompact space, the intersection of the free maximal ideals of $C(X)$ is the subring of $C(X)$ consisting of those mappings having compact support, i.e. those maps f in $C(X)$ for which $\text{cl}_X \text{Cz}(f)$ is compact [GJ, 8.20]. Denote this subring by $C_K(X)$. Since the free maximal ideals in $C(X)$ correspond to the points of X^* , we can write

$$\bigcap \{M^p : p \in X^*\} = C_K(X)$$

if X is realcompact.

Recall that a point p of X^* is a C-point if $p \notin \text{int}_* (\text{cl}_{\beta X} \text{Cz}(f) \cap X^*)$ for all f in $C(X)$.

LEMMA:

If X is realcompact, then

$$\text{int}_{X^*}(\text{cl}_{\beta X}(Z(f)) \cap X^*) = \text{int}_{\beta X}(\text{cl}_{\beta X} Z(f)) \cap X^*$$

for every mapping f in $C(X)$.

Proof: The containment

$$\text{int}_{\beta X}(\text{cl}_{\beta X} Z(f)) \cap X^* \subset \text{int}_{X^*}(\text{cl}_{\beta X}(Z(f)) \cap X^*)$$

is clear. Now let p be a point in $\text{int}_{X^*}(\text{cl}_{\beta X}(Z(f)) \cap X^*)$.

Since the closures in βX of the zero-sets of X form a base for the closed sets of βX , there exists g in $C(X)$ such that

$$p \in X^* \setminus (\text{cl}_{\beta X}(Z(g)) \cap X^*) \subset \text{cl}_{\beta X}(Z(f)) \cap X^*.$$

Then g fails to belong to M^p but the product fg belongs to $\cap \{M^q : q \in X^*\}$ since every point of X^* belongs to either $\text{cl}_{\beta X} Z(f)$ or $\text{cl}_{\beta X} Z(g)$. Thus, since X is realcompact, fg is in $C_K(X)$ and $K = \text{cl}_X C_Z(fg)$ is compact. Since K is contained in X , p cannot belong to K and since p also does not belong to $\text{cl}_{\beta X} Z(g)$, we have that

$$p \in \beta X \setminus (K \cup \text{cl}_{\beta X} Z(g)) \subset \text{cl}_{\beta X} Z(f),$$

so that p belongs to $\text{int}_{\beta X} \text{cl}_{\beta X} Z(f)$. $\square$

We saw in Example 4.28 that the converse of Proposition 4.27 was false for C^* -points. The importance of the preceding lemma will be to show that the converse is true for C -points when X is a metric space. The lemma can actually be proven in greater

generality. In 1966, S. M. Robinson showed that if X is completely uniformizable, then $\bigcap \{M^p : p \in X^*\} = C_K(X)$. Since if a measurable cardinal exists, there are completely uniformizable spaces which are not realcompact, Robinson's result is more general than the one cited for realcompact spaces. However, the realcompact hypothesis includes all metric spaces of non-measurable cardinal and in particular, all separable metric spaces. For a discussion of the role of the measurable cardinal and to see how unlikely it is that any metric space of interest fails to be realcompact, see [GJ, Chapter 12 and Theorems 15.20, 15.24].

4.39. The second lemma is important in the study of remote points since it provides many well-placed discrete subspaces in metric spaces. The lemma is taken from F. Hausdorff's book, Set Theory.

LEMMA:

In a metric space, the boundary of an open set can be represented as the set of cluster points of a discrete subspace contained in the open set.

Proof: Let X be a metric space with metric d and let F be the boundary of an open subset G of X . We can assume that F is non-empty, since the discrete subspace can be chosen to be the empty set if F is empty. For each x in G , let $\delta(x)$ be the distance of x from F . Then $\delta(x) > 0$. Now consider all subsets S of G which satisfy

$$d(x,y) \geq \frac{1}{2} \delta(x) + \frac{1}{2} \delta(y)$$

for x and y in S . Let A be a maximal set with respect to this property.

The cluster points of A are contained in F : A cluster point x of A is the limit of a sequence $\{x_n\}$ of distinct points of A . Since δ is continuous, $\delta(x_n) \rightarrow \delta(x)$, and in the inequality

$$d(x_n, x_{n+1}) \geq \frac{1}{2} \delta(x_n) + \frac{1}{2} \delta(x_{n+1}),$$

the left-hand side converges to 0 and the right-hand side to $\delta(x)$. Hence, $\delta(x) = 0$ and x is in F .

Every point of F is a cluster point of A : For a point z in F and $\epsilon > 0$, choose y in G such that $d(z,y) < \epsilon$. If y is in A , we are done. If not, the maximality of A implies that there is a point x in A for which

$$d(x,y) < \frac{1}{2} \delta(x) + \frac{1}{2} \delta(y).$$

Then we have

$$(*) \quad d(x,y) < \frac{1}{2} \delta(x) + \frac{1}{2} \delta(y) \leq \frac{1}{2} d(z,x) + \frac{1}{2} d(z,y)$$

from the definition of δ . Now we must show that x is close to z . Using the triangle inequality and (*), we have

$$d(z,x) \leq d(z,y) + d(x,y) < \frac{1}{2} d(z,x) + \frac{3}{2} d(z,y)$$

which yields

$$d(z,x) \leq 3d(z,y) < 3\epsilon.$$

Finally, the discreteness of A follows from the fact that all cluster points of A are contained in F . $\square$

4.40. We now relate remote points to C -points. Recall that in Section 1.32 we introduced the following notation for the z -ultrafilters on a space X : If p is in βX ,

$$A^p = \{Z(f) \in Z[X] : f \in M^p\}.$$

THEOREM:

Let p be in X^* where X is a metric space of non-measurable cardinal, and consider the following four conditions:

- (a) p is a C -point of X^* ,
- (b) A^p has no member which is nowhere dense.
- (c) $M^p = O^p$.
- (d) p is a remote point in βX .

Conditions (a), (b), and (c) are equivalent and are implied by (d).

All four conditions are equivalent if the set of isolated points of X has compact closure in X .

Proof: (a) $\Rightarrow$ (b): Let p be a C -point and let Z be in A^p . Then p is in $\text{int}_{X^*}(\text{cl}_{\beta X} Z \cap X^*)$. Thus by Lemma 4.38, p is in $V = \text{int}_{\beta X} \text{cl}_{\beta X} Z$. But then $V \cap X$ is a non-empty open set contained in Z and Z is not nowhere dense.

(b) $\Rightarrow$ (c): Let f be in M^p . Since X is a metric space, we can find a g in $C(X)$ such that $Z(g) = \text{cl}_X(X \setminus Z(f))$; thus $X = Z(f) \cup Z(g)$. If p is in $\text{cl}_{\beta X}(Z(g))$, g must be in M^p

and we must then have p in $cl_{\beta X}(Z(f) \cap Z(g)) = cl_{\beta X} \partial_X(Z(f))$. But this contradicts (b) since $\partial_X(Z(f))$ is nowhere dense. Thus, p is in $\beta X \setminus cl_{\beta X} Z(g) \subset cl_{\beta X} Z(f)$ which shows that f is in O^p .

(c) $\Rightarrow$ (a): This follows from Proposition 4.27.

(d) $\Rightarrow$ (b): Suppose that A^p has a nowhere dense member Z . Then Z is the boundary of its complement and by Lemma 4.39, there exists a discrete space $D \subset X \setminus Z$ such that $D \cup Z = cl_X D$. Thus, p is in $cl_{\beta X} Z \subset cl_{\beta X} D$, so p is not a remote point.

Now assuming in addition that the set L of isolated points of X has compact closure in X , we can show (b) $\Rightarrow$ (d):

It is sufficient to show that if p is not a remote point, then some member of A^p is nowhere dense. Let D be a discrete subspace of X such that p is in $cl_{\beta X} D$. Then $cl_X D$ is in A^p and if $cl_X D$ is not nowhere dense, then there exists a point in $D \cap int_X cl_X D$. But such a point must be isolated because D is discrete. Since $cl_{\beta X}(cl_X L \cap cl_X D)$ is contained in X because $cl_X L$ is compact, p belongs to $cl_{\beta X}(D \setminus L)$, and thus $cl_X(D \setminus L)$ is a nowhere dense member of A^p . $\blacksquare$

The equivalence of (b) and (d) was shown in 1962 by Fine and Gillman and the proof that (b) implies (c) was contributed by M. Mandelker. The theorem appears in D. Plank's 1969 paper.

4.41. By combining the hypotheses of the preceding theorem with those of Theorem 4.23, we obtain the following result concerning the existence of remote points. The separability hypothesis shows that the cardinality of $C(X)$ is c and also that the cardinality of X is non-measurable so that X is realcompact. (Note that

Exercise 1D also shows that X is realcompact.)

PROPOSITION:

If X is a separable, locally compact, non-compact metric space in which the set of isolated points has compact closure, then βX contains a set of 2^c remote points which is a dense subspace of X^* .

4.42. The proposition establishes the existence of 2^c remote points of $\beta \mathbb{R}$ which form a dense subspace of $\mathbb{R}^*$, but the local compactness hypothesis prevents an application to such basic examples as $\mathbb{Q}$. In order to eliminate the need for local compactness, it is necessary to use the methods developed by Fine and Gillman in 1962. The modification of their argument which will be employed in the proof of the following theorem was suggested by Plank in 1969. The crucial tool in the proof is Lemma 4.19 which will be used in each step of a transfinite induction argument.

THEOREM [CH]:

If X is a non-compact separable metric space in which the set of isolated points has compact closure, then βX contains 2^c remote points which form a dense subspace of X^* .

Proof: Since X is a separable metric space, it has a countable base of open sets [D, p. 187] and therefore under the assumption of the Continuum Hypothesis there can be no more than $\aleph_1$ dense open subsets of X . Let $\{U_\alpha : \omega_0 \leq \alpha < \omega_1\}$ be the family of dense open sets. Since a separable metric space has non-measurable

cardinal, X is realcompact and a point p of X^* is a remote point if and only if no member of A^p is nowhere dense (Theorem 4.40). Note that Exercise 1D also shows that X is realcompact.

Now let V be a closed βX -neighborhood of a point q in X^* . Since the set of isolated points of X has compact closure in X , we may assume that $V \cap X$ contains no isolated points. In order to exhibit a remote point in V , it is sufficient to show the existence of a free z -ultrafilter A^p on X such that no member of A^p is nowhere dense and the corresponding point p belongs to V . We will show that such z -ultrafilters exist by constructing a family $\mathfrak{F}$ of zero-sets contained in $V \cap X$ such that $\mathfrak{F}$ has the finite intersection property and any z -ultrafilter containing $\mathfrak{F}$ has no nowhere dense member.

(a) There exists a family $\mathfrak{F} = \{F_\alpha : \alpha < \omega_1\}$ of non-void zero-sets of X such that:

- (1) $\mathfrak{F}$ has the finite intersection property.
- (2) $\mathfrak{F}$ has empty intersection.
- (3) Every member of $\mathfrak{F}$ is contained in $V \cap X$.
- (4) Every dense open subset of X contains a member of $\mathfrak{F}$.

The family $\mathfrak{F}$ is constructed by transfinite induction. Since X is realcompact and q is not in X , there exists h in $C(X)$ such that h is unbounded on $V \cap X$. Choose a non-negative g in $C(\beta X)$ such that $g(q) = 1$ and g vanishes on $\beta X \setminus V$. Define $f = h \cdot (g|X)$. Then f belongs to $C(X)$ and is unbounded on $V \cap X$. Now for $n < \omega_0$, choose the sets E_n for the mapping f as in Lemma 4.19. Then the lemma shows that $F_n = \bigcup \{E_i : i \geq n\}$

is a zero-set, $\text{int } E_i \neq \emptyset$ for each i , and $\text{int } F_n = \bigcup \{\text{int } E_i : i \geq n\}$.

The family $\{F_n : n < \omega_0\}$ completes the construction of $\mathfrak{F}$ for the natural numbers. Since $\bigcap F_n = \emptyset$, $\mathfrak{F}$ will have empty intersection. We will now construct the remainder of the family $\mathfrak{F}$, i.e. $\{F_\alpha : \omega_0 \leq \alpha < \omega_1\}$, in such a way that each F_α is contained in the corresponding U_α .

Consider any ordinal α with $\omega_0 \leq \alpha < \omega_1$ and assume that for each $\beta < \alpha$ we have defined a zero-set F_β such that the family $\{\text{int } F_\beta : \beta < \alpha\}$ has the finite intersection property. To define F_α , first rearrange the countable family $\{\text{int } F_\beta : \beta < \alpha\}$ into a sequence $\{T_n : n < \omega_0\}$. Next define an increasing sequence of integers $\{n_i\}$ and a sequence of zero-sets $\{Z_j\}$ as follows: Let $k < \omega_0$ and assume that the n_i have been defined for all $i < k$ and that the Z_j have been defined for all $j \leq n_{k-1}$. By the induction hypothesis, $T_1 \cap \dots \cap T_k$ meets $\text{int } F_{n_{k-1}} + 1$ and hence meets $\text{int } E_{n_k}$ for some $n_k > n_{k-1}$. Since U_α is open and dense, we can choose a zero-set Z_{n_k} with non-void interior such that

$$Z_{n_k} \subset T_0 \cap \dots \cap T_k \cap E_{n_k} \cap U_\alpha.$$

Finally, put $Z_j = \emptyset$ for $n_{k-1} < j < n_k$ to complete the definition of the sequence $\{Z_j\}$. Now define $F_\alpha = \bigcup Z_j$. Since Z_j is contained in E_j for each j , Lemma 4.19 shows that F_α is a zero-set having non-empty interior. To complete the induction step, note that any finite intersection of the sets $\{\text{int } F_\beta\}_{\beta < \alpha}$ contains a set of the form $T_0 \cap \dots \cap T_k$. Therefore

it contains Z_{n_k} and hence meets $\text{int } F_\alpha$. Thus, the collection $\{\text{int } F_\beta : \beta \leq \alpha\}$ has the finite intersection property. Note that each F_α is contained in $V \cap X$ and that by definition, F_α is contained in U_α for $\omega_0 \leq \alpha < \omega_1$. Finally, $\mathfrak{F}$ has the finite intersection property since the interiors of the members of $\mathfrak{F}$ do.

(b) $V \cap X^*$ contains a remote point: Since $\mathfrak{F}$ has the finite intersection property, $\mathfrak{F}$ is contained in some z -ultrafilter A^p and A^p must be free since $\mathfrak{F}$ has empty intersection. Since the members of $\mathfrak{F}$ are contained in V , the point p is in $V \cap X^*$. Now let

$$S = \{p \in \beta X : \mathfrak{F} \subset A^p\} = \bigcap_{Z \in \mathfrak{F}} \text{cl}_{\beta X} Z.$$

Then S is a non-void compact subset of $V \cap X^*$. For p in S , A^p can contain no member which is nowhere dense since the complement of such a member would necessarily contain a member of $\mathfrak{F}$ which would contradict the fact that A^p is a z -ultrafilter containing $\mathfrak{F}$. Thus, by Theorem 4.40, the set S consists of remote points of βX .

(c) The set of remote points is dense in X^* and contains 2^c points: Since V was chosen as an arbitrary neighborhood of the arbitrary point q , it is clear that the set of remote points is dense in X^* . To obtain a lower bound on the number of remote points, observe that since $V \cap X$ contains no isolated points every open subset of $V \cap X$ contains two disjoint zero-sets having non-empty interiors. Thus, at each stage of the construction, there were two possible choices of zero-sets. Hence, assuming the Continuum Hypothesis, there are at least

2^c remote points. On the other hand, since X is separable, βX is an image of $\beta \mathbb{N}$ and thus can contain no more than 2^c points since the cardinality of $\beta \mathbb{N}$ is 2^c . **|**

4.43. EXAMPLE:

Theorem 4.42 shows that Q^* contains a dense set of 2^c remote points of βQ . Note that Lemma 4.33 shows that Q^* also contains a dense set of 2^c points which fail to be remote points of βQ . If p is a remote point of βQ , then the subspace $Q \cup \{p\}$ shows that there exists a countable space without isolated points such that one of the points is not the limit of any discrete subspace.

A similar example can be obtained by regarding Q as a subspace of $\beta \mathbb{R}$ and considering $Q \cup \{q\}$ where q is a remote point of $\beta \mathbb{R}$.

EXERCISES

4A. P-POINTS OF $\mathbb{N}^*$

Let p be in $\mathbb{N}^*$. Then p is a P-point of $\mathbb{N}^*$ if and only if for every countable family $\{Z_n\}$ contained in A^p there exists Z in A^p such that $|Z \setminus Z_n|$ is finite for every n .

Reference: N. Hindman, Thesis, 1969.

4B. COMPACT F-SPACES

If X is a compact F-space and S is a subspace of X such that $|X \setminus S| < 2^c$, then S is pseudocompact. [Proposition 1.64.]

Reference: Fine and Gillman, 1960.

CHAPTER FIVE:
CELLULARITY OF GROWTHS

5.1. In this chapter we consider relationships between the existence of families of open subsets of a space X and the cellularity of $X^* = \beta X \setminus X$. Recall that the cellularity of a space Y is the smallest cardinal number m for which each pairwise disjoint family of non-empty open sets of Y has m or fewer members. The density of a space Y is the smallest cardinal number which can be the cardinal number of some dense subspace of Y . It is clear that the density of a space is at least as great as the cellularity. Most of the results will take the form of providing a lower bound for the cellularity of X^* by demonstrating the existence of families of pairwise disjoint open sets in X^* . The methods used will be reminiscent of that used in Chapter 3 to show that the cellularity of $\mathbb{N}^*$ is c . In the last sections of the chapter, we will show that there exists a point in $\mathbb{N}^*$ which belongs to the closure of each member of a family of c pairwise disjoint open sets of $\mathbb{N}^*$.

LOWER BOUNDS FOR THE CELLULARITY OF X^*

5.2. The central results on the cellularity of X^* were obtained by W. W. Comfort and H. Gordon in 1964. The following theorem appears in a more general form in the 1928 paper of A. Tarski and will play a role similar to that played by Proposition 3.21 in the determination of the cellularity of $\mathbb{N}^*$.

Recall that a family of sets is said to be almost disjoint if any two members of the family meet in a finite set.

THEOREM: (Tarski)

If m and n are cardinal numbers with n infinite,
then $m \leq n^{\aleph_0}$ if and only if a set having cardinality n admits
an almost disjoint collection of cardinality m consisting of
infinite subsets.

Proof: Let D be a set with cardinality n and assume that $\mathcal{C}$ is a collection of subsets satisfying the required conditions. With each E in $\mathcal{C}$, associate a countably infinite subset S_E of E . Since the members of $\mathcal{C}$ are almost disjoint, the assignment $E \mapsto S_E$ is one-to-one from $\mathcal{C}$ into the set of countably infinite subsets of D , and $m \leq n^{\aleph_0}$.

To establish the converse, it is sufficient to show that there exists an almost disjoint family of cardinality $n^{\aleph_0}$ of infinite subsets of D . Let $\mathfrak{F}$ be the set of all finite sequences of distinct elements of D and let $\mathcal{G}$ be the set of all countably infinite sequences of distinct elements of D . Then the cardinality of $\mathfrak{F}$ is n and of $\mathcal{G}$ is $n^{\aleph_0}$. The proof will be completed by finding a collection of almost disjoint infinite subsets of $\mathfrak{F}$ indexed by $\mathcal{G}$. For each G in $\mathcal{G}$, let I_G be the set of all F in $\mathfrak{F}$ such that F is an initial segment of G . Then each set I_G is infinite and any distinct pair of sets of the form I_G have only finitely many elements of $\mathfrak{F}$ in common. Thus, $\mathcal{C} = \{I_G : G \in \mathcal{G}\}$ is an almost disjoint family of infinite subsets of $\mathfrak{F}$ and the cardinality

of $\mathcal{E}$ is $n^{\aleph_0}$. $\square$

5.3. A family of subsets of a space X is called locally finite if every point of X has a neighborhood which meets only finitely many members of the family. A subset is called relatively compact if its closure is compact. We now apply Tarski's Theorem to obtain a lower bound for the cellularity of X^* related to the existence of a locally finite family of open subsets of X .

THEOREM: (Comfort and Gordon)

If X admits an infinite locally finite collection of n non-empty relatively compact open sets, then the cellularity of X^* is at least $n^{\aleph_0}$.

Proof: Let $\mathcal{J}$ be the locally finite collection of relatively compact open sets. We first show that the members of $\mathcal{J}$ can be shrunk to obtain a pairwise disjoint family of n compact sets with non-void interiors. Let g be a function from $\mathcal{J}$ to X such that $g(T)$ is in T for T in $\mathcal{J}$, and let $A = g[\mathcal{J}]$. For each a in A , choose T_a in $\mathcal{J}$ such that $g(T_a) = a$. The function $a \mapsto T_a$ is one-to-one and onto a subset of $\mathcal{J}$ and the local finiteness of $\mathcal{J}$ guarantees that the cardinality of A is n . Using the local finiteness of $\mathcal{J}$ again and the regularity of X , we can find compact sets U_a and Y_a for each a in A such that

$$a \in \text{int } U_a \subset U_a \subset \text{int } Y_a \subset Y_a \subset T_a,$$

and such that $Y_a \cap A = \{a\}$. Now define

$$V_a = U_a \setminus \bigcup \{ \text{int } Y_b : b \in A, b \neq a \}.$$

Then V_a is a compact neighborhood of a . Further, if a and b are distinct points of A and if x is in V_a , then x is not in $\text{int } Y_b$ so that V_a and V_b are disjoint since V_b is contained in $\text{int } Y_b$. Thus,

$$\mathcal{V} = \{V_a : a \in A\}$$

is a pairwise disjoint and locally finite collection of n compact sets each having non-void interior.

Now apply Tarski's Theorem to find a collection $\mathcal{E}$ of $n^{\aleph_0}$ infinite subsets of A such that any two members of $\mathcal{E}$ have finite intersection. For each E in $\mathcal{E}$, define

$$W_E = \beta X \setminus \text{cl}_{\beta X} \left(X \setminus \bigcup_{a \in E} V_a \right).$$

Since each set W_E is open, the proof will be completed by showing that

$$\{W_E \cap X^* : E \in \mathcal{E}\}$$

is a pairwise disjoint family of non-void sets.

We begin by showing that the family is pairwise disjoint. Choose distinct elements E_1 and E_2 of $\mathcal{E}$. Then X can be written

$$(*) \quad X = \left(X \setminus \bigcup_{a \in E_1} V_a \right) \cup \left(X \setminus \bigcup_{a \in E_2} V_a \right) \cup \left(\bigcup_{a \in E_1} V_a \cap \bigcup_{a \in E_2} V_a \right).$$

Call the last of these sets K . Then K can be written

$K = \bigcup_{a \in E_1 \cap E_2} V_a$ and is therefore a compact subset of X . Taking the closure in βX of (*) yields

$$\beta X = \text{cl}_{\beta X}(X \setminus \bigcup_{a \in E_1} V_a) \cup \text{cl}_{\beta X}(X \setminus \bigcup_{a \in E_2} V_a) \cup K.$$

Taking complements in βX now gives

$$W_{E_1} \cap W_{E_2} \cap (\beta X \setminus K) = \emptyset$$

so that $W_{E_1} \cap W_{E_2} \subset K \subset X$. Thus, $(W_{E_1} \cap X^*) \cap (W_{E_2} \cap X^*) = \emptyset$.

To show that $W_E \cap X^*$ is non-empty, begin by associating to each a in A a continuous function f_a of X to $[0,1]$ such that $f_a(a) = 1$ and $f_a(x) = 0$ for all x in $X \setminus V_a$. For E in $\mathcal{C}$, define $f_E = \sum_{a \in E} f_a$. The local finiteness of $\mathcal{V}$ guarantees that f_E is continuous. Let

$$F_E = \{x \in X : f_E(x) = 1\}$$

and note that $E \subset F_E \subset X$. Since F_E is the union of a locally finite family of closed subsets of X , F_E is closed in X .

But E contains the infinite discrete subset E which is not closed in βX . Hence, there is a cluster point p_E of F_E in X^* and $\beta(f_E)(p_E) = 1$. Since f_E vanishes on $X \setminus \bigcup_{a \in E} V_a$, $\beta(f_E)$ vanishes on the βX -closure of this set. Hence, p_E is in $W_E \cap X^*$. $\blacksquare$

5.4. EXAMPLES:

The theorem shows that the locally compact spaces ω_1 and the Deleted Tychonoff Plank $T = (\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}$

admit no infinite locally finite collections of open subsets. More generally, this will be the case for any space X for which X^* contains at most one point.

On the other hand, an infinite discrete space D of cardinality n admits a locally finite family of n finite open sets, and hence D^* has cellularity at least $n^{\aleph_0}$.

5.5. The property of not admitting infinite locally finite collections of open sets is characteristic of the class of pseudocompact spaces and will allow us to obtain a lower bound for the cellularity of the growth of a non-pseudocompact locally compact space.

PROPOSITION:

A space is pseudocompact if and only if it admits no infinite locally finite collection of non-empty open subsets.

Proof: Suppose that a space X admits a countably infinite locally finite collection of non-empty open sets $\{U_n\}$. Choose a sequence of distinct points $\{x_n\}$ such that x_n belongs to U_n . Choose a real-valued continuous function f_n such that $f_n(x_n) = n$ and $f_n(y) = 0$ for y in $X \setminus U_n$. Then $f = \sum f_n$ is continuous because $\{U_n\}$ is locally finite and X is not pseudocompact since f is unbounded.

On the other hand, if X is not pseudocompact, then there exists an unbounded real-valued continuous function f on X such that $f[X]$ contains a sequence $\{r_n\}$ which has no accumulation point in $\mathbb{R}$. Choose a locally finite family of open intervals

$\{V_n\}$ in $\mathbb{R}$ such that r_n is in V_n . Then $\{f^{\leftarrow}(V_n)\}$ is an infinite locally finite sequence of non-empty open sets of X . |

5.6. In a locally compact space, any locally finite family of open sets can be assumed to be composed of relatively compact sets. From this observation, the following corollary is immediate since $c = \aleph_0$.

COROLLARY:

The cellularity of the growth of a locally compact but non-pseudocompact space is at least c .

Thus, $\mathbb{R}^*$ has cellularity of at least c . Theorem 5.8 below will show that c is also an upper bound for the cellularity of c .

5.7. Theorem 5.3 shows that the existence of a locally finite family of n open subsets of a space X implies the existence of a family of $n^{\aleph_0}$ pairwise disjoint open subsets of X^* . The following theorem shows that the existence of such a locally finite family is not a necessary condition by showing that for any cardinal number m , there exists a pseudocompact space whose growth has cellularity m . However, by Proposition 5.5, the pseudocompact space can admit only a finite locally finite collection of open subsets.

THEOREM: (Comfort and Gordon)

If m is any cardinal number, there exists a locally compact space S such that:

- (1) The cellularity of S^* is m .
- (2) Any space having the same growth as S is pseudocompact.

Proof: If m is finite, choose X to be the discrete space consisting of m points. Otherwise, choose X to be the one point compactification of the discrete space of cardinality m .

In either case, the cellularity of X is m . In Proposition 4.3 we saw that there exists a space S such that S^* is homeomorphic to X . It remains to show that any space with growth X is pseudocompact. Theorem 4.34 shows that the growth of a non-pseudocompact locally compact space contains infinitely many non-P-points. Since X can contain at most one non-P-point, any space with X as its growth must be pseudocompact. $\blacksquare$

5.8. In Theorem 3.22, we used an almost disjoint family of c infinite subsets of $\mathbb{N}$ in showing that the cellularity of $\mathbb{N}^*$ is c . The following theorem describes an analogous situation. The requirement that the members of the family of sets be infinite is replaced by requiring each set of the family to contain a non-compact zero-set and the almost disjoint requirement is replaced by asking that any two members of the family meet in a relatively compact set, i.e. in a set having compact closure.

THEOREM: (Comfort and Gordan)

If m is a cardinal number, then the cellularity of X^* is at least m if and only if X admits a collection of m cozero-sets $\{U_\alpha\}$ such that:

- (1) Each U_α contains a non-compact zero-set, and
- (2) $U_\alpha \cap U_\beta$ is relatively compact for $\alpha \neq \beta$.

Proof: Assume that there is a collection $\{V_\alpha\}$ of m pairwise disjoint non-empty open subsets of X^* . For each α , choose p_α in V_α and an open subset W_α of βX such that $W_\alpha \cap X^* = V_\alpha$. Next, for each α choose a mapping f_α of βX into I with $f_\alpha(p_\alpha) = 1$ and $f_\alpha(x) = 0$ for all x not in W_α . Define

$$U_\alpha = \{x \in X : f_\alpha(x) > \frac{1}{2}\}$$

and

$$Z_\alpha = \{x \in X : f_\alpha(x) \geq \frac{2}{3}\}.$$

To show that Z_α is the non-compact zero-set required in (1), it suffices to show that p_α is in Z_α^* . But this is clear since p_α is not in $(X \setminus Z_\alpha)^*$ because $f_\alpha(p_\alpha) = 1$ and $f_\alpha \leq \frac{2}{3}$ on $(X \setminus Z_\alpha)^*$. Now suppose that $\{U_\alpha\}$ fails to satisfy (2). Then for some pair α and β , $\alpha \neq \beta$, there exists a point p of X^* for which

$$p \in \text{cl}_{\beta X} \text{cl}_X(U_\alpha \cap U_\beta) = \text{cl}_{\beta X}(U_\alpha \cap U_\beta).$$

But then $f_\alpha(p) \geq \frac{1}{2}$ and $f_\beta(p) \geq \frac{1}{2}$ and it follows that p is in $W_\alpha \cap W_\beta \cap X^* = V_\alpha \cap V_\beta = \emptyset$, which is a contradiction.

To show the converse, assume that X admits a collection

$\{U_\alpha\}$ of cozero-sets satisfying (1) and (2). For each α , let Z_α be a non-compact zero-set contained in U_α . Since βX is normal and disjoint zero-sets of X have disjoint closures in βX , we can choose a mapping f_α of βX into I for each α such that $f_\alpha(p) = 0$ for all p in $cl_{\beta X}(X \setminus U_\alpha)$ and $f_\alpha(q) = 1$ for all q in $cl_{\beta X} Z_\alpha$. Let

$$V_\alpha = \{p \in \beta X : f_\alpha(p) > 0\}.$$

To complete the proof, we need to show that $V_\alpha \cap X^* \neq \emptyset$ for each α and that $V_\alpha \cap V_\beta \cap X^* = \emptyset$ for $\alpha \neq \beta$. First, note that $\emptyset \neq Z_\alpha^* \subset V_\alpha \cap X^*$ so that $V_\alpha \cap X^*$ is non-empty. Now suppose that p is in $V_\alpha \cap V_\beta$ for distinct α and β . Then $(f_\alpha f_\beta)(p) > 0$ so that

$$p \notin cl_{\beta X}(X \setminus U_\alpha) \cup cl_{\beta X}(X \setminus U_\beta).$$

Hence, p is in $cl_{\beta X}(U_\alpha \cap U_\beta)$ so that p belongs to X since $U_\alpha \cap U_\beta$ is relatively compact. Thus, $V_\alpha \cap V_\beta \cap X^*$ is empty. |

5.9. EXAMPLES:

The only relatively compact open subset of Q is the empty set, hence the only families of open sets of Q satisfying the hypothesis of Theorem 5.8 are pairwise disjoint families.

Because Q will admit only countable families of pairwise disjoint open sets, the theorem shows that the cellularity of Q^* is $\aleph_0$.

Because $\mathbb{R}$ is locally compact and not pseudocompact,

the cellularity of $\mathbb{R}^*$ is at least c by Corollary 5.6. Since $\mathbb{R}$ has a countable base for the open sets, Theorem 5.8 shows that the cellularity of $\mathbb{R}^*$ is exactly c .

5.10. Since every subset of a discrete space is both a zero-set and a cozero-set and the only compact subsets are finite, the preceding theorem and Tarski's Theorem yield the following corollary for discrete spaces.

COROLLARY:

For the infinite discrete space D of cardinality n , the cellularity of D^* is $n^{\aleph_0}$.

Thus, we see again that the cellularity of $\mathbb{N}^*$ is c . Since $c^{\aleph_0} = c$, we have the interesting result that the cellularity of D^* is no larger when the cardinality of D is c .

5.11. COROLLARY:

If D is the discrete space of cardinality c , the cellularity of D^* is c .

5.12. It is particularly simple to construct a dense subspace of the growth of a discrete space and hence we can determine the density of such a growth. The following result showing that the density and cellularity are equal was obtained in 1965 by W. W. Comfort.

THEOREM: (Comfort)

The density of the growth of the discrete space of infinite cardinality n is $n^{\aleph_0}$.

Proof: Let D be the discrete space of cardinality n . In Corollary 5.10, we saw that the cellularity of D^* is $n^{\aleph_0}$ so that the density of D^* is at least $n^{\aleph_0}$.

To obtain the reverse inequality, first observe that for any open set of D^* there is a countable set M of D such that the open set contains M^* . Let C be the collection of all countably infinite subsets of D and choose a point p_M in M^* for each M belonging to C . Then $\{p_M : M \in C\}$ is a dense subset of D^* having cardinality $n^{\aleph_0}$. $\square$

n-POINTS AND UNIFORM ULTRAFILTERS

5.13. If n is a cardinal number, a point is called an n -point if the point lies simultaneously in the closures of a pairwise disjoint family of n open sets which do not contain the point. In 1967, R. S. Pierce asked if the existence of a 3-point in $\mathbb{N}^*$ could be demonstrated without using the Continuum Hypothesis. Pierce's question was motivated by his use of sheaf theory to relate questions involving regular rings and modules to topological problems. In 1969, N. Hindman showed without the Continuum Hypothesis that not only did a 3-point exist in $\mathbb{N}^*$, but that in fact there is a c -point in $\mathbb{N}^*$. This section will be devoted to Hindman's work and the following section to Pierce's investigation of 2- and 3-points in totally

disconnected compact spaces.

An ultrafilter on a discrete space is called a uniform ultrafilter if every member of the ultrafilter has the same cardinality as the space. If D is an infinite discrete space, let uD denote the subspace of βD consisting of the points p of βD such that A^p is a uniform ultrafilter. It is clear that uD is contained in D^* and that $u\mathbb{N} = \mathbb{N}^*$. The following theorem shows that the cardinality of uD is the same as that of βD .

THEOREM:

If D is the infinite discrete space of cardinality n , uD has cardinality 2^{2^n} .

Proof: We first show that there exists a uniform ultrafilter on D . Let $\mathfrak{F}$ be the collection of all subsets of D whose complements have cardinality strictly less than n . Then $\mathfrak{F}$ is a filter and any ultrafilter containing $\mathfrak{F}$ must be uniform.

In order to show that $|uD| = 2^{2^n}$, we will show that uD contains a copy of βD . It is first necessary to show that uD is closed. If p does not belong to uD , then p is in $cl_{\beta D} Z$ for some Z of cardinality less than n . But then $cl_{\beta D} Z$ is a neighborhood of p which misses uD , and uD is closed in βD . Now express D as the union of n disjoint subsets F_α where each F_α has cardinality n . Since $cl_{\beta D} F_\alpha = \beta F_\alpha$ and is homeomorphic to βD , there exists a point in uF_α and such a point must belong to uD . Now we may obtain a copy of βD in uD by choosing x_α in $uD \cap F_\alpha^*$ and

setting $B = \{x_\alpha\}$. A mapping f in $C^*(B)$ can be extended to βD by defining a function g on D by letting g be constantly equal to $f(x_\alpha)$ on F_α . Then the extension of g to βD extends f . Thus, $cl_{\beta D} B$ is a copy of βD and is contained in uD since uD is closed. $\square$

5.14. In the proof of the preceding theorem, we have verified the following useful result.

PROPOSITION:

Every subset of the infinite discrete space D having the same cardinality as D belongs to some uniform ultrafilter on D .

5.15. If A_1 and A_2 are distinct members of an almost disjoint family of infinite subsets of $\mathbb{N}$, then A_1^* and A_2^* are disjoint open subsets of $\mathbb{N}^*$. This property of almost disjoint families of subsets of $\mathbb{N}$ has an analogue in the case of uD when D is the discrete space of infinite cardinality n . If $\mathcal{G}$ is a family of subsets of D each having cardinality n and such that any two members of $\mathcal{G}$ meet in a set of cardinality smaller than n , then $A_1^* \cap A_2^* \cap uD$ is empty for A_1 and A_2 in $\mathcal{G}$. Such a family thus yields disjoint open subsets of uD . The following lemma is the first of a sequence of three results which will ultimately demonstrate the existence of a 2^n -point in uD for certain cardinal numbers n . The lemma shows that for any family of m disjoint open subsets of uD obtained from a family $\mathcal{G}$ as above, there is a point p in uD such that every neighborhood of p meets m of the disjoint open sets.

LEMMA:

If an infinite collection $\mathcal{G}$ of subsets of the infinite discrete space D of cardinality n satisfies

$$(1) \quad |A| = n \text{ for all } A \text{ in } \mathcal{G}$$

(2) $|A_1 \cap A_2| < n$ for A_1 and A_2 distinct members of $\mathcal{G}$, then there exists a uniform ultrafilter $\mathcal{A}^p$ on D such that for each Z in $\mathcal{A}^p$,

$$|\{A \in \mathcal{G} : |Z \cap A| = n\}| = |\mathcal{G}|.$$

Proof: For each A in $\mathcal{G}$, choose x_A in $\text{cl}_{\beta D} A \cap uD$ and let $B = \{x_A : A \in \mathcal{G}\}$. If $A_1 \neq A_2$, then $x_{A_1} \neq x_{A_2}$ since x_{A_1} and x_{A_2} are in uD if x_{A_1} were the same point as x_{A_2} , we would have x_{A_1} in $\text{cl}_{\beta D} A_1 \cap \text{cl}_{\beta D} A_2 = \text{cl}_{\beta D} (A_1 \cap A_2)$ and $|A_1 \cap A_2| < n$, which contradicts the defining property of uD . Hence, B and $\mathcal{G}$ are of the same cardinality.

Now note that the subspace uD is compact since we saw in the proof of Theorem 5.13 that uD is closed in βD . We will use the compactness of uD to show that

(a) There is a point p in uD such that every neighborhood of p contains $|\mathcal{G}|$ points of B : If not, there is a finite open covering of uD each member of which contains fewer than $|\mathcal{G}|$ points of B . But this is impossible since B is infinite.

(b) $\mathcal{A}^p$ is the required ultrafilter: If Z is in $\mathcal{A}^p$ and $|Z \cap A| < n$, then $x_A \notin \text{cl}_{\beta D} Z \cap uD$. However, since $\text{cl}_{\beta D} Z$ is a neighborhood of p , there must be $|\mathcal{G}|$ members of $\mathcal{G}$ for which this does not happen and for these members, $|Z \cap A| = n$. $\square$

5.16. The next result appears in W. Sierpinski's 1928 paper and shows that for an infinite cardinal n such that $2^m \leq n$ for all $m < n$, a set of cardinality n will admit a family of 2^n subsets satisfying the hypothesis of the preceding lemma. Note that $\aleph_0$ satisfies the requirement on the cardinal number and that in this case, the theorem shows the existence of an almost disjoint family of c infinite subsets of $\mathbb{N}$. Thus, Proposition 3.21 is a special case of the theorem.

THEOREM: (Sierpinski)

If n is an infinite cardinal number such that $2^m \leq n$ for all $m < n$, then any set of cardinality n admits a family of 2^n subsets each having cardinality n such that the cardinality of the intersection of any two members of the family is strictly smaller than n .

Proof: Let $\mathfrak{F} = \{2^m : m < n\}$, i.e. $\mathfrak{F}$ is the set of all m -sequences of 0's and 1's where m runs through the ordinals $< n$. Then we have

$$|\mathfrak{F}| = \sum_{m < n} 2^m \leq n \cdot n = n$$

and it is clear that the cardinality of $\mathfrak{F}$ is at least n .

Thus, the cardinality of $\mathfrak{F}$ is n .

Now let $\mathcal{Q}$ be the set of all n -sequences of 0's and 1's and for each y in $\mathcal{Q}$, let E_y be the elements of $\mathfrak{F}$ which are initial segments of y . Then each E_y has cardinality n . If x and y are distinct elements of $\mathcal{Q}$, then there is a least ordinal $m < n$ at which x and y differ. Hence, E_x and E_y

have only m elements in common. Thus, the set $\mathcal{F}$ of cardinality n has a family of subsets of the required type and the proof is complete. $\square$

It is easy to see that a cofinal class of cardinal numbers will satisfy the theorem. For any cardinal n_0 , define inductively $n_{i+1} = 2^{n_i}$. Then $n = \sup\{n_i : i \in \omega_0\}$ is such a cardinal.

5.17. The preceding two results show that for some cardinal numbers n , uD contains a collection of 2^n disjoint open subsets and a point p such that every neighborhood of p meets 2^n of the open subsets. However, this does not yet establish p as a 2^n -point of uD since each neighborhood may meet a different subfamily of 2^n open subsets. The next theorem shows that the open sets may be chosen so that each neighborhood of p meets the same 2^n subsets.

THEOREM: (Hindman)

If D is the discrete space of cardinality n and $2^m \leq n$ for all $m < n$, there is a 2^n point in uD .

Proof: By the preceding theorem, there is a family $\mathcal{G}$ of 2^n subsets of D which satisfy the conditions of Lemma 5.15. By the same lemma, there exists a uniform ultrafilter $A^p = \{Z_\alpha : \alpha < 2^n\}$ on D such that every member of A^p meets 2^n sets of $\mathcal{G}$ in a set of cardinality n .

(a) It is possible to choose a subset X_α of Z_α for each α less than 2^n such that $|X_\alpha| = n$, $|X_\alpha \cap X_\gamma| < n$ if

$\alpha \neq \gamma$, and each X_α is the intersection of Z_α with a member of $\mathcal{Q}$. For Z_1 , choose A_1 in $\mathcal{Q}$ such that $|Z_1 \cap A_1| = n$. Now assume that for each $\sigma < \alpha$, we have chosen A_σ in $\mathcal{Q}$ such that $|Z_\sigma \cap A_\sigma| = n$ and $A_\sigma \neq A_\gamma$ for all $\gamma < \sigma$. Now since the lemma shows that there are 2^n such members of $\mathcal{Q}$ for each Z_α , there is a member A_α of $\mathcal{Q}$ such that $|Z_\alpha \cap A_\alpha| = n$ and $A_\alpha \neq A_\sigma$ for all $\sigma < \alpha$. Since the cardinality of the intersection of distinct members of $\mathcal{Q}$ is less than n , setting $X_\alpha = Z_\alpha \cap A_\alpha$ completes the proof of (a).

Since $|X_\alpha| = n$ for each α , there is a collection of subsets $\{X_{\tau\alpha} : \tau < 2^n\}$ of X_α such that each $X_{\tau\alpha}$ has cardinality n and $|X_{\tau\alpha} \cap X_{\sigma\alpha}| < n$ if $\tau \neq \sigma$. For each $\tau < 2^n$, set

$$U_\tau = (U\{cl_{\beta D} X_{\tau\alpha} : \alpha < 2^n\}) \cap uD.$$

It is clear that each U_τ is open in uD . We will show that $\{U_\tau\}$ is a pairwise disjoint family and that each U_τ contains p in its closure.

(b) $\{U_\tau\}$ is pairwise disjoint: If q is in $U_\sigma \cap U_\tau$, then there is a pair α and γ such that

$$q \in cl_{\beta D} X_{\sigma\alpha} \cap cl_{\beta D} X_{\tau\gamma} = cl_{\beta D} (X_{\sigma\alpha} \cap X_{\tau\gamma}).$$

However, since $|X_{\sigma\alpha} \cap X_{\tau\gamma}| < n$, this cannot occur since q belongs to uD .

(c) Every uD neighborhood of p meets each U_τ : Each such neighborhood is of the form $cl_{\beta D} Z_\alpha \cap uD$ for some Z_α in A^p so that it is sufficient to show that $cl_{\beta D} Z_\alpha \cap uD \cap U_\tau \neq \emptyset$

for all $\tau < 2^n$. Since $|X_{\tau\alpha}| = n$ and $X_{\tau\alpha} \subset X_\alpha \subset Z_\alpha$, both $X_{\tau\alpha}$ and Z_α belong to some uniform ultrafilter A^q and

$$q \in \text{cl}_{\beta D} Z_\alpha \cap \text{cl}_{\beta D} X_{\tau\alpha} \cap uD \subset \text{cl}_{\beta D} Z_\alpha \cap \text{cl}_{\beta D} U_\tau \cap uD.$$

Hence, p is a 2^n -point of uD . $\blacksquare$

5.18. Since $u\mathbb{N}$ is $\mathbb{N}^*$, the following corollary is immediate.

COROLLARY:

There is a c-point in $\mathbb{N}^*$.

This result is particularly interesting since we saw from Proposition 3.24 that two disjoint cozero-sets in $\mathbb{N}^*$ have disjoint closures.

5.19. In the case of the countable discrete space $\mathbb{N}$, Lemma 5.15 shows that with every almost disjoint family of n infinite subsets of $\mathbb{N}$ we can associate a free ultrafilter A^p on $\mathbb{N}$ such that p is a potential n -point of $\mathbb{N}^*$. By assuming the Continuum Hypothesis, we will be able to reverse the steps of the lemma. By assuming that the elements of any free ultrafilter A^p are indexed by $\omega_1 = c$, we can construct an almost disjoint family of infinite subsets of $\mathbb{N}$ such that each member of A^p contains a member of $\mathcal{G}$. The proof of Theorem 5.17 can then be adapted to A^p and the family $\mathcal{G}$ to show that p is a c -point of $\mathbb{N}^*$.

THEOREM [CH]: (Hindman)

Every point of $\mathbb{N}^*$ is a c-point of $\mathbb{N}^*$.

Proof: Using the Continuum Hypothesis, let the members of any free ultrafilter A^p be indexed by ω_1 , i.e. let $A^p = \{Z_\alpha : \alpha < \omega_1\}$. We show that infinite sets X_α can be chosen such that X_α is a subset of Z_α and $X_\alpha \cap X_\gamma$ is finite if $\alpha \neq \gamma$ as in (a) of the proof of Theorem 5.17. The remainder of the proof is the same as in Theorem 5.17.

Choose any infinite subset X_1 of Z_1 such that X_1 is not in A^p . For $\alpha < \omega_1$, assume that an infinite subset X_σ of Z_σ has been chosen for all $\sigma < \alpha$ such that X_σ is not in A^p and $\{X_\sigma : \sigma < \alpha\}$ is an almost disjoint family. Let $B = \{\sigma < \alpha : X_\sigma \cap Z_\alpha \text{ is infinite}\}$. If B is finite, then $\bigcup\{X_\sigma : \sigma \in B\}$ cannot be in A^p since no single set X_σ is in A^p . Thus, any infinite subset of $Z_\alpha \setminus \bigcup\{X_\sigma : \sigma \in B\}$ which is not in A^p may be chosen to be X_α . If B is infinite, we may index B by the natural numbers and write $B = \{\sigma_n\}$. Since $X_{\sigma_k} \cap Z_\alpha$ is infinite, it is possible to choose

$$x_k \in (X_{\sigma_k} \cap Z_\alpha) \setminus \bigcup_{j < k} X_{\sigma_j}.$$

Let X_α be any infinite subset of $\{x_k\}$ which does not belong to A^p . Then $|X_\alpha \cap X_{\sigma_k}| \leq k < \aleph_0$ and if σ is not in B , $|X_\alpha \cap X_\sigma| \leq |X_\sigma \cap Z_\alpha| < \aleph_0$ by the definition of B . $\square$

N-POINTS AND COMPACTIFICATIONS OF $\mathbb{N}$

5.20. Recall from Chapter 2 that an extremely disconnected space is one in which the closure of every open set is open. It can easily be shown that a space is extremely disconnected if and only if disjoint open sets of the space have disjoint closures, i.e. if and only if the space contains no 2-points. It is easy to see that $\beta\mathbb{N}$ is extremely disconnected.

In his 1967 paper, R. S. Pierce was particularly interested in 2- and 3-points of compact totally disconnected spaces. He showed that in the presence of the Continuum Hypothesis, the P-points of $\mathbb{N}^*$ are 3-points. He then utilized this result to show that any totally disconnected compact space contains a closed subspace which admits a 3-point. In the process, Pierce characterized those compactifications of $\mathbb{N}$ which do not admit a 3-point and also showed that every compactification other than $\beta\mathbb{N}$ does contain a 2-point. Recall that any compactification K of $\mathbb{N}$ is the image of $\beta\mathbb{N}$ under a mapping ϕ which sends $\mathbb{N}^*$ onto $K \setminus \mathbb{N}$ and leaves points of $\mathbb{N}$ fixed. We will call ϕ the canonical mapping of $\beta\mathbb{N}$ onto K . A fiber of a mapping is the inverse image of a point.

