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Variational Techniques for Water Waves and Singular Perturbations

BY

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To my family

Abstract

This thesis aims to provide a variational framework for the study of two problems that arise from fluid dynamics and continuum mechanics. The first part concerns a free boundary approach for the existence of periodic water waves. This is a notoriously hard problem as the only variational solutions of the unconstrained problem are waves with flat profiles. Nevertheless, it is shown that by considering an additional Dirichlet condition on part of the lateral boundary, nontrivial solutions can be found among minimizers of the classical Alt-Caffarelli functional. The second part of the thesis focuses on a regularization by singular perturbations of a mixed Dirichlet-Neumann boundary value problem. The asymptotic behavior of the solutions to the perturbed problems is studied by means of an asymptotic development by Gamma-convergence, recovering classical results in the literature.

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Contents

Chapter 1

Introduction

In this thesis we study two (essentially unrelated) problems with techniques borrowed from the theory of partial differential equations and the calculus of variations.

The first part of the thesis focuses on formulating a framework for the study of periodic water waves. The results presented here are mostly contained in the papers [56] and [57].

In the second part we are concerned with the study of a regularization for a mixed Dirichlet-Neumann boundary value problem. This contribution is contained in [55].

1.1 A free boundary approach for water waves

In the classical paper [2], Alt and Caffarelli studied the existence and regularity of solutions to the one-phase free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap \{u > 0\}, \\ u = 0 & \text{on } \Omega \cap \partial\{u > 0\}, \\ |\nabla u| = Q & \text{on } \Omega \cap \partial\{u > 0\}, \\ u = u_0 & \text{on } \Gamma, \end{cases} \tag{1.1.1}$$

using a variational approach. Here Ω is an open connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary and Q is a nonnegative measurable function. Solutions to (1.1.1) are critical points for the functional

$$\mathcal{J}(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}} Q^2\right) d\boldsymbol{x}, \quad u \in \mathcal{K}, \tag{1.1.2}$$

where

$$\mathcal{K} := \{u \in H^1_{\mathrm{loc}}(\Omega) : u = u_0 \text{ on } \Gamma\}, \tag{1.1.3}$$

with $\Gamma \subset \partial\Omega$ a measurable set with $\mathcal{H}^{N-1}(\Gamma) > 0$ and $u_0 \in H^1_{\mathrm{loc}}(\Omega)$ a nonnegative function satisfying

$$\mathcal{J}(u_0) < \infty. \tag{1.1.4}$$

The equality $u = u_0$ on Γ is in the sense of traces. Under the assumption that Q is a Hölder continuous function satisfying

$$0 < Q_{\min} \le Q(\boldsymbol{x}) \le Q_{\max} < \infty, \tag{1.1.5}$$

Alt and Caffarelli proved local Lipschitz regularity of local minima and showed that the free boundary $\partial\{u > 0\}$ is a $C^{1,\alpha}_{\mathrm{loc}}$ regular curve in Ω if $N = 2$, while if $N \ge 3$ they proved that the reduced free boundary is a hypersurface of class $C^{1,\alpha}_{\mathrm{loc}}$ in Ω, for some $0 < \alpha < 1$. See also [5] for the quasi-linear case and [45] for the case of the p-Laplace operator.

We remark that while the regularity of minimizers is optimal, the regularity of the free boundary for $N \ge 3$ was improved by Weiss in [91]. Weiss, following an approach closely related to the theory of minimal surfaces and by means of a monotonicity formula, proved the existence of a maximal dimension $k^* \ge 3$ such that for $N < k^*$ the free boundary is a hypersurface of class $C^{1,\alpha}_{\mathrm{loc}}$ in Ω, for $N = k^*$ the singular set consists at most of isolated points, and if $N > k^*$ then $\mathcal{H}^s(\{\text{singular set}\}) = 0$ for every $s > N - k^*$. In [27], Caffarelli, Jerison and Kenig proved the full regularity of the free boundary in dimension $N = 3$, thus showing that $k^* \ge 4$. They also conjectured that $k^* \ge 7$. In a later work De Silva and Jerison exhibited an example of a global energy minimizer with non-smooth free boundary in dimension 8 (see [49]); their result implies that $k^* \le 7$. As it was remarked in [2], if $N = 3$ the energy functional admits a critical point with a point singularity in the free boundary. Similar results have been obtained for two-phase free boundary problems (see [7], [22], [24], [23]).

It is important to observe that the regularity of the free boundary is strongly related to the assumption $0 < Q_{\min} \le Q(\boldsymbol{x})$ in (1.1.5). Indeed, in the recent paper [15], Arama and Leoni showed that for $N = 2$ and in the special case in which

$$Q(x, y) = \sqrt{(h - y)_+} \quad \text{for some } h > 0, \tag{1.1.6}$$

if a local minimizer u has support below the line $\{y = h\}$ and if there exists a point $\boldsymbol{x}_0 = (x_0, h) \in \partial\{u > 0\}$, then

$$|\nabla u(\boldsymbol{x})| \le C r^{1/2}, \quad \text{for } \boldsymbol{x} \in B_r(\boldsymbol{x}_0) \tag{1.1.7}$$

(see Remark 3.5 in [15]). On the other hand, using a monotonicity formula and a blow up method, Varvaruca and Weiss (see Theorem A in [90]) proved that for a suitable definition of solution if the constant C in (1.1.7) is one then the rescaled function

$$\frac{u(\boldsymbol{x}_0 + r\boldsymbol{x})}{r^{3/2}} \to \frac{\sqrt{2}}{3}\rho^{3/2}\cos\left(\frac{3}{2}\left(\min\left\{\max\left\{\theta, -\frac{5\pi}{6}\right\}, -\frac{\pi}{6}\right\} + \frac{\pi}{2}\right)\right) \quad \text{as } r \to 0^+,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ and locally uniformly on $\mathbb{R}^2$, where $(x, y) = (\rho\cos\theta, \rho\sin\theta)$, and near $\boldsymbol{x}_0$ the free boundary $\partial\{u > 0\}$ is the union of two C^1 graphs with right and left tangents at $\boldsymbol{x}_0$ (see also [93]). This type of singular solutions are related to Stokes' conjecture on the existence of extreme water waves (see [87]). The existence of extreme waves and the corner singularity have been proved in a series of papers (see [9], [10], [79], [84], [88]; see also [38], [65], [78], [85]) using a hodograph transformation to map the set $\{u > 0\}$ onto an annulus.

Note that for planar water waves of finite depth it is customary to set $N = 2$ and define

$$\Omega := (-\lambda/2, \lambda/2) \times (0, \infty), \qquad \Gamma := (-\lambda/2, \lambda/2) \times \{0\}, \qquad u_0 \equiv m \tag{1.1.8}$$

(see (1.1.1), (1.1.3)) and choose Q to be as in (1.1.6), where $\lambda > 0$ is the wavelength, and $m, h > 0$ are renormalized constants related to mass flux and hydraulic head, respectively (see, for example, [35]). Indeed, solutions to this problem correspond to steady periodic water waves moving on the free surface of an irrotational flow above a flat impermeable bed.

The main drawback in proving the existence of regular and extreme water waves using the variational setting of (1.1.2) is that global minimizers of the energy functional $\mathcal{J}$ specialized to the case (1.1.6), (1.1.8) are one dimensional functions of the form $u = u(y)$, which correspond to flat profiles (see Theorem 3.2.2). For this reason the paper [15] gives interesting results only for local minimizers or when the Dirichlet boundary datum u_0 is not constant on the bottom, a situation which is not compatible with water waves. Necessary and sufficient minimality conditions in terms of the second variation of $\mathcal{J}$ have been derived by Fonseca, Leoni and Mora in [50]. We refer to the papers [37], [36], [39], [30], [31], [52], [66], [89] and the references therein for alternative approaches to water waves.

To be precise, in the following we assume that $N = 2$ and focus on the study of solutions to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cap \{u > 0\}, \\ u(x+\lambda, y) = u(x,y) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega \cap \partial\{u > 0\}, \\ |\nabla u| = \sqrt{(h-y)_+} & \text{on } \Omega \cap \partial\{u > 0\}, \\ u = m & \text{on } \{y = 0\}, \end{cases} \tag{1.1.9}$$

where Ω is a half infinite strip, i.e.,

$$\Omega := \left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right) \times (0, \infty). \tag{1.1.10}$$

One of our main results is showing that by considering an additional Dirichlet boundary condition on part of the lateral boundary it is possible to construct solutions to (1.1.9), which are not of the form $u = u(y)$. We define the Sobolev space

$$H^1_{\lambda,\mathrm{loc}}(\Omega) := \left\{u \in H^1_{\mathrm{loc}}(\mathbb{R}^2_+) : u(x+\lambda, y) = u(x,y) \text{ for } \mathcal{L}^2\text{-a.e. } \boldsymbol{x} = (x,y) \in \mathbb{R}^2_+\right\}, \tag{1.1.11}$$

and for $m, h > 0$ consider the energy functional

$$\mathcal{J}_h(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}}(h-y)_+\right) d\boldsymbol{x}, \quad \text{for } u \in \mathcal{K}_\gamma, \tag{1.1.12}$$

where

$$\mathcal{K}_\gamma := \left\{u \in H^1_{\lambda,\mathrm{loc}}(\Omega) : u(\cdot, 0) = m \text{ and } u(\pm\lambda/2, y) = 0 \text{ for } y \geq \gamma\right\}. \tag{1.1.13}$$

Here γ is a positive constant, and the boundary conditions are satisfied in the sense of traces. Choosing γ opportunely has the effect of eliminating trivial solutions from the domain of $\mathcal{J}_h$. This is made precise in the following theorem.

Theorem 1.1.1. *Given $m, \lambda, h > 0$, let Ω and $\mathcal{J}_h$ be defined as in* $(1.1.10)$ *and* $(1.1.12)$*, respectively. Let*

$$h^{\#} := 3\left(\frac{m}{2}\right)^{2/3}, \quad h^* := 3\left(\frac{m}{\sqrt{2}}\right)^{2/3}, \tag{1.1.14}$$

and, for $h > h^{\#}$, let t_h be the first positive root of the cubic polynomial

$$t^3 - ht^2 + m^2 = 0.$$

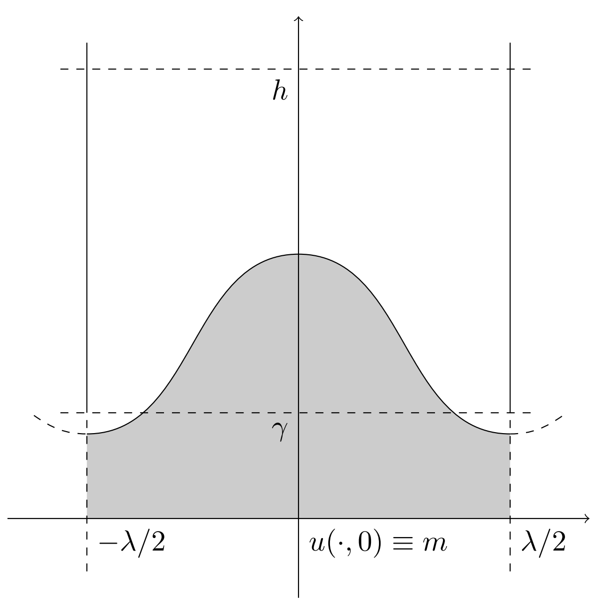
Furthermore, for $h \in (h^{\#}, h^*)$, let $\tau_h > t_h$ be the unique value such that

$$\frac{m^2}{t_h} + \frac{h^2 - (h - t_h)^2}{2} = \frac{m^2}{\tau_h} + \frac{h^2 - (h - \min\{h, \tau_h\})^2}{2},$$

and set $\tau_h = t_h = 2h/3$ if $h = h^{\#}$. Then every global minimizer uof $\mathcal{J}_h$ in $\mathcal{K}_\gamma$ is not of the form $u = u(y)$ provided

$$\begin{cases} \gamma \in (0, \infty) & \text{if } h < h^{\#}, \\ \gamma \in (0, t_h) \cup (\tau_h, \infty) & \text{if } h^{\#} \le h < h^*, \\ \gamma \in (0, t_h) & \text{if } h \ge h^*. \end{cases} \tag{1.1.15}$$

Remark 1.1.2. *The numbers $h^{\#}, h^*, t_h$, and τ_h arise naturally from the study of the minimization problem for a one dimensional version of $\mathcal{J}_h$, as discusses in detail in Section 3.2.*



We then study qualitative properties of global minimizers as we vary the height h. By adapting to our setting the monotonicity techniques developed in Section 5 in [3], Theorem 10.1 in [53], and the non-degeneracy result of Lemma 3.4 in [2], we are able to prove an analogue of Theorem 5.6 in [15].

Theorem 1.1.3 (Existence of a critical height)**.** *Given $m, \lambda > 0$, let $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function such that*

$$\theta(h) = \gamma_h, \tag{1.1.16}$$

where for every h the number γ_h is chosen as in (1.1.15)*. Furthermore, let Ω, $\mathcal{J}_h$ and $\mathcal{K}_{\gamma_h}$ be defined as in* (1.1.10)*,* (1.1.12)*, and* (1.1.13)*, respectively. Then there exists a critical height $0 < h_{\text{cr}} < \infty$ with the property that*

(i) if $h_{\text{cr}} < h < \infty$ then every global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_h}$ has support strictly below the line $\{y = h\}$;

(ii) if $0 < h < h_{\text{cr}}$ then every global minimizer is positive in $(-\lambda/2, \lambda/2) \times [h, \infty)$.

Notice that for $h > h_{\text{cr}}$ we are in a position to apply the regularity result of [2] to conclude that the free boundary $\partial\{u > 0\}$ of every global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_h}$ is locally an analytic curve in Ω. Theorem 3.3.5 shows that the result holds for $h > h_{\text{cr}}$ as well. We also remark that Theorem 1.1.3 shows that the critical height h_{cr} is the only value of the parameter h for which the free boundaries of global minimizers of $\mathcal{J}_h$ can touch the line $\{y = h\}$ without crossing it, and that every such minimizer is a Stokes wave. By the comparison principle in Theorem 3.3.6 and the convergence of minimizers of Corollary 4.1.4, it follows that by letting $h \nearrow h_{\text{cr}}$ there exists a global minimizer $u^- \in \mathcal{K}_{\gamma_{h_{\text{cr}}}}$ of $\mathcal{J}_{h_{\text{cr}}}$ whose support (restricted to Ω) is contained in $(-\lambda/2, \lambda/2) \times [0, h_{\text{cr}}]$, while if $h \searrow h_{\text{cr}}$ then there exists another global minimizer u^+ of $\mathcal{J}_{h_{\text{cr}}}$ with $u^- \leq u^+$ and whose support cannot be strictly below the line $\{y = h_{\text{cr}}\}$ (see also Theorem 3.3.15). We have not been able to prove that the support of any global minimizer touches the critical height. This would follow if we had uniqueness at this level (see Theorem 3.3.14).

Concerning the value of h_{cr}, we are able to show that for all m

$$h_{\text{cr}} \leq h^*,$$

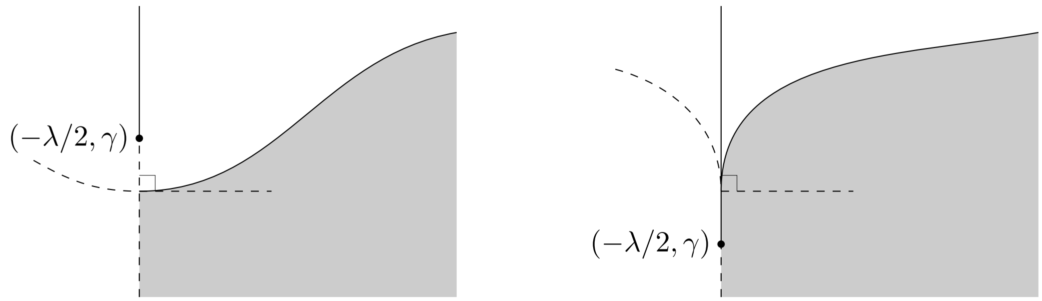
and, under mild assumptions on the function θ in the statement of Theorem 1.1.1, that

$$h_{\text{cr}} \geq k h^{\#},$$

where $k > 0$ is a constant (we refer to Lemma 3.3.8 and Theorem 3.3.9 for more details). In particular, we find the scaling law

$$h_{\text{cr}} \sim m^{2/3}.$$

Finally, it is important to notice that while the additional Dirichlet condition in (1.1.13) allows us to construct nontrivial solutions to (1.1.9), it comes with the disadvantage of potentially destroying the regularity near the fixed boundary. The regularity at the boundary for global minimizers and their free boundaries away from the points $(\pm\lambda/2, \gamma)$ is well understood. Indeed, due to the periodic boundary conditions below the line $\{y = \gamma\}$, if the free boundary $\partial\{u > 0\}$ of a global minimizer touches the fixed boundary strictly below the line $\{y = \gamma\}$ then the regularity follows from the classical interior regularity of [2]. On the other hand, if the free boundary touches the fixed boundary strictly above that line, then it follows from a recent results of Chang-Lara and Savin (see Theorem 1.1 in [29]; see also [4], [8], [63], and [92]) that its free boundary detaches tangentially from the fixed boundary and is a $C^{1,1/2}$ regular curve locally in a neighborhood of $\partial\Omega$.



We refer to the work of Raynor [86] for a variational proof of the Lipschitz continuity of global minimizers of $\mathcal{J}$ near a Neumann fixed boundary.

In Section 3.4 we analyze the behavior of the free boundary near the points $(\pm\lambda/2, \gamma)$ for a certain class of global minimizers of $\mathcal{J}_h$.

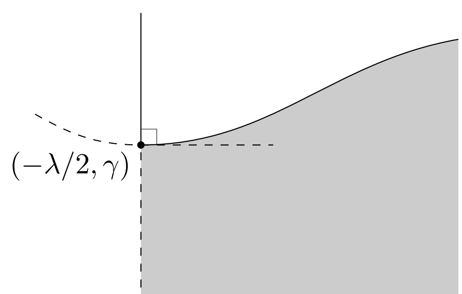
Definition 1.1.4. *Let u be a global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$. We say that u is a symmetric minimizer if for $\mathcal{L}^1$-a.e. $y \in \mathbb{R}_+$ the map $x \mapsto u(x, y)$ is even and nondecreasing in $(-\lambda/2, 0)$.*

The main result of the section can be stated as follows.

Theorem 1.1.5. *Given $m, \lambda, h > 0$ and $\gamma < h$, let Ω, $\mathcal{J}_h$, and $\mathcal{K}_\gamma$ be defined as in (1.1.10), (1.1.12) and (1.1.13), respectively. Let $u \in \mathcal{K}_\gamma$ be a symmetric global minimizer of $\mathcal{J}_h$ in the sense of Definition 1.1.4 and assume that $\boldsymbol{x}_0 = (-\lambda/2, \gamma)$ is an accumulation point for the free boundary on $\partial\Omega$, i.e.,*

$$\boldsymbol{x}_0 \in \overline{\partial\{u > 0\} \cap \Omega}.$$

Then the free boundary $\partial\{u > 0\}$ meets the fixed boundary at the point $\boldsymbol{x}_0$ with horizontal tangent.



Hence, the results of Section 3.4 show that it is possible to construct a family of gravity waves, i.e., solutions to (1.1.9), whose free boundaries are non flat and meet the fixed boundary either with a vertical tangent or with a horizontal tangent. We remark that if one was able to prove that for some choice of the parameters λ, m, h, γ there exists a global minimizer with the property that every contact point in the set $\{y \leq \gamma\}$, as a Corollary of Theorem 1.1.5, we would obtain a variational proof of the existence of regular water waves which does not rely on Nekrasov's equation (see the classical paper of Keady and Norbury [65]).

The starting point of our analysis is Theorem 3.4.1, where we prove a uniform estimate on the gradient of a symmetric minimizer in a neighborhood of the point $\boldsymbol{x}_0$. This kind of result is commonly referred to as the bounded gradient lemma (see, for example, Lemma 8.1 and 8.2 in [4], Lemma 2.1 and 2.2 in [26], and 3.7 Theorem in [1]). Our main contribution is proving that the estimate holds up to the fixed Dirichlet boundary, uniformly with respect to the distance from the point $\boldsymbol{x}_0$. This is accomplished through the use of a boundary Harnack principle (see Theorem 2.2.9). The relevance of Theorem 3.4.1 is that it allows us to consider blow-up limits. Indeed, as it is often the case for this kind of regularity results (see for example [63], [11], [12], and [19]), the proof of the main result will rely heavily on the complete characterization of blow-up solutions (see Theorem 3.4.13). This, in turn, is derived from a monotonicity formula. To be precise, we show that the boundary monotonicity formula of Weiss (see Theorem 3.3 and Corollary 3.4 in [92], see also [91] and [90]) holds at the point $\boldsymbol{x}_0$ for global minimizers of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$ with bounded gradients.

1.2 Higher-order Gamma-limits for singularly perturbed Dirichlet-Neumann problems

Mixed Dirichlet-Neumann boundary value problems arise naturally from a wide range of applications. Examples are the problem of a rigid punch or stamp making contact with an elastic body (see [40], [41], [94], and the references therein), the steady flow of an ideal inviscid and incompressible fluid through an aperture in a reservoir (see [80], [94], and the references therein), as well as free boundary problems (see, e.g., [2]).

The prototype for this kind of problems is given by

$$\begin{cases} \Delta u_0 = f \ \text{ in } \Omega, \\ \partial_\nu u_0 = 0 \ \text{ on } \Gamma_N, \\ u_0 = g \ \text{ on } \Gamma_D, \end{cases} \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open set with sufficiently smooth boundary and Γ_D, Γ_N are disjoint sets such that

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}.$$

It is well known (see [47], [58], [68], and [76]) that solutions to mixed boundary problems are in general not smooth near the points on the boundary of the domain where two different conditions meet. Indeed, when $N = 2$ in (1.2.1), $f = 0$, $g = 0$, and Ω is given in polar coordinates by

$$\{(r, \theta) : r > 0, 0 < \theta < \pi\},$$

the function $S \colon \Omega \to \mathbb{R}$ given is polar coordinates by[1]

$$\bar{S}(r, \theta) := r^{1/2} \sin(\theta/2) \tag{1.2.2}$$

is a solution to (1.2.1), where Γ_D and Γ_N correspond to the positive real axis and the negative real axis, respectively. However, S fails to be in H^2 in any neighborhood of the origin.

In dimension $N = 2$ it turns out that functions of the type (1.2.2) completely characterize the behavior of solutions to (1.2.1). Indeed, we have the following classical result (see [47], [58], [68], and [76]).

[1]In what follows, given a function $v = v(\boldsymbol{x})$ where $\boldsymbol{x} = (x, y)$, we denote by $\bar{v}$ the function $\bar{v}(r, \theta) := v(r\cos\theta, r\sin\theta)$, and with a slight abuse of notation we write $v = \bar{v}(r, \theta)$.

Theorem 1.2.1. *Let $N = 2$, and let Ω be an open, bounded, and connected subset of $\mathbb{R}^2$, with $\partial\Omega$ of class $C^{1,1}$. Assume that Γ_D and Γ_N are nonempty, relatively open, and connected subsets of $\partial\Omega$ with*

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \quad \text{and} \quad \overline{\Gamma_D} \cap \overline{\Gamma_N} = \{\boldsymbol{x}_1, \boldsymbol{x}_2\},$$

and that $\partial\Omega \cap B_\rho(\boldsymbol{x}_i)$ is a segment for $i = 1, 2$ and for some $0 < \rho < \min\{1, |\boldsymbol{x}_1 - \boldsymbol{x}_2|/2\}$. Let $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$, and let $u \in H^1(\Omega)$ be a weak solution to (1.2.1). Then u admits the decomposition

$$u = u_{\text{reg}} + \sum_{i=1}^{2} c_i S_i,$$

where $u_{\text{reg}} \in H^2(\Omega)$ and the c_i are coefficients that only depend on u. The singular functions S_i are given by the formula

$$\bar{S}_i(r_i, \theta_i) = \bar{\varphi}(r_i) r_i^{1/2} \sin(\theta_i/2),$$

where (r_i, θ_i) are polar coordinates centered at $\boldsymbol{x}_i$ such that

$$\Omega \cap B_\rho(\boldsymbol{x}_i) = \{\boldsymbol{x}_i + (r_i, \theta_i) : 0 < r_i < \rho, 0 < \theta_i < \pi\},$$

$$\Gamma_D \cap B_\rho(\boldsymbol{x}_i) = \{\boldsymbol{x}_i + (r_i, 0) : 0 < r_i < \rho\},$$

and $\bar{\varphi} \in C^\infty([0, \infty))$ is such that $\bar{\varphi} \equiv 1$ in $[0, \rho/2]$ and $\bar{\varphi} \equiv 0$ outside $[0, \rho]$. Furthermore, there exists a constant c, which only depends on the geometry of Ω, such that

$$\|u_{\text{reg}}\|_{H^2(\Omega)} + \sum_{i=1}^{2} |c_i| \leq c \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)} \right).$$

An approach that often proved to be successful for the study of ill-posed problems, and in general for problems that present singularities of some kind, is to consider a small perturbation, typically chosen with an opportunely regularizing effect, and then carry out a careful analysis on the convergence of solutions of the regularized problems to solutions of the original one. This procedure often requires to prove estimates that are independent of the parameter of the regularization. We refer to the classical monograph of Lions [73] for more details.

In the second part of this thesis we regularize the problem (1.2.1) by introducing a family of mixed Neumann-Robin boundary value problems parametrized by $\varepsilon > 0$. To be precise, we consider

$$\begin{cases} \Delta u_\varepsilon = f \;\text{ in } \Omega, \\ \partial_\nu u_\varepsilon = 0 \;\text{ on } \Gamma_N, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g \;\text{ on } \Gamma_D. \end{cases} \tag{1.2.3}$$

The convergence of solutions to (1.2.3) to solutions of (1.2.1) has been studied by Costabel and Dauge in [40] using classical PDE expansions (see [73]), who proved the following result.

Theorem 1.2.2 (Costabel-Dauge). *Let $N = 2$, Ω be as in Theorem 1.2.1, $f = 0$, $g \in H^{1+\delta}(\Gamma_D)$ for some $\delta > 0$, and let u_ε and u_0 be solutions to (1.2.3) and (1.2.1) (with $f = 0$), respectively. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon \log \varepsilon),$$

$$\|u_\varepsilon - u_0\|_{H^{1+s}(\Omega)} = \mathcal{O}(\varepsilon^{1/2-s}), \textit{ for } s \in \left(-\frac{1}{2}, \frac{1}{2}\right), \tag{1.2.4}$$

$$\left\|(u_\varepsilon - u_0)_{|\Gamma_D}\right\|_{L^2(\Gamma_D)} = \mathcal{O}(\varepsilon\sqrt{|\log\varepsilon|}). \tag{1.2.5}$$

Moreover, these estimates cannot be improved in general.

We refer to [40] for the precise statement in the case $f \neq 0$. This problem was also previously considered by Colli Franzone in [32], where the author proved estimates on the difference $u_\varepsilon - u_0$ in certain Sobolev norms (see also the work of Aubin [16] and Lions [73]).

The question of convergence of solutions to the family of problems (1.2.3) to the solution to (1.2.1) is of significance for the numerical approximations of (1.2.1). We refer to [17], [21], [41], [33], [34], and the references therein for more information on this topic.

In the second part of this thesis we present an alternative proof of the estimates (1.2.4) with $s = 0$ and (1.2.5) using the variational structure of (1.2.3). Indeed, solutions to (1.2.3) are minimizers of the functional

$$\int_\Omega \left(\frac{1}{2}|\nabla v|^2 + fv\right)\,d\boldsymbol{x} + \frac{1}{2\varepsilon}\int_{\Gamma_D}(v-g)^2\,d\mathcal{H}^1, \quad v \in H^1(\Omega). \tag{1.2.6}$$

Thus a natural approach is to use the notion of Gamma-convergence (Γ-convergence in what follows) introduced by De Giorgi in [48] (for more information see also [20] and [42]).

The powerfulness of asymptotic expansions by Γ-convergence has been shown in the recent papers [43], [71], [72], and [82], where the authors completely characterized the second order asymptotic expansion of the Modica-Mortola functional and used it to obtain new important results on the slow motion of interfaces for the mass-preserving Allen-Cahn equation and the Cahn-Hilliard equation in higher dimensions.

We investigate asymptotic expansions by Γ-convergence for the functionals (1.2.6) with respect to convergence in $L^2(\Omega)$, and thus we define $\mathcal{F}_\varepsilon\colon L^2(\Omega) \to (-\infty, \infty]$ via

$$\mathcal{F}_\varepsilon(v) := \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{2}|\nabla v|^2 + fv\right)\,d\boldsymbol{x} + \frac{1}{2\varepsilon}\int_{\Gamma_D}(v-g)^2\,d\mathcal{H}^1 & \text{if } v \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2.7}$$

We begin by studying the Γ-convergence of order zero of (1.2.7).

Theorem 1.2.3 (0th order Γ-convergence)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, connected set with Lipschitz continuous boundary, and let $\Gamma_D \subset \partial\Omega$ be non-empty and relatively open. Assume that $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Then the family of functionals $\{\mathcal{F}_\varepsilon\}_\varepsilon$ defined in* (1.2.7) *Γ-converges in $L^2(\Omega)$ to the functional*

$$\mathcal{F}_0(v) := \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{2}|\nabla v|^2 + fv\right)\,d\boldsymbol{x} & \textit{if } v \in V, \\ +\infty & \textit{otherwise,} \end{cases} \tag{1.2.8}$$

where

$$V := \{v \in H^1(\Omega) : v = g \textit{ on } \Gamma_D\}. \tag{1.2.9}$$

Since the first asymptotic development by Γ-convergence of (1.2.7) strongly relies on Theorem 1.2.1, in what follows we assume $N = 2$. We begin with a compactness result.

Theorem 1.2.4 (Compactness)**.** *Let $N = 2$, Ω be as in Theorem 1.2.1, $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$, $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ be the functionals defined in* (1.2.7) *and* (1.2.8)*, respectively, and define*

$$\mathcal{F}_\varepsilon^{(1)} := \frac{\mathcal{F}_\varepsilon - \min \mathcal{F}_0}{\varepsilon|\log\varepsilon|}. \tag{1.2.10}$$

If $\varepsilon_n \to 0^+$ and $v_n \in L^2(\Omega)$ are such that

$$\sup\{\mathcal{F}_{\varepsilon_n}^{(1)}(v_n) : n \in \mathbb{N}\} < \infty,$$

then there exist a subsequence $\{v_{n_k}\}_k$ of $\{v_n\}_n$, $r_0 \in H^1(\Omega)$ and $v_0 \in L^2(\Gamma_D)$ such that

$$\frac{v_{n_k} - u_0}{\sqrt{\varepsilon_{n_k}|\log\varepsilon_{n_k}|}} \rightharpoonup r_0 \quad \textit{in } H^1(\Omega), \tag{1.2.11}$$

$$\frac{v_{n_k} - u_0}{\varepsilon_{n_k}\sqrt{|\log\varepsilon_{n_k}|}} \rightharpoonup v_0 \quad \textit{in } L^2(\Gamma_D), \tag{1.2.12}$$

where u_0 is the solution to (1.2.1).

Theorem 1.2.5 (1st order Γ-convergence)**.** *Under the assumptions of Theorem 1.2.4, the family $\{\mathcal{F}_\varepsilon^{(1)}\}_\varepsilon$ Γ-converges in $L^2(\Omega)$ to the functional*

$$\mathcal{F}_1(v) := \begin{cases} -\dfrac{1}{8}\displaystyle\sum_{i=1}^{2} c_i^2 & \textit{if } v = u_0, \\ +\infty & \textit{otherwise}, \end{cases} \tag{1.2.13}$$

where the coefficients $c_i = c_i(u_0)$ are as in Theorem 1.2.1. In particular, if $u_\varepsilon \in H^1(\Omega)$ is a solution to (1.2.3)*, then*

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}_0(u_0) + \varepsilon|\log\varepsilon|\mathcal{F}_1(u_0) + o\left(\varepsilon|\log\varepsilon|\right). \tag{1.2.14}$$

To characterize the second order asymptotic development by Γ-convergence of the family of functionals $\{\mathcal{F}_\varepsilon\}_\varepsilon$, we introduce the auxiliary functional

$$\begin{aligned} \mathcal{J}_i(w) := {}& \int_{\mathbb{R}^2_+} |\nabla w(\boldsymbol{x})|^2 \, d\boldsymbol{x} + \int_0^1 \Big(w(x,0)^2 - c_i x^{-1/2} w(x,0) \Big) \, dx \\ & + \int_1^\infty \Big(w(x,0) - \frac{c_i}{2} x^{-1/2} \Big)^2 \, dx \end{aligned} \tag{1.2.15}$$

defined in

$$H := \{w \in H^1_{\mathrm{loc}}(\mathbb{R}^2_+) : w \in H^1(B_R^+(\mathbf{0})) \text{ for every } R > 0\}, \tag{1.2.16}$$

where $w(\cdot,0)$ indicates the trace of w on the positive real axis. Let[2]

$$A_i := \inf\{\mathcal{J}_i(w) : w \in H\}, \tag{1.2.17}$$

$$B_i := \frac{1}{2}\int_0^\rho \bar\varphi(r_i) r_i^{-1/2} \overline{\partial_\nu u^0_{\text{reg}}}^{(i)}(r_i,0)\,dr_i, \tag{1.2.18}$$

$$C_\varphi := \frac{1}{8}\int_{\rho/2}^1 \left(1-\bar\varphi(x)^2\right) x^{-1}\,dx, \tag{1.2.19}$$

$$\bar\psi_i(r_i) := \frac{1}{2}\bar\varphi(r_i) r_i^{-1/2}. \tag{1.2.20}$$

As shown in Proposition 4.3.4, there exists $w_i \in H$ such that $\mathcal{J}_i(w_i) = A_i$, and thus w_i satisfies

$$\begin{cases} \Delta w_i = 0 & \text{in } \mathbb{R}^2_+, \\ \partial_\nu w_i = 0 & \text{on } (-\infty,0)\times\{0\}, \\ \partial_\nu w_i + w_i = \frac{c_i}{2}x^{-1/2} & \text{on } (0,\infty)\times\{0\}. \end{cases} \tag{1.2.21}$$

Observe that if $c_i = 0$ then $\mathcal{J}_i \geq 0$ and so $w_i = 0$ and $A_i = 0$. Finally, let $u_1 \in H^1(\Omega)$ be the solution to the Dirichlet-Neumann problem

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega, \\ \partial_\nu u_1 = 0 & \text{on } \Gamma_N, \\ u_1 = -\partial_\nu u^0_{\text{reg}} & \text{on } \Gamma_D. \end{cases} \tag{1.2.22}$$

Theorem 1.2.6 (Compactness)**.** *Let $N = 2$, Ω be as in Theorem 1.2.1, $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$, $\mathcal{F}_\varepsilon$, $\mathcal{F}_0$, $\mathcal{F}^{(1)}_\varepsilon$, $\mathcal{F}_1$, $\mathcal{J}_i$ be as in* (1.2.7)*,* (1.2.8)*,* (1.2.10)*,* (1.2.13)*, and* (1.2.15)*, respectively, and define*

$$\mathcal{F}^{(2)}_\varepsilon := \frac{\mathcal{F}^{(1)}_\varepsilon - \min\mathcal{F}_1}{1/|\log\varepsilon|} = \frac{\mathcal{F}_\varepsilon - \min\mathcal{F}_0}{\varepsilon} - |\log\varepsilon|\min\mathcal{F}_1. \tag{1.2.23}$$

If $\varepsilon_n \to 0^+$, $w_n \in L^2(\Omega)$ are such that

$$\sup\{\mathcal{F}^{(2)}_{\varepsilon_n}(w_n) : n \in \mathbb{N}\} < \infty,$$

and $W_{i,n} \in H$ is defined as

$$\bar W_{i,n}(r_i,\theta_i) := \bar\varphi(r_i\varepsilon_n)\frac{\bar w_n^{(i)}(r_i\varepsilon_n,\theta_i) - \bar u_0^{(i)}(r_i\varepsilon_n,\theta_i) - \varepsilon_n \bar u_1^{(i)}(r_i\varepsilon_n,\theta_i)}{\sqrt{\varepsilon_n}} \tag{1.2.24}$$

[2] In what follows, given a function $v = v(\boldsymbol{x})$, we denote by $\bar v^i$ the function $\bar v^{(i)}(r_i,\theta_i) := v(\boldsymbol{x}_i + r_i(\cos\theta_i,\sin\theta_i))$, for polar coordinates (r_i,θ_i) given as in Theorem 1.2.1.

for (r_i, θ_i) *polar coordinates as in Theorem 1.2.1, then there exist a subsequence* $\{w_{n_k}\}_k$ *of* $\{w_n\}_n$*,* $w_0 \in H^1(\Omega)$ *and* $q_0 \in L^2_{\mathrm{loc}}(\Gamma_D)$ *such that*

$$\frac{w_{n_k} - u_0 - \varepsilon_{n_k} u_1}{\sqrt{\varepsilon_{n_k}}} \rightharpoonup w_0 \quad \text{in } H^1(\Omega), \tag{1.2.25}$$

$$\frac{w_{n_k} - u_0}{\varepsilon_{n_k}} - u_1 - \sum_{i=1}^{2} c_i \psi_i [1 - \chi_{B_{\varepsilon_{n_k}}(\boldsymbol{x}_i)}] \rightharpoonup q_0 - \sum_{i=1}^{2} c_i \psi_i \quad \text{in } L^2(\Gamma_D), \tag{1.2.26}$$

where ψ_i *is the function given in polar coordinates by* (1.2.20) *and* u_1 *is the solution to* (1.2.22)*. Furthermore, for every* $R > 0$*,*

$$W_{i,n_k} \rightharpoonup W_i \quad \text{in } H^1(B^+_R(\mathbf{0})), \quad \nabla W_{i,n_k} \rightharpoonup \nabla W_i \quad \text{in } L^2(\mathbb{R}^2_+; \mathbb{R}^2)), \tag{1.2.27}$$

$$W_{i,n_k}(\cdot, 0) \rightharpoonup W_i(\cdot, 0) \qquad\qquad \text{in } L^2((0,1) \times \{0\}), \tag{1.2.28}$$

$$W_{i,n_k}(\cdot, 0) - \frac{c_i}{2} x^{-1/2} \rightharpoonup W_i(\cdot, 0) - \frac{c_i}{2} x^{-1/2} \quad \text{in } L^2((1,\infty) \times \{0\}), \tag{1.2.29}$$

for some $W_i \in H$ *such that* $\mathcal{J}_i(W_i) < \infty$*, where* $W_{i,n_k}(\cdot, 0)$ *and* $W_i(\cdot, 0)$ *indicate the trace of* W_{i,n_k} *and* W_i *on the positive real axis.*

Theorem 1.2.7 (2nd order Γ-convergence)**.** *Under the assumptions of Theorem 1.2.6, the family* $\{\mathcal{F}^{(2)}_\varepsilon\}_\varepsilon$ Γ*-converges in* $L^2(\Omega)$ *to the functional*

$$\mathcal{F}_2(v) := \begin{cases} \sum_{i=1}^{2} \left(\frac{A_i}{2} + B_i c_i + C_\varphi c_i^2 \right) - \frac{1}{2} \int_{\Gamma_D} \left(\partial_\nu u^0_{\mathrm{reg}} \right)^2 \, d\mathcal{H}^1 & \text{if } v = u_0, \\ +\infty & \text{otherwise}, \end{cases}$$

where the numbers A_i, B_i*, and* C_φ *are defined in* (1.2.17)*,* (1.2.18)*, and* (1.2.19)*, respectively. In particular, if* $u_\varepsilon \in H^1(\Omega)$ *is a solution to* (1.2.3) *then*

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}_0(u_0) + \varepsilon |\log \varepsilon| \mathcal{F}_1(u_0) + \varepsilon \mathcal{F}_2(u_0) + o(\varepsilon). \tag{1.2.30}$$

As a consequence of our results, we obtain an alternative proof of the sharp estimates (1.2.4) for $s = 0$ and (1.2.5) in Theorem 1.2.2. Indeed, we have the following theorem.

Theorem 1.2.8. *Let* $N = 2$*,* Ω *as in Theorem 1.2.1,* $f \in L^2(\Omega)$*,* $g \in H^{3/2}(\partial\Omega)$*, and let* u_ε *and* u_0 *be solutions to* (1.2.3) *and* (1.2.1)*, respectively. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Gamma_D)} = \mathcal{O}\left(\varepsilon \sqrt{|\log \varepsilon|} \right), \tag{1.2.31}$$

$$\|\nabla (u_\varepsilon - u_0)\|_{L^2(\Omega;\mathbb{R}^2)} = \mathcal{O}\left(\varepsilon^{1/2} \right). \tag{1.2.32}$$

In contrast to the work of Costabel and Dauge [40], our results rely on the variational structure of the mixed Neumann-Robin problem (1.2.3), rather than the PDE. In particular, the compactness results in Theorem 1.2.4 and Theorem 1.2.6 are valid for energy bounded sequences and not just for minimizers, and thus are completely new. A key ingredient in the proof of compactness is the following Hardy-type inequality on balls due to Machihara, Ozawa and Wadade (see Corollary 6 in [75]).

Theorem 1.2.9. *Let $B_R(\mathbf{0})$ be the ball of $\mathbb{R}^2$ with radius $R > 0$ and center at the origin. Then*

$$\left(\int_{B_R(\mathbf{0})} \frac{h(\boldsymbol{x})^2}{|\boldsymbol{x}|^2\,(1+\log R - \log|\boldsymbol{x}|)^2}\,d\boldsymbol{x}\right)^{1/2} \le \frac{\sqrt{2}}{R}\left(\int_{B_R(\mathbf{0})} h(\boldsymbol{x})^2\,d\boldsymbol{x}\right)^{1/2} + 2(1+\sqrt{2})\left(\int_{B_R(\mathbf{0})} \left|\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot\nabla h(\boldsymbol{x})\right|^2 d\boldsymbol{x}\right)^{1/2}$$

for every $h \in H^1(B_R(\mathbf{0}))$.

We remark that our results rely heavily on the decomposition of Theorem 1.2.1 and on the Hardy-type inequality (Theorem 1.2.9) and thus hold only for $N = 2$. The extension to dimension $N \geq 3$ seems to be highly non-trivial and, in particular, the correct scalings in the asymptotic development by Γ-convergence are not clear and may depend in a significant way on the geometry of the domain (see, for example, [77] for a discussion on the mixed Dirichlet-Neumann problem in a three-dimensional dihedron).

It also important to observe that the asymptotic development by Γ-convergence leads naturally to the asymptotic expansion of the solutions u_ε to (1.2.3), and does not require an a priori ansatz of this expansion. Thus it could be applied to a large class of problems, including the p-Laplacian mixed problem

$$\begin{cases} \operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) = f & \text{in } \Omega, \\ |\nabla u_0|^{p-2}\partial_\nu u_0 = 0 & \text{on } \Gamma_N, \\ u_0 = g & \text{on } \Gamma_D. \end{cases}$$

In the seminal paper [18], Berestycki, Caffarelli and Nirenberg considered the family of elliptic equations

$$Lu_\varepsilon = \beta_\varepsilon(u_\varepsilon) \tag{1.2.33}$$

to approximate (as $\varepsilon \to 0^+$) a one-phase free boundary problem. Here the family $\{\beta_\varepsilon\}_\varepsilon$ is an approximate identity and the term $\beta_\varepsilon(u_\varepsilon)$ is non-zero only for values of u_ε less than ε. In particular, the region $\{u_\varepsilon < \varepsilon\}$ can be thought of as an approximation of the free boundary of the solution to the limiting problem. One-phase free boundary problems with mixed boundary conditions are strongly related to problems arising in fluid-dynamics (see [56]).