PROPOSITION: (R. S. Pierce)

A compactification K of $\mathbb{N}$ contains no three points exactly when no fiber of the canonical map ϕ of $\beta\mathbb{N}$ onto K contains more than two points. Further, any point of K which has two points in its fiber is a 2-point of K .

Proof: Let K be a compactification of $\mathbb{N}$ which contains no 3-points. We will show that no fiber of the canonical map contains more than two points. By Corollary 1.12, any point of K whose fiber contains more than a single point must lie in $K \setminus \mathbb{N}$ and the fiber must be contained in $\mathbb{N}^*$. Suppose that the fiber of a point x contains distinct points p_1, p_2 , and p_3 . Choose disjoint subsets Z_i for $i = 1, 2, 3$ such that Z_i belongs to A^{p_i} . Then $\phi[Z_i]$ is open in K and by the continuity of ϕ , x belongs to $\text{cl } \phi[Z_i]$ for each i , contradicting the assumption that K has no 3-point.

To prove the converse, assume that $\phi^{\leftarrow}(x) = \{p, q\}$ where p and q are not necessarily distinct.

(a) If V and W are neighborhoods of p and q , respectively, then x belongs to $\text{int } \phi(V \cup W)$: Suppose not. Then x is in $\text{cl } \phi(\beta\mathbb{N} \setminus (V \cup W))$. Put $F = \phi^{\leftarrow}(\phi(\beta\mathbb{N} \setminus (V \cup W)))$. Then $F \cap (V \cup W) = \emptyset$. But since $\beta\mathbb{N}$ is compact, ϕ is a closed map and

$$\phi[\text{cl } F] = \text{cl } \phi[F] = \text{cl}(K \setminus \phi(V \cup W)).$$

Since x belongs to $\text{cl } \phi[F]$, either p or q belongs to $\text{cl } F$. But this is impossible since V and W are neighborhoods of p and q and $F \cap (V \cup W) = \emptyset$.

(b) K has no 3-points: Assume the contrary. Let U_1, U_2 , and U_3 be disjoint open sets of K such that a point x of K belongs to none of the U_i but does belong to the closures of all three. The sets $\phi^{\leftarrow}(U_i \cap \mathbb{N})$ are disjoint open sets of $\mathbb{N}$ and therefore have disjoint closures in $\beta\mathbb{N}$.

Thus, if $\{p, q\}$ is the fiber of x , there is some $i = 1, 2$ or 3 such that p and q both fail to belong to $\text{cl } \varphi^{\leftarrow}(U_i \cap \mathbb{N})$. Let V and W be neighborhoods of p and q , respectively, such that V and W both miss $\varphi^{\leftarrow}(U_i \cap \mathbb{N})$. Then $\varphi(V \cup W) \cap U_i \cap \mathbb{N} = \emptyset$. By (a), x belongs to $\text{int } \varphi(U \cup V)$, and hence x cannot belong to $\text{cl}(U_i \cap \mathbb{N}) = \text{cl}(U_i)$. Thus, x is not a 3-point.

(c) If $\varphi^{\leftarrow}(x) = \{p, q\}$ and p and q are distinct, then x is a 2-point: Choose disjoint subsets Z and W of $\mathbb{N}$ such that Z belongs to A^p and W to A^q . Then by the continuity of φ , x belongs to $\text{cl } \varphi[U] \cap \text{cl } \varphi[W]$ and thus is a 2-point. $\blacksquare$

5.21. If K is any compactification of $\mathbb{N}$ other than $\beta\mathbb{N}$, it is clear that the canonical map φ of $\beta\mathbb{N}$ onto K identifies some pair of points and their common image is a 2-point of K . Thus, we have verified the following

COROLLARY:

$\beta\mathbb{N}$ is the unique extremally disconnected compactification of $\mathbb{N}$.

5.22. EXAMPLE:

Proposition 5.20 provides a way to construct interesting compactifications of $\mathbb{N}$. For example, define a permutation σ of $\mathbb{N}$ by

$$\sigma(n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Then $\sigma^2 = 1_{\mathbb{N}}$ and if $\beta(\sigma)$ is the homeomorphism of $\beta\mathbb{N}$ which

extends σ , $\beta(\sigma)^2 = 1_{\beta \mathbb{N}}$. The orbit of any point of $\mathbb{N}^*$ under $\beta(\sigma)$ consists of exactly two points. If K is the compactification of $\mathbb{N}$ which results from identifying each orbit contained in $\mathbb{N}^*$ to a single point, then every point of $K \setminus \mathbb{N}$ is a 2-point of K , but K contains no 3-points. K is easily seen to be totally disconnected by showing that distinct points of $K \setminus \mathbb{N}$ have complementary neighborhoods.

5.23. Pierce proved the next theorem under the assumption of the Continuum Hypothesis because the proof uses the existence of a 3-point in $\mathbb{N}^*$ which he had earlier demonstrated by using that assumption. We can now establish the result without the Continuum Hypothesis by virtue of Corollary 5.18.

THEOREM:

Any infinite totally disconnected compact space contains a closed subspace which admits a 3-point in the subspace topology.

Proof: Since X is an infinite Hausdorff space, it contains a copy N of $\mathbb{N}$. Let $K = \text{cl } N$. If K contains a 3-point, we are done. Otherwise, we will show that K contains a copy of $\mathbb{N}^*$ which will be the required subspace. Because K is a compactification of $\mathbb{N}$ without 3-points, each fiber of the canonical mapping ϕ of $\beta \mathbb{N}$ onto K contains at most two points. Let W be the subspace of $\beta \mathbb{N}$ consisting of singleton fibers, i.e.

$$W = \{p \in \beta \mathbb{N} : \phi^{-1}(\{\phi(p)\}) = \{p\}\}.$$

W clearly contains $\mathbb{N}$. If $W \setminus \mathbb{N}$ contains an infinite closed

subspace of $\mathbb{N}^*$, then Theorem 3.4 shows that $W \setminus \mathbb{N}$ contains a copy of $\mathbb{N}^*$ which is embedded into K by ϕ . Since $\mathbb{N}^*$ contains a 3-point, the copy of $\mathbb{N}^*$ is the required closed subspace of X .

If $W \setminus \mathbb{N}$ fails to contain an infinite closed subspace of $\mathbb{N}^*$, then $S = \beta \mathbb{N} \setminus W$ is dense in $\mathbb{N}^*$. For each point p of S , let $\hat{p}$ denote the point of S such that $\phi^{\leftarrow}(\{\phi(p)\}) = \{p, \hat{p}\}$. Note that $\hat{\hat{p}} = p$. Choose p_0 in S and let U_0 and V_0 be disjoint clopen neighborhoods in $\beta \mathbb{N}$ of p_0 and $\hat{p}_0$, respectively, such that $U_0 \cup V_0$ does not contain $\mathbb{N}^*$. Suppose that points $\{p_0, \dots, p_n\}$ have been chosen in S such that $\phi(p_i) \neq \phi(p_j)$ for $i \neq j$ and that disjoint clopen sets U_n and V_n have been found such that U_n contains $\{p_0, \dots, p_n\}$, V_n contains $\{\hat{p}_0, \dots, \hat{p}_n\}$, and $U_n \cup V_n$ does not contain $\mathbb{N}^*$. Now since S is dense in $\mathbb{N}^*$, there exists a point p in $S \setminus (U_n \cup V_n)$ and it is clear that $\phi(p)$ is distinct from all the $\phi(p_i)$. There are three distinct possibilities for the location of $\hat{p}$: $\hat{p}$ belongs to U_n , $\hat{p}$ belongs to V_n , or $\hat{p}$ fails to belong to $U_n \cup V_n$. In the first case, put $p_{n+1} = \hat{p}$. In the other cases, put $p_{n+1} = p$. The clopen sets U_n and V_n can be enlarged, if necessary, to disjoint clopen sets U_{n+1} and V_{n+1} to include p_{n+1} and $\hat{p}_{n+1}$ such that $U_{n+1} \cup V_{n+1}$ does not contain $\mathbb{N}^*$. Thus, by induction, we can obtain sequences of points $\{p_i\}$ and $\{\hat{p}_i\}$ such that

$$\{p_i : i < \omega_0\} \subset U = \bigcup U_i$$

and

$$\{\hat{p}_i : i < \omega_0\} \subset V = \bigcup V_i,$$

$\varphi(p_i) \neq \varphi(p_j)$ for $i \neq j$, and U and V are disjoint. Now put $C = \text{cl}_K \varphi(U \cap \mathbb{N}) \cap \text{cl}_K \varphi(V \cap \mathbb{N})$. Then C is an infinite closed subspace of $K \setminus \mathbb{N}$ and the fiber of each point of C consists of two points -- one in $\text{cl } U \setminus \mathbb{N}$ and one in $\text{cl } V \setminus \mathbb{N}$. Hence, C is homeomorphic to an infinite closed subspace of $\mathbb{N}^*$ and thus contains a copy of $\mathbb{N}^*$ which contains a 3-point. **|**

EXERCISE

5A. n-POINTS

Let x be a non-isolated point of a totally disconnected space X such that there is a well-ordered family $\{U_\alpha : \alpha < \lambda\}$ of clopen sets of X satisfying:

- (i) If $\alpha < \beta < \lambda$, then $U_\beta \subset U_\alpha$.
- (ii) $\bigcap_{\alpha < \lambda} U_\alpha = \{x\}$.

1. λ is a limit ordinal.
2. For each positive integer n , x is an n -point. [For $\alpha < \lambda$, put $P_\alpha = U_\alpha \setminus U_{\alpha+1}$. For $i = 0, 1, \dots, n-1$, put $W_i = \bigcup \{P_\alpha : \alpha < \lambda \text{ and } \alpha \equiv i \pmod{n}\}$. Then $x \in \text{cl } W_i$.]
3. Assume the Continuum Hypothesis. Any P -point of $\mathbb{N}^*$ is a n -point.

Reference: R. S. Pierce, 1967.

CHAPTER SIX:

MAPPINGS OF βX TO X^*

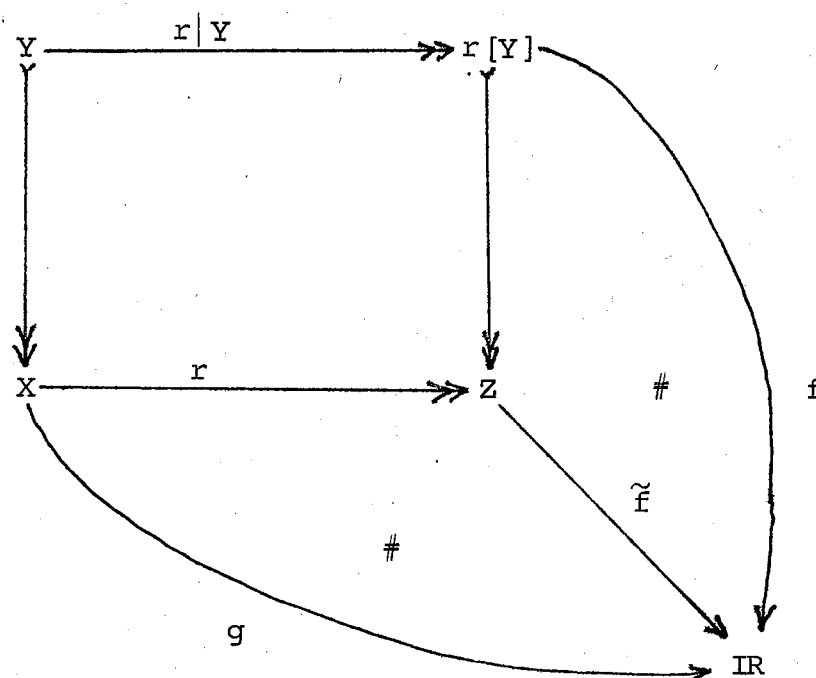
6.1. This chapter is devoted to the investigation of various types of mappings of βX to X^* . The chapter will be in four parts. We will first consider a sequence of preliminary results giving sufficient conditions to ensure that the continuous image of a dense, C^* -embedded subspace will be C^* -embedded in the image of the whole space. These results will then be applied to mappings of βX onto X^* and in particular to necessary conditions for the existence of a retraction of βX onto X^* . Next, the growths of the compactifications of a locally compact space X will be characterized as the continuous images of X^* . Finally, we will consider mappings of Stone-Ćech compactifications of discrete spaces.

 C^* -EMBEDDING OF IMAGES

6.2. The results of this and the succeeding section on retractive spaces appear in the 1965 paper of W. W. Comfort.

PROPOSITION:

If r is an open or closed mapping of X onto Z and Y is a dense subspace of X such that any fiber of r containing distinct points meets Y , then $r[Y]$ is C -embedded (resp. C^* -embedded) in Z if Y is C -embedded (resp. C^* -embedded) in X .



As we will see in Chapter 10, the diagram which illustrates the proposition indicates that this result has some aspects of an adjunction or pushout. The proof will be given for C -embedding and is similar for C^* -embedding.

Proof: To simplify the notation, write r_1 for $r|_Y$. Let f be in $C(r[Y])$ so that $f \circ r_1$ belongs to $C(Y)$. Since Y is C -embedded in X , $f \circ r_1$ has an extension g in $C(X)$. The first step toward defining an extension of f to Z is to show that:

(a) g is constant on any fiber of r : For any fiber containing more than one point, we choose a point y of Y lying in the fiber and show that $g(x) = g(y)$ for any other point x in the fiber. Since Y is dense in X , there is a net $\{y_\alpha\}$ in Y such that

$$x = \lim_{\alpha} y_{\alpha}.$$

Then

$$g(x) = \lim_{\alpha} g(y_{\alpha}) = \lim_{\alpha} (f \circ r_1)(y_{\alpha}).$$

Since r is continuous on X , we have that

$$\lim_{\alpha} r(y_{\alpha}) = r(x) = r(y) \in r[Y].$$

Since f is continuous on $r[Y]$,

$$\begin{aligned} g(y) &= (f \circ r_1)(y) = f(\lim_{\alpha} r(y_{\alpha})) = \lim_{\alpha} f(r(y_{\alpha})) \\ &= \lim_{\alpha} (f \circ r_1)(y_{\alpha}) = g(x). \end{aligned}$$

Since r is onto Z , we can use (a) to define $\tilde{f}$ on Z by

$$\tilde{f}(r(x)) = g(x).$$

(b) $\tilde{f}$ extends f : For $r(y)$ in $r[Y]$,

$$\tilde{f}(r(y)) = g(y) = (f \circ r_1)(y) = f(r_1(y)) = f(r(y)).$$

(c) $\tilde{f}$ is continuous: From (a), it follows that for any subset S of $\mathbb{R}$, $\tilde{f}^{\leftarrow}(S) = r[g^{\leftarrow}(S)]$. Then if r is open, taking S any open subset of $\mathbb{R}$ shows that $\tilde{f}$ is continuous. Taking S to be closed if r is closed completes the proof. $\square$

6.3. The assumption that r is a closed or open mapping in the proposition will not be overly restrictive since all of our applications will be to cases where X is compact and a mapping of a compact space into a Hausdorff is always closed.

COROLLARY:

If r maps a compact space K onto L and every fiber of r which meets the closure of a C^* -embedded subspace Y of K in more than a single point meets Y , then $r[Y]$ is C^* -embedded in L .

Proof: Since a mapping defined on a compact space is closed, applying the proposition with $X = cl_K Y$ and $Z = r[X]$ yields $r[Y]$ C^* -embedded in $r[X]$. It then follows that $r[Y]$ is C^* -embedded in L since the compact space $r[X]$ is C^* -embedded in L by Proposition 1.47. $\blacksquare$

6.4. The hypotheses of the preceding corollary are satisfied when K is chosen to be βX and the restriction of r to X^* is required to be one-to-one.

COROLLARY:

If r maps βX onto Z such that r restricted to X^* is one-to-one, then the image of a closed (in X) C^* -embedded subspace Y of X is C^* -embedded in Z and $\beta(r[Y]) = r[\beta Y]$.

Proof: It follows from the previous corollary that $r[Y]$ is C^* -embedded in Z . Since Z is compact and r is closed,

$$\beta(r[Y]) = cl_Z r[Y] = r[cl_{\beta X} Y] = r[\beta Y]. \quad \blacksquare$$

RETRACTIVE SPACES

6.5. A mapping of a space Y onto a subspace S of Y is called a retraction if the mapping leaves points of S fixed.

The subspace S is then called a retract of Y and it is easily

seen that S is closed whenever Y is a Hausdorff space [D, p. 321]. We will call a space X retractive if there is a retraction of βX onto X^* . Since a retraction of βX onto X^* is one-to-one on X^* , we can restate Corollary 6.4 for retractive spaces.

COROLLARY:

If r is a retraction of βX onto X^* and Y is a closed (in X) and C^* -embedded subspace of X , then $r[Y]$ is C^* -embedded in X^* and $\beta(r[Y]) = r[\beta Y]$. In particular, $\beta(r[X]) = X^*$.

6.6. Observe in the preceding corollary that if Y is both closed and C^* -embedded in X , then $Y^* = cl_{\beta X} Y \setminus X$ is the growth of Y in βY and is in the closure of both Y and $r[Y]$. In the next theorem, we will apply the corollary to show that in the presence of the Continuum Hypothesis, a retractive space X cannot contain a closed C^* -embedded copy of $\mathbb{N}$. The proof will be accomplished by contradiction. We will first show that any P -point of $\mathbb{N}^*$ must also be a P -point of X^* and then use the fact that $\beta(r[\mathbb{N}])$ is $r[\beta \mathbb{N}]$ to show that a P -point of $\mathbb{N}^*$ cannot be a P -point of X^* if X is retractive. Since $\mathbb{N}$ is not required to be C -embedded but only closed and C^* -embedded, we will first require the following improvement of Lemma 4.32.

LEMMA:

If $\mathbb{N}$ is closed and C^* -embedded in a locally compact space X , then every P-point of $\mathbb{N}^*$ is a P-point of X^* .

Proof: $\mathbb{N}^*$ is contained in X^* as in the proof of Lemma 4.32.

Let f be the extension to X of the mapping $n \mapsto \frac{1}{n}$ of $\mathbb{N}$ onto the subspace $\{\frac{1}{n} : n \in \mathbb{N}\}$ of $\mathbb{R}$. Then by choosing a family of pairwise disjoint neighborhoods of the $\frac{1}{n}$ points in $\mathbb{R}$, we can take inverses to obtain a family $\{V_n\}$ of pairwise disjoint neighborhoods of the points of $\mathbb{N}$ and each V_n can be assumed to be compact. For each n , let g_n be a member of $C^*(X)$ such that $g_n(n) = 0$, $g_n[X \setminus V_n] = \{\frac{1}{2^n}\}$, and $0 \leq g_n(x) < \frac{1}{2^n}$ for all x . Then the Weierstrass M-test shows that there exists g in $C^*(X)$ such that $g(n) = 0$ for each n and $g[X \setminus \bigcup V_n] = \{1\}$. Put $h = 1 - g$. The proof is now completed as in Lemma 4.32. $\blacksquare$

6.7. THEOREM [CH]: (Comfort)

A retractive space is locally compact and contains no closed C^* -embedded copy of $\mathbb{N}$.

Proof: Consider a retractive space X and let r be a retraction of βX onto X^* . Then X^* is closed in βX and X is therefore locally compact.

If X contains a countably infinite C^* -embedded closed discrete subspace M , Corollary 6.5 shows that

$$\beta(r[M]) = r[\beta M] = r[M] \cup M^*.$$

$\beta(r[M])$ is the closure in βX of $r[M]$ and contains M^* so that

$r[M]$ cannot be finite. Moreover, infinitely many points of $r[M]$ lie outside of M^* since otherwise $r[M] \cap M^*$ would be a countable dense subspace of M^* , which is impossible since the cellularity of M^* is c by Corollary 5.10. Choose a countably infinite discrete subspace $K = \{p_1, p_2, \dots\}$ of $r[M]$ which misses M^* . Choose x_n in $r^{-1}(p_n) \cap M$ for each n and define $N = \{x_n\}$. Since N is a subspace of M , it is clear that N is a countably infinite C^* -embedded closed subspace of X . Because r is a retraction, $K = r[N]$ is C^* -embedded in X^* and $N^* = K^*$.

We now use the fact that $N^* = K^*$ with N a subspace of X and K a subspace of X^* to reach a contradiction. If the Continuum Hypothesis is assumed, Corollary 4.30 shows that there exists a P -point of N^* . By the lemma, every P -point of N^* is a P -point of X^* since X is locally compact. However, the mapping f defined on K by $f(p_n) = \frac{1}{n}$ for all n has a continuous extension $\tilde{f}$ to X^* and $\tilde{f}$ vanishes on K^* . But $\tilde{f}$ cannot vanish on any X^* neighborhood of a point of K^* since every such neighborhood meets K . Hence, no point of K^* is a P -point of X^* , which is a contradiction showing that X can contain no closed, C^* -embedded copy of $\mathbb{N}$. $\square$

6.8. However, there are retractive spaces which do contain non-closed C^* -embedded copies of $\mathbb{N}$.

EXAMPLE:

The complement in $\beta \mathbb{N}$ of any finite subspace of $\mathbb{N}^*$ is a retractive space which contains $\mathbb{N}$ as a non-closed, C^* -embedded subspace. This is evident since $\beta \mathbb{N}$ can be written as the disjoint union of neighborhoods of the finitely many points and the retraction consists of shrinking each neighborhood to the respective point.

6.9. Since any space which fails to be pseudocompact contains a closed C -embedded copy of $\mathbb{N}$ by Corollary 4.5, the following corollary is immediate.

COROLLARY [CH]:

A retractive space is pseudocompact and locally compact.

The preceding results show that an infinite discrete space fails to be retractive if the Continuum Hypothesis is assumed. This result can also be obtained without the Continuum Hypothesis as outlined in Exercise 6A.

6.10. Since any pseudocompact metrizable or realcompact space is compact, we have the

COROLLARY [CH]:

No realcompact or metrizable space is retractive.

6.11. EXAMPLE:

If $\mathbb{N}$ is considered as a subspace of $\mathbb{R}$, then $\beta \mathbb{N} = \text{cl}_{\beta \mathbb{R}} \mathbb{N}$ and $\mathbb{N}^*$ is a subspace of $\mathbb{R}^*$. Consider $\Lambda = \beta \mathbb{R} \setminus \mathbb{N}^*$. If f is an unbounded real-valued mapping on Λ , then f must be

unbounded on some closed subspace S of $\mathbb{R}$ which misses $\mathbb{N}$. Thus, f must be unbounded on $cl_{\beta\mathbb{R}} S$, which is impossible. Hence, Λ is pseudocompact. But $\mathbb{N}$ is a closed and C^* -embedded subspace of Λ . Thus, Corollary 6.9 is weaker than Theorem 6.7.

Not only does Λ fail to be retractive, but the only maps which do exist from $\beta\Lambda = \beta\mathbb{R}$ into $\Lambda^* = \mathbb{N}^*$ are the constant maps. This is easily seen since $\beta\Lambda$ contains $\mathbb{R}$ as a dense subspace and is therefore connected whereas Λ^* is totally disconnected. The space Λ was introduced by M. Katětov in 1951 and is also described in [GJ, ex. 6P, 9E].

GROWTHS OF COMPACTIFICATIONS

6.12. Corollary 1.12 showed that βX is the maximal compactification of the space X in the sense that if K is another compactification of X , then the embedding of X into K extends to a mapping of βX onto K . We will now see that X^* can be thought of as the maximal growth of X since all the points of X^* are sent to points of $K \setminus X$ by the extension, i.e. every growth is the continuous image of X^* .

Recall from 1.44 that if f is a mapping of X to Y and $\mathfrak{F}$ is a z -filter on X , then the collection of zero-sets in Y whose inverse images belong to $\mathfrak{F}$ is a z -filter on Y and is denoted by $f^\# \mathfrak{F}$.

PROPOSITION:

The growth of any compactification of a space X is a continuous image of X^* .

Proof: Let K be any compactification of X . Then the embedding h of X into K has an extension $\beta(h)$ of βX onto K . Let p be a point of βX and suppose that $\beta(h)(p) = x$, where x is a point of X . Then $(\beta(h))^{\#}A^p = h^{\#}A^p = A^p$ converges to x as does A^x . Since each point of X is the limit of a unique z -ultrafilter, $x = p$. Hence, p belongs to X and the points of X^* must all be mapped to $K \setminus X$ by $\beta(h)$. $\blacksquare$

6.13. An immediate corollary occurs in the case of a retractive space.

COROLLARY:

The growth of any compactification of a retractive space X is a continuous image of βX .

6.14. The following theorem appears in the 1966 paper of K. D. Magill and shows that the converse of the Proposition 6.12 holds for a locally compact space X , i.e. that every continuous image of X^* is a growth of X in some compactification. A map f of X to Y is a quotient map whenever a subset U of Y is open if and only if $f^{\leftarrow}(U)$ is open in X . In the proof of Magill's result we will construct a compactification K of X by identifying the fibers of a function on βX to points and defining K to be the resulting quotient space.

THEOREM: (Magill)

The growths of the compactifications of a locally compact space X are the continuous images of X^* .

Proof: Proposition 6.12 showed that every growth of X is a continuous image of X^* . To obtain the converse, let f be a mapping of X^* onto Y and assume that X and Y are disjoint. Define a function g from βX to $X \cup Y$ by

$$\begin{cases} g(p) = p & \text{if } p \in X \\ g(p) = f(p) & \text{if } p \in X^*. \end{cases}$$

The function g induces an equivalence relation on βX by defining two points of βX to be equivalent if g takes the same value on the two points. Let K be the quotient space obtained by identifying the equivalence classes to points and let ϕ be the quotient map sending βX onto K . We will show that K is a compactification of X in which the growth of X is homeomorphic to Y .

Call a subset H of βX saturated if H contains the equivalence class of any point belonging to H . Since the locally compact space X is an open saturated subspace of βX and the restriction of g to X is a homeomorphism, $\phi[X]$ is homeomorphic to X . Because X^* is a closed saturated subspace of βX and the restriction of g to X^* is a closed mapping onto Y , $\phi[X^*]$ is homeomorphic to Y [D, p. 130]. Thus, the image of X is a copy of X which is easily seen to be dense in K and the complement of the image of X in K is homeomorphic to Y .

Since a continuous image of a compact space is compact, it remains to show that K is Hausdorff. There are three cases to be considered to show that distinct points of K may be separated by disjoint open sets. In each case, we will use that the image under ϕ of a saturated open subset of βX is open in K . In the first case, consider distinct points x_1 and x_2 belonging to $\phi[X]$. The fibers $\phi^{\leftarrow}(x_1)$ and $\phi^{\leftarrow}(x_2)$ are singletons and have disjoint open neighborhoods U_1 and U_2 in X . Because X is open in βX , U_1 and U_2 are also open in βX and since both sets are saturated, their images are disjoint neighborhoods of x_1 and x_2 in K .

In the second case, if y_1 and y_2 are distinct points of $\phi[X^*]$, $\phi^{\leftarrow}(y_1)$ and $\phi^{\leftarrow}(y_2)$ are disjoint closed subsets of the compact space X^* and are therefore contained in disjoint open sets U_1 and U_2 of X^* . Because the restriction of ϕ to X^* is a closed map onto the regular space $\phi[Y]$, there exist closed neighborhoods V_1 and V_2 of y_1 and y_2 in $\phi[X^*]$ such that

$$\phi^{\leftarrow}(y_i) \subset \phi^{\leftarrow}(\text{int } V_i) \subset \phi^{\leftarrow}(V_i) \subset U_i$$

for each value of i [D, p. 86]. Since the closed sets $\phi^{\leftarrow}(V_1)$ and $\phi^{\leftarrow}(V_2)$ are contained in disjoint open subsets of βX , there exists a pair of disjoint open sets W_1 and W_2 of βX such that

$$W_i \cap X^* = \phi^{\leftarrow}(\text{int } V_i)$$

for both values of i . But then W_1 and W_2 are saturated open sets of βX and their images under ϕ are the required

disjoint neighborhoods of y_1 and y_2 .

In the final case, let x be a point of $\phi[X]$ and y be a point of $\phi[X^*]$. Then $\phi^{\leftarrow}(x)$ and $\phi^{\leftarrow}(y)$ are contained in disjoint open sets U and V , respectively, and as in the preceding case, we can obtain a saturated X^* neighborhood W of $\phi^{\leftarrow}(y)$ such that

$$\phi^{\leftarrow}(y) \subset W \subset V \cap X^*.$$

Let G be an open set of βX such that $G \cap X^* = W$. Then $U \cap X$ and $G \cap V$ are disjoint open saturated neighborhoods of $\phi^{\leftarrow}(x)$ and $\phi^{\leftarrow}(y)$, respectively, and their images under ϕ are disjoint neighborhoods of x and y . $\square$

6.15. Magill's Theorem gives an added importance to any technique which will describe mappings having as their domain the growth of a locally compact space. One such method using Boolean algebras was used in Theorem 3.31 to establish Parovičenko's result that every compact space having weight at most $\aleph_1$ is a continuous image of $\mathbb{N}^*$. That result together with Magill's Theorem yields the

COROLLARY:

Every compact space of weight at most $\aleph_1$ is the growth of $\mathbb{N}$ in some compactification.

A second technique was illustrated in Example 5.22 where we constructed a compactification of $\mathbb{N}$ by identifying the orbits of points of $\mathbb{N}^*$ under an automorphism of $\beta \mathbb{N}$.

6.16. Corollary 6.15 implies that there exists a compactification of $\mathbb{N}$ whose growth is a copy of the ordinal space $\omega_1 + 1$. We will describe such a compactification by using properties of the clopen subsets of $\mathbb{N}^*$ to construct a mapping of $\mathbb{N}^*$ onto $\omega_1 + 1$. The construction will utilize a modification of the technique used to prove Urysohn's Lemma. Recall that a normal space is one in which every neighborhood of a closed set contains a closed neighborhood of the set. Urysohn's Lemma is the statement that disjoint closed subsets of a normal space are completely separated. If F and H are the disjoint closed subsets of the normal space X , then the definition of normality is applied successively to the neighborhood $X \setminus H$ of F to obtain a sequence of open neighborhoods $\{U_r\}$ where the indexing set is the set of rationals of $[0,1]$ and

$$F \subset U_0 \subset \text{cl } U_0 \subset \dots \subset U_r \subset \text{cl } U_r \subset U_s \subset \dots \subset X \setminus H = U_1$$

whenever $r < s$. Then putting

$$f(x) = \inf\{r : x \in U_r\}$$

yields a mapping which completely separates F from H . The details of the proof can be found in [GJ, 3.12].

The following application of Magill's Theorem appears in the 1971 paper of S. P. Franklin and M. Rajagopalan and uses a similar technique where the sets involved are clopen subsets of $\mathbb{N}^*$ and are indexed by ω_1 . Recall from Corollary 3.27 that every non-empty G_δ in $\mathbb{N}^*$ has non-empty interior and therefore contains a non-empty clopen set.

COROLLARY: (Franklin and Rajagopalan)

There exists a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ such that $\gamma\mathbb{N}\setminus\mathbb{N}$ is homeomorphic to $\omega_1 + 1$.

Proof: We first construct a strictly increasing ω_1 -sequence $\{U_\alpha : \alpha < \omega_1\}$ of non-empty clopen subsets of $\mathbb{N}^*$. Choose any proper non-empty clopen set of $\mathbb{N}^*$ as U_0 . If U_α has been defined for each $\alpha < \beta$, the complements $\mathbb{N}^* \setminus U_\alpha$ form a decreasing sequence of clopen subsets of $\mathbb{N}^*$. Therefore their intersection contains a non-empty clopen set A (Corollary 3.27). Write $A = B \cup C$ where B and C are disjoint non-empty clopen sets. Choose the complement of B to be U_β . Then C is contained in $U_\beta \setminus U_\alpha$ for all $\alpha < \beta$ so that U_α is properly contained in U_β for all $\alpha < \beta$. Further, the complement of U_β is B , which is non-empty allowing the process to continue on to ω_1 .

We now use this ω_1 -sequence to define a mapping f of $\mathbb{N}^*$ onto $\omega_1 + 1$. Imitating the Urysohn Lemma technique, define

$$f(p) = \sup\{\alpha < \omega_1 : p \notin U_\alpha\}.$$

Points in U_0 are thus sent to 0. Since the complements of the U_α form a decreasing sequence of compact sets, there exists at least one point outside the union of the U_α 's so that f takes on the value ω_1 . Moreover, f is onto since the sequence is strictly increasing. To prove that f is continuous, note that intervals of the form $[0, \beta)$ and $(\beta, \omega_1]$ for $0 \leq \beta < \omega_1$ form a subbase for the topology of $\omega_1 + 1$. Thus, it is sufficient to show that $f^{-1}([0, \beta))$ and $f^{-1}((\beta, \omega_1])$

are open. Note that $f(p) < \beta$ if and only if p is in U_α for some $\alpha < \beta$. Then $f^{-1}([0, \beta)) = \bigcup \{U_\alpha : \alpha \leq \beta\}$ and is open. Similarly, $f(p) > \beta$ if and only if p is not in U_α for some $\alpha > \beta$ so that $f^{-1}((\beta, \omega_1]) = \bigcup \{\mathbb{N}^* \setminus U_\alpha : \beta < \alpha < \omega_1\}$, and f is continuous. $\blacksquare$

6.17. The compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ is useful as a source of examples. If $\mathcal{u}$ and $\mathcal{v}$ are covers of a space, then $\mathcal{v}$ is a refinement of $\mathcal{u}$ if every member of $\mathcal{v}$ is a subset of some member of $\mathcal{u}$. A space is said to be paracompact if every open cover has a locally finite open refinement. Every paracompact space is also normal [D, p. 163], however, the ordered space ω_1 is a normal space which fails to be paracompact since the open covering of ω_1 consisting of initial segments fails to have a locally finite open refinement.

A second countable normal space is metrizable (Exercise 6J), and hence is paracompact [D, p. 186]. It is natural to ask if second countable can be replaced by separable in this statement, i.e. is there a separable normal space which fails to be paracompact? In 1956, M. E. Rudin and L. F. McAuley each gave fairly complicated examples of such a space. The compactification $\gamma\mathbb{N}$ was used by Franklin and Rajagopalan to give a simpler example. Since ω_1 is embedded as a closed subspace of $\gamma\mathbb{N} \setminus \{\omega_1\}$ and a closed subspace of a paracompact space must be paracompact, we can use the fact that ω_1 is not paracompact to show that $\gamma\mathbb{N} \setminus \{\omega_1\}$ is a separable normal space which fails to be paracompact.

EXAMPLE:

$\gamma \mathbb{N} \setminus \{\omega_1\}$ is a separable, totally disconnected locally compact space which is normal but which is not paracompact. It is immediate that $\gamma \mathbb{N} \setminus \{\omega_1\}$ is separable and locally compact and not paracompact.

(a) $\gamma \mathbb{N}$ is totally disconnected: Since the points of $\mathbb{N}$ are isolated, we need only show that points of $\omega_1 + 1$ can be separated by clopen sets. Because $\omega_1 + 1$ is zero-dimensional, if α and β are distinct points of $\omega_1 + 1$, then there is a clopen set U of $\omega_1 + 1$ which contains α but not β . Thus, if ϕ is the canonical map of $\beta \mathbb{N}$ onto $\gamma \mathbb{N}$, then $\phi^{-1}(U)$ is clopen in $\mathbb{N}^*$ and $\phi^{-1}(U) = Z^*$ for some subset Z of $\mathbb{N}$. The map ϕ is closed since $\beta \mathbb{N}$ is compact and therefore ϕ is a quotient map. Since $\text{cl } Z$ is saturated with respect to ϕ and is clopen, $\phi[\text{cl } Z]$ is a clopen subset of $\gamma \mathbb{N}$ which contains α but not β .

(b) Of any pair of disjoint closed subsets of ω_1 , one of the sets must be bounded: Let R and S be closed subsets of ω_1 and assume that both are unbounded. Then we can choose unbounded sequences $\{r_n\}$ and $\{s_n\}$ contained in R and S respectively and such that $r_n < s_n < r_{n+1}$ for every n . Then the two sequences have the same supremum which must then belong to both R and S .

(c) $\gamma \mathbb{N} \setminus \{\omega_1\}$ is normal: Let F and H be disjoint closed subsets of $\gamma \mathbb{N} \setminus \{\omega_1\}$. Then one of them, say F , meets ω_1 in a closed and bounded set. Since a closed and bounded subspace

of ω_1 is compact, there is a clopen subset U of $\gamma\mathbb{N}\setminus\{\omega_1\}$ which contains $F \cap \omega_1$ and misses H . Then $V = U \cup (F \cap \mathbb{N})$ is open since only isolated points have been added and is also closed since any cluster points of $F \cap \mathbb{N}$ are contained in $F \cap \omega_1$. Thus, V is clopen, contains F , and misses H .

6.18. EXAMPLE:

The construction of $\gamma\mathbb{N}$ shows that the space $\Lambda = \beta\mathbb{R}\setminus\mathbb{N}^*$ described in Example 6.11 also has a compactification with growth homeomorphic to $\omega_1 + 1$.

MAPPINGS OF βD

6.19. Although no infinite discrete space is retractive, it is possible to map βD onto D^* whenever the cardinality n of D satisfies $n = n^{\aleph_0}$.

THEOREM: (Comfort)

If D is the discrete space of infinite cardinality n , then D^* is a continuous image of βD if and only if $n = n^{\aleph_0}$.

Proof: Recall from Theorem 5.12 that the density character of D^* is $n^{\aleph_0}$. Then if f is a mapping of βD onto D^* , $f[D]$ is dense in D^* and $n^{\aleph_0} \leq |f[D]| \leq n$.

Conversely, map D onto a dense subset of D^* having cardinality $n = n^{\aleph_0}$. Since D^* is compact, the mapping extends to βD and the extended mapping is onto D^* since the image of D is dense in D^* . $\square$

6.20. EXAMPLES:

The theorem shows that IN^* is not a continuous image of βIN . However, if D is the discrete space of cardinality c , D^* is a continuous image of βD since $c^{\aleph_0} = c$.

6.21. By observing that $\aleph_1^{\aleph_0} = c$, the following equivalent form of the Continuum Hypothesis follows from Theorem 6.19.

COROLLARY:

If D is the discrete space of cardinality $\aleph_1$, the Continuum Hypothesis holds if and only if D^* is a continuous image of βD .

6.22. A fixed point of a function f is a point x such that $f(x) = x$. A set S is said to be f -invariant if $f[S]$ is contained in S . In the next two results we will see that if f is a mapping of a discrete space D into itself, then the set of fixed points of the extension $\beta(f) : \beta D \rightarrow \beta D$ consists of the closure of the set of fixed points of f . The proposition and its corollary are taken from the 1967 paper of M. Katětov.

PROPOSITION:

If X is a non-empty set and f is a function of X into X which has no fixed point, then X is the union of disjoint sets X_0, X_1 , and X_2 such that $f[X_i] \cap X_i = \emptyset$ for each i .

Proof: Put $f^0(x) = x$ and for x and y in X , define

$x \sim y$ if $f^n(x) = f^m(y)$ for some pair of non-negative integers n and m . It is clear that $\sim$ is an equivalence relation and

that if $R(z)$ denotes the equivalence class of z , then $R(z)$ is f -invariant. Since the equivalence classes constitute a pairwise disjoint family of f -invariant sets, it is sufficient to show that each equivalence class can be written as the union of three sets as required. The three required sets X_0 , X_1 , and X_2 can then be obtained for X by taking unions of the corresponding sets for the equivalence classes.

Now consider an equivalence class $R(z)$ and choose a in $R(z)$. For each y in $R(z)$, let $m(y)$ denote the least non-negative integer m such that $f^m(a) = f^n(y)$. Because $R(z)$ is f -invariant, $f^{m(y)}(a)$ belongs to $R(z)$. Let $n(y)$ denote the least non-negative integer n such that $f^n(y) = f^{m(y)}(a)$. We will use the integer $m(y) + n(y)$ to define the sets X_0 , X_1 , and X_2 . There are three cases to consider.

(a) If $n(x) > 0$, then $m(x) = m(f(x))$ and $n(f(x)) = n(x) - 1$. This follows from the definitions of $m(x)$ and $n(x)$ and the equations

$$f^{m(x)}(a) = f^{n(x)}(x) = f^{n(x)-1}f(x)$$

and

$$f^{m(x)}(a) = f^{n(f(x))}(f(x)).$$

Now consider the cases where $n(x) = 0$, i.e. where

$$x = f^0(x) = f^{m(x)}(a).$$

Since $f(x) = f^{m(x)+1}(a)$, it is clear that $m(f(x)) \leq m(x) + 1$ and $n(f(x)) = 0$ whenever $n(x) = 0$.

(b) There can be at most one element d in $R(x)$ such that $m(f(d)) < m(d) + 1$ and $n(d) = 0$: If d is such an element, then $f(d) = f^{m(d)+1}(a)$ and $m(f(d)) \leq m(d)$. Since $n(d) = 0$, $d = f^{m(d)}(a)$. If $m(f(d)) = m(d)$, then we have $f(d) = f^{m(d)}(a) = d$. But this is impossible since f has no fixed point. Hence, $m(f(d)) < m(d)$ and $f(d) = f^k(a)$ for some $k < m(d)$. Thus, $f(d) = f(f^{m(d)}(a)) = f^k(a)$ and successive iterations of f applied to a yield only $m(d)$ distinct images:

$$a, f(a), \dots, f^k(a), \dots, f^{m(d)}(a) = d.$$

Further, $n(a^k) = 0$ for each $k \leq m(d)$. Thus, if $n(x) = 0$ and $x \neq d$, then $x = f^k(a)$, $k < m(d)$, and $m(f(x)) = m(f(f^k(a))) = k + 1 = m(x) + 1$. Hence, there is at most one element d such that $n(d) = 0$ and $m(f(d)) < m(d) + 1$.

(c) If no element exists satisfying (b), then the successive images of a are all distinct. Further, if $n(x) = 0$, $x = f(a^k)$ and $m(f(x)) = m(x) + 1$.

We can now describe the sets X_0 , X_1 , and X_2 . If there exists an element d satisfying (b), put $X_0 = \{d\}$.

For an element x of $R(z) \setminus X_0$, we have

$$m(f(x)) + n(f(x)) = m(x) + n(x) - 1$$

if $n(x) > 0$, and

$$m(f(x)) + n(f(x)) = m(x) + n(x) + 1$$

if $n(x) = 0$. In either case, the integers $m(f(x)) + n(f(x))$ and $m(x) + n(x)$ cannot both be odd or both be even. Let X_1

be the set of elements of $R(z) \setminus X_0$ such that $m(x) + n(x)$ is odd and X_2 be the set of elements of $R(z) \setminus X_0$ such that $m(x) + n(x)$ is even. Then it is clear from the construction that $R(z) = X_0 \cup X_1 \cup X_2$ is a disjoint union and that $f[X_i] \subset X_j \cup X_k$ for all distinct choices of i, j , and k .

The proof of the proposition is completed by choosing X_0 , X_1 , and X_2 for each equivalence class and taking the unions of the corresponding sets to obtain the required decomposition of X . $\square$

6.23. COROLLARY:

If D is an infinite discrete space and f maps D into itself, then the set of fixed points of the extension mapping $\beta(f)$ of βD into itself coincides with the closure of the set of fixed points of f .

Proof: Let F denote the set of fixed points of f . If a point p of βD is in the closure of F , then there is a net in F converging to p . Since the net is mapped identically onto itself by $\beta(f)$, continuity implies that p is a fixed point of $\beta(f)$.

Now assume that p in D^* is a fixed point of $\beta(f)$. We will show that p belongs to $\text{cl } F$ by showing that p cannot belong to $\text{cl}(D \setminus F)$. Using the proposition, write $D \setminus F$ as a disjoint union

$$D \setminus F = X_0 \cup X_1 \cup X_2$$

such that $f[X_i] \subset X_j \cup X_k \cup F$ for all distinct choices of i ,

j, and k. Then for each i, $\mathcal{C}[X_i]$ is contained in $\mathcal{C}(D \setminus X_i)$ so that no point of X_i^* can be a fixed point of $\beta(f)$. Hence, the fixed points must belong to $\mathcal{C}F$. $\square$

6.24. The next series of results is taken from the 1968A paper of Z. Frolík and is an investigation of fixed points of mappings of BD into D^* . The first proposition shows that any fixed point of a mapping of BD into itself has an f -invariant neighborhood. Before undertaking the proof, which will involve a transfinite induction argument, it is necessary to define odd and even ordinal numbers. It is shown in E. Kamke's book that every ordinal number α can be written in the form

$\alpha = \omega^\beta + n$ where n is a natural number, i.e. $n < \omega^0$. We

shall say that the ordinal α is even or odd according as the natural number n is even or odd. Observe that if α is a limit ordinal, n is equal to 0 so that all limit ordinals

are even. The next result will ultimately be applied to show that no embedding of BD into D^* can have a fixed point.

PROPOSITION:

If D is an infinite discrete space and if p is a fixed point of a mapping f of BD into itself, then there is an f -invariant neighborhood V of p . Further, V can be chosen to be the closure of a subspace of D .

Proof: If p belongs to D , then the statement is obvious since $\{p\}$ is such a neighborhood. Assume that p belongs to D^* and choose Z in the ultrafilter $\mathcal{A}_p$ so that $\mathcal{C}Z$ is a neighborhood of p . We will show that a subset Y of Z

can be obtained so that $V = \text{cl } Y$ is the required f -invariant neighborhood of p . The construction of Y will be accomplished by transfinite induction. If $f[Z]$ is contained in $\text{cl } Z$, then by continuity $f[\text{cl } Z]$ is contained in $\text{cl } Z$ and we can choose $Y = Z$. Assume that $f[Z]$ is not contained in $\text{cl } Z$. Then the set

$$X_1 = \{d \in Z : f(d) \notin \text{cl } Z\}$$

is non-empty. Now for every point d in $Z \setminus X_1$, $f(d)$ belongs to $\text{cl } Z$, however, $\text{cl}(Z \setminus X_1)$ is not necessarily an f -invariant neighborhood of p since $Z \setminus X_1$ may contain points d such that $f(d)$ belongs to $\text{cl } X_1$. Therefore, define

$$X_2 = \{d \in Z \setminus X_1 : f(d) \in \text{cl } X_1\}.$$

Now by the same argument as given for $Z \setminus X_1$, the set $Z \setminus (X_1 \cup X_2)$ still may not yield the required f -invariant neighborhood. However, we can continue the elimination of points of Z until the required f -invariant neighborhood is obtained as the closure of the remaining points of Z . For each ordinal $\alpha > 1$, put

$$P_\alpha = \bigcup \{X_\beta : \beta < \alpha, \beta \text{ is non-limit}\}$$

$$Q_\alpha = \bigcup \{X : \beta < \alpha\}.$$

Now as in the first two steps in the elimination, put

$$X_\alpha = \{d \in Z \setminus Q_\alpha : f(d) \in \text{cl } Q_\alpha\}$$

if α is not a limit ordinal and put

$$X_\alpha = \{d \in Z \setminus Q_\alpha : f(d) \in \text{cl } P_\alpha\}$$

if α is a limit ordinal. By construction, $\{X_\alpha\}$ is a family of disjoint subsets of Z . It is thus impossible for all of the X_α 's to be non-empty and therefore there must exist some non-limit ordinal $\alpha > 1$ such that X_α is empty. Note also that X_β is empty for all $\beta \geq \alpha$ whenever X_α is empty. Now consider a point d in $Z \setminus Q_\alpha$. The image $f(d)$ of d must belong to one of the following sets: $cl(Z \setminus Q_\alpha)$, $cl Q_\alpha$, or $cl(D \setminus Z)$. Because $X_\alpha = \emptyset$, $f(d)$ is not in $cl Q_\alpha$. Any d such that $f(d)$ is in $cl(D \setminus Z)$ must belong to X_1 . Hence, we have that

$$\{d \in Z \setminus Q_\alpha : f(d) \notin cl(Z \setminus Q_\alpha)\} = (Z \setminus Q_\alpha) \cap X_1 = \emptyset$$

since X_1 is a subset of Q_α . Thus, by putting $Y = Z \setminus Q_\alpha$, we see that $f[Y]$ is contained in $cl Y$ and that $cl Y$ is f -invariant by the continuity of f .

It remains to show that Y belongs to A^p , i.e. that p is in $cl Y$ so that $cl Y$ is a neighborhood of p . This will be accomplished by writing D as the union of five disjoint sets of which one will be Y and showing that p fails to belong to the closure of any of the other four. First observe that if W is a subset of D containing p in its closure, then the set rW defined by

$$rW = \{d \in W : f(d) \in cl W\}$$

also contains p in its closure since p is a fixed point of f and f is continuous. Now since X_β is empty for all $\beta \geq \alpha$,

$$Y = Z \setminus Q_\alpha = Z \setminus \left(\bigcup_{\beta \geq 1} X_\beta \right).$$

We can write Q_α as the disjoint union of three sets:

$$Z_0 = \cup\{X_\beta : \beta \text{ is a limit ordinal}\}$$

$$Z_1 = \cup\{X_\beta : \beta \text{ is odd}\}$$

$$Z_2 = \cup\{X_\beta : \beta \text{ is even and non-limit}\}.$$

Then D can be written as a disjoint union:

$$D = Y \cup Z_0 \cup Z_1 \cup Z_2 \cup (D \setminus Z).$$

From the definition of the family $\{X_\beta\}$, we have

$$f[Z_0] \subset \text{cl}(Z_1 \cup Z_2)$$

$$f[Z_1] \subset \text{cl}[(D \setminus Z) \cup Z_0 \cup Z_2]$$

$$f[Z_2] \subset \text{cl } Z_1.$$

Thus, we have that $rZ_0 = rZ_1 = rZ_2 = \emptyset$ and by our earlier observation concerning rW , p cannot belong to the closure of any of the five sets except Y . $\parallel$

6.25. As we saw in Example 3.43, the theory of types of ultrafilters can be applied to construct mappings of $\beta\mathbb{N}$ into $\mathbb{N}^*$. If two points p and q of $\mathbb{N}^*$ have the same relative types with respect to countable discrete subspaces X and Y , then there is a bijection of X with Y whose extension to a mapping of $\text{cl } X$ onto $\text{cl } Y$ sends p to q . Using this fact, we can describe a mapping of $\beta\mathbb{N}$ into $\mathbb{N}^*$ which has exactly one fixed point. The construction will be similar to that of Example 3.43 except for modifications which assure that the map

is one-to-one on $\mathbb{N}$.

EXAMPLE:

There exists a mapping f of $\beta\mathbb{N}$ into $\mathbb{N}^*$ such that

- (a) f restricted to $\mathbb{N}$ is one-to-one,
- (b) $f[\mathbb{N}]$ is a countable union of discrete subspaces, and
- (c) $f^2[\beta\mathbb{N}]$ is a singleton implying that f has a unique

fixed point.

To construct f , choose a discrete countable set Y_n of $\mathbb{N}^*$ for each $n \geq 1$ such that Y_{n+1} is a subspace of Y_n^* and choose a point p which belongs to $\bigcap Y_n^*$. Let $\{N_n\}$ be any decomposition of $\mathbb{N}$ into infinite sets and choose a sequence $X = \{x_n\}$ in $\mathbb{N}^*$ such that the type of x_n relative to N_n is the same as the type of p relative to Y_n . The sequence X is a discrete subspace of $\mathbb{N}^*$ so that there exists a homeomorphism h of $\beta\mathbb{N}$ onto $\text{cl } X$. Put $q = h(p)$. Then because h is a homeomorphism, the relative type of q with respect to the discrete subspace $h[Y_n]$ is the same as that of x_n with respect to N_n . Therefore, there exists a bijection of N_n with $h[Y_n]$ which sends the traces of neighborhoods of x_n on N_n to traces of neighborhoods of q on $h[Y_n]$ for each n . Thus, we have defined a bijection of $\mathbb{N}$ onto $\bigcup h[Y_n]$ such that if f is the Stone-Čech extension of the bijection, then $f(x_n) = q$ for each n . Since $f[\beta\mathbb{N}]$ is contained in the image of h , $f[\beta\mathbb{N}]$ is contained in $\text{cl } X$. Thus, by continuity,

$$f^2[\beta\mathbb{N}] \subset f[\text{cl } X] \subset \{q\},$$

and q is the unique fixed point of f .

6.26. Although we have now seen several examples of mappings of $\beta\mathbb{N}$ into $\mathbb{N}^*$ which do have fixed-points, none of the examples have been embeddings although there exist many embeddings of βD into D^* . For instance, note that in Theorem 5.13 we showed that uD , the subspace of points p in D^* such that the corresponding ultrafilter A^p is uniform, contains a copy of βD . Thus, there exists an embedding of βD into uD . However, neither this nor any other embedding of βD into D^* can have a fixed point. The proof relies on the existence of an invariant neighborhood of any fixed point of such a mapping.

THEOREM: (Frolík)

If D is an infinite discrete space, then no embedding of βD into D^* has a fixed point.

Proof: Assume that a point p of D^* is a fixed point of an embedding h of βD into D^* . Consider the collection $\mathcal{X}$ of all subsets X of D which admit a disjoint decomposition $X = X_1 \cup X_2 \cup X_3$ such that whenever i, j , and k are distinct,

$$h[X_i] \subset \text{cl}(X_j \cup X_k).$$

Since h is one-to-one, the set of non-fixed points of βD is h -invariant, and it follows from Proposition 6.22 that $\mathcal{X}$ is non-empty. It is clear that the union of any pairwise disjoint subcollection of $\mathcal{X}$ belongs to $\mathcal{X}$ and that p can not belong to the closure of any member of $\mathcal{X}$. By applying Zorn's Lemma to the set of pairwise disjoint subcollections of $\mathcal{X}$ partially

ordered by inclusion, there exists a maximal pairwise disjoint subcollection $\mathfrak{m}$ of $\mathcal{X}$. Let M be the union of $\mathfrak{m}$. The proof will be completed by using the assumption that p is a fixed point of h to exhibit a non-void X in $\mathcal{X}$ which misses M , thus contradicting the maximality of the collection $\mathfrak{m}$.

Since $\text{cl}(D \setminus M)$ is a neighborhood of p , by Proposition 6.24 there exists Y contained in $D \setminus M$ such that $\text{cl } Y$ is an h -invariant neighborhood of p . Because h is one-to-one and $\text{cl } Y$ is h -invariant, not every point of Y^* can be a fixed point. Choose q in Y^* such that $h(q) \neq q$. Then $h(q)$ and q have disjoint neighborhoods so that we may choose X_3 contained in Y such that q belongs to X_3^* , $h(q)$ belongs to $(h[X_3])^*$, and $h[X_3] \cap \text{cl } X_3 = \emptyset$. Now because $h[X_3]$ is contained in Y^* , there is a subset Z_1 of $Y \setminus X_3$ such that

$$\text{cl } Z_1 \cap h[D] = h[X_3].$$

Since $h[D]$ is C^* -embedded in βD , $h[Z_1] \setminus \text{cl } X_3$ is completely separated in βD from its complement in $h[D]$ and also from $\text{cl } X_3$. Therefore, there exists Z_2 contained in $Y \setminus (X_3 \cup Z_1)$ such that

$$\text{cl } Z_2 \cap h[D] = h[Z_1] \setminus \text{cl } X_3.$$

Continuing by induction, obtain a sequence $\{Z_n\}$ of subsets of Y such that

$$Z_{n+1} \subset Y \setminus (X_3 \cup (\cup \{Z_i : i \leq n\}))$$

and

$$\text{cl } Z_{n+1} \cap h[D] = h[Z_n] \setminus \text{cl } X_3.$$

Now put

$$X_1 = \cup\{Z_n : n \text{ is odd}\} \quad \text{and} \quad X_2 = \cup\{Z_n : n \text{ is even}\}.$$

From the definition of Z_1 ,

$$h[X_3] \subset \text{cl } Z_1 \subset \text{cl } X_1$$

and for i and j distinct choices of 1 and 2,

$$f[X_i] \subset \text{cl}(X_j \cup X_3).$$

Consequently, the set $X = X_1 \cup X_2 \cup X_3$ belongs to $\mathcal{X}$ and is contained in $D \setminus M$, which contradicts the maximality of m . Hence, p cannot be a fixed point of h . $\square$

6.27. In the particular case of $\beta\mathbb{N}$, the fact that no embedding of $\beta\mathbb{N}$ into $\mathbb{N}^*$ has a fixed point can be interpreted in terms of relative types of ultrafilters and the producing relation which was introduced in 3.48. This interpretation will then yield a quick proof of the non-homogeneity of $\mathbb{N}^*$.

COROLLARY:

The type of a point of $\mathbb{N}^*$ is not also a relative type of the point, i.e. no type produces itself.

Proof: Suppose that p in $\mathbb{N}^*$ is of type t and that the type of p relative to a discrete sequence X contained in $\mathbb{N}^*$ is also t . Then there exists a bijection of $\mathbb{N}$ onto X which sends the ultrafilter A^p onto p_X , the neighborhood

traces of p on X . But then the Stone-Cech extension of the bijection is a homeomorphism of $\beta\mathbb{N}$ into $\mathbb{N}^*$ having p as a fixed point. $\blacksquare$

6.28. COROLLARY:

$\mathbb{N}^*$ is not homogeneous.

Proof: If h is an automorphism of $\mathbb{N}^*$ and $h(p) = q$, then the relative types of p and q coincide. Thus, if p is of type t and q has t as a relative type, then we cannot have $h(p) = q$ for any h . $\blacksquare$

EXERCISES

6A. RETRACTIVE F-SPACES

Recall that any countable subspace of an F -space is C^* -embedded (Proposition 1.64), and that X is an F -space if and only if βX is an F -space (Proposition 1.60).

1. A retractive F -space is locally compact and admits no closed C^* -embedded copy of $\mathbb{N}$. Note that this result does not depend on the Continuum Hypothesis as does Theorem 6.7.
[Construct N and K as in the proof of Theorem 6.7. Then $N \cup K$ is not C^* -embedded in $N \cup K \cup N^*$.]
2. No P -space is retractive; hence no discrete space is retractive.

Reference: Comfort, 1965.

6B. RETRACTIONS OF βD

If D is an infinite discrete space and $f : \beta D \rightarrow D^*$ is continuous, then f is a retraction if and only if $f^2(d) = f(d)$ for every d in D .

Reference: Frolik, 1968A.

6C. RETRACTIVE SUBSPACES

A closed, non-compact C^* -embedded subspace of a retractive space is retractive.

6D. AN APPLICATION OF CELLULARITY

1. A space X whose cellularity is at most $\aleph_0$ and which admits a mapping of βX onto X^* is pseudocompact.
[Consider the inverse image under such a mapping of a family of disjoint open sets provided by Corollary 5.6.]
2. If X is realcompact and separable, then there is no mapping of βX onto X^* . [Corollary 1.58.]

Reference: Comfort, 1965.

6E. THE CONVERSE OF MAGILL'S THEOREM

If a space X is such that every continuous image of X^* is homeomorphic to the growth of X in some Hausdorff compactification of X , then X is locally compact. [Shrink X^* to a point.]

6F. GROWTHS AND THE UNIT INTERVAL

Let X be a locally compact normal space which contains a closed, infinite, discrete subspace.

1. X contains a closed, C^* -embedded copy of $\mathbb{N}$.
2. X^* contains a copy of $\mathbb{N}^*$.
3. There is a countably infinite discrete space Z such that βZ is contained in X^* . [$\mathbb{N}^*$ contains a copy of $\mathbb{N}$.]
4. $I = [0,1]$ is a continuous image of X^* . [Map Z to the rationals of I and use Proposition 1.47.]
5. Any continuous image of I is a growth of X in some

compactification of X . [Theorem 6.14.]

6. All the hypotheses are needed. [Deleted Tychonoff Plank, ω_1 , any non-locally compact space.]

References: Magill, 1966. Hahn, in 1914, and Mazurkiewicz, in 1920, characterized the continuous images of I as Peano spaces, i.e. as compact, connected, locally connected metric spaces. For a proof of this result, see Willard, 1970.

6G. NON-PSEUDOCOMPACTNESS AND GROWTHS

If X is a locally compact space which fails to be pseudocompact then any continuous image of the unit interval is homeomorphic to the growth of X in some compactification. [Do the preceding Exercise first.]

6H. GROWTHS AS GRAPH CLOSURES

Let X be a locally compact but non-compact space and let K be a compact space. Let $\alpha X = X \cup \{\infty\}$ be the one point compactification of X and let f map X into K such that the image of the trace on X of every neighborhood of ∞ is dense in K .

1. X is homeomorphic to the graph of f in $X \times K$.
[$h(x) = (x, f(x))$.]
2. If the graph of f is considered as a subspace of $\alpha X \times K$, then no point of the form (x, k) where x belongs to X

and $f(x) \neq k$ is the closure of the graph. [K is Hausdorff and f is continuous.]

3. Points of the form (∞, k) are in the closure of the graph.
4. X has a compactification with growth homeomorphic to K .
5. The half-open interval $(0, 1]$ has a compactification with growth homeomorphic to $[-1, 1]$. [$f(x) = \sin(\frac{1}{x})$.]
6. If X is an infinite discrete space and K is compact with density at most the cardinality of X , then X has a compactification with growth homeomorphic to K . [Write X as a union of ω_0 disjoint subsets each having the same cardinality as X . Map each of these subsets to a dense subspace of K .]
7. Not all compactifications of $\mathbb{N}$ can be obtained as in 6. [$\mathbb{N}^*$ has density character c .]
8. The result 6F.5 can be obtained as a corollary of 4 above.

Reference: A. K. Steiner and E. F. Steiner, 1968. For some examples in the special case $X = [0, \infty)$, particularly with K the torus $S^1 \times S^1$, see B. Simon, 1969.

6I. LOCALLY COMPACT EXTREMALLY DISCONNECTED SPACES

1. A space X is extremally disconnected if and only if βX is extremally disconnected.
2. If X is locally compact and extremally disconnected, then βX is the unique extremally disconnected compactification of X .

6J. SECOND COUNTABLE NORMAL SPACES

A second countable normal space is metrizable, and hence is paracompact. [Apply the fact that disjoint closed subsets of a normal space are completely separated to pairs (U,V) of basic open sets such that $c \not\in U \subset V$ to obtain a countable family $\mathfrak{F}$ of mappings. Then use the Embedding Lemma, 1.5.]

Reference: In 1925A, P. Urysohn proved the more general result that a regular T_1 -space is metrizable. See also [D, p. 195].

CHAPTER SEVEN:

 $\beta \mathbb{N}$ REVISITED

7.1. This brief visitation to $\beta \mathbb{N}$ is made possible by Corollary 4.30 in which we saw that P-points exist in $\mathbb{N}^*$ under the assumption of the Continuum Hypothesis. Here we will establish two main results which depend upon the existence of P-points in $\mathbb{N}^*$ and therefore were not included in Chapter 3. The first is that the subspace $\mathbb{N}^* \setminus \{p\}$ of $\mathbb{N}^*$ is not normal. Second, we will show that if p and q are P-points of $\mathbb{N}^*$, then there is an automorphism of $\mathbb{N}^*$ which will send p to q . As an immediate consequence of this result we will see that $\mathbb{N}^*$ admits precisely 2^c automorphisms. The Continuum Hypothesis will be assumed for most of the chapter.

 $\mathbb{N}^* \setminus \{p\}$ IS NOT NORMAL

7.2 If p is any point of $\mathbb{N}^*$ and the Continuum Hypothesis is assumed, then $\mathbb{N}^* \setminus \{p\}$ is not normal. The proof of this result was given by L. Gillman for the case where p is a non-P-point of $\mathbb{N}^*$ and by M. Rajagopalan where p is a P-point. We will consider Gillman's argument as it appeared in the 1968 paper of W. W. Comfort and S. Negrepontis.

Recall from Section 1.65 that a non-P-point must belong to the boundary of a zero-set. That fact together with the following result form the key to demonstrating the non-normality of $\mathbb{N}^* \setminus \{p\}$ when p is not a P-point.

PROPOSITION [CH]: (Gillman)

The boundary of a non-open zero-set of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$.

Proof: Let Z be a non-open zero-set of $\mathbb{N}^*$ so that its boundary ∂Z is non-empty. We will show that ∂Z is a copy of $\mathbb{N}^*$ by showing that it satisfies the hypotheses of Theorem 3.32. It is clear that ∂Z is compact and totally disconnected. Because ∂Z is compact, it is C^* -embedded in $\mathbb{N}^*$ and therefore has a base of c clopen sets. Further, since $\mathbb{N}^*$ is an F -space (Proposition 1.62 or 3.24), the C^* -embedded subspace ∂Z is also an F -space. To show that the zero-sets of ∂Z are regular closed and that ∂Z has no isolated points, we will show that ∂Z is the growth of a locally compact, realcompact space (Proposition 4.21). The cozero-set $C = \mathbb{N}^* \setminus Z$ is C^* -embedded in $\mathbb{N}^*$ since $\mathbb{N}^*$ is an F -space. Hence, $\beta C = cl_{\mathbb{N}^*} C$ and $C^* = \partial Z$. A cozero-set in a realcompact space is realcompact [GJ, 8.14], so that C is realcompact. C is locally compact since it is open in $\mathbb{N}^*$. Proposition 4.21 now implies that the zero-sets of ∂Z are regular closed. Finally, in the proof of Theorem 4.23 we showed that the growth of a realcompact locally compact space has no isolated points. Hence, with the assumption of the Continuum Hypothesis, Theorem 3.32 implies that $cl Z$ is homeomorphic to $\mathbb{N}^*$. $\blacksquare$

7.3. THEOREM [CH]: (Gillman)

$\mathbb{N}^* \setminus \{p\}$ is not normal if p is a non-P-point of $\mathbb{N}^*$.

Proof: Since p is a non-P-point, there is a zero-set Z

in $\mathbb{N}^*$ such that p belongs to ∂Z . Since ∂Z is a copy of $\mathbb{N}^*$ and the Continuum Hypothesis implies that dense subspaces of $\mathbb{N}^*$ are not C^* -embedded (Proposition 3.30), $\partial Z \setminus \{p\}$ is not C^* -embedded in ∂Z . We will show that the assumption that $\mathbb{N}^* \setminus \{p\}$ is normal will contradict this fact. $\partial Z \setminus \{p\}$ is a closed subspace of $\mathbb{N}^* \setminus \{p\}$. Hence, if $\mathbb{N}^* \setminus \{p\}$ is normal, a bounded, real-valued mapping f on $\partial Z \setminus \{p\}$ will extend to a mapping g on $\mathbb{N}^* \setminus \{p\}$. As in the previous proposition, if $C = \mathbb{N}^* \setminus Z$, then $C^* = \partial Z$. Hence, the restriction $g|_C$ extends to βC and $\beta(g|_C)|_{\partial Z}$ extends f to ∂Z . But this is a contradiction. |

7.4. The case where p is a P -point is more complicated. The result was announced in 1969 by N. C. MacMaster and will also appear in a forthcoming paper by M. Rajogopalan. Our proof will be the one given by Rajogopalan. We will show that $\mathbb{N}^* \setminus \{p\}$ can be mapped onto the space $\omega_1 \times (\omega_1 + 1)$ by a closed mapping. Since $\omega_1 \times (\omega_1 + 1)$ is not normal [GJ, ex. 8M] and a closed continuous image of a normal space must be normal [D, p. 145], this will show that $\mathbb{N}^* \setminus \{p\}$ is not normal. (A proof that $\omega_1 \times (\omega_1 + 1)$ is not normal will be provided in Example 8.37.) The construction of the closed mapping will be based on the construction of the mapping of $\mathbb{N}^*$ onto $\omega_1 + 1$ as described in the proof of Corollary 6.16.

THEOREM [CH]:

$\mathbb{N}^* \setminus \{p\}$ is not normal if p is a P -point.

Proof: We begin by modifying the construction of the mapping f of $\mathbb{N}^*$ onto $\omega_1 + 1$ which was described in the proof of Corollary 6.16. Recall that we constructed a strictly increasing ω_1 -sequence $\{U_\alpha : \alpha < \omega_1\}$ of proper clopen subsets of $\mathbb{N}^*$. The definition $f(p) = \sup\{\alpha < \omega_1 : p \notin U_\alpha\}$ yields a mapping of $\mathbb{N}^*$ onto $\omega_1 + 1$ such that $f^{-1}(\{\omega_1\}) = \bigcap\{\mathbb{N}^* \setminus U_\alpha : \alpha < \omega_1\}$. Assuming the Continuum Hypothesis, there exists a P-point p in $\mathbb{N}^*$ and the base of c clopen neighborhoods of p can be indexed by ω_1 . Begin the construction of f as before by choosing U_0 to be a non-empty clopen set which misses p . Then because every G_δ containing p is a neighborhood of p , at each stage of the construction U_α can be chosen so that the complement of U_α is a basic neighborhood of p . Hence, $\bigcap\{\mathbb{N}^* \setminus U_\alpha : \alpha < \omega_1\} = \{p\}$ and if f is defined as above, $f^{-1}(\{\omega_1\}) = \{p\}$.

Using the mapping f , we will describe a partition of $\mathbb{N}^* \setminus \{p\}$ which induces a quotient mapping of $\mathbb{N}^* \setminus \{p\}$ onto $\omega_1 \times (\omega_1 + 1)$. For each α , put $W_\alpha = f^{-1}([1, \alpha])$ and $V_\alpha = W_\alpha \setminus (\bigcup\{W_\beta : \beta < \alpha\})$ so that $f^{-1}(\{\alpha\}) = V_\alpha$. Now we will define a family of subsets $\{V_{\alpha\beta} : \alpha < \omega_1, \beta \leq \omega_1\}$ so that:

(a) For every $\alpha < \omega_1$, the subcollection $\{V_{ij} : i \leq \alpha, j \leq \omega_1\}$ partitions W_α , and

(b) The partition of W_α induces a quotient mapping g_α of W_α onto $\alpha \times (\omega_1 + 1)$ and $g_{\alpha+1}|_{W_\alpha} = g_\alpha$.

The family will be constructed by transfinite induction. We will construct the partition for the non-limit ordinals first.

If α is non-limit, $U\{W_\beta : \beta < \alpha\}$ is clopen so that $V_\alpha = f_\alpha^{\leftarrow}(\{\alpha\})$ is the difference of clopen sets and is therefore clopen. Hence, there exists a P-point p_α in V_α since the P-points are dense in $\mathbb{N}^*$ (Corollary 4.30). Since the clopen set V_α is a copy of $\mathbb{N}^*$ (Proposition 3.16), there exists a mapping $f_\alpha : V_\alpha \rightarrow \omega_1 + 1$ such that $f_\alpha^{\leftarrow}(\{\omega_1\}) = \{p_\alpha\}$. Put $V_{\alpha\beta} = f_\alpha^{\leftarrow}(\{\beta\})$ for $\beta \leq \omega_1$. This completes the construction for non-limit ordinals.

The situation is somewhat more complicated for the limit ordinal case. Let α be a limit ordinal and assume that the partition of W_δ has been defined for all $\delta < \alpha$. To define $V_{\alpha\beta}$ for each $\beta < \omega_1$ we will construct an increasing sequence of clopen sets dominated by a decreasing sequence and use DuBois-Reymond separability to define $V_{\alpha\beta}$ (Section 2.20). Let $\{\alpha_n\}$ be a sequence of non-limit ordinals whose supremum is α . To obtain $V_{\alpha 0}$, begin by defining the following two sequences:

$$A_1 = U\{V_{i0} : i \leq \alpha_1\}, \dots, A_n = U\{V_{i0} : i \leq \alpha_n\}, \dots$$

Thus, $U A_n$ includes all points of $U\{W_\beta : \beta < \alpha\}$ which are mapped to points having second coordinate equal to 0.

$$B_1 = (W_\alpha \setminus (U\{W_i : i \leq \alpha_1\})) \cup A_1, \dots, B_n = (W_\alpha \setminus (U\{W_i : i \leq \alpha_n\})) \cup A_n, \dots$$

B_{n+1} is contained in B_n since the family $\{W_i\}$ is increasing and for any index i , $\alpha_n < i \leq \alpha_{n+1}$, the points of A_i which are included in B_{n+1} also belong to B_n . Hence, $\bigcap B_n$ excludes all points of $U\{W_\beta : \beta < \alpha\}$ which are sent to points having

second coordinate greater than 0. Since $A_n = g_{\alpha_n}^{\leftarrow}(\alpha_n \times \{0\})$, each A_n is clopen. Similarly, $B_n = (f_{\alpha_n}^{\leftarrow}([0, \alpha_n]) \setminus f_{\alpha_n}^{\leftarrow}([0, \alpha_n])) \cup A_n$ and is also clopen. Thus, we have two sequences of clopen sets such that

$$A_1 \subset \dots \subset A_n \subset \dots \subset B_n \subset \dots \subset B_1.$$

Since $CO(\mathbb{N}^*)$ is DuBois-Reymond separable (Corollary 3.25), there exists a clopen set H such that $A_n \subset H \subset B_n$ for all n . Put $V_{\alpha 0} = H \cap V_{\alpha}$. If $\{y_n\}$ is a sequence of points such that y_n is in $V_{\alpha_n 0}$ for each n , then the cluster points of $\{y_n\}$ must belong to $H \cap V_{\alpha}$ so that $V_{\alpha 0}$ is non-empty.

Now suppose that for some $\beta < \omega_1$, $V_{\alpha \gamma}$ has been defined for all $\gamma < \beta$. We define our sequences $\{A_n\}$ and $\{B_n\}$ somewhat as before. However, now we want A_n to include those points which g_{α_n} maps to points with second coordinate at most β . B_n will exclude those points which are mapped by g_{α_n} to points having second coordinate greater than β . Thus, define

$$A_n = \bigcup \{V_{ij} : i \leq \alpha_n, j \leq \beta\}$$

and

$$B_n = (W_{\alpha} \setminus W_{\alpha_n}) \cup A_n.$$

Then again we have sequences of clopen sets such that

$$A_1 \subset \dots \subset A_n \subset \dots \subset B_n \subset \dots \subset B_1$$

and there exists a clopen set H such that $A_n \subset H \subset B_n$ for all n . Put $V_{\alpha \beta} = (H \cap V_{\alpha}) \setminus (\bigcup \{V_{\alpha j} : j < \beta\})$. The set $V_{\alpha \beta}$

is seen to be non-empty by choosing a sequence $\{y_n\}$ as before, this time with y_n in $V_{\alpha_n \beta}$.

Finally, put $V_{\alpha \omega_1} = V_\alpha \setminus (\cup \{V_{\alpha j} : j < \omega_1\})$. Recall that for each of the indices α_n , $V_{\alpha_n \omega_1} = \{p_{\alpha_n}\}$ where p_{α_n} is a P-point. $V_{\alpha \omega_1}$ is non-empty since it must contain the cluster points of the sequence $\{p_{\alpha_n}\}$. Thus, we have defined a partition of W_α and it is clear that the quotient map g_α induced by the partition maps W_α onto $\alpha \times (\omega_1 + 1)$.

Now consider the partition of $\mathbb{N}^*$ given by

$$\Pi = \{V_{\alpha \beta} : \alpha < \omega_1, \beta \leq \omega_1\} \cup \{p\}.$$

It remains to show that the quotient map $g_{\omega_1} : \mathbb{N}^* \setminus \{p\} \rightarrow \omega_1 \times (\omega_1 + 1)$ is closed. To do this, we show that the partition Π of $\mathbb{N}^*$ induces a Hausdorff quotient space Y . Thus, the quotient map on $\mathbb{N}^*$ is closed and g_{ω_1} is just the restriction to a saturated set and hence is also closed.

Since each W_α is clopen and saturated, $g_{\omega_1}[W_\alpha]$ is clopen in Y . Hence, because the images of any two points other than p are contained in some $g_{\omega_1}[W_\alpha]$, they can be separated by open sets in Y . On the other hand, any point of Y other than the image of p is separated from the image of p by some $g_{\omega_1}[W_\alpha]$ and its complement. $\square$

TYPES, $\mathbb{N}^*$ -TYPES, AND P-POINTS

7.5. The remainder of the chapter will be devoted to a discussion of the P-points of $\mathbb{N}^*$ in the context of types of ultrafilters

as discussed in Chapter 3. There we defined two points of $\mathbb{N}^*$ to be of the same type if one can be mapped to the other by an automorphism of $\beta\mathbb{N}$. We now define two points of $\mathbb{N}^*$ to be of the same $\mathbb{N}^*$ -type if there is an automorphism of $\mathbb{N}^*$ which maps one to the other. All of the tools necessary to demonstrate that there are also 2^c $\mathbb{N}^*$ -types were actually developed in the proof of Corollary 3.51 where we showed that $\mathbb{N}^*$ is not homogeneous. Note that this result does not depend on the Continuum Hypothesis.

PROPOSITION: (Frolík)

There are 2^c $\mathbb{N}^*$ -types of points in $\mathbb{N}^*$ and $\mathbb{N}^*$ contains a dense subset of each type.

Proof: If p is a point of $\mathbb{N}^*$, let T_p denote the set of relative types of p as in Corollary 3.51. Consider the family $\mathcal{Q} = \{T_p : p \in \mathbb{N}^*\}$ of all such sets. Theorem 3.50 implies that each set T_p contains at most c types. Since $\mathcal{Q}$ is a cover of the set of all types and there are 2^c types, $\mathcal{Q}$ must contain 2^c distinct sets.

Since the set of relative types is an invariant under automorphisms of $\mathbb{N}^*$, T_p must equal T_q if p and q are of the same $\mathbb{N}^*$ -type. Hence, because $\mathcal{Q}$ contains 2^c distinct members, there must be a least 2^c $\mathbb{N}^*$ -types. There can be no more since $\mathbb{N}^*$ has just 2^c points. The density of the points of any given $\mathbb{N}^*$ -type is clear since the orbit of any point is dense in $\mathbb{N}^*$ (Corollary 3.20). $\blacksquare$

In her 1966 paper, M. E. Rudin proved a similar theorem in the presence of the Continuum Hypothesis. The present version is from Z. Frolik's 1967B paper.

7.6. If a point is a P-point of $\mathbb{N}^*$, then every point of the same type or $\mathbb{N}^*$ -type must also be a P-point. Since there are 2^c P-points and only c points of each type, there are 2^c distinct types of P-points. However, in 1956 W. Rudin showed that all P-points are of the same $\mathbb{N}^*$ -type. The proof is accomplished by showing that for any two P-points there is an automorphism of the Boolean algebra $\text{CO}(\mathbb{N}^*)$ which interchanges the filters determined by the two P-points. The resultant automorphism of $\mathbb{N}^*$ therefore interchanges the two P-points. The construction of the Boolean algebra automorphism is similar to that given in the proof of Theorem 3.31.

THEOREM [CH]: (W. Rudin)

All P-points of $\mathbb{N}^*$ are of the same $\mathbb{N}^*$ -type. Hence, $\mathbb{N}^*$ admits precisely 2^c automorphisms.

Proof: There can be no more than 2^c automorphisms since distinct automorphisms of $\mathbb{N}^*$ induce distinct permutations of the clopen sets of $\mathbb{N}^*$ and there are only 2^c such permutations. Because $\mathbb{N}^*$ has 2^c P-points, the first statement of the theorem implies the second.

Now let p and q be P-points of $\mathbb{N}^*$. Each of the points is contained in c clopen subsets of $\mathbb{N}^*$. Using the Continuum Hypothesis, let the families $\{S_\alpha : \alpha < \omega_1\}$ and $\{T_\alpha : \alpha < \omega_1\}$ of

all clopen subsets containing p and q , respectively, be indexed by the countable ordinals. Further, assume that $S_0 = T_0 = \mathbb{N}^*$. We now construct an automorphism σ of $\text{CO}(\mathbb{N}^*)$ so that $\sigma[\{S_\alpha\}] = \{T_\alpha\}$. Note that it is sufficient to define σ for the family $\{S_\alpha\}$ since the requirement that σ preserve the Boolean operations dictates the value of σ on the other clopen sets which are merely complements of the S_α 's. The construction will be carried out by induction. We will show that if σ has been defined for a countable field of sets not containing a set S_α , then σ can be extended to a countable field containing S_α . As in Theorem 3.31, Lemma 2.26 will be used to define the extension.

Begin by putting $\sigma(S_0) = T_0$ and $\sigma(\emptyset) = \emptyset$ so that the family for which σ is initially defined forms a field. Now assume that σ has been defined for a countable field $\mathcal{C}$ and that α is the least ordinal for which $\sigma(S_\alpha)$ has not been defined. Consider the intersection $S_\alpha \cap (\bigcap \{S_\beta : \sigma(S_\beta) \text{ has been defined}\})$. Since p is a P -point, there exists an S_γ with $\gamma > \alpha$ such that S_γ is contained in this intersection. Similarly, since q is a P -point and each $\sigma(S_\beta)$ is a clopen set containing q , there exists a set T_γ with $T_\gamma \subset \bigcap \{\sigma(S_\beta)\}$. Define $\sigma(S_\gamma) = T_\gamma$ and $\sigma(\mathbb{N}^* \setminus S_\gamma) = \mathbb{N}^* \setminus T_\gamma$. Let $\mathcal{C}'$ be the field generated by S_γ and $\mathcal{C}$. Extend σ to $\mathcal{C}'$ by the requirement that σ preserve the Boolean operations. Thus, σ is now defined on a countable field which includes a subset of S_α . Write $\mathcal{C}'$ as the union of three subfamilies $\{F_i\}$, $\{G_i\}$, and $\{C_i\}$ such that $F_i \subset S_\alpha$, $S_\alpha \subset G_i$, and no inclusion relation holds between S_α and C_i .

for all values of i . Put $A_n = F_1 \cup \dots \cup F_n$ and $B_n = G_1 \cap \dots \cap G_n$ so that

$$A_1 \subset \dots \subset A_n \subset \dots \subset S_\alpha \subset \dots \subset B_n \subset \dots \subset B_1$$

and no C_i is contained in any A_n nor contains any B_n . Since σ preserves the Boolean operations, we have

$$\sigma(A_1) \subset \dots \subset \sigma(A_n) \subset \dots \subset \sigma(B_n) \subset \dots \subset \sigma(B_1)$$

and no $\sigma(C_i)$ is contained in any $\sigma(A_n)$ and no $\sigma(C_i)$ contains any $\sigma(B_n)$. Since $\text{CO}(\mathbb{N}^*)$ is dense in itself and Cantor and DuBois-Reymond separable, Lemma 2.26 implies that there exists a set T_α such that

$$\sigma(A_1) \subset \dots \subset \sigma(A_n) \subset \dots \subset T_\alpha \subset \dots \subset \sigma(B_n) \subset \dots \subset \sigma(B_1)$$

and T_α is unrelated by inclusion to any $\sigma(C_n)$. Define $\sigma(S_\alpha) = T_\alpha$ and $\sigma(\mathbb{N}^* \setminus S_\alpha) = \mathbb{N}^* \setminus T_\alpha$. Let C'' be the field generated by $C' \cup \{S_\alpha\}$. The requirement that σ preserve the Boolean operations dictates a unique extension of σ to C'' .

Now having defined $\sigma(S_\alpha)$, $\sigma^{\leftarrow}$ satisfies the same induction hypothesis and by the same procedure, we can define $\sigma^{\leftarrow}(T_\delta)$ where δ is the least ordinal for which T_δ does not belong to the range of σ . The process is continued to yield the automorphism σ .

Finally, the automorphism σ induces an automorphism of $\mathbb{N}^*$ as in Theorem 3.31. Since the induced automorphism exchanges the points of $\mathbb{N}^*$ corresponding to the maximal filters exchanged by σ , the automorphism sends p to q . $\blacksquare$

7.7. Proposition 3.49 shows that for a type to be produced by another type, it is necessary for a point of the type to be produced to be a cluster point of a countable subset of $\mathbb{N}^*$.

Since this is impossible for any P-point, we have verified the

PROPOSITION:

No type which consists of P-points is produced by any type.

This is not a characteristic property of P-points, however.

In 1970, K. Kunen announced that the Continuum Hypothesis implies the existence of a non-P-point in $\mathbb{N}^*$ which is not a cluster point of any countable subset of $\mathbb{N}^*$. In 1966, M. E. Rudin showed the existence of a point of $\mathbb{N}^*$ which is not a cluster point of any countable set of P-points of $\mathbb{N}^*$.

7.8. The problem of giving a necessary and sufficient condition for two points of $\mathbb{N}^*$ to be of the same type or $\mathbb{N}^*$ -type is unsolved. Here we have included only the very basic results relating P-points to this question. The most recent work on the question appears in M. E. Rudin's 1971 paper which considers partial orders on types of points. In that paper, she expresses the opinion that the work of D. D. Booth indicates that a solution independent of a logical assumption seems unlikely.

CHAPTER EIGHT:
PRODUCT THEOREMS

8.1. If $\{X_\alpha\}$ is any family of spaces, then $\times (\beta X_\alpha)$ is a compactification of $\times X_\alpha$. In Example 1.67 we observed that in the case of $\mathbb{R} \times \mathbb{R}$, $\beta \mathbb{R} \times \beta \mathbb{R}$ is not the Stone-Ćech compactification of $\mathbb{R} \times \mathbb{R}$. The present chapter will be largely devoted to determining when it will be true that $\times (\beta X_\alpha)$ is $\beta(\times X_\alpha)$, i.e. when $\times X_\alpha$ will be C^* -embedded in $\times (\beta X_\alpha)$. I. Glicksberg showed in 1959 that for infinite spaces, this will be the case exactly when the product $\times X_\alpha$ is pseudocompact.

Our program will be to first investigate conditions under which the product of two spaces will be pseudocompact and to obtain the Glicksberg Theorem for finite products. We will then find that the proof for a pseudocompact product of infinitely many spaces can be reduced to a situation involving the product of two spaces. At the end of the chapter we will consider some assorted product theorems which are somewhat related to the Glicksberg Theorem.

8.2. The following proposition eliminates the trivial case by showing that β will always distribute over the product of two spaces if one of them is finite.

PROPOSITION:

If X is finite, then $X \times Y$ is C^* -embedded in $X \times \beta Y$,
i.e. $\beta(X \times Y) = X \times \beta Y$.

Proof: Let $X = \{x_1, \dots, x_n\}$ and let f belong to $C^*(X \times Y)$. For each i , $f|_{\{x_i\} \times Y} \equiv f(x_i, y)$ belongs to $C^*(Y)$ and therefore has a continuous extension g_i to βY . Define $\beta(f)$ on $X \times \beta Y$ by $\beta(f)(x_i, p) = g_i(p)$ for each x_i in X and p in βY . The function $\beta(f)$ is continuous since its restriction to each set $\{x_i\} \times \beta Y$ is continuous and the family $\{\{x_i\} \times \beta Y\}$ is a finite closed cover of $X \times \beta Y$. $\square$

GLICKSBERG'S THEOREM FOR FINITE PRODUCTS

8.3. Glicksberg proved that $\beta(X \times Y)$ is $\beta X \times \beta Y$ precisely when $X \times Y$ is a pseudocompact product of two infinite spaces. We will first show that the product of two pseudocompact spaces may fail to be pseudocompact. The next result appeared in the 1962 paper of Fine and Gillman and describes a useful family of pseudocompact spaces.

LEMMA:

Let Y be a realcompact space, let H be a subspace of Y^* , and put $X = Y \cup H$. If X is pseudocompact, then H is dense in Y^* . Conversely, if H is dense in Y^* and Y is also locally compact, then X is pseudocompact.

Proof: We first prove the contrapositive of the first statement. If H is not dense in Y^* , then there exists an open set U of Y^* which misses H and $U = V \cap Y^*$ for some open set V of βY . Choose p in U . Then there exists a zero-set Z_1 of βY such that p belongs to Z_1 and Z_1 is contained in V .

Because Y is realcompact, by Theorem 1.53, there exists a zero-set Z_2 of βY which contains p and misses Y . Then $Z_1 \cap Z_2$ is a zero-set which contains p and is contained in $Y^* \setminus H = X^*$. Hence, X fails to be pseudocompact by Theorem 1.57.

The converse is also demonstrated by showing the contrapositive. If X is not pseudocompact, then there exists a non-empty zero-set Z of βX which is contained in X^* and hence in Y^* as well. But because Y is both realcompact and locally compact, Lemma 4.21 shows that the interior of Z is a non-empty open subset of Y^* which misses H . Hence, H is not dense in Y^* . $\blacksquare$

8.4. The pseudocompact spaces described in the lemma make it easy to construct two pseudocompact spaces whose product fails to be pseudocompact.

EXAMPLE:

Choose two points p and q of $\mathbb{N}^*$ which are of different types. Let Σ_p and Σ_q denote their orbits under automorphisms of $\beta \mathbb{N}$. Then Σ_p and Σ_q are disjoint dense subspaces of $\mathbb{N}^*$. Hence $\mathbb{N} \cup \Sigma_p$ and $\mathbb{N} \cup \Sigma_q$ are both pseudocompact. However, $(\mathbb{N} \cup \Sigma_p) \times (\mathbb{N} \cup \Sigma_q)$ contains the diagonal $\Delta = \{(n, n) : n \in \mathbb{N}\}$ as a clopen and therefore C -embedded copy of $\mathbb{N}$ and so is not pseudocompact.

Several examples similar to the preceding one have appeared in the literature. Perhaps the first two appeared in the 1952 paper of H. Terasaka and the 1953B paper of J. Novák. Novák's example was simplified by Z. Frolík in his 1959 paper. The

simplified example is described in R. Engelking's book, 1968. Terasaka's example is described in [GJ, 9.15]. The spaces in both examples are subspaces of $\beta \mathbb{N}$.

8.5. The preceding example shows that we must require more than that the factor spaces be pseudocompact in order to ensure that the product will be pseudocompact. In order to describe the proper additional conditions, we will need different characterizations of pseudocompactness than those given in Chapters 1, 4, and 5. A point is said to be a cluster point of a family of subsets of a space if every neighborhood of the point meets infinitely many subsets of the family.

PROPOSITION:

The following are equivalent for any space X :

- (1) X is pseudocompact.
- (2) Each member of $C^*(X)$ assumes its supremum and infimum.
- (3) Each sequence of non-empty open subsets of X has a cluster point.

Proof: (1) $\iff$ (3) is merely a restatement of Proposition 5.5.

(1) $\implies$ (2): If X is pseudocompact and f in $C^*(X)$ fails to assume its supremum (or infimum) r , then the mapping defined by $g(x) = 1/(f(x)-r)$ is an unbounded member of $C(X)$, which is a contradiction.

(2) $\implies$ (1): If f belongs to $C(X)$, then the mapping $g = 1/(|f| \vee 1)$ belongs to $C^*(X)$ and thus obtains its infimum.

Hence, f belongs to $C^*(X)$ and X is pseudocompact. $\square$

8.6. The next theorem is a compilation of nine equivalent conditions which have been introduced in order to investigate pseudocompact products and related topics. We will see shortly that any of the nine conditions on $X \times Y$ is both necessary and sufficient for the product space to be pseudocompact whenever both of the factors are pseudocompact. This theorem appears, together with several additional conditions, in the comprehensive 1971 paper of W. W. Comfort and A. W. Hager which forms the basis of the present discussion. Some terminology must be introduced before stating the theorem and we will also take this opportunity to mention the sources of some of the nine conditions.

A z-closed mapping is one in which the image of every zero-set is closed. Condition (1) of the theorem states that the projection $\pi_X : X \times Y \rightarrow X$ is z-closed and was apparently introduced for the first time by H. Tamano in 1960A. Tamano's result will be Theorem 8.8 below. Condition (1) has also been used in the 1968 and 1969 papers of Stephenson and in the 1967 and 1969 papers of T. Isiwata.

Let M be a metric space with metric d . For each y in M and $\epsilon > 0$, let $B(y, \epsilon) = \{z \in M : d(y, z) < \epsilon\}$. A family $\mathfrak{F}$ of mappings from X into M is said to be equicontinuous at a point x_0 of X if for every $\epsilon > 0$, there exists a neighborhood U of x_0 such that $f[U]$ is contained in $B(f(x_0), \epsilon)$ for every f in $\mathfrak{F}$. We will simply say that $\mathfrak{F}$ is equicontinuous if it is equicontinuous at each point of X .

To each real-valued mapping f on a product space $X \times Y$ we can associate several related mappings. Two such mappings are defined by

$$f_y(x) = f(x, y) \text{ for each } y \text{ in } Y,$$

and

$${}_x f(y) = f(x, y) \text{ for each } x \text{ in } X,$$

and belong to $C(X)$ and $C(Y)$, respectively. With these definitions, (6) of the theorem states that the family $\{f_y : y \in Y\}$ is equicontinuous on X for each f in $C^*(X \times Y)$.

The topology of uniform convergence on $C^*(Y)$ is the topology induced by the sup norm, i.e. the norm defined by $\|g\| = \sup\{|g(y)| : y \in Y\}$ for each g in $C^*(Y)$. Condition (7) of the theorem states that for each f in $C^*(X \times Y)$ the assignment

$$\Phi(f)(x) = {}_x f$$

is a continuous mapping of X into $C^*(Y)$, where $C^*(Y)$ has the topology of uniform convergence.

Glicksberg's proof of the product theorem used lemmas which showed that $X \times Y$ being pseudocompact implies (6) which in turn implies that $X \times Y$ is C^* -embedded in $X \times \beta Y$. This last statement is condition (3) of the theorem.

A pseudometric on a set X is a function ϕ on $X \times X$ into the non-negative reals which satisfies, for all x, y , and z in X :

- (1) $\phi(x, x) = 0,$
- (2) $\phi(x, y) = \phi(y, x),$
- (3) $\phi(x, y) \leq \phi(x, z) + \phi(z, y).$

If X is a topological space, then a pseudometric ϕ on X is said to be a continuous pseudometric if it is continuous on $X \times X$. Any pseudometric satisfies the inequality:

$$|\phi(x_1, y_1) - \phi(x, y)| \leq \phi(x_1, x) + \phi(y_1, y).$$

Using this inequality, we can show that any pseudometric which is separately continuous on $X \times X$ is continuous. If ϕ is separately continuous and $\epsilon > 0$ is given, choose neighborhoods U_1 of x_1 and U_2 of x_2 such that the mappings $y \mapsto \phi(x_1, y)$ and $y \mapsto \phi(x_2, y)$ vary by less than $\epsilon/2$ on U_1 and U_2 , respectively. Then if (y, z) belongs to the neighborhood $U_1 \times U_2$ of (x_1, x_2) , we have

$$|\phi(x_1, x_2) - \phi(y, z)| \leq \phi(x_1, y) + \phi(x_2, z) < \epsilon$$

so that ϕ is jointly continuous at (x_1, x_2) . Pseudometrics are discussed in detail in [GJ, Chapter 15].

Other conditions which can be stated in terms of uniformities can be added to the nine which are described here. These conditions are presented in the previously mentioned paper of Comfort and Hager. The interested reader should also consult the 1969B paper of N. Noble, Chapter 17 of J. R. Isbell's book, Uniform Spaces, and the 1969A paper of Hager. C - and C^* -embedding

have also been treated in terms of extensions of continuous pseudometrics in the 1969 paper of R. A. Alo and the 1970 paper of Alo and H. L. Shapiro.

Condition (5) of the theorem states that any member f of $C^*(X \times Y)$ induces a continuous pseudometric γ on X by means of the definition:

$$\gamma(x_1, x_2) \equiv \sup\{|f(x_1, y) - f(x_2, y)| : y \in Y\}.$$

Condition (4) is closely related and states that the function defined by

$$F(x) \equiv \sup\{f(x, y) : y \in Y\}$$

belongs to $C^*(X)$.

In 1960B, Z. Frolik reproved Glicksberg's Theorem and showed that if $X \times Y$ is pseudocompact, then conditions (4), (5), and (7) are satisfied. He also showed that those members of $C^*(X \times Y)$ which extend continuously to $\beta X \times \beta Y$ are characterized by a condition similar to (9). A variant of Frolik's result will be given in Theorem 8.14.

One final bit of notation is needed before stating the theorem. If f belongs to $C^*(X)$ and S is a subspace of Y , then we put

$$\text{osc}_S(f) = \sup\{|f(s) - f(t)| : s, t \in S\}.$$

With this definition, we can write that a bounded real-valued function is continuous on X if whenever x in X and $\epsilon > 0$ are given, then there is a neighborhood U of x such that

$$\text{osc}_U(f) < \epsilon.$$

We can now state the theorem.