Our original motivation for considering the family of problems (1.2.3) was the study of the regularized problem

$$\begin{cases} \Delta u_\varepsilon = \frac{1}{2}\beta_\varepsilon(u_\varepsilon)Q^2 & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Gamma_N, \\ \varepsilon\partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \Gamma_D, \end{cases}$$

where $\{\beta_\varepsilon\}_\varepsilon$ is a family of approximate identities as in (1.2.33) and Q is a nonnegative function in $L^2_{\mathrm{loc}}(\Omega)$. Solutions u_ε of this problem converge to a solution u of the one-phase free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0, \ |\nabla u| = Q & \text{on } \Omega \cap \partial\{u > 0\}, \\ \partial_\nu u = 0 & \text{on } \Gamma_N, \\ u = g & \text{on } \Gamma_D. \end{cases}$$

The asymptotic development by Γ-convergence of the corresponding family of functionals

$$\int_\Omega \left(|\nabla v|^2 + B_\varepsilon(v) Q^2\right) \, d\boldsymbol{x} + \frac{1}{\varepsilon} \int_{\Gamma_D} (v - g)^2 \, d\mathcal{H}^{N-1}, \quad v \in H^1(\Omega)$$

is ongoing work. Here B_ε is a primitive of β_ε.

As a warm-up problem, we begin Chapter 4 with the study of the simpler case in which $\Gamma_D = \partial\Omega$, so that (1.2.3) reduces to

$$\begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega, \\ \varepsilon \partial_\nu u_\varepsilon + u_\varepsilon = g & \text{on } \partial\Omega. \end{cases} \tag{1.2.34}$$

Under suitable regularity assumptions on the set Ω, we characterize the complete asymptotic expansion by Γ-convergence of $\{\mathcal{F}_\varepsilon\}_\varepsilon$, still defined as in (1.2.7), but with Γ_D replaced by $\partial\Omega$ (see Theorem 4.2.2, Theorem 4.2.4, and Theorem 4.2.6). In Corollary 4.2.5 and Corollary 4.2.7 we address the question of the convergence of u_ε to u_0, i.e., the unique variational solution to the Dirichlet problem

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega, \\ u_0 = g & \text{on } \partial\Omega. \end{cases} \tag{1.2.35}$$

To be precise, we show that the asymptotic expansion

$$u_\varepsilon = \sum_{i=1}^{\infty} \varepsilon^i u_i$$

holds, where for every $i \in \mathbb{N}$ the function u_i is a solution to the Dirichlet problem

$$\begin{cases} \Delta u_i = 0 & \text{in } \Omega, \\ u_i = -\partial_\nu u_{i-1} & \text{on } \partial\Omega. \end{cases}$$

We remark that Corollary 4.2.7 fully recovers the results of Theorem 2.3 in [40] and that the auxiliary problems for u_i arise naturally during the study of higher order Γ-limits of $\mathcal{F}_\varepsilon$ (see, for example, the proof of Theorem 4.2.4). The case of a Robin boundary condition that transforms into a Dirichlet boundary condition for Helmholtz equation was considered by Kirsch in [67].

Chapter 2

Preliminaries

2.1 Gamma-convergence and asymptotic developments

Γ-convergence, introduced by De Giorgi in 1975 (see [48]), is a notion of convergence which is particularly suited for analyzing the convergence of variational problems. For more information we refer to the monographs of Braides [20] and Dal Maso [42].

Definition 2.1.1. *Given a metric space X and a family of functions $\mathcal{F}_\varepsilon \colon X \to \overline{\mathbb{R}}$, $\varepsilon > 0$, we say that $\{\mathcal{F}_\varepsilon\}_\varepsilon$ Γ-converges to $\mathcal{F}_0 \colon X \to \overline{\mathbb{R}}$ as $\varepsilon \to 0^+$, and we write $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}_0$, if for every sequence $\varepsilon_n \to 0^+$ the following two conditions hold:*

(i) liminf inequality*: for every $x \in X$ and every sequence $\{x_n\}_n$ of elements of X such that $x_n \to x$,*
$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(x_n) \geq \mathcal{F}_0(x);$$

(ii) limsup inequality*: for every $x \in X$, there is a sequence $\{x_n\}_n$ of elements of X such that $x_n \to x$ and*
$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(x_n) \leq \mathcal{F}_0(x).$$

A sequence $\{x_n\}_n$ as in (ii) is called a recovery sequence *for x.*

Proposition 2.1.2 (Proposition 1.42 in [20])**.** *Let X be a separable metric space and let $\{\mathcal{F}_n\}_n$ be a sequence of functions $\mathcal{F}_n \colon X \to \overline{\mathbb{R}}$. Then there exist a subsequence $\{\mathcal{F}_{n_k}\}_k$ of $\{\mathcal{F}_n\}$ and a function $\mathcal{F} \colon X \to \overline{\mathbb{R}}$ such that $\{\mathcal{F}_{n_k}\}_k$ Γ-converges to $\mathcal{F}$.*

Definition 2.1.3 (Coercive and equi-midly coercive)**.** *A function $\mathcal{F} \colon X \to \overline{\mathbb{R}}$ is* coercive *if for all $t \in \mathbb{R}$ the set $\{\mathcal{F} \leq t\}$ is precompact, and is* midly coercive *if there exists a non-empty compact subset K such that $\inf\{\mathcal{F}(x) : x \in X\} = \inf\{\mathcal{F}(x) : x \in K\}$. A family of functions $\{\mathcal{F}_\varepsilon\}_\varepsilon$ is* equi-midly coercive *if there exists a a non-empty compact subset K such that $\inf\{\mathcal{F}_\varepsilon(x) : x \in X\} = \inf\{\mathcal{F}_\varepsilon(x) : x \in K\}$ for all $\varepsilon > 0$.*

Theorem 2.1.4 (Theorem 1.21 in [20])**.** *Let X be a metric space and let $\{\mathcal{F}_\varepsilon\}_\varepsilon$ be a family of functions where $\mathcal{F}_\varepsilon \colon X \to \overline{\mathbb{R}}$ for all $\varepsilon > 0$. Suppose that the family $\{\mathcal{F}_\varepsilon\}_\varepsilon$ is equi-mildly coercive and that $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}$ for some $\mathcal{F} \colon X \to \overline{\mathbb{R}}$. Then there exists $x_0 \in X$ such that*
$$\mathcal{F}(x_0) = \inf\{\mathcal{F}(x) : x \in X\} = \lim_{\varepsilon\to 0} \inf\{\mathcal{F}_\varepsilon(x) : x \in X\}.$$

Moreover, if $\{x_\varepsilon\}_\varepsilon$ is precompact and such that

$$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(x_\varepsilon)=\lim_{\varepsilon\to 0}\inf\{\mathcal{F}_\varepsilon(x):x\in X\},\tag{2.1.1}$$

then every accumulation point of $\{x_\varepsilon\}_\varepsilon$ is a minimum point for $\mathcal{F}$.

As explained in [13], the Γ-limit might fail to completely characterize the asymptotic behavior of the family $\{\mathcal{F}_\varepsilon\}_\varepsilon$. Indeed, consider $X=\mathbb{R}$ and let $\mathcal{F}_\varepsilon(x)=\varepsilon|x|$; then, as one can readily check, $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}$, where $\mathcal{F}$ is identically equal to zero. In turn, every $x\in\mathbb{R}$ is a minimizer for $\mathcal{F}$ while $\{x_\varepsilon:\mathcal{F}_\varepsilon(x_\varepsilon)=\inf_{\mathbb{R}}\mathcal{F}_\varepsilon\}=\{0\}$. This shows that in general the inclusion

$$\{\text{limits of minimizers}\}\subset\{\text{minimizers of the }\Gamma\text{-limit}\}$$

can be a proper inclusion.

Definition 2.1.5. *We say that the* asymptotic development by Γ-convergence of order k

$$\mathcal{F}_\varepsilon=\mathcal{F}_0+\omega_1(\varepsilon)\mathcal{F}_1+\cdots+\omega_k(\varepsilon)\mathcal{F}_k$$

holds if there are functions $\mathcal{F}_i\colon X\to\overline{\mathbb{R}}$, $i=0,\dots k$, such that $\mathcal{F}_\varepsilon\xrightarrow{\Gamma}\mathcal{F}_0$ and for $i\geq 1$

$$\mathcal{F}_\varepsilon^{(i)}:=\left(\mathcal{F}_\varepsilon^{(i-1)}-\inf\{\mathcal{F}_{i-1}(x):x\in X\}\right)\frac{\omega_{i-1}(\varepsilon)}{\omega_i(\varepsilon)}\xrightarrow{\Gamma}\mathcal{F}_i,$$

where $\mathcal{F}_\varepsilon^{(0)}:=\mathcal{F}_\varepsilon$, $\omega_0\equiv 1$ and for $i\geq 1$, $\omega_i\colon\mathbb{R}_+\to\mathbb{R}_+$ is a suitably chosen function such that both ω_i and ω_i/ω_{i-1} converge to zero as $\varepsilon\to 0^+$.

Remark 2.1.6. *For $\omega_i(\varepsilon):=\varepsilon^i$ one has the standard power series asymptotic expansion*

$$\mathcal{F}_\varepsilon=\mathcal{F}_0+\varepsilon\mathcal{F}_1+\cdots+\varepsilon^k\mathcal{F}_k.$$

Asymptotic developments by Γ-convergence provide a selection criteria for minimizers of $\mathcal{F}_0$. This is the content of the following result.

Theorem 2.1.7 (Proposition 1.3 in [14])**.** *Let X be a metric space and let $\{\mathcal{F}_\varepsilon\}_\varepsilon$ be a family of functions where $\mathcal{F}_\varepsilon\colon X\to\overline{\mathbb{R}}$ for all $\varepsilon>0$. Suppose that $\mathcal{F}_\varepsilon$ admits an asymptotic expansion as in Definition 2.1.5. For each $\varepsilon>0$ let x_ε be a minimizer of $\mathcal{F}_\varepsilon$ and assume that $x_n\to x_0\in X$ for some subsequence $\varepsilon_n\to 0^+$. Then the following holds:*

(i) *x_0 is a minimizer of $\mathcal{F}_0,\dots,\mathcal{F}_k$,*

(ii) *if we let $\mathcal{U}_i$ be the set of minimizers of $\mathcal{F}_i$ then*

$$\mathcal{F}_i(x)=\infty\quad\text{for } x\in X\setminus\mathcal{U}_i.$$

In particular,

$$\{\text{limits of minimizers of }\mathcal{F}_\varepsilon\}\subset\mathcal{U}_k\subset\cdots\subset\mathcal{U}_1\subset\mathcal{U}_0.$$

(iii) *If m_ε denotes the infimum of $\mathcal{F}_\varepsilon$ and m_i denotes the infimum of $\mathcal{F}_i$, then*

$$m_\varepsilon=m_0+\omega_1(\varepsilon)m_1+\cdots+\omega_k(\varepsilon)m_k+o(\omega_k(\varepsilon)).$$

2.2 Second order linear elliptic equations

In this section we collect a few selected results on the existence and regularity of solutions to linear elliptic partial differential equations of the second order that will be used throughout the following chapters.

2.2.1 Solvability of the classical Dirichlet problem

We call a set Ω a domain in $\mathbb{R}^N$ if Ω is a connected, non-empty, proper open subset of $\mathbb{R}^N$.

Definition 2.2.1. *Let Ω be a domain in $\mathbb{R}^N$ and let $u \in C^2(\Omega)$. The function u is called* harmonic *in Ω if it satisfies $\Delta u = 0$ everywhere in Ω.*

Definition 2.2.2. *Let Ω be a domain in $\mathbb{R}^N$ and let $v\colon \Omega \to \mathbb{R}\cup\{-\infty\}$ be an upper semi-continuous function. The function v is called* subharmonic *in Ω if for every closed ball $\overline{B_r(\boldsymbol{x})}$ contained in Ω and every harmonic function u in $B_r(\boldsymbol{x})$ that satisfies $v \le u$ on $\partial B_r(\boldsymbol{x})$ we have $v \le u$ in $B_r(\boldsymbol{x})$. A function w is called* superharmonic *in Ω if $-w$ is subharmonic in Ω.*

Definition 2.2.3. *Let Ω be a domain in $\mathbb{R}^N$ and let ξ be a point of $\partial\Omega$. A continuous function $w \in C(\overline{\Omega})$ is called a* barrier *at $\boldsymbol{x}$ relative to Ω if:*

(i) *w is superharmonic in Ω;*

(ii) *$w > 0$ in $\overline{\Omega} \setminus \{\boldsymbol{x}\}$, $w(\boldsymbol{x}) = 0$.*

Furthermore, a boundary point will be called regular if there exists a barrier at that point.

Theorem 2.2.4 (Theorem 2.14 in [54])**.** *Let Ω be a bounded domain in $\mathbb{R}^N$. Then, for every continuous function $\varphi \in C(\partial\Omega)$, there exists a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

if and only if every boundary point is regular.

Remark 2.2.5. *We recall that, if Ω is a domain in $\mathbb{R}^2$ and $\boldsymbol{x} \in \partial\Omega$, then $\boldsymbol{x}$ is a regular point if it is the endpoint of a simple arc lying in the exterior of Ω. Thus, if $N = 2$, the Dirichlet problem is solvable in every bounded domain whose boundary points are accessible from the exterior by a simple arc. On the other hand, if $N \ge 3$, a simple sufficient condition for the solvability in a bounded domain is that Ω satisfies the exterior ball condition, i.e. for every point $\boldsymbol{x} \in \partial\Omega$ there exists a ball $B_R(\boldsymbol{y})$ such that $\overline{\Omega} \cap \overline{B_R(\boldsymbol{y})} = \{\boldsymbol{x}\}$. This is satisfied, in particular, if $\partial\Omega$ is of class C^2. For more details we refer to the discussion at the end of Section 2.8 in [54].*

A similar result to that of Theorem 2.2.4 holds also for Poisson's equation.

Theorem 2.2.6 (Theorem 4.3 in [54])**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ and suppose that every boundary point is regular. Then, if f is a bounded and locally Hölder continuous function in Ω, the classical Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable for any continuous function φ.

2.2.2 Harnack's and Carleson's inequalities

The classical Harnack inequality can be stated as follows.

Theorem 2.2.7 (Theorem 2.5 in [54])**.** *Let u be a nonnegative harmonic function in $\Omega \subset \mathbb{R}^N$. Then for every bounded domain Ω' which is compactly supported in Ω there exists a positive constant C, depending only on N, Ω and Ω' such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

The following two results are a version of Harnack inequality when Ω and Ω' are two concentric balls, particularly useful due to the fact that the constant is explicit, and an up to the boundary version of Harnack's inequality, respectively. The latter is also referred to as Carleson estimate in the literature. These will prove instrumental when addressing the issue of boundary regularity in the following chapter.

Theorem 2.2.8 (Problem 2.6 in [54])**.** *Let u be a nonnegative harmonic function in $B_R(\mathbf{0})$. Then*

$$\frac{R^{N-2}(R-|\boldsymbol{x}|)}{(R+|\boldsymbol{x}|)^{N-1}}u(\mathbf{0}) \leq u(\boldsymbol{x}) \leq \frac{R^{N-2}(R+|\boldsymbol{x}|)}{(R-|\boldsymbol{x}|)^{N-1}}u(\mathbf{0}).$$

for every $\boldsymbol{x} \in B_R(\mathbf{0})$.

Theorem 2.2.9 (Theorem 11.5 in [28])**.** *Let $u \in C^2(B_r^+(\mathbf{0})) \cap C(\overline{B_r^+(\mathbf{0})})$ be a nonnegative harmonic function in $B_R^+(\mathbf{0})$, vanishing on $\{x_N = 0\}$. Then there exists a positive constant $M = M(N)$ such that*

$$u(\boldsymbol{x}) \leq Mu\left(\frac{R}{2}\boldsymbol{e}_N\right)$$

for every $\boldsymbol{x} \in B^+_{R/2}(\mathbf{0})$.

2.2.3 Interior and boundary estimates in concentric balls

Following the presentation of Chapter 4 in [54], we introduce the following notation: for $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,1]$, and $D \subset \mathbb{R}^N$ let $d := \operatorname{diam}(D)$ and define

$$|D^k u|_{0;D} := \sup_{|\boldsymbol{\beta}|=k} \sup_{D} |D^{\boldsymbol{\beta}} u|,$$

$$[D^k u]_{\alpha;D} := \sup_{\boldsymbol{\beta}=k} \sup_{\boldsymbol{x}\neq\boldsymbol{y}\in D} \frac{|D^{\boldsymbol{\beta}}u(\boldsymbol{x}) - D^{\boldsymbol{\beta}}u(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|},$$

$$|u|_{k,\alpha;D} := \sum_{j=0}^{k} |D^j u|_{0,D} + [D^k u]_{\alpha,D},$$

$$|u|'_{k,\alpha;D} := \sum_{j=0}^{k} d^j |D^j u|_{0,D} + d^{k+\alpha}[D^k u]_{\alpha,D}.$$

Theorem 2.2.10 (Theorem 4.6 in [54]). *Let Ω be a domain in $\mathbb{R}^N$ and let $u \in C^2(\Omega)$, $f \in C^\alpha(\Omega)$, satisfy Poisson's equation $\Delta u = f$ in Ω. Then for any two concentric balls $B_1 := B_R(\boldsymbol{x})$, $B_2 := B_{2R}(\boldsymbol{x})$ compactly contained in Ω we have*

$$|u|'_{2,\alpha;B_1} \leq C\left(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2}\right)$$

where $C = C(N, \alpha) > 0$.

Theorem 2.2.11 (Theorem 4.11 in [54]). *Let $B_1^+ := B_R(\mathbf{0}) \cap \{x_N > 0\}$, $B_2^+ := B_{2R}(\mathbf{0}) \cap \{x_N > 0\}$, and $u \in C^2(B_2^+) \cap C(\overline{B_2^+})$ satisfy*

$$\begin{cases} \Delta u = f \ \text{ in } B_2^+, \\ \quad u = 0 \ \text{ on } \{x_N = 0\}, \end{cases}$$

where $f \in C^\alpha(\overline{B_2^+})$. Then $u \in C^{2,\alpha}(\overline{B_1^+})$ and we have

$$|u|'_{2,\alpha;B_1^+} \leq C\left(|u|_{0;B_2^+} + R^2|f|'_{0,\alpha;B_2^+}\right)$$

where $C = C(N, \alpha) > 0$.

2.2.4 Boundary estimates in more general domains

While a satisfactory interior regularity theory for solutions to Laplace's or Poisson's equations essentially follows from Theorem 2.2.10, the regularity up to the boundary, which requires straightening a portion of the boundary, is based on the study of more general elliptic equations. For sufficiently smooth domain, i.e. if $\partial\Omega$ is of class $C^{2,\alpha}$, one can rely on Schauder's theory for equations of the form

$$Lu = a^{ij}(\boldsymbol{x})\partial^2_{ij}u + b^i(\boldsymbol{x})\partial_i u + c(\boldsymbol{x})u = f(\boldsymbol{x}),$$

where repeated indices are summed, and the coefficients satisfy

$$a^{ij}(\boldsymbol{x})\xi_i\xi_j \geq \lambda|\boldsymbol{\xi}| \tag{2.2.1}$$

for every $\boldsymbol{x} \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{R}^N$, and

$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \Lambda.$$

Here it is assumed that Λ, λ are positive constants.

Theorem 2.2.12 (Corollary 6.7 and Lemma 6.18 in [54]). *Let $\Omega \subset \mathbb{R}^N$ be a domain with a $C^{2,\alpha}$ boundary portion T, and let $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Suppose that u is a $C(\overline{\Omega}) \cap C^2(\Omega)$ function satisfying*

$$\begin{cases} Lu = f \ \text{ in } \Omega, \\ \quad u = \varphi \ \text{ on } T, \end{cases}$$

where L is as above and $f \in C^\alpha(\overline{\Omega})$. Then $u \in C^{2,\alpha}(\Omega \cup T)$ and furthermore, if $\boldsymbol{x}_0 \in T$ and $B := B_\rho(\boldsymbol{x}_0)$ is a ball with radius $\rho < \operatorname{dist}(\boldsymbol{x}_0, \partial\Omega \setminus T)$, we have

$$|u|_{2,\alpha;B\cap\Omega} \leq C\left(|u|_{0;\Omega} + |\varphi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}\right),$$

where $C = C(N, \alpha, \lambda, \Lambda, B \cap \Omega) > 0$.

The issue of boundary regularity for domains that are only of class $C^{1,\alpha}$ can be addressed by the theory of weak solutions for operators whose principal part is in divergence form, i.e.,

$$Lu = \partial_i \left(a^{ij}(\boldsymbol{x})\partial_j u + b^i(\boldsymbol{x})u \right) + c^i(\boldsymbol{x})\partial_i u + d(\boldsymbol{x}).$$

Here the coefficients a^{ij}, b^i, c^i, d are assumed to be measurable functions on Ω that satisfy (2.2.1) and

$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c^i|_{0;\Omega} + |d|_{0;\Omega} \le K$$

for some positive constant K.

Theorem 2.2.13 (Corollary 8.36 in [54])**.** *Let $\Omega \subset \mathbb{R}^N$ be a domain with a $C^{1,\alpha}$ boundary portion T, and suppose $u \in H^1(\Omega)$ is a weak solution of*

$$\begin{cases} Lu = g + \partial_i f^i & \text{in } \Omega, \\ u = \varphi & \text{on } T, \end{cases}$$

where $g \in L^\infty(\Omega)$, $f^i \in C^\alpha(\overline{\Omega})$, $\varphi \in C^{1,\alpha}(\overline{\Omega})$, and the boundary condition is satisfied in the sense of traces. Then $u \in C^{1,\alpha}(\Omega \cup T)$, and for any Ω' compactly contained in $\Omega \cup T$ we have

$$|u|_{1,\alpha;\Omega'} \le C \left(|u|_{0;\Omega} + |g|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{1,\alpha;\Omega} \right),$$

where $C = C(N, \lambda, K, \operatorname{dist}(\Omega', \partial\Omega \setminus T), T) > 0$.

2.3 Symmetric rearrangements

Let $U \subset \mathbb{R}^2$ be an open subset of the strip $S_a := (-a, a) \times \mathbb{R}$, $U_t := U \cap \{y = t\}$, and define

$$U_t^* := \left\{ (x, t) : -\frac{1}{2}\mathcal{L}^1(U_t) < x < \frac{1}{2}\mathcal{L}^1(U_t) \right\}.$$

Definition 2.3.1. *The set*

$$U^* := \bigcup_{t \in \mathbb{R}} U_t^*$$

is called the Steiner symmetrization of U with respect to the y-axis.

Remark 2.3.2. *Analogous definitions can be given for a closed subset of S_a. Furthermore, it is not difficult to see that if U is an open (closed) subset of S_a, then U^* is also open (closed) and $\mathcal{L}^2(U) = \mathcal{L}^2(U^*)$.*

Let $R_{a,b} := (-a, a) \times (0, b)$ and consider a function $u \colon R_{a,b} \to \mathbb{R}$ which is continuous, nonnegative, and symmetric about the y-axis. For $y \in (0, b)$, let $v_y := u(\cdot, y)$ and consider the segments

$$\{v_y \ge c\}^* := \left\{ x \in (-a, a) : |x| \le \frac{1}{2}\mathcal{L}^1(\{v_y \ge c\}) \right\},$$

$$\{v_y > c\}^* := \left\{ x \in (-a, a) : |x| < \frac{1}{2}\mathcal{L}^1(\{v_y > c\}) \right\}.$$

Definition 2.3.3. *The function $u^*\colon R_{a,b} \to \mathbb{R}$ defined via*

$$u^*(x,y) = c \quad \text{if } x \in \{v_y \geq c\}^* \setminus \{v_y > c\}^*$$

is called the Steiner symmetrization of u in the variable x or with respect to the y-axis.

Lemma 2.3.4. *Let u be as above and u^* be its Steiner symmetrization in variable x. Then u^* is also symmetric with respect to the y-axis and is monotone decreasing in x for $x \geq 0$. Moreover, for any real numbers c and d such that $0 < c < d$, we have*

$$\mathcal{L}^2(\{\boldsymbol{x} : u^*(\boldsymbol{x}) \in (c,d)\}) = \mathcal{L}^2(\{\boldsymbol{x} : u(\boldsymbol{x}) \in (c,d)\}).$$

In addition, if u is decreasing in x for $x \geq 0$ then $u = u^*$.

We conclude this section with the following version of the Pólya-Szegő inequality.

Theorem 2.3.5. *Let $u \in C(R_{a,b}) \cap H^1(R_{a,b})$ be nonnegative, symmetric in the variable x, and let u^* be its Steiner symmetrization with respect to the y-axis. Then $u^* \in H^1(R_{a,b})$ and*

$$\int_{R_{a,b}} |\nabla u^*|^2 \, d\boldsymbol{x} \leq \int_{R_{a,b}} |\nabla u|^2 \, d\boldsymbol{x}.$$

Proof. The proof is essentially a corollary of Theorem 2.31 in [64]. Indeed, since $R_{a,b}$ satisfies the assumptions of Remark 2.44c and Remark 2.32 in [64], one can then reason as in the proof of Corollary 2.14 in [64]; we omit the details. □

2.4 Derivation of Bernoulli's free boundary problem

For the convenience of the reader we present here the derivation of the one-phase free boundary problem (1.1.1) from the equations of motion of a planar periodic wave. The content of this section is adapted from [35] (see also [38] and [39]). To be precise, we consider a two-dimensional, inviscid, incompressible fluid which undergoes a steady motion in a vertical plane over a flat, horizontal, impermeable bed. By steady we mean that the flow propagates in a fixed direction at constant speed c. We assume that the gravity is the only restoring force. Choosing a moving frame of reference we can eliminate the time variable and rewrite the equations of conservation of momentum as

$$\begin{cases} \rho((u-c)\partial_x u + v\partial_y u) = -\partial_x P, \\ \rho((u-c)\partial_x v + v\partial_y v) = -\partial_y P - \rho g, \end{cases} \tag{2.4.1}$$

where $\boldsymbol{u} = (u,v)$ if the flow velocity field, P is the pressure and g is the gravitational constant. The conservation of mass condition can be written as

$$\nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{2.4.2}$$

Henceforth we assume that the density ρ is everywhere equal to one in the fluid, and so the incompressibility condition (2.4.2) simplifies to

$$\partial_x u + \partial_y v = 0. \tag{2.4.3}$$

Furthermore, we assume that the free surface is the graph of a function η; thus, the region occupied by the fluid is given by

$$D_\eta \coloneqq \{(x,y) : x \in \mathbb{R} \text{ and } 0 < y < \eta(x)\}.$$

In the following we require that the unknowns u, v, P, η are periodic in the x variable of period λ. The equations of motion (2.4.1) and (2.4.3) are complemented by the kinematic and dynamic boundary conditions on the free surface

$$\begin{cases} v = (u-c)\eta' & \text{on } y = \eta(x), \\ P = P_{\text{atm}} & \text{on } y = \eta(x), \end{cases} \tag{2.4.4}$$

as well as the kinematic boundary condition on the flat bottom:

$$v = 0 \quad \text{on } y = 0. \tag{2.4.5}$$

Notice that if ψ is a stream function for the flow (defined up to a constant), i.e.,

$$\partial_y \psi = u - c, \quad \partial_x \psi = -v, \tag{2.4.6}$$

then the irrotationality of the flow implies that

$$\Delta \psi = 0 \tag{2.4.7}$$

in the fluid domain D_η. Moreover, by $(2.4.4)_1$ and (2.4.5), we see that both the free surface $\{(x, \eta(x))\}$ and the bottom $\{y = 0\}$ must be streamlines, i.e. level sets of ψ. Assuming that $\psi(x, \eta(x)) = 0$, it follows from (2.4.6) that

$$\psi(x,y) = p_0 + \int_0^y (u(x,s) - c) \, ds, \tag{2.4.8}$$

where p_0 is a constant. Finally, observe that Bernoulli's condition

$$\frac{(u-c)^2 + v^2}{2} + gy + P = \text{const}$$

holds along the free surface, where it can be rewritten in terms of the stream function ψ as

$$|\nabla \psi| = \sqrt{\text{const} - 2P_{\text{atm}} - 2gy}. \tag{2.4.9}$$

After an opportune renormalization, the free boundary problem (1.1.1), for Q as in (1.1.6), is equivalent to (2.4.7), (2.4.8), (2.4.9).

Chapter 3

Variational methods for water waves

3.1 Existence and regularity of global minimizers via regularization

Throughout the section we assume that Ω is an open connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary. We remark that Ω may be unbounded. For the convenience of the reader, in this section we recall some fundamentally well known results concerning the minimization problem for $\mathcal{J}$ in $\mathcal{K}$, defined as in (1.1.2) and (1.1.3) respectively. Here it is only assumed that

$$Q \in L^2_{\mathrm{loc}}(\Omega), \quad Q \geq 0. \tag{3.1.1}$$

Following the approach of [18], we introduce the family of approximate identities β_ε, defined as

$$\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right), \tag{3.1.2}$$

where

$$\beta \in C(\mathbb{R}; [0, \infty)), \quad \operatorname{supp} \beta \subset [0, 1], \quad \int_0^\infty \beta(s)\, ds = \int_0^1 \beta(s)\, ds = 1. \tag{3.1.3}$$

We also define B_ε by

$$B_\varepsilon(t) := \int_0^t \beta_\varepsilon(s)\, ds. \tag{3.1.4}$$

It follows that B_ε is nonnegative, increasing, Lipschitz continuous, with

$$B_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \displaystyle\int_0^{t/\varepsilon} \beta(s)\, ds & \text{if } 0 < t < \varepsilon, \\ 1 & \text{if } t \geq \varepsilon. \end{cases} \tag{3.1.5}$$

Finally, we consider the functional

$$\mathcal{J}_\varepsilon(u) := \int_\Omega \left(|\nabla u|^2 + B_\varepsilon(u) Q^2\right) d\boldsymbol{x} \tag{3.1.6}$$

defined for $u \in \mathcal{K}$. We refer to [25], [45], [46], [59], [60], [62], [69], [81] and the references therein for some of the recent literature on this type of singularly perturbed free boundary problems.

3.1.1 Gamma convergence and global minimizers

The proof of the existence of a global minimizer for $\mathcal{J}_\varepsilon$ in the next theorem is adapted from Theorem 3.1 in [2].

Theorem 3.1.1. *Let Ω be an open and connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary, and assume that* (1.1.4), (3.1.1), (3.1.3) *hold. Let $\mathcal{J}_\varepsilon$ and $\mathcal{K}$ be defined as in* (3.1.6) *and* (1.1.3), *respectively. Then there exists a global minimizer $u_\varepsilon \in \mathcal{K}$ of the functional $\mathcal{J}_\varepsilon$. Furthermore, u_ε is a weak solution of the mixed Dirichlet-Neumann problem*

$$\begin{cases} \Delta u_\varepsilon = \frac{1}{2}\beta_\varepsilon(u_\varepsilon)Q^2 & \text{in } \Omega, \\ u_\varepsilon = u_0 & \text{on } \Gamma, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega \setminus \Gamma, \end{cases} \tag{3.1.7}$$

where ν is the outward unit normal vector to $\partial\Omega$.

Proof. We claim that for every $u \in \mathcal{K}$,

$$\mathcal{J}_\varepsilon(u) \le \mathcal{J}(u), \tag{3.1.8}$$

where $\mathcal{J}$ is the functional defined in (1.1.2). Indeed, by (3.1.3) and (3.1.5) we have that for every $u \in L^1_{\mathrm{loc}}(\Omega)$,

$$B_\varepsilon(u(\boldsymbol{x})) \le \chi_{\{u>0\}}(\boldsymbol{x}) \text{ for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \Omega,$$

and the claim follows. In particular, we see from (1.1.4) and (3.1.8) that $\mathcal{J}_\varepsilon(u_0) < \infty$.

We now let $\alpha := \inf\{\mathcal{J}_\varepsilon(u) : u \in \mathcal{K}\}$ and $\{u_{k,\varepsilon}\}_k \subset \mathcal{K}$ be a minimizing sequence, that is,

$$\lim_{k\to\infty} \mathcal{J}_\varepsilon(u_{k,\varepsilon}) = \alpha.$$

Then $\{\nabla u_{k,\varepsilon}\}_k$ is bounded in $L^2(\Omega;\mathbb{R}^N)$. Let $\Omega_r := \Omega \cap B_r(\mathbf{0})$, where r is such that $\mathcal{H}^{N-1}(B_r(\mathbf{0}) \cap \Gamma) > 0$. Then by Poincaré's inequality we have that

$$\int_{\Omega_r} |u_{k,\varepsilon} - u_0|^2 \, d\boldsymbol{x} \le C(\Gamma, \Omega_r) \int_{\Omega_r} |\nabla u_{k,\varepsilon} - \nabla u_0|^2 \, d\boldsymbol{x}.$$

Therefore $\{u_{k,\varepsilon}\}_k$ is bounded in $H^1(\Omega_r)$ and hence, up to extraction of a subsequence (not relabeled), we can assume that $u_{k,\varepsilon} \to u_\varepsilon$ in $L^2(\Omega_r)$ and pointwise almost everywhere as $k \to \infty$ to some $u_\varepsilon \in H^1_{\mathrm{loc}}(\Omega_r)$. By letting $r \nearrow \infty$ and by using a diagonal argument, up to extraction of a further subsequence, we have that

$$\begin{aligned} \nabla u_{k,\varepsilon} &\rightharpoonup \nabla u_\varepsilon && \text{in } L^2(\Omega, \mathbb{R}^N), \\ u_{k,\varepsilon} &\to u_\varepsilon && \text{in } L^2_{\mathrm{loc}}(\Omega), \\ u_{k,\varepsilon} &\to u_\varepsilon && \text{pointwise almost everywhere in } \Omega. \end{aligned} \tag{3.1.9}$$

Moreover, since B_ε is Lipschitz continuous and nonnegative (see (3.1.3) and (3.1.4)), by the weakly lower semicontinuity of the L^2-norm and Fatou's lemma, we have that

$$\int_\Omega \left(|\nabla u_\varepsilon|^2 + B_\varepsilon(u_\varepsilon)Q^2\right) d\boldsymbol{x} \leq \liminf_{k\to\infty} \int_\Omega \left(|\nabla u_{k,\varepsilon}|^2 + B_\varepsilon(u_{k,\varepsilon})Q^2\right) d\boldsymbol{x} = \alpha.$$

To conclude, notice that $u_\varepsilon \in \mathcal{K}$ since $\mathcal{K}$ is closed with respect to the convergence in (3.1.9). Moreover, one can check that u_ε is a weak solution of (3.1.7) by considering variations of the functional $\mathcal{J}_\varepsilon$. We omit the details. □

Corollary 3.1.2. *Let $u_\varepsilon \in \mathcal{K}$ be a global minimizer of the functional $\mathcal{J}_\varepsilon$. Then, under the assumptions of Theorem 3.1.1,*

$$0 \leq u_\varepsilon(\boldsymbol{x}) \leq \|u_0\|_{L^\infty(\Gamma)}$$

for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \Omega$, provided ε is small enough.

Proof. To prove the upper bound, we can assume without loss of generality that $M := \|u_0\|_{L^\infty(\Gamma)} < \infty$, since otherwise there is nothing to prove. For every $0 < \varepsilon < M$ and for every $\eta > 0$, let $v_\varepsilon := \max\{u_\varepsilon - M, 0\}$ and consider $u_\varepsilon^\eta := u_\varepsilon - \eta v_\varepsilon$. Then $u_\varepsilon^\eta \in \mathcal{K}$ and

$$B_\varepsilon(u_\varepsilon(\boldsymbol{x})) = B_\varepsilon(u_\varepsilon^\eta(\boldsymbol{x})) \tag{3.1.10}$$

for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \Omega$. Indeed, the equality holds almost everywhere in $\{v_\varepsilon = 0\}$, while for almost every $\boldsymbol{x}$ such that $v_\varepsilon(\boldsymbol{x}) > 0$ we have that

$$u_\varepsilon(\boldsymbol{x}) > u_\varepsilon^\eta(\boldsymbol{x}) = (1-\eta)u_\varepsilon(\boldsymbol{x}) + \eta M > (1-\eta)M + \eta M > \varepsilon.$$

Therefore (3.1.10) follows from (3.1.5). This, together with the minimality of u_ε, implies that

$$\int_\Omega |\nabla u_\varepsilon|^2 \, d\boldsymbol{x} \leq \int_\Omega |\nabla u_\varepsilon^\eta|^2 \, d\boldsymbol{x}.$$

Expanding the square on the right-hand side, rearranging the terms, and dividing by η in the previous inequality yields

$$2\int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, d\boldsymbol{x} \leq \eta \int_\Omega |\nabla v_\varepsilon|^2 \, d\boldsymbol{x} = \eta \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, d\boldsymbol{x},$$

where in the last equality we have used the fact that $\nabla u_\varepsilon = \nabla v_\varepsilon$ a.e. in the set $\{u_\varepsilon > M\}$ while $\nabla v_\varepsilon = 0$ a.e. in the set $\{u_\varepsilon \leq M\}$. Taking $\eta < 2$, since Ω is connected, we have that $v_\varepsilon \equiv c_\varepsilon$ for some constant c_ε. In turn, its trace is c_ε, but since $u_\varepsilon = u_0 \leq M$ on Γ, necessarily $c_\varepsilon = 0$. Thus $u_\varepsilon \leq M$ as desired.

The proof that u_ε is nonnegative is similar taking $u_\varepsilon^\eta := u_\varepsilon - \eta \min\{u_\varepsilon, 0\}$ and therefore we omit it. □

Theorem 3.1.3 (Compactness)**.** *Let Ω be an open and connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary, and let $\mathcal{J}_\varepsilon$ and $\mathcal{K}$ be defined as in* (3.1.6) *and* (1.1.3)*, respectively. Assume that* (3.1.1)*,* (3.1.3) *hold. Given $\varepsilon_n \to 0^+$ and $\{u_n\}_n \subset \mathcal{K}$ such that*

$$\sup\{\mathcal{J}_{\varepsilon_n}(u_n) : n \in \mathbb{N}\} < \infty, \tag{3.1.11}$$

there are a subsequence $\{\varepsilon_{n_k}\}_k$ of $\{\varepsilon_n\}_n$ and $u \in \mathcal{K}$ such that $u_n \to u$ in $L^2_{\mathrm{loc}}(\Omega)$.

Proof. Since $\{\nabla u_n\}_n$ is bounded in $L^2(\Omega;\mathbb{R}^N)$ by (3.1.11) and $B_\varepsilon \geq 0$, the desired convergence follows as in the proof of (3.1.9). We omit the details. $\square$

In view of the previous theorem, we study the Γ-convergence of the family of functionals defined as in (3.1.6) with respect to convergence in $L^2_{\mathrm{loc}}(\Omega)$ (see Definition 2.1.1). In the following, although with a slight abuse of notation, we consider the functionals $\mathcal{J}_\varepsilon, \mathcal{J}\colon L^2_{\mathrm{loc}}(\Omega) \to [0,\infty]$, extended to infinity outside of $\mathcal{K}$.

Theorem 3.1.4. *Let Ω be an open and connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary, and let $\mathcal{J}_\varepsilon$ and $\mathcal{J}$ be defined as in (3.1.6) and (1.1.2), respectively. Assume that* (3.1.1), (3.1.3) *hold. Then $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}$ with respect to L^2_{loc} convergence.*

Proof. Let $u_n \to u$ in $L^2_{\mathrm{loc}}(\Omega)$. Without loss of generality, we may assume that

$$\liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n) = \lim_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n) < \infty,$$

since otherwise there is nothing to prove. By extracting successive subsequences, we may find a subsequence $\{\varepsilon_{n_k}\}_k$ of $\{\varepsilon_n\}_n$ such that $\sup\{\mathcal{J}_{\varepsilon_{n_k}}(u_{n_k}) : k \in \mathbb{N}\} < \infty$, $u_{n_k} \to u$ pointwise $\mathcal{L}^N$-a.e. in Ω and the following limits exist and are finite

$$\lim_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, d\boldsymbol{x}, \quad \lim_{k\to\infty} \int_\Omega B_{\varepsilon_{n_k}}(u_{n_k}) Q^2 \, d\boldsymbol{x}.$$

In turn,

$$\int_\Omega |\nabla u|^2 \, d\boldsymbol{x} \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, d\boldsymbol{x} = \lim_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, d\boldsymbol{x}. \tag{3.1.12}$$

Now fix $\delta > 0$ and let K be any compact set contained in $\{u > \delta\}$. By Egorov's theorem, for every $\eta > 0$ there exists a compact set $K_\eta \subset K$ such that $\mathcal{L}^N(K \setminus K_\eta) \leq \eta$ and $\{u_{n_k}\}_k$ converges uniformly to u on K_η. Notice that $\{B_{\varepsilon_{n_k}}(u_{n_k})\}_k$ is bounded in $L^\infty(\Omega)$ and hence admits a further subsequence (not relabeled) that converges in the weak star topology to some function $\xi \in L^\infty(\Omega)$. By uniform convergence, we can find $\overline{k}$ such that $u_{n_k} \geq \delta/2$ on K_η for $k \geq \overline{k}$. Moreover, if $\varepsilon_{n_k} \leq \delta/2$, $B_{\varepsilon_{n_k}}(u_{n_k}(\boldsymbol{x})) = 1$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x}$ in K_η by (3.1.5), and hence

$$0 = \int_{K_\eta} (B_{\varepsilon_{n_k}}(u_{n_k}) - 1) u_{n_k} \, d\boldsymbol{x} \to \int_{K_\eta} (\xi - 1) u \, d\boldsymbol{x}.$$

Since $u > 0$ on K_η, then necessarily $\xi = 1$ $\mathcal{L}^N$-a.e. in K_η. Letting $\eta \searrow 0$, $K \nearrow \{u > \delta\}$ and $\delta \searrow 0$ we conclude that $\xi = 1$ $\mathcal{L}^N$-a.e. in $\{u > 0\}$ and hence

$$\xi(\boldsymbol{x}) \geq \chi_{\{u>0\}}(\boldsymbol{x}) \text{ for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \Omega.$$

Let now D be a compact subset of Ω. By the previous inequality, the fact that $Q^2 \in L^1(D)$ and $B_{\varepsilon_{n_k}}(u_{n_k}) \overset{*}{\rightharpoonup} \xi$ in $L^\infty(\Omega)$,

$$\int_D \chi_{\{u>0\}} Q^2 \, d\boldsymbol{x} \leq \int_D \xi Q^2 \, d\boldsymbol{x} = \lim_{k\to\infty} \int_D B_{\varepsilon_{n_k}}(u_{n_k}) Q^2 \, d\boldsymbol{x} \leq \lim_{k\to\infty} \int_\Omega B_{\varepsilon_{n_k}}(u_{n_k}) Q^2 \, d\boldsymbol{x}.$$

Finally, letting $D \nearrow \Omega$ we get

$$\int_\Omega \chi_{\{u>0\}} Q^2 \, d\boldsymbol{x} \le \lim_{k\to\infty} \int_\Omega B_{\varepsilon_{n_k}}(u_{n_k}) Q^2 \, d\boldsymbol{x},$$

which together with (3.1.12) proves that $\mathcal{J}(u) \le \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n)$.