THEOREM:

The following conditions on the product space $X \times Y$ are equivalent:

- (1) The projection map π_X from $X \times Y$ onto X is z -closed.
- (2) If Z is a zero-set in $X \times Y$, then

$$\text{cl } Z = \bigcup \{ \text{cl}(Z \cap (\{x\} \times Y)) : x \in X \}$$

where the closures are taken in $X \times \beta Y$.

- (3) $X \times Y$ is C^* -embedded in $X \times \beta Y$.
- (4) If f belongs to $C^*(X \times Y)$, then the function defined by

$$F(x) \equiv \sup \{ f(x, y) : y \in Y \}$$

is continuous on X (as is $\inf \{ f(x, y) : y \in Y \})$.

- (5) If f belongs to $C^*(X \times Y)$, then the function defined by

$$\gamma(x_1, x_2) \equiv \sup \{ |f(x_1, y) - f(x_2, y)| : y \in Y \}$$

is a continuous pseudometric on X .

- (6) If f belongs to $C^*(X \times Y)$, then $\{f_y : y \in Y\}$ is an equicontinuous family on X .
- (7) If f belongs to $C^*(X \times Y)$, then $\Phi(f)(x) = \sup_x f$ defines a continuous mapping $\Phi(f)$ from X into $C^*(Y)$.

- (8) If f belongs to $C^*(X \times Y)$, x_0 is a point of X , and $\epsilon > 0$, then there is a neighborhood U of x_0 and g in $C^*(Y)$ such that $|f(x,y) - g(y)| < \epsilon$ whenever (x,y) is in $U \times Y$.
- (9) If f belongs to $C^*(X \times Y)$ and $\epsilon > 0$, then there is an open cover $\mathcal{u}$ of X , and for each U in $\mathcal{u}$ a finite open cover $\mathcal{v}(U)$ of Y such that $\text{osc}_{U \times V}(f) < \epsilon$ whenever U is in $\mathcal{u}$ and V is in $\mathcal{v}(U)$.

Proof: (1) $\Rightarrow$ (2): Denote the set $Z \cap (\{x\} \times Y)$ by Z_x . If (2) fails, then for some (x_0, p) in $X \times \beta Y$,

$$(x_0, p) \in \text{cl } Z \setminus \bigcup \{\text{cl } Z_x : x \in X\},$$

and in particular, (x_0, p) does not belong to $\text{cl } Z_{x_0}$. We will now obtain a contradiction of (1) by constructing a zero-set Z_1 of $X \times Y$ such that (x_0, p) is in $\text{cl } Z_1$ but $\pi_X[Z_1]$ misses x_0 . There is a real-valued mapping f on $X \times \beta Y$ such that f is identically equal to 1 on $\text{cl } Z_{x_0}$ and is identically equal to 0 on some neighborhood of (x_0, p) . Now let $Z_1 = Z \cap Z(f)$. The zero-set Z_1 contains (x_0, p) in its closure in $X \times \beta Y$, but $\pi_X[Z_1]$ misses x_0 . Applying (1) gives a contradiction since

$$x_0 \in \pi_X[\text{cl } Z_1] \subset \text{cl}_X[Z_1] = \pi_X[Z_1].$$

(2) $\Rightarrow$ (3): From Theorem 1.46 it follows that a dense subspace S of a space T is C^* -embedded in T if and only if disjoint zero-sets of S have disjoint closures in T . We

will use (2) to show that this condition is satisfied with $S = X \times Y$ and $T = X \times \beta Y$. If Z_1 and Z_2 are zero-sets of $X \times Y$, then

$$\begin{aligned} cl Z_1 \cap cl Z_2 &= [U\{cl(Z_1)_x : x \in X\}] \cap [U\{cl(Z_2)_x : x \in X\}] \\ &= U\{cl(Z_1)_x \cap cl(Z_2)_x : x \in X\}. \end{aligned}$$

Each $(Z_i)_x$ can be viewed as a zero-set of Y so that if Z_1 and Z_2 are disjoint, (5) of Theorem 1.46 shows that $cl(Z_1)_x$ and $cl(Z_2)_x$ are disjoint for each x . Hence, Z_1 and Z_2 have disjoint closures in $X \times \beta Y$.

(3) $\Rightarrow$ (4): Given f in $C^*(X \times Y)$, let F be defined as in (4) and let g be the continuous extension of f to $X \times \beta Y$. To show that F is continuous at a point x_0 of X , let $\epsilon > 0$ be given and for each point p of βY find a basic neighborhood $U_p \times V_p$ of (x_0, p) on which g varies less than ϵ . The cover $\{V_q\}$ of βY admits a finite subcover $\{V_{q_i} : 1 \leq i \leq n\}$. Then F varies by less than ϵ on the neighborhood $\cap\{V_{q_i} : 1 \leq i \leq n\}$.

(4) $\Rightarrow$ (5): Because γ is a pseudometric, joint continuity of γ will follow from separate continuity. To show that γ is separately continuous, fix a point x_2 in X and note that the function defined by

$$(x_1, y) \mapsto |f(x_1, y) - f(x_2, y)|$$

is continuous on $X \times Y$. Then the associated sup function defined as in (4) is continuous on X and its value at x_1 is just $\gamma(x_1, x_2)$.

(5) $\Rightarrow$ (6): To show that the family $\{f_y : y \in Y\}$ is equicontinuous at a point x_0 of X , let $\epsilon > 0$ be given and use the continuity of the pseudometric γ of (5) to choose a neighborhood U of x_0 such that

$$\sup\{|f(x,y) - f(x_0,y)| : y \in Y\} < \epsilon$$

whenever x is in U . Then U is the required neighborhood of x_0 .

(6) $\Leftrightarrow$ (7): This equivalence is clear because both conditions may be stated as follows: For each point x_0 of X and each $\epsilon > 0$, there is a neighborhood U of x_0 such that

$$|f(x,y) - f(x_0,y)| < \epsilon$$

for all y in Y and all x in U .

(6) $\Rightarrow$ (8): Given f in $C^*(X \times Y)$, x_0 in X , and $\epsilon > 0$, use the equicontinuity of the family $\{f_y\}$ to obtain a neighborhood U of x_0 such that

$$|_Y f(x) - _Y f(x_0)| < \epsilon$$

for all y in Y and all x in U . Then the required member g of $C^*(Y)$ is obtained by defining $g(y) = f(x_0, y)$.

(8) $\Rightarrow$ (9): If f in $C^*(X \times Y)$ and $\epsilon > 0$ are given, then for every x_0 in X we must find a neighborhood U of x_0 and a finite open cover $\mathcal{V}$ of Y such that $\text{osc}_{U \times V}(f) < \epsilon$ whenever V is an element of $\mathcal{V}$. By (8), there exists g in $C^*(Y)$ and a neighborhood U of x_0 such that

$$|f(x,y) - g(y)| < \frac{\epsilon}{3}$$

whenever x is in U and y is in Y . Because g is bounded, there is a finite open cover $\mathcal{U}$ of Y such that $\text{osc}_V(g) < \frac{\epsilon}{3}$ for each V in $\mathcal{U}$ and an application of the triangle inequality shows that $\mathcal{U}$ is the required covering.

(9) $\Rightarrow$ (1): Let $Z = Z(g)$ be a zero-set of $X \times Y$ and suppose that x_0 is not in $\pi_X[Z]$. We will produce a neighborhood of x_0 which misses $\pi_X[Z]$. Since $\{x_0\} \times Y$ is closed, we have that the restriction $g|_{\{x_0\} \times Y}$ is bounded away from zero. Hence, putting

$$f(x,y) = |g(x,y)/g(x_0,y)| \wedge 1$$

yields a member of f of $C^*(X \times Y)$ which has $Z(g)$ as its zero-set and which is identically equal to 1 on $\{x_0\} \times Y$. Choose $\epsilon = \frac{1}{2}$ and apply (9) to obtain a neighborhood U of x_0 such that $f(x,y) > \frac{1}{2}$ for all (x,y) in $U \times Y$. Hence, U is a neighborhood of x_0 missing $\pi_X[Z]$, and therefore $\pi_X[Z]$ is closed in X . $\blacksquare$

8.7. Of the eight conditions in Theorem 8.6 which are equivalent to $X \times Y$ being C^* -embedded in $X \times \beta Y$, condition (1) is one of the easiest to verify. Observe that if both projections π_X and π_Y on $X \times Y$ are z -closed, then $X \times Y$ is C^* -embedded in both $\beta X \times Y$ and $X \times \beta Y$. In their 1966 paper, W. W. Comfort and S. Negrepointis call such a pair of spaces X and Y a C^* -pair. If in addition, $X \times Y$ is not C^* -embedded in

$\beta X \times \beta Y$, X and Y are called a proper C^* -pair. They presented the following example of a proper C^* -pair.

EXAMPLE:

Two copies of $\mathbb{N}$ constitute a proper C^* -pair. It is easy to see that $\mathbb{N} \times \mathbb{N}$ is C^* -embedded in both $\beta \mathbb{N} \times \mathbb{N}$ and $\mathbb{N} \times \beta \mathbb{N}$ since both projections must be z -closed. However, the Kronecker delta function defined by

$$\delta(n,m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

will not extend to $\beta \mathbb{N} \times \beta \mathbb{N}$ since any neighborhood of a point (p,p) in $\mathbb{N}^* \times \mathbb{N}^*$ must meet infinitely many elements which lie on the diagonal of $\mathbb{N} \times \mathbb{N}$ as well as infinitely many off diagonal elements.

8.8. The effort devoted to Theorem 8.6 now begins to pay dividends in the proof of Tamano's 1960A characterization of pseudocompact products. The proof given here is a Comfort-Hager modification of a Glicksberg-Frolík argument for necessity and of a 1968 Stephenson argument for sufficiency.

THEOREM: (Tamano)

The product $X \times Y$ is pseudocompact if and only if both X and Y are pseudocompact and the projection map π_X of $X \times Y$ onto X is z -closed.

Proof: Necessity: X and Y are pseudocompact since both are continuous images of $X \times Y$. If π_X fails to be z -closed,

then there is a zero-set $Z = Z(f)$ in $X \times Y$ and a point x_0 of X belonging to $\text{cl } \pi_X[Z] \setminus \pi_X[Z]$. As in the proof of (9) $\Rightarrow$ (1) in Theorem 8.6, we can assume that $f(x_0, y) = 1$ for all y in Y . The proof will be completed by constructing a sequence of open sets of $X \times Y$ which will cluster at a point at which f will fail to be continuous, yielding a contradiction. Observe that for any y in Y , every neighborhood of the point (x_0, y) meets the open set $\{(x, y) \in X \times Y : f(x, y) < 1/3\}$. Thus, it is possible to choose a sequence $\{(x_n, y_n)\}$ of points of Z , and neighborhoods $\{(U_n \times V_n)\}$ and $\{(W_n \times V_n)\}$ of (x_n, y_n) and (x_0, y_n) , respectively, such that:

$$U_n \times V_n \subset \{(x, y) : f(x, y) < 1/3\}$$

$$W_n \times V_n \subset \{(x, y) : f(x, y) > 2/3\}$$

$$W_{n+1} \cup U_{n+1} \subset U_n.$$

Since $X \times Y$ is pseudocompact, the sequence $\{U_n \times V_n\}$ clusters at a point (w, z) and $f(w, z) \leq 1/3$. However, by construction, every neighborhood of (w, z) meets $W_n \times V_n$ so that $f(w, z) \geq 2/3$, which is a contradiction.

Sufficiency: To show that $X \times Y$ is pseudocompact, it suffices to show that for f in $C^*(X \times Y)$ such that f is everywhere positive, then

$$\inf\{f(x, y) : (x, y) \in X \times Y\} > 0.$$

Set $F(x) = \inf\{f(x, y) : y \in Y\}$. By assuming (4) of Theorem 8.6, F belongs to $C^*(X)$. Since X is pseudocompact, Proposition

8.5 implies that

$$\inf\{f(x,y) : (x,y) \in X \times Y\} = \inf\{F(x) : x \in X\} > 0. \quad \blacksquare$$

Observe that in the light of Theorem 8.6, Tamano's Theorem is actually nine characterizations rolled into one, a fact which will frequently be useful.

8.9. As Example 8.7 showed, a projection on a product space may be z -closed without either factor being pseudocompact. However, in such a case the image of the projection must be a P -space. The following result has appeared in a number of contexts. The present proof appears in Comfort and Hager's paper and is credited there to N. J. Fine.

LEMMA:

If the projection π_X of $X \times Y$ onto X is z -closed, then either X is a P -space or Y is pseudocompact.

Proof: If X fails to be a P -space, then there exists h in $C^*(X)$ and a point x_0 in $Z(h)$ such that h is not identically equal to 0 on any neighborhood of x_0 . If Y fails to be pseudocompact, then by Proposition 8.5 there is an everywhere positive map g in $C^*(Y)$ such that $\inf\{g(y) : y \in Y\} = 0$. Define f in $C^*(X \times Y)$ by $f(x,y) = g(y)^{h(x)}$. Theorem 8.6 demands that $F(x) = \inf\{f(x,y) : y \in Y\}$ define a continuous function on X . Since $h(x_0) = 0$, we have $F(x_0) = 1$. However, because g is not bounded away from zero, every neighborhood of x_0 contains a point x such that $F(x) < \frac{1}{2}$. Hence, (4)

of Theorem 8.6 fails, and π_X thus cannot be z -closed. |

8.10. The lemma has a number of corollaries. The first is a modification of Tamano's Theorem which is obtained by requiring that one of the factor spaces be infinite. An infinite pseudocompact space cannot be a P -space by Proposition 1.65. Thus, if the projection map π_X of $X \times Y$ onto X is z -closed, the other factor Y must be pseudocompact. When combined with Theorem 8.8, this verifies the

COROLLARY:

If X is infinite, then the product space $X \times Y$ is pseudocompact if and only if X is pseudocompact and the projection map π_X of $X \times Y$ onto X is z -closed.

8.11. From Theorem 8.6, we obtain

COROLLARY:

If $X \times Y$ is C^* -embedded in $X \times \beta Y$, then either Y is pseudocompact or X is a P -space.

Other corollaries will follow Glicksberg's Theorem or be contained in Exercises.

8.12. The following proof of Glicksberg's Theorem is taken from Comfort and Hager's 1971 paper.

THEOREM: (Glicksberg)

If X and Y are infinite, then the product space $X \times Y$ is pseudocompact if and only if $X \times Y$ is C^* -embedded in

$\beta X \times \beta Y$, i.e. $\beta(X \times Y) = \beta X \times \beta Y$.

Proof: Necessity: Because $X \times Y$ is pseudocompact, Theorem 8.8 implies that the projection π_X is z -closed. Theorem 8.6 then requires that $X \times Y$ be C^* -embedded in $X \times \beta Y$. Since the pseudocompact space $X \times Y$ is dense in $X \times \beta Y$, the space $X \times \beta Y$ is also pseudocompact. Hence, with βY playing the role of X and X playing the role of Y , Theorem 8.6 implies that $X \times \beta Y$ is C^* -embedded in $\beta X \times \beta Y$. Therefore, $X \times Y$ is C^* -embedded in $\beta X \times \beta Y$.

Sufficiency: $\beta(X \times Y) = \beta X \times \beta Y$ implies that $X \times Y$ is C^* -embedded in both $\beta X \times Y$ and $X \times \beta Y$. Theorem 8.6 then implies that both projections π_X and π_Y are z -closed. Hence, applying Lemma 8.9, we see that either X is pseudocompact or βX is a P -space. But neither of the infinite compact spaces βX and βY are P -spaces, so that both X and Y are pseudocompact. Then Theorem 8.8 implies that the product $X \times Y$ is pseudocompact. $\blacksquare$

Partial results along the lines of the preceding theorem had been obtained prior to Glicksberg's solution. In 1952, T. Isiwata showed that if $\beta(X \times X)$ is $\beta X \times \beta X$, then X is totally bounded in any uniform structure on X , which is equivalent to X being pseudocompact [GJ, ex. 15Q]. In 1957B, M. Henrikson and J. R. Isbell showed that if $\beta(X \times Y) = \beta X \times \beta Y$, then $X \times Y$ is pseudocompact and they also obtained partial results in the converse direction.

8.13. If X and Y form a proper C^* -pair (Section 8.7), then both projections must be z -closed. If either space were pseudocompact, then Glicksberg's Theorem and Corollary 8.10 would imply that the pair is not proper. Combining these facts with Corollary 8.11 verifies the

COROLLARY:

If X and Y are a proper C^* -pair, then both X and Y are P -spaces.

8.14. In his 1960B paper, Z. Frolík offered another proof of Glicksberg's Theorem and introduced a condition similar to (9) of Theorem 8.6. A mapping f in $C^*(X \times Y)$ is said to satisfy the rectangle condition if given $\epsilon > 0$, there is a finite covering $\{U_i \times V_i\}$ of $X \times Y$ by open rectangles such that for each i , $\text{osc}_{U_i \times V_i}(f) < \epsilon$. Frolík showed that if every f in $C^*(X \times Y)$ satisfies the rectangle condition, then $X \times Y$ is pseudocompact, and conversely. In his 1966 paper, A. W. Hager showed that the rectangle condition characterizes those maps in $C^*(X \times Y)$ which will extend continuously to $\beta X \times \beta Y$. Here we prove a version of Hager's result which requires the rectangles to be products of cozero-sets.

THEOREM:

A member f of $C^*(X \times Y)$ admits a continuous extension to $\beta X \times \beta Y$ if and only if for any $\epsilon > 0$, there exists a finite open cover $\{U_i \times V_i : 1 \leq i \leq n\}$ of $X \times Y$ where each U_i and V_i are cozero-sets and such that $\text{osc}_{U_i \times V_i}(f) < \epsilon$ for each i .

Proof: Sufficiency: The equivalence of (3) and (9) of Theorem 8.6 shows that f has a continuous extension g to $X \times \beta Y$. Following [GJ, 16.10], we now extend the cozero-set cover $\{V_i\}$ of Y to an open cover of βY . Put

$$V_i^\beta = \beta Y \setminus \text{cl}_{\beta Y}(Y \setminus V_i).$$

Since $\{V_i\}$ is a cover of Y , $\cap(Y \setminus V_i) = \emptyset$. Because each $Y \setminus V_i$ is a zero-set, (5) of Theorem 1.46 implies that

$$\beta Y \setminus \bigcup V_i^\beta = \cap \text{cl}_{\beta Y}(Y \setminus V_i) = \emptyset$$

so that $\{V_i^\beta\}$ covers βY . Now each point of $U_i \times V_i^\beta$ is a limit point of $U_i \times V_i$. Hence,

$$\text{osc}_{U_i \times V_i^\beta}(g) = \text{osc}_{U_i \times V_i}(f) < \epsilon.$$

Hence, applying the equivalence of (3) and (9) of Theorem 8.6 again yields an extension h of g to $\beta X \times \beta Y$.

Necessity: If h is the extension of f to $\beta X \times \beta Y$, then the compactness of $\beta X \times \beta Y$ implies that there is a finite covering $\{U_i \times V_i\}$ of $\beta X \times \beta Y$ consisting of products of cozero-sets which satisfies the required oscillation condition. Intersecting these sets with $X \times Y$ yields the result. $\blacksquare$

8.15. COROLLARY:

The product space $X \times Y$ is pseudocompact if and only if for every f in $C^*(X \times Y)$ and $\epsilon > 0$, there exists a finite open cover $\{U_i \times V_i : 1 \leq i \leq n\}$ consisting of products of cozero-sets such that $\text{osc}_{U_i \times V_i}(f) < \epsilon$ for each i .

8.16. In addition to characterizing the members of $C^*(X \times Y)$ which are continuously extendable to $\beta X \times \beta Y$ in terms of the rectangle condition, in his 1966 paper Hager also characterized this set of mappings in terms of tensor products of function rings. The tensor product of the rings $C^*(X)$ and $C^*(Y)$ is denoted by $C^*(X) \otimes C^*(Y)$ and consists of all members of $C^*(X \times Y)$ of the form

$$h(x,y) = \sum f_i(x)g_i(y)$$

where the sum is finite and each f_i and g_i belongs to $C^*(X)$ and $C^*(Y)$, respectively. In the following theorem, uniform closure refers to closure in the topology on $C^*(X \times Y)$ induced by the sup norm.

THEOREM: (Hager)

The set of mappings contained in $C^*(X \times Y)$ which extend continuously to $\beta X \times \beta Y$ coincides with the uniform closure of $C^*(X) \otimes C^*(Y)$ in $C^*(X \times Y)$.

Proof: Let $h = \sum f_i \cdot g_i$ be in $C^*(X) \otimes C^*(Y)$ where f_i and g_i belong to $C^*(X)$ and $C^*(Y)$, respectively. Then each f_i and g_i extend to the appropriate compactification βX or βY and $\iota = \sum \beta(f_i) \cdot \beta(g_i)$ is the appropriate extension of h . Now let the sequence $\{h_n\}$ contained in $C^*(X) \otimes C^*(Y)$ converge uniformly to b in $C^*(X \times Y)$. Then the corresponding extensions $\{\iota_n\}$ form a Cauchy sequence in $C^*(\beta X \times \beta Y)$ since $X \times Y$ is dense in $\beta X \times \beta Y$. The limit of the sequence $\{\iota_n\}$ is the appropriate extension of b to $\beta X \times \beta Y$.

Conversely, the tensor product $C(\beta X) \otimes C(\beta Y)$ is easily seen to separate points of $\beta X \times \beta Y$ and to contain constant mappings. Therefore, by the Stone-Weierstrass Theorem, [GJ, 16.4] or [D, p. 282], the subring $C(\beta X) \otimes C(\beta Y)$ is dense in $C(\beta X \times \beta Y)$ so that every element f in $C(\beta X \times \beta Y)$ is the uniform limit of a sequence $\{f_n\}$ contained in $C(\beta X) \otimes C(\beta Y)$. Hence, $f|_{X \times Y}$ is the uniform limit of $\{f_n|_{X \times Y}\}$ and $f_n|_{X \times Y}$ belongs to $C^*(X) \otimes C^*(Y)$. $\parallel$

8.17. In addition to the characterization of pseudocompact products given in Theorem 8.8, in 1960A Tamano also gave a characterization in terms of the tensor product $C^*(X) \otimes C^*(Y)$. His result can now be stated as a corollary of the theorems of Glicksberg and Hager.

COROLLARY: (Tamano)

The product space $X \times Y$ is pseudocompact if and only if $C^*(X) \otimes C^*(Y)$ is dense in $C^*(X \times Y)$.

8.18. Observe that in the statement of Glicksberg's Theorem, we first asserted that $X \times Y$ is C^* -embedded in $\beta X \times \beta Y$ and then wrote that $\beta(X \times Y)$ is equal to $\beta X \times \beta Y$. It is important to clarify what is intended by equality in this setting. We mean equality in the partially ordered set of compactifications as discussed in Section 1.12. Thus, we mean that there is a homeomorphism of $\beta X \times \beta Y$ with $\beta(X \times Y)$ under which $X \times Y$ is pointwise fixed. The following example shows that it is not enough to merely ask that $\beta X \times \beta Y$ be homeomorphic with $\beta(X \times Y)$.

The example is taken from the 1959 paper of Gillman and Jerison.

EXAMPLE:

Let $X \oplus Y$ denote the disjoint topological sum of two spaces X and Y and let $\underline{3}$ denote the discrete space having 3 points. Let N_1 and N_2 denote the odd and even integers, respectively, and put $E = N_1 \oplus \beta N_2$. Then $\beta E = \beta \mathbb{N}$. We will show that $\beta(E \times E)$ and $\beta E \times \beta E$ are homeomorphic. However, since E is not pseudocompact, it is clear from Glicksberg's Theorem that they are not equal. The statement that X is homeomorphic with Y will be written $X \approx Y$.

(a) $\beta(\mathbb{N} \times \beta \mathbb{N}) \approx \beta \mathbb{N}$: $\mathbb{N} \times \mathbb{N}$ is dense and C^* -embedded in $\mathbb{N} \times \beta \mathbb{N}$ and therefore in $\beta(\mathbb{N} \times \beta \mathbb{N})$. Hence, $\beta(\mathbb{N} \times \mathbb{N})$ is equal to $\beta(\mathbb{N} \times \beta \mathbb{N})$. Since $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$, we have that $\beta(\mathbb{N} \times \beta \mathbb{N}) \approx \beta(\mathbb{N} \times \mathbb{N}) \approx \beta \mathbb{N}$.

(b) $\beta(E \times E) \approx \beta E \times \beta E$: Since $E \approx \mathbb{N} \oplus \beta \mathbb{N}$, we have:

$$\begin{aligned} E \times E &\approx (\beta \mathbb{N} \oplus \mathbb{N}) \times (\beta \mathbb{N} \oplus \mathbb{N}) \\ &\approx \beta \mathbb{N} \times \beta \mathbb{N} \oplus \mathbb{N} \times \beta \mathbb{N} \oplus \beta \mathbb{N} \times \mathbb{N} \oplus \mathbb{N} \times \mathbb{N} \\ &\approx \beta \mathbb{N} \times \beta \mathbb{N} \oplus \mathbb{N} \times \beta \mathbb{N} \oplus \mathbb{N} \times \beta \mathbb{N} \oplus \mathbb{N}. \end{aligned}$$

Since β will distribute over finite disjoint sums, this gives:

$$\beta(E \times E) \approx \beta(\beta \mathbb{N} \times \beta \mathbb{N}) \oplus \beta(\mathbb{N} \times \beta \mathbb{N}) \oplus \beta(\mathbb{N} \times \beta \mathbb{N}) \oplus \beta \mathbb{N}.$$

By (a), this now becomes:

$$\begin{aligned}
\beta(E \times E) &\approx \beta \mathbb{N} \times \beta \mathbb{N} \oplus \beta \mathbb{N} \oplus \beta \mathbb{N} \oplus \beta \mathbb{N} \\
&= \beta \mathbb{N} \times (\beta \mathbb{N} \oplus \underline{3}) \\
&\approx \beta \mathbb{N} \times \beta \mathbb{N} \\
&\approx \beta E \times \beta E.
\end{aligned}$$

Hence, $\beta(E \times E)$ and $\beta E \times \beta E$ are homeomorphic.

8.19. In the paper containing the previous example, Gillman and Jerison showed that if both X and Y are first countable, then it is possible to replace equal by homeomorphic in Glicksberg's Theorem. The key to the proof is that no point of X^* can have a countable neighborhood base in βX for any X . This fact follows from Corollary 3.8.

PROPOSITION:

If X and Y are both first countable spaces, then $\beta(X \times Y)$ and $\beta X \times \beta Y$ are homeomorphic if and only if they are equal.

Proof: We need only prove necessity. Let h be a homeomorphism from $\beta(X \times Y)$ onto $\beta X \times \beta Y$. Since each point of $X \times Y$ has a countable neighborhood base in $\beta(X \times Y)$, each point of $h[X \times Y]$ will have a countable neighborhood base in $\beta X \times \beta Y$. Therefore, h maps the subspace $X \times Y$ of $\beta(X \times Y)$ onto the subspace $X \times Y$ of $\beta X \times \beta Y$. Since $X \times Y$ is C^* -embedded in $\beta(X \times Y)$, its homeomorphic image $X \times Y$ is C^* -embedded in $\beta X \times \beta Y$. Thus, $\beta(X \times Y)$ and $\beta X \times \beta Y$ are equal. $\blacksquare$

8.20. Combining the previous result with Glicksberg's Theorem gives an immediate corollary.

COROLLARY:

If X and Y are infinite first countable spaces, then $\beta(X \times Y)$ and $\beta X \times \beta Y$ are homeomorphic if and only if $X \times Y$ is pseudocompact.

8.21. Glicksberg's Theorem ignited an interest in determining sufficient conditions for the product of two spaces to be pseudocompact. The following proposition is one of the simpler and most useful of the results to be obtained.

PROPOSITION:

The product of two pseudocompact spaces one of which is also locally compact is pseudocompact.

Proof: Let X and Y be pseudocompact spaces and let Y be locally compact. For a point y of Y , choose a compact neighborhood V of y . Then the projection $\pi_X : X \times V \rightarrow X$ is closed since V is compact [D, p. 227] and hence, π_X is z -closed. Therefore, Theorem 8.8 shows that $X \times V$ is pseudocompact. From Glicksberg's Theorem and (7) of Theorem 8.6, we have that if g belongs to $C^*(X \times Y)$, then the assignment

$$\Phi(g)(y) = g_y$$

defines a continuous mapping from V into $C^*(X)$. But it follows from the fact that Y is locally compact that $\Phi(g)$ is continuous on all of Y since we have shown that it is

continuous on a neighborhood of each point. Now the equivalence of (1) and (7) of Theorem 8.6 shows that the projection

$$\pi_X : X \times Y \rightarrow X$$

is z -closed and $X \times Y$ is pseudocompact by Theorem 8.8. $\blacksquare$

Frolík, 1960B, and Glicksberg, 1959, both consider sufficient conditions for a product $X \times Y$ to be pseudocompact as do Stephenson, 1968, and Tamano, 1960A. In addition, Frolík obtained a characterization for the class of spaces such that the product of any space in the class with any pseudocompact space is pseudocompact. In 1960A, Frolík considers the analogous class of spaces for countably compact products. The classes of spaces introduced by Frolík in 1960A and 1960B are also discussed in Isiwata's 1964 paper and Noble's 1969C paper.

8.22. Recall that a space X is said to be a retractive space if X^* is a retract of βX (Section 6.5). In Theorem 6.7, we saw, modulo the Continuum Hypothesis, that any retractive space is pseudocompact. The preceding result together with Glicksberg's Theorem show that the class of retractive spaces is closed under products with compact spaces.

COROLLARY [CH]:

The product of a retractive space and a compact space is retractive.

Proof: Let X be retractive and Y be compact. Then the product $X \times Y$ is pseudocompact and we have that

$\beta(X \times Y) = \beta X \times \beta Y = \beta X \times Y$. Then if r is a retraction of βX onto X^* , the required retraction of $\beta(X \times Y)$ onto $(X \times Y)^* = X^* \times Y$ is given by $(p, q) \mapsto (r(p), q)$. $\square$

8.23. However, the product of two retractive spaces need not be retractive. The following example appears in Comfort's 1965 paper and is based on an argument used in Exercise 4E of Kelley, 1955. Since the example will be both locally compact and pseudo-compact, it also shows that the converse to Corollary 6.9 is false.

EXAMPLE:

The space ω_1 is surely retractive since its growth in $\beta\omega_1 = \omega_1 + 1$ is a single point. Glicksberg's Theorem together with the fact that ω_1 is locally compact show that $\beta(\omega_1 \times \omega_1) = (\omega_1 + 1) \times (\omega_1 + 1)$. Denote the sets $\{\omega_1\} \times \omega_1$ and $\omega_1 \times \{\omega_1\}$ by E and F , respectively, so that the growth of $\omega_1 \times \omega_1$ is $E \cup F \cup \{(\omega_1, \omega_1)\}$.

Now assume that r is a retraction of $\beta(\omega_1 \times \omega_1)$ onto $(\omega_1 \times \omega_1)^*$. Choose $x_1 < \omega_1$. Then the point $(\omega_1, x_1) = r(\omega_1, x_1)$ does not belong to the closed set $F \cup \{(\omega_1, \omega_1)\}$. Thus, the image of some neighborhood of (ω_1, x_1) misses $F \cup \{(\omega_1, \omega_1)\}$ and we can therefore find x_2 such that $x_1 < x_2 < \omega_1$ and $r(x_2, x_1)$ belongs to E . Since $r(x_2, \omega_1) = (x_2, \omega_1)$ misses $E \cup \{(\omega_1, \omega_1)\}$, we can find x_3 such that $x_2 < x_3 < \omega_1$ and $r(x_2, x_3)$ belongs to F .

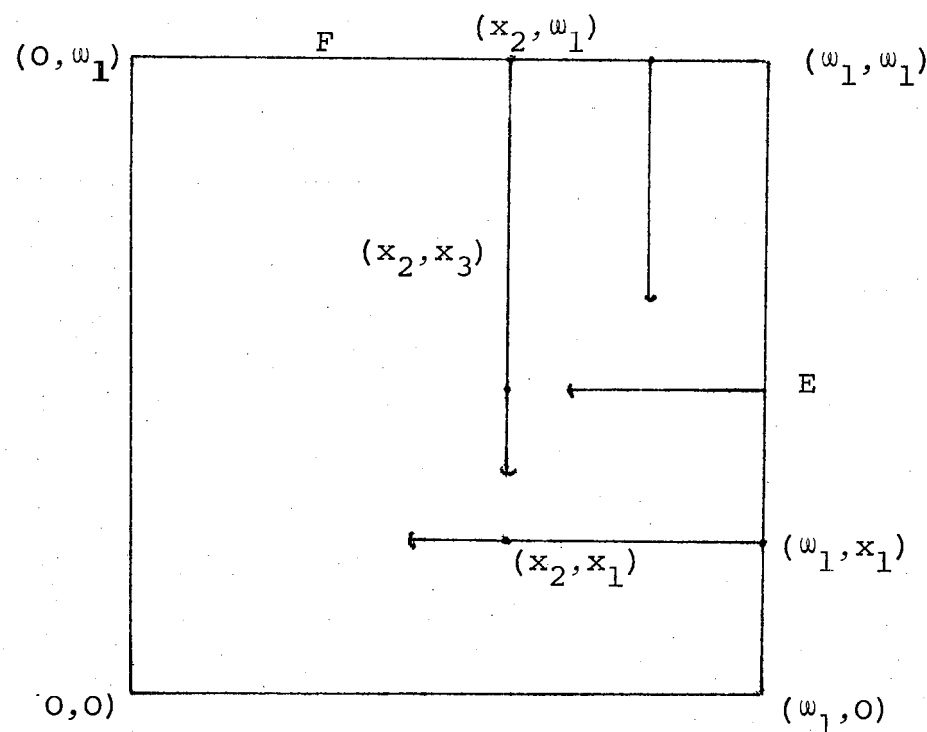


Figure 8.1

By continuing this process, if we assume that for all $n < 2k$ we have chosen x_n , we can choose x_{2k} such that $x_{2k-1} < x_{2k} < \omega_1$ and $r(x_{2k}, x_{2k-1})$ belongs to E . Then we can choose x_{2k+1} such that $x_{2k} < x_{2k+1} < \omega_1$ and $r(x_{2k}, x_{2k+1})$ belongs to F . By putting $\sigma = \sup\{x_n\}$, we see that

$$\lim(x_{2k}, x_{2k-1}) = \lim(x_{2k}, x_{2k+1}) = (\sigma, \sigma).$$

By continuity, we have that $r(\sigma, \sigma)$ belongs to $\text{cl } E \cap \text{cl } F$ and therefore that $r(\sigma, \sigma) = (\omega_1, \omega_1)$. Thus the countable sequence $r(x_{2k}, x_{2k-1})$ must have the limit (ω_1, ω_1) . However, no countable sequence is cofinal in E because E is a copy of ω_1 [GJ, 5.12]. Thus, the assumption that r is a retraction leads to a contradiction.

THE PRODUCT THEOREM FOR INFINITE PRODUCTS

8.24. Glicksberg's Theorem for finite pseudocompact products can be extended to a pseudocompact product involving an arbitrarily large index set. The extension will be accomplished by applying the theorem for finite products to a finite partial product. The following result is an analogue of (8) in Theorem 8.6 and will be used to single out an appropriate finite partial product. The proof of both the lemma and the necessity portion of the theorem are as in Glicksberg's 1959 paper.

LEMMA:

If the product space $\times \{X_\alpha : \alpha \in A\}$ is pseudocompact, f is in $C(\times X_\alpha)$, and $\epsilon > 0$, then there exists a finite set F of indices such that whenever the coordinates of two points x and y agree for each element of F , then $|f(x) - f(y)| < \epsilon$.

Proof: Suppose that no such finite set exists. Then if F_0 is any finite set of indices, there are points x_1 and y_1 whose coordinates agree for the indices of F_0 and such that $|f(x_1) - f(y_1)| > \epsilon$. Choose disjoint basic neighborhoods of x_1 and y_1 which are contained in $f^{-1}(f(x_1) - \epsilon/4, f(x_1) + \epsilon/4)$ and $f^{-1}(f(y_1) - \epsilon/4, f(y_1) + \epsilon/4)$, respectively. Since basic neighborhoods in a product space restrict only finitely many coordinates, we can assume that x_1 and y_1 differ in only finitely many coordinates, say those belonging to F_1 , by asking only that $|f(x_1) - f(y_1)| > \epsilon/2$.

Now by considering the finite set $F_0 \cup F_1$, we can in the same way obtain two points x_2 and y_2 such that their coordinates

agree in $F_0 \cup F_1$, differ in only a finite set F_2 , and such that $|f(x_2) - f(y_2)| > \epsilon/2$. Continue this process to obtain sequences $\{x_i\}$, $\{y_i\}$, and $\{F_i\}$ such that $|f(x_i) - f(y_i)| > \epsilon/2$, x_i and y_i differ only in those coordinates belonging to F_i , and the F_i are pairwise disjoint finite sets of indices.

By continuity, we can obtain basic open neighborhoods U_i and V_i of x_i and y_i , respectively, such that $|f(x) - f(y)| > \epsilon/4$ whenever x belongs to U_i and y belongs to V_i , and such that U_i and V_i restrict each factor space X_α equally except for those whose index belongs to F_i . Since the product is pseudo-compact, $\{U_i\}$ has a cluster point x_0 (Proposition 8.5). If W is any basic neighborhood of x_0 , then the set E of coordinates which are restricted by W is finite, and so for some $n \geq 1$,

$$E \cap (\cup F_i) \subseteq \cup \{F_i : 1 \leq i \leq n\}.$$

Thus, W meets both U_i and V_i for all $i > n$ so that W contains points x and y such that $|f(x) - f(y)| > \epsilon/4$, which contradicts the continuity of f . $\blacksquare$

8.25. In the proof of Glicksberg's Theorem, we will use the lemma to write $\times X_\alpha = X \times Y$ where Y is a partial product over a finite index set. We can then apply the theorem for the finite case to Y . The proof is then completed by using the equivalence of (1), (3), (6), and (7) in Theorem 8.6.

THEOREM: (Glicksberg)

If $\{X_\alpha : \alpha \in A\}$ is any family of spaces such that
 $\times \{X_\alpha : \alpha \in A, \alpha \neq \alpha_0\}$ is infinite for each α_0 , then $\times X_\alpha$ is

pseudocompact if and only if $\times X_\alpha$ is C^* -embedded in $\times (\beta X_\alpha)$,
i.e. $\beta(\times X_\alpha) = \times (\beta X_\alpha)$.

Proof: Necessity: The proof will be by contradiction. We will show that any member of $C^*(\times X_\alpha)$ which fails to extend continuously to $\times (\beta X_\alpha)$ cannot itself be continuous. If f in $C^*(\times X_\alpha)$ does not admit a continuous extension to $\times (\beta X_\alpha)$, there exists a net $\{x_\delta\}$ in $\times X_\alpha$ converging to a point p in $\times (\beta X_\alpha)$ such that

$$\limsup\{f(x_\delta)\} - \liminf\{f(x_\delta)\} = a > 0.$$

Let $0 < 3\epsilon < a$ and let $\{\alpha_1, \dots, \alpha_n\}$ be the finite set of indices obtained for this f by the lemma. Put

$$X = \times \{X_\alpha : \alpha \neq \alpha_i, 1 \leq i \leq n\} \text{ and } Y = \times \{X_{\alpha_i} : 1 \leq i \leq n\}.$$

Theorem 8.8 implies that the projection $\pi_X : X \times Y \rightarrow X$ is z -closed. Therefore, (3) of Theorem 8.6 shows that f extends to a mapping g in $C^*(X \times \beta Y)$. The Glicksberg Theorem for finite products, (8.12), shows that

$$X \times \beta Y = X \times \beta(\times X_{\alpha_i}) = X \times (\times (\beta X_{\alpha_i})).$$

The product $X \times \beta Y$ is also pseudocompact and we can again apply Theorem 8.6, this time with βY in the role of X and X in the role of Y . By (6) of Theorem 8.6, the family $\{g_x : x \in X\}$ is an equicontinuous family on βY . By (7) of Theorem 8.6, if h belongs to $C^*(X \times \beta Y)$ and p is a point of βY , then the assignment $\Phi(h)(p) = h_p$ defines a mapping of βY into $C^*(X)$. Thus, there exists a neighborhood U of the point $\{p_{\alpha_1}, \dots, p_{\alpha_n}\}$ in βY such that both $\text{osc}_U(g) < \epsilon$ and $\text{osc}_{X \times U}(g) < \epsilon$.

Further, the neighborhood $X \times U$ of p contains points $(x_{\delta_1}, y_{\delta_1})$ and $(x_{\delta_2}, y_{\delta_2})$ of the net such that

$$\begin{aligned} a - \epsilon &< |f(x_{\delta_1}, y_{\delta_1}) - f(x_{\delta_2}, y_{\delta_2})| \\ &\leq |f(x_{\delta_1}, y_{\delta_1}) - f(x_{\delta_1}, y_{\delta_2})| + |f(x_{\delta_1}, y_{\delta_2}) - f(x_{\delta_2}, y_{\delta_2})| \\ &= |x_{\delta_1} g(y_{\delta_1}) - x_{\delta_1} g(y_{\delta_2})| + |g(x_{\delta_1}, y_{\delta_2}) - g(x_{\delta_2}, y_{\delta_2})| \\ &< 2\epsilon. \end{aligned}$$

Hence, we obtain $3\epsilon < a < 3\epsilon$, which contradicts the assumption that f had no extension to $\times (\beta X_\alpha)$.

Sufficiency: Choose any proper subfamily B of A and write $X = \times \{X_\alpha : \alpha \in B\}$ and $Y = \times \{X_\alpha : \alpha \in A \setminus B\}$. Then if y_0 is any point of Y , $X \times \{y_0\}$ is homeomorphic with X and is easily seen to be C^* -embedded in $\times X_\alpha$. Hence, $X \times \{y_0\}$ is C^* -embedded in $\times (\beta X_\alpha)$ and therefore in $\beta X \times Y$. In a similar fashion, $X \times Y$ is C^* -embedded in $X \times \beta Y$. Thus, both projections π_X and π_Y are z -closed by Theorem 8.6, and $X \times Y$ is shown to be pseudocompact as in Theorem 8.12. $\square$

8.26. As did his theorem for finite products, Glicksberg's Theorem for infinite products stimulated interest in finding conditions under which products would be pseudocompact. Glicksberg conducted such an investigation and one of his results is the following which gives a criterion for the pseudocompactness of a product in terms of countable partial products.

PROPOSITION:

The pseudocompactness of the product is equivalent to the pseudocompactness of every countable partial product.

Proof: If the product is pseudocompact, then every partial product is the continuous image of the product and therefore must be pseudocompact.

Now assume that every countable partial product of $\times X_\alpha$ is pseudocompact. We will show that any sequence of non-void open subsets of $\times X_\alpha$ has a cluster point and that therefore Proposition 8.5 establishes that the product is pseudocompact. We can assume that the members of the sequence of open sets are basic open sets. Therefore, each set restricts only finitely many factor spaces. Let $\{\alpha_i\}$ be the countable set of indices corresponding to the factor spaces restricted by one or more of the open sets. Then the projection $\pi : \times X_\alpha \rightarrow \times X_{\alpha_i}$ sends the open sets to open sets in the countable partial product and the sequence of images therefore has a cluster point $\{x_{\alpha_i}\}$. An arbitrary choice of coordinates in the remaining factors yields a cluster point of the original sequence. $\blacksquare$

8.27. Example 8.4 showed that a subspace of $\beta \mathbb{N}$ consisting of $\mathbb{N}$ and the orbit Σ_p of a point p of $\mathbb{N}^*$ is pseudocompact. By choosing two points of different types, two pseudocompact spaces were obtained whose product failed to be pseudocompact. That argument can be extended to show that the preceding criterion for pseudocompactness of a product cannot be improved, i.e. we will construct a non-pseudocompact product

such that every finite partial product is pseudocompact. The first step is to demonstrate that any finite power of the space $\mathbb{IN} \cup \Sigma_p$ is pseudocompact. The proposition and subsequent example both appear in W. W. Comfort's 1967A paper.

PROPOSITION:

If p belongs to $\mathbb{IN}^*$ and m is a positive integer,
then $(\mathbb{IN} \cup \Sigma_p)^m$ is pseudocompact.

Proof: The proof will be by induction on m . The first step of the argument has already been accomplished in Example 8.4. Now assume that the theorem has been demonstrated for all integers less than m and that it fails for m . Then there is an unbounded real-valued mapping f on $(\mathbb{IN} \cup \Sigma_p)^m$ and a sequence $\{x_n\}$ can be chosen from $\mathbb{IN}^m$ such that $f(x_n) > n$. The set $\{x_n^1\}$ of first coordinates of the sequence is infinite. If not, infinitely many of the points have the same first coordinate so that $(\mathbb{IN} \cup \Sigma_p)^m$ contains a copy of $(\mathbb{IN} \cup \Sigma_p)^{m-1}$ on which f is unbounded. By considering only a subsequence if necessary, we may assume that $x_i^1 \neq x_j^1$ if $i \neq j$. Repeating this argument $(m-1)$ times, we can assume that for all $1 \leq k \leq m$, we have $x_i^k \neq x_j^k$ whenever $i \neq j$. Finally, by discarding infinitely many points of the sequence, we may also assume that for each k , the set $\{x_n^k\}$ has infinite complement in $\mathbb{IN}$.

We will now use the properties of our adjusted sequence to locate a point q of $(\mathbb{IN} \cup \Sigma_p)^m$ such that f is unbounded on every neighborhood of q , thus contradicting the assumption that f is continuous. Since the sets $\{x_n^1\}$ and $\{x_n^k\}$ are both

infinite and both have infinite complements in $\mathbb{N}$, we can find a permutation σ_k of $\mathbb{N}$ such that $\sigma_k(x_n^1) = x_n^k$. Because Σ_p is dense in $\mathbb{N}^*$, there is a point q^1 of Σ_p in the closure of $\{x_n^1\}$. Put $q^k = \beta(\sigma_k)(q^1)$ for each k , $1 \leq k \leq m$, and let q be the point whose k -th coordinate is q^k . If $U = U_1 \times \dots \times U_m$ is any basic neighborhood of q , put $V_k = \beta(\sigma_k)^{\leftarrow}(U_k)$ so that $V = \cap\{V_k : 1 \leq k \leq m\}$ is a neighborhood of q^1 . Then there are infinitely many integers n for which x_n^1 belongs to V , and for every such n , we have

$$x_n^k = \sigma_k(x_n^1) \in \beta(\sigma_k)[V_k] = U_k$$

so that the point x_n belongs to U . Hence, f is unbounded on U , so that f cannot be continuous at q . $\blacksquare$

8.28. EXAMPLE:

Choose a sequence $\{t_n\}$ of distinct types of ultrafilters and for each n , choose p_n in $\mathbb{N}^*$ such that $\mathcal{J}(p_n) = t_n$. Then the definition of type shows that $\Sigma_{p_i} \cap \Sigma_{p_j} = \emptyset$ whenever $i \neq j$. For each n , put $X_n = \mathbb{N} \cup (\cup\{\Sigma_{p_i} : i \neq n\})$. We will show that $X = \times X_n$ is not pseudocompact but that every finite partial product is pseudocompact.

(a) X is not pseudocompact: Let $A_k = \{x \in X : x^n = k, 1 \leq n \leq k\}$. The sequence $\{A_k\}$ consists of disjoint sets so that the following function on X is well-defined:

$$f(x) = \begin{cases} k & \text{if } x \in A_k \\ 0 & \text{if } x \notin \cup A_k. \end{cases}$$

The function f is unbounded and is continuous at each point of $\bigcup A_k$ since f is constant on each open set A_k . To show that f is continuous, it remains to show that $f^{-1}(0)$ is open. Choose x in $f^{-1}(0)$ and let m be the smallest integer such that $x^m \neq x^1$. Such an integer must exist since x cannot belong to A_k and because $\bigcap \Sigma_{p_i} = \emptyset$, no point of X can have every coordinate equal to the same point of $\mathbb{N}^*$. Choose disjoint open neighborhoods U and V of x^1 and x^m , respectively. Then the set

$$W = (\pi_1^{-1}(U) \cap \pi_m^{-1}(V)) \setminus \bigcup \{A_k : 1 \leq k \leq m\}$$

is open because each A_k is closed. Further, it is clear that x belongs to W and that f is constantly equal to 0 on W . Hence, f is continuous and X fails to be pseudocompact.

To show that each of the finite partial products is pseudocompact, it suffices to show that

(b) $\times \{X_n : 1 \leq n \leq m\}$ is pseudocompact: We need only to produce a dense pseudocompact subspace. For any integer $k > m$, $(\mathbb{N} \cup \Sigma_{p_k})^m$ is such a subspace as was shown in the preceding proposition.

8.29. If $\{X_\alpha\}$ is an uncountable family of spaces and x is a point of $\times X_\alpha$, then the Σ -product of $\{X_\alpha\}$ with base point x is the subspace of $\times X_\alpha$ consisting of those points which differ from x at only countably many coordinates. Observe that the restriction to an uncountable family is necessary only to assure that a Σ -product will be a proper subspace of the

product. This terminology was introduced in 1959 by H. H. Corson, who investigated normality of Σ -products and gave an example of a normal space X whose Hewitt-Nachbin realcompactification νX (Section 1.53) fails to be normal.

By combining the methods which he used to prove Lemma 8.24 and Theorem 8.25, Glicksberg showed that the Stone-Čech compactification of a Σ -product of a family of non-trivial compact spaces is the product of the family. A Σ -product of compact spaces proves to be countably compact which assures that each sequence must cluster. Thus, in the proof we will see that the technique of obtaining a family of open sets which must have a cluster point in a pseudocompact space as was done in Lemma 8.24 can be replaced by sequences in Σ -products.

THEOREM: (Glicksberg)

If S is a Σ -product of an uncountable family $\{X_\alpha\}$ of compact spaces each of which has at least two points, then

$$\beta S = \times X_\alpha.$$

Proof: S is clearly dense in the compact space $\times X_\alpha$. S is countably compact since any sequence in S has a cluster point in $\times X_\alpha$, and it is clear that the cluster point can differ from the base point in only countably many coordinates and therefore belongs to S .

As in Lemma 8.24, for any member f of $C(S)$ and $\epsilon > 0$, there is a finite set F of indices such that if points x and y agree on the coordinate set F , then $|f(x) - f(y)| < \epsilon$. If not, since we may alter countably many coordinates of a point in S and remain in S , we can obtain sequences $\{x_i\}$ and $\{y_i\}$ of points

of S and a sequence $\{F_i\}$ of disjoint finite sets of coordinates such that $|f(x_i) - f(y_i)| > \epsilon/2$ and x_i and y_i differ only in those coordinates which belong to F_i as in the proof of Lemma 8.24. Because S is countably compact, the sequence $\{x_i\}$ must cluster to a point x_0 of S . Because any basic neighborhood W of x_0 is the restriction to S of a basic neighborhood in the product topology, W restricts only a finite set E of coordinates. Hence, for some $n \geq 1$,

$$E \cap (\cup F_i) \subseteq \cup \{F_i : 1 \leq i \leq n\}.$$

Hence, if $j > n$ and W contains x_j , W must also contain y_j so that x_0 is also a cluster point of the sequence $\{y_i\}$. Thus, W contains points x and y such that $|f(x) - f(y)| > \epsilon/4$, which contradicts the continuity of f .

We use the result of the previous paragraph to show that S is C^* -embedded in $\times X_\alpha$ as was done in the proof of Theorem 8.25. If a member f of $C^*(S)$ fails to admit a continuous extension to $\times X_\alpha$, then there exists a point x_1 in $\times X_\alpha$ and a net $\{x_\delta\}$ in S converging to x_1 such that

$$\limsup \{x_\delta\} - \liminf \{x_\delta\} = a > 0.$$

We choose $\epsilon > 0$ such that $0 < 3\epsilon < a$ and obtain the finite set $\{\alpha_i\}$ of indices as outlined above. Put $Y = \times \{X_{\alpha_i} : 1 \leq i \leq n\}$. Let X be the Σ -product of the remaining spaces X_α where $\alpha \neq \alpha_i$ for any i and the base point is the restriction of the base point of S . Then we can write $S = X \times Y$. Since Y is

compact, $\beta Y = Y = \times \{X_{\alpha_i} : 1 \leq i \leq n\}$. Because $S = X \times Y$ is countably compact and therefore pseudocompact, the proof can now be completed exactly as in Theorem 8.25. $\square$

8.30. Some of the more interesting applications of the previous result occur when each X_{α} is a compact topological group. Then the usual topological product is also a topological group when given the algebraic structure of the group product. If the base point is obtained by choosing the identity element in each coordinate, then the Σ -product is easily seen to also be a topological group whose Stone-Ćech compactification is the whole product. One interesting modification of such an example enabled Glicksberg to describe a space which is homeomorphic to its growth.

EXAMPLE:

Let G be an uncountable product of the two element discrete group $\{0,1\}$ and let S be the Σ -product with base point the identity element of the product group. Then $\beta S = G$. Let the element of G which has every coordinate equal to 1 be denoted by u and let H be a maximal subgroup of G containing S but missing u . Then for any element x not in H , $\{x\} \cup H$ generates G so that $u = x + h$ for some element h of H . Thus, $G \setminus H$ is the coset $x + H$ and is also the growth H^* of H . Because translation by any element of the group is a homeomorphism in a topological group, H^* is homeomorphic to H since H^* is just the translation of H by x . A further consequence of the fact that translation is a homeomorphism is that any

topological group is homogeneous. Thus, G , S , H , and H^* are all homogeneous.

8.31. The preceding example also yields information about density. Recall from Chapter 5 that the density of a space Y is the smallest cardinal number which can be the cardinality of some dense subspace of Y . If X is dense in Y , then it is apparent that the density of X is at least as great as that of Y . The following adaptation of the preceding example is due to Comfort in 1963 and shows that the density of X may be strictly greater than that of βX .

EXAMPLE:

Let G and S be as in the preceding example and assume in addition that the cardinality of the indexing set of the product is c . Then we have shown that $\beta S = G$. Since G is the product of c separable spaces, G is separable [D, p. 175]. It remains to show that S is not separable so that the density of S is uncountable. Let $\{s_i\}$ be any countable subset of S . We will show that $\{s_i\}$ cannot be dense in S . There is some index α for which $s_i^\alpha = 0$ for all i . Let $U = \{g \in G : g^\alpha = 1\}$. Then U is an open set of G which meets S but which fails to contain any member of the set $\{s_i\}$. Hence, S is not separable.

8.32. Before leaving the subject of pseudocompact products, we should remark that there is one class of pseudocompact spaces which is closed under products. In their 1966 paper, W. W. Comfort and K. Ross showed that any product of pseudocompact

topological groups is pseudocompact. By applying Glicksberg's Theorem for finite products to the square of a pseudocompact group and using the continuity of the group operations, one can show that the Stone-Čech compactification of a pseudocompact group is a topological group. Thus, Example 8.30 is one instance of a more general theorem since both S and G are topological groups and $\beta S = G$.

ASSORTED PRODUCT THEOREMS

8.33. We have seen that many spaces contain closed copies of $\mathbb{N}$. In particular, the spaces which fail to be countably compact can be characterized as those spaces which contain closed copies of $\mathbb{N}$. Since the product of a family of such spaces will contain a closed subspace consisting of products of copies of $\mathbb{N}$, we can expect that the following result will be useful in the investigation of such products. The result appears in A. H. Stone's 1948 paper.

THEOREM: (A. H. Stone)

An uncountable product of copies of $\mathbb{N}$ fails to be normal.

Proof: We can think of the product of copies of $\mathbb{N}$ indexed by an uncountable index set A as the set $\mathbb{N}^A$, i.e. the functions on A taking their values in $\mathbb{N}$. We will assume that a basic neighborhood U of a point $x = \{x^\alpha\}$ in $\mathbb{N}^A$ is obtained by restricting a finite set $F(U) = \{\alpha_1, \dots, \alpha_j\}$ of coordinates to

a single point x^{α_i} .

For each positive integer k , let B_k be the set of all points $x = \{x^\alpha\}$ such that for each integer n other than k , $x^\alpha = n$ for at most one α in A . The sets B_k are closed and pairwise disjoint. Thus, if $\mathbb{N}^A$ were normal, there would exist disjoint open sets V and W containing B_1 and B_2 , respectively. We will show that this assumption leads to a contradiction.

We will define inductively three sequences: a sequence $\{x_n\}$ of points of B_1 , a sequence $\{m_n\}$ of strictly increasing positive integers, and a sequence $\{\alpha_n\}$ of elements of A . Define x_1 to be the point which is constantly equal to 1. Then x_1 belongs to B_1 and therefore there is a basic neighborhood U_1 of x_1 contained in V . Let $F(U_1) = \{\alpha_1, \dots, \alpha_{m_1}\}$ be the set of coordinates restricted by U_1 and m_1 be the number of coordinates restricted. Now suppose that x_n and U_n have been obtained so that x_n belongs to B_1 , U_n is a basic neighborhood of x_n contained in V , and $F(U_n) = \{\alpha_1, \dots, \alpha_{m_1}, \dots, \alpha_{m_n}\}$ is the set of coordinates restricted by U_n . Define x_{n+1} as follows:

$$x_{n+1}^\alpha = \begin{cases} k & \text{if } \alpha = \alpha_k \text{ for some } k, 1 \leq k \leq m_n \\ 1 & \text{otherwise.} \end{cases}$$

Then x_{n+1} belongs to B_1 and there is a basic neighborhood U_{n+1} of x_{n+1} contained in V . We can also assume that $F(U_{n+1})$ contains $F(U_n)$ as a proper subset so that $F(U_{n+1})$ has m_{n+1} elements and $m_{n+1} > m_n$. Write

$$F(U_{n+1}) \setminus F(U_n) = \{\alpha_{m_n+1}, \dots, \alpha_{m_{n+1}}\}.$$

The induction is now complete.

Define a point y in $\mathbb{N}^A$ by:

$$y^\alpha = \begin{cases} k & \text{if } \alpha = \alpha_k \text{ for some } k \geq 1 \\ 2 & \text{otherwise.} \end{cases}$$

Thus, y belongs to B_2 and hence there is a basic neighborhood O of y contained in W . Since $F(O)$ is finite, there is some integer n such that α_k is not in $F(O)$ for all $k > m_n$. Finally, define z in $\mathbb{N}^A$ by

$$z^\alpha = \begin{cases} k & \text{if } \alpha = \alpha_k \text{ for some } k, k \leq m_n \\ 1 & \text{if } \alpha = \alpha_k \text{ for some } k, m_n < k \leq m_{n+1} \\ 2 & \text{otherwise.} \end{cases}$$

Then because z agrees with both y and x_{n+1} on the coordinates $\alpha_1, \dots, \alpha_{m_{n+1}}$, we have that

$$z \in U_{n+1} \cap O \subset U \cap V,$$

which is a contradiction. $\parallel$

8.34. Because an uncountable product of non-countably compact spaces will contain $\mathbb{N}^A$ as a closed, non-normal subspace, we have verified the

COROLLARY: (A. H. Stone)

If a product of non-empty T_1 -spaces is normal, all but at most a countable number of the factor spaces is countably compact.

8.35. The previous result together with Glicksberg's Theorem for infinite products enables us to establish the following result which appeared in N. Noble's 1971 paper and shows that any T_1 -space such that every power of the space is normal must be compact. The proof appears in a joint paper of S. P. Franklin and the present author.

THEOREM: (N. Noble)

If each power of a T_1 -space is normal then the space is compact.

Proof: Let X be such a space and let X^m be an uncountable power of X . Since $(X^m)^m = X^m$, the preceding corollary shows that X^m is countably compact and therefore is pseudocompact. Because X is a T_1 -space and is normal, X is completely regular and we can consider βX . Glicksberg's Theorem shows that $\beta(X^m) = (\beta X)^m$. The proof will be completed by choosing a particular uncountable index set.

If X fails to be compact, there is a point p in X^* and we can identify p with the unique free z -ultrafilter on X which converges to p . Consider the product set X^p which consists of the functions defined on the collection of zero-sets belonging to p and taking their values in X . The proof is completed by showing that the assumption that p is free will provide disjoint closed subsets of X^p which cannot be completely separated. Write Δ for the diagonal, i.e. the set of all constant functions. Write C for the set of choice functions, i.e.

$$C = \{x \in X^p : x_z \in Z \text{ for } z \in p\}.$$

The sets Δ and C are closed and are disjoint since p is free. Therefore they are completely separated by a mapping f in $C(X^p)$. But f cannot extend continuously to $\beta(X^p)$ since the function in $(\beta X)^p = \beta(X^p)$ which is constantly equal to p belongs to the closure in $(\beta X)^p$ of both Δ and C . Thus, it must be that $\Delta \cap C \neq \emptyset$, i.e. that p is fixed and X is compact. $\blacksquare$

8.36. Recall from Chapter 6 that a paracompact space is one in which every open cover has a locally finite open refinement. Since a finite subcover is also a locally finite refinement, the class of paracompact spaces is another class containing the compact spaces. The equivalence of the first two conditions in the following theorem was demonstrated by H. Tamano in 1962. The methods used in the proof will be reminiscent of those used earlier in the chapter.

THEOREM: (Tamano)

The following are equivalent for any space X :

- (1) $X \times \beta X$ is normal.
- (2) X is paracompact.
- (3) If K is a compact space, $X \times K$ is paracompact.

Proof: The following technical condition is also equivalent to the three stated in the theorem. It is included to smooth the proof of $(1) \Rightarrow (2)$.

- (4) For each compact subspace F of X^* , there is a locally finite open cover $\{U_\alpha : \alpha \in A\}$ of X such that $(cl_{\beta X} U_\alpha) \cap F = \emptyset$ for each α in A .

(2) $\Rightarrow$ (3): Assume that X is paracompact and let $\mathcal{u}$ be an open cover of $X \times K$. For each point x of X , there is a finite subfamily $\{U_i^x : 1 \leq i \leq n_x\}$ which covers the compact space $\{x\} \times K$. Since K is compact, we can choose an open neighborhood V_x of x such that $V_x \times K$ is contained in $\bigcup \{U_i^x : 1 \leq i \leq n_x\}$ [D, p. 228]. Repeating this for each point of X yields an open cover $\{V_x : x \in X\}$ of X . Since X is paracompact, the cover has a locally finite open refinement $\{W_\beta : \beta \in B\}$. For each W_β choose x in X such that W_β is contained in V_x and consider the open sets

$$(W_\beta \times K) \cap (\bigcup \{U_i^x : 1 \leq i \leq n_x\}).$$

Running through the set B of indices yields an open cover $\mathcal{v}$ of $X \times K$. For each (x, k) in $X \times K$, there is a neighborhood U of x which meets only finitely many elements of $\{W_\beta\}$. Hence, $U \times K$ is a neighborhood of (x, k) which meets only finitely many sets of $\mathcal{v}$. Thus, $\mathcal{v}$ is a locally finite open refinement of $\mathcal{u}$ and $X \times K$ is paracompact.

(3) $\Rightarrow$ (1): This is immediate from the observation that a paracompact space is normal.

(1) $\Rightarrow$ (4): Assume that $X \times \beta X$ is normal and that F is a compact subset of X^* . Then the diagonal $\Delta = \{(x, x) \in X \times \beta X : x \in X\}$ and $X \times F$ are disjoint closed subsets of $X \times \beta X$. Thus there

is a map $f : X \times \beta X \rightarrow I$ such that f is constantly 0 on Δ and 1 on $X \times F$. Placing βX in the role of Y , the equivalence of (3) and (5) of Theorem 8.6 implies that

$$\gamma(x_1, x_2) = \sup\{|f(x_1, p) - f(x_2, p)| : p \in \beta X\}$$

defines a continuous pseudometric on X . Thus γ induces a pseudometric topology on X which is weaker than the original topology. The open cover of X consisting of the spheres

$$B(x, 1/2) = \{y \in X : \gamma(x, y) < 1/2\}$$

has a locally finite open refinement $\{U_\alpha : \alpha \in A\}$ in the pseudometric topology which is also open and locally finite in the original topology. If y belongs to $B(x, 1/2)$, then $\gamma(x, y) < 1/2$ so that

$${}_x f(y) = |f(x, y) - f(y, y)| < 1/2.$$

Thus, ${}_x f(p) \leq 1/2$ for each p in $cl_{\beta X} B(x, 1/2)$ and since ${}_x f(p) = 1$ for each point p of F , $cl_{\beta X} B(x, 1/2) \cap F = \emptyset$. Hence, each member of the locally finite open cover $\{U_\alpha\}$ satisfies $cl_{\beta X} U_\alpha \cap F = \emptyset$.

(4) $\Rightarrow$ (2): Let $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ be an open cover of X .

For every V_α , choose an open set W_α of βX such that $V_\alpha = X \cap W_\alpha$. Put $F = \bigcap \{\beta X \setminus W_\alpha : \alpha \in A\}$. Then F is a compact subspace of X^* so that by (2) there exists a locally finite open cover $\{U_\beta : \beta \in B\}$ of X such that $cl_{\beta X} U_\beta \cap F = \emptyset$ for each U_β . Thus, $cl_{\beta X} U_\beta$ is contained in $\bigcup \{W_\alpha : \alpha \in A\}$ and since $cl_{\beta X} U_\beta$ is compact, it can be covered by a finite

subfamily $\{W_{\alpha_1}, \dots, W_{\alpha_{n_\beta}}\}$. Define $\mathcal{U}$ to be the cover
 $\{U_\beta \cap V_{\alpha_i} : \beta \in B, \alpha_1, \dots, \alpha_{n_\beta}\}$. $\mathcal{U}$ is then easily seen to be the
 required locally finite refinement of $\mathcal{U}$. $\blacksquare$

8.37. EXAMPLE:

In Section 6.17, we noted that the ordinal space ω_1 is normal but not paracompact. Since $\beta\omega_1 = \omega_1 + 1$ [GJ, 5.12], Theorem 8.36 implies that $\omega_1 \times (\omega_1 + 1)$ is not normal. Hence, we also see that the product of a normal space with a compact space is not normal.

8.38. The rather intriguing condition (1) of Theorem 8.36 is made even more so by observing that there is an analogous characterization of P-spaces:

THEOREM: (Negrepontis)

X is a P-space if and only if $X \times \beta X$ is an F-space.

The theorem appears in the 1969B paper of S. Negrepontis. Since the infinite compact space βX cannot be a P-space, one direction of the proof is immediate from Proposition 1.65. The other direction of the proof requires machinery which we have not developed and will not be given.

THE $\mathcal{U}$ -ANALOGUE: AN OPEN QUESTION

8.39. Once Glicksberg's Theorem has been proven, it is natural to seek a characterization of pairs of spaces X and Y such that $\mathcal{U}(X \times Y) = \mathcal{U}X \times \mathcal{U}Y$. Because a space S is pseudocompact if and only if $\mathcal{U}S = \beta S$ (Corollary 1.58), we can interpret Glicksberg's Theorem as providing one sufficient condition. However, the pseudocompactness of the product is hardly necessary since $\mathbb{R} \times \mathbb{R}$ is not pseudocompact and yet

$$\mathcal{U}\mathbb{R} \times \mathcal{U}\mathbb{R} = \mathbb{R} \times \mathbb{R} = \mathcal{U}(\mathbb{R} \times \mathbb{R}).$$

The problem appears to be substantially more difficult than the corresponding one for the Stone-Ćech compactification. One complication is the relationship between realcompactness and measurable cardinals which is described in [GJ, Chapters 12 and 15]. A discrete space is realcompact if and only if its cardinality is non-measurable and this fact causes cardinality restrictions to enter into the problem. One approach to the problem based on the assumption that a measurable cardinal exists will be discussed briefly in Example 10.24(g).

The problem is approachable from the point of view of uniform spaces as is shown in the 1960 paper of N. Onuchic and the 1969 paper of A. W. Hager. In his 1964 text, Uniform Spaces, J. R. Isbell proves Glicksberg's Theorem from the point of view of uniformities. A uniform space is said to be fine if the uniformity is the finest one which is compatible with the topology of the space. Isbell discusses the question of when the product of two

fine spaces will be fine, which is a related question.

In his 1968 paper, W. W. Comfort shows that if X is a k -space (see Exercise 8H), $\mathcal{O}Y$ is locally compact, and neither X nor $\mathcal{O}Y$ contains a compact subspace of measurable cardinal, then $\mathcal{O}X \times \mathcal{O}Y = \mathcal{O}(X \times Y)$. Comfort's results together with other related results are also contained in the thesis of M. Weir. In his paper to appear, Hušek approaches the problem from the point of view of function spaces.

In 1969, T. Isiwata showed that $\mathcal{O}$ will distribute over $X \times Y$ if X is a continuous image of a first countable space, Y is pseudocompact, and Y has non-measurable cardinal. In his 1970 paper, W. G. McArthur characterizes the pairs X and Y for which $\mathcal{O}X \times \mathcal{O}Y = \mathcal{O}(X \times Y)$ in terms of a condition similar to that of Theorem 8.14.

EXERCISES

8A. PSEUDOCOMPACT SUBSPACES OF βX

For H contained in $\beta X \setminus \cup X$, define $Y = X \cup H$.

1. If Y is pseudocompact, then H is dense in $\beta X \setminus \cup X$.
2. If H is dense in $\beta X \setminus \cup X$ and if $\cup X$ is locally compact, then Y is pseudocompact.

Reference: In his 1967B paper, W. W. Comfort characterizes those spaces X for which $\cup X$ is locally compact.

8B. CLOSED PROJECTIONS

Let X be a P -space and Y be Lindelöf.

1. The projection $\pi_X : X \times Y \rightarrow X$ is a closed map.
2. $X \times Y$ is C^* -embedded in $X \times \beta Y$.

Reference: Comfort and Negrepontis, 1966. The thesis and paper of A. K. Misra contain more results concerning P -spaces and Lindelöf spaces.

8C. PROPER C^* -PAIRS

1. Let $S \subset X \subset \beta S$ and suppose that X and Y are a proper C^* -pair for some space Y . Then $X \subset \cup S$. [Theorem 1.53, Corollary 8.13.]

2. Let S be realcompact and $S \subset X \subset \beta S$. Then if X and Y are a proper C^* -pair for some space Y , X is equal to S .
3. Let X be σ -compact and $S \subset X \subset \beta S$. Then if X and Y are a proper C^* -pair for some space Y , $X = S = \mathbb{IN}$.

[Proposition 1.65.]

Reference: Comfort and Negrepontis, 1966.

8D. SUBSPACES OF $\beta(X \times Y)$

Suppose that $\beta(X \times Y)$ has a subspace S such that $\beta(X \times Y) \setminus S$ is an F -space and $|S| < |\beta Y|$.

1. If e is an embedding of $\beta X \times \beta Y$ into $\beta(X \times Y)$, then there exists a point q of βY such that $\beta X \times \{q\}$ misses $e^{-1}(S)$.
2. If X is not an F -space, then $\beta(X \times Y)$ contains no copy of $\beta X \times \beta Y$.
3. $\beta(\mathbb{IR} \times \mathbb{IR})$ contains no copy of $\beta \mathbb{IR} \times \beta \mathbb{IR}$.

Reference: Gillman and Jerison, 1959, contains this result and several similar results.

8F. $\beta \mathbb{IN}$ CONTAINS 2^C DISJOINT COPIES OF $\beta \mathbb{IN}$

1. If p belongs to $\beta \mathbb{IN}$, then $\mathbb{IN} \times \{p\}$ is C^* -embedded in $\mathbb{IN} \times \beta \mathbb{IN}$.
2. If p and q are distinct, then $\mathbb{IN} \times \{p\}$ and $\mathbb{IN} \times \{q\}$ are completely separated in $\mathbb{IN} \times \beta \mathbb{IN}$.

3. $\beta\mathbb{N}$ contains $2^{\mathbb{C}}$ mutually disjoint copies of itself.

[8.18(a).]

Reference: Gillman and Jerison, 1959. Note that in the proof of Theorem 3.50, we showed that $\mathbb{N}^*$ contains $2^{\mathbb{C}}$ disjoint of $\beta\mathbb{N}$ by a different method.

8G. C-EMBEDDING OF $X \times Y$ IN $\cup X \times \cup Y$

A subspace X of S is said to be G_δ -dense in S if every non-empty G_δ of S meets X .

1. If X is G_δ -dense in S and Y is G_δ -dense in T , then $X \times Y$ is G_δ -dense in $S \times T$.
2. If $X \times Y$ is C^* -embedded in $\cup X \times \cup Y$, then $X \times Y$ is C -embedded in $\cup X \times \cup Y$. [Theorem 1.3.]

Reference: Comfort and Negrepontis, 1966.

8H. PSEUDOCOMPACT PRODUCTS AND k -SPACES

A space X is a k -space if each subset F of X such that $F \cap K$ is closed in K for every compact subspace K is closed in F .

1. A function is continuous on a k -space if and only if its restriction to every compact subspace is continuous.
2. If X and Y are pseudocompact and Y is a k -space, then $X \times Y$ is pseudocompact. [Modify 8.21.]

3. Any first countable space is a k -space. [A convergent sequence together with its limit point is compact.]
4. Any locally compact space is a k -space.
5. Let p belong to $\mathbb{N}^*$. The subspace $\mathbb{N} \cup \{p\}$ of $\beta\mathbb{N}$ is not a k -space. [$\mathbb{N}$ is not closed.]

Reference: (2) appears in Tamano, 1960A. k -spaces are discussed in [D, pp. 247-249].

8I. PRODUCTS OF METRIC SPACES

1. A normal space is pseudocompact if and only if it is countably compact.
2. A metric space is pseudocompact if and only if it is compact.
3. The following are equivalent for a product of non-empty metric spaces:
 - (a) The product is normal.
 - (b) The product is paracompact.
 - (c) At most countably many of the factors fail to be compact.

Reference: (3) appears in A. H. Stone, 1948.

8J. PSEUDOCOMPACT SUBSPACES OF $\beta\mathbb{N}$

Every pseudocompact subspace of $\beta\mathbb{N}$ which contains $\mathbb{N}$ has cardinality at least c . [8.3 and 5.12.]

Reference: S. Mrowka, 1959.

CHAPTER NINE:
SOME CONNECTEDNESS RESULTS

9.1. A space is connected if it is not the union of two disjoint non-empty closed sets. An equivalent condition is that there does not exist a continuous mapping of the space onto the discrete space having two points. From the latter condition it is easy to see that a space X is connected if and only if βX is connected.

A space is said to be locally connected if every point has a base of connected neighborhoods. The relationship between local connectedness of a space and the local connectedness of its Stone-Čech compactification is not as clear-cut as for connectedness. For example, the space $\mathbb{N}$ is locally connected, but it is clear that $\beta\mathbb{N}$ is not locally connected. In this chapter, we will use the lattice structure of $C^*(X)$ to characterize the class of spaces which have locally connected Stone-Čech compactifications.

In the last sections of the chapter, we will consider an example of a space whose growth is an indecomposable continuum, i.e a compact connected space which is not the union of two proper closed connected subspaces.

COMPACTIFICATIONS OF LOCALLY CONNECTED SPACES

9.2. Suppose that a space X is the disjoint union of n non-empty closed subsets F_i , $1 \leq i \leq n$. Then each F_i is

is also open and the characteristic function χ_i of each F_i belongs to $C^*(X)$. Thus, if f is in $C^*(X)$, we can write $f_i = \chi_i f$ so that $f = f_1 + \dots + f_n$. Further, no two of the mappings $\{f_i\}$ are non-zero at the same point, and hence we have $|f_i| \wedge |f_j| = 0$ whenever $i \neq j$. When f can be expressed in this form, the set $\{f_i : 1 \leq i \leq n\}$ is called a decomposition of f . Our characterization of the class of spaces having locally connected Stone-Čech compactifications will be in terms of a certain type of decomposition.

Let E be a normed linear lattice. A subset $\{p_i : 1 \leq i \leq n\}$ of E is said to be a decomposition of a point x of E if $x = p_1 + \dots + p_n$ and $|p_i| \wedge |p_j| = 0$ whenever $i \neq j$. A member x of E is said to be well behaved if for every $\epsilon > 0$, x admits a decomposition $\{p_i : 1 \leq i \leq n\}$ such that for $i = 1, \dots, n-1$, p_i admits no proper decomposition and $\|p_n\| < \epsilon$. The lattice E is called well behaved if every point of E is well behaved.

If $f = p_1 + \dots + p_n$ is a decomposition of a member f of $C^*(X)$, then the cozero-set of f is the disjoint union of the cozero-sets of the p_i , and therefore is not connected. We will ultimately use the existence of mappings which do not admit proper decompositions to produce a base of connected zero-sets.

If S is C^* -embedded in a space X , then it is easy to see that a mapping in $C^*(S)$ can be extended to one in $C^*(X)$ in such a way that the norm of the mapping is preserved. Hence, $C^*(X)$ contains a lattice isomorphic copy of $C^*(S)$. Since we frequently deal with C^* -embedded copies of $\mathbb{N}$, the following

example of a normed linear lattice which is isomorphic to $C^*(\mathbb{N})$ will be useful.

EXAMPLE:

The normed linear lattice l_∞ consisting of all bounded sequences taken with the sup norm is not well behaved. This is clear since any sequence containing more than two non-zero terms will admit a proper decomposition. Note also that because l_∞ is isomorphic to $C^*(\mathbb{N})$, it is also isomorphic to $C(\beta\mathbb{N})$.

9.3. We now show that βX is locally connected exactly when X is both locally connected and pseudocompact. In his 1956 paper, B. Banaschewski showed that X must be locally connected and pseudocompact whenever βX is locally connected. The converse was established by M. Henriksen and J. R. Isbell in their 1957B paper. The characterization of pseudocompact locally connected spaces in terms of well behaved linear lattices was introduced by D. E. Wulbert in 1969.

THEOREM:

The following are equivalent:

- (1) X is locally connected and pseudocompact.
- (2) $C^*(X)$ is a well behaved linear lattice.
- (3) βX is locally connected.

The proof will use the characterization of locally connected spaces as those spaces in which the components of each open subspace are open [D, p. 113].

Proof: (1) $\Rightarrow$ (2): Let X be locally connected and pseudo-compact and let f belong to $C^*(X)$. Let $\mathcal{Q}$ be the family of components of the cozero-set of f . Then $\mathcal{Q}$ is a family of open sets because X is locally connected. For $\epsilon > 0$, consider the family

$$\mathfrak{F} = \{G \cap \{x \in X : |f(x)| > \epsilon\} : G \in \mathcal{Q}\}.$$

The family $\mathfrak{F}$ is locally finite: If $f(x) = 0$, then $\{x \in X : |f(x)| < \epsilon\}$ is a neighborhood of x which misses each member of $\mathfrak{F}$. If $|f(x)| > 0$, then x has a neighborhood which is contained in some member of $\mathcal{Q}$ and which therefore meets at most one member of $\mathfrak{F}$.

Now because X is pseudocompact, Proposition 5.5 implies that the locally finite family $\mathfrak{F}$ can contain only finitely many non-empty sets. Suppose that f will admit no well behaved decomposition. Then we can generate an infinite sequence $\{f_i\}$ of members of $C^*(X)$ such that each f_i is a restriction of f , each has norm greater than or equal to ϵ , and the cozero-sets $\{Cz(f_i)\}$ form a disjoint family. But then the non-empty members of the family

$$\{Cz(f_i) \cap F : F \in \mathfrak{F}, i = 1, 2, \dots\}$$

form an infinite, locally finite family of non-empty open sets, which cannot exist in a pseudocompact space. Hence, $C^*(X)$ is well behaved.

(2) $\Rightarrow$ (1): Assume that $C^*(X)$ is a well behaved linear lattice. We will first show that X is pseudocompact. Any

non-pseudocompact space contains a closed, C^* -embedded copy of $\mathbb{N}$ (Lemma 4.5). Hence, $C^*(X)$ contains a lattice isomorphic copy of ℓ_∞ , the space of all bounded sequences. Since we have seen that ℓ_∞ is not well behaved, $C^*(X)$ cannot be well behaved. Hence, X is pseudocompact.

Now we establish that X is locally connected. Let x belong to X and let U be a neighborhood of x . We will exhibit a connected neighborhood of x which is contained in U . There exists f in $C^*(X)$ such that $f(x) = \|f\| = 1$ and such that f vanishes on the complement of U . Further, since f is well behaved, we can assume that f admits no proper decomposition. Hence, the cozero-set of f is a connected neighborhood of x contained in U , and X is locally connected.

(2) $\iff$ (3): This is immediate since $C^*(X)$ and $C^*(\beta X)$ are lattice isomorphic. $\blacksquare$

9.4. EXAMPLES:

$\beta\mathbb{R}$ is connected, but not locally connected.

$\beta\mathbb{Q}$ is neither connected nor locally connected.

9.5. The class of locally connected pseudocompact spaces has several interesting properties which are not shared by the locally connected spaces. Note that the continuous image of a locally connected space need not be locally connected. For instance, any discrete space is locally connected, but the discrete space of cardinality 2^c will map onto $\beta\mathbb{N}$ and $\beta\mathbb{N}$ fails to be locally connected.

However, a quotient of a locally connected space is locally connected [D, p. 125]. The following result appears in the 1967 paper of J. de Groot and R. H. McDowell and shows that the quotient hypothesis is not needed in the class of pseudocompact locally connected spaces.

PROPOSITION:

A continuous image of a pseudocompact, locally connected space is locally connected.

Proof: Let X be pseudocompact and locally connected and let f map X onto Y . Then βX is locally connected and βY is a continuous image of βX under the extension $\beta(f)$. Because βX is compact, $\beta(f)$ is a closed map and is therefore a quotient mapping. But a quotient of a locally connected space is locally connected, hence βY is locally connected. But then we have seen that Y must be locally connected. $\square$

9.6. COROLLARY:

If K_1 and K_2 are compactifications of X and K_2 is greater than K_1 , then K_1 is locally connected if K_2 is.

9.7. The next result is also due to de Groot and McDowell.

THEOREM:

The following are equivalent:

- (1) X is locally connected and pseudocompact.
- (2) Every space in which X is dense is locally connected.
- (3) Every compactification of X is locally connected.

Proof: $(1) \Rightarrow (2)$: Let X be dense in Y . Then X is also dense in βY and therefore βY is a continuous image of βX . But βX is locally connected, so that Proposition 9.5 implies that βY is locally connected. Hence, Y is also locally connected. $(2) \Rightarrow (3)$ is immediate and Theorem 9.3 shows that (1) follows from (3). $\blacksquare$

9.8. Certain results concerning the local connectedness of X and βX can be achieved without assuming that either space is locally connected. The next three results appear in Henriksen and Isbell's 1957 paper. The first one was established for normal spaces by A. D. Wallace in 1951.

LEMMA:

An open subset U of βX is connected if and only if $X \cap U$ is connected.

Proof: The proof will be accomplished by proving both contrapositives.

If U is not connected, then $U = V \cup W$ where V and W are disjoint non-empty open sets. Then $U \cap X = (V \cap X) \cup (W \cap X)$, so that $U \cap X$ is also disconnected.

Now suppose that $U \cap X$ is disconnected, i.e. that $U \cap X = V \cup W$ where V and W are disjoint, non-empty open subsets of X . Since X is dense in βX , U is contained in $\text{cl}_{\beta X} V \cup \text{cl}_{\beta X} W$. If no point of U belongs to both of these closures, then we have

$$U = ((\text{cl } V) \cap U) \cup ((\text{cl } W) \cap U)$$

and U is not connected. Now suppose that a point p of U belongs to both closures. Let g in $C(\beta X)$ be such that $g(p) = 0$ and $g[\beta X \setminus U] = \{1\}$. Define a function f on X by:

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \in W \text{ and } g(x) < \frac{1}{2} \\ g(x) & \text{otherwise.} \end{cases}$$

Because V and W are complementary clopen subsets of $U \cap X$, it is easy to verify that f is continuous. Now the extension $\beta(f)$ of f must coincide with g on $cl_{\beta X} V$ and be greater than or equal to $1/2$ on $cl_{\beta X} W$. But this contradicts the assumption that p belongs to both closures. Hence, U is not connected. $\blacksquare$

9.9. The following corollary is immediate and corresponds to Exercise 2F.2 which states an analogous result for points of first countability.

COROLLARY:

βX is locally connected at a point of X if and only if X is locally connected at the point.

9.10. In Corollary 1.58, we saw that $\alpha X = \beta X$ if and only if X is pseudocompact. Thus, the following result shows that βX cannot be locally connected whenever αX is not equal to βX and indicates the importance of pseudocompactness in (1) of Theorem 9.3. Notice that the preceding Corollary combined with the following proposition yield another proof that X must be locally connected and pseudocompact whenever βX is locally connected.

PROPOSITION:

βX is not locally connected at any point of $\beta X \setminus \alpha X$.

Proof: If p belongs to $\beta X \setminus \alpha X$, then there is an f in $C(X)$ such that $f^\alpha(p) = \infty$ (Section 1.53). For $i = 0, 1, 2, 3$, let Z_i be the set of all x in X such that $n \leq f(x) \leq n+1$ for some integer n such that $n \equiv i \pmod{4}$. The sets Z_i cover X so that p belongs to $\text{cl}_{\beta X} Z_j$ for some j . Then let $k = 0, 1, 2$, or 3 be congruent to $j + 2$ modulo 4. Then Z_k and Z_j are disjoint zero-sets and therefore p cannot belong to $\text{cl}_{\beta X} Z_k$. Thus, there is a neighborhood U of p which misses Z_k . If βX is locally connected at p , then there is a connected neighborhood V of p such that V is contained in U . By Lemma 9.8, $V \cap X$ is connected. However, f is necessarily unbounded on $V \cap X$ and $V \cap X$ misses Z_k . Thus, $f|_{V \cap X}$ cannot take any values between $4n + k$ and $4n + k + 1$ for any n , so that the image of $V \cap X$ is not connected. This contradicts the assumption that $V \cap X$ is connected and therefore that βX is locally connected at p . $\square$

9.11. EXAMPLES:

$\beta \mathbb{N}$, $\beta \mathbb{Q}$, and $\beta \mathbb{R}$ are not locally connected at any point in their respective growths.

A NON-METRIC INDECOMPOSABLE CONTINUUM

9.12. A continuum is a compact, connected set. Common examples of continua are the closed unit interval I and the circle S^1 . In the remaining sections of the chapter, we will consider one of

several recent results concerning Stone-Čech compactifications whose growths are continua. A continuum is said to be decomposable if it can be written as the union of two proper subcontinua. Otherwise, a continuum is said to be indecomposable. Both of the examples given above are decomposable metric continua. We will show that the growth of interval $[1, \infty)$ is a non-metric indecomposable continuum. We will first need a preliminary result on the intersection of continua. The proof is taken from the text of S. Willard.

PROPOSITION:

The intersection of a decreasing family of continua is a continuum.

Proof: Let $\{K_\alpha : \alpha < \beta\}$ be a family of continua in a space X such that $K_{\alpha+1}$ is contained in K_α for each α . Then $\bigcap K_\alpha$ is a closed subset of each K_α and is therefore compact. Now suppose that the intersection fails to be connected, i.e. that $\bigcap K_\alpha = F \cup H$ where F and H are disjoint, non-empty closed sets. F and H are contained in K_1 so that we can replace X by K_1 and assume that X is compact. Thus, there exist disjoint open sets U and V of X containing F and H , respectively. For each α , K_α can not be contained in $U \cup V$ since $U \cap K_\alpha$ and $V \cap K_\alpha$ would disconnect K_α . Hence, we can choose a point x_α in $K_\alpha \setminus (U \cup V)$ for each α and the resulting net $\{x_\alpha : \alpha < \beta\}$ must cluster at some point z of X . If W is any neighborhood of z , then W meets each K_α so that z belongs to $\text{cl } K_\alpha = K_\alpha$ for each α . But then we have

$$z \in \bigcap K_\alpha \subset U \cup V,$$

and this is a contradiction since the net $\{x_\alpha\}$ is never in the neighborhood $U \cup V$ of z . $\square$

9.13. The growth of $\beta\mathbb{R}$ consists of two pieces, one at each end of $\mathbb{R}$. It is easy to see that the two pieces are the growths of the rays $A = [1, \infty)$ and $B = (-\infty, -1]$. The next result shows that each piece is a non-metric indecomposable continuum. The theorem appears in the thesis of D. P. Bellamy and also in his 1971 paper.

THEOREM: (D. P. Bellamy)

$A^* = \beta A \setminus A$ is a non-metric indecomposable continuum.

Proof: A^* is a continuum: Put $A_n = [n, \infty)$. Then $cl_{\beta\mathbb{R}} A_n$ is a continuum since it is compact and is the closure of a connected set. Then we have

$$A^* = \bigcap cl_{\beta\mathbb{R}} A_n$$

and the previous proposition implies that A^* is a continuum.

A^* is non-metric: Because A is σ -compact and locally compact, Proposition 1.62 implies that A^* is a compact F -space. A^* is clearly infinite, and so contains a copy of $\beta\mathbb{N}$ (Proposition 1.64). No point of $\mathbb{N}^*$ is a G_δ in $\beta\mathbb{N}$ (Corollary 3.8) so that no point of the growth $\mathbb{N}^*$ is a G_δ in A^* . Hence, A^* fails to be first countable and therefore cannot be metric. A^* is indecomposable: Suppose that F and

H are proper closed subsets of A^* such that $A^* = F \cup H$. To show that A^* is not the union of two proper subcontinua, it will be sufficient to show that F is not connected. We will construct a mapping of F onto the discrete space $\{0,1\}$ to establish that F fails to be connected.

Choose x in $A^* \setminus H$ and y in $A^* \setminus F$. Then there exist open βA -neighborhoods U and V of x and y , respectively, such that

$$cl_{\beta A} U \cap cl_{\beta A} V = (cl_{\beta A} U) \cap H = (cl_{\beta A} V) \cap F = \emptyset.$$

Note also that $U \cap A$ and $V \cap A$ are both unbounded. We now choose by induction three infinite sequences of distinct points of A to enable us to construct the previously mentioned mapping. Choose p_1 in $U \cap A$ and $q_1 > p_1$ with q_1 in $V \cap A$. Because V is open, it is possible to choose $s_1 > q_1$ so that the interval (q_1, s_1) is contained in V . Now assume that p_k , q_k , and s_k have been chosen for all $k < n$ such that for each k ,

- (1) $p_k \in U$,
- (2) The interval (q_k, s_k) is contained in V ,
- (3) $p_k < q_k < s_k$, and if $k < n-1$, then $s_k < p_{k+1}$.

Since $U \cap A$ is unbounded, we can choose p_n in $U \cap A$ such that $p_n > s_{n-1}$. Because $V \cap A$ is unbounded, we can choose q_n in $V \cap A$ such that $q_n > p_n$. Finally, because $V \cap A$ is open, we can choose $s_n > q_n$ such that (q_n, s_n) is contained in V , and this completes the induction.

Each of the sequences is unbounded, since otherwise they have a common supremum which must lie in $cl_{\beta A} U \cap cl_{\beta A} V$, which is impossible. Now define $f : A \rightarrow I$ as follows:

$$(1) \quad f(r) = \begin{cases} 0 & \text{if } r \leq q_1 \\ 0 & \text{if } r \in [s_{k-1}, q_k] \text{ for } k \text{ odd} \\ 1 & \text{if } r \in [s_{k-1}, q_k] \text{ for } k \text{ even} \end{cases}$$

(2) f is linear on the intervals $[q_k, s_k]$.

A portion of the mapping f is illustrated below:

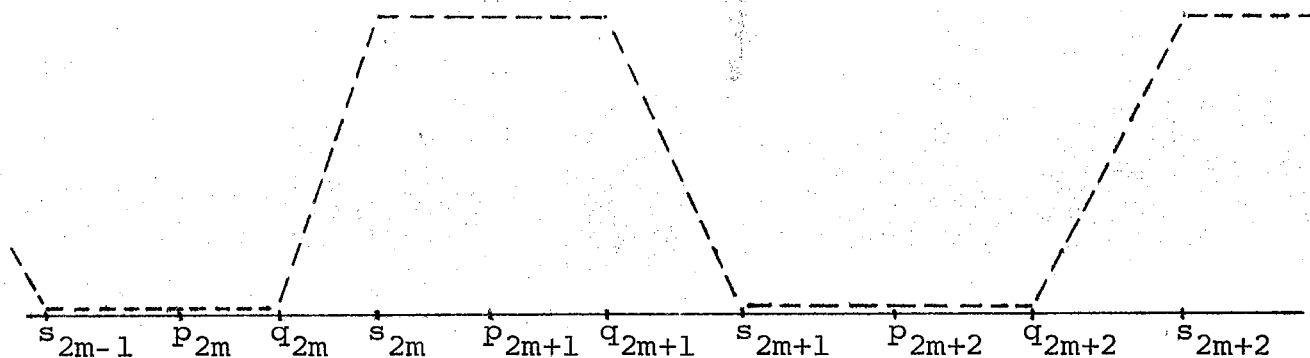


Figure 9.1

Consider the extension $\beta(f)$ of f . We will show that $\beta(f)$ takes only the values 0 and 1 on F . The zero-set $Z(\beta(f))$ is a closed set containing the subsequence $\{p_{2k+1}\}$. Thus, $Z(\beta(f))$ meets A^* . Further, any limit point of $\{p_{2k+1}\}$ is in $cl_{\beta A} U$ and therefore is not in H . Hence, $Z(\beta(f))$ meets F .

By a similar argument, $\beta(f)^{\leftarrow}(1)$ also meets F . Now suppose that a is a point of A^* such that $\beta(f)(a)$ belongs to $(0,1)$. Then a is a limit point of $f^{\leftarrow}(0,1)$ and we also have that

$$f^{\leftarrow}(0,1) = \bigcup \{(q_k, s_k) : k = 1, 2, \dots\} \subset V$$

so that a belongs to $cl_{\beta A} V$. But then a is not in F so that $\beta(f)[F] = \{0,1\}$, and F is not connected. $\blacksquare$

9.14. The previous result was also obtained by R. G. Woods in his thesis. Woods also showed that if n is greater than 1, then the growth of $\mathbb{R}^n$ is a decomposable continuum.

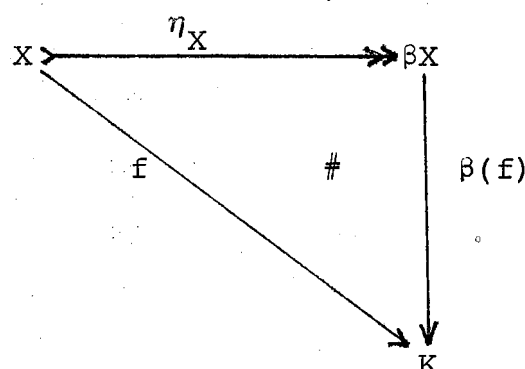
In his 1971 paper, R. F. Dickman, Jr. characterizes the locally compact connected metric spaces for which the growth will be an indecomposable continuum.

The space A^* is also the subject of M. E. Rudin's 1970 paper.

CHAPTER TEN:

 βX IN CATEGORICAL PERSPECTIVE

10.1. Extensions of topological structures abound. In Chapter 1 we investigated the extensions βX and νX . Metric and uniform completions are also well known constructions, which, although not strictly topological, are similar. All satisfy a factorization property similar to the requirement that a mapping f of X to a compact space K extend to βX :



In the first portion of this chapter, we will relate the existence of such a diagram to properties of the appropriate classes of spaces. A more complete treatment of this area is given in H. Herrlich's 1969 notes.

In the second part of the chapter, we will place the familiar construction of the adjunction space in a categorical context. We will apply our findings to the construction of certain Stone-Čech compactifications.

In the third part of the chapter, we will examine the class of perfect mappings from a categorical standpoint and consider classes of spaces which are inversely preserved under perfect

mappings.

In the final portion of the chapter, we will consider projective spaces and projective covers. This final topic will serve to relate the Stone-Čech compactification to Boolean algebras and the class of extremely disconnected spaces.

Though much of the language of the chapter will be categorical in nature, most of the proofs will be topological. Much of the chapter is based on the 1970 notes of S. P. Franklin. A comprehensive survey of the area is also provided in the 1971 paper of H. Herrlich.

For much of the chapter, we will be dealing with Hausdorff spaces and it will be necessary to suspend the presumption that all spaces mentioned are completely regular.

CATEGORIES AND FUNCTORS

10.2. All the categories to be considered here will be made up of a class of spaces together with the mappings between spaces belonging to the class. For example, we will be speaking of the categories of completely regular spaces, of compact spaces, of realcompact spaces, et cetera. It is helpful when considering the following definition to keep such examples in mind.

A category C consists of two classes, a class oC called the objects of C and a class called the morphisms of C , together with the following axioms which link the two classes:

- (1) The composition $h \circ g \circ f$ of three morphisms is defined whenever the compositions $h \circ g$ and $g \circ f$

are defined.