To prove the existence of a recovery sequence, we let $u \in L^2_{\text{loc}}(\Omega)$ and define $u_n \equiv u$. If $\mathcal{J}(u) = \infty$, then there is nothing to prove. Thus, assume that $\mathcal{J}(u) < \infty$. By (3.1.8) we have $\mathcal{J}_{\varepsilon_n}(u_n) \le \mathcal{J}(u)$ and therefore the result follows. □

Corollary 3.1.5. *Let Ω be an open and connected subset of $\mathbb{R}^N$ with locally Lipschitz continuous boundary, and assume that* (1.1.4)*,* (3.1.1)*,* (3.1.3) *hold. Let $\mathcal{J}$ and $\mathcal{K}$ be defined as in* (1.1.2) *and* (1.1.3) *respectively. Then there exists a global minimizer $u \in \mathcal{K}$ of the functional $\mathcal{J}$. Furthermore, every global minimizer of $\mathcal{J}$ in $\mathcal{K}$ is locally Lipschitz continuous in Ω, and solves* (1.1.1)*, where the free boundary condition is satisfied in a distributional sense.*

Proof. Let $\varepsilon_n \to 0^+$. By Theorem 3.1.1, for every $n \in \mathbb{N}$ we can find u_n, a global minimizer of $\mathcal{J}_{\varepsilon_n}$. Then by (1.1.4) we have

$$\sup\{\mathcal{J}_{\varepsilon_n}(u_n) : n \in \mathbb{N}\} \le \mathcal{J}(u_0) < \infty.$$

Let $\{\varepsilon_{n_k}\}_k$ and $u \in \mathcal{K}$ be given as in Theorem 3.1.3. Then, by Theorem 3.1.4, u is a global minimizer of $\mathcal{J}$. The rest is classical, see Lemma 2.4, Theorem 2.5, and Corollary 3.3 in [2]. □

Remark 3.1.6. *In view of the previous corollary, given a global minimizer $u \in \mathcal{K}$ of the functional $\mathcal{J}$, we can work with the precise representative*

$$u(\boldsymbol{x}) = \lim_{r\to 0^+} \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, d\boldsymbol{y}, \quad \boldsymbol{x} \in \Omega.$$

3.1.2 Uniform gradient estimates and boundary regularity

In view of Corollary 3.1.2, we study uniform properties of nonnegative and uniformly bounded solutions of (3.1.7). In particular (see Corollary 3.1.10), combining the results of [18] with the ones of [59] and [61], we show that under certain regularity conditions on $\partial\Omega$ and u_0, if u_ε is a global minimizer of $\mathcal{J}_\varepsilon$ in $\mathcal{K}$ (see (1.1.3), Theorem 3.1.1 and (3.1.6)), then the family $\{u_\varepsilon\}_\varepsilon$ satisfies a uniform-in-ε Lipschitz estimate away from $\partial\Gamma$, where $\partial\Gamma$ denotes the boundary of Γ as a subspace of $\partial\Omega$. In this subsection we work with sets that have the uniform C^2-regularity property.

Definition 3.1.7 (Definition 4.1 in [51])**.** *Let Ω be an open subset of $\mathbb{R}^N$. We say that Ω has the* uniform C^2-regularity property *if there exist a locally finite open cover $\{U_s\}_s$ of $\partial\Omega$, and corresponding C^2 homeomorphisms $\boldsymbol{\phi}_s$, such that:*

(i) *for each s, $\boldsymbol{\phi}_s(U_s) = B_1(\mathbf{0})$ and $\boldsymbol{\phi}_s(\Omega \cap U_s) = B_1^+(\mathbf{0})$;*

(ii) $\bigcup_s \boldsymbol{\phi}_s^{-1}(B_{1/2}(\mathbf{0})) \supset \{\boldsymbol{x} \in \overline{\Omega} : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \le \tau\}$*, for some $\tau > 0$;*

(iii) *there exists an integer R such that any $R+1$ distinct sets U_s have empty intersection;*

(iv) *for some sequence of points* $\{c_s\}_s \subset \mathbb{R}^N$*,*

$$\|\boldsymbol{\phi}_s\|_{C^2(\overline{U_s};\mathbb{R}^N)} \leq M, \quad \|\boldsymbol{\phi}_s^{-1} - c_s\|_{C^2(\overline{B_1(\mathbf{0})};\mathbb{R}^N)} \leq M,$$

for some M *independent of* s*.*

Remark 3.1.8. (i) *Definition 3.1.7 is standard in the treatment of regularity results for PDEs in unbounded domains. We remark that it is equivalent to the definition of boundary uniformly of class* C^2 *(see Definition 3.4 and Theorem 4.2 in [51]). Moreover, it is also equivalent to Property P in [18].*

(ii) *For any given* $d > 0$*, eventually replacing* R *with a larger number, we can assume without loss of generality that* $\operatorname{diam} U_s \leq d$*.*

Theorem 3.1.9. *Let* Ω *be an open connected subset of* $\mathbb{R}^N$ *with boundary* $\partial\Omega$ *uniformly of class* C^2*, and let* $u_0 \in C^{1,\alpha}(\overline{\Omega})$*,* $0 < \alpha < 1$*. Let* $\{u_\varepsilon\}_\varepsilon \subset W^{2,p}_{\mathrm{loc}}(\Omega)$*,* $N < p < \infty$*, be a family of nonnegative uniformly bounded solutions of* (3.1.7) *where* Q*, in addition to* (3.1.1)*, is assumed to be locally bounded in* $\overline{\Omega}$*. Then, for every* K *compactly contained in* $\overline{\Omega} \setminus \partial\Gamma$*, there exists a constant* C *such that*

$$|\nabla u_\varepsilon(\boldsymbol{x})| \leq C, \quad \boldsymbol{x} \in K, \tag{3.1.13}$$

where C *only depends on* $N, p, K, \|Q\|_{L^\infty(K)}, \|\beta\|_{L^\infty(\mathbb{R})}, \|u_0\|_{C^{1,\alpha}(\overline{\Omega})}, \sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}$ *and* $\partial\Omega$ *through* τ, R *and* M *as in Definition 3.1.7.*

Proof. Let K be a compact subset of $\overline{\Omega} \setminus \partial\Gamma$. If $K \subset \Omega$ the desired result follows directly from Theorem 3.1 (a) in [18]. Thus, assume that $K \cap \partial\Omega$ is non-empty and let $d_K := \operatorname{dist}(K, \partial\Gamma)$. Let $\{U_s\}_s$ be as in Definition 3.1.7 with $\operatorname{diam} U_s \leq d_K/2$ (see Remark 3.1.8 (ii)). By a compactness argument, we can find an integer S such that $K \cap U_s$ is empty for every $s > S$. Then there are $\mathcal{D}, \mathcal{N} \subset \mathbb{N}$ such that:

(i) $\mathcal{D}, \mathcal{N}$ are disjoint and $\mathcal{D} \cup \mathcal{N} = \{1, \ldots, S\}$;

(ii) $U_i \cap \partial\Omega \subset \Gamma$ for every $i \in \mathcal{D}$ and $U_j \cap \partial\Omega \subset \partial\Omega \setminus \Gamma$ for every $j \in \mathcal{N}$;

(iii) $\displaystyle\bigcup_{s \in \mathcal{D} \cup \mathcal{N}} \boldsymbol{\phi}_s^{-1}(B_{1/2}(\mathbf{0})) \supset K \cap \{\boldsymbol{x} \in \overline{\Omega} : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \leq \tau\}$, where τ is as in Definition 3.1.7;

(iv) $\displaystyle\bigcup_{s \in \mathcal{D} \cup \mathcal{N}} U_s \cap \overline{\Omega} \subset \overline{\Omega} \setminus \{\boldsymbol{x} \in \overline{\Omega} : \operatorname{dist}(\boldsymbol{x}, \partial\Gamma) < d_K/2\}$.

Notice that we are in a position to apply Theorem 3.1 in [61] in $U_i \cap \Omega$, $i \in \mathcal{D}$, and Theorem 3.1 (b) in [18] in $U_j \cap \Omega$, $j \in \mathcal{N}$. Therefore, there exists a constant C (depending on the other parameters of the problem, but independent of ε) such that

$$|\nabla u_\varepsilon(\boldsymbol{x})| \leq C, \quad \boldsymbol{x} \in \bigcup_{s \in \mathcal{D} \cup \mathcal{N}} \boldsymbol{\phi}_s^{-1}(B_{1/2}(\mathbf{0})).$$

Moreover, again by Theorem 3.1 (a) in [18], a similar estimate holds in $K \cap \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \geq \tau/2\}$ and hence, by (iii), everywhere in K. $\square$

Corollary 3.1.10. *Let Ω be an open and connected subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ uniformly of class C^2, and assume that* (1.1.4), (3.1.1), (3.1.3) *hold. In addition, we assume that $u_0 \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, and that Q, in addition to* (3.1.1), *is locally bounded in $\overline{\Omega}$. Let $\mathcal{J}_\varepsilon, \mathcal{J}$ and $\mathcal{K}$ be defined as in* (3.1.6), (1.1.2) *and* (1.1.3) *respectively. Then, given $\varepsilon_n \to 0^+$ and $\{u_n\}_n \subset \mathcal{K}$ such that u_n is a global minimizer of $\mathcal{J}_{\varepsilon_n}$ for every $n \in \mathbb{N}$, we have that $\{u_n\}_n \subset W^{2,p}_{\mathrm{loc}}(\Omega)$, $N < p < \infty$, and moreover there exists a subsequence $\{\varepsilon_{n_k}\}_k$ such that $\{u_{n_k}\}_k$ converges locally uniformly in $\overline{\Omega} \setminus \partial\Gamma$ to a function u that is a global minimizer of $\mathcal{J}$ in $\mathcal{K}$. In particular, u is locally Lipschitz continuous in $\overline{\Omega} \setminus \partial\Gamma$.*

Proof. By Theorem 3.1.1, for every $n \in \mathbb{N}$, u_n is a weak solution of (3.1.7) with $\varepsilon = \varepsilon_n$. Moreover, by Corollary 3.1.2, the sequence $\{u_n\}_n$ is nonnegative and uniformly bounded from above by $\|u_0\|_{L^\infty(\Gamma)}$, which is finite by assumption. By standard elliptic regularity theory, $\{u_n\}_n \subset W^{2,p}_{\mathrm{loc}}(\Omega)$, $N < p < \infty$ (see, e.g., [54] and [83]). Let $\{\varepsilon_{n_k}\}_k$, u be given as in Theorem 3.1.3. Then, reasoning as in the proof of Corollary 3.1.5, we obtain that u is a global minimizer of $\mathcal{J}$ in $\mathcal{K}$. Notice that by Theorem 3.1.9, we are in a position to apply the Ascoli-Arzelà Theorem to $\{u_{n_k}\}_k$. This proves the existence of a further subsequence (which we don't relabel) that converges uniformly to u on every compact subsets of $\overline{\Omega} \setminus \partial\Gamma$. To conclude, it is enough to notice that u inherits the gradient estimates on every compact subset of $\overline{\Omega} \setminus \partial\Gamma$ from the weak star convergence in L^∞ of (a subsequence of) $\{\nabla u_{n_k}\}_k$. □

Remark 3.1.11. (i) *Under the slightly more restrictive assumptions that $\partial\Omega$ is smooth and $u_0 \in C^{2,\alpha}(\overline{\Omega})$, an estimate up to the boundary near the Dirichlet fixed boundary can be obtained as in Section 2.3 of [59].*

(ii) *One of the main results presented in this thesis is the study of the boundary regularity for a certain class of global minimizers, with special emphasis given to the regularity of the free boundary. We refer to Section 3.4 for more information.*

3.2 Existence of nontrivial minimizers

In this section we shift the attention to our suggested framework for the study of planar gravity waves. For the convenience of the reader, we recall that throughout the rest of the chapter we will assume that $N = 2$ and let Ω be a half infinite strip, i.e.

$$\Omega := \left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right) \times (0, \infty).$$

Furthermore, we define the Sobolev space

$$H^1_{\lambda,\mathrm{loc}}(\Omega) := \left\{u \in H^1_{\mathrm{loc}}(\mathbb{R}^2_+) : u(x+\lambda, y) = u(x,y) \text{ for } \mathcal{L}^2\text{-a.e. } \boldsymbol{x} = (x,y) \in \mathbb{R}^2_+\right\},$$

and for $m, h > 0$ consider the energy functional

$$\mathcal{J}_h(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}}(h-y)_+\right) d\boldsymbol{x}, \qquad \text{for } u \in \mathcal{K}_\gamma,$$

where

$$\mathcal{K}_\gamma := \left\{u \in H^1_{\lambda,\mathrm{loc}}(\Omega) : u(\cdot, 0) = m \text{ and } u(\pm\lambda/2, y) = 0 \text{ for } y \geq \gamma\right\}.$$

Here γ is a positive constant, and the boundary conditions are satisfied in the sense of traces.

Theorem 3.2.1. *Let $\Omega, \mathcal{J}_h$, and $\mathcal{K}_\gamma$ be defined as above. Then there exists a global minimizer $u \in \mathcal{K}_\gamma$ of the functional $\mathcal{J}_h$. Furthermore, every global minimizer is locally Lipschitz continuous away from the points $\{(\pm\lambda/2, \gamma)\}$, and solves* (1.1.1)*, for Q as in* (1.1.6)*, where the free boundary condition is satisfied in a distributional sense.*

Proof. Notice that if we let

$$u_0(\boldsymbol{x}) := m\left(1 - \frac{y}{\gamma}\right)_+$$

then $u_0 \in \mathcal{K}_\gamma$ and $\mathcal{J}_h(u_0) < \infty$. Consequently, the existence of global minimizers in $\mathcal{K}_\gamma$ for $\mathcal{J}_h$ can be adapted from the results of the previous subsections (see Corollary 3.1.5), essentially without change. We omit the details. Since the boundary regularity for $y > \gamma$ is a consequence of Corollary 3.1.10, we are left to show that if u is a global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$ then u is Lipschitz continuous in $\overline{\Omega} \cap \{y < \gamma\}$. To see this it is enough to notice that u is a minimizer of $\mathcal{J}_h$ over the set

$$\mathcal{K}(u) := \{v \in H^1(B_r(\boldsymbol{x}) \cap \{y > 0\}) : v = u \text{ on } \partial(B_r(\boldsymbol{x}) \cap \{y > 0\})\}$$

for every $\boldsymbol{x} \in \overline{\Omega}$ and $\mathcal{L}^1$-a.e. $r \in (0, \lambda/2)$ such that $\overline{B_r(\boldsymbol{x})} \subset \{y < \gamma\}$. The desired result then follows from both the interior and the boundary regularity, as above. □

The rest of this section is dedicated to the proof of Theorem 1.1.1. Since this requires a precise understanding of the energy landscape of trivial solutions, i.e., solutions of the form $u = u(y)$, for the convenience of the reader we recall some preliminary definitions and results concerning the one-dimensional minimization problem for the functional

$$\mathcal{I}_h(v) := \int_0^\infty \left(v'(t) + \chi_{\{v>0\}}(t)(h-t)_+\right) dt, \tag{3.2.1}$$

defined over the set

$$\mathcal{K}_{\gamma,\text{1-d}} := \{v \in H^1_{\text{loc}}((0,\infty)) : v(0) = m \text{ and } v(\gamma) = 0\}, \tag{3.2.2}$$

where m, h, γ are positive numbers, and $H^1_{\text{loc}}((0,\infty))$ is the space of all functions $v \in L^2_{\text{loc}}((0,\infty))$ such that $v \in H^1((0,r))$ for every $r > 0$. Indeed, if $u \in \mathcal{K}_\gamma$ is a trivial solution to (1.1.9), then $u(x,y) = v(y)$ for $\mathcal{L}^1$-a.e. $x \in (-\lambda/2, \lambda/2)$ and $\mathcal{L}^1$-a.e. $y \in (0,\infty)$, where $v(0) = m$ and $v(\gamma) = 0$, and by Tonelli's theorem we have

$$\mathcal{J}_h(u) = \int_{-\lambda/2}^{\lambda/2}\int_0^\infty \left(|v'(y)|^2 + \chi_{\{v>0\}}(h-y)_+\right) dydx = \lambda\mathcal{I}_h(v). \tag{3.2.3}$$

For a fixed $m > 0$ and for every $h > 0$, we define $g_h \colon \mathbb{R}_+ \to \mathbb{R}$ by

$$g_h(t) := \frac{m^2}{t} + \frac{h^2 - (h - \min\{h,t\})^2}{2}, \tag{3.2.4}$$

and let $v_t \colon \mathbb{R}_+ \to \mathbb{R}$ be the function

$$v_t(s) := \frac{m}{t}(t-s)_+ \tag{3.2.5}$$

be defined for $t \in \mathbb{R}_+$. Observe that $g_h \in C^1(\mathbb{R}_+)$.

Theorem 3.2.2. *Given $m, h, \gamma > 0$, let $\mathcal{I}_h$ and $\mathcal{K}_{\gamma,1\text{-}d}$ be as in* (3.2.1) *and* (3.2.2)*, respectively, let $h^{\#}, h^*$ be given as in* (1.1.14)*, and g_h, v_t be given as above. Then*

$$\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = \inf\{g_h(t) : 0 < t < \gamma\}, \tag{3.2.6}$$

and the following hold:

(i) *if $h \leq h^{\#}$ then g_h is decreasing and v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$,*

(ii) *if $h^{\#} < h < h^*$ then g_h has two critical points, t_h, T_h,*

$$0 < t_h < \frac{2h}{3} < T_h < h, \tag{3.2.7}$$

which correspond to a point of local minimum and a point of local maximum of g_h, respectively. Moreover, there exists a unique $\tau_h > T_h$ such that $g_h(t_h) = g_h(\tau_h)$. In this case we have that

(a) *if $0 < \gamma \leq t_h$ then g_h is decreasing in $(0, \gamma)$ and v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$;*

(b) *if $t_h < \gamma < \tau_h$ then $\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = g_h(t_h)$ and v_{t_h} is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$;*

(c) *if $\gamma = \tau_h$ then $\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = g_h(t_h) = g_h(\tau_h)$ and v_{t_h}, v_{τ_h} are the only global minimizers of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$;*

(d) *if $\gamma > \tau_h$ then $\inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,1\text{-}d}\} = g_h(\gamma)$ and v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$;*

(iii) *if $h \geq h^*$ then t_h is a point of absolute minimum for g_h. Moreover, v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,1\text{-}d}$ if $0 < \gamma \leq t_h$, while if $t_h < \gamma$ then the only global minimizer is given by v_{t_h}.*

Proof. We divide the proof into several steps.
Step 1: By Corollary 3.1.5 we have that there exists a global minimizer v of $\mathcal{I}_h$ in $\mathcal{K}_{\gamma,1\text{-d}}$. We claim that v is linear on $\{v > 0\}$. Indeed, the minimality of v implies that the set $\{v > 0\}$ is connected; the claim follows recalling that v is harmonic in $\{v > 0\}$ (see Corollary 3.1.5). Thus, v is of the form $v = v_t$ for some $0 < t < \gamma$ and so (3.2.6) follows by noticing that

$$\mathcal{I}_h(v_t) = g_h(t). \tag{3.2.8}$$

Thus it remains to study $\inf\{g_h(t) : 0 < t < \gamma\}$.
Step 2: Since

$$g_h'(t) = \begin{cases} -\frac{m^2}{t^2} + h - t & \text{if } t \leq h, \\ -\frac{m^2}{t^2} & \text{if } t > h, \end{cases}$$

we have that $g_h'(t) < 0$ if $t \geq h$. Moreover, $g_h'(t) \leq 0$ for $t < h$ if and only if

$$\psi_h(t) := -m^2 + ht^2 - t^3 \leq 0. \tag{3.2.9}$$

Since ψ_h has a global maximum in $(0,h)$ at the point $t = 2h/3$, it follows that

$$\psi_h(2h/3) = -m^2 + \frac{4}{27}h^3 \leq 0 \tag{3.2.10}$$

if and only if $h \leq h^{\#}$, where $h^{\#}$ is the number given in $(1.1.14)_1$. Consequently, if $h \leq h^{\#}$ then g_h is decreasing and so

$$\inf\{g_h(t) : 0 < t < \gamma\} = g_h(\gamma),$$

which, together with (3.2.6) and (3.2.8), shows that v_γ is the only global minimizer of $\mathcal{I}_h$ in the class $\mathcal{K}_{\gamma,\text{1-d}}$.

Step 3: If $h > h^{\#}$, then in view of (3.2.9) and (3.2.10) there exist

$$0 < t_h < \frac{2h}{3} < T_h < h$$

such that g_h strictly decreases in $(0,t_h)$ and in (T_h,∞), and strictly increases in (t_h,T_h). It follows that

$$\inf\{g_h(t) : 0 < t < \gamma\} = \begin{cases} g_h(\gamma) & \text{if } 0 < \gamma \leq t_h, \\ g_h(t_h) & \text{if } t_h < \gamma \leq T_h, \\ \min\{g_h(t_h), g_h(\gamma)\} & \text{if } \gamma > T_h. \end{cases} \tag{3.2.11}$$

Hence, in what follows, it remains to treat the case $\gamma > T_h$. Notice that

$$\inf\{g_h(t) : 0 < t < \gamma\} = g_h(t_h) \leq \lim_{t\to\infty} g_h(t) = \frac{h^2}{2} \tag{3.2.12}$$

if and only if

$$2m^2 \leq \sup\{f_h(t) : 0 < t < h\},$$

where $f_h(t) \coloneqq t(h-t)^2$. The function f_h has a maximum at $t = h/3$, and so,

$$2m^2 \leq f_h(h/3),$$

or equivalently $h \geq h^*$, where h^* is the number given in $(1.1.14)_2$. Hence by (3.2.12) if $h \geq h^*$ then $g_h(t_h) < g_h(\gamma)$, which, by (3.2.6), (3.2.8), and (3.2.11), proves (iii), while if $h < h^*$ then by (3.2.12) there exists $T_h < \tau_h$ such that $g_h(t_h) = g_h(\tau_h)$.

Properties $(a), (b), (c), (d)$ now follow again by (3.2.6),(3.2.8), and (3.2.11). $\square$

Proof of Theorem 1.1.1. **Step 1:** For $\gamma, \delta > 0$, consider the function

$$w(x,y) \coloneqq m\left(1 - \frac{y\lambda}{(\gamma+\delta)\lambda - 2|x|\delta}\right)_+,$$

defined first for $\boldsymbol{x} = (x,y) \in \Omega$ and then extended by periodicity to $\mathbb{R} \times (0,\infty)$. Notice that the support of w in Ω corresponds to the polygonal region with vertices $(\pm\lambda/2, 0)$, $(\pm\lambda/2, \gamma)$, and $(0, \gamma+\delta)$, and therefore w belongs to the class $\mathcal{K}_\gamma$. A direct computations shows that

$$\int_\Omega |\nabla w|^2 \, d\boldsymbol{x} = m^2 \left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta}\right) \log\left(1 + \frac{\delta}{\gamma}\right). \tag{3.2.13}$$

In the following, if $\gamma < h$, we will only consider values of δ for which $\gamma + \delta \leq h$. Once again by means of a direct computation, we see that the contribution from the second term in the energy for w is given by

$$\begin{cases} \dfrac{\lambda h^2}{2} & \text{if } h \leq \gamma, \\ \dfrac{\lambda h^2}{2} - \dfrac{\lambda(h-\gamma)^2}{2} + \dfrac{\lambda(h-\gamma)\delta}{2} - \dfrac{\lambda\delta^2}{6} & \text{if } \gamma < \gamma + \delta \leq h. \end{cases} \tag{3.2.14}$$

Next, we will show that if $\delta > 0$ is chosen opportunely then

$$\mathcal{J}_h(w) < \lambda g_h(\gamma), \tag{3.2.15}$$

where g_h is the function defined in (3.2.4). Notice that this implies the desired result; indeed, if γ is chosen as in (1.1.15), then it follows from Theorem 3.2.2 and (3.2.3) that

$$\lambda g_h(\gamma) = \lambda \inf\{\mathcal{I}_h(v) : v \in \mathcal{K}_{\gamma,\text{1-d}}\} \leq \inf\{\mathcal{J}_h(u) : u \in \mathcal{K}_\gamma, u = u(y)\}, \tag{3.2.16}$$

and so (3.2.15) implies that solutions of the form $u = u(y)$ cannot be found among global minimizers of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$.

Step 2: In this step we address the case in which γ, chosen as in (1.1.15), is such that $\gamma \geq h$. Using the inequality

$$\log(1+t) < t - \frac{t^2}{2} + \frac{t^3}{3}, \quad t > 0, \tag{3.2.17}$$

it follows from (3.2.4), (3.2.13) and (3.2.14) that to prove (3.2.15) it is enough to show that for a certain choice of δ

$$\left(\frac{4\delta}{3\lambda^2} + \frac{1}{\delta}\right)\left(\frac{\delta}{\gamma} - \frac{\delta^2}{2\gamma^2} + \frac{\delta^3}{3\gamma^3}\right) < \frac{1}{\gamma}. \tag{3.2.18}$$

Notice that the previous inequality is equivalent to

$$-\frac{1}{2\gamma^2} + \delta\left(\frac{4}{3\lambda^2\gamma} + \frac{1}{3\gamma^3} - \frac{2\delta}{3\lambda^2\gamma^2} + \frac{4\delta^2}{9\lambda^2\gamma^3}\right) < 0, \tag{3.2.19}$$

which is satisfied if δ is sufficiently small.

Step 3: We now turn our attention to the case $\gamma < h$. Since we only consider values of δ for which $\gamma + \delta \leq h$, by (3.2.4), (3.2.13) and (3.2.14) we find that (3.2.15) is equivalent to

$$m^2\left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta}\right)\log\left(1 + \frac{\delta}{\gamma}\right) + \frac{\lambda(h-\gamma)\delta}{2} - \frac{\lambda\delta^2}{6} < \frac{\lambda m^2}{\gamma}. \tag{3.2.20}$$

By (3.2.17), and reasoning as in the previous step, we see that it is enough to prove that the inequality

$$-\frac{m^2}{2\gamma^2} + m^2\delta\left(\frac{4}{3\lambda^2\gamma} + \frac{1}{3\gamma^3} - \frac{1}{6m^2} - \frac{2\delta}{3\lambda^2\gamma^2} + \frac{4\delta^2}{9\lambda^2\gamma^3}\right) + \frac{(h-\gamma)}{2} \leq 0 \tag{3.2.21}$$

holds for an opportune choice of δ. By Theorem 3.2.2 we see that if γ is as in (1.1.15) then

$$-\frac{m^2}{\gamma^2} + h - \gamma = g_h'(\gamma) < 0. \tag{3.2.22}$$

Consequently, also in this case, (3.2.21) is satisfied if we choose δ small enough (with respect to the other parameters in the problem). $\square$

Lemma 3.2.3. *Given $m > 0$, let t_h and τ_h be defined as in Theorem 3.2.2. Then t_h is decreasing as a function of h, while τ_h is increasing.*

Proof. By the implicit function theorem, we have that the maps $h \mapsto t_h$ and $h \mapsto \tau_h$ are differentiable, and we write t'_h and τ'_h to denote the derivatives. Then we see that

$$t'_h = -\frac{t_h}{2h - 3t_h} < 0, \quad \text{for } h > h^{\#},$$

from which we conclude that t_h is decreasing as a function of h. To prove the statement about τ_h, we first assume that $\tau_h < h$. Then τ_h is defined by

$$\frac{m^2}{t_h} - \frac{(h - t_h)^2}{2} = \frac{m^2}{\tau_h} - \frac{(h - \tau_h)^2}{2},$$

which in turn implies that

$$\frac{m^2 t'_h}{t_h^2} + (h - t_h)(1 - t'_h) = \frac{m^2 \tau'_h}{\tau_h^2} + (h - \tau_h)(1 - \tau'_h). \tag{3.2.23}$$

The definition of t_h can now be used to simplify the left-hand side of (3.2.23):

$$\frac{m^2 t'_h}{t_h^2} + (h - t_h)(1 - t'_h) = t'_h \left(\frac{m^2}{t_h^2} - h + t_h \right) + h - t_h = h - t_h.$$

Therefore we can rewrite (3.2.23) as

$$\left(\frac{m^2}{\tau_h^2} - h + \tau_h \right) \tau'_h = \tau_h - t_h,$$

and the conclusion follows recalling that $t_h < \tau_h$ and $m^2 - h\tau_h^2 + \tau_h^3 > 0$. The proof for the case $\tau_h \geq h$ is similar but simpler. $\square$

Remark 3.2.4. *The result of Theorem 1.1.1 cannot be improved for $h < h^{\#}$ and $h \geq h^*$. However, it is still unclear whether the result is optimal also for $h^{\#} \leq h < h^*$ (see figure below).*

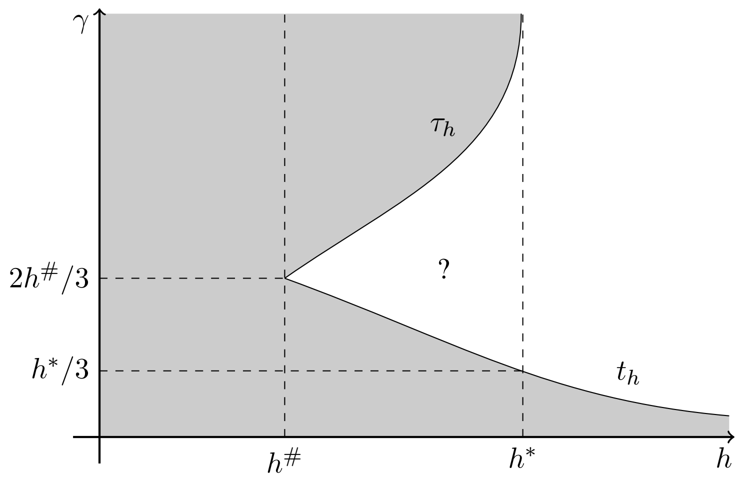
Remark 3.2.5. *For $h > h^{\#}$ and $0 < t < h$, the cubic equation $t^3 - ht^2 + m^2 = 0$ has three real solutions, two of which are positive. Setting*

$$\theta := \arccos\left(1 - \frac{3^3}{2}\frac{m^2}{h^3}\right)$$

so that $0 < \theta < \pi$, the two positive solutions are given by

$$t_h := \frac{2h}{3} \cos\frac{\theta + 4\pi}{3} + \frac{h}{3} \in \left(0, \frac{2h}{3}\right),$$

$$T_h := \frac{2h}{3} \cos\frac{\theta}{3} + \frac{h}{3} \in \left(\frac{2h}{3}, h\right).$$



We also know that

$$t_h < 2^{1/3} m^{2/3} < T_h.$$

Indeed, for every $\eta \in (0, h - h^{\#})$, by (3.2.7) *and Lemma 3.2.3 we have*

$$t_h < t_{h-\eta} < \frac{2}{3}(h - \eta) < T_{h-\eta} < T_h.$$

To conclude, let $\eta \to h - h^{\#}$.

We conclude the section with a result which states that not only trivial solutions, but also flat free boundaries cannot be observed if γ is chosen as in Theorem 1.1.1.

Lemma 3.2.6. *For γ as in Theorem 1.1.1, let $u \in \mathcal{K}_\gamma$ be a global minimizer of $\mathcal{J}_h$. Then $\partial\{u > 0\} \cap \Omega$ does not coincide with a horizontal line segment.*

Proof. Assume for the sake of contradiction that $\partial\{u > 0\}$ coincides with the line $\{y = k\}$, for some $k > 0$. Then $k \leq h$. Assume first that $k \in (0, \gamma]$. We claim that

$$v(x, y) = \frac{m}{k}(k - y)_+$$

satisfies $\mathcal{J}_h(v) = \mathcal{J}_h(u)$. Notice that since by assumption $k \leq \gamma$ we have that $v \in \mathcal{K}_\gamma$ and the claim would imply that v is a global minimizer of $\mathcal{J}_h$, which would be in contradiction with the choice of γ. To prove the claim, it is enough to observe that Tonelli's theorem and Jensen's inequality yield

$$\int_\Omega |\nabla u|^2 \, d\boldsymbol{x} \geq \int_{-\lambda/2}^{\lambda/2} \int_0^k (\partial_y u)^2 \, dy dx \geq \int_{-\lambda/2}^{\lambda/2} \frac{1}{k} \left(\int_0^k \partial_y u \, dy \right)^2 dx = \frac{\lambda m^2}{k} = \int_\Omega |\nabla v|^2 \, d\boldsymbol{x},$$

and that the functions u and v have the same support. On the other hand, since the free boundary detaches tangentially from a smooth portion of the Dirichlet fixed boundary (see Theorem 1.1 in [29]), k cannot be larger than γ, and the result is thus proved. $\square$

3.3 The shape of global minimizers

The aim of this section is to carry out the study of additional properties of global minimizers of the functional $\mathcal{J}_h$, defined as in (1.1.12). In particular, our main interest lies in understanding how the shape of global minimizers is influenced by the parameter h. To this end, throughout the rest of this section we fix a non-increasing function $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\theta(h) = \gamma_h, \tag{3.3.1}$$

where for every h the number γ_h is chosen in accordance with Theorem 1.1.1. We then consider solutions to the minimization problem for $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_h}$.

3.3.1 Existence of minimizers with bounded support

Proposition 3.3.1. *Given $m, \lambda, h, \gamma > 0$ and $k \in (0,1)$, there exists a positive constant $C_{\min}(k)$ such that for every minimizer u of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$ and for every ball $B_r(\boldsymbol{x}) \subset \Omega$, if*

$$\frac{1}{r}\fint_{\partial B_r(\boldsymbol{x})} u \, d\mathcal{H}^1 \leq C_{\min}(k)\sqrt{(h-y-kr)_+},$$

then $u \equiv 0$ in $B_{kr}(\boldsymbol{x})$. Moreover, if $0 < r < \lambda$, the result is still valid for balls not contained in Ω, provided $\overline{B_r(\boldsymbol{x})} \subset \{y > \gamma\}$ or $\overline{B_r(\boldsymbol{x})} \subset \{y < \gamma\}$.

For a proof of Proposition 3.4.3 we refer to Lemma 3.4 and Remark 3.5 in [2]; see also Theorem 3.6 and Remark 5.2 in [15].

Theorem 3.3.2. *Given $m, \lambda > 0$, let $\mathcal{J}_h$ and $\mathcal{K}_{\gamma_h}$ be defined as in* (1.1.12), (1.1.13)*, respectively, where for every h the value of γ_h is given as in* (3.3.1)*. Then for every $\bar{y} > 0$ there exists $h_0 = h_0(\bar{y})$ such that if $h \geq h_0$ then the support of every global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_h}$ is contained in the set $\{y < h\}$. In particular, the free boundary of every such minimizer is locally of analytic in Ω.*

Proof. Let $\bar{y} > 0$ be given. Assume first that there exists h_1 such that $\gamma_{h_1} \leq 3\bar{y}/2$ and let $r := \bar{y}/4$. Then for every $x \in (-\lambda/2, \lambda/2)$,

$$\overline{B_r(x,\bar{y})} \subset \{y > \gamma_h\}$$

for every $h \geq h_1$. Moreover, for every global minimizer u,

$$\frac{1}{r\sqrt{h-\bar{y}-r/2}}\fint_{\partial B_r(x,\bar{y})} u \, d\mathcal{H}^1 \leq \frac{m}{r\sqrt{h-\bar{y}-r/2}} = \frac{4m}{\bar{y}\sqrt{h-\frac{9}{8}\bar{y}}},$$

where the first inequality follows from Corollary 3.1.2. We can then find $h_0 \geq h_1$ such that if $h \geq h_0$ then

$$\frac{4m}{\bar{y}\sqrt{h-\frac{9}{8}\bar{y}}} \leq C(1/2).$$

Thus we are in a position to apply Proposition 3.4.3 to show that u is identically equal to zero in the set $(-\lambda/2, \lambda/2) \times [7\bar{y}/8, 9\bar{y}/8]$. Since by minimality the support of u is connected, we have that u must vanish in $(-\lambda/2, \lambda/2) \times [\bar{y}, \infty)$. Similarly, if $\gamma_h > 3\bar{y}/2$ for every h, we have that

$$\overline{B_r(x,\bar{y})} \subset \{y < \gamma_h\}$$

for every $h > 0$. Thus we can proceed as above. The last statement follows from Theorem 8.4 in [2]. $\square$

3.3.2 Existence of a critical height

The following result is inspired by Theorem 10.1 in [53] (see also Theorem 5.5 in [15]).

Theorem 3.3.3. (Monotonicity). *Given $m, \lambda > 0$, let $\mathcal{J}_h$ and $\mathcal{K}_{\gamma_h}$ be defined as in* (1.1.12), (1.1.13)*, respectively, where for every h the value of γ_h is given as in* (3.3.1)*. Consider $0 < d < h$ and let u_d, u_h be global minimizers of $\mathcal{J}_d$ and $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_d}$ and $\mathcal{K}_{\gamma_h}$ respectively. Then*

$$\{\boldsymbol{x} \in \Omega : u_h(\boldsymbol{x}) > 0\} \subset \{\boldsymbol{x} \in \Omega : u_d(\boldsymbol{x}) > 0\} \tag{3.3.2}$$

and

$$u_h \le u_d. \tag{3.3.3}$$

Moreover, if $\partial\{u_d > 0\} \neq \emptyset$ then $u_h < u_d$ in $\{\boldsymbol{x} \in \Omega : u_d(\boldsymbol{x}) > 0\}$.

Proof. **Step 1:** Define $v_1 := \min\{u_d, u_h\}$ and $v_2 := \max\{u_d, u_h\}$. Since $h \mapsto \gamma_h$ is decreasing, we have that $v_1 \in K_{\gamma_h}$ and $v_2 \in \mathcal{K}_{\gamma_d}$, and so

$$\mathcal{J}_d(u_d) + \mathcal{J}_h(u_h) \le \mathcal{J}_d(v_2) + \mathcal{J}_h(v_1). \tag{3.3.4}$$

Notice that

$$\begin{aligned}
\int_\Omega \left(|\nabla v_1|^2 + |\nabla v_2|^2\right) \, d\boldsymbol{x} &= \int_{\{u_h > u_d\}} \left(|\nabla v_1|^2 + |\nabla v_2|^2\right) d\boldsymbol{x} + \int_{\{u_h \le u_d\}} \left(|\nabla v_1|^2 + |\nabla v_2|^2\right) d\boldsymbol{x} \\
&= \int_{\{u_h > u_d\}} \left(|\nabla u_d|^2 + |\nabla u_h|^2\right) d\boldsymbol{x} + \int_{\{u_h \le u_d\}} \left(|\nabla u_h|^2 + |\nabla u_d|^2\right) d\boldsymbol{x} \\
&= \int_\Omega \left(|\nabla u_d|^2 + |\nabla u_h|^2\right) \, d\boldsymbol{x}.
\end{aligned}$$

Therefore we can rewrite (3.3.4) canceling out the gradient terms and by rearranging the remaining terms we obtain

$$\int_{\{u_h > u_d\}} \left(\chi_{\{u_h > 0\}} - \chi_{\{u_d > 0\}}\right) \left((h - y)_+ - (d - y)_+\right) \, d\boldsymbol{x} \le 0. \tag{3.3.5}$$

Since the integrand is nonnegative in the set $\{u_h > u_d\}$, and recalling that u_d and u_h are continuous in Ω, we have that

$$\{u_h > 0\} \cap \{y < h\} \cap \{u_h > u_d\} \subset \{u_d > 0\} \cap \{y < h\} \cap \{u_h > u_d\},$$

which together with the fact that

$$\{u_h > 0\} \cap \{u_h \le u_d\} \subset \{u_d > 0\} \cap \{u_h \le u_d\}$$

yields

$$\{u_h > 0\} \cap \{y < h\} \subset \{u_d > 0\} \cap \{y < h\}. \tag{3.3.6}$$

We now notice that if $\operatorname{supp} u_h \subset (-\lambda/2, \lambda/2) \times [0, d]$ then (3.3.2) follows from (3.3.6), while if it is not the case, again by (3.3.6) we get that there is $\boldsymbol{x} \in (-\lambda/2, \lambda/2) \times (d, \infty)$ such that $u_d(\boldsymbol{x}) > 0$.

Furhermore, we see that $u_d > 0$ in $(-\lambda/2, \lambda/2) \times (d, \infty)$, and so the desired inclusion is also satisfied in $(-\lambda/2, \lambda/2) \times [h, \infty)$. This concludes the proof of (3.3.2).

Step 2: We observe that since the equality holds in (3.3.5), then the equality necessarily holds in (3.3.4) as well, and so v_1 and v_2 are global minimizers of $\mathcal{J}_h$ and $\mathcal{J}_d$ in $\mathcal{K}_{\gamma_h}$ and $\mathcal{K}_{\gamma_d}$ respectively. We now claim that if there is $\boldsymbol{x}_0 \in \Omega$ such that $u_d(\boldsymbol{x}_0) = u_h(\boldsymbol{x}_0) > 0$, then $u_d = u_h$ everywhere in Ω. To see this, we notice that in a neighborhood of $\boldsymbol{x}_0$ the functions $u_d - v_2$ and $u_h - v_2$ are harmonic, nonpositive and attain a maximum at an interior point. Then, by the maximum principle, $u_d - v_2 = u_h - v_2 \equiv 0$ in the connected component of $\{u_h > 0\}$ that contains $\boldsymbol{x}_0$; since $\{u_h > 0\}$ is connected by minimality, this proves the claim.

To prove (3.3.3), assume by contradiction that there is $\boldsymbol{x} \in \Omega$ such that $u_h(\boldsymbol{x}) > u_d(\boldsymbol{x})$. If there is $\boldsymbol{y} \in \{u_h > 0\}$ such that $u_d(\boldsymbol{y}) > u_h(\boldsymbol{y})$, then by the connectedness of $\{u_h > 0\}$, together with the fact that u_h and u_d are continuous, we have that there is $\boldsymbol{z} \in \Omega$ such that $u_h(\boldsymbol{z}) = u_d(\boldsymbol{z}) > 0$. By the claim we just proved, this would imply that $u_h = u_d$, a contradiction. Hence $u_d \leq u_h$ in $\{u_h > 0\}$, which together with (3.3.2) implies that

$$\{u_h > 0\} = \{u_d > 0\}. \tag{3.3.7}$$

In turn,

$$\begin{aligned} \int_\Omega \chi_{\{u_h>0\}}(h-y)_+ \, d\boldsymbol{x} &= \int_\Omega \chi_{\{u_d>0\}}(h-y)_+ \, d\boldsymbol{x}, \\ \int_\Omega \chi_{\{u_d>0\}}(d-y)_+ \, d\boldsymbol{x} &= \int_\Omega \chi_{\{u_h>0\}}(d-y)_+ \, d\boldsymbol{x}. \end{aligned} \tag{3.3.8}$$

From (3.3.7) we also see that $u_d \in \mathcal{K}_{\gamma_h}$. Since $h \mapsto \gamma_h$ is decreasing, we also have that $u_h \in \mathcal{K}_{\gamma_d}$ and hence we can conclude that $\mathcal{J}_h(u_h) \leq \mathcal{J}_h(u_d)$ and $\mathcal{J}_d(u_d) \leq \mathcal{J}_d(u_h)$, which, together with (3.3.8), implies that

$$\int_\Omega |\nabla u_h|^2 \, d\boldsymbol{x} = \int_\Omega |\nabla u_d|^2 \, d\boldsymbol{x}.$$

Consider $v := \frac{1}{2}u_h + \frac{1}{2}u_d \in \mathcal{K}_{\gamma_h}$. By the strict convexity of the Dirichlet energy, we have

$$\mathcal{J}_h(v) < \int_\Omega \left(\frac{1}{2}|\nabla u_h|^2 + \frac{1}{2}|\nabla u_d|^2 + \chi_{\{v>0\}}(h-y)_+ \right) d\boldsymbol{x} = \mathcal{J}_h(u_h),$$

a contradiction to the minimality of u_h, and (3.3.3) is hence proved.