- (2) Composition of morphisms is associative, i.e.
 $(h \circ g) \circ f = h \circ (g \circ f)$ and both compositions are defined if either is defined.
- (3) There is a bijection which assigns to each object X an identity morphism 1_X , and for each morphism f there are two identity morphisms 1_X and 1_Y such that $f = f \circ 1_X$ and $f = 1_Y \circ f$.

10.3. The objects in many common categories are sets together with an algebraic or topological structure. The appropriate morphisms are usually just the structure preserving functions between the sets. Our examples will be of this type.

EXAMPLES:

- (a) Any class of topological spaces together with the continuous functions between them form a category. Most of our discussion will center around the category where the spaces are Hausdorff.
- (b) The classes of all sets and functions form a category.
- (c) Any of the classes of algebraic objects, such as groups, rings, or vector spaces will form a category when the class of morphisms is taken to be the appropriate class of homomorphisms.
- (d) Topological structure can be combined with an algebraic structure to yield such examples as the category of topological groups or topological vector spaces. For instance, a topological group is a group $(G, +)$ together with a topology τ on G such that the group operation $+$ from $G \times G$ to G and the formation

of inverses are both continuous. The appropriate morphisms are the continuous group homomorphisms.

(e) The class of Boolean algebras together with Boolean algebra homomorphisms forms a category.

10.4. A category $\mathcal{B}$ is a subcategory of a category $\mathcal{C}$ if every object and morphism of $\mathcal{B}$ is also an object or morphism of $\mathcal{C}$. The subcategory $\mathcal{B}$ is said to be a full subcategory if every morphism in $\mathcal{C}$ between two objects of $\mathcal{B}$ is also a morphism in $\mathcal{B}$. We will consider two examples of non-full subcategories below, otherwise, all of the subcategories which we will discuss will be presumed to be full. Because our examples consist mainly of subclasses of topological spaces together with all the mappings between them, this will not be a serious restriction.

EXAMPLES:

(a) Throughout most of the preceding chapters, we have been restricting our attention to the full subcategory of the category of all topological spaces which consists of all completely regular spaces and all mappings between them. In the present chapter we will be concerned mainly with subcategories of the category of Hausdorff spaces.

(b) The category of all complete Boolean algebras together with all complete Boolean algebra homomorphisms is a subcategory of the category of all Boolean algebras and all homomorphisms. Note that one must verify that the composition of complete homomorphisms is complete in order to establish the previous statement. We will show that this provides an example of a

non-full subcategory. Let L be the Boolean algebra of all subsets of $\beta\mathbb{N}$ and consider the inclusion of $\text{CO}(\beta\mathbb{N})$ into L . The algebra L is clearly complete and $\text{CO}(\beta\mathbb{N})$ is complete because $\beta\mathbb{N}$ is both zero-dimensional and extremely disconnected. (Proposition 2.5.) However, in L the supremum is merely the union of the sets, so that the inclusion is clearly not a complete homomorphism.

(c) The category of all Abelian topological groups and all continuous homomorphisms is a subcategory of the category of topological groups.

10.5. Let $f : A \rightarrow B$ be a morphism in a category C . A morphism $r : B \rightarrow A$ such that $f \circ r = 1_B$ is called a right inverse of f . A left inverse of f is a morphism $\iota : B \rightarrow A$ such that $\iota \circ f = 1_A$. An isomorphism is a morphism which has both a left and a right inverse. If f is an isomorphism, then the following string of equalities shows that the inverses r and ι of f must be equal:

$$\iota = \iota \circ 1_B = \iota \circ (f \circ r) = (\iota \circ f) \circ r = 1_A \circ r = r.$$

Thus, if f is an isomorphism, we will write $f^{\leftarrow} = \iota = r$ and refer to $f^{\leftarrow}$ as the inverse of f . There are many familiar examples of isomorphisms.

EXAMPLES:

(a) We will mainly be considering the four categories of topological spaces obtained by specifying that the objects be Hausdorff, completely regular, compact, or realcompact. Since

each of these properties is a topological invariant, the isomorphisms in each category are the homeomorphisms.

(b) In the category of metric spaces and uniformly continuous maps, the isomorphisms are the isometrics, i.e. the distance preserving homeomorphisms. Note therefore that this is a non-full subcategory of the category of metric spaces and all mappings between metric spaces. This situation arises because the property of being a metric space is not a topological invariant. However, the property of being metrizable is a topological invariant and in the category of metrizable spaces, the isomorphisms are the homeomorphisms.

(c) In the category of groups, the isomorphisms are the bijective homomorphisms.

10.6. A subcategory $\mathcal{B}$ of $\mathcal{C}$ is a replete subcategory of $\mathcal{C}$ if every object A of $\mathcal{C}$ which is isomorphic to an object B of $\mathcal{B}$ must belong to $\mathcal{B}$ as does the isomorphism. Thus, any subcategory of the category of topological spaces which is obtained by specifying that the objects possess a topological invariant is replete. For this reason, all of the subcategories which we will discuss will be presumed to be replete.

However, just to note that there are subcategories which fail to be replete, we observe that two of our previous examples are not replete.

EXAMPLES:

(a) The subcategory of metric spaces and uniformly continuous maps is not replete because two metric spaces can be topologically

isomorphic without the homeomorphism being an isometry.

(b) The subcategory of complete Boolean algebras and complete homomorphisms is not replete. Observe that if the injection discussed in Example 10.4(b) is thought of as a homomorphism onto its image, then neither it nor its image is complete.

10.7. A functor from a category $\mathcal{U}$ to a category $\mathcal{C}$ is a rule F which assigns to each object A and morphism f of $\mathcal{U}$ an object FA and morphism $F(f)$ such that:

- (1) F preserves identities, i.e. $F(1_A) = 1_{FA}$,
- (2) F preserves composition, i.e. if $f \circ g$ is defined in $\mathcal{U}$, then $F(f) \circ F(g)$ is defined in $\mathcal{C}$ and is equal to $F(f \circ g)$.

There are many examples of functors, several of which we are already acquainted with.

EXAMPLES:

(a) For every set S , let DS be the topological space obtained by putting the discrete topology on S . Then if $f : S \rightarrow T$ is a function between sets, if $D(f)$ is the same function regarded as a mapping of DS to DT , then D is a functor from the category of sets to the category of topological spaces.

(b) For any topological space X , let UX be the points or underlying set of X and for a mapping, let U "forget" the continuity of the map leaving only a function. Then U is a functor from the category of topological spaces to that of sets.

U is frequently called a forgetful functor or underlying set functor.

(c) In Sections 1.52 and 1.55 we saw that the Stone-Čech compactification and the Hewitt-Nachbin realcompactification are both functorial, i.e. both can be used to define a functor. It is also easy to see that the completely regular space ρX described in Theorem 1.6 can be used to define a functor from the category of all topological spaces to the category of completely regular spaces. We will see that these three functors have much in common. These and similar functors will be the principal objects of investigation in this chapter.

10.8. The definition of functor requires that a functor preserves compositions, i.e. that $F(f \circ g) = F(f) \circ F(g)$. If the definition is altered by requiring that compositions be reversed, i.e. that $F(f \circ g) = F(g) \circ F(f)$, then F is called a contravariant functor.

EXAMPLES:

(a) Our discussion in Section 2.10 shows that the assignment of the Stone space to a Boolean algebra and the mapping of the Stone spaces induced by a Boolean algebra homomorphism defines a contravariant functor from the category of Boolean algebras and homomorphisms to the category of compact totally disconnected spaces and the mappings between them.

(b) If X is any set, let $\mathcal{P}X$ denote the power set of X . Then for any function $f : X \rightarrow Y$ between two sets, define $\mathcal{P}(f)$ to be the function $f^{\leftarrow} : \mathcal{P}Y \rightarrow \mathcal{P}X$. Then $\mathcal{P}$ is a contravariant functor from the category of sets to itself and is called the

power set functor. Observe that if X and Y are topological spaces, then the statement that f is continuous is a statement about the image of f under the power set functor.

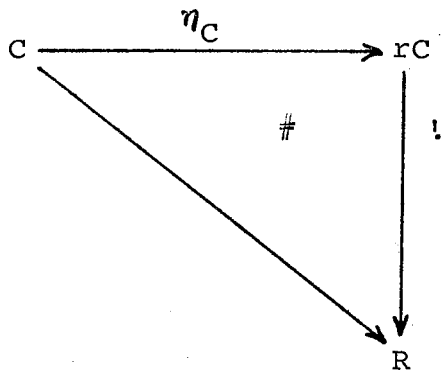
10.9. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be morphisms in a category $\mathcal{C}$. If $F : \mathcal{C} \rightarrow \mathcal{C}$ is a functor and $f \circ g = 1_B$ and $g \circ f = 1_A$, we must have that $F(f) \circ F(g) = 1_{FB}$ and $F(g) \circ F(f) = 1_{FA}$. A similar statement can be made for a contravariant functor, so that we have verified the

PROPOSITION:

A functor or contravariant functor preserves isomorphisms.

REFLECTIVE SUBCATEGORIES OF THE CATEGORY OF HAUSDORFF SPACES

10.10. We now consider a particular type of functor which includes the Stone-Ćech compactification and the Hewitt-Nachbin realcompactification as examples. A functor r from a category $\mathcal{C}$ to a subcategory $\mathcal{R}$ of $\mathcal{C}$ is a reflective functor if there is a morphism $\eta_C : C \rightarrow rC$ and every morphism from C to an object R of $\mathcal{R}$ factors uniquely through rC via η_C so that the following diagram commutes:



If $r : \mathcal{C} \rightarrow \mathcal{R}$ is a reflective functor, the subcategory $\mathcal{R}$ is called a reflective subcategory. The object rC is called the reflection of C in $\mathcal{R}$. The symbol "!" indicates that the morphism is required to be unique.

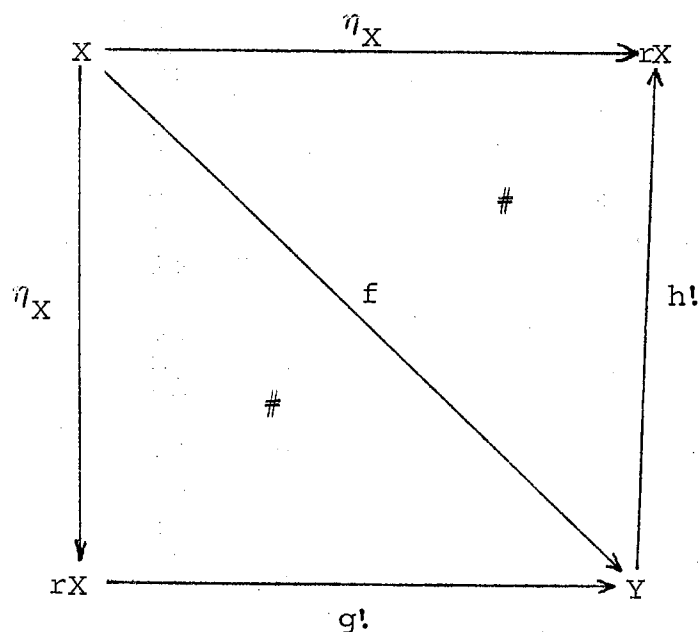
We have verified in Chapter 1 that βX and νX can be viewed as the images of objects under reflective functors. In this chapter, we will place these two reflective functors in a common categorical context and describe the topological characteristics of the categories of compact and realcompact spaces which give rise to their similarity. We will also reexamine the role of the category of completely regular spaces from a categorical standpoint.

10.11. We will find that many of the properties of reflective functors and subcategories have already been verified in the case of β and the category of compact spaces. The following result is such a property.

PROPOSITION:

The reflection of an object is unique up to isomorphism.

Proof: Let $\mathcal{R}$ be a reflective subcategory of $\mathcal{C}$. Let rX be the reflection of an object X . Let Y be any object of $\mathcal{R}$ and let $f : X \rightarrow Y$ be a morphism such that any morphism from X to an object of $\mathcal{R}$ factors uniquely through Y via f . Then we have the following commutative diagram in which the morphisms h and g exist and are unique by the reflective properties of Y and rX , respectively. Because rX is an object of $\mathcal{R}$, the morphism η_X must factor uniquely through η_X . One such factorization is l_{rX} and the diagram shows that $h \circ g$



is another factorization. Thus, we must have $h \circ g = l_{rX}$ since there can be only one factorization. Repeating the argument with rX and Y interchanged will yield $g \circ h = l_Y$. Hence, rX and Y are isomorphic. $\square$

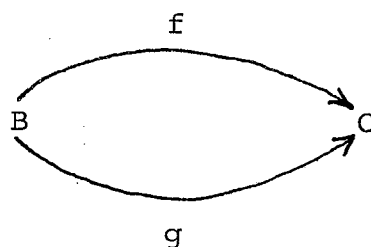
10.12. If R is an object of $\mathcal{R}$, any morphism will factor through the identity l_R . Hence, the following corollary is immediate:

COROLLARY:

Any object in a reflective subcategory is isomorphic to its reflection.

10.13. In Chapter 1, we saw that the factorization of a mapping of X through the mapping $\eta_X : X \rightarrow \beta X$ is unique because $\eta_X[X]$ is dense in βX and two maps which agree on a dense subspace of a Hausdorff space are equal. We will see that η_X is a specific instance of a particular type of morphism. A morphism $e : A \rightarrow B$ is an epimorphism if for every pair of

morphisms



the equality $f \circ e = g \circ e$ implies that $f = g$. The double-headed arrow will be used to indicate an epimorphism. A reflective functor r is said to be epi-reflective if the morphism $\eta_X : X \rightarrow rX$ is an epimorphism. Thus, the preceding discussion shows that the mappings having dense images are epimorphisms in the category of Hausdorff spaces. Our previous considerations also show that β is an epi-reflective functor and that the category of compact spaces is an epi-reflective subcategory of the category of completely regular spaces. However, a closer analysis of the category of Hausdorff spaces will show that the compact spaces actually form an epi-reflective subcategory of the category of Hausdorff spaces. Of course, if X fails to be a completely regular space, the mapping η_X must fail to be an embedding since a subspace of a compact space must be completely regular.

10.14. The investigation of epi-reflective subcategories of the category of Hausdorff spaces will require an examination of properties of epimorphisms. The following result shows that one need only show that an epimorphism has a left-inverse in order to show that it is an isomorphism.

PROPOSITION:

An epimorphism with a left inverse is an isomorphism.

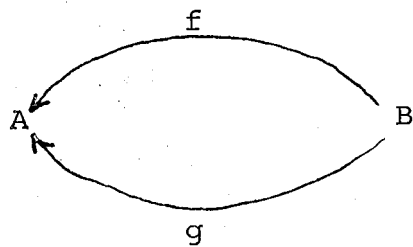
Proof: Let $e : X \twoheadrightarrow Y$ be an epimorphism in a category $\mathcal{C}$ and let $\iota : Y \rightarrow X$ be a left inverse for e . Then we have that:

$$l_Y \circ e = l_Y \circ e \circ l_X = e \circ (\iota \circ e) = (e \circ \iota) \circ e.$$

Now since e is an epimorphism, the equality of $l_Y \circ e$ and $(e \circ \iota) \circ e$ implies that $l_Y = e \circ \iota$ so that ι is also a right inverse of e . Hence, e is an isomorphism. $\blacksquare$

Every concept in category theory has a dual concept which is obtained by reversing the arrows in the appropriate diagram.

The dual of an epimorphism is a monomorphism. A morphism $m : C \rightarrow B$ is a monomorphism if for every pair of morphisms



the equality $m \circ f = m \circ g$ implies that $f = g$. Thus a monomorphism is a morphism that is left-cancellable while an epimorphism is one that is right-cancellable. A double-headed arrow indicates an isomorphism while a tail on the arrow signals a monomorphism.

The relationship between epimorphisms and isomorphisms has its dual for monomorphisms.

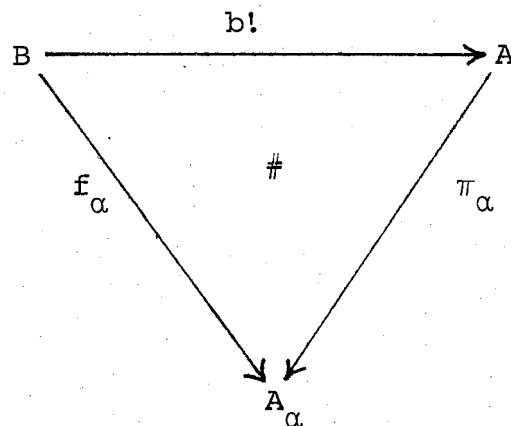
DUAL PROPOSITION:

A monomorphism with a right inverse is an isomorphism.

The preceding results and definitions can be recalled and motivated from the familiar situation in the category of sets and functions. A function is an epimorphism (resp. monomorphism) if and only if it is onto (resp. one-to-one). Further, a function is onto (resp. one-to-one) exactly when it has a right (resp. left) inverse. Thus, the concepts of epimorphism and monomorphism are generalizations of the familiar onto and one-to-one functions, respectively.

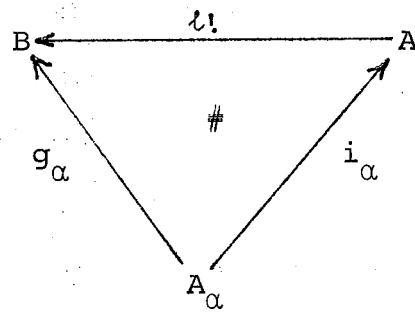
We will frequently not mention the dual of various concepts which we will consider, although a further discussion of duals will be carried out in the exercises.

10.15. We will, however, need to discuss both the concept of the product and its dual, the coproduct. Let $\{A_\alpha : \alpha \in Q\}$ be any family of objects of a category C . Then an object A of C together with a family of morphisms $\{\pi_\alpha : A \rightarrow A_\alpha\}$ is called the product of the family $\{A_\alpha\}$ if for every object B of C and family of morphisms $\{f_\alpha : B \rightarrow A_\alpha\}$, there is a unique morphism $b : B \rightarrow A$ such that the diagram



commutes for each α . The product of the family $\{A_\alpha\}$ will be denoted by $\times A_\alpha$. In the category of topological spaces, the product is the usual topological product.

Dually, an object A of $\mathcal{C}$ together with a family of morphisms $\{i_\alpha : A_\alpha \rightarrow A\}$ is called the coproduct of the family $\{A_\alpha\}$ if for every object B and family of morphisms $\{g_\alpha : A_\alpha \rightarrow B\}$ there exists a unique morphism $\iota : A \rightarrow B$ such that the diagram



commutes for each α . The coproduct of the family $\{A_\alpha\}$ will be denoted by $\oplus A_\alpha$. In the category of topological spaces, the coproduct is the disjoint topological sum.

It may happen that products or coproducts exist in both a category $\mathcal{C}$ and a subcategory $\mathcal{Q}$ of $\mathcal{C}$, but that the constructions of the product or coproduct in the two categories are different. For example, we have seen that the compact spaces form a reflective subcategory of the category of completely regular spaces. It is easy to see that the disjoint topological sum will be the coproduct for completely regular spaces, but not for compact spaces. However, the following result shows that coproducts do exist in the category of compact spaces.

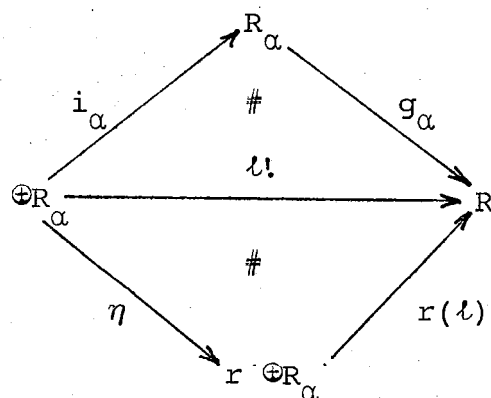
PROPOSITION:

If $r : \mathcal{C} \rightarrow \mathcal{R}$ is a reflective functor, then the coproduct

in $\mathcal{R}$ of a family $\{R_\alpha\}$ of $\mathcal{R}$ is $r(\oplus R_\alpha)$ where $\oplus R_\alpha$ is the
coproduct of the family in $\mathcal{C}$.

Proof: To show that $r(\oplus R_\alpha)$ is the coproduct in $\mathcal{R}$ of the family $\{R_\alpha\}$, we must exhibit a family of morphisms $\{j_\alpha : R_\alpha \rightarrow r(\oplus R_\alpha)\}$ such that if R is any object of $\mathcal{R}$ and $\{g_\alpha : R_\alpha \rightarrow R\}$ is any family of morphisms, then there exists a unique morphism $h : r(\oplus R_\alpha) \rightarrow R$ such that $g_\alpha = h \circ j_\alpha$ for each α .

Let $\{i_\alpha : R_\alpha \rightarrow \oplus R_\alpha\}$ be the coproduct morphisms in $\mathcal{C}$. Then there exists a unique morphism l such that $g_\alpha = l \circ i_\alpha$ for each α . Since R is an object of $\mathcal{R}$ and $r(\oplus R_\alpha)$ is the reflection of $\oplus R_\alpha$, we have the following diagram:



Thus, we have $g_\alpha = r(l) \circ \eta \circ i_\alpha$ for each α . Hence, the family $\{\eta \circ i_\alpha\}$ is the required family of morphisms and $r(l)$ is the unique morphism which must exist to show that $r(\oplus R_\alpha)$ is the coproduct in $\mathcal{R}$. $\square$

The proposition shows that the coproduct of a family of compact spaces in the category of compact spaces is the Stone-Čech compactification of their disjoint topological sum. Hence,

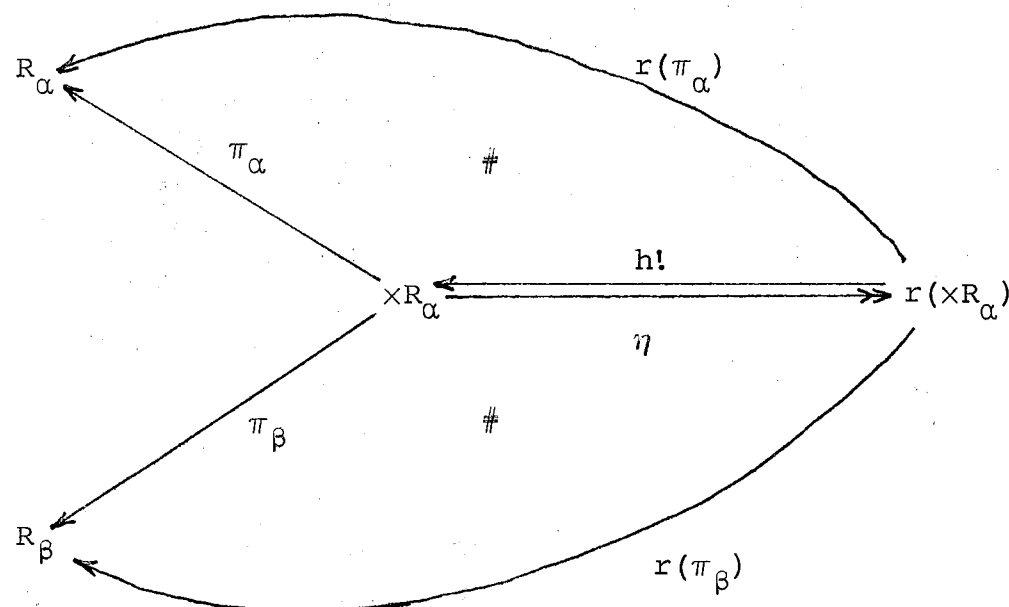
$\beta \mathbb{N}$ can be interpreted as the coproduct of a family of countably many singletons.

10.16. However, we now show that the product in $\mathcal{C}$ of a family in $\mathcal{R}$ is also the product in $\mathcal{R}$.

PROPOSITION:

If $\mathcal{R}$ is an epi-reflective subcategory of $\mathcal{C}$, then the product in $\mathcal{C}$ of objects of $\mathcal{R}$ belongs to $\mathcal{R}$.

Proof: Let $\{R_\alpha\}$ be a family of objects of $\mathcal{R}$. We show that $\times R_\alpha$ is in $\mathcal{R}$ by showing that it is isomorphic to its reflection $r(\times R_\alpha)$. We have the following diagram:



The morphism $r(\pi_\alpha)$ exists and satisfies $\pi_\alpha = r(\pi_\alpha) \circ \eta$ for each α because $r(\times R_\alpha)$ is the reflection of $\times R_\alpha$. Then the definition of product implies that there exists a unique morphism h such that $r(\pi_\alpha) = \pi_\alpha \circ h$. We will show that h and η are inverses of each other. For every α , $r(\pi_\alpha) \circ \eta$ is morphism from $\times R_\alpha$ to R_α . Thus, from the defining property of the

product, there exists a unique morphism $b : \times R_\alpha \rightarrow \times R_\alpha$ such that $r(\pi_\alpha) \circ \eta = \pi_\alpha \circ b$ for every α . However, we have that

$$r(\pi_\alpha) \circ \eta = \pi_\alpha \circ (h \circ \eta)$$

and also that

$$r(\pi_\alpha) \circ \eta = \pi_\alpha \circ 1_{\times R_\alpha}.$$

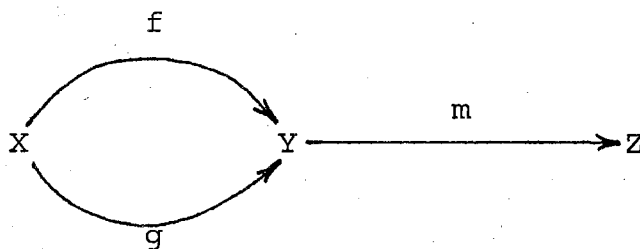
Hence, we must have $h \circ \eta = 1_{\times R_\alpha}$. Thus, η is an epimorphism with a left inverse and is therefore an isomorphism (Proposition 10.14). $\parallel$

10.17. Further description of epi-reflective subcategories of the category of Hausdorff spaces will rely on the categorical properties of three types of mappings: one-to-one maps, maps with dense range, and closed embeddings. We first show that the one-to-one maps are exactly the monomorphisms.

PROPOSITION:

The monomorphisms in the category of Hausdorff spaces are the one-to-one mappings.

Proof: We first show that a one-to-one mapping is a monomorphism. In the following diagram, let m be a one-to-one map and assume that $m \circ g = m \circ f$.

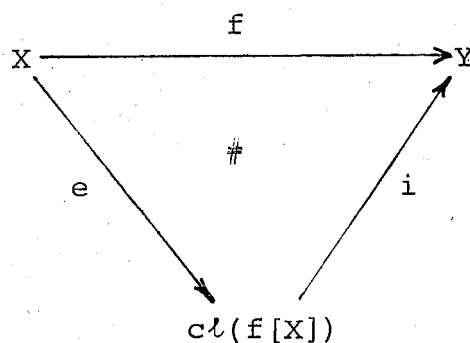


Then for every x in X , we have $m(f(x)) = m(g(x))$. Since m is one-to-one, this implies that $f(x) = g(x)$. Hence, $f = g$ and m is a monomorphism.

Conversely, assume in the diagram that m is a monomorphism. Let y_1 and y_2 be distinct points of Y and let X be the singleton space. Let f send the point of X to y_1 and g send the point of X to y_2 . Because m is a monomorphism, $m \circ f \neq m \circ g$. Hence, $m(y_1) \neq m(y_2)$ and m is one-to-one. $\square$

Note that the preceding proof does not use the Hausdorff hypothesis and so we have actually shown that the one-to-one maps are also the monomorphisms in the larger category of all topological spaces.

10.18. Any map f between two spaces can be factored through the closure of its image as is illustrated by the diagram:



In Theorem 10.21 we will see that this factorization will be important in the characterization of epi-reflective subcategories. We have already seen that if Y is Hausdorff, then e is an epimorphism. We now see that every epimorphism in the category of Hausdorff spaces is of this form, i.e. is a map with dense

range.

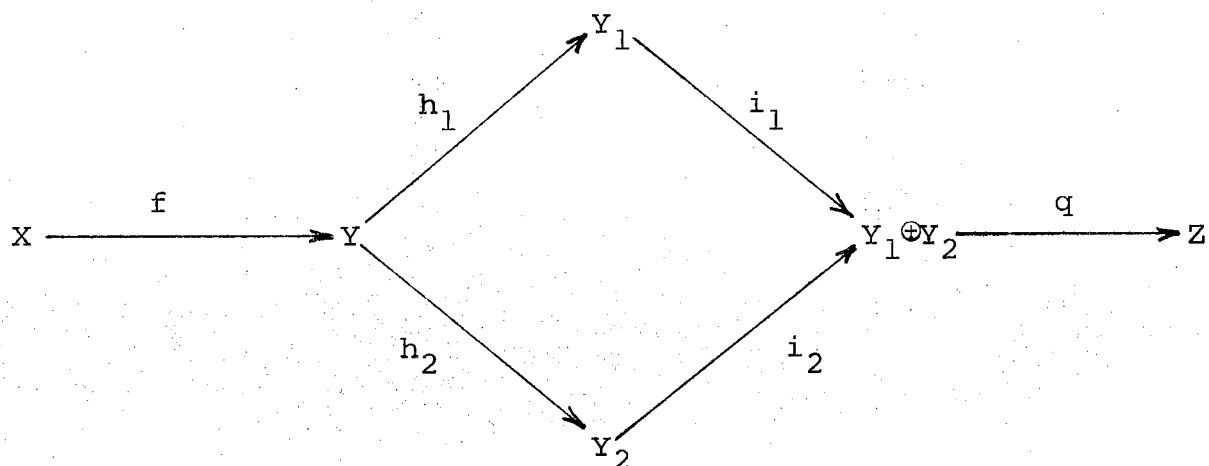
PROPOSITION:

The epimorphisms in the category of Hausdorff spaces are the mappings with dense range.

Proof: We have already seen that the maps with dense range are epimorphisms because two maps which agree on a dense subspace of a Hausdorff space are equal.

Conversely, suppose that $f : X \rightarrow Y$ is a mapping such that $f[X]$ is not dense in Y , i.e. such that $\text{cl}(f[X])$ is a proper subspace of Y . We will show that f cannot be an epimorphism by constructing a space Z and two maps ι_1 and ι_2 of Y into Z which agree on $f[X]$ but which are not equal. To construct Z , we first make two disjoint copies of Y by writing $Y_1 = Y \times \{1\}$ and $Y_2 = Y \times \{2\}$. Both Y_1 and Y_2 taken with the product topology are homeomorphic to Y . Let $h_i : Y \rightarrow Y_i$ be defined by $h_i(y) = (y, i)$ for each $i = 1, 2$. Let $Y_1 \oplus Y_2$ be the disjoint sum or coproduct of Y_1 and Y_2 and for $i = 1, 2$ let $i_k : Y_k \rightarrow Y_1 \oplus Y_2$ be the inclusion map. In each of the spaces Y_i , $h_i[\text{cl}(f[X])]$ is a copy of $\text{cl}(f[X])$. Thus, $i_1 \circ h_1[\text{cl}(f[X])] \cup i_2 \circ h_2[\text{cl}(f[X])]$ is the union of two copies of $\text{cl}(f[X])$ which are contained in $Y_1 \oplus Y_2$. We will take Z to be the space obtained by joining Y_1 and Y_2 along the corresponding copies of $\text{cl}(f[X])$. More precisely, let Z be the image of the quotient map q obtained by identifying $i_1 \circ h_1(y) = i_1(y, 1)$ and $i_2 \circ h_2(y) = i_2(y, 2)$ if y belongs to $\text{cl}(f[X])$.

Then we have the following diagram:



Now if x is a point of X , the two maps $i_1 \circ h_1$ and $i_2 \circ h_2$ split the point $f(x)$ in two and the map q joins the halves together again. Thus, we see that $((q \circ i_1 \circ h_1) \circ f)(x) = ((q \circ i_2 \circ h_2) \circ f)(x)$. Hence, $(q \circ i_1 \circ h_1) \circ f = (q \circ i_2 \circ h_2) \circ f$. However, any point lying outside of $\text{cl}(f[X])$ in Y is split by $i_1 \circ h_1$ and $i_2 \circ h_2$ but is not joined again by q . Hence, $q \circ i_1 \circ h_1 \neq q \circ i_2 \circ h_2$.

This would show that f cannot be an epimorphism and would complete the proof except that we have not shown that the construction of Z has kept us within the category of Hausdorff spaces. Hence it remains to show that the quotient space Z is Hausdorff.

Let p and r be distinct points of Z . We will find disjoint neighborhoods of p and r in each of five cases. However, only three of the cases will require distinct arguments. Case 1: p and r both belong to $(q \circ i_1 \circ h_1)[Y \setminus \text{cl}(f[X])]$: Since $\text{cl}(f[X])$ is closed, there exist open sets U_p and U_r of Y

containing $(q \circ i_1 \circ h_1)^{\leftarrow}(p)$ and $(q \circ i_1 \circ h_1)^{\leftarrow}(r)$, respectively, and missing $\text{cl}(f[X])$. Since Y is Hausdorff, there exist disjoint neighborhoods V_p and V_r of $(q \circ i_1 \circ h_1)^{\leftarrow}(p)$ and $(q \circ i_1 \circ h_1)^{\leftarrow}(r)$, respectively. Then $(q \circ i_1 \circ h_1)[V_p \cap U_p]$ and $(q \circ i_1 \circ h_1)[V_r \cap U_r]$ are the required disjoint neighborhoods of p and r .

Case 2: p and r both belong to $(q \circ i_2 \circ h_2)[Y \setminus \text{cl}(f[X])]$:

This is the same as Case 1 with only the index changed.

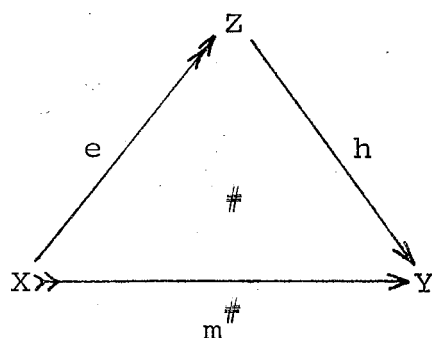
Case 3: p belongs to $(q \circ i_1 \circ h_1)[Y \setminus \text{cl}(f[X])]$ and r belongs to $(q \circ i_2 \circ h_2)[Y \setminus \text{cl}(f[X])]$: This case is easy since the two given sets containing the points are already disjoint open sets.

Case 4: $p \in (q \circ i_1 \circ h_1)[Y \setminus \text{cl}(f[X])]$ and $r = q \circ i_1 \circ h_1(y)$ for some $y \in \text{cl}(f[X])$: Note that the subscript does not matter for r because the two maps must agree on $\text{cl}(f[X])$. There exist disjoint open subsets U and V in Y with $(q \circ i_1 \circ h_1)^{\leftarrow}(p)$ in U and y in V with U missing $\text{cl}(f[X])$. Now p is in $(q \circ i_1 \circ h_1)[U]$ and r belongs to $q(i_1 \circ h_1[V] \cup i_2 \circ h_2[V])$ and these sets are disjoint. But they are also both open since $i_1 \circ h_1[U]$ and $i_1 \circ h_1(V) \cup i_2 \circ h_2(V)$ are both open and saturated in $Y_1 \oplus Y_2$.

Case 5: $p \in (q \circ i_2 \circ h_2)[Y \setminus \text{cl}(f[X])]$ and $r = q \circ i_1 \circ h_1(y)$ for some $y \in \text{cl}(f[X])$: This is the same as Case 4 with an index changed.

Hence, Z is Hausdorff. $\blacksquare$

10.19. When a mapping is factored through the closure of its image, the second factor is a closed embedding. These maps also have a categorical characterization in the category of Hausdorff spaces. A monomorphism $m^\#$ is an extremal monomorphism if whenever $m^\#$ can be factored as illustrated

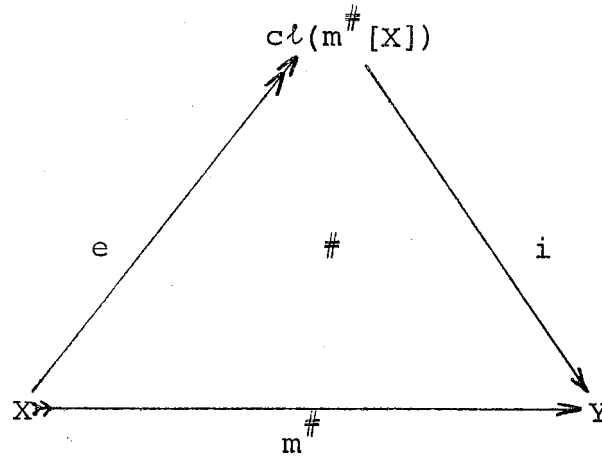


so that e is an epimorphism, then e is an isomorphism. An extremal monomorphism will be indicated by the "double tail" and a "#" on the name of the morphism. In the diagram, the object X is said to be an extremal subobject of Y . We will show that the extremal monomorphisms in the category of Hausdorff spaces are the closed embeddings. Thus, the extremal subobjects in the category of Hausdorff spaces are the closed subspaces.

PROPOSITION:

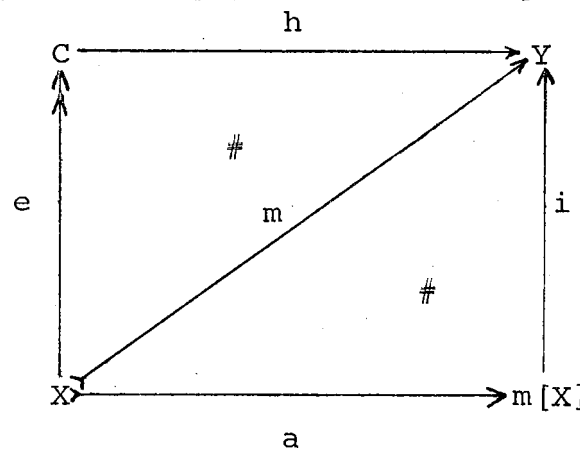
The extremal monomorphisms in the category of Hausdorff spaces are the closed embeddings.

Proof: We first show that an extremal monomorphism $m^\# : X \rightarrow Y$ is a closed embedding. We can factor $m^\#$ through the closure of its range:



Since the range of e is dense, e is an epimorphism. But then e must be an isomorphism since $m^\#$ is an extremal monomorphism. Thus, $m^\#$ is a closed embedding.

Now let $m : X \rightarrow Y$ be a closed embedding. Assume that $m = h \circ e$ is a factorization of m where e is an epimorphism. The map m is a monomorphism because it is one-to-one. Thus, it remains to show that e is an isomorphism. We can also factor m through its image, thus obtaining the diagram:



We will show that the epimorphism e is an isomorphism by obtaining a left-inverse for e . Because e has dense range and $m[X]$ is closed in Y , $h[C]$ is contained in $m[X]$. Thus, if we define $h^\sharp : C \rightarrow m[X]$ by $h^\sharp(x) = h(x)$, we have that $h = i \circ h^\sharp$. But then we also have

$$i \circ h' \circ e = h \circ e = i \circ a$$

where i is a monomorphism. Therefore, $h' \circ e = a$. Since a is an isomorphism, we can write

$$l_X = (a^{-1} \circ h') \circ e.$$

Thus, e is an epimorphism with a left-inverse and is therefore an isomorphism (Proposition 10.14). $\square$

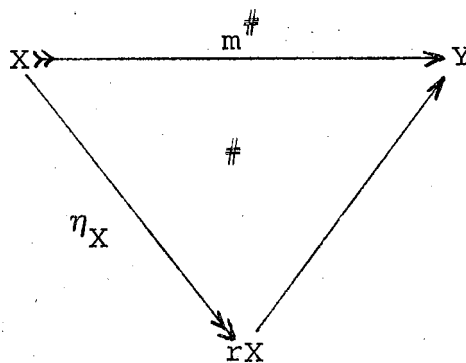
10.20. In Proposition 10.16, we saw that an epi-reflective subcategory is closed under the formation of products. The next result shows that an epi-reflective subcategory is also closed under extremal subobjects.

PROPOSITION:

Epi-reflective subcategories are closed under extremal subobjects. Hence, epi-reflective subcategories of the category of Hausdorff spaces are closed hereditary.

Proof: Let $\mathcal{W}$ be an epi-reflective subcategory of $\mathcal{C}$. Let Y belong to $\mathcal{W}$ and let $m^\# : X \rightarrow Y$ be an extremal monomorphism..

Then we have the following commutative diagram:



But η_X is therefore an isomorphism because $m^\#$ is an extremal monomorphism. $\square$

10.21. The following theorem characterizes the epi-reflective subcategories of the category of Hausdorff spaces as the productive and closed hereditary subcategories. The present form of the theorem appears in the 1965 paper of J. F. Kennison. More general categorical formulations of the theorem appear in the following papers: J. R. Isbell, 1964B; Kennison, 1965; and S. Baron, 1969. Reflections and coreflections in topology are discussed in detail in the 1968 Lecture Notes of H. Herrlich.

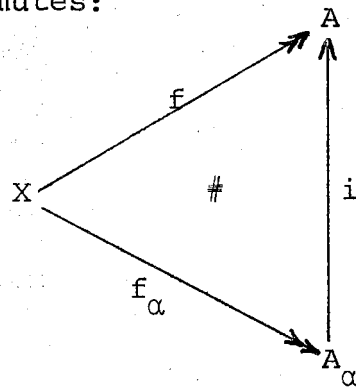
THEOREM:

The epi-reflective subcategories of the category of Hausdorff spaces are those subcategories which are closed under products and closed subspaces.

Proof: The necessity of the conditions follows from Propositions 10.16 and 10.20.

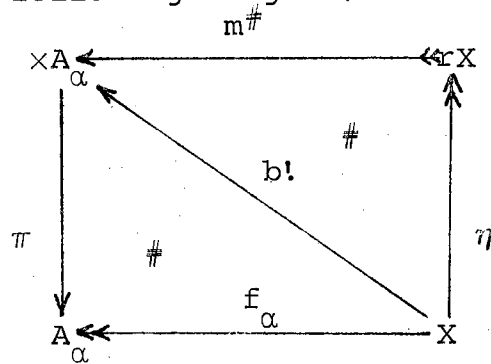
For sufficiency, assume that $\mathcal{C}$ is a productive and closed hereditary subcategory of the category of Hausdorff spaces and let X be a Hausdorff space. We will obtain a space rX in $\mathcal{C}$ which will be the reflection of X . Because X is Hausdorff, there is only a set of pairwise non-homeomorphic Hausdorff spaces which can contain a dense image of X . This follows from the facts that if $e : X \rightarrow Y$ has dense range, then $|Y| \leq 2^{2^{|X|}}$ and there are only a set of topologies on a set of given cardinality. Thus, there exists a set of epimorphisms $\{f_\alpha : X \rightarrow A_\alpha\}$ with A_α in $\mathcal{C}$ such that if $f : X \rightarrow A$ is an epimorphism with A in $\mathcal{C}$,

then there is a homeomorphism i and an index α such that the following diagram commutes:



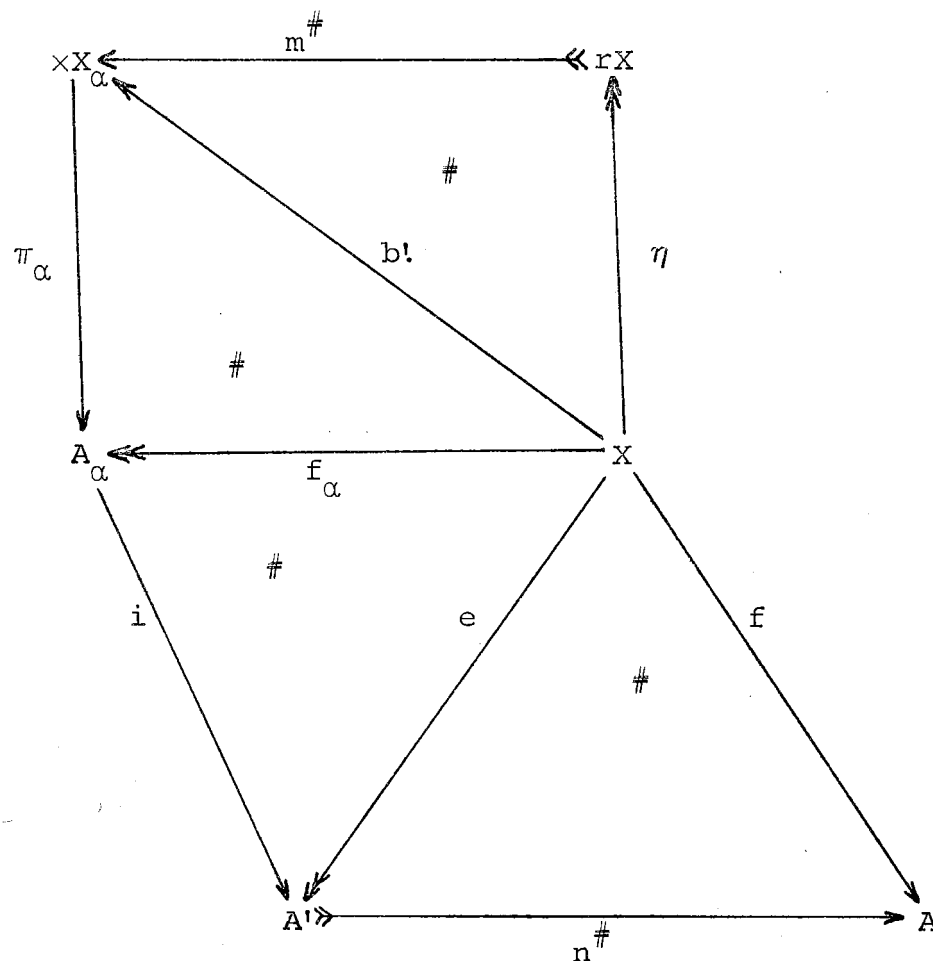
Such a family is called a skeleton of epimorphisms.

Consider the following diagram:



The mapping b exists and is unique from the definition of a product. The maps η and $m^\#$ are obtained by factoring b through the closure of its image. Because rX is a closed subspace of a product of objects of $\mathcal{Q}$, our hypotheses guarantee that rX is in $\mathcal{Q}$.

We will show that rX is the required reflection of X in $\mathcal{Q}$. Let $f : X \rightarrow A$ be a mapping with A belonging to $\mathcal{Q}$. The following diagram illustrates the procedure to construct the factorization of f through rX :



We first factor f through the closure of its image in order to express f as an epimorphism e followed by an extremal monomorphism $n^\#$. Then because the subcategory $\mathcal{U}$ is closed under extremal subobjects, A' belongs to $\mathcal{U}$. Now because $\{f_\alpha\}$ is a skeleton of epimorphisms, there is an isomorphism i between some A_α and A' such that $e = i \circ f_\alpha$.

It remains to show that $n^\# \circ i \circ \pi_\alpha \circ m^\# \circ \eta$ is a unique factorization of f . We can write

$$n^\# \circ i \circ \pi_\alpha \circ m^\# \circ \eta = n^\# \circ i \circ \pi_\alpha \circ b$$

by the factorization of b . From the defining condition of a

product, we have

$$n^{\#} \circ i \circ \pi_{\alpha} \circ b = n^{\#} \circ i \circ f_{\alpha}.$$

The factorization of e through the skeleton member f_{α} yields

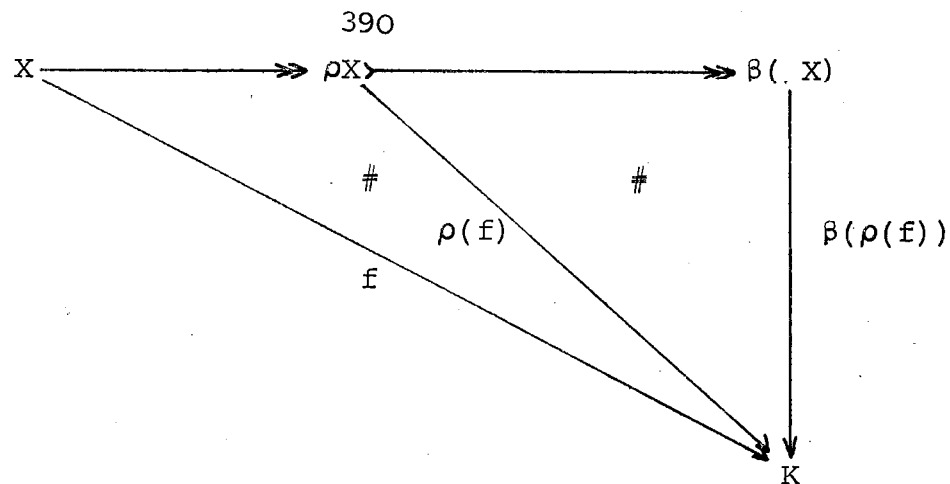
$$n^{\#} \circ i \circ f_{\alpha} = n^{\#} \circ e = f.$$

Finally, the factorization is unique because η is an epimorphism. $\blacksquare$

10.22. Since topological products and closed subspaces of compact spaces are compact, the preceding theorem shows that the category of compact spaces is epireflective in the category of Hausdorff spaces. We must reconcile this fact with the fact that we have defined the Stone-Čech compactification only for completely regular spaces.

Observe that since the category of completely regular spaces is productive and hereditary, the theorem also shows that the completely regular spaces form an epi-reflective subcategory of the Hausdorff spaces. In Corollary 1.8, we saw that the space ρX is the completely regular reflection of a space X . Actually, we showed that ρX can be defined for an arbitrary space, a fact which will be placed in categorical context in Exercise 10A.

Now let X be a Hausdorff space and let f map X to a compact space K . Then we have the following diagram:



Thus, the reflection of a Hausdorff space X is the Stone-Čech compactification of ρX . Note that if we replace K by a realcompact space, then we can repeat the diagram with β replaced by ν to show that the realcompact spaces are also epi-reflective in the category of Hausdorff spaces.

We can carry the similarity between compact and realcompact spaces further. Recall that the compact spaces are just the closed subspaces of products of copies of the unit interval I . In 1948, E. Hewitt proved the following analogous result for realcompact spaces with I replaced by the real line $\mathbb{R}$.

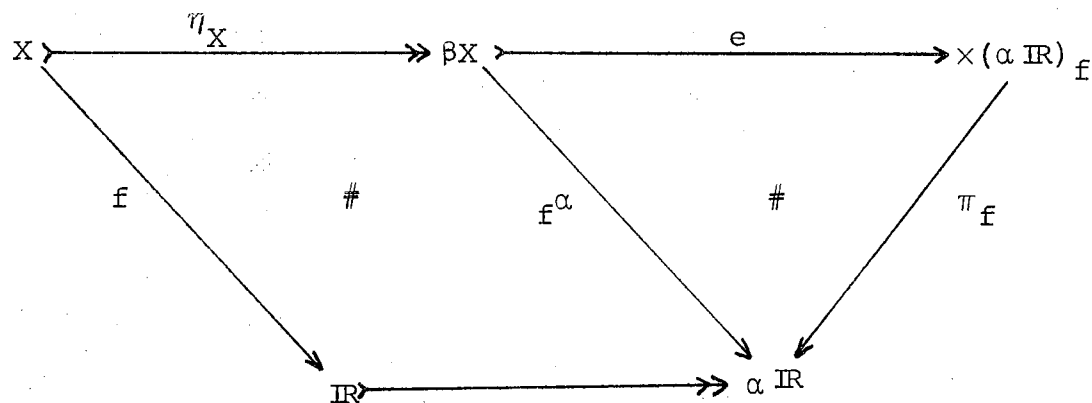
THEOREM: (Hewitt)

The realcompact spaces are the closed subspaces of products of copies of $\mathbb{R}$.

Proof: We have already shown that the realcompact spaces form an epi-reflective subcategory of the category of Hausdorff spaces and that $\mathbb{R}$ is realcompact (Example 1.54). Thus, a closed subspace of a product of real lines must be realcompact.

Now assume that X is realcompact. For each f in $C(X)$, let f^α denote the extension of f to a mapping of βX into the one-point compactification $\alpha\mathbb{R}$ of $\mathbb{R}$. The family of all

such extensions separates points and points and closed sets of βX . Hence, the Embedding Lemma 1.5 shows that βX can be embedded as a subspace of a product of copies of $\alpha \mathbb{R}$ indexed by $C(X)$ and because βX is compact, its image under the embedding is closed. Thus, we have the following diagram:



The points of ηX correspond to the subspace of $e[X]$ obtained by restricting the coordinates to the points of $\mathbb{R}$ (Section 1.53). Thus, ηX is a restriction of the closed subspace $e[\beta X]$ to a product of real lines. Hence, $X = \eta X$ is homeomorphic to a closed subspace of a product of real lines. $\square$

10.23. Now assume that $\mathcal{R}$ is an epi-reflective subcategory of the Hausdorff spaces and that I or $\mathbb{R}$ is an object of $\mathcal{R}$. Then Theorem 10.21 shows that $\mathcal{R}$ must contain all compact or all realcompact spaces, respectively. Hence, either space is contained in a smallest epi-reflective subcategory of the Hausdorff spaces which consists of all closed subspaces of products of copies of the space. More generally, if $\mathcal{G}$ is any subcategory of the category of Hausdorff spaces, then the intersection of the productive and closed hereditary subcategories containing $\mathcal{G}$ is

also productive and closed hereditary. Note that the Hausdorff spaces form such a category so that the intersection is non-empty. Hence, $\mathcal{Q}$ is contained in a smallest epi-reflective subcategory of the Hausdorff spaces. We have verified the

PROPOSITION:

Any subcategory $\mathcal{Q}$ of the category of Hausdorff spaces is contained in a smallest epi-reflective subcategory of the Hausdorff spaces and the objects of this subcategory are the closed subspaces of products of spaces in $\mathcal{Q}$.

The epi-reflective subcategory described in the previous theorem is denoted by $\text{HAUS}(\mathcal{Q})$ and is called the epi-reflective hull of $\mathcal{Q}$. $\text{HAUS}(\mathcal{Q})$ is said to be generated by $\mathcal{Q}$ and is said to be simply generated if $\mathcal{Q}$ has only a single object.

10.24. The systematic study of simply generated epi-reflective categories was begun in the 1958 paper of R. Engelking and S. Mrówka. The 1966 and 1968 papers of Mrówka continue the investigation as does the thesis of R. Blefko. Let E be a topological space. In the terminology introduced by Engelking and Mrówka, E-completely regular spaces are defined to be subspaces of products of copies of E and closed subspaces of such products are defined to be E-compact spaces. If E is a Hausdorff space, then the E-compact spaces form a simply generated subcategory of the category of Hausdorff spaces. The E-completely regular spaces are precisely those spaces which are densely embedded in their "E-compactifications". Thus, the E-completely regular spaces

are related to the E -compact spaces in the same way that the completely regular spaces are related to the compact spaces. The category of E -compact spaces has been extensively investigated for several specific choices of E .

EXAMPLES:

(a) The I -completely regular spaces are the completely regular spaces and the I -compact spaces are the compact spaces.

(b) The IR -completely regular spaces are the completely regular spaces and the IR -compact spaces are the realcompact spaces.

(c) Let $\mathbb{2}$ denote the two point discrete space. In his 1955 paper, B. Banaschewski showed that the $\mathbb{2}$ -completely regular spaces are the zero-dimensional spaces and the $\mathbb{2}$ -compact spaces are the compact zero-dimensional spaces. The associated reflective functor is called ζ .

(d) The category of IN -compact spaces has been the subject of much recent investigation and there are several unsolved problems in this area. In 1967B, H. Herrlich showed that a zero-dimensional space is IN -compact if and only if every ultrafilter of clopen sets which has the countable intersection property is fixed. This result also appears in K. Chew's 1970 paper. The IN -compact spaces are contained between two subcategories of realcompact spaces. Because products and subspaces of zero-dimensional spaces are zero-dimensional, Theorem 10.21 shows that the category of zero-dimensional realcompact spaces is epi-reflective. However, it is not known if this category is simply

generated. It is easy to see that every IN -compact space is zero-dimensional and realcompact. The converse would show that the category of zero-dimensional realcompact spaces is simply generated, however, the converse is false. In his thesis, P. Nyikos describes a zero-dimensional realcompact space Δ which fails to be IN -compact. A very brief account of Δ and its significance is also given in his 1971 paper. The space Δ is a highly complicated example originally introduced by P. Roy in his 1962 and 1968 papers to show that small inductive dimension and Lebesgue covering dimension are not the same in metric spaces.

The category of IN -compact spaces contains the category of strongly zero-dimensional realcompact spaces (Sections 3.33-3.34). This result was first shown by H. Herrlich in his 1967B paper. It is not known if this containment is proper. In fact, it is not yet known if the category of strongly zero-dimensional spaces is even finitely productive or closed hereditary.

The epi-reflective functor associated with the category of IN -compact spaces is denoted by ν .

(e) The categories of Q -compact spaces and IP -compact spaces are both identical with the category of IN -compact spaces. We will sketch a proof of this fact, showing the following containments:

$$\text{HAUS}(\text{IN}) \subseteq \text{HAUS}(\text{Q}) \subseteq \text{HAUS}(\text{IP}) \subseteq \text{HAUS}(\text{IN}).$$

The first containment is clear since IN is a closed subspace of Q .

The proof that $\text{HAUS}(Q)$ is contained in $\text{HAUS}(\mathbb{I}\mathbb{P})$ is based on the fact that any countable dense subset of $\mathbb{I}\mathbb{R}$ is homeomorphic to Q . This fact is then used to embed Q as the diagonal of a product of copies of $\mathbb{I}\mathbb{P}$. Let Q_D denote the dyadic rationals. For each i in $\mathbb{I}\mathbb{P}$, put $\mathbb{I}\mathbb{P}_i = \mathbb{I}\mathbb{R} \setminus (Q_D \cup \{i\})$. Then $\bigcap \{\mathbb{I}\mathbb{P}_i : i \in \mathbb{I}\mathbb{P}\} = Q \setminus Q_D$ is homeomorphic to Q and each $\mathbb{I}\mathbb{P}_i$ is a copy of $\mathbb{I}\mathbb{P}$. Hence, the diagonal of $\times \{\mathbb{I}\mathbb{P}_i : i \in \mathbb{I}\mathbb{P}\}$ is homeomorphic to Q and is closed since the diagonal of a product of Hausdorff spaces is closed [D, p. 138]. Hence, Q is homeomorphic to a closed subspace of a product of copies of $\mathbb{I}\mathbb{P}$. Thus, Q is an object of $\text{HAUS}(\mathbb{I}\mathbb{P})$ and $\text{HAUS}(Q)$ is contained in $\text{HAUS}(\mathbb{I}\mathbb{P})$.

Finally, $\mathbb{I}\mathbb{P}$ is homeomorphic to a countable product of copies of $\mathbb{I}\mathbb{N}$. A proof of this result is outlined in Exercise 24K of Willard, 1970. Thus, $\text{HAUS}(\mathbb{I}\mathbb{P})$ is contained in $\text{HAUS}(\mathbb{I}\mathbb{N})$.

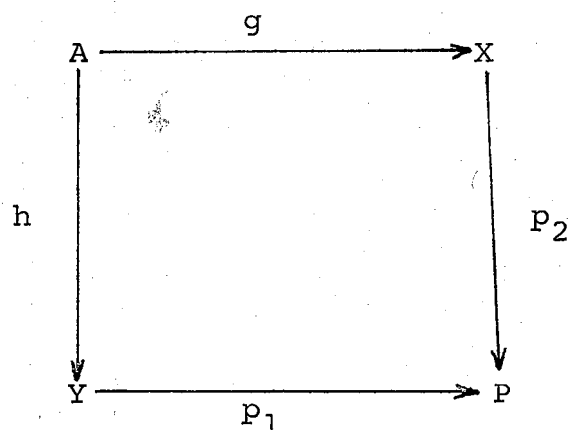
(f) Let k be a cardinal number. A completely regular space is said to be k -compact if every z -ultrafilter such that the intersection of every subfamily of less than k members is non-empty is fixed. These categories were introduced by H. Herrlich in his 1967A paper, and were shown to be simply generated by M. Hušek in 1969.

(g) In his 1971 paper, M. Hušek assumes that measurable cardinals exist and describes a metric space S having measurable cardinality. He then relates the category of S -compact spaces to the question of an $\mathcal{O}$ -analog for the Glicksberg Theorem. A family of subsets of a space is said to be discrete if every point of the space has a neighborhood meeting at most one of the sets. If m is the least measurable cardinal, then a space

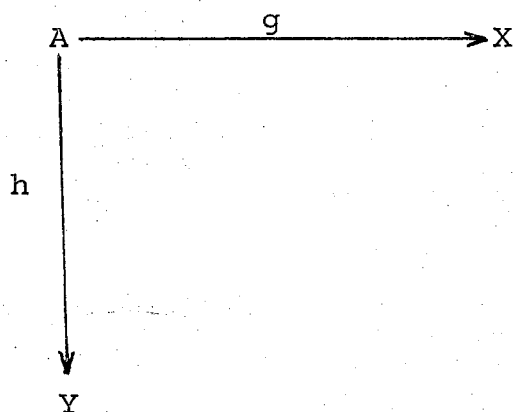
is said to be pseudo-m-compact if every discrete family of open subspaces has non-measurable cardinal. Hušek shows that S plays a role for pseudo-m-compactness analogous to that played by $\mathbb{R}$ for pseudocompactness. Denote the S -compact reflection of a completely regular space X by $\beta_S X$. Then he shows that $\beta_S X = \cup X$ if and only if X is pseudo-m-compact. Further, if $\cup X \times \cup Y$ is equal to $\cup(X \times Y)$, then $X \times Y$ is pseudo-m-compact. However, Hušek also shows that the converse is false.

ADJUNCTIONS IN REFLECTIVE SUBCATEGORIES

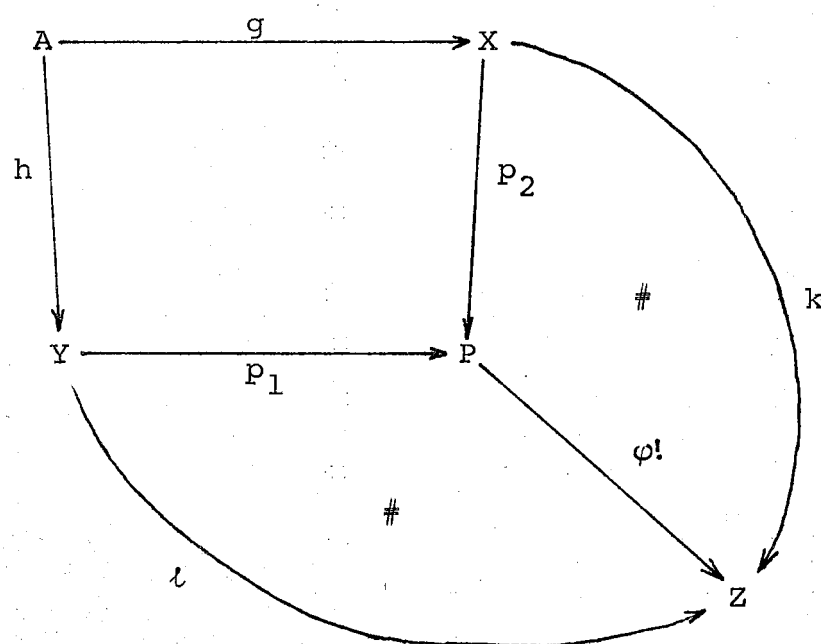
10.25. Consider the following diagram in a category $\mathcal{C}$:



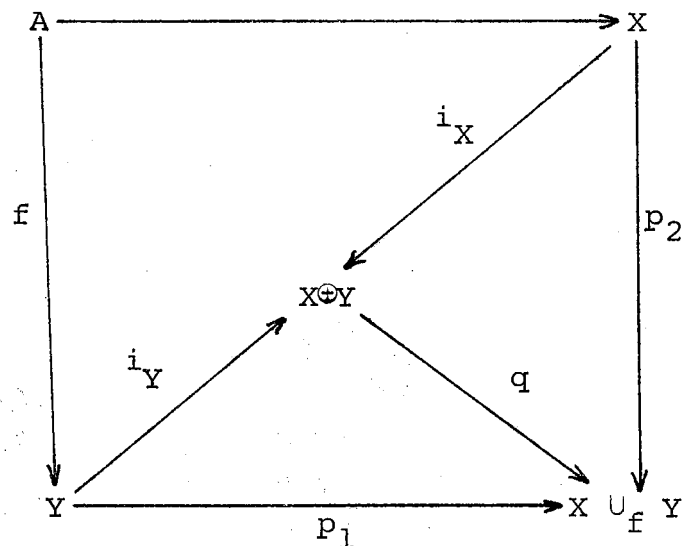
The object P together with the morphisms p_1 and p_2 is said to be the pushout of the diagram



if for every pair of morphisms $\iota : Y \rightarrow Z$ and $k : X \rightarrow Z$ such that $\iota \circ h = k \circ g$, there is a unique morphism φ such that the following diagram commutes:



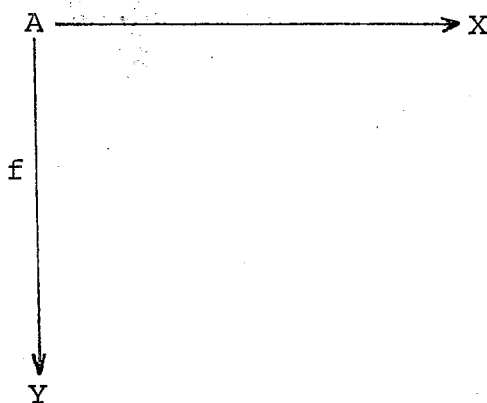
One familiar example of a pushout is the adjunction space in the category of topological spaces. Let A be a closed subspace of a space X and let f map A to Y . Then the adjunction space $X \cup_f Y$ formed by "attaching X to Y by f " is the quotient space of $X \oplus Y$ formed by identifying each point a of A with its image $f(a)$ in Y . In the following diagram, q denotes the quotient map and p_1 and p_2 are the compositions of q with the embeddings of Y and X into $X \oplus Y$.



An account of the basic properties of adjunction spaces is given in the 1964 text of S. T. Hu.

PROPOSITION:

The adjunction space $X \cup_f Y$ is the pushout of the diagram



Proof: Let $\iota: Y \rightarrow Z$ and $k: X \rightarrow Z$ be maps such that $\iota \circ f = k|_A$, i.e. such that k agrees with $\iota \circ f$ on A . Then we can define a function φ from $X \cup_f Y$ to Z by

$$\varphi(r) = \begin{cases} k(x) & \text{if } r = q(x) \text{ for } x \in X \setminus A \\ \iota(y) & \text{if } r = q(y) \text{ for } y \in Y \setminus f[A] \\ k(a) = (\iota \circ f)(a) & \text{if } r = q(a) = q(f(a)) \text{ for } a \in A. \end{cases}$$

The definition of φ implies that $\varphi \circ p_1 = \iota$ and $\varphi \circ p_2 = k$. It remains to show that φ is continuous. Let U be open in Z . Then $\iota^{\leftarrow}(U)$ and $k^{\leftarrow}(U)$ are open in Y and X , respectively. Further, $f^{\leftarrow}(\iota^{\leftarrow}(U)) = k^{\leftarrow}(U) \cap A$. Thus, $i_X[\iota^{\leftarrow}(U)] \cup i_Y[k^{\leftarrow}(U)]$ is a saturated open set of $X \oplus Y$. Since $\iota^{\leftarrow}(U) = p_1^{\leftarrow}(\varphi^{\leftarrow}(U))$ and $k^{\leftarrow}(U) = p_2^{\leftarrow}(\varphi^{\leftarrow}(U))$, we have that

$$\varphi^{\leftarrow}(U) = q[i_X[\iota^{\leftarrow}(U)] \cup i_Y[k^{\leftarrow}(U)]],$$

which is open in the quotient space $X \cup_f Y$. $\blacksquare$

10.26. The next lemma emphasizes that $X \cup_f Y$ is actually a copy of X joined to a copy of Y along the subspace $f[A]$.

LEMMA:

Y and $X \setminus A$ are embedded as subspaces of $X \cup_f Y$ and are closed and open, respectively.

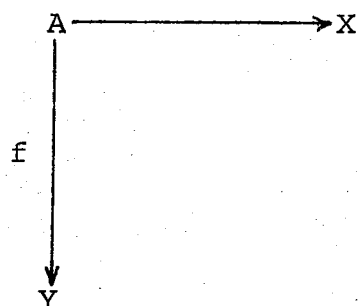
Proof: $X \setminus A$ is embedded as an open subspace of $X \oplus Y$ and $q|_{X \setminus A}$ is one-to-one. Therefore, $q|_{X \setminus A}$ is an open continuous bijection onto its image, and hence is an embedding. Similarly, Y is embedded as a closed subspace of $X \oplus Y$ and $q|_Y$ is a closed continuous bijection onto its image. $\blacksquare$

10.27. Thus far we have spoken only about pushouts in the category of all topological spaces. By modifying the construction, we can show that pushouts exist in reflective subcategories of

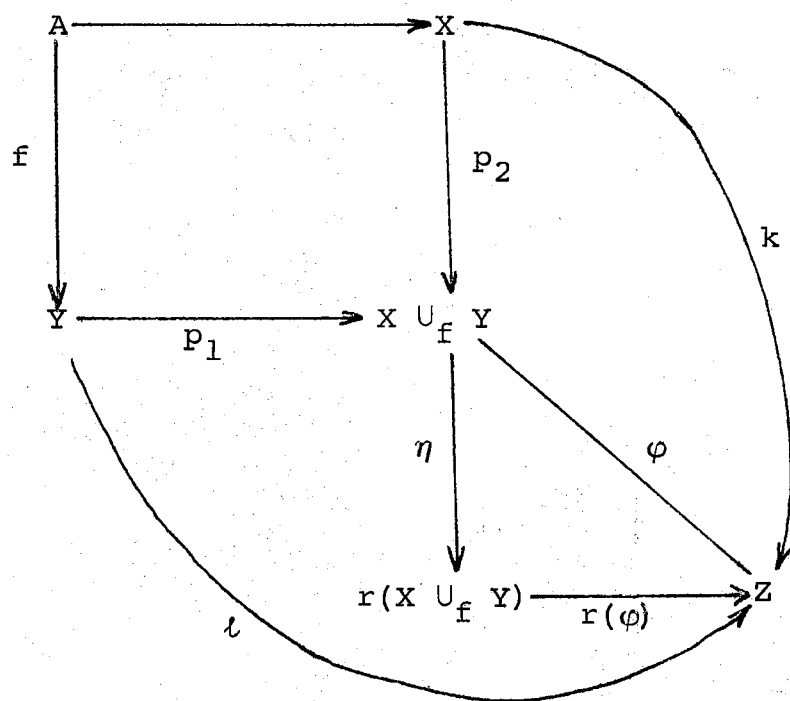
the category of topological spaces.

PROPOSITION:

Let $\mathcal{R}$ be a reflective subcategory of the category of topological spaces. Then if A is a closed subspace of X , the following diagram in $\mathcal{R}$ has a pushout in $\mathcal{R}$:



Proof: Let r be the reflective functor associated with $\mathcal{R}$. We will show that the pushout is $r(X \cup_f Y)$. Let Z belong to $\mathcal{R}$ and let $\iota : Y \rightarrow Z$ and $k : X \rightarrow Z$ be such that $\iota \circ f$ agrees with k on A . Then we have the following diagram:

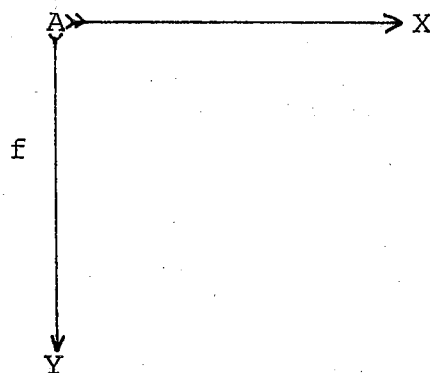


The map φ exists because $X \cup_f Y$ is the pushout in the category of topological spaces. Then because Z is an object of $\mathcal{R}$, φ will factor through η yielding the map $r(\varphi)$ such that $k = r(\varphi) \circ \eta \circ p_2$ and $l = r(\varphi) \circ \eta \circ p_1$. Thus, $r(X \cup_f Y)$ together with the morphisms $\eta \circ p_1$ and $\eta \circ p_2$ is the pushout in $\mathcal{R}$. $\blacksquare$

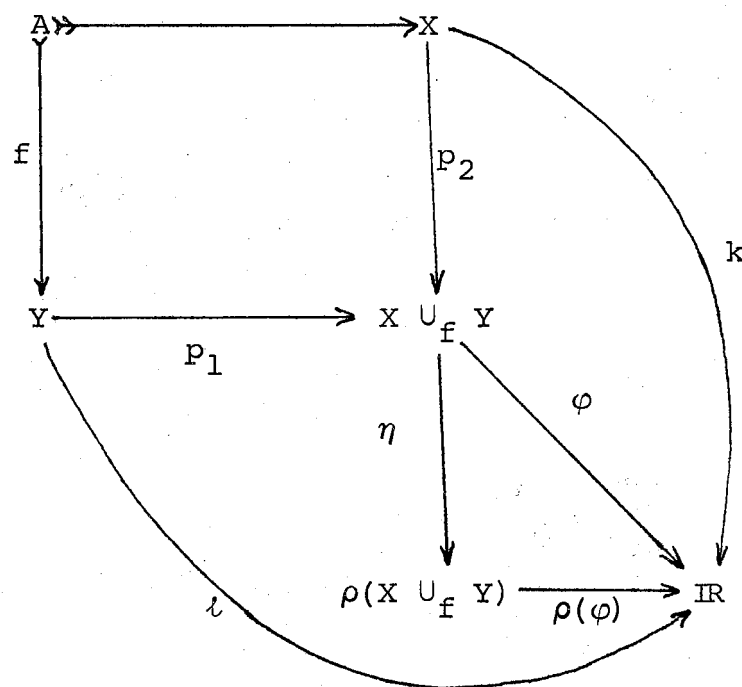
10.28. In particular, we have seen in Section 10.22 that the category of completely regular spaces is reflective in the category of topological spaces. In this case, the preceding proposition yields $\rho(X \cup_f Y)$ as the pushout. This construction can be applied to describe the Stone-Ćech compactification of certain spaces. The next lemma is a crucial step in that direction because it deals with C^* -embedded subspaces of $\rho(X \cup_f Y)$.

LEMMA:

Consider the following diagram in the category of completely regular spaces. If f is a C^* -embedding of A into Y , then X is C^* -embedded in the pushout $\rho(X \cup_f Y)$.



Proof: Let k belong to $C^*(X)$ and let ι be the extension to Y of $k|_A$. Then we have the following diagram:



We first show that X is C^* -embedded in $X \cup_f Y$. Lemma 10.26 shows that $X \setminus A$ and Y are both embedded in $X \cup_f Y$. Since we also have here that f is an embedding, $p_2|_A$ is just the composition of the embeddings p_1 and f so that p_2 is an embedding of all of X into $X \cup_f Y$. Thus, we have shown that X is C^* -embedded in the adjunction space $X \cup_f Y$ since φ is an extension of the mapping k .

To show that X is C^* -embedded in $\rho(X \cup_f Y)$, we will show that $\eta|_{p_2[X]}$ is a one-to-one open map onto its range and hence is an embedding.

Recall from Theorem 1.6 that for any space S , the map η of S onto ρS identifies those points of S which cannot be

separated by members of $C^*(S)$. We will show that every distinct pair of points of $p_2[X]$ is separated so that $\eta|_{p_2[X]}$ is one-to-one. Let $p_2(x)$ and $p_2(z)$ be distinct points of $p_2[X]$. Then there exists a member k of $C^*(X)$ which separates x from z . But then we have the mapping ϕ such that $k = \phi \circ p_2$. Therefore, ϕ separates $p_2(x)$ and $p_2(z)$.

Now we show that $\eta|_{p_2[X]}$ is an open map onto its range. Because $p_2[X]$ is completely regular, the cozero-sets of $p_2[X]$ form a base for the topology and it is sufficient to show that the image of a cozero-set is open. Because $p_2[X]$ is C^* -embedded in $X \cup_f Y$, every cozero-set in $p_2[X]$ is the trace of a cozero-set of $X \cup_f Y$. Let $U = Cz(g) \cap p_2[X]$ be such a trace. Then we can write $g = \rho(g) \circ \eta$ where $\rho(g)$ belongs to $C^*(\rho(X \cup_f Y))$. But this factorization shows that $\eta[U] = Cz(\rho(g)) \cap (\eta \circ p_2)[X]$, and hence $\eta[U]$ is open in $(\eta \circ p_2)[X]$.

Hence, $\eta \circ p_2$ is an embedding and X is C^* -embedded in $\rho(X \cup_f Y)$. $\blacksquare$

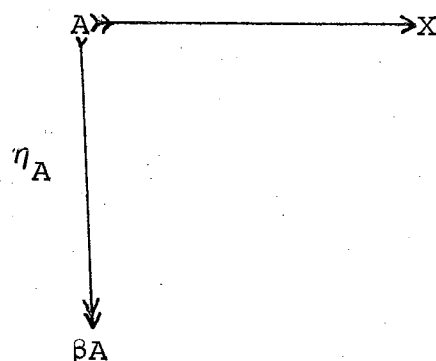
Observe that the analogous result will hold for C -embedding.

10.29. We now use the lemma to describe the Stone-Čech compactification of a completely regular space which contains a closed subspace that meets every non-compact zero-set.

THEOREM:

Let A be a closed subspace of a completely regular space X such that every non-compact zero-set of X meets A . Then βX is the pushout of the following diagram in the category of completely

regular spaces:



Proof: The lemma shows that X is C^* -embedded in the pushout $\rho(X \cup_{\eta_A} \beta A)$. Because A is dense in βA , X is also dense in the pushout. It remains to show compactness.

Denote the following composition by g :

$$X \oplus \beta A \longrightarrow X \cup_{\eta_A} \beta A \longrightarrow \rho(X \cup_{\eta_A} \beta A).$$

Let $\mathcal{U}$ be a z -ultrafilter on $\rho(X \cup_{\eta_A} \beta A)$. We will show that $\mathcal{U}$ is fixed. If there exists a zero-set Z in $\mathcal{U}$ such that $g^{\leftarrow}(Z)$ misses A , then the zero-set $g^{\leftarrow}(Z)$ is contained in $(\beta A \setminus A) \oplus (X \setminus A)$. Hence, the compactness of βA and our hypothesis imply that $g^{\leftarrow}(Z)$ is compact. Therefore, $Z = g[g^{\leftarrow}(Z)]$ is compact. Since no compact zero-set can belong to a free z -ultrafilter, $\mathcal{U}$ is fixed. If no such Z exists in $\mathcal{U}$, then $g^{\leftarrow}(Z)$ meets A for every Z in $\mathcal{U}$, and the family $\{g^{\leftarrow}(Z) : Z \in \mathcal{U}\}$ is a z -filter on A . This z -filter on A must cluster in βA , and the image of the cluster point under g is a cluster point of $\mathcal{U}$. Since the z -ultrafilter $\mathcal{U}$ converges to any cluster point (Proposition 1.28), $\mathcal{U}$ is fixed and $\rho(X \cup_{\eta_A} \beta A)$ is compact.

Hence, $\rho(X \cup_{\eta_A} \beta A) = \beta X$. $\square$

10.30. We now consider two applications of the previous theorem to particular spaces. The following example was introduced by J. R. Isbell and is described in [GJ, ex. 5I].

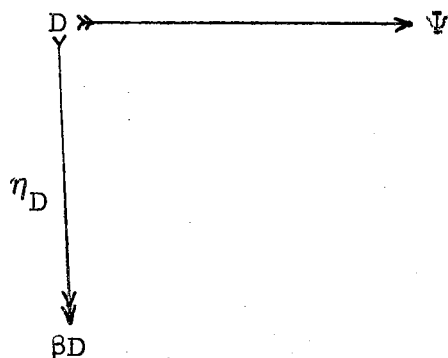
EXAMPLE:

In Proposition 3.21 we showed that $\mathbb{N}$ admits an almost disjoint family of c infinite subsets. By enlarging $\mathcal{C}$ if necessary, assume that $\mathcal{C}$ is a maximal almost-disjoint family of infinite subsets. For each E in $\mathcal{C}$, choose a new point ω_E and let $\Psi = \mathbb{N} \cup \{\omega_E : E \in \mathcal{C}\}$. Let the points of $\mathbb{N}$ be isolated and let a basic neighborhood of a point ω_E be any subset containing ω_E and all but finitely many points of E . Ψ is clearly Hausdorff, and since the subspace $E \cup \{\omega_E\}$ is a compact neighborhood of ω_E , Ψ is locally compact. Thus, Ψ is completely regular [D, p. 238].

Put $D = \{\omega_E : E \in \mathcal{C}\}$. Since points of $\mathbb{N}$ are isolated, D is closed. Because the neighborhood $E \cup \{\omega_E\}$ of ω_E meets D in the singleton $\{\omega_E\}$, D is a discrete subspace of cardinality c . Because Ψ is separable, it can admit only c real-valued mappings. Hence, the closed, discrete subspace D cannot be C^* -embedded in Ψ since D admits 2^c real-valued mappings. Hence, Ψ is not normal.

Now we apply the theorem to describe $\beta\Psi$. The maximality of $\mathcal{C}$ shows that any closed set which misses D is a finite subset of $\mathbb{N}$ and is compact. Hence, every non-compact zero-set of Ψ meets the closed subspace D . Thus, the theorem implies

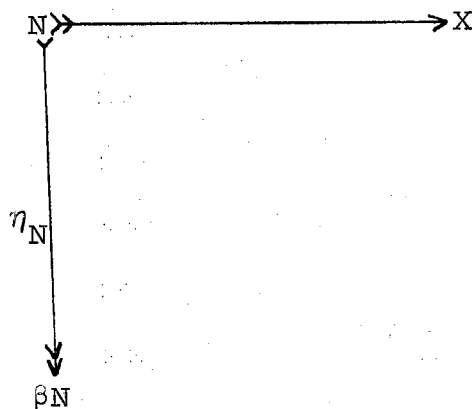
that $\beta\Psi$ is $\rho(\Psi \cup_{\eta_D} \beta D)$, the pushout of the following diagram in the category of completely regular spaces:



Note also that the pushout of the diagram in the category of topological spaces, i.e. $\Psi \cup_{\eta_D} \beta D$, is not $\beta\Psi$. If the adjunction space were compact and Hausdorff, then the compact subspace βD would be C^* -embedded and as a consequence, so would D . But this would imply that D is C^* -embedded in Ψ , which we have seen is impossible. However, it is possible to show, using the maximality of $\mathcal{E}$, that every open cover of $\Psi \cup_{\eta_D} \beta D$ has a finite subcover. Hence, it must be the case that $\Psi \cup_{\eta_D} \beta D$ fails to be Hausdorff.

10.31. EXAMPLE:

Consider the Tychonoff Plank, $T = (\omega_1 + 1) \times (\omega_0 + 1)$ as discussed in Sections 4.1 and 4.4 and let X be the subspace obtained from T by omitting the point $\{(\omega_0, 0)\}$ from the "bottom row". Then every non-compact zero-set of X meets the set $N = \{(n, 0) : n < \omega_0\}$ and N is homeomorphic with $\mathbb{N}$. Hence, Theorem 10.29 shows that βX is the pushout in the category of completely regular spaces of the following diagram:



However, in this case both X and βN are normal and the adjunction space $X \cup_{\eta_N} \beta N$ is therefore Hausdorff [D, p. 145]. One can also show that the adjunction space is compact, so that in this case we have that $\beta X = X \cup_{\eta_N} \beta N$. Thus, βX is obtained by attaching a copy of $\mathbb{N}^*$ and it is not actually necessary to consider the reflection of the adjunction space.

PERFECT MAPPINGS

10.32. A mapping $f : X \rightarrow Y$ is called a compact mapping if its fibers are compact, i.e. if $f^{-1}(y)$ is compact for every y in Y . The mapping f is said to be perfect if it is both closed and compact. In this section, we will investigate the relationships between perfect mappings and the Stone-Ćech compactification.

There are numerous examples of perfect maps. Any map from a compact space to a Hausdorff space is perfect as is any closed embedding. A projection parallel to a compact factor is closed [D, p. 227] and has fibers which are homeomorphic to the compact factor. Hence, such a projection is perfect.

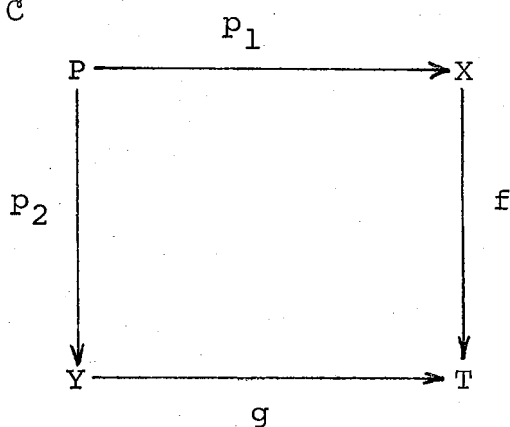
If $\{f_\alpha : X_\alpha \rightarrow Y_\alpha\}$ is a family of perfect maps, then the

product mapping,

$$\times f_{\alpha} : \times X_{\alpha} \longrightarrow \times Y_{\alpha}$$

defined by $(\times f_{\alpha})((x_{\alpha}))_{\alpha} = f_{\alpha}(x_{\alpha})$ is easily seen to be perfect.

10.33. We will find that the dual notion of a pushout will be useful in the description of perfect maps. The following square in a category $\mathcal{C}$



is said to be a pullback in $\mathcal{C}$ if for every pair of morphisms $\iota : Z \rightarrow X$ and $k : Z \rightarrow Y$ such that $f \circ \iota = g \circ k$ there exists a unique morphism $h : Z \rightarrow P$ such that $p_1 \circ h = \iota$ and $p_2 \circ h = k$. Following the example of pushouts, we say that P together with p_1 and p_2 is the pullback of f and g .

Pullbacks are easily described in the category of topological spaces.

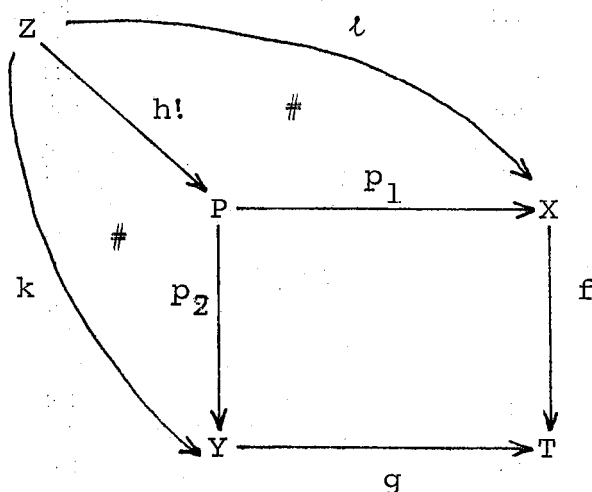
PROPOSITION:

Pullbacks exist in the category of topological spaces.

Proof: Let $f : X \rightarrow T$ and $g : Y \rightarrow T$ be maps and let

$P = \{(x, y) \in X \times Y : f(x) = g(y)\}$. Let $p_1 : P \rightarrow X$ and

$p_2 : P \rightarrow Y$ be the restrictions of the appropriate projections. If $\iota : Z \rightarrow X$ and $k : Z \rightarrow Y$ are maps such that $f \circ \iota = g \circ k$, then the definition $h(z) = (\iota(z), k(z))$ makes the following diagram commute since the image of ι is easily seen to be contained in P :



Thus, P together with the maps p_1 and p_2 is the pullback of f and g . $\square$

The structure of P in the preceding proof indicates that pullbacks exist in many subcategories of the category of topological spaces. For instance, pullbacks will exist in any full subcategory which is productive and hereditary.

10.34. We now consider characterizations of perfect maps which relate them to the Stone-Čech compactification.

PROPOSITION:

If X and Y are completely regular spaces, then the following are equivalent for a map $f : X \rightarrow Y$:

- (1) f is perfect.

- (2) If $\mathcal{u}$ is an ultrafilter on X and if $f[\mathcal{u}]$ converges to y in Y , then $\mathcal{u}$ converges (necessarily to some x in $f^{\leftarrow}(y)$).
- (3) $\beta(f)$ takes growth to growth, i.e. $\beta(f)[X^*]$ is contained in Y^* .

(4)
$$\begin{array}{ccc} X & \xrightarrow{\quad} & \beta X \\ f \downarrow & & \downarrow \beta(f) \\ Y & \xrightarrow{\eta_Y} & \beta Y \end{array} \quad \text{is a pullback.}$$

Condition (2) in the proposition was introduced by N. Bourbaki. Condition (4) was discovered independently by S. P. Franklin and H. Herrlich and perhaps others.

Proof: (1) $\Rightarrow$ (2): Let f be perfect and $\mathcal{u}$ be an ultrafilter on X such that $f[\mathcal{u}]$ converges to y in Y . Because f is continuous, if $\mathcal{u}$ converges it must converge to a point of $f^{\leftarrow}(y)$. If $\mathcal{u}$ fails to converge, then for every x in $f^{\leftarrow}(y)$, there is an open neighborhood U_x of x such that U_x is not in $\mathcal{u}$. Since $f^{\leftarrow}(y)$ is compact, it is covered by a finite subfamily $\{U_{x_i}\}$. The open set $V = \bigcup U_{x_i}$ does not belong to $\mathcal{u}$ because $\mathcal{u}$ is an ultrafilter. Thus, $X \setminus V$ does belong to $\mathcal{u}$ so that $f[X \setminus V]$ belongs to $f[\mathcal{u}]$. Because f is a closed map, $Y \setminus f[X \setminus V]$ is a neighborhood of y which fails to belong to $f[\mathcal{u}]$. This contradicts the assumption that $f[\mathcal{u}]$ converges to y .

Hence, $\mathcal{u}$ must converge.

(2) $\Rightarrow$ (1): We first show that f has compact fibers. Let $\mathcal{v}$ be an ultrafilter on $f^{\leftarrow}(y)$ for some y in Y . Let $\mathcal{u}$ be an ultrafilter on X which contains $\mathcal{v}$. Then $f[\mathcal{u}]$ converges to y in Y . Hence, $\mathcal{u}$ and therefore $\mathcal{v}$ converge to a point in $f^{\leftarrow}(y)$ and $f^{\leftarrow}(y)$ is compact because every ultrafilter on $f^{\leftarrow}(y)$ converges.

Now we show that f is closed. Let F be a closed subset of X and let $\mathcal{v}$ be an ultrafilter on $f[F]$ converging to a point y of Y . For every V in $\mathcal{v}$, $f(f^{\leftarrow}(V) \cap F) = V \cap f[F]$ is non-empty. Hence, the family $\{f^{\leftarrow}(V) \cap F : V \in \mathcal{v}\}$ is contained in a ultrafilter $\mathcal{u}$ on X . Then $f[\mathcal{u}]$ converges to y , and therefore $\mathcal{u}$ converges to a point x in $f^{\leftarrow}(y)$. Since F is closed, x belongs to F . Hence, $y = f(x)$ and $f[F]$ is closed.

(2) $\Rightarrow$ (3): Let p belong to βX . Because X is dense in βX , there is an ultrafilter $\mathcal{u}$ on X which converges to p . Continuity implies that $\beta(f)[\mathcal{u}] = f[\mathcal{u}]$ converges to a point q of βY . If q belongs to Y , then $\mathcal{u}$ converges to a point x in $f^{\leftarrow}(y)$. Because βX is Hausdorff, we must have $p = x$. Thus, the only points of βX which are mapped to points of Y^* are the points of X^* .

(3) $\Rightarrow$ (2): Suppose that $\mathcal{u}$ is an ultrafilter on X such that $f[\mathcal{u}]$ converges to y in Y . Because X is dense in βX , $\mathcal{u}$ converges to a point p of βX . Then continuity implies that $\beta(f)[\mathcal{u}] = f[\mathcal{u}]$ converges to y in Y and p belongs

to $\beta(f)^{\leftarrow}(y)$. Since $\beta(f)$ sends X^* into Y^* , p must belong to X .

(3) $\Rightarrow$ (4): Suppose that $h : Z \rightarrow \beta X$ and $g : Z \rightarrow Y$ are mappings such that $\beta(f) \circ h = \eta_Y \circ g$. Since $\eta_Y \circ g[Z]$ is contained in βY and $\beta(f)$ sends X^* into Y^* , we have that $h[Z]$ is contained in X . Hence, defining $\iota : Z \rightarrow X$ by $\iota(z) = h(z)$ shows that the square is a pullback.

(4) $\Rightarrow$ (3): Choose p in βX and assume that $\beta(f)(p) = y$ belongs to Y . Then let h be the map which embeds $\{p\}$ into βX and g be the map from the subspace $\{p\}$ which sends p to $\beta(f)(p)$. Then $\beta(f) \circ h = \eta_Y \circ g$ so that there exists a map $\iota : \{p\} \rightarrow X$ such that $h = \eta_X \circ \iota$. Hence, p belongs to X . $\blacksquare$

10.35. Since the extension of a perfect map sends growth to growth, it is clear that the inverse image of a compact space under a perfect mapping is compact. Thus, the fibers of a composition of perfect maps will be compact and we have verified the

COROLLARY:

The inverse image of a compact space under a perfect mapping is compact. Hence, the composition of perfect maps is perfect.

10.36. A subcategory $\mathcal{G}$ of the completely regular spaces is said to be left-fitting if whenever Y is an object of $\mathcal{G}$ and $f : X \rightarrow Y$ is a perfect mapping, then X must also be an object of $\mathcal{G}$. Right-fitting subcategories are defined analogously.

EXAMPLES:

We have just seen that the subcategory of compact spaces is left-fitting as a consequence of the fact that the extension of a perfect map sends growth to growth. By using this same condition on the extension we can show that the subcategories of locally compact, realcompact, and σ -compact spaces are also left-fitting.

Since the growth of a space is closed if and only if the space is locally compact, it is clear from this condition that the subcategory of locally compact spaces is both left and right-fitting.

Recall that a space X is realcompact if and only if every point of X^* is contained in a zero-set which misses X (Theorem 1.53). Then the condition on the growths shows that the category of realcompact spaces is left-fitting since the inverse image of a zero-set is a zero-set. In 1958B, S. Mrówka gave an example to show that a perfect image of a realcompact space need not be realcompact. This example has also been discussed by M. Weir in his thesis.

The same condition also shows that the category of σ -compact spaces is left-fitting. (See Exercise 1B.) Since the continuous image of a compact space is compact, this subcategory is also right-fitting.

10.37. Now consider any subcategory $\mathcal{C}$ of completely regular spaces. Let $\mathcal{B}$ be the subcategory of completely regular spaces whose objects are the family of spaces which map perfectly

onto a space belonging to $\mathcal{G}$. Then $\mathcal{B}$ contains $\mathcal{G}$ since an identity map is perfect. Further, $\mathcal{B}$ is left-fitting since perfect maps compose. Thus, every subcategory of completely regular spaces is contained in a left-fitting subcategory. Left-fitting subcategories are easily seen to be closed under intersections, and therefore there is a smallest left-fitting subcategory containing any subcategory $\mathcal{G}$. We will call this smallest left-fitting subcategory the left-fitting hull of $\mathcal{G}$.

A more complete description of the spaces in the left-fitting hull of $\mathcal{G}$ is provided by the following result.

PROPOSITION:

Let $\mathcal{G}$ be a subcategory of the category of completely regular spaces. Then the following are equivalent for any completely regular space X :

- (1) X belongs to the left-fitting hull of $\mathcal{G}$.
- (2) There exists a space A in $\mathcal{G}$ and a perfect map $f : X \rightarrow A$.
- (3) X can be embedded as a closed subspace of a product of a compact space and a space in $\mathcal{G}$.

If $\mathcal{G}$ is countably productive, so is its left-fitting hull.

Proof: Let $\mathcal{L}$ be the left-fitting hull of $\mathcal{G}$.

(1) $\Rightarrow$ (2): The subcategory of $\mathcal{L}$ whose objects satisfy (2) is easily seen to contain $\mathcal{G}$ and is left-fitting because the composition of perfect maps is again perfect. Because $\mathcal{L}$ is the smallest left-fitting subcategory containing $\mathcal{G}$, the subcategory of $\mathcal{L}$ just described must be all of $\mathcal{L}$.

(2) $\Rightarrow$ (1): If a space X maps perfectly to a space of $\mathcal{Q}$, then X must belong to the smallest left-fitting subcategory containing $\mathcal{Q}$. Hence, X belongs to $\mathcal{L}$.