Step 3: Finally, assume by contradiction that there is $\boldsymbol{x}_0 \in \{u_d > 0\}$ such that $u_h(\boldsymbol{x}_0) = u_d(\boldsymbol{x}_0)$, so that $u_h = u_d$ in Ω. Since by assumption $\partial\{u_d > 0\} \neq \emptyset$, and since γ_d is chosen in such a way that u_d is not a one-dimensional profile, it must be the case that there exists $\boldsymbol{x} \in \partial\{u_d > 0\} \cap \{y < d\}$. Consequently, for every such $\boldsymbol{x}$ we have

$$\sqrt{h-y} = \frac{\partial u_h}{\partial \nu}(\boldsymbol{x}) = \frac{\partial u_d}{\partial \nu}(\boldsymbol{x}) = \sqrt{d-y}.$$

This is a contradiction since by assumption $d < h$. Hence $u_h < u_d$ in $\{u_d > 0\}$ and the proof is complete. $\square$

Proof of Theorem 1.1.3. Let

$$h_{\mathrm{cr}} := \inf\{h > 0 : \text{there is a global minimizer } u_h \in \mathcal{K}_{\gamma_h} \text{ of } \mathcal{J}_h \text{ s.t. } \operatorname{supp} u_h \subset \{y \le h\}\}. \quad (3.3.9)$$

By Theorem 3.3.2 we have that $h_{\mathrm{cr}} < \infty$. Assume for the sake of contradiction that $h_{\mathrm{cr}} = 0$. Then for every $h > 0$ there exists a global minimizer $u_h \in \mathcal{K}_{\gamma_h}$ with the property that the support of u_h is entirely contained in the set $\{y \le h\}$. Reasoning as in the proof of Lemma 3.2.6, we see that

$$\mathcal{J}_h(u_h) > \int_\Omega |\nabla u_h|^2 \, d\boldsymbol{x} \ge \frac{\lambda m^2}{h}.$$

Since by assumption the function θ is non-increasing, there exists $\bar{h}$ such that if $h \le \bar{h}$ then

$$h \le \theta(h) = \gamma_h.$$

For every such h we let w be function defined in the proof of Theorem 1.1.1, where for simplicity we fix $\delta = 1$. Then, it follows from (3.2.13) and (3.2.14) that

$$\mathcal{J}_h(w) = m^2 \left(\frac{4}{3\lambda} + \lambda \right) \log \left(1 + \frac{1}{\gamma_h} \right) + \frac{\lambda h^2}{2}.$$

In particular, notice that for every h small enough (with respect to the other parameters in the problem)

$$m^2 \left(\frac{4}{3\lambda} + \lambda \right) \log \left(1 + \frac{1}{\gamma_h} \right) + \frac{\lambda h^2}{2} \le \frac{\lambda m^2}{h}.$$

Since by definition $w \in \mathcal{K}_\gamma$, this gives a contradiction with the minimality of u_h. In turn, we have shown that $h_{\mathrm{cr}} > 0$. Properties (i) and (ii) follow immediately from Theorem 3.3.3; we omit the details. □

Remark 3.3.4. *By Theorem 8.4 in [2], it follows that if $u \in \mathcal{K}_{\gamma_h}$ is a global minimizer of $\mathcal{J}_h$ for $h > h_{\mathrm{cr}}$, then $\partial\{u > 0\}$ is analytic locally in Ω. The following result shows that the result is true also for $h < h_{\mathrm{cr}}$.*

Theorem 3.3.5. *Under the assumptions of Theorem 1.1.3, let $h > h_{\mathrm{cr}}$. Then the free boundary of every global minimizer is analytic locally in Ω.*

Proof. Since by assumption there exists $\boldsymbol{x} \in \Omega$ with $y > h$ such that $u(\boldsymbol{x}) > 0$, it follows from the maximum principle that $u > 0$ in $(-\lambda/2, \lambda/2) \times (h, \infty)$. For the sake of contradiction, assume that the free boundary of a global minimizer u intersects the line $\{y = h\}$ at a point $\boldsymbol{z}$. Then, by Remark 3.5 in [15], there exists a constant C such that

$$|\nabla u(\boldsymbol{y})| \le C\sqrt{r} \quad (3.3.10)$$

for all $\boldsymbol{y} \in B_r(\boldsymbol{z})$, where $r > 0$ is sufficiently small. Let B_ρ be any ball in $(-\lambda/2, \lambda/2) \times (h, \infty)$ such that $\boldsymbol{x} \in \partial B_\rho$. Since $u(\boldsymbol{y}) > u(\boldsymbol{z}) = 0$ for every $\boldsymbol{y} \in B_\rho$, we have that (3.3.10) is in contradiction with Hopf's Lemma. To conclude it is then enough to invoke Theorem 8.4 in [2]. □

3.3.3 Scaling of the critical height

Theorem 3.3.6. (Comparison principle). *Given $m, \lambda > 0$, let $\mathcal{J}_h$ and $\mathcal{K}_{\gamma_h}$ be defined as in* (1.1.12), (1.1.13)*, respectively, where for every h the value of γ_h is given as in* (3.3.1)*. Let u and w be a global minimizers of $\mathcal{J}_h$ in $\mathcal{K}_\delta$ and $\mathcal{K}_\gamma$ respectively, where $\mathcal{K}_\delta, \mathcal{K}_\gamma$ are defined as in* (1.1.13)*. Then either*

$$\{u > 0\} \subset \{w > 0\} \text{ and } u \le w$$

or

$$\{w > 0\} \subset \{u > 0\} \text{ and } w \le u.$$

Proof. Assume without loss of generality that $\delta \le \gamma$. As in the proof of Theorem 3.3.3, we consider $v_1 := \min\{u, w\}$ and $v_2 := \max\{u, w\}$. Then $v_1 \in \mathcal{K}_\delta$, $v_2 \in \mathcal{K}_\gamma$ and in particular we have

$$\mathcal{J}_h(u) + \mathcal{J}_h(w) = \mathcal{J}_h(v_1) + \mathcal{J}_h(v_2).$$

Therefore v_1 and v_2 are global minimizers of $\mathcal{J}_h$ in $\mathcal{K}_\delta$ and $\mathcal{K}_\gamma$ respectively. Reasoning as in the proof of Theorem 3.3.3, we have that if there exists a point $\boldsymbol{x}_0$ such that $u(\boldsymbol{x}_0) = w(\boldsymbol{x}_0) > 0$ then $u = w$ everywhere in Ω. Next, we assume by contradiction that the supports of u and w do not satisfy the inclusions as in the statement, i.e., there exist $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ such that $u(\boldsymbol{x}) > 0$, $w(\boldsymbol{y}) > 0$ and $u(\boldsymbol{y}) = w(\boldsymbol{x}) = 0$. Let $\boldsymbol{z} \in \Omega$ be such that $u(\boldsymbol{z}) > 0$ and $w(\boldsymbol{z}) > 0$ (such a point $\boldsymbol{z}$ exists since by minimality we have that $\mathcal{J}_h(u)$ and $\mathcal{J}_h(w)$ are both finite). We assume first that $w(\boldsymbol{z}) > u(\boldsymbol{z})$. Then, since by minimality $\{u > 0\}$ is open and connected and thus path-wise connected, we can find a continuous curve $\boldsymbol{\varphi}\colon [0,1] \to \Omega$ joining $\boldsymbol{z}$ to $\boldsymbol{x}$, with support contained in Ω. Define

$$v(t) := w(\boldsymbol{\varphi}(t)) - u(\boldsymbol{\varphi}(t)).$$

Notice that by construction $v(0) = w(\boldsymbol{z}) - u(\boldsymbol{z}) > 0$ and $v(1) = w(\boldsymbol{x}) - u(\boldsymbol{x}) < 0$, and so there exists $t_0 \in (0,1)$ such that $v(t_0) = 0$. Thus $0 < u(\boldsymbol{\varphi}(t)) = w(\boldsymbol{\varphi}(t))$, which in turn implies that $u = w$, a contradiction. Similarly, if $u(\boldsymbol{z}) > w(\boldsymbol{z})$, we arrive to a contradiction by considering a continuous curve $\boldsymbol{\psi}\colon [0,1] \to \Omega$ that joins $\boldsymbol{z}$ with $\boldsymbol{y}$ and with support contained in $\{w > 0\}$. The rest of the proof is analogous to the proof of (3.3.3). ☐

Remark 3.3.7. *Notice that in Theorem 3.3.6 we also allow for the case where $\delta = \gamma$.*

In this lemma we show that h_{cr} in Theorem 1.1.3 is less than the value h^* given in $(1.1.14)_2$.

Lemma 3.3.8. *Under the assumptions of Theorem 1.1.3, we have that*

$$h_{\text{cr}} \le h^* = 3\left(\frac{m}{\sqrt{2}}\right)^{2/3}.$$

Proof. Assume by contradiction that $h_{\text{cr}} > h^*$, and let $h^* < h < h_{\text{cr}}$. By Tonelli's theorem and Theorem 3.2.2 (iii) we have that $w\colon \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$w(\cdot, y) := v_{t_h}(y)$$

is the unique global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_{t_h}$. Let $u \in \mathcal{K}_{\gamma_h}$ be a global minimizer of $\mathcal{J}_h$. Since $h > h^*$, by Theorem 1.1.1 we have that $\gamma_h < t_h$, and hence $u(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \{\pm\lambda/2\} \times (\gamma_h, \infty)$. By continuity, we can find $x_0 \in (-\lambda/2, \lambda/2)$ such that

$$u(x_0, \gamma_h) < \frac{m}{t_h}(t_h - \gamma_h) = w(x_0, \gamma_h).$$

Then, by Theorem 3.3.6, $u \leq w$ and

$$\{u > 0\} \subset \{w > 0\} = \{x_N < t_h\}.$$

Thus, by (3.2.7), u has bounded support in Ω, a contradiction to the definition of h_{cr}. $\square$

The following result shows that for certain choices of θ as in (3.3.1), the critical height h_{cr} is greater or equal to a constant multiple of $h^{\#}$, where $h^{\#}$ is given as in $(1.1.14)_1$.

Theorem 3.3.9. *Under the assumptions of Theorem 1.1.3, fix $k_1, k_2 > 0$ such that*

$$\frac{1}{k_2} < 1 - \frac{k_1^3}{2}, \quad \textit{and let} \quad k = \frac{2^{2/3} k_1}{3}$$

Notice that $k_2 > 1$ and $k \in (0, 2/3)$. Furthermore, assume that $\gamma_h \geq k_2 h$ for $h < h^{\#}$. Then $h_{\text{cr}} \geq k h^{\#}$.

Proof. Assume for the sake of contradiction that $h_{\text{cr}} < k h^{\#}$. Then, every global minimizer of J_h in $\mathcal{K}_{\gamma_h}$ for $h = k h^{\#}$ is a regular solution. In turn,

$$\mathcal{J}_h(u) \geq \int_{\Omega} |\nabla u|^2 \, d\boldsymbol{x} \geq \frac{\lambda m^2}{h}.$$

Let w be the function given in Theorem A, then

$$\mathcal{J}_h(w) = m^2 \left(\frac{4\delta}{3\lambda} + \frac{\lambda}{\delta} \right) \log \left(1 + \frac{\delta}{\gamma} \right) + \frac{\lambda h^2}{2}.$$

Notice that we obtain a contradiction if we show that

$$\mathcal{J}_h(w) < \mathcal{J}_h(u).$$

In particular, it is enough to prove that, if δ is chosen appropriately,

$$m^2 \left(\frac{4\delta}{3\lambda^2} + \frac{1}{\delta} \right) \left(\frac{\delta}{\gamma} - \frac{\delta^2}{2\gamma^2} + \frac{\delta^3}{3\gamma^3} \right) + \frac{h^2}{2} \leq \frac{m^2}{h}. \tag{3.3.11}$$

For δ small we can rewrite (3.3.11) as

$$\frac{m^2}{\gamma} - \frac{m^2}{h} + \frac{h^2}{2} + \mathcal{O}(\delta) \leq 0. \tag{3.3.12}$$

Notice that $h = k h^{\#} = k_1 m^{2/3}$ and since $\gamma \geq k_2 h = k_1 k_2 m^{2/3}$ we see that to prove (3.3.12) it is enough to prove the following inequality

$$\frac{1}{k_1 k_2} - \frac{1}{k_1} + \frac{k_1^2}{2} < 0,$$

which is true by assumption. $\square$

Remark 3.3.10. *Under the assumptions of Theorem 3.3.9 we have that $h_{\text{cr}} \sim m^{2/3}$.*

3.3.4 Convergence and uniqueness of global minimizers

Theorem 3.3.11. *Given $m, \lambda > 0$, let $\mathcal{J}_h$ and $\mathcal{K}_{\gamma_h}$ be defined as in* (1.1.12), (1.1.13)*, respectively, where for every h the value of γ_h is given as in* (3.3.1)*. In addition, assume that θ is continuous and let $\{h_n\}_n \subset (0, \infty)$ be a strictly increasing (respectively, decreasing) sequence converging to h. Then there exists $u \in \mathcal{K}_{\gamma_h}$ which is a global minimizer of $\mathcal{J}_h$, with the property that for if u_n is a global minimizer of $\mathcal{J}_{h_n}$ in $\mathcal{K}_{\gamma_{h_n}}$ for every $n \in \mathbb{N}$, then $u_n \to u \in H^1_{\mathrm{loc}}(\Omega)$ and uniformly on compact subsets of Ω. Moreover, if $\{\ell_n\}_n \subset (0, \infty)$ is another strictly increasing (respectively, decreasing) sequence converging to h and $v_n \in \mathcal{K}_{\gamma_{\ell_n}}$ are global minimizers of $\mathcal{J}_{\ell_n}$, then $v_n \to u$ in $H^1_{\mathrm{loc}}(\Omega)$ and uniformly on compact subsets of Ω.*

We begin by proving a preliminary lemma.

Lemma 3.3.12. *Under the assumptions of Theorem 3.3.11, let $w \in \mathcal{K}_{\gamma_h}$ be such that $J_h(w) < \infty$. Then there exists a sequence $\{w_n\}_n$ such that $w_n \in \mathcal{K}_{\gamma_{h_n}}$ for every $n \in \mathbb{N}$ and $\mathcal{J}_{h_n}(w_n) \to \mathcal{J}_h(w)$ as $n \to \infty$.*

Proof. Notice that if $h_n \nearrow h$ then $w \in \mathcal{K}_{\gamma_{h_n}}$ for every $n \in \mathbb{N}$ and the result follows by considering the constant sequence $w_n = w$. Hence we assume that $h_n \searrow h$, set

$$\sigma_n := \frac{\gamma_h}{\gamma_{h_n}},$$

and define the rescaled function $w_n(x, y) := w(x, \sigma_n y)$. We then notice that $w_n \in \mathcal{K}_{\gamma_{h_n}}$ and by a change of variables

$$\begin{aligned}\int_\Omega |\nabla w_n|^2 \, d\boldsymbol{x} &= \int_\Omega (\partial_x w(x, \sigma_n y))^2 + (\sigma_n \partial_y w(x, \sigma_n y))^2 \, d\boldsymbol{x} \\ &= \int_\Omega \left((\partial_x w(x, z))^2 + (\sigma_n \partial_y w(x, z))^2\right) \sigma_n^{-1} \, dxdz \\ &\to \int_\Omega |\nabla w(x, z)|^2 \, dxdz,\end{aligned}$$

where in the last step we have used the fact that $\sigma_n \searrow 1$. Similarly one can show that

$$\int_\Omega \chi_{\{w_n > 0\}} (h_n - y)_+ \, d\boldsymbol{x} \to \int_\Omega \chi_{\{w > 0\}} (h - y)_+ \, d\boldsymbol{x},$$

and the result follows. $\square$

Proof of Theorem 3.3.11. Assume that $h_n \searrow h$. We divide the proof into several steps.
Step 1: We begin by showing that there exists a subsequence of $\{u_n\}_n$ that converges weakly in $H^1_{\mathrm{loc}}(\Omega)$ to a function u that is a global minimizer of $\mathcal{J}_h$ in the class $\mathcal{K}_{\gamma_h}$. To this end, let $v \colon \mathbb{R}^2_+ \to \mathbb{R}$ be defined by

$$v(\cdot, y) := \frac{m}{\gamma_{h_1}} (\gamma_{h_1} - y)_+$$

(see (3.2.5)). Then $v \in \mathcal{K}_{\gamma_{h_n}}$ for every $n \in \mathbb{N}$ and in particular

$$\int_\Omega |\nabla u_n|^2 \, d\boldsymbol{x} \leq \mathcal{J}_{h_n}(u_n) \leq \mathcal{J}_{h_n}(v) \leq \mathcal{J}_{h_1}(v) < \infty.$$

Hence $\{\nabla u_n\}_n$ is bounded in $L^2(\Omega;\mathbb{R}^N)$. Moreover since $u_n - v = 0$ on $(-\lambda/2, \lambda/2)\times\{0\}$, by Poincaré's inequality we obtain

$$\int_{\Omega_r} |u_n - v|^2\, d\boldsymbol{x} \leq C(\Omega_r)\int_{\Omega_r} |\nabla u_n - \nabla v|^2\, d\boldsymbol{x},$$

where $\Omega_r \coloneqq \Omega \cap \{y < r\}$, with $r > 0$. This shows that $\{u_n\}_n$ is bounded in $H^1(\Omega_r)$ and thus, up to the extraction of a subsequence, $u_n \rightharpoonup u^r$ in $H^1(\Omega_r)$. If we now let $s > r$, up to extraction of a further subsequence, we have that $u_n \rightharpoonup u^r$ in $H^1(\Omega_r)$ and $u_n \rightharpoonup u^s$ in $H^1(\Omega_s)$. By the uniqueness of the weak limit we conclude that

$$u^r(\boldsymbol{x}) = u^s(\boldsymbol{x}) \quad \text{for } \mathcal{L}^2\text{-a.e. } \boldsymbol{x} \in \Omega_r.$$

By letting $r \nearrow \infty$ and by a diagonal argument, up to the extraction of a consecutive subsequences, this defines a function u such that for some $\{n_k\}_k \subset \mathbb{N}$

$$\begin{aligned} \nabla u_{n_k} &\rightharpoonup \nabla u \quad \text{in } L^2(\Omega, \mathbb{R}^2),\\ u_{n_k} &\to u \quad \text{in } L^2_{\text{loc}}(\Omega),\\ u_{n_k} &\to u \quad \text{pointwise almost everywhere in } \Omega,\\ u_{n_k} &\to u \quad \text{in } L^2_{\text{loc}}(\partial\Omega), \end{aligned} \tag{3.3.13}$$

In particular, this shows that $u \in \mathcal{K}_{\gamma_h}$. Moreover, we claim that up to the extraction of a subsequence which we don't relabel, $\{\chi_{\{u_{n_k}>0\}}\}_k$ converges weakly star in $L^\infty(\Omega)$ to a function ξ such that

$$\xi(\boldsymbol{x}) \geq \chi_{\{u>0\}}(\boldsymbol{x}) \text{ for } \mathcal{L}^2\text{-a.e. } \boldsymbol{x} \in \Omega. \tag{3.3.14}$$

Indeed, arguing as in the proof of Theorem 3.1.4, we observe that for every D compactly contained in $\{u > 0\}$

$$0 = \int_D (\chi_{\{u_{n_k}>0\}} - 1)u_{n_k}\, d\boldsymbol{x} \to \int_D (\xi - 1)u\, d\boldsymbol{x}.$$

Since $u > 0$ in D, then necessarily $\xi = 1$ $\mathcal{L}^2$-a.e. in D and hence in $\{u > 0\}$.

To prove that u is a global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_h}$, fix $r > 0$, let $w \in \mathcal{K}_{\gamma_h}$. If $\mathcal{J}_h(w) = \infty$ there is nothing to show, hence we assume that $\mathcal{J}_h(w) < \infty$ and consider $\{w_n\}_n$ as in Lemma 3.3.12. Then

$$\begin{aligned} \int_{\Omega_r} \left(|\nabla u|^2 + \chi_{\{u>0\}}(h-y)_+\right)\, d\boldsymbol{x} &\leq \int_{\Omega} \left(|\nabla u|^2 + \xi(h-y)_+\right)\, d\boldsymbol{x}\\ &\leq \liminf_{k\to\infty} \mathcal{J}_{h_{n_k}}(u_{n_k}) \leq \lim_{k\to\infty} \mathcal{J}_{h_{n_k}}(w_{n_k})\\ &= \mathcal{J}_h(w). \end{aligned} \tag{3.3.15}$$

Letting $r \nearrow \infty$, we conclude that $\mathcal{J}_h(u) \leq \mathcal{J}_h(w)$ for every $w \in \mathcal{K}_{\gamma_h}$.

Step 2: Taking $w = u$ in (3.3.15) yields

$$\int_{\Omega_r} \left(|\nabla u|^2 + \chi_{\{u>0\}}(h-y)_+\right) d\boldsymbol{x} \leq \liminf_{k\to\infty} \mathcal{J}_{h_{n_k}}(u_{n_k}) \leq \limsup_{k\to\infty} \mathcal{J}_{h_{n_k}}(u_{n_k}) \leq \mathcal{J}_h(u).$$

Letting $r \nearrow \infty$ we conclude that

$$\mathcal{J}_h(u) = \lim_{k\to\infty} \mathcal{J}_{h_{n_k}}(u_{n_k}).$$

On the other hand, by the lower semicontinuity of the L^2-norm and (3.3.14)

$$\int_\Omega |\nabla u|^2 \, d\boldsymbol{x} \le \liminf_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 \, d\boldsymbol{x},$$

and

$$\int_\Omega \chi_{\{u>0\}}(h-y)_+ \, d\boldsymbol{x} \le \liminf_{k\to\infty} \int_\Omega \chi_{\{u_{n_k}>0\}}(h-y)_+ \, d\boldsymbol{x}.$$

Thus the previous two inequalities are necessarily equalities and therefore $u_{n_k} \to u$ in $H^1_{\mathrm{loc}}(\Omega)$. Moreover, by Theorem 3.3.3, $\{u_{n_k}\}_k$ is an increasing sequence of continuous functions with a continuous pointwise limit (see (3.3.13)). Hence, by Dini's convergence theorem, the convergence is uniform on compact subsets of Ω.

Step 3: Suppose by contradiction that the entire sequence does not converge to u in $H^1_{\mathrm{loc}}(\Omega)$. Then there are another subsequence $\{u_{n_j}\}_j$ and a minimizer w of $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_h}$ such that $u_{n_j} \to w$ in $H^1_{\mathrm{loc}}(\Omega)$ and uniformly on compact subsets of Ω. By Theorem 3.3.3 we have that $u_{n_k} \le w$ and $u_{n_j} \le u$. Let x and r be such that $B_r(\boldsymbol{x})$ is compactly contained in the support of u. Then, passing to the limit as $k \to \infty$ and $j \to \infty$ in the previous inequalities we obtain $u = w$ in $B_r(\boldsymbol{x})$ and in particular $0 < u(\boldsymbol{x}) = w(\boldsymbol{x})$. Reasoning as in the proof of Theorem 3.3.3 we obtain that $u = w$ in Ω.

The same technique can be used to show the independence of the limiting minimizer on the sequences $\{h_n\}_n$ and $\{u_n\}_n$. This concludes the proof. $\square$

Corollary 3.3.13. *Under the assumptions of Theorem 3.3.11, for every $h > 0$ there are two (possibly equal) global minimizers u_h^+, u_h^- of J_h in $\mathcal{K}_{\gamma_h}$ such that $u_h^- \le u_h^+$ and if w is another global minimizer then $u_h^- \le w \le u_h^+$.*

Theorem 3.3.14. *Given $m, \lambda > 0$, let $\mathcal{J}_h$ and $\mathcal{K}_{\gamma_h}$ be defined as in* (1.1.12), (1.1.13)*, respectively, where for every h the value of γ_h is given as in* (3.3.1)*. In addition, assume that θ is continuous. Then there is a unique global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_h}$ for all but countably many values of h.*

Proof. We define

$$\Lambda := \{h \in \mathbb{R}_+ : \text{ the minimization problem for } \mathcal{J}_h \text{ in } \mathcal{K}_{\gamma_h} \text{ has at least two distinct solutions}\}.$$

We claim that

$$\Lambda = \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ h \in (j-1, j] : \sup\{|u_h^+(\boldsymbol{x}) - u_h^-(\boldsymbol{x})| : \boldsymbol{x} \in (-\lambda/4, \lambda/4) \times (0, \gamma_h/2)\} \ge \frac{1}{n} \right\}.$$

We recall that by Corollary 3.3.13, $h \in \Lambda$ if and only if $u_h^- \ne u_h^+$. To prove the claim it is enough to notice that if $u_h^- = u_h^+$ in $(-\lambda/4, \lambda/4) \times (0, \gamma_h/2)$ then the equality holds everywhere in Ω. Let

$$\Lambda_{j,n} := \left\{ h \in (j-1, j] : \sup\{|u_h^+(\boldsymbol{x}) - u_h^-(\boldsymbol{x})| : \boldsymbol{x} \in (-\lambda/4, \lambda/4) \times (0, \gamma_h/2)\} \ge \frac{1}{n} \right\};$$

we observe that it is enough to show that $\Lambda_{1,n}$ is countable for every $n \in \mathbb{N}$ and that $\Lambda_{j,n}$ is finite for every $j, n \in \mathbb{N}$ with $j \geq 2$. Fix $j, n \in \mathbb{N}$ with $j \geq 2$ and assume by contradiction that $\Lambda_{j,n}$ has infinite cardinality. Then we can find a sequence $\{h_i\}_i \subset h_{j,n}$ and $h \in [j-1, j]$ such that $\{h_i\}_i$ converges strictly monotonically to h. By Theorem 3.3.11, there exists a function u such that $u^-_{h_i}, u^+_{h_i} \to u$ in $H^1_{\text{loc}}(\Omega)$ and uniformly in the compact set of $[-\lambda/4, \lambda/4] \times [0, \gamma_h/2]$. In turn, for i large enough we have that

$$|u^+_i(\boldsymbol{x}) - u^-_i(\boldsymbol{x})| \leq |u^+_i(\boldsymbol{x}) - u(\boldsymbol{x})| + |u(\boldsymbol{x}) - u^-_i(\boldsymbol{x})| < \frac{1}{n}$$

for all $\boldsymbol{x} \in (-\lambda/4, \lambda/4) \times (0, \gamma_h/2)$. We notice that this is in contradiction with the definition of $h_{j,n}$. On the other hand, if $j = 1$ we can write

$$\Lambda_{1,n} = \bigcup_{i=2}^{\infty} \left\{ h \in \left(\frac{1}{i}, 1\right] : \sup\{|u^+_h(\boldsymbol{x}) - u^-_h(\boldsymbol{x})| : \boldsymbol{x} \in (-\lambda/4, \lambda/4) \times (0, \gamma_h/2)\} \geq \frac{1}{n} \right\}.$$

We can then set

$$\Lambda_{1,n,i} \coloneqq \left\{ h \in \left(\frac{1}{i}, 1\right] : \sup\{|u^+_h(\boldsymbol{x}) - u^-_h(\boldsymbol{x})| : \boldsymbol{x} \in (-\lambda/4, \lambda/4) \times (0, \gamma_h/2)\} \geq \frac{1}{n} \right\}$$

and repeat the same argument as above to prove that $\Lambda_{1,n,i}$ is finite for every $i \geq 2$. This concludes the proof. □

Having established the convergence of monotone sequences of minimizers in Theorem 3.3.11, we now investigate the type of convergence of the associated free boundaries. Our proof is inspired by standard techniques commonly used in the study of blow-up limits (see, for example, 4.7 Blow-up limits in [2]).

Theorem 3.3.15. *Under the assumptions of Theorem 3.3.11, let $\{h_n\}_n \subset (0, \infty)$ be a monotone sequence that converges to $h > 0$. For every $n \in \mathbb{N}$, let u_n be a global minimizer of $\mathcal{J}_{h_n}$ in $\mathcal{K}_{\gamma_{h_n}}$ and consider u^+_h, u^-_h as in Corollary 3.3.13. The following statements hold:*

- (i) *if $h_n \searrow h$ then $\partial\{u_n > 0\} \to \partial\{u^-_h > 0\}$ in Hausdorff distance locally in Ω;*
- (ii) *if $h_n \nearrow h$ then $\partial\{u_n > 0\} \to \partial\{u^+_h > 0\}$ in Hausdorff distance locally in $(-\lambda/2, \lambda/2) \times (0, h)$;*
- (iii) *if $h_n \searrow h$ (respectively $h_n \nearrow h$) then $\chi_{\{u_n > 0\}} \to \chi_{\{u^-_h > 0\}}$ (respectively to $\chi_{\{u^+_h > 0\}}$) in $L^1_{\text{loc}}((-\lambda/2, \lambda/2) \times (0, h))$.*

Proof. (i) Let $h_n \searrow h > 0$ and consider a ball $B_r(\boldsymbol{x}) \subset \Omega$ such that $B_r(\boldsymbol{x}) \cap \partial\{u^-_h > 0\} = \emptyset$. Then either $u^-_h \equiv 0$ in $B_r(\boldsymbol{x})$ or $u^-_h > 0$ in $B_r(\boldsymbol{x})$. By Theorem 3.3.3 we have that for every $n \in \mathbb{N}$ $\{u_n > 0\} \subset \{u^-_h > 0\}$; thus if $u^-_h \equiv 0$ in $B_r(\boldsymbol{x})$ so does u_n for every $n \in \mathbb{N}$. In particular, this implies that

$$B_{r/2}(\boldsymbol{x}) \cap \partial\{u_n > 0\} = \emptyset. \tag{3.3.16}$$

On the other hand, if $u^-_h > 0$ in $B_r(\boldsymbol{x})$, since by Theorem 3.3.11 we have that $\{u_n\}_n$ converges uniformly to u^-_h in $B_{r/2}(\boldsymbol{x})$, then for n sufficiently large

$$u_n(\boldsymbol{x}) \geq \frac{1}{2} \min\left\{u^-_h(\boldsymbol{y}) : \boldsymbol{y} \in \overline{B_{r/2}(\boldsymbol{x})}\right\} > 0$$

for every $\boldsymbol{x} \in B_{r/2}(\boldsymbol{x})$ and hence (3.3.16) is satisfied.

Conversely, if $B_r(\boldsymbol{x}) \cap \partial\{u_n > 0\} = \emptyset$ then for all n sufficiently large we have that either $u_n > 0$ in $B_r(\boldsymbol{x})$ or $u_n = 0$ in $B_r(\boldsymbol{x})$. Assume first that $u_m > 0$ in $B_r(\boldsymbol{x})$ for some $m \in \mathbb{N}$. Then, by Theorem 3.3.3, $u_n > 0$ in $B_r(\boldsymbol{x})$ for every $n \geq m$ and therefore u_h^- is harmonic in $B_{r/2}(\boldsymbol{x})$ being the uniform limit of harmonic functions. Consequently, either $u_h^- > 0$ in $B_{r/2}(\boldsymbol{x})$ or $u_h^- = 0$ in $B_{r/2}(\boldsymbol{x})$. In both cases

$$B_{r/2}(\boldsymbol{x}) \cap \partial\{u_h^- > 0\} = \emptyset. \tag{3.3.17}$$

On the other hand, if $u_n \equiv 0$ in $B_{r/2}(\boldsymbol{x})$ for every $n \in \mathbb{N}$ then also $u_h^- \equiv 0$ in $B_{r/2}(\boldsymbol{x})$. This shows that (3.3.17) is also satisfied in case. By a standard compactness argument one can show that $\partial\{u_n > 0\} \to \partial\{u_h^- > 0\}$ in Hausdorff distance locally in Ω.

(ii) Let $h_n \nearrow h$ and consider a ball $B_r(\boldsymbol{x}) \subset (-\lambda/2, \lambda/2) \times (0, h)$ such that $B_r(\boldsymbol{x}) \cap \partial\{u_h^+ > 0\} = \emptyset$. As before, either $u_h^+ \equiv 0$ in $B_r(\boldsymbol{x})$ or $u_h^+ > 0$ in $B_r(\boldsymbol{x})$. If $u_h^+ > 0$ in $B_r(\boldsymbol{x})$, by Theorem 3.3.3, $u_n > 0$ in $B_r(\boldsymbol{x})$ for every $n \in \mathbb{N}$. Therefore (3.3.17) holds. On the other hand, if $u^+ = 0$, for every $\delta > 0$ we can find m such that $u_n \leq \delta$ in $B_{3r/4}(\boldsymbol{x})$ for every $n \geq m$. Hence, for $\delta = \delta(r)$ sufficiently small and $n \geq m$,

$$\frac{1}{\frac{3}{4}r} \fint_{B_{3r/4}(\boldsymbol{x})} u_n \, d\mathcal{H}^1 \leq \frac{4\delta}{3r} \leq C(2/3)\sqrt{h - y - \frac{2}{3}\frac{3}{4}r}.$$

Then we can conclude from Proposition 3.4.3 that $u_n \equiv 0$ in $B_{r/2}(\boldsymbol{x})$, proving that (3.3.16) holds. The rest of the proof follows as in the previous case, therefore we omit the details.

(iii) Let $h_n \searrow h > 0$ and let K be a compact subset of $(-\lambda/2, \lambda/2) \times (0, h)$. If $\operatorname{dist}(K, \partial\{u_h^- > 0\}) > 0$ then either $u_h^- \equiv 0$ in K or $u_h^- > 0$ in K. Reasoning as the proof of (i), we can conclude that either $u_n \equiv 0$ in K for every n or $u_n > 0$ in K for n sufficiently large; hence in this case there is nothing to prove. Therefore, we can assume that $K \cap \partial\{u_h^- > 0\} \neq \emptyset$. By (i), for every $0 < \eta < d_K := \operatorname{dist}(K, \partial((-\lambda/2, \lambda/2) \times (0, h)))$ we can find $m = m(\eta, K)$ such that if $n \geq m$ then

$$\partial\{u_n > 0\} \cap K \subset \mathcal{N}_\eta(\partial\{u_h^- > 0\}),$$

where for any set $A \subset \Omega$, $\mathcal{N}_\eta(A)$ represents the tubular neighborhood of A of width η, i.e.

$$\mathcal{N}_\eta(A) := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, A) < \eta\}.$$

Observe that by Proposition 3.4.3, for every ball $B_r(\boldsymbol{x}) \subset K$ with center on $\partial\{u_h^- > 0\}$

$$\frac{1}{r} \fint_{\partial B_r(\boldsymbol{x})} u_h^- \, d\mathcal{H}^1 \geq C(1/2)\sqrt{(h - y - r/2)_+} > C(1/2)\sqrt{d_K}.$$

Similarly, by Lemma 3.2 in [2] (see also Theorem 3.1 in [15]), there is a constant $C_{\max}$ such that

$$\frac{1}{r} \fint_{\partial B_r(\boldsymbol{x})} u_h^- \, d\mathcal{H}^1 \leq C_{\max}\sqrt{h - y + r} < C_{\max}\sqrt{2h}.$$

Hence we are in a position to apply Theorem 4.5 in [2] to conclude that

$$\mathcal{H}^1(\partial\{u_h^- > 0\} \cap K) < \infty.$$

Since $\chi_{\{u_n>0\}} \to \chi_{\{u_h^->0\}}$ in $L^1(K \setminus \mathcal{N}_\eta(\partial\{u_h^- > 0\}))$ and since

$$\mathcal{L}^2(\mathcal{N}_\eta(\partial\{u_h^- > 0\}) \cap K) \leq 2\eta\mathcal{H}^1(\partial\{u_h^- > 0\} \cap K),$$

letting $\eta \to 0^+$ in the previous estimate concludes the proof.

The proof of (iii) for a monotonically increasing sequence $h_n \nearrow h$ is almost identical, thus we omit the details. $\square$

3.3.5 Symmetric global minimizers

In this section we prove the existence of a global minimizer which is symmetric with respect to the y-axis and monotone in the x variable in the half-strips $(-\lambda/2, 0) \times (0, \infty)$ and $(0, \lambda/2) \times (0, \infty)$. The results of this section are inspired by Section 7 in Chapter 3 of [53] and Theorem 5.10 in [15].

Theorem 3.3.16. *Given $m, \lambda, h > 0$ and γ as in Theorem 1.1.1 there exists a symmetric global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$ in the sense of Definition 1.1.4.*

Proof. Let $u \in \mathcal{K}_\gamma$ be a global minimizer of $\mathcal{J}_h$ and set

$$v(\boldsymbol{x}) := \begin{cases} u(x,y) & \text{if } x \geq 0, \\ u(-x,y) & \text{if } x < 0, \end{cases}$$

and

$$w(\boldsymbol{x}) := \begin{cases} u(x,y) & \text{if } x \leq 0, \\ u(-x,y) & \text{if } x > 0. \end{cases}$$

Notice that $v, w \in \mathcal{K}_\gamma$ and that the minimality of u implies that $\mathcal{J}_h(u) = \mathcal{J}_h(v) = \mathcal{J}_h(w)$. Therefore v and w are two minimizers that are even in the x-variable. In particular, the support of v and w in Ω coincides with their Steiner symmetrizations with the respect to the y-axis (see Definition 2.3.1). Let v^* be the Steiner symmetrization of v with respect to the variable x (see Definition 2.3.3). Then $v^* \in \mathcal{K}_\gamma$ and by the Pólya-Szegö inequality (see Theorem 2.3.5), together with Tonelli's theorem, we obtain

$$\int_\Omega |\nabla v^*|^2 \, d\boldsymbol{x} \leq \int_\Omega |\nabla v|^2 \, d\boldsymbol{x}.$$

By definition of v^*, for every $y > 0$

$$\int_{-\lambda/2}^{\lambda/2} \chi_{\{v^*>0\}}(x,y) \, dx = \int_{-\lambda/2}^{\lambda/2} \chi_{\{v>0\}}(x,y) \, dx$$

and thus, again by Tonelli's theorem,

$$\begin{aligned} \int_\Omega \chi_{\{v^*>0\}}(h-y)_+ \, d\boldsymbol{x} &= \int_0^h (h-y)_+ \int_{-\lambda/2}^{\lambda/2} \chi_{\{v^*>0\}}(x,y) \, dx \, dy \\ &= \int_0^h (h-y)_+ \int_{-\lambda/2}^{\lambda/2} \chi_{\{v>0\}}(x,y) \, dx \, dy \end{aligned}$$

$$= \int_\Omega \chi_{\{v>0\}}(h-y)_+ \, d\boldsymbol{x}.$$

Consequently, $\mathcal{J}_h(v^*) \le \mathcal{J}_h(v)$ and similarly $\mathcal{J}_h(w^*) \le \mathcal{J}_h(w)$. In turn, this implies that v^* and w^* are also minimizers of $\mathcal{J}_h$. This concludes the proof. □

Corollary 3.3.17. *Let u_h^+, u_h^- be as in Corollary 3.3.13. Then u_h^+, u_h^- are symmetric in the sense of Definition 1.1.4.*

Proof. Let $h \in \mathbb{R}_+ \setminus \Lambda$, where Λ is the set defined in the proof of Theorem 3.3.14, and let u_h be the unique global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_{\gamma_h}$. Then $u_h^+ = u_h^- = u_h = u_h^*$. On the other hand, if $h \in \Lambda$, consider a sequence $\{h_n\}_n \subset \mathbb{R} \setminus \Lambda$ such that $h_n \nearrow h$ and let u_n be the unique minimizer of $\mathcal{J}_{h_n}$ in $\mathcal{K}_{\gamma_{h_n}}$. Then, $u_{h_n} = u_{h_n}^*$ and by Theorem 3.3.11 it follows that u_h^+ has all the desired properties. The result for u_h^- follows similarly by considering a sequence $\{h_n\}_n \subset \mathbb{R} \setminus \Lambda$ such that $h_n \searrow h$. □

Remark 3.3.18. *Notice that the free boundary $\partial\{u > 0\}$ can be described by the graph of a function $x = g(y)$, where $g \colon (0, h) \to [-\lambda/2, 0]$ is defined via*

$$g(y) := \inf\{x \in (-\lambda/2, 0) : u(x, y) > 0\}. \tag{3.3.18}$$

Proposition 3.3.19. *For γ as in Theorem 1.1.1, let $u \in \mathcal{K}_\gamma$ be a symmetric minimizer of $\mathcal{J}_h$ and let g be defined as above. Then g is a continuous function.*

Proof. **Step 1:** We begin by showing that if $\partial\{u > 0\}$ contains the line segment S of endpoints $(\ell, k), (L, k)$, with $\ell < L$ and $k < h$, then $\partial\{u > 0\} = \{y = k\} \cap \Omega$, a contradiction to Lemma 3.2.6. Without loss of generality, we can assume that S is maximal, i.e., for every line segment S' such that $S \subset S'$ and $S' \subset \partial\{u > 0\}$, it must be that $S = S'$. If $\ell = -\lambda/2$ and $L = \lambda/2$ there is nothing to do. Then assume without loss that $L < \lambda/2$. Since $k < h$, we are in a position to apply Theorem 8.4 in [2], which gives a number $\rho > 0$, an analytic function f, and a set of local coordinates such that the free boundary $\partial\{u > 0\}$ coincides with the graph of f in $B_\rho((L, k))$ in the local coordinates. In turn, f agrees with an affine function on a subinterval of its domain, and so by analyticity it must be equal to the same affine function on its whole domain; this contradicts the maximality of S.
Step 2: Next we show that both one-sided limits

$$\lim_{y \to \bar{y}^+} g(y) \quad \text{and} \quad \lim_{y \to \bar{y}^-} g(y)$$

exist for every $\bar{y} \in (0, h)$. To see this, suppose that

$$L := \limsup_{y \to \bar{y}^+} g(y) > \liminf_{y \to \bar{y}^+} g(y) =: \ell;$$

then we can find two sequences $\{y_n\}_n, \{z_n\}_n$ such that $y_n \searrow \bar{y}$, $z_n \in (y_{n+1}, y_n)$ and

$$\lim_{n \to \infty} g(y_n) = L, \qquad \lim_{n \to \infty} g(z_n) = \ell.$$

Let $\boldsymbol{y} := (L, \bar{y})$. We claim that $\boldsymbol{y} \in \partial\{u > 0\}$. To prove the claim, first observe that there exists a $\delta > 0$ such that $B_\delta(\boldsymbol{y}) \subset \Omega$, and notice that $u(\boldsymbol{y}) = 0$ since u is continuous in Ω and by assumption

$u(g(y_n), y_n) = 0$ for every $n \in \mathbb{N}$. Given $\eta > 0$, if $n \in \mathbb{N}$ is large enough then $(g(y_n), y_n) \in B_\eta(\boldsymbol{y})$ and since by assumption $(g(y_n), y_n) \in \partial\{u > 0\}$ then there exists $\boldsymbol{x}_n \in B_\eta((g(y_n), y_n))$ such that $u(\boldsymbol{x}_n) > 0$. This shows that $\boldsymbol{y} \in \partial\{u > 0\}$. By Theorem 8.1 in [2] there exists $\rho > 0$ such that $B_\rho \cap \partial\{u > 0\}$ is the graph of a $C^{1,\alpha}$ function (in an opportunely defined set of coordinates centered at the point $\boldsymbol{y}$). Let κ be the Lipschitz constant of this function in $(-\rho, \rho)$. Then the length of $\partial\{u > 0\}$ in $B_\rho(\boldsymbol{y})$ cannot exceed $2\rho\sqrt{1+\kappa^2}$. But on the other hand, observe that for n large enough one also has that $(g(y_n), y_n) \in B_{\rho/2}(\boldsymbol{y})$ and $(g(z_n), z_n) \notin B_\rho(\boldsymbol{y})$, thus showing that the length of $\partial\{u > 0\}$ cannot be finite. We have therefore arrived at a contradiction. The proof in the other case is similar and therefore we omit the details.