(2) $\Rightarrow$ (3): Let $f : X \rightarrow A$ be perfect with A belonging to $\mathcal{Q}$ and let $\eta_X : X \rightarrow \beta X$ be the usual embedding. Then the Embedding Lemma, 1.2, implies that the evaluation map

$$e : X \rightarrow X \times \beta X$$

defined by $e(x) = (x, \eta_X)$ is an embedding. Further, $e[X]$ is the graph of η_X and is therefore closed in the product space because βX is Hausdorff [D, p. 140]. The mapping

$$f \times l_{\beta X} : X \times \beta X \rightarrow A \times \beta X$$

is perfect because it is the product of two perfect maps. Also, $f \times l_{\beta X}$ is clearly one-to-one. Hence, $(f \times l_{\beta X}) \circ e$ is a closed embedding since it is a continuous, closed bijection onto its range.

(3) $\Rightarrow$ (2): Let $e : X \rightarrow A \times K$ be a closed embedding where A is an object of $\mathcal{Q}$ and K is compact. The closed embedding e is perfect as is the projection π_A of $A \times K$ onto A . Hence, $\pi_A \circ e$ is the required perfect map from X to an object of $\mathcal{Q}$. $\blacksquare$

10.38. We now see that condition (3) of the previous proposition allows us to characterize left-fitting subcategories in a result similar to Theorem 10.21 for epi-reflective subcategories.

PROPOSITION:

The left-fitting subcategories of the category of completely regular spaces are the closed hereditary subcategories which are closed under products with compact spaces.

Proof: Let $\mathcal{G}$ be a left-fitting subcategory. Then $\mathcal{G}$ is closed hereditary because a closed embedding is a perfect map. The product of a space in $\mathcal{G}$ and a compact space is again in $\mathcal{G}$ because the projection parallel to a compact factor is a perfect map [D, p. 227].

Conversely, let $\mathcal{G}$ be a closed hereditary subcategory which is closed under products with compact spaces. Let Y be an object of $\mathcal{G}$ and $f : X \rightarrow Y$ be a perfect map. Then Proposition 10.37 implies that X is homeomorphic to a closed subspace of $A \times K$ where K is a compact space and A is an object of $\mathcal{G}$. The hypotheses on $\mathcal{G}$ then imply that X is an object of $\mathcal{G}$, so that $\mathcal{G}$ is left-fitting. $\parallel$

10.39. EXAMPLES:

(a) In Proposition 8.36, we saw that the product of a paracompact space and a compact space is paracompact. It is easy to see that a closed subspace of a paracompact space is paracompact, so that the subcategory of paracompact spaces is left-fitting.

(b) In a similar way, one can show that the Lindelöf spaces, the countably compact spaces, the metacompact spaces, and the countably paracompact spaces all form left-fitting subcategories.

(c) In 1964, K. Morita introduced the category of M -spaces

in connection with the problem of characterizing those spaces whose products with metric spaces are normal. He showed that this subcategory is left-fitting. A survey of the theory of M-spaces is given in the 1971 paper of Morita.

(d) In his 1961 paper, Z. Frolík showed that the left-fitting hull of the category of completely metrizable spaces is the category of paracompact, topologically complete spaces.

(e) Frolík also showed that the left-fitting hull of the category whose objects are the open subspaces of $\mathbb{R}$ is the category of σ -compact, locally compact spaces.

(f) It is easy to see that the left-fitting hull of the category whose only object is the singleton space is the category of all compact spaces.

(g) In 1964, K. Morita showed that the left-fitting hull of the category of metric spaces is the category of paracompact M-spaces.

10.40. Earlier, in Section 10.24, we considered the epi-reflective hull of a single space. Now we will describe the spaces which belong to the epi-reflective hull of a left-fitting subcategory of completely regular spaces. We first need a preliminary result.

PROPOSITION:

The intersection of a family of subspaces belonging to an epi-reflective subcategory $\mathcal{R}$ of the Hausdorff spaces also belongs to $\mathcal{R}$.

Proof: Let $\{R_\alpha\}$ be a family of subspaces of a space X with each R_α in $\mathcal{R}$. Then the map $(e(y))_\alpha = y$ embeds Y as the diagonal of $\times R_\alpha$. Because the diagonal of a product of Hausdorff spaces is closed, Theorem 10.21 shows that Y belongs to $\mathcal{R}$. $\square$

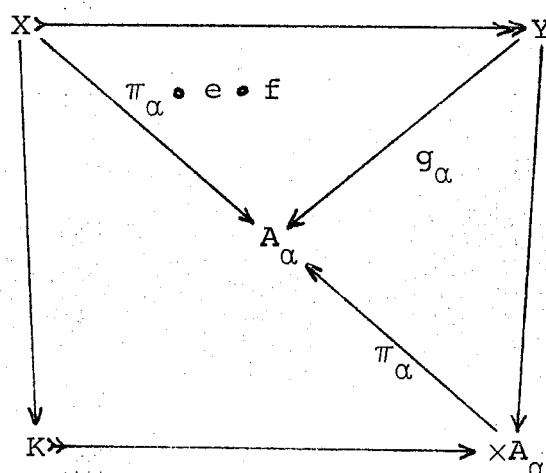
10.41. Following the terminology of Section 10.24, if $\mathcal{G}$ is a subcategory of completely regular spaces, then the $\mathcal{G}$ -completely regular spaces are those spaces which are subspaces of products of spaces of $\mathcal{G}$. The $\mathcal{G}$ -compact spaces are the $\mathcal{G}$ -completely regular spaces which are closed subspaces of such products. In Chapter 1 we saw that the reflection βX of an I -completely regular space X in the category of I -compact spaces can be characterized as that compactification of X to which every mapping of X into I will extend. We now mimic the proof of Corollary 1.11 to derive the analogous result for the $\mathcal{G}$ -compact reflection of an $\mathcal{G}$ -completely regular space. Call a subspace S of X $\mathcal{G}$ -embedded if every map from S to an object of $\mathcal{G}$ will extend to X .

PROPOSITION:

The reflection αX of an $\mathcal{G}$ -completely regular space X in the category of $\mathcal{G}$ -compact spaces is the unique $\mathcal{G}$ -compact space in which X is dense and $\mathcal{G}$ -embedded.

Proof: Since any object of $\mathcal{G}$ is $\mathcal{G}$ -compact, αX must clearly have this property. Now suppose that X is dense and $\mathcal{G}$ -embedded in an $\mathcal{G}$ -compact space Y . We will show that Y is αX . Let

f map X to an $\mathcal{G}$ -compact space K . We must show that f extends to Y . Since a reflection must be unique (Proposition 10.11), this will show that Y is αX . Because K is $\mathcal{G}$ -compact, there is a closed embedding e of K into a product $\times A_\alpha$ of spaces of $\mathcal{G}$. Then the composition $\pi_\alpha \circ e \circ f$ maps X to A_α and therefore extends to a mapping g_α of Y into A_α . If h from Y to $\times A_\alpha$ is defined by $h(y)_\alpha = g_\alpha(y)$, we have the following diagram:



Since K is closed in $\times A_\alpha$, $h[Y] \subset e[K]$ as in Theorem 1.11 and $e \circ h$ is the required extension of f . $\blacksquare$

10.42. If $\mathcal{G}$ is a left-fitting subcategory of completely regular spaces, then the following result which appears in the 1971 paper of S. P. Franklin shows that αX is a subspace of βX .

THEOREM: (Franklin)

If $\mathcal{G}$ is a left-fitting subcategory of completely regular spaces, then the reflection αX of a completely regular space X in $\text{HAUS}(\mathcal{G})$ is the intersection of the subspaces of βX which contain X and belong to $\mathcal{G}$.

The key to the proof will lie in the fact that any mapping $f : X \rightarrow Y$ can be extended to a subspace of βX so that the extended map is perfect and retains the same codomain as f . This observation follows from the fact that a map is perfect if its Stone-Ćech extension sends growth to growth. Thus, $\beta(f) | \beta(f)^{\leftarrow}(Y)$ is the required perfect extension of f .

Proof: Let Y be the intersection of all subspaces of βX which contain X and belong to $\mathcal{G}$. Then X is dense in Y and Proposition 10.40 shows that Y is an object of $\text{HAUS}(\mathcal{G})$. By Proposition 10.41, we need only show that every map f of X into a space A of $\mathcal{G}$ will extend to Y . But $\beta(f) | \beta(f)^{\leftarrow}(A)$ is a perfect map into A and hence $\beta(f)^{\leftarrow}(A)$ is an object of $\mathcal{G}$ since $\mathcal{G}$ is left-fitting. Then by definition, Y is a subspace of $\beta(f)^{\leftarrow}(A)$ so that $\beta(f) | Y$ is the required extension. |

10.43. Note that we have shown in the previous result that if $\mathcal{G}$ is left-fitting, then every completely regular space is $\mathcal{G}$ -completely regular. Since any $\mathcal{G}$ -compact space is homeomorphic to its $\mathcal{G}$ -compact reflection, the following corollary is immediate.

COROLLARY:

If $\mathcal{G}$ is a left-fitting subcategory of completely regular spaces, then every completely regular space is $\mathcal{G}$ -completely regular. A space is $\mathcal{G}$ -compact if and only if it is the intersection of a family of subspaces of βX which belong to $\mathcal{G}$.

10.44. The definition of realcompact given in Section 1.53 shows that αX is the intersection of perfect inverse images of the real line $\mathbb{R}$. Many of the examples of left-fitting subcategories are themselves subcategories of the category of realcompact spaces and contain $\mathbb{R}$. Thus, they have the realcompact spaces as their epi-reflective hull. Examples of such subcategories are the Lindelöf spaces, the σ -compact spaces, and the σ -compact locally compact spaces. In addition, one can show that the epi-reflective hull of the paracompact spaces and the paracompact M -spaces will be the category of realcompact spaces if and only if no measurable cardinal exists.

10.45. In Theorem 10.41, we saw that the reflection of a space X in $\text{HAUS}(\mathcal{C})$ can be characterized by the existence of extensions of mappings of X into spaces belonging to $\mathcal{C}$. We now show that if we know $\mathcal{C}$ to be a left-fitting subcategory and that $\mathcal{C}$ is the left-fitting hull of a subcategory $\mathcal{B}$, then there is an alternate characterization of αX in terms of extensions of maps of X into spaces belonging to the smaller subcategory $\mathcal{B}$.

THEOREM:

If $\mathcal{C}$ is the left-fitting hull of $\mathcal{B}$, then αX is characterized as a space Y with the following properties:

- (a) Y is an object of $\text{HAUS}(\mathcal{A})$ and $X \subset Y \subset \beta X$.
- (b) Any mapping of X into a space in $\mathcal{B}$ extends to Y .

Proof: It is clear from Theorem 10.42 that αX satisfies (a) and (b) and also that

$$aX = \bigcap \{ \beta(f)^{\leftarrow} (A) : f : X \rightarrow A \text{ with } A \in \mathcal{U} \}.$$

Since the embedding of X into an $\mathcal{U}$ -compact subspace of βX must extend to aX , if Y satisfies (a) we must have aX contained in Y . We will exhibit a subspace Z of βX satisfying (a) and (b) such that any subspace Y of βX that also satisfies (a) and (b) will be contained in Z . Then we will have $aX \subset Y \subset Z$ and the proof will be completed by showing that $aX = Z$. Define Z by

$$Z = \bigcap \{ \beta(f)^{\leftarrow} (B) : f : X \rightarrow B \text{ with } B \in \mathcal{B} \}.$$

It is clear that Z satisfies (a) and we can see that Z satisfies (b) by considering restrictions of extensions to βX . The definition of Z shows that Z contains any subspace Y satisfying (a) and (b).

To show that $aX = Z$, we will show that the two families of subspaces of βX whose intersections define aX and Z are actually the same. By Proposition 10.37, if A is an object of $\mathcal{U}$, then there exists a perfect map $g : A \rightarrow B$ where g is onto and B is an object of $\mathcal{B}$. Now if $f : X \rightarrow A$, we have two extensions to βX : $\beta(f)$ and $\beta(g \circ f)$. Further, the extension $\beta(g) : \beta A \rightarrow \beta B$ satisfies $\beta(g)[A^*] \subset B^*$. Thus, we have

$$\beta(g \circ f)^{\leftarrow} (B) = \beta(f)^{\leftarrow} (\beta(g)^{\leftarrow} (B)) = \beta(f)^{\leftarrow} (A).$$

Thus, every subspace of βX which appears in the intersection aX is also a subspace involved in the intersection Z . Hence,

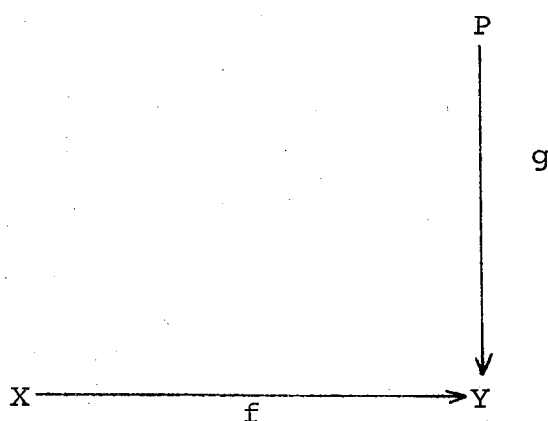
$Y = \alpha X.$ $\square$

Observe that condition (a) in the theorem plays an essential role. If $\mathcal{C}$ is the category of all compact spaces, then we must have $\alpha X = \beta X$. But $\mathcal{C}$ is the left-fitting hull of the singleton space and every constant map of X will of course extend to any compactification of X . Hence, it is necessary to require that Y be a subspace of βX .

PROJECTIVES

10.46. In the remainder of the chapter, we will adapt the definition of a projective object from homological algebra and module theory to investigate an analogous definition in a topological context. We will see that the adaptation is a fruitful one in that it provides additional information about the relationship between extremely disconnected spaces and complete Boolean algebras described in Proposition 2.5. There we saw that the Boolean algebra of clopen sets of a zero-dimensional space is complete precisely when the space is extremely disconnected.

Consider the following diagram in a category $\mathcal{C}$ of topological spaces and continuous mappings. We will say that the space P is a projective object in $\mathcal{C}$ if there exists a mapping $\psi : P \rightarrow X$ such that $f \circ \psi = g$ whenever f is a perfect onto map.



Recall from Section 2.5 that an extremely disconnected space is one in which the closure of every open subset is open. We will show that all projective objects in certain subcategories are extremely disconnected. Call a full subcategory $\mathcal{C}$ of Hausdorff spaces acceptable if

- (a) Whenever Y is an object of $\mathcal{C}$ and $\{0,1\}$ is the two-point discrete space, then $A \times \{0,1\}$ is an object of $\mathcal{C}$.
- (b) A closed subspace of an object of $\mathcal{C}$ is also an object of $\mathcal{C}$.

Observe that the categories of Hausdorff spaces, completely regular spaces, and compact Hausdorff spaces are all acceptable. In his 1958 paper, A. M. Gleason proved the following result which shows that a projective object in any of these three categories is extremely disconnected.

PROPOSITION: (Gleason)

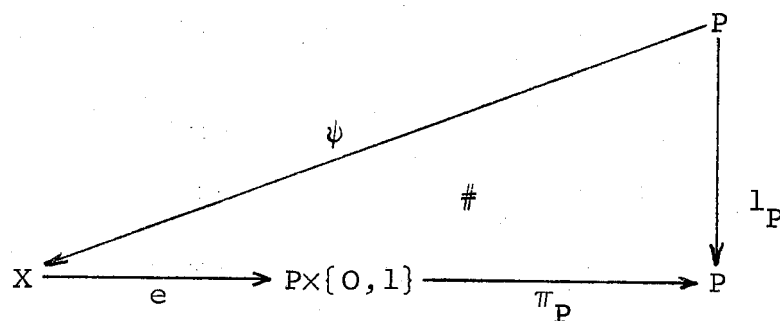
Any projective object in an acceptable category is extremely disconnected.

Proof: Let P be a projective object in an acceptable category $\mathcal{C}$ and let U be an open subspace of P . We must show that

$c\ell U$ is open. Let X be the closed subspace of $P \times \{0,1\}$ defined by

$$X = ((P \setminus U) \times \{0\}) \cup ((c\ell U) \times \{1\}).$$

Because $\mathcal{C}$ is acceptable, X is an object of $\mathcal{C}$. Let π_P be the projection of $P \times \{0,1\}$ onto P and let e be the embedding of X into $P \times \{0,1\}$. Both π_P and e are perfect so that the composition $\pi_P \circ e$ is perfect. Since $P \subseteq (P \setminus U) \cup c\ell U$, $\pi_P \circ e$ is also onto. Since P is a projective object of $\mathcal{C}$, there exists a mapping ψ such that the following diagram commutes:



Because $\pi_P \circ e$ is one-to-one on $U \times \{1\}$, we must have $\psi(p) = (p,1)$ for p in U . Hence, by continuity, $\psi(p) = (p,1)$ for p in $c\ell U$. Similarly, if p is not in $c\ell U$, we have $\psi(p) = (p,0)$. Thus, $c\ell U = \psi^{-1}(c\ell U \times \{1\})$ and because $c\ell U \times \{1\}$ is clopen in X , $c\ell U$ is open in P . $\blacksquare$

10.47. The proposition indicates that the category of extremely disconnected spaces will play an important role in the investigation of projectives. The next sequence of results describe the properties of extremely disconnected spaces which we will need later. The following proposition is based on [GJ, ex. 1H,6M].

PROPOSITION:

The following are equivalent:

- (1) X is extremely disconnected.
- (2) Disjoint open subsets of X have disjoint closures.
- (3) βX is extremely disconnected.
- (4) Every dense subspace of X is C^* -embedded.

Proof: (1) $\Rightarrow$ (2): Let U and V be disjoint open subsets of X . Then $\text{cl } V \cap U = \emptyset$ because U is open. Similarly, $\text{cl } V \cap \text{cl } U = \emptyset$ because $\text{cl } V$ is open.

(2) $\Rightarrow$ (1): Let U be open in X . Then U and $X \setminus \text{cl } U$ have disjoint closures whose union is X . Hence, $\text{cl } U$ is open.

(2) $\Rightarrow$ (3): Let U be an open subset of βX . Then $U \cap X$ and $X \setminus \text{cl}(U \cap X)$ have disjoint closures whose union is X . Thus, $\text{cl}(U \cap X)$ is clopen in X and therefore, $\text{cl}_{\beta X} U = \text{cl}_{\beta X}(\text{cl}_X(U \cap X))$ is clopen in βX . Hence, βX is extremely disconnected.

(3) $\Rightarrow$ (1): Let U be open in X . Then $U = X \cap V$ for some open set V of βX . Because X is dense in βX , $\text{cl}_X U = X \cap \text{cl}_{\beta X} V$, and $\text{cl}_X U$ is thus clopen in X .

(1) and (2) $\Rightarrow$ (4): Let Y be dense in X . We will show that completely separated subsets of Y are completely separated in X . Thus, Urysohn's Embedding Lemma, 1.2, will show that Y is C^* -embedded. Completely separated subsets of Y are contained in disjoint open subsets of Y . Disjoint open subsets of Y are easily seen to be the traces on Y of disjoint open subsets of X . Then (1) and (2) imply that completely separated subsets of Y are contained in disjoint clopen subsets of X and are thus completely separated in X .

(4) $\Rightarrow$ (1): Let U be open in X . Then $Y = U \cup (X \setminus \text{cl } U)$ is dense in X , and hence is C^* -embedded. Since U and $X \setminus \text{cl } U$ are completely separated in Y , they are completely separated in X . But then their closures in X are disjoint closed sets whose union is X . Hence, $\text{cl } U$ is open and X is extremely disconnected. $\blacksquare$

10.48. A mapping f of X onto Y is said to be irreducible if Y is not the image of any proper closed subspace of X . The following results on irreducible mappings are taken from the 1967 paper of D. P. Strauss. Gleason proved similar lemmas under more restrictive assumptions. First we show that every compact map has an irreducible restriction.

LEMMA:

If f is a compact mapping of a space X onto a space Y , then there is a closed subspace F of X such that $f|_F$ is an irreducible mapping onto Y .

Proof: Let $\mathfrak{F}$ be the family of closed subspaces of X which are mapped onto Y by f . Let $\{H_\alpha : \alpha < \beta\}$ be a descending family of members of $\mathfrak{F}$. For a point y of Y , $\{H_\alpha \cap f^{-1}(y)\}$ is a descending family of non-empty compact subsets and hence, $\bigcap (H_\alpha \cap f^{-1}(y))$ is not empty [D, p. 225]. Therefore, $f[\bigcap H_\alpha] = Y$ and Zorn's Lemma shows the existence of the subspace F . $\blacksquare$

10.49. LEMMA:

Let $f : X \rightarrow Y$ be an irreducible map. Then if U is an open subset of X ,

$$F[U] \subset \text{cl}(Y \setminus f[X \setminus U]).$$

Proof: We can assume that U is non-empty, since otherwise there is nothing to prove. Let y belong to $f[U]$ and let V be a neighborhood of y . It will suffice to show that V meets $Y \setminus f[X \setminus U]$. Because $U \cap f^{\leftarrow}(V)$ is a non-empty open set and f is irreducible, there is a point z belonging to $Y \setminus f[X \setminus (U \cap f^{\leftarrow}(V))]$. Choose x such that $f(x) = z$. Then $z = f(x)$ is in $V = f[f^{\leftarrow}(V)]$ so that we have $z \in V \cap (Y \setminus f[X \setminus U])$. $\blacksquare$

10.50. Now we relate irreducible maps to extremely disconnected spaces.

LEMMA:

A closed irreducible mapping of a Hausdorff space onto an extremely disconnected space is a homeomorphism.

Proof: Let $f : X \rightarrow Y$ be closed and irreducible and let Y be extremely disconnected. We will show that distinct points x and y of X have distinct images. Let U and V be disjoint open neighborhoods of x and y , respectively. Since U and V are disjoint, every point of Y belongs to either $f[X \setminus U]$ or $f[X \setminus V]$, so that we have $Y = f[X \setminus U] \cup f[X \setminus V]$. Hence, $Y \setminus f[X \setminus V]$ and $Y \setminus f[X \setminus U]$ are disjoint open sets, and since Y is extremely disconnected, they have disjoint closures. But $f(x)$ belongs to $\text{cl}(Y \setminus f[X \setminus U])$ and $f(y)$ belongs to $\text{cl}(Y \setminus f[X \setminus V])$ by Lemma 10.49 so that $f(x) \neq f(y)$. Hence, f is a closed, continuous bijection and is therefore a homeomorphism. $\blacksquare$

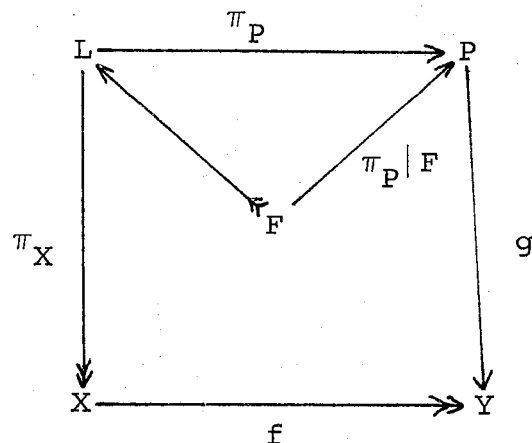
10.51. We are now able to characterize the projective objects in the category of compact spaces.

THEOREM: (Gleason)

The projective objects in the category of compact spaces are precisely the extremely disconnected spaces.

Proof: Proposition 10.46 shows that the projective objects are all extremely disconnected. We must show that every extremely disconnected space is projective.

Let P be an extremely disconnected compact space. Let f map a compact space X onto another compact space Y . Then f must be perfect. Let g map P to Y . The pullback $L = \{(x, p) : f(x) = g(p)\}$ is a closed subspace of $X \times P$ and is therefore compact. Because f is onto, the projection π_P carries L onto P . Lemma 10.48 yields a closed subspace F of L such that $\pi_P|_F$ is an irreducible mapping onto P . Hence, we have the following diagram:



Since P is extremely disconnected, Lemma 10.50 shows that $\pi_P|_F$ is a homeomorphism. Then $h = \pi_X \circ (\pi_P|_F)^{\leftarrow}$ is the required map: If p is in P , then

$$f \circ h(p) = g \circ h(p) = g(p)$$

because L is the pullback. Hence, P is a projective object. |

10.52. The previous result for compact spaces can be applied, together with a characterization of perfect maps, to obtain the analogous result for completely regular spaces.

COROLLARY:

The projective objects in the category of completely regular spaces are precisely the extremely disconnected spaces.

Proof: It follows from Proposition 10.46 that a projective space is extremely disconnected.

Now we show that an extremely disconnected space is projective. Let P be extremely disconnected and let $f : X \rightarrow Y$ be a perfect-onto mapping between two completely regular spaces. Let $g : P \rightarrow Y$ be any mapping. Then by taking Stone-Čech compactifications and using the preceding theorem, there exists a mapping ψ making the following diagram commute since βP is extremely disconnected (Proposition 10.47):

$$\begin{array}{ccc}
 & \beta P & \\
 \psi \swarrow & & \searrow \beta(g) \\
 \beta X & \xrightarrow{\beta(f)} & \beta Y
 \end{array}$$

#

Because f is perfect, $\beta(f)$ sends X^* to Y^* . Hence,
 $f \circ (\psi|_P) = g$ showing that P is projective. $\square$

10.53. Every space is the continuous image of the discrete space having the same cardinality. Hence, every space is the continuous image of a projective space. However, the discrete space does not reflect the topological structure of the original space. We would like to associate with each completely regular space X a projective space $E(X)$ and a mapping g of $E(X)$ onto X so that $E(X)$ reflects the structure of X . In particular, we would like $E(X)$ to be homeomorphic to X whenever X is already projective, i.e. when X is extremely disconnected. Lemma 10.50 indicates that this would be the case if we require g to be a closed, irreducible mapping of $E(X)$ onto X . We will be able to accomplish this and also make the restriction that the fibers of g be compact, i.e. that g will also be a compact mapping. We will call the projective space $E(X)$ together with the perfect irreducible mapping g the projective cover of X . Before considering the general question of the existence of projective covers, we look at a specific case.

EXAMPLE:

Following the terminology of Example 3.44, denote by S_1 the space consisting of a convergent sequence $\{x_n : n \geq 1\}$ and its limit x_0 . Define a map $g : \beta\mathbb{N} \rightarrow S_1$ by sending n to x_n and points of the growth to x_0 . We have seen that $\beta\mathbb{N}$ is extremely disconnected and g is easily seen to be perfect

and irreducible. Thus, $E(S_1) = \beta\mathbb{N}$.

10.54. Our procedure to show the existence of projective covers will follow the lines of the characterization of projectives. We will show the compact case first and then establish the completely regular case as a corollary. To show that every compact space X has a projective cover, we must associate to each compact space X an extremely disconnected compact space $E(X)$ and an irreducible mapping of $E(X)$ onto X . The extremely disconnected space can be obtained in a natural way. The Boolean algebra $R(X)$ of regular closed subsets of X is complete (Proposition 2.3) and therefore its Stone space $S(R(X))$ is extremely disconnected (Proposition 2.5). Moreover, because the points of $S(R(X))$ are the maximal filters of $R(X)$ and X is compact, there is a natural way to define a mapping of $S(R(X))$ onto X . Each maximal filter $\mathfrak{F}$ of regular closed sets is a family of closed subsets of X having the finite intersection property and therefore has non-empty intersection. Further, X has a base of regular closed sets which makes it easy to see that the intersection of each maximal filter $\mathfrak{F}$ is a single point of X . Thus, to each maximal filter $\mathfrak{F}$ in $R(X)$ we can associate the unique point of X belonging to $\bigcap \mathfrak{F}$. To show that $S(R(X))$ is the projective cover of X , it remains only to show that the function just described is an irreducible mapping onto X and that any other compact extremely disconnected space which maps irreducibly onto X is homeomorphic to $S(R(X))$.

THEOREM: (Gleason)

The projective cover of a compact space X is $S(R(X))$ together with the mapping which assigns to each maximal filter of $R(X)$ its limit in X . Further, any compact extremely disconnected space which maps irreducibly onto X is homeomorphic to $S(R(X))$.

The proof will require the use of the Boolean algebra isomorphism between $R(X)$ and the clopen subsets of $S(R(X))$. Recall from Theorem 2.10 that this isomorphism associates to each regular closed subset of X the family of maximal filters which contain it. We will use this assignment to show that the function described above is both continuous and irreducible.

Proof: Let $g : S(R(X)) \rightarrow X$ be the function described above. Note that if U is any open set containing the point $g(\mathfrak{F})$, then $\text{cl } U$ belongs to $\mathfrak{F}$. This follows from the maximality of $\mathfrak{F}$ since $(\text{cl } U)^c$ does not contain $g(\mathfrak{F})$ and hence cannot belong to $\mathfrak{F}$.

We now show that g is continuous. Let $\mathfrak{F}$ belong to $S(R(X))$ and let U be a neighborhood of $g(\mathfrak{F})$. Because X is regular, there exists an open set V such that

$$g(\mathfrak{F}) \subset V \subset \text{cl } V \subset U.$$

Now $\text{cl } V$ is a regular closed subset of X and hence determines a clopen set W of $S(R(X))$ via the Stone isomorphism h of $R(X)$ with the clopen sets of $S(R(X))$. Thus, we have

$$W = h(\text{cl } V) = \{Q \in S(R(X)) : \text{cl } V \in Q\}.$$

But then the definition of g implies that if Q belongs to W ,

$$g(Q) \in \bigcap Q \subset \text{cl } V \subset U.$$

Thus, $g[W]$ is contained in U and g is continuous.

To show that g maps $S(R(X))$ onto X , we must show that every point x of X is the limit of a maximal filter of regular closed sets. But this is clear since the set of regular closed sets containing x in their interior is closed under finite meets and hence is contained in a maximal filter.

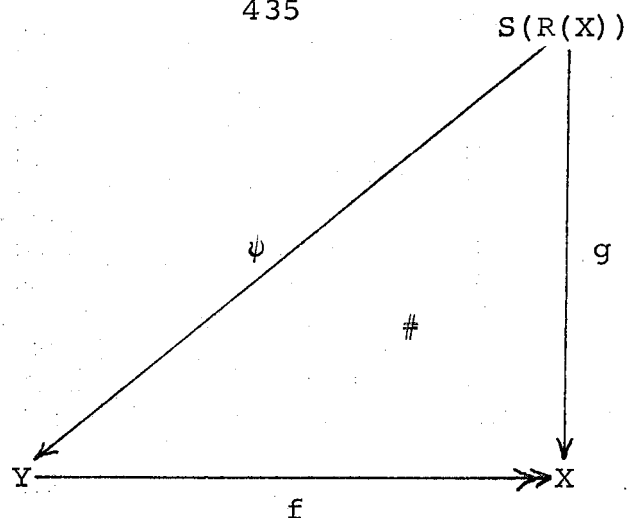
Finally, we show that g is irreducible. The complement of any proper closed subset F of $S(R(X))$ contains a non-empty basic clopen set W . The Stone Representation Theorem implies that there exists a non-empty regular closed subset K of X such that

$$W = h(K) = \{Q \in S(R(X)) : K \in Q\}.$$

Now if $g(Q)$ is in $\text{int } K$, we have seen that K must belong to Q and that Q is in W . Hence, $g[F]$ misses $\text{int } K$, and g is irreducible.

Hence, $S(R(X))$ and g form a projective cover of X .

Now suppose that Y is a compact extremely disconnected space which is mapped irreducibly onto X by f . Then we have a map ψ such that the following diagram commutes:



Because $S(R(X))$ is compact, ψ is closed. Because $f \circ \psi[S(R(X))] = X$ and f is irreducible, the closed subspace $\psi[S(R(X))]$ of Y must be all of Y . On the other hand, if F is a proper closed subspace of $S(R(X))$, $g[F] = f \circ \psi[F]$ cannot be all of X because g is irreducible. Hence, $\psi[F]$ is a proper subspace of Y and ψ is therefore irreducible. Therefore, Lemma 10.50 implies that ψ is a homeomorphism because Y is extremely disconnected. $\blacksquare$

10.55. Note that the preceding theorem together with Example 10.53 shows that the Stone space of $R(S_1)$ is $\beta \mathbb{N}$.

10.56. As was the case for the characterization of projective objects, we now apply the compact case to obtain the projective cover for an arbitrary completely regular space.

COROLLARY:

Every completely regular space has a projective cover which is unique up to homeomorphism.

Proof: Let X be completely regular and let $g : S(R(\beta X)) \rightarrow \beta X$ be the projective cover of βX . Put $E(X) = g^{-1}(X)$. The restriction $g|E(X)$ is easily seen to be perfect and irreducible because it is the restriction to a saturated set. If U is a non-empty open subset of $S(R(\beta X))$ missing $E(X)$, then $g[S(R(\beta X)) \setminus U]$ is a closed subspace of βX containing X , and hence, is all of βX . Thus, $S(R(\beta X)) \setminus U = S(R(\beta X))$ since g is irreducible, and U therefore is empty. Hence, $E(X)$ is dense in $S(R(\beta X))$. Proposition 10.47 implies that $E(X)$ is C^* -embedded in the extremely disconnected space $S(R(\beta X))$. Hence, $\beta(E(X)) = S(R(\beta X))$ so that Proposition 10.47 now implies that $E(X)$ is extremely disconnected. Hence, $g|E(X) : E(X) \rightarrow X$ is the projective cover of X . The uniqueness of $E(X)$ follows from that of $S(R(\beta X))$. $\blacksquare$

10.57. Note that in the preceding proof we showed that $\beta(E(X)) = S(R(\beta X))$. Since $S(R(\beta X))$ is $E(\beta X)$, we have established that E and β commute.

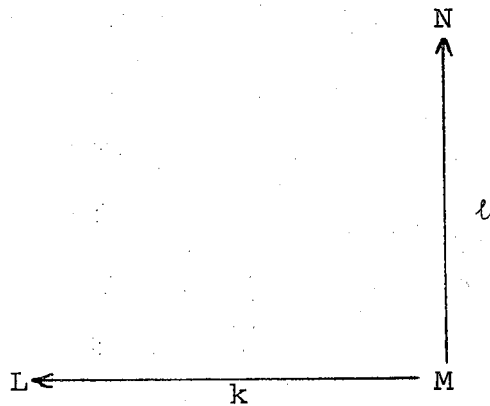
PROPOSITION:

If X is any completely regular space, then $\beta(E(X)) = E(\beta X)$.

10.58. The 1967 paper of Strauss contains a construction of projective covers using filters of open sets. The 1959 paper of J. Rainwater describes an additional construction. The 1966 and 1968 papers of B. Banaschewski describe projective covers in more general categorical terms. The 1971 paper of A. W. Hager provides a fourth proof of the existence of a projective cover

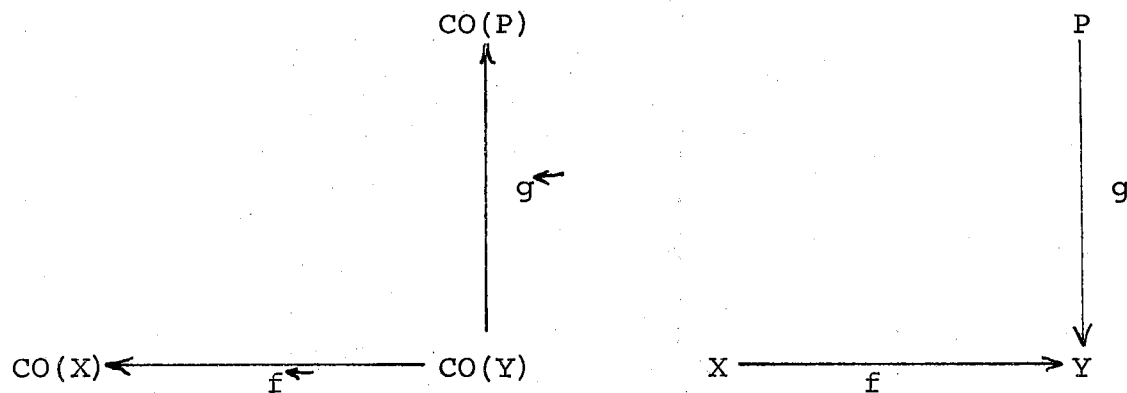
for a compact space.

10.59. Now we will borrow again from module theory and briefly consider injective Boolean algebras. Consider the following diagram in the category of Boolean algebras and Boolean algebra homomorphisms:



The Boolean algebra N is said to be injective if there exists a homomorphism $\varphi : L \rightarrow N$ such that $\varphi \circ k = l$ whenever k is a monomorphism, i.e. when k is one-to-one.

We will use the duality between Boolean algebras and compact totally disconnected spaces to characterize the injective Boolean algebras. If we express the Boolean algebras as algebras of clopen sets of compact totally disconnected spaces, then we have the following two diagrams:



Further, we can easily see from Proposition 2.8 that $f^{\leftarrow}$ is a monomorphism exactly when f is onto. Since X is compact and Y is Hausdorff, f will always be perfect.

Since a mapping $\psi : P \rightarrow X$ such that $f \circ \psi = g$ will always exist precisely when P is extremally disconnected, we see that the homomorphism $\varphi = \psi^{\leftarrow}$ will exist precisely when $CO(P)$ is complete (Proposition 2.5). Hence, we see that the injective Boolean algebras are just the complete ones.

EXERCISES

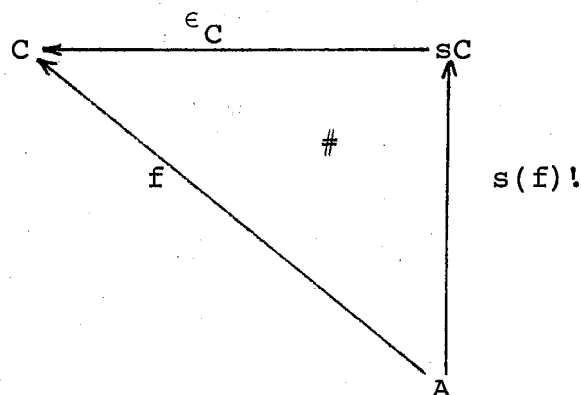
10A. EPI-REFLECTIVE SUBCATEGORIES

The category of all topological spaces and all mappings has the following properties:

1. The epimorphisms are the onto mappings.
2. The extremal monomorphisms are the embeddings.
3. Every mapping factors into an epimorphism followed by an extremal monomorphism.
4. The epi-reflective subcategories are the productive and hereditary subcategories.
5. The completely regular spaces form an epi-reflective subcategory.
6. A space is functionally Hausdorff if distinct points are completely separated. The reflection is the quotient space obtained by identifying points which are not completely separated.

10B. COREFLECTIVE SUBCATEGORIES

A functor $s : C \rightarrow Q$ is a coreflective functor if Q is a subcategory of C and there is a morphism $\epsilon_C : sC \rightarrow C$ such that any morphism $f : A \rightarrow C$ with A an object of Q factors uniquely through ϵ_C . Q is called a coreflective subcategory of C and sC is called the coreflection of C in Q . We will give three examples of coreflective subcategories of the



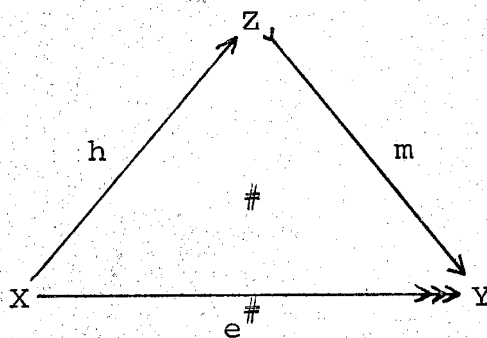
category of all topological spaces by describing the coreflection.

1. The category of P-spaces (Section 1.65) is a coreflective subcategory and the coreflection is obtained by enlarging the topology on the space to include all G_δ sets as open sets.
2. The P-space coreflection of a completely regular space can be obtained by enlarging the topology to include the zero-sets.
3. The category of k-spaces is a coreflective subcategory.
[The coreflection is obtained by enlarging the topology in a suitable way. See Exercise 8H.]
4. A subset of a space is called sequentially open if every sequence converging to a point in the subset is eventually in the subset. Any open set is sequentially open but not conversely. [S_1 or S_2 , 3.44.]
5. A sequential space is one in which every sequentially open set is open. The sequential spaces form a coreflective subcategory. The coreflection is obtained by enlarging the topology to include the sequentially open sets.

References: Sequential spaces are discussed in S. P. Franklin, 1965, 1967A, and S. Baron, 1968. The notion of topological coreflections are also discussed in Herrlich, 1968.

10C. CHARACTERIZATION OF COREFLECTIVE SUBCATEGORIES

A coreflective subcategory is called mono-coreflective if the mapping $\epsilon_C : sC \rightarrow C$ is a monomorphism. An epimorphism $e^\#$ is an extremal epimorphism if whenever $e^\#$ can be factored as illustrated



so that m is a monomorphism, then m is an isomorphism.

Extremal epimorphisms will be indicated by "triple-headed arrows" and a "#" on the name of the morphism. The category of all topological spaces has the following properties:

1. The monomorphisms are the one-to-one mappings.
2. The extremal epimorphisms are the quotient mappings.
3. Every mapping factors into an extremal epimorphism followed by a monomorphism. [Identify fibers of the mapping.]
4. There is only a set of pairwise non-homeomorphic spaces which can be mapped to a given space by a monomorphism.
5. The monoreflective subcategories are those which are closed under quotients and disjoint topological sums. [Dualize 10.21.]

10D. LOCALLY CONNECTED SPACES

A space is locally connected if each point has a base of connected neighborhoods.

1. A space is locally connected if and only if the components of its open sets are open.
2. A quotient of a locally connected space is locally connected.
3. A disjoint topological sum of a family of locally connected spaces is locally connected.
4. The locally connected spaces form a mono-coreflective subcategory of all topological spaces.

Reference: A. M. Gleason, 1963. One might expect from Exercise 10B that the coreflection is obtained by enlarging the topology to include components of open sets. The coreflection is actually obtained by iterating this process.

10E. THE CATEGORY OF PSEUDOCOMPACT SPACES

1. The product of a pseudocompact space and a compact space is compact. [8.21.]
2. A regular closed subspace of a pseudocompact space is pseudocompact. [8.5.]
3. A closed subspace of a pseudocompact space need not be pseudocompact. Hence, the category of pseudocompact spaces is not left-fitting. [6.11 and 10.38.]
4. Let $\mathcal{P}$ be the class of spaces such that if Y is in $\mathcal{P}$, then $X \times Y$ is pseudocompact for any pseudocompact space X .

If Y is a pseudocompact space such that every point of Y has a neighborhood belonging to $\mathcal{P}$, then Y is in $\mathcal{P}$.

[Modify 8.21.]

5. $\mathcal{P}$ is finitely productive.

Reference: (4) appears in Frolik, 1960B.

10F. ONE-POINT COMPACTIFICATION AS FUNCTOR

1. Let X and Y be non-compact, locally compact spaces. A mapping $f : X \rightarrow Y$ is perfect if and only if the extension $\alpha(f) : \alpha X \rightarrow \alpha Y$ defined by

$$\alpha(f)(x) = \begin{cases} f(x) & \text{for } x \in X \\ \infty & \text{for } x = \infty \end{cases}$$

is continuous.

2. Let $\mathcal{C}$ be the (non-full) category of all non-compact, locally compact spaces and perfect mappings. The one-point compactification defines a functor from $\mathcal{C}$ to the category of compact spaces.

10G. LEFT-FITTING AND EPI-REFLECTIVE

1. An epi-reflective subcategory of the category of Hausdorff spaces is left-fitting if and only if it contains all the compact spaces. [10.21 and 10.38]
2. $\cup X$ is the smallest realcompact subspace of βX which contains X .

10H. ANOTHER DEFINITION OF PROJECTIVE

Define an object P of a category $\mathcal{C}$ of topological spaces and mappings to be projective if whenever $f : X \rightarrow Y$ is an onto mapping and $g : P \rightarrow Y$ is any mapping, then there exists a map $\psi : P \rightarrow X$ such that $f \circ \psi = g$. With this definition, the projective objects in any full subcategory of topological spaces and mappings which contains the discrete spaces are just the discrete spaces.

10I. AN EQUIVALENT OF THE AXIOM OF CHOICE

In Lemma 10.48, we used a form of the Axiom of Choice, Zorn's Lemma, to show that any compact onto mapping has an irreducible restriction. Show that the existence of an irreducible restriction for any compact onto mapping implies the Axiom of Choice. [Let $\{F_\alpha : \alpha \in Q\}$ be any family of non-empty sets. Define a topology on each F_α by letting the closed subsets be the finite subsets. Map $\bigoplus F_\alpha$ onto the index set Q with Q having the discrete topology.]

Reference: S. P. Franklin and B. V. S. Thomas, 1971.

10J. COPRODUCTS OF PROJECTIVE OBJECTS

If $\mathcal{C}$ is any full subcategory in which coproducts exist, then a coproduct of a family of objects is projective if and only if each object in the family is projective.

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LIST OF SYMBOLS

- [D] reference to Dugundji, 1966
- [GJ] reference to Gillman and Jerison, 1960
- $C(X)$ ring of real-valued continuous functions, 1.2
- $C^*(X)$ ring of bounded real-valued continuous functions, 1.2
- $m(A)$ maximal ideal space, 1.20
- $m(X)$ maximal ideal space of $C(X)$, 1.23
- $m^*(X)$ maximal ideal space of $C^*(X)$, 1.23
- M^P maximal ideal of $C(X)$, 1.30
- M^{*P} maximal ideal of $C^*(X)$, 1.22
- f^{-1} reciprocal of f in $C(X)$, 1.2
- $f^{\leftarrow}$ inverse function, 1.2
- $Z(f)$ the zero-set of f , i.e. $f^{\leftarrow}(0)$, 1.2
- $Cz(f) = X \setminus Z(f)$ the cozero-set of f , 1.2
- $\text{pos}(f) = \{x : f(x) > 0\}$ 1.60
- $\text{neg}(f) = \{x : f(x) < 0\}$ 1.60
- $\text{osc}_S(f) = \sup\{|f(s) - f(t)| : s, t \in S\}$ 8.6
- $\mathcal{B}(x)$ neighborhood filter of x , 1.18
- $\text{cl } S$ closure of S
- $\text{int } S$ interior of S
- ∂S boundary of S
- $X \approx Y$ homeomorphic, 8.18
- $|S|$ cardinality of the set S
- $|r|$ absolute value of the number r
- $g \circ f$ composition, 1.2
- c the cardinality of the continuum
- $\oplus X_\alpha$ coproduct, 10.15
- $\times X_\alpha$ product, 10.15

- $\tilde{r}$ constant real-valued mapping with value r , 1.2
- $x \vee y$ least upper bound, 2.2
- $x \wedge y$ greatest lower bound, 2.2
- $\#$ signals a commutative diagram, 1.2
- $!$ signals uniqueness of morphism, 10.10
- $\hookrightarrow$ embedding, 1.9
- $\twoheadrightarrow$ closed embedding, 1.9
- $\rightarrow$ map with dense range, 1.9
- $\mathcal{I}(p)$ type of ultrafilter, 3.41
- $\mathcal{I}_X(p)$ relative type, 3.42
- $Z[X]$ the collection of all zero-sets of X , 1.2
- A^D z -ultrafilter, 1.32
- $CZ[X]$ the collection of all cozero-sets of X , 1.2
- βX Stone-Čech compactification of X , 1.1
- ρX the completely regular reflection of X , 1.6
- νX Hewitt-Nachbin realcompactification, 1.53
- αX one-point compactification
- X^* the growth $\beta X \setminus X$, 1.53
- $R(X)$ Boolean algebra of regular closed subsets of X , 2.3
- $CO(X)$ Boolean algebra of clopen subsets of X , 2.4
- $S(L)$ Stone space of the Boolean algebra L , 2.10
- uD subspace of uniform ultrafilters, 5.13
- $HAUS(\mathcal{C})$ epireflective hull of $\mathcal{C}$, 10.23
- $E(X)$ projective cover of X , 10.53
- $\omega_X(\mathcal{B})$ space of $\mathcal{B}$ -ultrafilters, 1.19
- $\omega(Z[X])$ space of z -ultrafilters, 1.42
- Z^ω basic closed set in $\omega_X(\mathcal{B})$, 1.19
- U^ω basic open set of $\omega_X(\mathcal{B})$, 1.38

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- $\mathcal{Q}$ -completely regular spaces, 10.41
- $\mathcal{Q}$ -embedded, 10.41
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