Step 3: The previous step we shows that g cannot have essential discontinuities. To exclude jump discontinuities it is enough to notice that these would correspond to horizontal line segments in the free boundary of u, a behavior that is ruled out in the first step. Finally, in view of Corollary 3.6 in [2], we see that removable discontinuities are also not possible. This concludes the proof. □

3.4 Boundary regularity

As remarked in the introduction, the behavior of the free boundary near contact points away from $(\pm\lambda/2, \gamma)$ is well understood. This section is devoted to the study of the remaining case in which the free boundary hits the fixed boundary exactly at $(\pm\lambda/2, \gamma)$.

3.4.1 The bounded gradient lemma

The following result states that a symmetric minimizer is Lipschitz continuous in a neighborhood of the contact point $\boldsymbol{x}_0$. This fact will be of fundamental importance in the following sections.

Theorem 3.4.1. *Given $m, \lambda, h > 0$ and $\gamma < h$, let $u \in \mathcal{K}_\gamma$ be a symmetric global minimizer of $\mathcal{J}_h$ in the sense of Definition 1.1.4 and assume that $\boldsymbol{x}_0 = (-\lambda/2, \gamma)$ is an accumulation point for the free boundary on $\partial\Omega$, i.e.,*

$$\boldsymbol{x}_0 \in \overline{\partial\{u > 0\} \cap \Omega}. \tag{3.4.1}$$

Then ∇u is bounded in a neighborhood of $\boldsymbol{x}_0$.

Proof. It is enough to show that there exists a constant C such that for every $\mu > 0$ sufficiently small (with respect to λ, h, γ and $h - \gamma$)

$$|\nabla u(\boldsymbol{y})| \leq C \tag{3.4.2}$$

for $\boldsymbol{y} \in \Omega \cap B_{2\mu}(\boldsymbol{x}_0) \setminus B_\mu(\boldsymbol{x}_0)$. For $\boldsymbol{x} \in B_8(\mathbf{0})$ and μ small enough, let w be the rescaled function

$$w(\boldsymbol{x}) := \frac{u(\boldsymbol{x}_0 + \mu\boldsymbol{x})}{\mu}.$$

Then w is harmonic in $\{w > 0\}$ and for $\boldsymbol{x} = (x, y) \in \partial\{w > 0\}$, by Theorem 8.1 in [2], we have that the Bernoulli condition is satisfied in a classical sense, i.e.

$$\partial_\nu w(\boldsymbol{x}) = \partial_\nu u(\boldsymbol{x}_0 + \mu\boldsymbol{x}) = \sqrt{h - \gamma - \mu y}, \tag{3.4.3}$$

where ν is the interior unit normal vector to $\{w > 0\}$ at $\boldsymbol{x}$. Then, to prove (3.4.2) is enough to show that

$$|\nabla w(\boldsymbol{x})| \leq C \tag{3.4.4}$$

for $\boldsymbol{x} \in \{w > 0\} \cap B_2^+(\mathbf{0}) \setminus B_1^+(\mathbf{0})$, where $B_r^+(\mathbf{0}) := B_r(\mathbf{0}) \cap \{x > 0\}$. For $\boldsymbol{x} \in B_8(\mathbf{0})$ we define

$$d(\boldsymbol{x}) := \operatorname{dist}(\boldsymbol{x}, \partial\{w > 0\}), \quad D(\boldsymbol{x}) := \operatorname{dist}(\boldsymbol{x}, \{(0, y) : y \geq 0\}). \tag{3.4.5}$$

The proof of (3.4.4) is divided into several steps.

Step 1: In this first step we show that in order to obtain (3.4.4), it is enough to prove that for every $\boldsymbol{x} \in B_2^+(\mathbf{0}) \setminus B_1^+(\mathbf{0})$ either

$$w(\boldsymbol{x}) \leq c \min\{d(\boldsymbol{x}), D(\boldsymbol{x})\} \tag{3.4.6}$$

or there exists $\rho > 0$ such that $\boldsymbol{x} = (x, y) \in B_{\rho/2}^+(0, y)$ and for every $\boldsymbol{y} \in B_\rho^+(0, y)$

$$w(\boldsymbol{y}) \leq c\rho. \tag{3.4.7}$$

Indeed, assume that $\boldsymbol{x} \in B_2^+(\mathbf{0}) \setminus B_1^+(\mathbf{0})$ is such that (3.4.6) is satisfied. Then, if let $\delta(\boldsymbol{x}) := \min\{d(\boldsymbol{x}), D(\boldsymbol{x})\}$, we have that w is harmonic in $B_{\delta(\boldsymbol{x})}(\boldsymbol{x})$ and

$$|\nabla w(\boldsymbol{x})| \leq \sup\{|\nabla w(\boldsymbol{y})| : \boldsymbol{y} \in B_{\delta(\boldsymbol{x})/2}(\boldsymbol{x})\} \leq \frac{4}{\delta(\boldsymbol{x})} \sup\{w(\boldsymbol{y}) : \boldsymbol{y} \in B_{\delta(\boldsymbol{x})}(\boldsymbol{x})\} \leq 4c,$$

where the second inequality follows from the standard interior gradient estimates (see Theorem 2.10 in [54]). Similarly, for every $\boldsymbol{x} = (x, y) \in B_2^+(\mathbf{0}) \setminus B_1^+(\mathbf{0})$ such that (3.4.7) holds we see that

$$|\nabla w(\boldsymbol{x})| \leq \sup\{|\nabla w(\boldsymbol{y})| : \boldsymbol{y} \in B_{\rho/2}^+(0, y)\} \leq \frac{K}{\rho} \sup\{w(\boldsymbol{y}) : \boldsymbol{y} \in B_\rho^+(0, y)\} \leq Kc,$$

where in the second inequality we have used the result of Theorem 2.2.11.

Step 2: Let $c_0 > 3 \log 2\sqrt{h}$. We claim that for every $\boldsymbol{x} \in B_4^+(\mathbf{0})$ for which $d(\boldsymbol{x}) < D(\boldsymbol{x})$ then

$$w(\boldsymbol{x}) \leq c_0 d(\boldsymbol{x}). \tag{3.4.8}$$

Notice that if $w(\boldsymbol{x}) = 0$ then there is nothing to do, therefore, we can assume that $w(\boldsymbol{x}) > 0$. Since $B_{d(\boldsymbol{x})}(\boldsymbol{x}) \subset \{w > 0\}$ we have that w is harmonic in $B_{d(\boldsymbol{x})}(\boldsymbol{x})$ and by definition there must be $\bar{\boldsymbol{x}} \in \partial B_{d(\boldsymbol{x})}(\boldsymbol{x}) \cap \partial\{w > 0\}$. Suppose that

$$w(\boldsymbol{x}) > c_0 d(\boldsymbol{x}). \tag{3.4.9}$$

Then, by Harnack's inequality (see Excercise 2.6 in [54]),

$$w(\boldsymbol{y}) \geq \frac{w(\boldsymbol{x})}{3} > \frac{c_0 d(\boldsymbol{x})}{3}$$

for every $\boldsymbol{y} \in B_{d(\boldsymbol{x})/2}(\boldsymbol{x})$. Let v be the harmonic function in the annulus $B_{d(\boldsymbol{x})}(\boldsymbol{x}) \setminus B_{d(\boldsymbol{x})/2}(\boldsymbol{x})$ which satisfies the boundary conditions

$$\begin{cases} v = c_0 d(\boldsymbol{x})/3 & \text{on } \partial B_{d(\boldsymbol{x})/2}(\boldsymbol{x}), \\ v = 0 & \text{on } \partial B_{d(\boldsymbol{x})}(\boldsymbol{x}). \end{cases}$$

Writing v in polar coordinates centered at $\boldsymbol{x}$, v must be the radial function

$$r \mapsto \frac{c_0 d(\boldsymbol{x})}{3\log 2} \log\left(\frac{d(\boldsymbol{x})}{r}\right)$$

By the maximum principle for harmonic functions, $v \leq w$ in the annulus, and since the equality holds at $\bar{\boldsymbol{x}}$, it follows that

$$\frac{2c_0}{3} = \partial_\nu v(\bar{\boldsymbol{x}}) \leq \partial_\nu w(\bar{\boldsymbol{x}}) \leq \sqrt{h},$$

where in the last inequality we have used (3.4.3). In turn,

$$c_0 \leq 3\log 2\sqrt{h},$$

which is a contradiction.

Step 3: Let

$$U_0 \coloneqq \{\boldsymbol{x} = (x, 0) : 1 < x < 4\}.$$

In this step we show that there exists a constant $c_1 \geq c_0$, independent of μ, such that for every $\boldsymbol{x} \in U_0 \cap \{w > 0\}$ with $1 \leq D(\boldsymbol{x}) \leq d(\boldsymbol{x})$ the following inequality holds

$$w(\boldsymbol{x}) \leq c_1 D(\boldsymbol{x}). \tag{3.4.10}$$

Let $\boldsymbol{z}_0 = (s_0, t_0)$ be any point on $\partial\{w > 0\} \cap B^+_{1/4}(\mathbf{0})$. Then, for every s such that $(s, t_0) \in B^+_4(\mathbf{0}) \setminus B^+_1(\mathbf{0})$, we must have that $d(s, t_0) < D(s, t_0)$. Consequently,

$$w(s, t_0) \leq c_0 d(s, t_0),$$

where c_0 is the constant given in the previous step. Notice that by assumption $d(\boldsymbol{x}) \geq D(\boldsymbol{x}) \geq 1$, and therefore $B_{1/2}(\boldsymbol{x}) \subset \{w > 0\}$. Moreover, the ball $B_{1/4}(\boldsymbol{x})$ contains the point (x, t_0) and Harnack's inequality then yields

$$w(\boldsymbol{x}) \leq 3w(x, t_0). \tag{3.4.11}$$

On the other hand,

$$w(x, t_0) \leq c_0 d(x, t_0) < c_0 D(x, t_0) \leq c_0\sqrt{x^2 + t_0^2} \leq \frac{\sqrt{17}c_0 x}{4} = \frac{\sqrt{17}c_0 D(\boldsymbol{x})}{4}, \tag{3.4.12}$$

where in the last inequality we have used the fact that $|t_0| \leq 1/4 \leq x/4$. The desired inequality (3.4.10) follows directly from (3.4.11) and (3.4.12).

Step 4: The purpose of this step is to show that (3.4.10) holds, possibly with a larger constant, at every point $\boldsymbol{x} \in B^+_4(\mathbf{0}) \setminus B^+_1(\mathbf{0})$, such that $y < 0$ and $D(\boldsymbol{x}) \leq d(\boldsymbol{x})$. We begin by considering the case

$$\boldsymbol{x} \in U_1 \coloneqq \left\{\boldsymbol{z} = (s, t) \in B^+_4(\mathbf{0}) \setminus B^+_1(\mathbf{0}) : t < 0 \text{ and } \operatorname{dist}(\boldsymbol{z}, U_0) < \frac{1}{4}\right\}.$$

Reasoning as in the previous step, we see that since $d(\boldsymbol{x}) \geq D(\boldsymbol{x}) \geq 1$ we are in a position to apply Harnack's inequality in $B_{1/4}(\boldsymbol{x}) \subset B_{1/2}(\boldsymbol{x}) \subset \{w > 0\}$ to conclude that

$$w(\boldsymbol{x}) \leq 3w(\boldsymbol{z}_1)$$

for every $\boldsymbol{z}_1 \in U_0$ such that $|\boldsymbol{x}-\boldsymbol{z}_1|<1/4$. Additionally, it follows from the first two steps that

$$w(\boldsymbol{z}_1) \leq c_1 \min\{d(\boldsymbol{z}_1), D(\boldsymbol{z}_1)\} \leq c_1 D(\boldsymbol{z}_1) \leq 4c_1 \leq 4c_1 D(\boldsymbol{x}).$$

Define the sets U_i, $i \geq 2$, recursively via

$$U_i := \left\{ \boldsymbol{z} = (s,t) \in \left(B_4^+(\mathbf{0}) \setminus B_1^+(\mathbf{0})\right) \setminus \bigcup_{j=1}^{i-1} U_j : t < 0 \text{ and } \operatorname{dist}(\boldsymbol{z}, U_{i-1}) < \frac{1}{4} \right\},$$

and notice that, by simple geometric considerations,

$$\left(B_4^+(\mathbf{0}) \setminus B_1^+(\mathbf{0})\right) \cap \{t \leq 0\} = \bigcup_{j=0}^{16} U_i.$$

In particular, if $\boldsymbol{x} \in U_i$ is such that $D(\boldsymbol{x}) \leq d(\boldsymbol{x})$ then an iteration of the argument above yields

$$w(\boldsymbol{x}) \leq 12^i c_1 D(\boldsymbol{x}). \tag{3.4.13}$$

Step 5: We are left to consider the case where $\boldsymbol{x} = (x,y) \in B_2^+(\mathbf{0}) \setminus B_1^+(\mathbf{0})$ is such that $y > 0$, $\boldsymbol{x} \in \{w > 0\}$, and $D(\boldsymbol{x}) \leq d(\boldsymbol{x})$. Suppose that there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset B_4^+(\mathbf{0}) \setminus B_1^+(\mathbf{0})$ such that $\boldsymbol{x}_n \to \boldsymbol{x}$ and such that $d(\boldsymbol{x}_n) < D(\boldsymbol{x}_n)$ for every n. Then necessarily $d(\boldsymbol{x}) = D(\boldsymbol{x})$ and by (3.4.8)

$$w(\boldsymbol{x}) = \lim_{n\to\infty} w(\boldsymbol{x}_n) \leq \lim_{n\to\infty} c d(\boldsymbol{x}_n) = cD(\boldsymbol{x}). \tag{3.4.14}$$

Hence, we can assume that such a sequence does not exist. Then there is $0 < \delta < y$ such that for every $t \in (y-\delta, y+\delta)$ the point (x,t) is such that $D(x,t) \leq d(x,t)$, and in particular $w(s,t) > 0$ for every $0 < s < x$. We define

$$a := \inf\{t \leq y : \text{ for every } t < \bar{t} < y+\delta, w(s,\bar{t}) > 0 \text{ for every } s \text{ small}\},$$

$$b := \sup\{t \geq y : \text{ for every } y-\delta < \bar{t} < t, w(s,\bar{t}) > 0 \text{ for every } s \text{ small}\}.$$

Notice that by (3.4.1), $a \geq 0$. Moreover, $y \in (a,b)$, and it follows from the definition that if $b < \infty$, every point of the form (s,a) and (s,b), $s > 0$, is the limit a sequence of points $\{\boldsymbol{x}_n\}_{n\in\mathbb{N}}$ with the property that $d(\boldsymbol{x}_n) < D(\boldsymbol{x}_n)$. In turn, (3.4.8) and (3.4.14) imply that

$$w(s,a) \leq cs, \quad w(s,b) \leq cs, \tag{3.4.15}$$

for every $s > 0$ such that the points $(s,a),(s,b) \in B_4^+(\mathbf{0})$. Assume first that $y-a < b-y$ and fix $\varepsilon > 0$ small enough so that

$$1 - \tan\theta \leq \frac{1}{4}, \quad \theta := \frac{\pi}{4} - \varepsilon. \tag{3.4.16}$$

Case 1: Assume that $y - a \leq x\tan\theta$. Let $\bar{\boldsymbol{x}} = (x,a)$ and notice that

$$|\boldsymbol{x} - \bar{\boldsymbol{x}}| = y - a \leq x\tan\theta < x = D(\boldsymbol{x}).$$

Since by assumption $D(\boldsymbol{x}) \leq d(\boldsymbol{x})$ we have that $B_{x\tan\theta}(\boldsymbol{x}) \subset B_{D(\boldsymbol{x})}(\boldsymbol{x}) \subset \{w > 0\}$ and by Harnack's inequality we can find a constant $c = c(\varepsilon)$ such that

$$w(\boldsymbol{x}) \leq cw(\bar{\boldsymbol{x}}) \leq cx = cD(\boldsymbol{x}),$$

where in the last inequality we have used (3.4.15).
Case 2: Assume that $\tan\theta < y - a \leq x$ and let $\widehat{\boldsymbol{x}} = (x, a + x\tan\theta)$. By (3.4.16) we see that

$$|\boldsymbol{x} - \widehat{\boldsymbol{x}}| \leq x\,(1 - \tan\theta) \leq \frac{x}{4}.$$

In turn, $B_{x/2}(\boldsymbol{x}) \subset \{w > 0\}$, and similarly to above, by Harnack's inequality,

$$w(\boldsymbol{x}) \leq 3w(\widehat{\boldsymbol{x}}) \leq cD(\boldsymbol{x}), \tag{3.4.17}$$

where in the last inequality follows from the fact that $\widehat{\boldsymbol{x}}$ satisfies the conditions of Case 1.
Case 3: Assume that

$$\frac{3}{4}(y - a) \leq x < y - a.$$

Since $y - a < b - y$ it follows that $B_{\frac{1}{2}(y-a)}(\boldsymbol{x}) \subset \{w > 0\}$, and therefore

$$w(\boldsymbol{x}) \leq 3w(y - a, y) \leq c(y - a) \leq \frac{4}{3}cx = \frac{4}{3}cD(\boldsymbol{x}),$$

where in the second inequality we have used the fact that the point $(y - a, y)$ satisfies the conditions of Case 2.
Case 4: Assume that

$$\frac{1}{2}(y - a) \leq x < \frac{3}{4}(y - a).$$

Then $(3(y - a)/4, y)$ satisfies of the conditions of Case 3, and so, reasoning as above, we obtain that

$$w(\boldsymbol{x}) \leq cw(3(y - a)/4, y - a) \leq c(y - a) \leq 2cx = 2cD(\boldsymbol{x}).$$

Case 5: Finally, assume that $x < (y - a)/2$. Notice that $B^+_{y-a}(0, y) \subset \{w > 0\}$ by the non-decreasing property of symmetric minimizers. Then, for every $\boldsymbol{y} \in B^+_{(y-a)/2}(0, y)$, by the boundary Harnack principle (see Theorem 11.5 in [28]) we have that

$$w(\boldsymbol{y}) \leq Mw(y - a, y) \leq Mc(y - a),$$

where in the last inequality we used (3.4.17). If $y - a > b - y$ then $4 \geq 2y - a > b$ and therefore we can repeat the same argument as above. This concludes the proof. $\square$

Remark 3.4.2. *Note that Theorem 3.4.1 holds also for $\gamma = h$, with minor changes in the argument for the first step. To be precise, let $\boldsymbol{x}$ be such that $d(\boldsymbol{x}) < D(\boldsymbol{x})$, and consider $\bar{\boldsymbol{x}}$ as in the first step. If $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y})$, with $\bar{y} < 0$ then (3.4.3) holds and we can proceed as above. On the other hand, if $\bar{\boldsymbol{x}} = (\bar{x}, 0)$ we must replace (3.4.3) with the estimate*

$$|\nabla u(\boldsymbol{x})| \leq C\sqrt{3r},$$

which holds for every $\boldsymbol{x}$ in $B_r(\bar{\boldsymbol{x}})$, as shown in Remark 3.5 (i) in [15]. The rest follows without changes.

3.4.2 Blow-up limits

Given a global minimizer $u \in \mathcal{K}_\gamma$, consider a sequence $\rho_n \to 0^+$, a real number $R > 0$, and for every $n \in \mathbb{N}$ sufficiently large define the rescaled functions

$$u_n(\boldsymbol{z}) := \frac{u(\boldsymbol{x}_0 + \rho_n \boldsymbol{z})}{\rho_n}, \tag{3.4.18}$$

where $\boldsymbol{z} \in B_R(\mathbf{0})$, and $\boldsymbol{x}_0 = (-\lambda/2, \gamma)$. Notice that if ∇u is bounded in a neighborhood of $\boldsymbol{x}_0$ (a condition that is guaranteed by Theorem 3.4.1 if, in addition, u is symmetric, $\gamma < h$, and $\boldsymbol{x}_0$ is an accumulation point for the free boundary $\partial\{u > 0\}$), then

$$|\nabla u_n(\boldsymbol{z})| = |\nabla u(\boldsymbol{x}_0 + \rho_n \boldsymbol{z})| \le C,$$

where C is a positive constant independent of n and $\boldsymbol{z}$. Since $u_n(\mathbf{0}) = 0$ for every $n \in \mathbb{N}$, it follows that there exist a subsequence (which we don't relabel) and a function $w \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^2)$ such that for every $R > 0$,

$$\begin{aligned} u_n \to w \qquad & \text{in } C^{0,\alpha}(B_R(\mathbf{0})) \text{ for all } 0 < \alpha < 1, \\ \nabla u_n \overset{*}{\rightharpoonup} \nabla w \quad & \text{in } L^\infty(B_R(\mathbf{0}); \mathbb{R}^2). \end{aligned} \tag{3.4.19}$$

The function w is called a *blow-up limit* of u at $\boldsymbol{x}_0$ with respect to the sequence $\{\rho_n\}_n$.

Non-degeneracy properties of blow-up limits

Proposition 3.4.3. *Given $m, \lambda, h, \gamma > 0$ and $k \in (0,1)$, there exists a positive constant $C_{\min}(k)$ such that for every minimizer u of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$ and for every ball $B_r(\boldsymbol{x}) \subset \Omega$, $\boldsymbol{x} = (x, y)$, if*

$$\frac{1}{r} \fint_{\partial B_r(\boldsymbol{x})} u \, d\mathcal{H}^1 \le C_{\min}(k) \sqrt{(h - y - kr)_+},$$

then $u \equiv 0$ in $B_{kr}(\boldsymbol{x})$. Moreover, if $0 < r < \lambda$, the result is still valid for balls not contained in Ω, provided $B_r(\boldsymbol{x}) \cap \partial\Omega \subset \{y > \gamma\}$.

For a proof of Proposition 3.4.3 we refer to Lemma 3.4 and Remark 3.5 in [2]; see also Theorem 3.6 and Remark 5.2 in [15].

Lemma 3.4.4. *Given $m, \lambda, h > 0$ and $\gamma < h$, let $u \in \mathcal{K}_\gamma$ be a global minimizer of $\mathcal{J}_h$ and let w be a blow-up limit of u at $\boldsymbol{x}_0 = (-\lambda/2, \gamma)$ with respect to the sequence $\{\rho_n\}_n$. Furthermore, assume that there exist a constant $\beta \ge 1$ and a sequence of points $\boldsymbol{x}_n \in \partial\{u > 0\} \cap \Omega$ such that*

$$\rho_n \le |\boldsymbol{x}_n - \boldsymbol{x}_0| \le \beta \rho_n \tag{3.4.20}$$

for every n large enough. Then w is not identically equal to zero.

Proof. By assumption, there exists a sequence of radii $\{r_n\}_n$, $1 \le r_n \le \beta$, such that

$$\partial B_{\rho_n r_n}(\boldsymbol{x}_0) \cap \partial\{u > 0\} \cap \Omega \ne \emptyset.$$

Thus, for every $n \in \mathbb{N}$ sufficiently large,

$$\boldsymbol{z}_n := \frac{\boldsymbol{x}_n - \boldsymbol{x}_0}{\rho_n} \in \partial B_{r_n}(\mathbf{0}) \cap \partial\{u_n > 0\} \cap \{s > 0\},$$

and furthermore we can assume that $h - \gamma - 2\beta\rho_n > 0$. Given $k \in (0,1)$, for every such n, consider the ball $B_{r_n}(\boldsymbol{z}_n)$ and observe that by the change of variables $\boldsymbol{x} = \boldsymbol{x}_0 + \rho_n \boldsymbol{z}$, (3.4.18), and Proposition 3.4.3

$$\begin{aligned}\frac{1}{r_n} \fint_{\partial B_{r_n}(\boldsymbol{z}_n)} u_n \, d\mathcal{H}^1 &= \frac{1}{2\pi \rho_n r_n^2} \int_{\partial B_{r_n}(\boldsymbol{z}_n)} u(\boldsymbol{x}_0 + \rho_n \boldsymbol{z}) \, d\mathcal{H}^1(\boldsymbol{z}) \\ &= \frac{1}{2\pi \rho_n^2 r_n^2} \int_{\partial B_{\rho_n r_n}(\boldsymbol{x}_n)} u(\boldsymbol{x}) \, d\mathcal{H}^1(\boldsymbol{x}) \\ &= \frac{1}{\rho_n r_n} \fint_{\partial B_{\rho_n r_n}(\boldsymbol{x}_n)} u \, d\mathcal{H}^1 \geq C_{\min}(k) \sqrt{(h - y_n - k\rho_n r_n)_+}.\end{aligned}$$

In addition, we notice that by (3.4.20), $y_n \leq \gamma + \rho_n r_n$, and therefore

$$\frac{1}{r_n} \fint_{\partial B_{r_n}(\boldsymbol{z}_n)} u_n \, d\mathcal{H}^1 \geq C_{\min}(k) \sqrt{h - \gamma - 2\beta\rho_n}. \tag{3.4.21}$$

Let $\bar{\boldsymbol{z}}_n$ be such that $u_n(\bar{\boldsymbol{z}}_n) = \sup\{u_n(\boldsymbol{z}) : \boldsymbol{z} \in \partial B_{r_n}(\boldsymbol{z}_n)\}$. Then, by (3.4.21) we see that

$$u_n(\bar{\boldsymbol{z}}_n) \geq \fint_{\partial B_{r_n}(\boldsymbol{z}_n)} u_n \, d\mathcal{H}^1 \geq r_n C_{\min}(k) \sqrt{h - \gamma - 2\beta\rho_n}. \tag{3.4.22}$$

Eventually extracting a subsequence (which we don't relabel), we can find a point $\bar{\boldsymbol{z}}$ such that $\bar{\boldsymbol{z}}_n \to \bar{\boldsymbol{z}}$. Consequently, by the uniform convergence of u_n to w, (3.4.22), and the fact that $r_n \geq 1$ for every n, we obtain

$$w(\bar{\boldsymbol{z}}) = \lim_{n\to\infty} u_n(\bar{\boldsymbol{z}}_n) \geq \lim_{n\to\infty} r_n C_{\min}(k) \sqrt{h - \gamma - 2\beta\rho_n} \geq C_{\min}(k) \sqrt{h - \gamma} > 0.$$

This concludes the proof. □

Some preliminary results

The following classical lemma, due to Alt and Caffarelli, is a consequence of Proposition 3.4.3; for a proof we refer to Section 4.7 in [2].

Lemma 3.4.5. *Given $m, \lambda, h > 0$ and $\gamma < h$, let $u \in \mathcal{K}_\gamma$ be a global minimizer of $\mathcal{J}_h$ and let w be a blow-up limit of u at $\boldsymbol{x}_0$. Then, if u_n is defined as in* (3.4.18),

(i) $\partial\{u_n > 0\} \to \partial\{w > 0\}$ *locally in Hausdorff distance in* $\mathbb{R}^2 \setminus \{(0, y) : y \geq 0\}$,

(ii) $\chi_{\{u_n > 0\}} \to \chi_{\{w>0\}}$ *in* $L^1_{\text{loc}}(\mathbb{R}^2 \setminus \{(0, y) : y \geq 0\})$.

Theorem 3.4.6. *Given $m, \lambda, h > 0$ and $\gamma < h$, let $u \in \mathcal{K}_\gamma$ be a global minimizer of $\mathcal{J}_h$ and let w be a blow-up limit of u at $\boldsymbol{x}_0 = (-\lambda/2, \gamma)$. Then, for every $R > 0$, w is a global minimizer of*

$$\mathcal{F}_h(v) := \int_{B_R(\mathbf{0})} \left(|\nabla v(\boldsymbol{z})|^2 + \chi_{\{v>0\}}(\boldsymbol{z})(h - \gamma)\right) \, d\boldsymbol{z}, \tag{3.4.23}$$

over the set

$$\mathcal{K}(w, R) := \left\{v \in H^1_{\text{loc}}(\mathbb{R}^2) : v = w \text{ on } \partial B_R(\mathbf{0}) \text{ and } v(0, y) = 0 \text{ for } 0 < y < R\right\}. \tag{3.4.24}$$

The following proof is adapted from Lemma 5.4 in [2].

Proof. For u_n defined as in (3.4.18) and n large enough so that $0 < \gamma - R\rho_n < \gamma + R\rho_n < h$, let $\eta \in C_0^1(B_R(\mathbf{0}); [0,1])$ and for $v \in \mathcal{K}(w, R)$ set

$$v_n(\boldsymbol{z}) := v(\boldsymbol{z}) + (1 - \eta(\boldsymbol{z}))(u_n(\boldsymbol{z}) - w(\boldsymbol{z}))$$

and, for $\boldsymbol{x} \in B_{R\rho_n}(\boldsymbol{x}_0)$, define

$$w_n(\boldsymbol{x}) := \rho_n v_n\left(\frac{\boldsymbol{x} - \boldsymbol{x}_0}{\rho_n}\right).$$

Notice that by (3.4.18), $w_n = u$ on $\partial B_{R\rho_n}(\boldsymbol{x}_0)$ in the sense of traces and furthermore that $w(-\lambda/2, y) = 0$ for $\mathcal{L}^1$-a.e. $y \in (\gamma, \gamma + R\rho_n)$. Then the minimality of u implies that

$$\int_{B_{R\rho_n}(\boldsymbol{x}_0)} \left(|\nabla u|^2 + \chi_{\{u>0\}}(h-y)\right)\, d\boldsymbol{x} \le \int_{B_{R\rho_n}(\boldsymbol{x}_0)} \left(|\nabla w_n|^2 + \chi_{\{w_n>0\}}(h-y)\right)\, d\boldsymbol{x},$$

and the change of variables $\boldsymbol{x} = \boldsymbol{x}_0 + \rho_n \boldsymbol{z}$, $\boldsymbol{z} = (s,t)$, then yields

$$\begin{aligned}\int_{B_R(\mathbf{0})} &\left(|\nabla u_n|^2 + \chi_{\{u_n>0\}}(h - \gamma - \rho_n t)\right)\, d\boldsymbol{z} \\ &\le \int_{B_R(\mathbf{0})} \left(|\nabla v_n|^2 + \chi_{\{v_n>0\}}(h - \gamma - \rho_n t)\right)\, d\boldsymbol{z}.\end{aligned} \tag{3.4.25}$$

Since

$$\nabla v_n(\boldsymbol{z}) = \nabla v(\boldsymbol{z}) + (1 - \eta(\boldsymbol{z}))(\nabla u_n(\boldsymbol{z}) - \nabla w(\boldsymbol{z})) - \nabla \eta(\boldsymbol{z})(u_n(\boldsymbol{z}) - w(\boldsymbol{z})),$$

we observe that

$$\begin{aligned}|\nabla v_n|^2 - |\nabla u_n|^2 &= |\nabla v|^2 + |\nabla \eta|^2 |u_n - w|^2 - 2(u_n - w)\nabla\eta \cdot \nabla v + (1-\eta)^2 |\nabla u_n - \nabla w|^2 \\ &\quad + 2(1-\eta)(\nabla u_n - \nabla w)\cdot(\nabla v - \nabla \eta (u_n - w)) - |\nabla u_n|^2 \\ &\le |\nabla v|^2 + |\nabla \eta|^2 |u_n - w|^2 - 2(u_n - w)\nabla\eta \cdot \nabla v - 2\nabla u_n \cdot \nabla w + |\nabla w|^2 \\ &\quad + 2(1-\eta)(\nabla u_n - \nabla w)\cdot(\nabla v - \nabla \eta (u_n - w)),\end{aligned} \tag{3.4.26}$$

where in the last inequality we have used the fact that $(1-\eta)^2 \le 1$. Fix $\varepsilon > 0$ and let $R_\varepsilon := \{\boldsymbol{z} : \operatorname{dist}(\boldsymbol{z}, \{(0,y) : y \ge 0\}) < \varepsilon\}$. Then, by Lemma 3.4.5, it follows that

$$\int_{B_R(\mathbf{0})\setminus R_\varepsilon} \chi_{\{u_n>0\}}(h - \gamma - \rho_n t)\, d\boldsymbol{z} \to \int_{B_R(\mathbf{0})\setminus R_\varepsilon} \chi_{\{w>0\}}(h-\gamma)\, d\boldsymbol{z} \tag{3.4.27}$$

Using the fact that

$$\chi_{\{v_n>0\}} \le \chi_{\{v>0\}} + \chi_{\{\eta<1\}},$$

combining (3.4.25), (3.4.26), (3.4.27), letting $n \to \infty$, and using the fact that $u_n \rightharpoonup u$ in H^1, we deduce that

$$\int_{B_R(\mathbf{0})} |\nabla w|^2 \, d\boldsymbol{z} + \int_{B_R(\mathbf{0})\setminus R_\varepsilon} \chi_{\{w>0\}}(h-\gamma)\, d\boldsymbol{z}$$
$$\leq \int_{B_R(\mathbf{0})} \left(|\nabla v|^2 + (\chi_{\{v>0\}} + \chi_{\{\eta<1\}})(h-\gamma)\right)\, d\boldsymbol{z}.$$

Letting $\varepsilon \to 0^+$, by the monotone convergence theorem we see that

$$\mathcal{F}_h(w) \leq \int_{B_R(\mathbf{0})} \left(|\nabla v|^2 + (\chi_{\{v>0\}} + \chi_{\{\eta<1\}})(h-\gamma)\right)\, d\boldsymbol{z}.$$

The desired result follows from an application of the dominated convergence theorem, choosing a sequence of functions η_k such that $\eta_k \nearrow 1$. □

The next result is commonly referred to as a non-oscillation lemma (see, for example, Lemma 6.1 in [6], Lemma 5.2 in Chapter 3 of [53], and Lemma 2.4 in [74]).

Lemma 3.4.7. *Given $m, \lambda, h > 0$ and $\gamma < h$, let $u \in \mathcal{K}_\gamma$ be a global minimizer of $\mathcal{J}_h$ and let w be a blow-up limit of u at $\boldsymbol{x}_0 = (-\lambda/2, \gamma)$. Assume that there exists an open set G contained in $\{w > 0\}$, which is compactly supported in $\mathbb{R}^2 \setminus \{(0, y) : y \geq 0\}$ and bounded by the line segments*

$$\ell_i := \{(s_i, t) : t_i < t < t_i + \varepsilon_i\}, \quad i = 1, 2,$$

and two non intersecting arcs ϕ_i, $i = 1, 2$, contained in the free boundary $\partial\{w > 0\}$ and joining the points (s_1, t_1) with (s_2, t_2) and $(s_1, t_1 + \varepsilon_1)$ with $(s_2, t_2 + \varepsilon_2)$. Then

$$|s_2 - s_1| \leq \frac{(\varepsilon_1 + \varepsilon_2)\sup_G |\nabla w|}{2\sqrt{h-\gamma}}.$$

Proof. Observe that w is harmonic in G and therefore by the divergence theorem

$$0 = \int_{\partial G} \partial_\nu w \, d\mathcal{H}^1 = \sum_{i=1}^{2} \int_{\ell_i} \partial_\nu w \, d\mathcal{H}^1 + \sum_{i=1}^{2} \int_{\phi_i} \partial_\nu w \, d\mathcal{H}^1.$$

Notice that

$$-\int_{\phi_i} \partial_\nu w \, d\mathcal{H}^1 = \mathcal{H}^1(\phi_i)\sqrt{h-\gamma} \geq |s_2 - s_1|\sqrt{h-\gamma},$$

while

$$\int_{\ell_i} \partial_\nu w \, d\mathcal{H}^1 \leq \sup_G |\nabla w| \varepsilon_i.$$

Consequently

$$2|s_2 - s_1|\sqrt{h-\gamma} \leq -\sum_{i=1}^{2} \int_{\phi_i} \partial_\nu w \, d\mathcal{H}^1 = \sum_{i=1}^{2} \int_{\ell_i} \partial_\nu w \, d\mathcal{H}^1 \leq \sup_G |\nabla w|(\varepsilon_1 + \varepsilon_2),$$

and the desired result readily follows. □

Convergence of free boundaries for symmetric blow-up limits

Throughout this subsection we will work under the assumptions of Theorem 3.4.1. In particular, if w is a blow-up limit with respect to the sequence $\{\rho_n\}_n$ of the symmetric global minimizer u, then the map $s \mapsto w(s,t)$ is increasing in $[0,\infty)$ (and decreasing in $(-\infty,0]$) for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. In turn, its free boundary restricted to the half-plane $\{s > 0\}$ coincides with the set $\{(g_0(t),t) : t \in \mathbb{R}\}$, where $g_0 \colon \mathbb{R} \to [0,\infty]$ is defined via

$$g_0(t) \coloneqq \inf\{s > 0 : w(s,t) > 0\}. \tag{3.4.28}$$

We recall that by Proposition 3.3.19 the function g_0 is continuous in its effective domain. Furthermore, if we let

$$g_n(t) \coloneqq \frac{g(\gamma + \rho_n t) - g(\gamma)}{\rho_n} = \frac{g(\gamma + \rho_n t) + \frac{\lambda}{2}}{\rho_n} \tag{3.4.29}$$

for g defined as in (3.3.18), then we have that

$$g_n(t) = \inf\{s > 0 : u_n(s,t) > 0\}. \tag{3.4.30}$$

Thus the free boundary of u_n in $B_R(\mathbf{0}) \cap \{s > 0\}$ is given by the graph of g_n. It is then natural to ask whether g_n converges to g_0.

Lemma 3.4.8. *Let g_n, g_0 be given as above. Then for every $\tau \in \mathbb{R}$ such that $g_0(\tau) < \infty$ we have that g_0 is finite in a neighborhood of τ and*

$$g_0(\tau) = \lim_{n\to\infty} g_n(\tau). \tag{3.4.31}$$

Proof. **Step 1:** Let τ be as in the statement. We begin by proving that either $g_0(t) < \infty$ for every $t < \tau$ or $g_0(t) < \infty$ for every $t > \tau$. Indeed, assume for the sake of contradiction that there exist $t_1 < \tau < t_2$ such that $g_0(t_1) = g_0(t_2) = \infty$, so that $w(s,t_1) = w(s,t_2) = 0$ for every $s > 0$ by (3.4.28), and fix $s > g_0(\tau)$. For every $M > 0$, by the continuity of w, there exist $T_1, T_2 \in \mathbb{R}$, $\varepsilon_1, \varepsilon_2 > 0$ such that

$$t_1 \le T_i < \tau < T_i + \varepsilon_i \le t_2$$

and with the property that

$$\{(s,T_1),(s,T_1+\varepsilon_1),(s+M,T_2),(s+M,T_2+\varepsilon)\} \subset \partial\{w > 0\}.$$

Let G be the region bounded by the free boundary $\partial\{w > 0\}$ and the two vertical line segments that connect the points (s,T_1) with $(s,T_1+\varepsilon_1)$ and $(s+M,T_2)$ with $(s+M,T_2+\varepsilon_2)$. Then Lemma 3.4.7 yields

$$M \le \frac{C(t_2 - t_1)}{\sqrt{h-\gamma}},$$

a contradiction to the fact that M is arbitrary. Hence $g_0(t) < \infty$ for all $t \le \tau$ or for all $t \ge \tau$. Without loss of generality, we assume the latter. Arguing by contradiction, assume that there exists a sequence $t_n \to \tau^-$ such that $g_0(t_n) = \infty$. Reasoning as above we see that necessarily $g_0(t) = \infty$ for $t_1 \le t < \tau$. In turn, since w is continuous, it must be the case that $w(s,\tau) = 0$ for every $s > 0$,

a contradiction to the assumption that $g_0(\tau) < \infty$.
Step 2: Suppose that there exists $\varepsilon > 0$ such that

$$L := \limsup_{n\to\infty} g_n(\tau) \geq g_0(\tau) + \varepsilon. \tag{3.4.32}$$

By eventually extracting a subsequence we can assume that the limsup is achieved, and furthermore we notice that for every n sufficiently large, (3.4.30) and (3.4.32) imply that $u_n(L - \varepsilon/2, \tau) = 0$. Since the map $s \mapsto u_n(s, \tau)$ is increasing by assumption, we have that $u_n(s, \tau) = 0$ for every $s \leq L - \varepsilon/2$. In turn, passing to the limit in n, $w(s, \tau) = 0$ for every $s \leq L - \varepsilon/2$, which is in contradiction with the definition of g_0 (see (3.4.28) and (3.4.32)). This shows that

$$\limsup_{n\to\infty} g_n(\tau) \leq g_0(\tau).$$

Notice that if $g_0(\tau) = 0$ then there is nothing else to prove. Therefore, we can assume without loss that $g_0(\tau) > 0$. Assume for the sake of contradiction that for some $\varepsilon > 0$

$$\liminf_{n\to\infty} g_n(\tau) \leq g_0(\tau) - 2\varepsilon. \tag{3.4.33}$$

Since g_0 is continuous in a neighborhood of τ, there exists $\delta = \delta(\varepsilon, \tau) > 0$ such that if $|t - \tau| < \delta$ then

$$g_0(\tau) - \varepsilon \leq g_0(t).$$

Notice that without loss of generality we can assume that $4\varepsilon < g_0(\tau)$. Fix $r < \min\{\varepsilon, \delta\}$ and set $\sigma := g_0(\tau) - \varepsilon - r$. Then $B_r(\sigma, \tau) \subset \{w = 0\}$ and thus it follows from Proposition 3.4.3 that

$$B_{r/2}(\sigma, \tau) \subset \{u_n = 0\}$$

for every n sufficiently large. In particular, $u_n(s, \tau) = 0$ for every $s \leq \sigma + r/2$ and therefore

$$g_n(\tau) \geq \sigma + \frac{r}{2} \geq g_0(\tau) - \frac{3}{2}\varepsilon. \tag{3.4.34}$$

Since (3.4.34) is in contradiction with (3.4.33) we conclude that

$$\liminf_{n\to\infty} g_n(\tau) \geq g_0(\tau),$$

which completes the proof. □

3.4.3 A boundary monotonicity formula

In this section we show that the boundary monotonicity formula of Weiss (see Theorem 3.3 and Corollary 3.4 in [92]) holds at the point $\boldsymbol{x}_0 = (-\lambda/2, \gamma)$ for global minimizers of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$ with bounded gradient in a neighborhood of $\boldsymbol{x}_0$. In particular, in view of Theorem 3.4.1, the following theorem applies to symmetric global minimizers.

Theorem 3.4.9. *Given $m, \lambda, h > 0$ and $\gamma < h$, let u be a global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$. Furthermore, let $r_0 < \min\{\gamma, h - \gamma, \lambda\}$ be such that ∇u is bounded in $B_{r_0}(\boldsymbol{x}_0)$ and for $r \in (0, r_0)$ define*

$$\Phi(r) := r^{-2} \int_{B_r(\boldsymbol{x}_0)} \left(|\nabla u|^2 + \chi_{\{u>0\}}(h - y)\right)\, d\boldsymbol{x} - r^{-3} \int_{\partial B_r(\boldsymbol{x}_0)} u^2 \, d\mathcal{H}^1. \tag{3.4.35}$$

Then for $\mathcal{L}^1$-a.e. ρ and σ such that $0<\rho<\sigma<r_0$*,*

$$\Phi(\sigma)-\Phi(\rho)=\int_\rho^\sigma r^{-2}\int_{\partial B_r(\boldsymbol{x}_0)}2\left(\partial_\nu u-\frac{u}{r}\right)^2\,d\mathcal{H}^1dr-\int_\rho^\sigma r^{-3}\int_{B_r(\boldsymbol{x}_0)}\chi_{\{u>0\}}(y-\gamma)\,d\boldsymbol{x}dr.$$

Proof. **Step 1:** For simplicity we consider the translated function

$$w(\boldsymbol{x})=u(\boldsymbol{x}_0+\boldsymbol{x}).$$

We begin by showing that for $\mathcal{L}^1$-a.e. $r\in(0,r_0)$,

$$\begin{aligned}\int_{B_r(\mathbf{0})}&\chi_{\{w>0\}}(2h-2\gamma-3y)\,d\boldsymbol{x}\\&=\int_{\partial B_r(\mathbf{0})}r\left(|\nabla w|^2-2\left(\partial_\nu w\right)^2+\chi_{\{w>0\}}(h-\gamma-y)\right)\,d\mathcal{H}^1.\end{aligned}\tag{3.4.36}$$

To this end, we consider the functional

$$\mathcal{F}_h(v)\coloneqq\int_{B_{r_0}(\mathbf{0})}\left(|\nabla v|^2+\chi_{\{v>0\}}(h-\gamma-y)\right)\,d\boldsymbol{x},$$

defined for $v\in\mathcal{K}(w,r_0)$ (see (3.4.24)). By the minimality of u, w is necessarily a global minimizer of $\mathcal{F}_h$ in $\mathcal{K}(w,r_0)$, and in particular it must be the case that the first variation of $\mathcal{F}_h$ with respect to domain variations vanishes at w. To be precise, for every $\boldsymbol{\phi}=(\phi_1,\phi_2)\in C^1(B_{r_0}(\mathbf{0});\mathbb{R}^2)$ which is compactly supported in $B_{r_0}(\mathbf{0})\setminus\{(0,y):y\geq 0\}$, if we set $w_\varepsilon(\boldsymbol{x})\coloneqq w(\boldsymbol{x}+\varepsilon\boldsymbol{\phi}(\boldsymbol{x}))$ we have that $w_\varepsilon\in\mathcal{K}(w,r_0)$ for every ε sufficiently small and

$$\begin{aligned}0=&-\frac{d}{d\varepsilon}\mathcal{F}_h(w_\varepsilon)_{|\varepsilon=0}\\=&\int_{B_{r_0}(\mathbf{0})}\left(|\nabla w|^2\operatorname{div}\boldsymbol{\phi}-2\nabla wD\boldsymbol{\phi}\nabla w+\chi_{\{w>0\}}(h-\gamma-y)\operatorname{div}\boldsymbol{\phi}-\chi_{\{w>0\}}\phi_2\right)d\boldsymbol{x}.\end{aligned}\tag{3.4.37}$$

For $r\in(0,r_0)$ and $\delta>0$ define

$$\eta_\delta(\boldsymbol{x})\coloneqq\max\left\{0,\min\left\{1,\frac{1}{\delta}(r-|\boldsymbol{x}|)\right\}\right\},\tag{3.4.38}$$

$$\xi_\delta(\boldsymbol{x})\coloneqq\min\left\{1,\frac{1}{\delta}\operatorname{dist}(\boldsymbol{x},\{(0,y):y\geq 0\})\right\}.\tag{3.4.39}$$

Let $\boldsymbol{\phi}_\delta(\boldsymbol{x})\coloneqq\eta_\delta(\boldsymbol{x})\xi_\delta(\boldsymbol{x})\boldsymbol{x}$. By a standard density argument, for every $\delta>0$ we can find a sequence $\{\boldsymbol{\phi}_{\delta,\varepsilon}\}_\varepsilon$ of functions in $C_0^1(B_{r_0}(\mathbf{0});\mathbb{R}^2)$ with compact support in $B_{r_0}(\mathbf{0})\setminus\{(0,y):y\geq 0\}$ such that $\boldsymbol{\phi}_{\delta,\varepsilon}\to\boldsymbol{\phi}_\delta$ in $W^{1,\infty}(B_{r_0}(\mathbf{0}),\mathbb{R}^2)$. Using $\boldsymbol{\phi}_{\delta,\varepsilon}$ as test function in (3.4.37), letting $\varepsilon\to 0$, and noticing that

$$D\boldsymbol{\phi}_\delta=\eta_\delta\xi_\delta\operatorname{Id}+\eta_\delta\nabla\xi_\delta\otimes\boldsymbol{x}+\xi_\delta\nabla\eta_\delta\otimes\boldsymbol{x},$$

$$\operatorname{div}\boldsymbol{\phi}_\delta=2\eta_\delta\xi_\delta+\eta_\delta\nabla\xi_\delta\cdot\boldsymbol{x}+\xi_\delta\nabla\eta_\delta\cdot\boldsymbol{x},$$

we obtain the identity

$$I_1^\delta + I_2^\delta + I_3^\delta = 0, \tag{3.4.40}$$

where

$$I_1^\delta := \int_{B_{r_0}(\mathbf{0})} \eta_\delta \xi_\delta \chi_{\{w>0\}}(2h - 2\gamma - 3y)\, d\boldsymbol{x},$$

$$I_2^\delta := \int_{B_{r_0}(\mathbf{0})} \xi_\delta \Big(|\nabla w|^2 \nabla \eta_\delta \cdot \boldsymbol{x} - 2(\nabla w \cdot \boldsymbol{x})(\nabla w \cdot \nabla \eta_\delta) + \chi_{\{w>0\}}(h - \gamma - y)\nabla \eta_\delta \cdot \boldsymbol{x} \Big)\, d\boldsymbol{x},$$

$$I_3^\delta := \int_{B_{r_0}(\mathbf{0})} \eta_\delta \Big(|\nabla w|^2 \nabla \xi_\delta \cdot \boldsymbol{x} - 2(\nabla w \cdot \boldsymbol{x})(\nabla w \cdot \nabla \xi_\delta) + \chi_{\{w>0\}}(h - \gamma - y)\nabla \xi_\delta \cdot \boldsymbol{x} \Big)\, d\boldsymbol{x}.$$

By (3.4.38), (3.4.39), and the monotone convergence theorem we have that

$$I_1^\delta \to \int_{B_r(\mathbf{0})} \chi_{\{w>0\}}(2h - 2\gamma - 3y)\, d\boldsymbol{x}. \tag{3.4.41}$$

Observe that

$$\nabla \eta_\delta(\boldsymbol{x}) = \begin{cases} -\dfrac{\boldsymbol{x}}{\delta |\boldsymbol{x}|} & \text{in } B_r(\mathbf{0}) \setminus B_{r-\delta}(\mathbf{0}), \\ 0 & \text{otherwise.} \end{cases}$$

Thus we can rewrite I_2^δ as follows:

$$\begin{aligned} I_2^\delta &= -\frac{1}{\delta} \int_{B_r(\mathbf{0}) \setminus B_{r-\delta}(\mathbf{0})} \xi_\delta |\boldsymbol{x}| \left(|\nabla w|^2 - 2\left(\nabla w \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)^2 + \chi_{\{w>0\}}(h - \gamma - y) \right) d\boldsymbol{x} \\ &= -\frac{1}{\delta} \int_{r-\delta}^{r} \int_{\partial B_s(\mathbf{0})} \xi_\delta s \left(|\nabla w|^2 - 2\left(\nabla w \cdot \frac{\boldsymbol{x}}{s}\right)^2 + \chi_{\{w>0\}}(h - \gamma - y) \right) d\mathcal{H}^1(\boldsymbol{x}) ds. \end{aligned}$$

Consequently, by Fubini's theorem and Lebesgue's differentiation theorem, for $\mathcal{L}^1$-a.e. $0 < r < r_0$, we have that

$$I_2^\delta \to -\int_{\partial B_r(\mathbf{0})} r \left(|\nabla w|^2 - 2\left(\partial_\nu w\right)^2 + \chi_{\{w>0\}}(h - \gamma - y) \right) d\mathcal{H}^1 \tag{3.4.42}$$

as $\delta \to 0^+$. By (3.4.40), (3.4.41), and (3.4.42), it follows that to conclude the proof of (3.4.36) we are left to show that $I_3^\delta \to 0$ as $\delta \to 0^+$. To this end, we let

$$\Omega_\delta^+ := \left\{ \boldsymbol{x} \in B_r(\mathbf{0}) : x > 0,\ y > 0,\ \text{and } \operatorname{dist}(\boldsymbol{x}, \{(0, y) : y \geq 0\}) < \delta \right\},$$

$$\Omega_\delta^- := \left\{ \boldsymbol{x} \in B_r(\mathbf{0}) : x < 0,\ y > 0,\ \text{and } \operatorname{dist}(\boldsymbol{x}, \{(0, y) : y \geq 0\}) < \delta \right\},$$

$$\Omega_\delta^* := \left\{ \boldsymbol{x} \in B_\delta(\mathbf{0}) : y < 0 \right\},$$

and notice that

$$\nabla \xi_\delta(\boldsymbol{x}) = \begin{cases} (\pm\delta^{-1}, 0) & \text{in } \Omega_\delta^\pm, \\ \dfrac{\boldsymbol{x}}{\delta|\boldsymbol{x}|} & \text{in } \Omega_\delta^*, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.43}$$

From (3.4.43) and the fact that

$$\operatorname{dist}(\boldsymbol{x}, \{(0,y) : y \geq 0\}) = \begin{cases} |x| & \text{in } \Omega_\delta^\pm, \\ |\boldsymbol{x}| & \text{in } \Omega_\delta^*, \end{cases}$$

we see that $|\nabla \xi_\delta \cdot \boldsymbol{x}| \leq 1$ in $B_{r_0}(\mathbf{0})$, and consequently

$$\left| \int_{B_{r_0}(\mathbf{0})} \eta_\delta \left(|\nabla w|^2 + \chi_{\{w>0\}}(h - \gamma - y) \right) \nabla \xi_\delta \cdot \boldsymbol{x} \, d\boldsymbol{x} \right| \\ \leq \int_{\{\nabla \xi_\delta \neq 0\}} \left(|\nabla w|^2 + \chi_{\{w>0\}}(h - \gamma - y) \right) \, d\boldsymbol{x}.$$

Furthermore, the right-hand side in the previous inequality vanishes as $\delta \to 0^+$ by the dominated convergence theorem. It remains to show that

$$\int_{B_{r_0}(\mathbf{0})} \eta_\delta \left(x \partial_x w + y \partial_y w \right) \left(\nabla w \cdot \nabla \xi_\delta \right) \, d\boldsymbol{x} \to 0$$

as $\delta \to 0^+$. Since $|\nabla \xi_\delta||x| \leq 1$, reasoning as above we see that

$$\left| \int_{B_{r_0}(\mathbf{0})} \eta_\delta x \partial_x w \left(\nabla w \cdot \nabla \xi_\delta \right) \, d\boldsymbol{x} \right| \leq \int_{\{\nabla \xi_\delta \neq 0\}} |\partial_x w||\nabla w| \, d\boldsymbol{x} \to 0.$$

Fix $\varepsilon \in (0, r)$. Using (3.4.43) and the fact that η_δ vanishes outside $B_r(\mathbf{0})$, we see that

$$\left| \int_{\Omega_\delta^+} \eta_\delta y \partial_y w (\nabla w \cdot \nabla \xi_\delta) \, d\boldsymbol{x} \right| \leq \frac{1}{\delta} \int_{(0,\delta)\times(0,r)} y |\partial_y w||\partial_x w| \, d\boldsymbol{x} \\ \leq \frac{\varepsilon}{\delta} \int_{(0,\delta)\times(0,\varepsilon)} |\partial_y w||\partial_x w| \, d\boldsymbol{x} + \frac{r}{\delta} \int_{(0,\delta)\times(\varepsilon,r)} |\partial_y w||\partial_x w| \, d\boldsymbol{x}.$$

Since ∇w is bounded, the first term on the right-hand side can be bounded uniformly in δ, and so it vanishes as $\varepsilon \to 0^+$. By Theorem 1.1 in [29], we have that the extended free boundary

$$\overline{\partial\{w > 0\}} \cap \Omega_\delta^+ \setminus B_{\varepsilon/2}(\mathbf{0})$$

is of class $C^{1,1/2}$. In turn, it follows from Corollary 8.36 in [54] that

$$w \in C^{1,1/2}(\overline{\{w > 0\} \cap \Omega_\delta^+} \setminus B_\varepsilon(\mathbf{0})). \tag{3.4.44}$$

In particular, this implies that $\nabla_\tau w = 0$ on $\partial\{w > 0\} \setminus B_\varepsilon(\mathbf{0})$. Consequently, a change of variables and the dominated convergence theorem give

$$\frac{r}{\delta}\int_{(0,\delta)\times(\varepsilon,r)} |\partial_y w||\partial_x w|\,d\boldsymbol{x} = r\int_{(0,1)\times(\varepsilon,r)} |\partial_y w(\delta x, y)||\partial_x w(\delta x, y)|\,d\boldsymbol{x} \to 0$$

as $\delta \to 0^+$. Since similar estimates hold in Ω_δ^- and Ω_δ^*, this concludes the proof of (3.4.36).

Step 2: This step is dedicated to the proof of the integration by parts formula

$$\int_{B_r(\mathbf{0})} |\nabla w|^2\,d\boldsymbol{x} = \int_{\partial B_r(\mathbf{0})} w\partial_\nu w\,d\mathcal{H}^1, \tag{3.4.45}$$

which holds for $\mathcal{L}^1$-a.e. $r \in (0, r_0)$, and is in spirit very close to the result of Lemma 3.1 in [44]. Let

$$U_{\varepsilon,\eta} \coloneqq B_r(\mathbf{0}) \setminus (B_\varepsilon(\mathbf{0}) \cup \{\boldsymbol{x} : \operatorname{dist}(\boldsymbol{x}, \{(0,y) : y \geq 0\}) < \eta\}),$$

and observe that by the divergence theorem, together with the fact that $w = 0$ on $\partial\{w > 0\}$,

$$\int_{U_{\varepsilon,\eta}\cap\{w>0\}} |\nabla w|^2\,d\boldsymbol{x} = \int_{\partial U_{\varepsilon,\eta}\cap\{w>0\}} w\partial_\nu w\,d\mathcal{H}^1.$$

Next, using the fact that w is Lipschitz continuous in $B_{r_0}(\mathbf{0})$, that $w(0, y) = 0$ for $y > 0$, and (3.4.44), we obtain

$$\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\int_{U_{\varepsilon,\eta}\cap\{w>0\}} |\nabla w|^2\,d\boldsymbol{x} = \lim_{\varepsilon\to 0^+}\int_{\partial(B_r(\mathbf{0})\setminus B_\varepsilon(\mathbf{0}))} w\partial_\nu w\,d\mathcal{H}^1 = \int_{\partial B_r(\mathbf{0})} w\partial_\nu w\,d\mathcal{H}^1,$$

and the desired formula (4.3.1) follows immediately upon noticing that

$$\int_{B_r(\mathbf{0})} |\nabla w|^2\,d\boldsymbol{x} = \lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\int_{U_{\varepsilon,\eta}\cap\{w>0\}} |\nabla w|^2\,d\boldsymbol{x}.$$

Step 3: By a direct computation we see that for $\mathcal{L}^1$-a.e $r \in (0, r_0)$,

$$\begin{aligned}\Phi'(r) = r^{-2}&\int_{\partial B_r(\mathbf{0})} \left(|\nabla w|^2 + (h-\gamma-y)\chi_{\{w>0\}} + 2r^{-2}w^2 - 2r^{-1}w(\partial_\nu w)\right]\,d\mathcal{H}^1 \\ &- 2r^{-3}\int_{B_r(\mathbf{0})} \left(|\nabla w|^2 + \chi_{\{w>0\}}(h-\gamma-y)\right)\,d\boldsymbol{x}\end{aligned} \tag{3.4.46}$$

where Φ is defined in (3.4.35) and we recall that $w(\boldsymbol{x}) = u(\boldsymbol{x}_0 + \boldsymbol{x})$. Moreover, by (3.4.36) and (4.3.1), we can rewrite (3.4.46) as

$$\Phi'(r) = 2r^{-2}\int_{\partial B_r(\mathbf{0})} \left(\partial_\nu w - \frac{w}{r}\right)^2\,d\mathcal{H}^1 - r^{-3}\int_{B_r(\mathbf{0})} \chi_{\{w>0\}}y\,d\boldsymbol{x},$$

and the desired formula follows by integration. □

Remark 3.4.10. *Under the additional assumption that $\boldsymbol{x}_0$ is an isolated accumulation point for $\partial\{u > 0\}$ on $\partial\Omega$, the regularity result of [29] is not needed for the proof of Theorem 3.4.9.*

Corollary 3.4.11. *Let Φ be defined as in Theorem 3.4.9. Then Φ has finite right limit at zero, i.e.,*

$$\lim_{\rho\to 0^+}\Phi(\rho) =: \Phi(0^+) \in \mathbb{R}.$$

Proof. Fix $0 < \sigma < r_0$ and consider $\rho < \sigma$. By Theorem 3.4.9

$$\Phi(\sigma) = \Phi(\rho) + A(\rho,\sigma) + B(\rho,\sigma) + C(\rho,\sigma),$$

where

$$A(\rho,\sigma) := \int_\rho^\sigma r^{-2}\int_{\partial B_r(\boldsymbol{x}_0)} 2\left(\partial_\nu u(\boldsymbol{x}) - \frac{u(\boldsymbol{x})}{r}\right)^2 d\mathcal{H}^1(\boldsymbol{x})dr,$$

$$B(\rho,\sigma) := -\int_\rho^\sigma r^{-3}\int_{B_r(\boldsymbol{x}_0)} \chi_{\{u>0\}}(y-\gamma)\chi_{\{y\geq\gamma\}}\, d\boldsymbol{x}dr,$$

$$C(\rho,\sigma) := -\int_\rho^\sigma r^{-3}\int_{B_r(\boldsymbol{x}_0)} \chi_{\{u>0\}}(y-\gamma)\chi_{\{y<\gamma\}}\, d\boldsymbol{x}dr.$$

Notice that the maps $\rho \mapsto A(\rho,\sigma)$ and $\rho\mapsto C(\rho,\sigma)$ are decreasing, while $r\mapsto B(\rho,\sigma)$ is increasing. Then

$$\lim_{\rho\to 0^+} A(\rho,\sigma) + C(\rho,\sigma) = \sup\{A(\rho,\sigma)+C(\rho,\sigma) : 0<\rho<\sigma\},$$

$$\lim_{\rho\to 0^+} B(\rho,\sigma) = \inf\{B(\rho,\sigma) : 0<\rho<\sigma\} < \infty.$$

In turn, Φ admits a limit as $\rho\to 0^+$ as it was claimed. Moreover, the fact that $|\Phi(0^+)|<\infty$ follows upon recalling that u is Lipschitz continuous in a neighborhood of $\boldsymbol{x}_0$ and $u(\boldsymbol{x}_0) = 0$. Hence $u(\boldsymbol{x})^2 \leq C|\boldsymbol{x}-\boldsymbol{x}_0|^2$, and so Φ is bounded (see 3.4.35). □

Corollary 3.4.12. *Under the assumptions of Theorem 3.4.9, let w be a blow-up limit of u with respect to the sequence $\{\rho_n\}_n$. Then*

$$\nabla w(\boldsymbol{z})\cdot\boldsymbol{z} = w(\boldsymbol{z}) \quad \textit{for } \mathcal{L}^2\textit{-a.e. } \boldsymbol{z}\in\mathbb{R}^2. \tag{3.4.47}$$

Proof. For every $r>0$ and n large enough so that $\rho_n r < r_0$, by the change of variables $\boldsymbol{x} = \boldsymbol{x}_0 + \rho_n\boldsymbol{z}$ we see that (3.4.35) becomes

$$\Phi(\rho_n r) = r^{-2}\int_{B_r(\mathbf{0})}\left(|\nabla u_n|^2 + \chi_{\{u_n>0\}}(h-\gamma-\rho_n t)\right)d\boldsymbol{z} - r^{-3}\int_{\partial B_r(\mathbf{0})} u_n^2\, d\mathcal{H}^1,$$

where the functions u_n are defined as in (3.4.18). Therefore for every $0<R<S$ and n large enough we have the formula

$$\Phi(\rho_n S) - \Phi(\rho_n R) = \int_R^S r^{-2}\int_{\partial B_r(\mathbf{0})} 2\left(\partial_\nu u_n - \frac{u_n}{r}\right)^2 d\mathcal{H}^1 dr - \int_R^S r^{-3}\int_{B_r(\mathbf{0})}\chi_{\{u_n>0\}}\rho_n t\, d\boldsymbol{x}dr.$$

Letting $n \to \infty$, by Corollary 3.4.11, we obtain

$$0 = \lim_{n\to\infty} \Phi(\rho_n S) - \Phi(\rho_n R) = \liminf_{n\to\infty} \int_R^S r^{-2} \int_{\partial B_r(\mathbf{0})} 2\left(\partial_\nu u_n - \frac{u_n}{r}\right)^2 d\mathcal{H}^1 dr$$

$$\geq \int_{B_S(\mathbf{0})\setminus B_R(\mathbf{0})} 2|\boldsymbol{z}|^{-4} \left(\nabla w(\boldsymbol{z}) \cdot \boldsymbol{z} - w(\boldsymbol{z})\right)^2 d\boldsymbol{z}. \qquad (3.4.48)$$

In turn, the integrand in (3.4.48) must be zero $\mathcal{L}^2$-a.e. in $B_S(\mathbf{0}) \setminus B_R(\mathbf{0})$ By the arbitrariness of R, S, this concludes the proof. □

3.4.4 The proof of Theorem 1.1.5

Theorem 3.4.13. *Given $m, \lambda, h > 0$ and $\gamma < h$, let u be a symmetric global minimizer of $\mathcal{J}_h$ in $\mathcal{K}_\gamma$ in the sense of Definition 1.1.4, and w be a blow-up limit of u at $\boldsymbol{x}_0$. Then either w is identically equal to zero or $w(s,t) = (h-\gamma)(-t)_+$.*

Proof. **Step 1:** We begin by showing that w is a positively homogenous function of degree one. To see this let $\boldsymbol{z} \in \{w > 0\}$ and notice that

$$\frac{d}{dt}\left(\frac{1}{t} w(t\boldsymbol{z})\right) = \frac{1}{t}\nabla w(t\boldsymbol{z}) \cdot \boldsymbol{z} - \frac{1}{t^2} w(t\boldsymbol{z}) = \frac{1}{t^2}\left(\nabla w(t\boldsymbol{z}) \cdot t\boldsymbol{z} - w(t\boldsymbol{z})\right) = 0$$

for every $t > 0$ such that $w(t\boldsymbol{z}) > 0$, where in the last equality we have used (3.4.47). Consequently, it must be the case that $w(t\boldsymbol{z}) = tw(\boldsymbol{z})$ for every such that, and furthermore it follows that the entire ray $\{t\boldsymbol{z} : t \in \mathbb{R}_+\}$ must necessarily be contained in $\{w > 0\}$. In particular, each connected component of $\{w > 0\}$ is a sector with vertex at the origin. Next, we claim that the opening angle of every such sector is π, i.e. each connected component of $\{w > 0\}$ is a half-plane passing through the origin. To this end, we can find a rotation R, a set of polar coordinates (r, θ), and a function f in such a way that

$$f(r,\theta) = w(R(r\cos\theta, r\sin\theta)),$$

and

$$\begin{cases} \Delta f = 0 \ \text{ in } S_\alpha := \{(r,\theta) : 0 < r < \infty, 0 < \theta < \alpha\}, \\ \quad f = 0 \ \text{ on } \partial S_\alpha. \end{cases} \qquad (3.4.49)$$

Notice that the homogeneity of w implies that

$$f(r,\theta) = rf(1,\theta) = rh(\theta),$$

for a function h which satisfies

$$h''(\theta) + h(\theta) = 0.$$

In turn, $h(\theta) = c_1\cos\theta + c_2\sin\theta$. Moreover, the boundary conditions in (3.4.49) give that $c_1 = 0$ and $c_2 \sin\alpha = 0$. Since $f > 0$ in S_α, then it must be the case that $\alpha = \pi$.

Step 2: Since u is symmetric about the line $\{x = -\lambda/2\}$, then u_n, defined as in (3.4.18), is symmetric about the t-axis, and so is w. This, together with the fact that $w(0,t) = 0$ for $t \geq 0$, shows that if w is not identically equal to zero then either $w(s,t) = (h-\gamma)(-t)_+$ or $w(s,t) = (h-\gamma)|s|$. The desired result follows upon noticing that $w(s,t) = (h-\gamma)|s|$ does not minimize the functional $\mathcal{F}_h$ over the set $\mathcal{K}(w,1)$ (see 3.4.24), a contradiction to Theorem 3.4.6. □

Proof of Theorem 1.1.5. For g be defined as in (3.3.18), assume for the sake of contradiction that

$$\liminf_{y\to\infty} \frac{|g(y)-g(\gamma)|}{|y-\gamma|} = \alpha < \infty,$$

let $\{y_n\}_n$ be a sequence for which the limit is realized, and assume without loss of generality that $\{y_n\}_n$ is monotone. Let $\rho_n \coloneqq |y_n - \gamma|$ and notice that for n large enough

$$\rho_n \leq \sqrt{(g(y_n)-g(\gamma))^2 + (y_n-\gamma)^2} \leq \beta|y_n-\gamma| = \beta\rho_n, \quad \text{where } \beta \coloneqq \sqrt{\alpha^2+2}.$$

In turn, Lemma 3.4.4 gives that every blow up of u at $\boldsymbol{x}_0$ with respect to the sequence $\{\rho_n\}_n$ is not identically equal to zero. Then, it follows from Theorem 3.4.13 that the half-plane solution

$$w(s,t) = (h-\gamma)(-t)_+ \tag{3.4.50}$$

is the unique blow-up limit. Assume first that $y_n \to \gamma^+$, set $\rho_n \coloneqq y_n - \gamma$ and let u_n be defined as in (3.4.18). Notice that (3.4.29)

$$0 \leq g_n(1) = \frac{g(y_n)-g(\gamma)}{y_n-\gamma} = \frac{g(y_n)+\frac{\lambda}{2}}{y_n-\gamma} \to \alpha. \tag{3.4.51}$$

On the other hand, since $\{t \geq 0\} \subset \{w = 0\}$ by (3.4.50), it must be the case that $u_n \equiv 0$ in $B_{1/2}(\alpha+1, 1)$ by Lemma 3.4.5. This contradicts (3.4.51). Next, we assume that $y_n \to \gamma^-$. Then $g_n(-1) \to \alpha$ and by the uniform convergence of u_n to w we see that

$$0 = u_n(g_n(-1), -1) \to w(\alpha, -1) = h - \gamma > 0.$$

This concludes the proof. $\square$

Chapter 4

Singular perturbations of mixed Dirichlet-Neumann boundary value problems

4.1 Gamma-convergence of order zero and global minimizers

Throughout the section we study the mixed problem (1.2.3) and the associated minimization problem under the following assumptions on the set Ω and on Γ_D, namely the portion of the boundary where the Robin boundary condition is imposed:

$$\begin{cases} (i)\ \Omega \text{ is an open, bounded and connected subset of } \mathbb{R}^N, \\ (ii)\ \partial\Omega \text{ is Lipschitz continuous,} \\ (iii)\ \Gamma_D \text{ is a non-empty and relatively open subset of } \partial\Omega. \end{cases} \tag{H_0}$$

Furthermore, define $\Gamma_N \coloneqq \partial\Omega \setminus \overline{\Gamma_D}$. Notice that for the purposes of this section we do not assume that $\Gamma_N \neq \emptyset$; analogous results hold (with trivial changes) if $\Gamma_N = \emptyset$.

Theorem 4.1.1. *Let Ω be as in* (H_0)*, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, and $\varepsilon \in (0,1)$. Then the functional $\mathcal{F}_\varepsilon$, defined as in* (1.2.7)*, admits a unique minimizer $u_\varepsilon \in H^1(\Omega)$. Furthermore, u_ε is a weak solution to the mixed Neumann-Robin problem* (1.2.3).

The proof of Theorem 4.1.1 is based on the following well-known result.

Lemma 4.1.2. *Let Ω be as in (H_0) and for $u \in H^1(\Omega)$ set*

$$|||u|||_{H^1(\Omega)} \coloneqq \left(\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^N)} + \|u\|^2_{L^2(\Gamma_D)} \right)^{1/2}. \tag{4.1.1}$$

Then $|||\cdot|||_{H^1(\Omega)}$ defines a norm on $H^1(\Omega)$ that is equivalent to the standard norm, i.e., there are two constants κ_1, κ_2, which only depend on the geometry of Ω and Γ_D, such that for every $u \in H^1(\Omega)$,

$$\kappa_1 |||u|||_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq \kappa_2 |||u|||_{H^1(\Omega)}.$$

Proof of Theorem 4.1.1. By Hölder's inequality, we have that for every $\varepsilon \in (0,1)$ and for every $u \in H^1(\Omega)$,

$$\mathcal{F}_\varepsilon(u) \geq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^N)} - \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \frac{1}{2}\|u-g\|^2_{L^2(\Gamma_D)}. \tag{4.1.2}$$

Young's inequality then implies

$$\begin{aligned}\|u-g\|^2_{L^2(\Gamma_D)} &= \|u\|^2_{L^2(\Gamma_D)} + \|g\|^2_{L^2(\Gamma_D)} - 2\int_{\Gamma_D} ug\, d\mathcal{H}^{N-1} \\ &\geq \frac{1}{2}\|u\|^2_{L^2(\Gamma_D)} - 7\|g\|^2_{L^2(\Gamma_D)},\end{aligned} \tag{4.1.3}$$

and thus, combining the estimates (4.1.2) and (4.1.3) with Lemma 4.1.2, we obtain

$$\mathcal{F}_\varepsilon(u) \geq \frac{1}{4}|||u|||^2_{H^1(\Omega)} - \kappa_2\|f\|_{L^2(\Omega)}|||u|||_{H^1(\Omega)} - \frac{7}{2}\|g\|^2_{L^2(\Gamma_D)}.$$

In turn,

$$\inf\{\mathcal{F}_\varepsilon(u) : \varepsilon \in (0,1), u \in L^2(\Omega)\} > -\infty$$

and for every $\varepsilon \in (0,1)$ the functional $\mathcal{F}_\varepsilon$ is coercive. Since $\mathcal{F}_\varepsilon$ is lower semicontinuous with respect to weak convergence in $L^2(\Omega)$, the existence of a global minimizer u_ε follows from the direct method in the calculus of variations and the assertion about uniqueness is a consequence of the strict convexity of the functional $\mathcal{F}_\varepsilon$. Moreover, one can check that u_ε is a weak solution to (1.2.3) by considering variations of the functional $\mathcal{F}_\varepsilon$. We omit the details. □

Proposition 4.1.3 (Compactness)**.** *Under the assumptions of Theorem 1.2.3, if $\varepsilon_n \to 0^+$ and u_n are such that*

$$\sup\{\mathcal{F}_{\varepsilon_n}(u_n) : n \in \mathbb{N}\} < \infty,$$

then there exist a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$, $u \in V$ and $v \in L^2(\Gamma_D)$ such that

$$\begin{aligned} u_{n_k} &\rightharpoonup u \quad \text{in } H^1(\Omega), \\ \varepsilon_{n_k}^{-1/2}(u_{n_k} - g) &\rightharpoonup v \quad \text{in } L^2(\Gamma_D).\end{aligned}$$

Proof. Let $M := \sup_n \mathcal{F}_{\varepsilon_n}(u_n)$ and assume without loss of generality that $\varepsilon_1 \leq 1$. Reasoning as in the proof of Theorem 4.1.1, by Hölder's inequality we see that

$$M \geq \frac{1}{2}\|\nabla u_n\|^2_{L^2(\Omega;\mathbb{R}^N)} - \|f\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)} + \frac{1}{2\varepsilon_n}\|u_n - g\|^2_{L^2(\Gamma_D)} \tag{4.1.4}$$

for every $n \in \mathbb{N}$. Young's inequality, together with the fact that $\varepsilon_n \leq 1$, then implies that

$$\begin{aligned}\frac{1}{2\varepsilon_n}\|u_n - g\|^2_{L^2(\Gamma_D)} &\geq \frac{1}{4}\|u_n - g\|^2_{L^2(\Gamma_D)} + \frac{1}{4\varepsilon_n}\|u_n - g\|^2_{L^2(\Gamma_D)} \\ &\geq \frac{1}{8}\|u_n\|^2_{L^2(\Gamma_D)} - \frac{7}{4}\|g\|^2_{L^2(\Gamma_D)} + \frac{1}{4\varepsilon_n}\|u_n - g\|^2_{L^2(\Gamma_D)},\end{aligned} \tag{4.1.5}$$

and thus, combining the estimates (4.1.4) and (4.1.5) with Lemma 4.1.2, and using the notation (4.1.1), we arrive at

$$M \geq \frac{1}{8}|||u_n|||^2_{H^1(\Omega)} - \kappa_2\|f\|_{L^2(\Omega)}|||u_n|||_{H^1(\Omega)} - \frac{7}{4}\|g\|^2_{L^2(\Gamma_D)} + \frac{1}{4\varepsilon_n}\|u_n - g\|^2_{L^2(\Gamma_D)}.$$

Consequently, $\{u_n\}_n$ is bounded in $H^1(\Omega)$ by Lemma 4.1.2, and furthermore $\{\varepsilon_n^{-1/2}(u_n - g)\}_n$ is bounded in $L^2(\Gamma_D)$. Hence there are functions $u \in H^1(\Omega)$, $v \in L^2(\Gamma_D)$ and a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ as in the statement. To conclude we notice that $u_n \to g$ in $L^2(\Gamma_D)$, and so $u \in V$. $\square$

Proof of Theorem 1.2.3. Let $\varepsilon_n \to 0^+$ and $\{u_n\}_n$ be a sequence of functions in $L^2(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$. If $\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) = \infty$ there is nothing to prove. Hence, up to the extraction of a subsequence (not relabeled), we can assume without loss of generality that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) = \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) < \infty.$$

In particular, $\mathcal{F}_{\varepsilon_n}(u_n) < \infty$ for every n sufficiently large. Let $\{u_{n_k}\}_k$ and u be given as in Proposition 4.1.3, then

$$\begin{aligned}
\liminf_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}(u_{n_k}) &\geq \liminf_{k\to\infty} \int_\Omega \left(\frac{1}{2}|\nabla u_{n_k}|^2 + f u_{n_k}\right) d\boldsymbol{x} \\
&= \liminf_{k\to\infty} \frac{1}{2}\int_\Omega |\nabla u_{n_k}|^2 \, d\boldsymbol{x} + \lim_{k\to\infty} \int_\Omega f u_{n_k} \, d\boldsymbol{x} \\
&\geq \frac{1}{2}\int_\Omega |\nabla u|^2 \, d\boldsymbol{x} + \int_\Omega f u \, d\boldsymbol{x} = \mathcal{F}_0(u).
\end{aligned}$$

On the other hand, for every $u \in L^2(\Omega)$, the constant sequence $u_n = u$ is a recovery sequence. Indeed, $\mathcal{F}_{\varepsilon_n}(u) = \mathcal{F}_0(u)$ for every $u \in V$, while if $u \notin V$ then $\mathcal{F}_0(u) = \infty$ and hence there is nothing to prove. $\square$

Corollary 4.1.4. *Under the assumptions of Theorem 1.2.3, if $\varepsilon_n \to 0^+$ and $\{u_n\}_n$ is a sequence of functions in $L^2(\Omega)$ such that*

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \leq \inf\left\{\mathcal{F}_0(v) : v \in L^2(\Omega)\right\}$$

then $u_n \to u_0$ strongly in $H^1(\Omega)$, where u_0 is the unique global minimizer of $\mathcal{F}_0$. In particular, global minimizers u_{ε_n} of $\mathcal{F}_{\varepsilon_n}$ converge in $H^1(\Omega)$ to u_0.

Proof. Since $g \in H^{1/2}(\partial\Omega)$, by standard trace theorems (see Theorem 18.40 in [70]) the space V defined in (1.2.9) is nonempty. In turn, the strictly convex functional $\mathcal{F}_0$ given in (1.2.8) admits a unique minimizer u_0 which is a weak solution to (1.2.1). Let $\{u_n\}_n$ be a sequence of functions in $L^2(\Omega)$ such that

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \leq \mathcal{F}_0(u_0). \tag{4.1.6}$$

Given a subsequence $\{\varepsilon_{n_k}\}_k$ of $\{\varepsilon_n\}_n$, by Proposition 4.1.3 we can find a further subsequence $\{u_{n_{k_j}}\}_j$ and $v_0 \in V$ such that $u_{n_{k_j}} \to v_0$. By Γ-convergence

$$\mathcal{F}_0(u_0) \geq \limsup_{j\to\infty} \mathcal{F}_{\varepsilon_{n_{k_j}}}(u_{n_{k_j}}) \geq F_0(v_0),$$

which in turn implies that $v_0 = u_0$. Hence the full sequence $\{u_n\}_n$ converges in $L^2(\Omega)$ to u_0. Moreover, by (4.1.6)

$$\mathcal{F}_0(u_0) \geq \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \geq \limsup_{n\to\infty} \int_\Omega \left(\frac{1}{2}|\nabla u_n|^2 + f u_n\right) d\boldsymbol{x}$$

$$\geq \liminf_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla u_n|^2 \, d\boldsymbol{x} + \int_\Omega f u_0 \, d\boldsymbol{x} \geq \mathcal{F}_0(u_0),$$

and so

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 \, d\boldsymbol{x} = \int_\Omega |\nabla u_0|^2 \, d\boldsymbol{x}.$$

By the strict convexity of the L^2-norm it follows that $\nabla u_n \to \nabla u_0$ in $L^2(\Omega;\mathbb{R}^N)$. □

4.2 A problem without singularities

Following Costabel and Dauge [40], in this section we will be concerned with the study of the easier case of the non-mixed problem (1.2.34); to be precise, it is assumed throughout the section that $\Gamma_D = \partial\Omega$. Under this additional assumption we prove asymptotic developments by Γ-convergence of all orders for the family of functionals $\{\mathcal{F}_\varepsilon\}_\varepsilon$ and deduce a complete asymptotic expansion for u_ε, i.e., the solution to (1.2.34) (see Theorem 4.1.1). Throughout the section, we will make the following assumptions on the set Ω:

$$\begin{cases} (i)\ \Omega \text{ is an open, bounded and connected subset of } \mathbb{R}^N, \\ (ii)\ \partial\Omega \text{ is of class } C^{j,1}. \end{cases} \tag{H_j}$$

4.2.1 The non-mixed problem: Gamma-convergence of order one

In this section we prove a first order asymptotic expansion for $\mathcal{F}_\varepsilon$. We begin by studying the compactness properties of sequences with bounded energy.

Proposition 4.2.1 (Compactness)**.** *Let Ω be as in (H_1), $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$, $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ be the functionals defined in* (1.2.7) *and* (1.2.8) *(with $\Gamma_D = \partial\Omega$), respectively, and define*

$$\mathcal{F}_\varepsilon^{(1)} := \frac{\mathcal{F}_\varepsilon - \min \mathcal{F}_0}{\varepsilon}. \tag{4.2.1}$$

If $\varepsilon_n \to 0^+$ and $v_n \in L^2(\Omega)$ are such that

$$\sup\{\mathcal{F}_{\varepsilon_n}^{(1)}(v_n) : n \in \mathbb{N}\} < \infty,$$

then $u_n \to u_0$ in $H^1(\Omega)$ and there exist a subsequence $\{v_{n_k}\}_k$ of $\{v_n\}_n$, $r_0 \in H^1(\Omega)$ and $v_0 \in L^2(\partial\Omega)$ such that

$$\begin{aligned} \frac{v_{n_k} - u_0}{\sqrt{\varepsilon_{n_k}}} &\rightharpoonup r_0 \quad \text{in } H^1(\Omega), \\ \frac{v_{n_k} - u_0}{\varepsilon_{n_k}} &\rightharpoonup v_0 \quad \text{in } L^2(\partial\Omega), \end{aligned} \tag{4.2.2}$$

where u_0 is the solution to (1.2.35).

Proof. If we let $M := \sup\{\mathcal{F}^{(1)}_{\varepsilon_n}(v_n) : n \in \mathbb{N}\}$, then $\mathcal{F}_\varepsilon(v_n) \le \mathcal{F}_0(u_0) + \varepsilon_n M$. On the other hand,

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v_n) \ge \mathcal{F}_0(u_0)$$

by Theorem 1.2.3, and in turn $v_n \to u_0$ strongly in $H^1(\Omega)$ by Corollary 4.1.4.

For every $n \in \mathbb{N}$, let $r_n \in L^2(\Omega)$ be such that $v_n = u_0 + \varepsilon_n r_n$. Then $\mathcal{F}^{(1)}_{\varepsilon_n}(v_n)$ can be rewritten as

$$\mathcal{F}^{(1)}_{\varepsilon_n}(v_n) = \int_\Omega \left(\nabla u_0 \cdot \nabla r_n + \frac{\varepsilon_n}{2}|\nabla r_n|^2 + f r_n\right)\, d\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega} r_n^2 \, d\mathcal{H}^{N-1}. \tag{4.2.3}$$

Since $\partial\Omega$ is of class $C^{1,1}$, $f \in L^2(\Omega)$, and $g \in H^{3/2}(\partial\Omega)$, by standard elliptic regularity theory for (1.2.35), $u_0 \in H^2(\Omega)$ (see Theorem 2.4.2.5 in [58]) and by an application of the divergence theorem we have

$$\int_\Omega (\nabla u_0 \cdot \nabla r_n + f r_n)\, d\boldsymbol{x} = \int_{\partial\Omega} \partial_\nu u_0 r_n \, d\mathcal{H}^{N-1}. \tag{4.2.4}$$

Substituting (4.2.4) into (4.2.3) we arrive at

$$\begin{aligned} M \ge \mathcal{F}^{(1)}_{\varepsilon_n}(v_n) &= \frac{\varepsilon_n}{2}\int_\Omega |\nabla r_n|^2 \, d\boldsymbol{x} + \int_{\partial\Omega}\left(\frac{1}{2}r_n^2 + \partial_\nu u_0 r_n\right)\, d\mathcal{H}^{N-1} \\ &= \frac{\varepsilon_n}{2}\int_\Omega |\nabla r_n|^2 \, d\boldsymbol{x} + \frac{1}{2}\int_{\partial\Omega}\left[(r_n + \partial_\nu u_0)^2 - (\partial_\nu u_0)^2\right]\, d\mathcal{H}^{N-1}, \end{aligned} \tag{4.2.5}$$

and (4.2.2) is proved at once. □

Theorem 4.2.2 (1st order Γ-convergence)**.** *Under the assumptions of Proposition 4.2.1, the family* $\{\mathcal{F}^{(1)}_\varepsilon\}_\varepsilon$ *Γ-converges in* $L^2(\Omega)$ *to the functional*

$$\mathcal{F}_1(v) := \begin{cases} -\dfrac{1}{2}\displaystyle\int_{\partial\Omega} (\partial_\nu u_0)^2 \, d\mathcal{H}^{N-1} & \text{if } v = u_0, \\ +\infty & \text{otherwise}. \end{cases} \tag{4.2.6}$$

In particular, if $u_\varepsilon \in H^1(\Omega)$ *is the solution to* (1.2.34)*, then*

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}_0(u_0) + \varepsilon\mathcal{F}_1(u_0) + o(\varepsilon). \tag{4.2.7}$$

Proof. Let $\varepsilon_n \to 0^+$ and $\{v_n\}_n$ be a sequence of functions in $L^2(\Omega)$ such that $v_n \to v$ in $L^2(\Omega)$. Reasoning as in the proof of Theorem 1.2.3, we can assume without loss of generality that

$$\liminf_{n\to\infty} \mathcal{F}^{(1)}_{\varepsilon_n}(v_n) = \lim_{n\to\infty} \mathcal{F}^{(1)}_{\varepsilon_n}(v_n) < \infty.$$

In particular, $\mathcal{F}^{(1)}_{\varepsilon_n}(v_n) < \infty$ for every n sufficiently large. Let $\{v_{n_k}\}_k$ be as in Proposition 4.2.1. Then $v_n \to u_0$ in $H^1(\Omega)$ and from (4.2.5) we deduce that

$$\liminf_{n\to\infty} \mathcal{F}^{(1)}_{\varepsilon_n}(v_n) \ge -\frac{1}{2}\int_{\partial\Omega} (\partial_\nu u_0)^2 \, d\mathcal{H}^{N-1} = \mathcal{F}_1(u_0).$$

On the other hand, for every $v \in L^2(\Omega) \setminus \{u_0\}$ the constant sequence $v_n = v$ is a recovery sequence. If now $v = u_0$, since by assumption $\partial_\nu u_0 \in H^{1/2}(\partial\Omega)$, we can find $w \in H^1(\Omega)$ such that $w = -\partial_\nu u_0$ on $\partial\Omega$, where the equality holds in the sense of traces. Set $v_n := u_0 + \varepsilon_n w$. Then $v_n \to u_0$ in $H^1(\Omega)$ and again from (4.2.5) it follows that

$$\lim_{n\to\infty} \mathcal{F}^{(1)}_{\varepsilon_n}(v_n) = \lim_{n\to\infty} \frac{\varepsilon_n}{2} \int_\Omega |\nabla w|^2 \, d\boldsymbol{x} - \frac{1}{2} \int_{\partial\Omega} (\partial_\nu u_0)^2 \; d\mathcal{H}^{N-1} = \mathcal{F}_1(u_0).$$

This concludes the proof of the Γ-convergence. The energy expansion (4.2.7) follows from Theorem 2.1.7. $\square$

4.2.2 The non-mixed problem: Gamma-convergence of order two

In this section we prove a second order asymptotic expansion for $\mathcal{F}_\varepsilon$. As customary, we begin by investigating the compactness properties of sequences with bounded energy.

Proposition 4.2.3 (Compactness)**.** *Let Ω be as in (H_1), $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$, $\mathcal{F}_\varepsilon$, $\mathcal{F}_0$, $\mathcal{F}^{(1)}_\varepsilon$, and $\mathcal{F}_1$ be as in* (1.2.7), (1.2.8), (4.2.1), *and* (4.2.6), *respectively, and define*

$$\mathcal{F}^{(2)}_\varepsilon := \frac{\mathcal{F}^{(1)}_\varepsilon - \min \mathcal{F}_1}{\varepsilon} = \frac{\mathcal{F}_\varepsilon - \min \mathcal{F}_0 - \varepsilon \min \mathcal{F}_1}{\varepsilon^2}.$$

If $\varepsilon_n \to 0^+$ and $w_n \in L^2(\Omega)$ are such that

$$\sup\{\mathcal{F}^{(2)}_{\varepsilon_n}(w_n) : n \in \mathbb{N}\} < \infty,$$

then $w_n \to u_0$ in $H^1(\Omega)$ and there exist a subsequence $\{w_{n_k}\}_k$ of $\{w_n\}_n$, $w_0 \in H^1(\Omega)$ and $q_0 \in L^2(\partial\Omega)$ such that

$$\frac{w_{n_k} - u_0}{\varepsilon_{n_k}} \rightharpoonup w_0 \quad \text{in } H^1(\Omega),$$

$$\frac{w_{n_k} - u_0 + \varepsilon_{n_k}\partial_\nu u_0}{\varepsilon_{n_k}^{3/2}} \rightharpoonup q_0 \quad \text{in } L^2(\partial\Omega),$$

where u_0 is the solution to (1.2.35)*. In particular, $w_0 = -\partial_\nu u_0$ on $\partial\Omega$ in the sense of traces.*

Proof. By Corollary 4.1.4, we deduce that $w_n \to u_0$ in $H^1(\Omega)$. For every $n \in \mathbb{N}$, let $r_n \in L^2(\Omega)$ be such that $w_n = u_0 + \varepsilon_n r_n$. Then $\mathcal{F}^{(2)}_{\varepsilon_n}(w_n)$ can be rewritten as

$$\mathcal{F}^{(2)}_{\varepsilon_n}(w_n) = \frac{1}{2} \int_\Omega |\nabla r_n|^2 \, d\boldsymbol{x} + \frac{1}{2\varepsilon_n} \int_{\partial\Omega} (r_n + \partial_\nu u_0)^2 \, d\mathcal{H}^{N-1}. \tag{4.2.8}$$

We then proceed as in the proof of Proposition 4.1.3 with $f = 0$, $g = -\partial_\nu u_0$ and r_n in place of u_n. $\square$

Theorem 4.2.4 (2nd order Γ-convergence)**.** *Under the assumptions of Proposition 4.2.3, let $u_1 \in H^1(\Omega)$ be the unique solution to the Dirichlet problem*

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega, \\ u_1 = -\partial_\nu u_0 & \text{on } \partial\Omega. \end{cases}$$

Then the family $\{\mathcal{F}_\varepsilon^{(2)}\}_\varepsilon$ Γ*-converges in* $L^2(\Omega)$ *to the functional*

$$\mathcal{F}_2(v) \coloneqq \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u_1|^2\, d\boldsymbol{x} & \text{if } v = u_0, \\ +\infty & \text{otherwise}. \end{cases}$$

In particular, if $u_\varepsilon \in H^1(\Omega)$ *is the solution to* (1.2.34)*, then*

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}_0(u_0) + \varepsilon\mathcal{F}_1(u_0) + \varepsilon^2\mathcal{F}_2(u_0) + o\left(\varepsilon^2\right). \tag{4.2.9}$$

Proof. Let $\varepsilon_n \to 0^+$ and $\{w_n\}_n$ be a sequence of functions in $L^2(\Omega)$ such that $w_n \to w$ in $L^2(\Omega)$. Reasoning as in the proof of Theorem 1.2.3, we can assume without loss of generality that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}^{(2)}(w_n) = \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}^{(2)}(w_n) < \infty.$$

In particular, $\mathcal{F}_{\varepsilon_n}^{(2)}(w_n) < \infty$ for every n sufficiently large. Let $\{w_{n_k}\}_k$ and w_0 be as in Proposition 4.2.3. Then $w_n \to u_0$ in $H^1(\Omega)$ and from (4.2.8) we deduce that

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}^{(2)}(w_{n_k}) &\geq \liminf_{k\to\infty} \frac{1}{2}\int_\Omega |\nabla r_{n_k}|^2\, d\boldsymbol{x} \geq \frac{1}{2}\int_\Omega |\nabla w_0|^2\, d\boldsymbol{x} \\ &\geq \inf\left\{\frac{1}{2}\int_\Omega |\nabla p|^2\, d\boldsymbol{x} : p \in H^1(\Omega),\ p = -\partial_\nu u_0 \text{ on } \partial\Omega\right\} \\ &= \frac{1}{2}\int_\Omega |\nabla u_1|^2\, d\boldsymbol{x} = \mathcal{F}_2(u_0).\end{aligned}$$

We remark that the function u_1 exists (and is unique) by an application of Corollary 4.1.4.

On the other hand, for every $w \in L^2(\Omega) \setminus \{u_0\}$ the constant sequence $w_n = w$ is a recovery sequence. As one can check from (4.2.8), $w_n \coloneqq u_0 + \varepsilon_n u_1$ is a recovery sequence for u_0. This concludes the proof of the Γ-convergence. The energy expansion (4.2.9) follows from Theorem 2.1.7. □

Corollary 4.2.5. *Let* Ω *be as in* (H_1)*,* $f \in L^2(\Omega)$*,* $g \in H^{3/2}(\partial\Omega)$*, and let* u_ε *and* u_0 *be solutions to* (1.2.34) *and* (1.2.35)*, respectively. Then there exists a constant* $c > 0$*, independent of* ε*, such that*

$$\|u_\varepsilon - u_0\|_{H^1(\Omega)} \leq c\varepsilon\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)}\right),$$

$$\|u_\varepsilon - u_0 + \varepsilon\partial_\nu u_0\|_{L^2(\partial\Omega)} \leq c\varepsilon^{3/2}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)}\right).$$

Proof. If we let $w_\varepsilon \coloneqq u_0 + \varepsilon u_1$, for u_1 as in Theorem 4.2.4, then

$$\mathcal{F}_\varepsilon(w_\varepsilon) = \mathcal{F}_0(u_0) + \varepsilon\mathcal{F}_1(u_0) + \varepsilon^2\mathcal{F}_2(u_0)$$

and from the minimality of u_ε we deduce

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq \mathcal{F}_0(u_0) + \varepsilon\mathcal{F}_1(u_0) + \varepsilon^2\mathcal{F}_2(u_0).$$

Writing $r_\varepsilon := \frac{u_\varepsilon - u_0}{\varepsilon}$, expanding, and rearranging the terms in the previous inequality we arrive at

$$\frac{1}{2}\int_\Omega |\nabla r_\varepsilon|^2 \, d\boldsymbol{x} + \frac{1}{2\varepsilon}\int_{\partial\Omega} (r_\varepsilon + \varepsilon \partial_\nu u_0)^2 \, d\mathcal{H}^{N-1} \le \frac{\varepsilon^2}{2}\int_\Omega |\nabla u_1|^2 \, d\boldsymbol{x}. \tag{4.2.10}$$

Since $\partial\Omega$ is of class $C^{1,1}$, $f \in L^2(\Omega)$, and $g \in H^{3/2}(\partial\Omega)$, by standard elliptic estimates (see Theorem 2.4.2.5 in [58]) the solution $u_0 \in H^1(\Omega)$ to the Dirichlet problem (1.2.35) belongs to $H^2(\Omega)$ with

$$\|u_0\|_{H^2(\Omega)} \le k_1 \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\Omega)} \right).$$

In turn, by standard trace theorems (see Theorem 18.40 in [70]), we have that $\partial_\nu u_0 \in H^{1/2}(\partial\Omega)$, and so there is $z_0 \in H^1(\Omega)$ such that $z_0 = -\partial_\nu u_0$ on $\partial\Omega$ in the sense of traces and

$$\|z_0\|_{H^1(\Omega)} \le k_2 \|\partial_\nu u_0\|_{H^{1/2}(\partial\Omega)} \le k_3 \|u_0\|_{H^2(\Omega)} \le c \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\Omega)} \right).$$

Since $u_1 \in H^1(\Omega)$ is a minimizer of

$$v \mapsto \int_\Omega |\nabla v|^2 \, d\boldsymbol{x}$$

over all functions v with $v = -\partial_\nu u_0$ on $\partial\Omega$, we have that

$$\|\nabla u_1\|_{L^2(\Omega;\mathbb{R}^N)} \le \|\nabla z_0\|_{L^2(\Omega;\mathbb{R}^N)} \le c \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\Omega)} \right).$$

The previous estimate, together with (4.2.10), gives the desired result. □

4.2.3 The non-mixed problem: Gamma-convergences of all orders

In this section we prove asymptotic expansions by Γ-convergence of any order for $\mathcal{F}_\varepsilon$ and derive asymptotic expansions for u_ε, i.e., the solution to (1.2.34).

Theorem 4.2.6. *Given $k \in \mathbb{N}$, let $j \in \mathbb{N}$ be such that $k = 2j-1$ or $k = 2j$, Ω be as in (H_j), $f \in L^2(\Omega)$, $g \in H^{1/2+j}(\partial\Omega)$, and for every $m \in \{1, \dots, j\}$ let $u_m \in H^1(\Omega)$ be the solution to the Dirichlet problem*

$$\begin{cases} \Delta u_m = 0 & \text{in } \Omega, \\ u_m = -\partial_\nu u_{m-1} & \text{on } \partial\Omega, \end{cases} \tag{4.2.11}$$

where u_0 is the solution to (1.2.35). *Let $\mathcal{F}_\varepsilon^{(k+1)}$ be defined recursively by*

$$\mathcal{F}_\varepsilon^{(k+1)} := \frac{\mathcal{F}_\varepsilon^{(k)} - \mathcal{F}_k(u_0)}{\varepsilon},$$

where $\mathcal{F}_\varepsilon^{(1)}$ is given as in (4.2.1) *and the functionals $\mathcal{F}_i$, for $i \in \{1, \dots, k+1\}$, are given by*

$$\mathcal{F}_{2m+1}(v) := \begin{cases} -\dfrac{1}{2}\displaystyle\int_{\partial\Omega} (\partial_\nu u_m)^2 \, d\mathcal{H}^{N-1} & \text{if } v = u_0, \\ +\infty & \text{otherwise}, \end{cases}$$

and

$$\mathcal{F}_{2m}(v) := \begin{cases} +\dfrac{1}{2}\displaystyle\int_{\Omega} |\nabla u_m|^2 \, d\boldsymbol{x} & \text{if } v = u_0, \\ +\infty & \text{otherwise}. \end{cases}$$

Then the family $\{\mathcal{F}_\varepsilon^{(i)}\}_\varepsilon$ Γ*-converges in* $L^2(\Omega)$ *to the functional* $\mathcal{F}_i$ *for every* $i \in \{2, \dots, k+1\}$*. In particular, if* $u_\varepsilon \in H^1(\Omega)$ *is the solution to* (1.2.34)*, then*

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \sum_{i=0}^{k+1} \varepsilon^i \mathcal{F}_i(u_0) + o\left(\varepsilon^{k+1}\right).$$

Proof. Notice that for $k = 1$ we have that $j = 1$ and so the statement reduces to the one of Theorem 4.2.4. The result for $k \geq 2$ follows by induction from arguments similar to the ones of Theorem 4.2.2 and Theorem 4.2.4 (depending on the parity of k). We omit the details. □

Corollary 4.2.7. *Under the assumptions of Theorem 4.2.6, and for an odd value of* $k \in \mathbb{N}$*, let* u_ε*,* u_0*,* u_i *be solutions to* (1.2.34)*,* (1.2.35)*, and* (4.2.11)*, respectively. Then there exists a constant* $c > 0$*, independent of* ε*, such that for every* $j \in \{1, \dots, (k+1)/2\}$

$$\left\| u_\varepsilon - \sum_{i=0}^{j-1} \varepsilon^i u_i \right\|_{H^1(\Omega)} \leq C\varepsilon^j \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2+j}(\Omega)} \right),$$

$$\left\| u_\varepsilon - \sum_{i=0}^{j-1} \varepsilon^i u_i + \varepsilon \partial_\nu u_j \right\|_{L^2(\partial\Omega)} \leq C\varepsilon^{1/2+j} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2+j}(\Omega)} \right).$$

Proof. The proof is analogous to the one of Corollary 4.2.5 and therefore we omit the details. □

4.3 The case of mixed boundary conditions

In this section we prove our main results regarding the higher order Γ-limits for the functional $\mathcal{F}_\varepsilon$ defined as in (1.2.7).

4.3.1 Some technical results

Throughout the section Ω is assumed to be as in the statement of Theorem 1.2.1. We recall that we use the following notations: given a function $v = v(\boldsymbol{x})$ where $\boldsymbol{x} = (x, y)$, we denote by $\bar{v}$ the function

$$\bar{v}(r, \theta) := v(r\cos\theta, r\sin\theta), \tag{4.3.1}$$

and with a slight abuse of notation we write $v = \bar{v}(r, \theta)$. Moreover, we denote by $\bar{v}^{(i)}$ the function

$$\bar{v}^{(i)}(r_i, \theta_i) := v(\boldsymbol{x}_i + r_i(\cos\theta_i, \sin\theta_i)), \tag{4.3.2}$$

where the polar coordinates (r_i, θ_i) are as in Theorem 1.2.1. Furthermore, recall that $\bar{\varphi} \in C^\infty([0, \infty))$ is such that $\bar{\varphi} \equiv 1$ in $[0, \rho/2]$ and $\bar{\varphi} \equiv 0$ outside $[0, \rho]$.

Proposition 4.3.1. *Let $N = 2$, Ω be as in Theorem 1.2.1, $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$, and let $u_0 \in H^1(\Omega)$ be the solution to* (1.2.1). *Then*

$$\int_\Omega (\nabla u_0 \cdot \nabla\psi + f\psi)\, d\boldsymbol{x} = \int_{\Gamma_D} \partial_\nu u^0_{\mathrm{reg}}\psi\, d\mathcal{H}^1 - \sum_{i=1}^{2} \frac{c_i}{2} \int_0^\rho \bar\varphi(r_i) r_i^{-1/2} \bar\psi^{(i)}(r_i, 0)\, dr_i$$

for every $\psi \in H^1(\Omega)$, where u^0_{reg}, c_i and $\bar\varphi$ are given as in Theorem 1.2.1.

Proof. By Theorem 1.2.1, given $\psi \in H^1(\Omega)$, we get

$$\begin{aligned}\int_\Omega \nabla u_0 \cdot \nabla\psi\, d\boldsymbol{x} &= \int_\Omega \nabla u^0_{\mathrm{reg}} \cdot \nabla\psi\, d\boldsymbol{x} \\ &\quad + \sum_{i=1}^{2} c_i \int_0^\pi \int_0^\rho \left(\partial_{r_i}\bar S_i \partial_{r_i}\bar\psi^{(i)} r_i^{-2} \partial_{\theta_i}\bar S_i \partial_{\theta_i}\bar\psi^{(i)}\right) r_i\, dr_i d\theta_i.\end{aligned} \tag{4.3.3}$$

Since the function u^0_{reg} belongs to $H^2(\Omega)$ and satisfies a homogenous Neumann boundary condition on Γ_N, the divergence theorem yields

$$\int_\Omega \nabla u^0_{\mathrm{reg}} \cdot \nabla\psi\, d\boldsymbol{x} = \int_\Omega -\Delta u^0_{\mathrm{reg}}\psi\, d\boldsymbol{x} + \int_{\Gamma_D} \partial_\nu u^0_{\mathrm{reg}}\psi\, d\mathcal{H}^1. \tag{4.3.4}$$

To rewrite the second term on the right-hand side of (4.3.3), we consider the auxiliary function

$$\bar\Phi(r_i, \theta_i) \coloneqq r_i \partial_{r_i}\bar S_i(r_i, \theta_i)\bar\psi^{(i)}(r_i, \theta_i);$$

indeed, a simple computation shows that $\bar\Phi \in W^{1,1}((0,\rho) \times (0,\pi))$ and thus $\bar\Phi(\cdot, \theta_i)$ is absolutely continuous for $\mathcal{L}^1$-a.e. $\theta_i \in (0,\pi)$. For any such θ_i, by the fundamental theorem of calculus, we have that

$$\begin{aligned}0 = \bar\Phi(\rho, \theta_i) - \bar\Phi(0, \theta_i) &= \int_0^\rho \partial_{r_i}\bar\Phi(r_i, \theta_i)\, dr_i \\ &= \int_0^\rho \left(\partial_{r_i}\bar S_i \bar\psi^{(i)} + r_i \partial^2_{r_i}\bar S_i \bar\psi^{(i)} + r_i \partial_{r_i}\bar S_i \partial_{r_i}\bar\psi^{(i)}\right) dr_i.\end{aligned} \tag{4.3.5}$$

Similarly, noticing that the function $\bar\Psi(r_i, \theta_i) \coloneqq r_i^{-1}\partial_{\theta_i}\bar S_i(r_i, \theta_i)\bar\psi^{(i)}(r_i, \theta_i)$ belongs to the space $W^{1,1}((0,\rho) \times (0,\pi))$, and reasoning as above we find that

$$\begin{aligned}-\frac{1}{2}\bar\varphi(r_i) r_i^{-1/2}\bar\psi^{(i)}(r_i, 0) = \bar\Psi(r_i, \pi) - \bar\Psi(r_i, 0) &= \int_0^\pi \partial_{\theta_i}\bar\Psi(r_i, \theta_i)\, d\theta_i \\ &= \int_0^\pi r_i^{-1}\left(\partial^2_{\theta_i}\bar S_i \bar\psi^{(i)} + \partial_{\theta_i}\bar S_i \partial_{\theta_i}\bar\psi^{(i)}\right) d\theta_i\end{aligned} \tag{4.3.6}$$

holds for $\mathcal{L}^1$-a.e. $r_i \in (0,\rho)$. Combining the identities (4.3.5) and (4.3.6), we get

$$\int_0^\pi \int_0^\rho \left(\partial_{r_i}\bar S_i \partial_{r_i}\bar\psi^{(i)} + r_i^{-2}\partial_{\theta_i}\bar S_i \partial_{\theta_i}\bar\psi^{(i)}\right) r_i\, dr_i d\theta_i$$

$$= -\frac{1}{2}\int_0^\rho \bar{\varphi}(r_i) r_i^{-1/2}\bar{\psi}^{(i)}(r_i,0)\,dr_i - \int_0^\pi\int_0^\rho \bar{\psi}^{(i)}\left(\partial^2_{r_i}\bar{S}_i + r_i^{-1}\partial_{r_i}\bar{S}_i + r_i^{-2}\partial^2_{\theta_i}\bar{S}_i\right) r_i\,dr_i d\theta_i$$

$$= -\frac{1}{2}\int_0^\rho \bar{\varphi}(r_i) r_i^{-1/2}\bar{\psi}^{(i)}(r_i,0)\,dr_i - \int_0^\pi\int_0^\rho \bar{\psi}^{(i)}\Delta_{(r_i,\theta_i)}\bar{S}_i r_i\,dr_i d\theta_i.$$

Consequently, the desired formula follows from the previous equality, (4.3.3), (4.3.4), and upon noticing that

$$\int_\Omega f\psi\,d\boldsymbol{x} = \int_\Omega \Delta u^0_{\mathrm{reg}}\psi\,d\boldsymbol{x} + \sum_{i=1}^2 c_i\int_0^\pi\int_0^\rho \bar{\psi}^{(i)}\Delta_{(r_i,\theta_i)}\bar{S}_i r_i\,dr_i d\theta_i.$$

This concludes the proof. □

In the following theorem we present an estimate that will prove instrumental for the proofs of our compactness results, namely Theorem 1.2.4 and Theorem 1.2.6.

Theorem 4.3.2. *There exists a constant κ such that for any $R > 0$ and $h \in H^1(B_R^+(\mathbf{0}))$,*

$$\int_0^R x^{-1/2}|h(x,0)|\,dx \le \kappa\left(R\int_{B_R^+(\mathbf{0})}|\nabla h(\boldsymbol{x})|^2\,d\boldsymbol{x}\right)^{1/2} + \kappa\left(\int_0^R h(x,0)^2\,dx\right)^{1/2},$$

where $h(\cdot,0)$ indicates the trace of h on the positive real axis.

We begin by adapting Theorem 1.2.9 to our framework.

Lemma 4.3.3. *There exists a constant $\overline{\kappa}$ such that for any $R > 0$ and $h \in H^1(B_R^+(\mathbf{0}))$,*

$$\int_{B_R^+(\mathbf{0})}\frac{h(\boldsymbol{x})^2}{|\boldsymbol{x}|^2(1+\log R - \log|\boldsymbol{x}|)^2}\,d\boldsymbol{x} \le \overline{\kappa}\left(\int_{B_R^+(\mathbf{0})}|\nabla h(\boldsymbol{x})|^2\,d\boldsymbol{x} + \frac{1}{R}\int_0^R h(x,0)^2\,dx\right),$$

where $h(\cdot,0)$ indicates the trace of h on the positive real axis.

Proof. Since $B_R^+(\mathbf{0})$ is an extension domain, we can find $\hat{h} \in H^1(B_R(\mathbf{0}))$ such that $\hat{h}(\boldsymbol{x}) = h(\boldsymbol{x})$ for $\mathcal{L}^2$-a.e. $\boldsymbol{x} \in B_R^+(\mathbf{0})$ and with the property that

$$\|\hat{h}\|_{L^2(B_R(\mathbf{0}))} \le C_1\|h\|_{L^2(B_R^+(\mathbf{0}))},$$

$$\|\nabla\hat{h}\|_{L^2(B_R(\mathbf{0});\mathbb{R}^2)} \le C_1\|\nabla h\|_{L^2(B_R^+(\mathbf{0});\mathbb{R}^2)},$$

for some constant $C_1 > 0$ independent of R. Theorem 1.2.9 applied to the function $\hat{h}$ and the previous estimates yield

$$\int_{B_R^+(\mathbf{0})}\frac{h(\boldsymbol{x})^2}{|\boldsymbol{x}|^2(1+\log R - \log|\boldsymbol{x}|)^2}\,d\boldsymbol{x} \le C_2\left(\int_{B_R^+(\mathbf{0})}|\nabla h(\boldsymbol{x})|^2\,d\boldsymbol{x} + \frac{1}{R^2}\int_{B_R^+(\mathbf{0})}h(\boldsymbol{x})^2\,d\boldsymbol{x}\right),$$

for some constant $C_2 > 0$ independent of h and R. By Lemma 4.1.2, together with a simple rescaling argument, we deduce that

$$\frac{1}{R^2}\int_{B_R^+(\mathbf{0})}h(\boldsymbol{x})^2\,d\boldsymbol{x} \le C_3\left(\int_{B_R^+(\mathbf{0})}|\nabla h(\boldsymbol{x})|^2\,d\boldsymbol{x} + \frac{1}{R}\int_0^R h(x,0)^2\,dx\right)$$

for some constant $C_3 > 0$, which is again independent of both h and R. This concludes the proof. $\square$

Proof of Theorem 4.3.2. By the fundamental theorem of calculus,

$$\bar{h}(r,\theta) = \bar{h}(r,0) + \int_0^\theta \partial_\theta \bar{h}(r,\alpha)\,d\alpha,$$

and so, multiplying both sides by $r^{-1/2}$ and integrating over $B_R^+(\mathbf{0})$, we get

$$\begin{aligned}\int_0^R r^{-1/2}\bar{h}(r,0)\,dr &= \frac{1}{\pi}\int_0^\pi \int_0^R r^{-1/2}\bar{h}(r,\theta)\,drd\theta - \frac{1}{\pi}\int_0^\pi \int_0^R \int_0^\theta r^{-1/2}\partial_\theta \bar{h}(r,\alpha)\,d\alpha drd\theta \\ &= \frac{1}{\pi}\int_0^\pi \int_0^R r^{-1/2}\bar{h}(r,\theta)\,drd\theta - \int_0^\pi \int_0^R \frac{(\pi-\theta)}{\pi} r^{-1/2}\partial_\theta \bar{h}(r,\theta)\,drd\theta,\end{aligned}$$

where the last equality follows from Fubini's theorem. In particular,

$$\int_0^R r^{-1/2}|\bar{h}(r,0)|\,dr \le \frac{1}{\pi}\int_0^\pi \int_0^R r^{-1/2}|\bar{h}(r,\theta)|\,drd\theta + \int_0^\pi \int_0^R r^{-1/2}|\partial_\theta \bar{h}(r,\theta)|\,drd\theta, \tag{4.3.7}$$

and thus we proceed to estimate the terms on the right-hand side of (4.3.7). Passing to cartesian coordinates,

$$\begin{aligned}\int_0^\pi \int_0^R r^{-1/2}|\bar{h}(r,\theta)|\,drd\theta &= \int_{B_R^+(\mathbf{0})} \frac{|h(\boldsymbol{x})|}{|\boldsymbol{x}|\,(1+\log R - \log|\boldsymbol{x}|)} \frac{(1+\log R-\log|\boldsymbol{x}|)}{|\boldsymbol{x}|^{1/2}}\,d\boldsymbol{x} \\ &\le (5\pi R)^{1/2}\left(\int_{B_R^+(\mathbf{0})} \frac{h(\boldsymbol{x})^2}{|\boldsymbol{x}|^2\,(1+\log R-\log|\boldsymbol{x}|)^2}\,d\boldsymbol{x}\right)^{1/2},\end{aligned}$$

where in the last step we have used Hölder's inequality together with the fact that

$$\int_{B_R^+(\mathbf{0})} \frac{(1+\log R-\log|\boldsymbol{x}|)^2}{|\boldsymbol{x}|} d\boldsymbol{x} = \pi\int_0^R (1+\log R-\log r)^2\,dr = 5\pi R.$$

Then, from Lemma 4.3.3 we deduce that

$$\begin{aligned}&\int_0^\pi \int_0^R r^{-1/2}|\bar{h}(r,\theta)|\,drd\theta \\ &\qquad \le (5\pi\overline{\kappa})^{1/2}\left(R\int_{B_R^+(\mathbf{0})} |\nabla h(\boldsymbol{x})|^2\,d\boldsymbol{x} + \int_0^R h(x,0)^2\,dx\right)^{1/2}.\end{aligned} \tag{4.3.8}$$

On the other hand, Hölder's inequality yields

$$\begin{aligned}\int_0^\pi \int_0^R r^{-1/2}|\partial_\theta \bar{h}(r,\theta)|\,drd\theta &\le \left(\pi R\int_0^\pi \int_0^R r^{-1}|\partial_\theta \bar{h}(r,\theta)|^2\,drd\theta\right)^{1/2} \\ &\le \left(\pi R\int_{B_R^+(\mathbf{0})} |\nabla h(\boldsymbol{x})|^2\,d\boldsymbol{x}\right)^{1/2},\end{aligned} \tag{4.3.9}$$

and so the desired inequality follows from (4.3.7), (4.3.8), and (4.3.9). $\square$

4.3.2 Mixed boundary conditions: Gamma-convergence of order one

In this section we prove Theorem 1.2.4 and Theorem 1.2.5. We recall that we use the notations (4.3.1) and (4.3.2).

Proof of Theorem 1.2.4. By Corollary 4.1.4 we have that $v_n \to u_0$ in $H^1(\Omega)$. For every $n \in \mathbb{N}$, let $z_n \in L^2(\Omega)$ be such that $v_n = u_0 + \varepsilon_n\sqrt{|\log \varepsilon_n|}z_n$. Then $\mathcal{F}^{(1)}_{\varepsilon_n}(v_n)$ can be rewritten as

$$\mathcal{F}^{(1)}_{\varepsilon}(v_n) = \frac{1}{\sqrt{|\log \varepsilon_n|}}\int_\Omega (\nabla u_0 \cdot \nabla z_n + f z_n)\, d\boldsymbol{x} + \frac{\varepsilon_n}{2}\int_\Omega |\nabla z_n|^2\, d\boldsymbol{x} + \frac{1}{2}\int_{\Gamma_D} z_n^2\, d\mathcal{H}^1,$$

and an application of Proposition 4.3.1 yields

$$\begin{aligned}\mathcal{F}^{(1)}_{\varepsilon_n}(v_n) = \frac{1}{\sqrt{|\log \varepsilon_n|}}&\left(\int_{\Gamma_D} \partial_\nu u^0_{\mathrm{reg}} z_n\, d\mathcal{H}^1 - \sum_{i=1}^2 \frac{c_i}{2}\int_0^\rho \bar{\varphi}(r_i) r_i^{-1/2}\bar{z}_n^{(i)}(r_i,0)\, dr_i\right)\\ &+ \frac{\varepsilon_n}{2}\int_\Omega |\nabla z_n|^2\, d\boldsymbol{x} + \frac{1}{2}\int_{\Gamma_D} z_n^2\, d\mathcal{H}^1.\end{aligned} \tag{4.3.10}$$

For n large enough so that $2\varepsilon_n \le \rho$, we write

$$\int_0^\rho \bar{\varphi}(r_i) r_i^{-1/2}\bar{z}_n^{(i)}(r_i,0)\, dr_i = \int_0^{\varepsilon_n} r_i^{-1/2}\bar{z}_n^{(i)}(r_i,0)\, dr_i + \int_{\varepsilon_n}^\rho \bar{\varphi}(r_i) r_i^{-1/2}\bar{z}_n^{(i)}(r_i,0)\, dr_i \tag{4.3.11}$$

and proceed to estimate both terms on the right-hand side separately. By Theorem 4.3.2 we obtain

$$\begin{aligned}\int_0^{\varepsilon_n} r_i^{-1/2}|\bar{z}_n^{(i)}(r_i,0)|\, dr_i \le \kappa &\left(\varepsilon_n \int_{B_{\varepsilon_n}(\boldsymbol{x}_i)\cap\Omega} |\nabla z_n|^2\, d\boldsymbol{x}\right)^{1/2}\\ &+ \kappa\left(\int_0^{\varepsilon_n} \bar{z}_n^{(i)}(r_i,0)^2\, dr_i\right)^{1/2},\end{aligned} \tag{4.3.12}$$

while by Hölder's inequality we get

$$\int_{\varepsilon_n}^\rho \bar{\varphi}(r_i) r_i^{-1/2}|\bar{z}_n^{(i)}(r_i,0)|\, dr_i \le \sqrt{\log\rho + |\log \varepsilon_n|}\left(\int_{\varepsilon_n}^\rho \bar{z}_n^{(i)}(r_i,0)^2\, dr_i\right)^{1/2}. \tag{4.3.13}$$

Consequently, from (4.3.10), (4.3.12), and (4.3.13) we deduce that

$$\begin{aligned}\mathcal{F}^{(1)}_{\varepsilon_n}(v_n) \ge \frac{1}{2}\|z_n\|^2_{L^2(\Gamma_D)} &- \left(\frac{\|\partial_\nu u^0_{\mathrm{reg}}\|_{L^2(\Gamma_D)}}{\sqrt{|\log \varepsilon_n|}} + \frac{|c_i|(\kappa + \sqrt{\log\rho + |\log \varepsilon_n|})}{2\sqrt{|\log \varepsilon_n|}}\right)\|z_n\|_{L^2(\Gamma_D)}\\ &+ \frac{1}{2}\|\varepsilon_n^{1/2}\nabla z_n\|^2_{L^2(\Omega;\mathbb{R}^2)} - \frac{|c_i|\kappa}{2\sqrt{|\log \varepsilon_n|}}\|\varepsilon_n^{1/2}\nabla z_n\|_{L^2(\Omega;\mathbb{R}^2)},\end{aligned}$$

and so (1.2.11) and (1.2.12) are proved at once. □

Proof of Theorem 1.2.5. **Step 1:** Let $\varepsilon_n \to 0^+$ and $\{v_n\}_n$ be a sequence of functions in $L^2(\Omega)$ such that $v_n \to v$ in $L^2(\Omega)$. Reasoning as in the proof of Theorem 1.2.3, we can assume without loss of generality that

$$\liminf_{n\to\infty} \mathcal{F}^{(1)}_{\varepsilon_n}(v_n) = \lim_{n\to\infty} \mathcal{F}^{(1)}_{\varepsilon_n}(v_n) < \infty.$$

In particular, $\mathcal{F}^{(1)}_{\varepsilon_n}(v_n) < \infty$ for every n sufficiently large. Let $\{v_{n_k}\}_k$ be a subsequence of $\{v_n\}_n$ given as in Theorem 1.2.4 and define

$$\bar{\xi}^{(i)}_n(r_i) \coloneqq \frac{c_i}{2\sqrt{|\log \varepsilon_n|}}\bar{\varphi}(r_i) r_i^{-1/2}. \tag{4.3.14}$$

Arguing as in the proof of Theorem 1.2.4 (see (4.3.10) and (4.3.12)) we arrive at

$$\begin{aligned}
\mathcal{F}^{(1)}_{\varepsilon_{n_k}}(v_{n_k}) \geq{}& \frac{1}{2}\|z_{n_k}\|^2_{L^2(\Gamma_D)} - \left(\frac{\|\partial_\nu u^0_{\mathrm{reg}}\|_{L^2(\Gamma_D)}}{\sqrt{|\log \varepsilon_{n_k}|}} + \frac{|c_i|\kappa}{2\sqrt{|\log \varepsilon_{n_k}|}}\right)\|z_{n_k}\|_{L^2(\Gamma_D)} \\
&- \frac{|c_i|\kappa}{2\sqrt{|\log \varepsilon_{n_k}|}}\|\varepsilon^{1/2}_{n_k}\nabla z_{n_k}\|_{L^2(\Omega;\mathbb{R}^2)} + \frac{1}{2}\|\varepsilon^{1/2}_{n_k}\nabla z_{n_k}\|^2_{L^2(\Omega;\mathbb{R}^2)} \\
&- \sum_{i=1}^{2}\int_{\varepsilon_{n_k}}^{\rho} \bar{\xi}^{(i)}_{n_k}(r_i)\bar{z}^{(i)}_{n_k}(r_i,0)\,dr_i.
\end{aligned} \tag{4.3.15}$$

Then, as $k \to \infty$, we have

$$\begin{aligned}
\liminf_{k\to\infty} \mathcal{F}^{(1)}_{\varepsilon_{n_k}}(v_{n_k}) \geq{}& \liminf_{k\to\infty} \sum_{i=1}^{2}\int_{\varepsilon_{n_k}}^{\rho}\left(\frac{1}{2}\bar{z}^{(i)}_{n_k}(r_i,0)^2 - \bar{\xi}^{(i)}_{n_k}(r_i)\bar{z}^{(i)}_{n_k}(r_i,0)\right)dr_i \\
={}& \liminf_{k\to\infty}\sum_{i=1}^{2}\left[\frac{1}{2}\int_{\varepsilon_{n_k}}^{\rho}\left(\bar{z}^{(i)}_{n_k}(r_i,0) - \bar{\xi}^{(i)}_{n_k}(r_i)\right)^2 dr_i - \frac{1}{2}\int_{\varepsilon_{n_k}}^{\rho}\bar{\xi}^{(i)}_{n_k}(r_i)^2\,dr_i\right] \\
\geq{}& -\frac{1}{2}\sum_{i=1}^{2}\liminf_{k\to\infty}\int_{\varepsilon_{n_k}}^{\rho}\bar{\xi}^{(i)}_{n_k}(r_i)^2\,dr_i \\
={}& -\frac{1}{8}\sum_{i=1}^{2} c_i^2 \liminf_{k\to\infty}\frac{1}{|\log \varepsilon_{n_k}|}\int_{\varepsilon_{n_k}}^{\rho}\bar{\varphi}(r_i)^2 r_i^{-1}\,dr_i \\
\geq{}& -\frac{1}{8}\sum_{i=1}^{2} c_i^2 \liminf_{k\to\infty}\frac{1}{|\log \varepsilon_{n_k}|}\left(\log\rho + |\log \varepsilon_{n_k}|\right) = -\frac{1}{8}\sum_{i=1}^{2} c_i^2,
\end{aligned} \tag{4.3.16}$$

where in the second to last step we have used (4.3.14).
Step 2: For every $v \in L^2(\Omega)\setminus\{u_0\}$, the constant sequence $v_n = v$ is a recovery sequence. Then let $v = u_0$ and consider the radial function $\zeta_{i,n}$ given in polar coordinates at $\boldsymbol{x}_i$ by

$$\bar{\zeta}_{i,n}(r_i) \coloneqq \bar{\xi}^{(i)}_n(r_i)\left(1 - \bar{\varphi}\left(\frac{\rho}{\varepsilon_n}r_i\right)\right) = \frac{c_i}{2\sqrt{|\log \varepsilon_n|}}\bar{\varphi}(r_i)\left(1 - \bar{\varphi}\left(\frac{\rho}{\varepsilon_n}r_i\right)\right) r_i^{-1/2}, \tag{4.3.17}$$

where $\bar{\xi}_n^{(i)}$ is the function defined in (4.3.14). We define

$$z_n(\boldsymbol{x}) \coloneqq \begin{cases} \zeta_{i,n}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in B_r(\boldsymbol{x}_i) \cap \Omega \text{ with } r < \rho, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.18}$$

Notice that if we let

$$\bar{\Psi}_{i,n}(r_i) \coloneqq \bar{\varphi}(r_i) \left(1 - \bar{\varphi}\left(\frac{\rho}{\varepsilon_n} r_i\right)\right),$$

then $\bar{\Psi}_{i,n} \colon \mathbb{R}^+ \to [0,1]$ and satisfies

$$\begin{cases} \bar{\Psi}_{i,n}(r_i) = 1 & \text{if } \varepsilon_n \leq r_i < \rho/2, \\ \bar{\Psi}_{i,n}(r_i) = 0 & \text{if } 0 \leq r_i \leq \varepsilon_n/2 \text{ or } \rho \leq r, \\ |\bar{\Psi}'_{i,n}(r_i)| \leq \frac{c}{\varepsilon_n} & \text{if } \varepsilon_n/2 \leq r_i \leq \varepsilon_n, \\ |\bar{\Psi}'_{i,n}(r_i)| \leq c & \text{if } \rho/2 \leq r_i \leq \rho, \end{cases} \tag{4.3.19}$$

for some constant $c > 0$ independent of n. Finally, set

$$v_n \coloneqq u_0 + \varepsilon_n \sqrt{|\log \varepsilon_n|} z_n.$$

Notice that $v_n \to u_0$ in $L^2(\Omega)$ since the sequence $\{z_n\}_n$ is uniformly bounded in $L^2(\Omega)$, indeed

$$\int_\Omega z_n^2 \, d\boldsymbol{x} \leq \sum_{i=1}^2 \frac{c_i^2 \pi}{4|\log \varepsilon_n|} \int_{\varepsilon_n/2}^\rho r_i^{-1} \, dr_i = \frac{\pi(\log \rho + |\log \varepsilon_n| + \log 2)}{4|\log \varepsilon_n|} \sum_{i=1}^2 c_i^2.$$

Next, we claim that $\varepsilon_n^{1/2} \nabla z_n \to \mathbf{0}$ in $L^2(\Omega; \mathbb{R}^2)$. Indeed, using the notation above we have that

$$\bar{\zeta}_{i,n}(r_i) = \frac{c_i}{2\sqrt{|\log \varepsilon_n|}} \bar{\Psi}_{i,n}(r_i) r_i^{-1/2},$$

and therefore

$$\begin{aligned} \varepsilon_n \int_\Omega |\nabla z_n|^2 \, d\boldsymbol{x} &= \frac{\varepsilon_n}{|\log \varepsilon_n|} \left(\sum_{i=1}^2 \frac{c_i^2 \pi}{4}\right) \int_0^\rho \left(\bar{\Psi}'_{i,n}(r_i) r_i^{-1/2} - \frac{1}{2} r_i^{-3/2} \bar{\Psi}_{i,n}(r_i)\right)^2 r_i \, dr_i \\ &\leq \frac{\varepsilon_n}{|\log \varepsilon_n|} \left(\sum_{i=1}^2 \frac{c_i^2 \pi}{2}\right) \int_0^\rho \left(\bar{\Psi}'_{i,n}(r_i)^2 + \frac{1}{4} r_i^{-2} \bar{\Psi}_{i,n}(r_i)^2\right) dr_i. \end{aligned} \tag{4.3.20}$$

From (4.3.19) we see that

$$\int_0^\rho \bar{\Psi}'_{i,n}(r_i)^2 \, dr_i = \int_{\varepsilon_n/2}^{\varepsilon_n} \bar{\Psi}'_{i,n}(r_i)^2 \, dr_i + \int_{\rho/2}^\rho \bar{\Psi}'_{i,n}(r_i)^2 \, dr_i \leq c^2 \left(\frac{1}{2\varepsilon_n} + \frac{\rho}{2}\right) \tag{4.3.21}$$

and

$$\int_0^\rho r_i^{-2} \bar{\Psi}_{i,n}(r_i)^2 \, dr_i \leq \int_{\varepsilon_n/2}^\rho r_i^{-2} \, dr_i = \frac{2}{\varepsilon_n} - \frac{1}{\rho}. \tag{4.3.22}$$

Combining (4.3.20) with the estimates (4.3.21) and (4.3.22) we obtain

$$\varepsilon_n \int_\Omega |\nabla z_n|^2 \, d\boldsymbol{x} \le \frac{\varepsilon_n}{|\log \varepsilon_n|} \left(\sum_{i=1}^2 \frac{c_i^2 \pi}{2} \right) \left(\frac{c^2}{2\varepsilon_n} + \frac{c^2 \rho}{2} + \frac{1}{2\varepsilon_n} - \frac{1}{4\rho} \right) \to 0 \tag{4.3.23}$$

and the claim is proved. From (4.3.10), using (4.3.11), (4.3.12), (4.3.13), and (4.3.14) we have

$$\begin{aligned}
\mathcal{F}^{(1)}_{\varepsilon_n}(v_n) \le{} & \frac{1}{2}\|z_n\|^2_{L^2(\Gamma_D)} + \left(\frac{\|\partial_\nu u^0_{\mathrm{reg}}\|_{L^2(\Gamma_D)}}{\sqrt{|\log \varepsilon_n|}} + \frac{|c_i|\kappa}{2\sqrt{|\log \varepsilon_n|}} \right) \|z_n\|_{L^2(\Gamma_D)} \\
& + \frac{1}{2}\|\varepsilon_n^{1/2} \nabla z_n\|^2_{L^2(\Omega;\mathbb{R}^2)} \frac{|c_i|\kappa}{2\sqrt{|\log \varepsilon_n|}} \|\varepsilon_n^{1/2} \nabla z_n\|_{L^2(\Omega;\mathbb{R}^2)} \\
& - \sum_{i=1}^2 \int_{\varepsilon_n}^{\rho} \bar{\xi}^{(i)}_n(r_i) \bar{\zeta}_{i,n}(r_i) \, dr_i,
\end{aligned} \tag{4.3.24}$$

By (4.3.23) we have that the second, third, and fourth member on the right-hand side of the previous inequality vanish as $n \to \infty$. Since $\bar{\varphi}\left(\frac{\rho}{\varepsilon_n} r_i\right) = 0$ for $r_i \in [\varepsilon_n, \rho]$, by (4.3.14) and (4.3.17),

$$\bar{\zeta}_{i,n} = \bar{\xi}^{(i)}_n \quad \text{ in } [\varepsilon_n, \rho]. \tag{4.3.25}$$

Consequently, from (4.3.14), (4.3.25), (4.3.18), and the fact that $\bar{\varphi} \equiv 1$ in $[0, \rho/2]$,

$$\begin{aligned}
\limsup_{n\to\infty} \mathcal{F}^{(1)}_{\varepsilon_n}(v_n) \le{} & \limsup_{n\to\infty} \left\{ \frac{1}{2}\|z_n\|^2_{L^2(\Gamma_D)} - \sum_{i=1}^2 \int_{\varepsilon_n}^{\rho} \bar{\xi}^{(i)}_n(r_i) \bar{\zeta}_{i,n}(r_i) \, dr_i \right\} \\
={} & \limsup_{n\to\infty} \sum_{i=1}^2 \int_{\varepsilon_n}^{\rho} \left(\frac{1}{2}\bar{\zeta}_{i,n}(r_i)^2 - \bar{\xi}^{(i)}_n(r_i) \bar{\zeta}_{i,n}(r_i) \right) dr_i \\
={} & \limsup_{n\to\infty} \sum_{i=1}^2 -\frac{1}{2} \int_{\varepsilon_n}^{\rho} \bar{\xi}^{(i)}_n(r_i)^2 \, dr_i \\
\le{} & -\frac{1}{8} \sum_{i=1}^2 c_i^2 \liminf_{n\to\infty} \frac{1}{|\log \varepsilon_n|} \left(\int_{\varepsilon_n}^{\rho/2} r_i^{-1} \, dr_i + \int_{\rho/2}^{\rho} \bar{\varphi}(r_i)^2 r_i^{-1} \, dr_i \right) \\
={} & -\frac{1}{8} \sum_{i=1}^2 c_i^2.
\end{aligned} \tag{4.3.26}$$

The energy expansion (1.2.14) follows from Theorem 2.1.7. □

4.3.3 An auxiliary variational problem

In this section we study the functional

$$\mathcal{J}_i(w) := \int_{\mathbb{R}^2_+} |\nabla w(\boldsymbol{x})|^2 \, d\boldsymbol{x} + \int_0^1 \left(w(x,0)^2 - c_i x^{-1/2} w(x,0) \right) dx$$

$$+\int_1^\infty \left(w(x,0) - \frac{c_i}{2}x^{-1/2}\right)^2 dx$$

defined in

$$H := \left\{ w \in H^1_{\mathrm{loc}}(\mathbb{R}^2_+) : w \in H^1(B_R^+(\mathbf{0})) \text{ for every } R > 0 \right\},$$

where $w(\cdot, 0)$ indicates the trace of w on the positive real axis. This functional appears in the characterization of the second order Γ-convergence of $\mathcal{F}_\varepsilon$ (see (1.2.15), (1.2.16), (1.2.17), Theorem 1.2.6, and Theorem 1.2.7).

Proposition 4.3.4. *Let $\mathcal{J}_i$ and H be given as above. Then $A_i := \inf\{\mathcal{J}_i(w) : w \in H\} \in \mathbb{R}$ and there exists $w_i \in H$ such that $\mathcal{J}_i(w_i) = A_i$. Furthermore, w_i is a weak solution to the mixed problem* (1.2.21).

Proof. Let v be the function given in polar coordinates by

$$\bar{v}(r,\theta) := \begin{cases} \dfrac{c_i}{2\sqrt{r}} & \text{if } r > 1 \text{ and } 0 < \theta < \pi, \\[2ex] \dfrac{c_i}{2}\sqrt{r} & \text{if } r \leq 1 \text{ and } 0 < \theta < \pi, \end{cases}$$

where (r, θ) are polar coordinates centered at the origin of $\mathbb{R}^2$ and such that the set $\{(r, 0) : r > 0\}$ coincides with the positive real axis. Then $v \in H$ and $\mathcal{J}_i(v) < \infty$, indeed

$$\mathcal{J}_i(v) = \int_0^\pi \int_0^\infty r(\partial_r \bar{v})^2 \, dr d\theta + \int_0^1 \left(\bar{v}(r,0) - c_i \bar{v}(r,0)\right) dr = \frac{c_i^2(\pi - 3)}{8}.$$

In turn, this implies that $A_i < \infty$. On the other hand, by Theorem 4.3.2, we see that for every $w \in H$,

$$\mathcal{J}_i(w) \geq \int_{\mathbb{R}^2_+} |\nabla w(\boldsymbol{x})|^2 \, d\boldsymbol{x} + \int_0^1 w(x,0)^2 \, dx - |c_i|\kappa \left(\int_{B_1^+(\mathbf{0})} |\nabla w|^2 \, d\boldsymbol{x} \right)^{1/2}$$

$$- |c_i|\kappa \left(\int_0^1 w(x,0)^2 \, dx \right)^{1/2} + \int_1^\infty \left(w(x,0) - \frac{c_i}{2} x^{-1/2} \right)^2 dx,$$

and so $A_i > -\infty$. Furthermore, we deduce that for an infimizing sequence it must be the case that (eventually extracting a subsequence which we don't relabel)

$$\begin{aligned} \nabla w_n &\rightharpoonup \nabla w && \text{in } L^2(\mathbb{R}^2_+; \mathbb{R}^2), \\ w_n(\cdot, 0) &\rightharpoonup w(\cdot, 0) && \text{in } L^2((0,1) \times \{0\}), \\ w_n(\cdot, 0) - \frac{c_i}{2}x^{-1/2} &\rightharpoonup w(\cdot, 0) - \frac{c_i}{2}x^{-1/2} && \text{in } L^2((1,\infty) \times \{0\}), \end{aligned}$$

for some $w \in H$, where $w_n(\cdot, 0)$ and $w(\cdot, 0)$ indicate the trace of w_n and w on the positive real axis. To conclude, it is enough to show that $\mathcal{J}_i$ is lower semicontinuous for sequences converging as above. The lower semicontinuity is certainly true for the nonnegative terms in $\mathcal{J}_i$, thanks to Fatou's lemma. In order to pass to the limit in the remaining term we can argue as follows. First, we observe that by Lemma 4.1.2 $\{w_n\}_n$ in bounded in $H^1(B_1^+(\mathbf{0}))$ and in particular in $H^{1/2}((0,1)\times\{0\})$. Next, we recall that $H^{1/2}((0,1)\times\{0\})$ embeds continuously into $L^p((0,1)\times\{0\})$ for every $p\in[1,\infty)$. Consequently, up to the extraction of a further subsequence, we can assume that $w_n \rightharpoonup w$ in $L^p((0,1)\times\{0\})$, $p>2$. Therefore, we deduce that

$$\liminf_{n\to\infty}\int_0^1 x^{-1/2}w_n(x,0)\,dx = \int_0^1 x^{-1/2}w(x,0)\,dx.$$

This proves the existence of a global minimizer of $\mathcal{J}_i$ in H. The rest of proposition follows by considering variations of the functional $\mathcal{J}_i$; we omit the details. □

We remark that w_i doesn't necessarily belong to the space $L^2(\mathbb{R}^2_+)$, unless $c_i=0$, in which case $w_i\equiv 0$. In the following lemma we prove an estimate on the L^2-norm of global minimizers in an annulus that escapes to infinity. This estimate will be crucial for the construction of the recovery sequence for u_0 in the proof of Theorem 1.2.7.

Lemma 4.3.5. *Let $\varepsilon_n\to 0^+$ and w_i be given as in Proposition 4.3.4. Then*

$$\varepsilon_n^2\int_{B^+_{\rho/\varepsilon_n}(\mathbf{0})\setminus B^+_{\rho/2\varepsilon_n}(\mathbf{0})} w_i^2\,d\boldsymbol{x}\to 0$$

as $n\to\infty$.

Proof. By applying Lemma 4.1.2 and by a rescaling argument in $B_1^+(\mathbf{0})\setminus B^+_{1/2}(\mathbf{0})$ we can deduce that there exists a constant c, independent of n, such that

$$\int_{B^+_{\rho/\varepsilon_n}(\mathbf{0})\setminus B^+_{\rho/2\varepsilon_n}(\mathbf{0})} w^2\,d\boldsymbol{x}\le\frac{c}{\varepsilon_n^2}\left(\int_{B^+_{\rho/\varepsilon_n}(\mathbf{0})\setminus B^+_{\rho/2\varepsilon_n}(\mathbf{0})}|\nabla w|^2\,d\boldsymbol{x}+\varepsilon_n\int_{\rho/2\varepsilon_n}^{\rho/\varepsilon_n} w(x,0)^2\,dx\right)$$

for every $w\in H^1(B^+_{\rho/\varepsilon_n}(\mathbf{0})\setminus B^+_{\rho/2\varepsilon_n}(\mathbf{0}))$. If we apply the previous inequality to $w=\varepsilon_n w_i$ we obtain

$$\varepsilon_n^2\int_{B^+_{\rho/\varepsilon_n}(\mathbf{0})\setminus B^+_{\rho/2\varepsilon_n}(\mathbf{0})} w_i^2\,d\boldsymbol{x}\le c\left(\int_{B^+_{\rho/\varepsilon_n}(\mathbf{0})\setminus B^+_{\rho/2\varepsilon_n}(\mathbf{0})}|\nabla w_i|^2\,d\boldsymbol{x}+\varepsilon_n\int_{\rho/2\varepsilon_n}^{\rho/\varepsilon_n} w_i(x,0)^2\,dx\right).$$

The first term on the right-hand side vanishes as $n\to\infty$ since $\nabla w_i\in L^2(\mathbb{R}^2_+;\mathbb{R}^2)$, and the second term is shown to vanish by the following computation:

$$\begin{aligned}\varepsilon_n\int_{\rho/2\varepsilon_n}^{\rho/\varepsilon_n} w(x,0)^2\,dx &\le 2\varepsilon_n\int_{\rho/2\varepsilon_n}^{\rho/\varepsilon_n}\left(w_i(x,0)-\frac{c_i}{2}x^{-1/2}\right)^2dx+2\varepsilon_n\int_{\rho/2\varepsilon_n}^{\rho/\varepsilon_n}\frac{c_i^2}{4x}\,dx\\ &=2\varepsilon_n\int_{\rho/2\varepsilon_n}^{\rho/\varepsilon_n}\left(w_i(x,0)-\frac{c_i}{2}x^{-1/2}\right)^2dx+2\varepsilon_n\log 2\to 0\end{aligned}$$

since $w_i(\cdot,0)-\frac{c_i}{2}x^{-1/2}\in L^2((1,\infty))$. This concludes the proof. □

4.3.4 Mixed boundary conditions: Gamma-convergence of order two

In this section we prove Theorem 1.2.6 and Theorem 1.2.7. We recall that we use the notations (4.3.1) and (4.3.2).

Proof of Theorem 1.2.6. **Step 1:** By Corollary 4.1.4 we have that $w_n \to u_0$ in $H^1(\Omega)$. For every $n \in \mathbb{N}$, let $s_n \in L^2(\Omega)$ be such that

$$w_n = u_0 + \sqrt{\varepsilon_n} s_n. \tag{4.3.27}$$

Then, by (1.2.7), (1.2.8), (1.2.10), (1.2.13), and (1.2.23), $\mathcal{F}^{(2)}_{\varepsilon_n}(w_n)$ can be rewritten as

$$\mathcal{F}^{(2)}_{\varepsilon_n}(w_n) = \frac{1}{\sqrt{\varepsilon_n}} \int_\Omega (\nabla u_0 \cdot \nabla s_n + f s_n)\, d\boldsymbol{x} + \frac{1}{2}\int_\Omega |\nabla s_n|^2\, d\boldsymbol{x} + \frac{1}{2\varepsilon_n}\int_{\Gamma_D} s_n^2\, d\mathcal{H}^1 \\ + \frac{|\log \varepsilon_n|}{8} \sum_{i=1}^2 c_i^2,$$

and an application of Proposition 4.3.1 yields

$$\mathcal{F}^{(2)}_{\varepsilon}(w_n) = \frac{1}{\sqrt{\varepsilon_n}} \left(\int_{\Gamma_D} \partial_\nu u^0_{\mathrm{reg}} s_n \, d\mathcal{H}^1 - \sum_{i=1}^2 \frac{c_i}{2} \int_0^\rho \bar{\varphi}(r_i) r_i^{-1/2} \bar{s}_n^{(i)}(r_i, 0)\, dr_i \right) \\ + \frac{1}{2}\int_\Omega |\nabla s_n|^2 \, d\boldsymbol{x} + \frac{1}{2\varepsilon_n}\int_{\Gamma_D} s_n^2 \, d\mathcal{H}^1 + \frac{|\log\varepsilon_n|}{8}\sum_{i=1}^2 c_i^2.$$

Using the fact that $|\log \varepsilon_n| = \int_{\varepsilon_n}^1 r^{-1}\, dr$, grouping together the different contributions on $\Gamma_D \cap B_{\varepsilon_n}(\boldsymbol{x}_i)$, $\Gamma_D \cap (B_\rho(\boldsymbol{x}_i) \setminus B_{\varepsilon_n}(\boldsymbol{x}_i))$ and $\Gamma_D \setminus B_\rho(\boldsymbol{x}_i)$, and completing the squares we obtain

$$\mathcal{F}^{(2)}_{\varepsilon_n}(w_n) = \sum_{i=1}^2 \Bigg\{ \frac{1}{2}\int_{\varepsilon_n}^\rho \left(\frac{\bar{s}_n^{(i)}(r_i,0)}{\sqrt{\varepsilon_n}} + \overline{\partial_\nu u^0_{\mathrm{reg}}}^{(i)}(r_i,0) - \frac{c_i}{2}\bar{\varphi}(r_i) r_i^{-1/2} \right)^2 dr_i + B_{i,n} c_i + C_\varphi c_i^2 \\ + \int_0^{\varepsilon_n} \left(\overline{\partial_\nu u^0_{\mathrm{reg}}}^{(i)}(r_i,0) \frac{\bar{s}_n^{(i)}(r_i,0)}{\sqrt{\varepsilon_n}} - \frac{c_i}{2} r_i^{-1/2} \frac{\bar{s}_n^{(i)}(r_i,0)}{\sqrt{\varepsilon_n}} + \frac{\bar{s}_n^{(i)}(r_i,0)^2}{2\varepsilon_n} \right) dr_i \Bigg\} \\ + \frac{1}{2}\int_{\Gamma_D \setminus \bigcup_i B_\rho(\boldsymbol{x}_i)} \left(\frac{s_n}{\sqrt{\varepsilon_n}} + \partial_\nu u^0_{\mathrm{reg}} \right)^2 d\mathcal{H}^1 - \frac{1}{2}\int_{\Gamma_D \setminus \bigcup_i B_{\varepsilon_n}(\boldsymbol{x}_i)} \left(\partial_\nu u^0_{\mathrm{reg}}\right)^2 d\mathcal{H}^1 \\ + \frac{1}{2}\int_\Omega |\nabla s_n|^2\, d\boldsymbol{x},$$

where

$$B_{i,n} := \frac{1}{2}\int_{\varepsilon_n}^\rho \bar{\varphi}(r_i) r_i^{-1/2} \overline{\partial_\nu u^0_{\mathrm{reg}}}^{(i)}(r_i,0)\, dr_i, \tag{4.3.28}$$

and C_φ is given as in (1.2.19). Setting

$$z_n := s_n - \sqrt{\varepsilon_n} u_1, \tag{4.3.29}$$

where u_1 is the solution to (1.2.22), and using the fact that $u_1 = -\partial_\nu u^0_{\rm reg}$ on Γ_D we can rewrite the previous expression as

$$\begin{aligned}\mathcal{F}^{(2)}_{\varepsilon_n}(w_n) = \sum_{i=1}^{2}\Bigg\{&\frac{1}{2}\int_{\varepsilon_n}^{\rho}\left(\frac{\bar{z}^{(i)}_n(r_i,0)}{\sqrt{\varepsilon_n}} - \frac{c_i}{2}\bar{\varphi}(r_i)r_i^{-1/2}\right)^2 dr_i + B_{i,n}c_i + C_\varphi c_i^2\\ &+\frac{1}{2}\int_0^{\varepsilon_n}\left(\frac{\bar{z}^{(i)}_n(r_i,0)^2}{\varepsilon_n} - c_i r_i^{-1/2}\frac{\bar{z}^{(i)}_n(r_i,0)}{\sqrt{\varepsilon_n}}\right) dr_i\Bigg\} + \frac{1}{2}\int_{\Gamma_D\setminus\bigcup_i B_\rho(\boldsymbol{x}_i)}\frac{z_n^2}{\varepsilon_n}\,d\mathcal{H}^1\\ &-\frac{1}{2}\int_{\Gamma_D}\left(\partial_\nu u^0_{\rm reg}\right)^2 d\mathcal{H}^1 + \frac{1}{2}\int_\Omega |\nabla(z_n + \sqrt{\varepsilon_n}u_1)|^2\,d\boldsymbol{x}. \end{aligned}\tag{4.3.30}$$

Notice that all the terms in the previous expression are either positive or independent of n, with the only exception of $B_{i,n}c_i$, which converges to $B_i c_i$, and the fourth term on the right-hand side. However, by an application of Theorem 4.3.2 we get

$$-\int_0^{\varepsilon_n} c_i r_i^{-1/2}\frac{\bar{z}^{(i)}_n(r_i,0)}{\sqrt{\varepsilon_n}}\,dr_i \geq -|c_i|\kappa\left(\int_{B^+_{\varepsilon_n}(\boldsymbol{x}_i)}|\nabla z_n|^2\,d\boldsymbol{x}\right)^{1/2} - |c_i|\kappa\left(\int_0^{\varepsilon_n}\frac{\bar{z}^{(i)}_n(r_i,0)^2}{\varepsilon_n}\right)^{1/2},$$

and thus (1.2.25) and (1.2.26) are proved at once.
Step 2: Let $W_{i,n}$ be as in (1.2.24). Then

$$\bar{W}_{i,n}(r_i,\theta_i) = \bar{\varphi}(\varepsilon_n r_i)\bar{z}^{(i)}_n(\varepsilon_n r_i,\theta_i)\tag{4.3.31}$$

by (4.3.27) and (4.3.29), and thus by a change of variables and the fact that $\bar{\varphi}\equiv 1$ in $[0,\rho/2]$, if $\varepsilon_n < \rho/2$,

$$\int_0^1\left(\bar{W}_{i,n}(s,0)^2 - c_i s^{-1/2}\bar{W}_{i,n}(s,0)\right) ds = \int_0^{\varepsilon_n}\left(\frac{\bar{z}^{(i)}_n(r_i,0)^2}{\varepsilon_n} - c_i r_i^{-1/2}\frac{\bar{z}^{(i)}_n(r_i,0)}{\sqrt{\varepsilon_n}}\right) dr_i.$$

Similarly, for every $R>1$ and for every n such that $\varepsilon_n R < \rho/2$, we have

$$\int_1^R\left(\bar{W}_{i,n}(s,0) - \frac{c_i}{2}s^{-1/2}\right)^2 ds = \int_{\varepsilon_n}^{\varepsilon_n R}\left(\frac{\bar{z}^{(i)}_n(r_i,0)}{\sqrt{\varepsilon_n}} - \frac{c_i}{2}r_i^{-1/2}\right)^2 dr_i,$$

$$\int_{B^+_R(\mathbf{0})}|\nabla W_{i,n}|^2\,d\boldsymbol{y} = \int_{B^+_{\varepsilon_n R}(\boldsymbol{x}_i)}|\nabla z_n|^2\,d\boldsymbol{x}.$$

Hence, in view of (4.3.30)

$$\begin{aligned}M \geq \mathcal{F}^{(2)}_{\varepsilon_n}(w_n) \geq \sum_{i=1}^{2}\Bigg\{&\frac{1}{2}\int_0^1\left(\bar{W}_{i,n}(s,0)^2 - c_i s^{-1/2}\bar{W}_{i,n}(s,0)\right) ds + B_{i,n}c_i + C_\varphi c_i^2\\ &+\frac{1}{2}\int_1^R\left(\bar{W}_{i,n}(s,0) - \frac{c_i}{2}s^{-1/2}\right)^2 ds + \frac{1}{2}\int_{B^+_R(\mathbf{0})}|\nabla W_{i,n}|^2\,d\boldsymbol{y}\Bigg\}\end{aligned}$$

$$- \frac{1}{2} \int_{\Gamma_D} \left(\partial_\nu u^0_{\text{reg}}\right)^2 \, d\mathcal{H}^1 + \sqrt{\varepsilon_n} \int_\Omega \nabla z_n \cdot \nabla u_1 \, d\boldsymbol{x}. \tag{4.3.32}$$

Since $\{\nabla z_n\}_n$ is bounded in $L^2(\Omega; \mathbb{R}^2_+)$ (see (1.2.25)), it follows that

$$\int_{B^+_R(\mathbf{0})} |\nabla W_{i,n}|^2 \, d\boldsymbol{y} + \int_0^1 \left(\bar{W}_{i,n}(s,0)^2 - c_i s^{-1/2} \bar{W}_{i,n}(s,0)\right) ds$$
$$+ \int_1^R \left(\bar{W}_{i,n}(s,0) - \frac{c_i}{2} s^{-1/2}\right)^2 ds \le c,$$

for some constant $c > 0$ independent of n and R. To conclude, it is enough to send $R \to \infty$. □

Proof of Theorem 1.2.7. **Step 1:** Let $\varepsilon_n \to 0^+$ and $\{w_n\}_n$ be a sequence of functions in $L^2(\Omega)$ such that $w_n \to w$ in $L^2(\Omega)$. Reasoning as in the proof of Theorem 1.2.3, we can assume without loss of generality that

$$\liminf_{n\to\infty} \mathcal{F}^{(2)}_{\varepsilon_n}(w_n) = \lim_{n\to\infty} \mathcal{F}^{(2)}_{\varepsilon_n}(w_n) < \infty.$$

In particular, $\mathcal{F}^{(2)}_{\varepsilon_n}(w_n) < \infty$ for every n sufficiently large. Let $\{w_{n_k}\}_k$ be the subsequence of $\{w_n\}_n$ given in Theorem 1.2.6 and for every $k \in \mathbb{N}$ let z_{n_k} be such that $w_{n_k} = u_0 + \sqrt{\varepsilon_{n_k}} z_{n_k} + \varepsilon_{n_k} u_1$. Let $W_{i,n}$ be given as in (4.3.31), then by (4.3.30), taking $n = n_k$ in (4.3.32) and letting $k \to 0$ we obtain

$$\liminf_{k\to\infty} \mathcal{F}^{(2)}_{\varepsilon_{n_k}}(w_{n_k}) \ge \sum_{i=1}^{2} \left\{ \frac{1}{2} \int_0^1 \left(\bar{W}_i(s,0)^2 - c_i s^{-1/2} \bar{W}_i(s,0)\right) ds + B_i c_i + C_\varphi c_i^2 \right.$$
$$\left. + \frac{1}{2} \int_1^R \left(\bar{W}_i(s,0) - \frac{c_i}{2} s^{-1/2}\right)^2 ds + \frac{1}{2} \int_{B^+_R(\mathbf{0})} |\nabla W_i|^2 \, d\boldsymbol{y} \right\}$$
$$- \frac{1}{2} \int_{\Gamma_D} \left(\partial_\nu u^0_{\text{reg}}\right)^2 \, d\mathcal{H}^1,$$

where we have used (1.2.27), (1.2.28), (1.2.29), and the fact that $\{\nabla z_n\}_n$ is bounded in $L^2(\Omega; \mathbb{R}^2_+)$ (see (1.2.25)). By letting $R \to \infty$ in the previous inequality we get

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{F}^{(2)}_{\varepsilon_n}(w_n) &= \lim_{k\to\infty} \mathcal{F}^{(2)}_{\varepsilon_{n_k}}(w_{n_k}) \\
&\ge \sum_{i=1}^{2} \left\{ \frac{\mathcal{J}_i(W_i)}{2} + B_i c_i + C_\varphi c_i^2 \right\} - \frac{1}{2} \int_{\Gamma_D} \left(\partial_\nu u^0_{\text{reg}}\right)^2 \, d\mathcal{H}^1 \\
&\ge \mathcal{F}_2(w),
\end{aligned}$$

where in the last step we used the fact that $\mathcal{J}_i(W_i) \ge A_i$.

Step 2: For every $w \in L^2(\Omega) \setminus \{u_0\}$, the constant sequence $w_n = w$ is a recovery sequence. On the other hand, if $w = u_0$, let $w_i \in H$ be given as in Proposition 4.3.4. Let z_n be the function defined in $B_\rho(\boldsymbol{x}_i) \cap \Omega$ using polar coordinates around $\boldsymbol{x}_i$ (see (4.3.2)) via

$$\bar{z}^{(i)}_n(r_i, \theta_i) := \bar{\varphi}(r_i) \bar{W}_i\left(\frac{r_i}{\varepsilon_n}, \theta_i\right) \tag{4.3.33}$$

and $z_n(\boldsymbol{x}) := 0$ in $\Omega \setminus \bigcup_{i=1}^2 B_\rho(\boldsymbol{x}_i)$. Set

$$w_n := u_0 + \sqrt{\varepsilon_n} z_n + \varepsilon_n u_1.$$

We claim that $\{w_n\}_n$ is a recovery sequence for u_0. To prove the claim, we notice that (4.3.30) implies

$$\begin{aligned}\limsup_{n\to\infty} \mathcal{F}^{(2)}_{\varepsilon_n}(w_n) \le \sum_{i=1}^{2} \Bigg\{ &\limsup_{n\to\infty} \frac{1}{2} \int_0^{\varepsilon_n} \left(\frac{\bar{W}_i(r_i/\varepsilon_n, 0)^2}{\varepsilon_n} - c_i r_i^{-1/2} \frac{w\bar{W}_i(r_i/\varepsilon_n, 0)}{\sqrt{\varepsilon_n}} \right) dr_i \\ &+ B_i c_i + C_\varphi c_i^2 + \limsup_{n\to\infty} \frac{1}{2} \int_{\varepsilon_n}^{\rho} \varphi_i(r)^2 \left(\frac{w_i(r/\varepsilon_n, 0)}{\sqrt{\varepsilon_n}} - \frac{c_i}{2} r^{-1/2} \right)^2 dr \Bigg\} \\ &- \frac{1}{2} \int_{\Gamma_D} \left(\partial_\nu u^0_{\mathrm{reg}}\right)^2 d\mathcal{H}^1 + \limsup_{n\to\infty} \frac{1}{2} \int_\Omega |\nabla (z_n + \sqrt{\varepsilon_n} u_1)| \, d\boldsymbol{x}. \qquad (4.3.34)\end{aligned}$$

Letting $r = s\varepsilon_n$, we obtain

$$\begin{aligned}\int_0^{\varepsilon_n} &\left(\frac{\bar{W}_i(r_i/\varepsilon_n, 0)^2}{\varepsilon_n} - c_i r_i^{-1/2} \frac{\bar{W}_i(r_i/\varepsilon_n, 0)}{\sqrt{\varepsilon_n}} \right) dr_i \\ &= \int_0^1 \left(\bar{W}_i(s,0)^2 - c_i s^{-1/2} \bar{W}_i(s,0) \right) ds,\end{aligned} \qquad (4.3.35)$$

and similarly

$$\begin{aligned}\int_{\varepsilon_n}^{\rho} \varphi_i(r)^2 \left(\frac{\bar{W}_i(r_i/\varepsilon_n, 0)}{\sqrt{\varepsilon_n}} - \frac{c_i}{2} r_i^{-1/2} \right)^2 dr &= \int_1^{\rho/\varepsilon_n} \varphi_i(s\varepsilon_n)^2 \left(\bar{W}_i(s,0) - \frac{c_i}{2} s^{-1/2} \right)^2 ds \\ &\le \int_1^\infty \left(\bar{W}_i(s,0) - \frac{c_i}{2} s^{-1/2} \right)^2 ds. \qquad (4.3.36)\end{aligned}$$

Next, we compute the contribution to the energy coming from the gradient term. Since $\bar{\varphi} = 0$ outside of $[0, \rho]$, by (4.3.33) we have

$$\int_\Omega |\nabla z_n|^2 \, d\boldsymbol{x} = \sum_{i=1}^{2} \int_{B_\rho(\boldsymbol{x}_i)} |\nabla z_n|^2 \, d\boldsymbol{x}$$

and therefore

$$\begin{aligned}&\int_\Omega |\nabla z_n|^2 \, d\boldsymbol{x} \\ &\quad = \sum_{i=1}^{2} \int_0^\pi \int_0^\rho \left[r_i \left(\partial_{r_i} (\bar{\varphi}(r_i) \bar{W}_i(r_i/\varepsilon_n, \theta_i)) \right)^2 + \frac{1}{r_i} \bar{\varphi}(r_i)^2 \left(\partial_{\theta_i} \bar{W}_i(r_i/\varepsilon_n, \theta_i) \right)^2 \right] dr_i d\theta_i.\end{aligned}$$

We write

$$\int_0^\pi \int_0^\rho r \left(\partial_r (\varphi_i(r) w_i(r/\varepsilon_n, \theta)) \right)^2 dr d\theta$$

$$= \int_0^\pi \int_0^\rho r \left(\varphi_i'(r) w_i(r/\varepsilon_n, \theta) + \varphi_i(r)\varepsilon_n \partial_r w_i(r/\varepsilon_n, \theta)\right)^2 \, dr d\theta.$$

Expanding the square on the right-hand side of the previous identity we obtain three terms, which we study separately. By the change of variables $s = r_i/\varepsilon_n$ we obtain

$$\int_0^\pi \int_0^\rho r_i \bar{\varphi}'(r_i)^2 \bar{W}_i(r_i/\varepsilon_n, \theta_i)^2 \, dr_i d\theta_i = \int_0^\pi \int_0^{\rho/\varepsilon_n} s\varepsilon_n^2 \varphi_i'(s\varepsilon_n)^2 \bar{W}_i(s, \theta_i)^2 \, dr d\theta_i$$
$$\leq \frac{c}{\rho} \int_0^\pi \int_{\rho/2\varepsilon_n}^{\rho/\varepsilon_n} s\varepsilon_n^2 \bar{W}_i(s, \theta)^2 \, ds d\theta \to 0,$$

where in the last step we have used Lemma 4.3.5. Similarly,

$$\int_0^\pi \int_0^\rho r_i \bar{\varphi}(r_i)^2 (\partial_{r_i} \bar{W}_i(r_i/\varepsilon_n, \theta_i))^2 \, dr_i d\theta_i = \int_0^\pi \int_0^{\rho/\varepsilon_n} s\bar{\varphi}(s\varepsilon_n)^2 (\partial_s \bar{W}_i(s, \theta_i))^2 \, ds d\theta_i$$
$$\leq \int_0^\pi \int_0^{\rho/\varepsilon_n} s(\partial_s \bar{W}_i(s, \theta))^2 \, ds d\theta.$$

In turn, Hölder's inequality implies that

$$2\int_0^\pi \int_0^\rho r_i \bar{\varphi}'(r_i) \bar{W}_i(r_i/\varepsilon_n, \theta_i) \bar{\varphi}(r_i) \partial_{r_i} \bar{W}_i(r_i/\varepsilon_n, \theta_i) \, dr_i d\theta_i \to 0$$

as $n \to \infty$. The same change of variables $s = r_i/\varepsilon_n$ also yields

$$\int_0^\pi \int_0^\rho \frac{\bar{\varphi}(r_i)}{r_i} \left(\partial_{\theta_i} \bar{W}_i(r_i/\varepsilon_n, \theta_i)\right)^2 \, dr_i d\theta_i = \int_0^\pi \int_0^{\rho/\varepsilon_n} \frac{1}{s} \bar{\varphi}(s\varepsilon_n)^2 (\partial_{\theta_i} \bar{W}_i(s, \theta_i))^2 \, ds d\theta_i$$
$$\leq \int_0^\pi \int_0^{\rho/\varepsilon_n} \frac{1}{s} (\partial_{\theta_i} \bar{W}_i(s, \theta))^2 \, ds d\theta.$$

Thus

$$\limsup_{n\to\infty} \int_\Omega |\nabla(z_n + \sqrt{\varepsilon_n} u_1)|^2 \, d\boldsymbol{x} \leq \limsup_{n\to\infty} \int_\Omega |\nabla z_n|^2 \, d\boldsymbol{x} \leq \sum_{i=1}^2 \int_{\mathbb{R}^2_+} |\nabla W_i|^2 \, d\boldsymbol{x}, \tag{4.3.37}$$

which, together with (4.3.34), (4.3.35), and (4.3.36), concludes the proof of the Γ-limsup inequality.

The energy expansion (1.2.30) follows from Theorem 2.1.7. □

4.3.5 Sharp estimates

Proof of Theorem 1.2.8. Suppose by contradiction that (1.2.31) is not true. Then there exists a sequence $\varepsilon_n \to 0^+$ such that

$$\|u_{\varepsilon_n} - u_0\|_{L^2(\Gamma_D)} > n \left(\varepsilon_n \sqrt{|\log \varepsilon_n|}\right) \tag{4.3.38}$$

for every $n \in \mathbb{N}$. In view of (1.2.14), we have that

$$\sup\{\mathcal{F}^{(1)}_{\varepsilon_n}(u_{\varepsilon_n}) : n \in \mathbb{N}\} < \infty,$$

and thus by Theorem 1.2.4 there exist a subsequence $\{u_{\varepsilon_{n_k}}\}_k$ of $\{u_{\varepsilon_n}\}_n$ and $v_0 \in L^2(\Gamma_D)$ such that

$$\frac{u_{\varepsilon_n} - u_0}{\varepsilon_n \sqrt{|\log \varepsilon_n|}} \rightharpoonup v_0,$$

which is a contradiction to (4.3.38).

The proof of (1.2.32) follows analogously from (1.2.25) and (1.2.30). $\square$

4.4 More general Gamma-convergence results

Our results can be recast in a more general framework by decoupling the different scales in the asymptotic expansion of u_ε. Here we present in full detail the generalizations of Theorem 1.2.5 and Theorem 1.2.7; the results of Section 3 can be analogously reformulated. Throughout the section we assume that the domain Ω is given as in Theorem 1.2.1 and use the notations introduced in (4.3.1) and (4.3.2).

Theorem 4.4.1. *Under the assumptions of Theorem 1.2.4, let $\mathcal{K}_\varepsilon^{(1)} \colon L^2(\Omega) \times L^2(\Gamma_D) \to \overline{\mathbb{R}}$ be defined via*

$$\mathcal{K}_\varepsilon^{(1)}(u,v) := \begin{cases} \mathcal{F}_\varepsilon^{(1)}(u) & \text{if } u \in H^1(\Omega) \text{ and } \frac{u-u_0}{\varepsilon\sqrt{|\log \varepsilon|}} = v \text{ on } \Gamma_D, \\ +\infty & \text{otherwise}. \end{cases} \tag{4.4.1}$$

Then the family $\{\mathcal{K}_\varepsilon^{(1)}\}_\varepsilon$ Γ-converges in $L^2(\Omega) \times L^2(\Gamma_D)$ to the functional

$$\mathcal{K}_1(u,v) := \begin{cases} \frac{1}{2}\int_{\Gamma_D} v^2 \, d\mathcal{H}^1 - \frac{1}{8}\sum_{i=1}^{2} c_i^2 & \text{if } u = u_0 \text{ and } v \in L^2(\Gamma_D), \\ +\infty & \text{otherwise}, \end{cases}$$

where the coefficients c_i are as in Theorem 1.2.1.

Proof. **Step 1:** (*Compactness*) Let $\varepsilon_n \to 0^+$ and $(u_n, v_n) \in L^2(\Omega) \times L^2(\Gamma_D)$ such that

$$\sup\{\mathcal{K}_{\varepsilon_n}^{(1)}(u_n, v_n) : n \in \mathbb{N}\} < \infty.$$

Then by (4.4.1), $u_n \in H^1(\Omega)$, the function

$$v_n^* := \frac{u_n - u_0}{\varepsilon_n \sqrt{|\log \varepsilon_n|}}$$

belongs to $H^1(\Omega)$ and satisfies $v_n^* = v_n$ on Γ_D in the sense of traces. By Theorem 1.2.4, there exist a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$, $r \in H^1(\Omega)$ and $v \in L^2(\Gamma_D)$ such that

$$\varepsilon_{n_k}^{1/2} \nabla v_{n_k}^* \rightharpoonup r \quad \text{in } H^1(\Omega),$$

$$v_{n_k} \rightharpoonup v \quad \text{in } L^2(\Gamma_D).$$

Step 2: (*Liminf inequality*) Let $\varepsilon_n \to 0^+$ and $\{(u_n, v_n)\}_n$ be a sequence in $L^2(\Omega) \times L^2(\Gamma_D)$ such that $(u_n, v_n) \to (u, v)$. Reasoning as in the proof of Theorem 1.2.3, we can assume without loss of generality that

$$\liminf_{n\to\infty} \mathcal{K}^{(1)}_{\varepsilon_n}(u_n, v_n) = \lim_{n\to\infty} \mathcal{K}^{(1)}_{\varepsilon_n}(u_n, v_n) < \infty.$$

In particular, $\mathcal{K}^{(1)}_{\varepsilon_n}(u_n, v_n) < \infty$ for every n sufficiently large. Let $\{u_{n_k}\}_k$ be the subsequence of $\{u_n\}_n$ given as in the previous step and ξ^i_n be the function defined in polar coordinates as in (4.3.14). Then

$$\liminf_{k\to\infty} \mathcal{K}^{(1)}_{\varepsilon_{n_k}}(u_{n_k}, v_{n_k}) = \liminf_{k\to\infty} \mathcal{F}^{(1)}_{\varepsilon_{n_k}}(u_{n_k})$$

and so, reasoning as in the proof of Theorem 1.2.5 (by (4.3.15) and (4.3.16) with v_{n_k} and z_{n_k} replaced by u_{n_k} and $v^*_{n_k}$, respectively), we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \mathcal{K}^{(1)}_{\varepsilon_{n_k}}(u_{n_k}, v_{n_k}) &\geq \liminf_{k\to\infty} \left\{ \frac{1}{2}\int_{\Gamma_D} v^2_{n_k}\, d\mathcal{H}^1 - \int_{\Gamma_D \setminus \bigcup_i B_{\varepsilon_{n_k}}(\boldsymbol{x}_i)} v_{n_k}(\xi^1_{n_k} + \xi^2_{n_k})\, d\mathcal{H}^1 \right\} \\
&\geq \liminf_{k\to\infty} \int_{\Gamma_D \setminus \bigcup_i B_{\varepsilon_{n_k}}(\boldsymbol{x}_i)} \left[\frac{1}{2} v^2_{n_k} - v_{n_k}(\xi^1_{n_k} + \xi^2_{n_k}) \right] d\mathcal{H}^1 \\
&= \liminf_{k\to\infty} \frac{1}{2} \int_{\Gamma_D \setminus \bigcup_i B_{\varepsilon_{n_k}}(\boldsymbol{x}_i)} \left[\left(v_{n_k} - \xi^1_{n_k} - \xi^2_{n_k} \right)^2 - (\xi^1_n)^2 - (\xi^2_{n_k})^2 \right] d\mathcal{H}^1 \\
&\geq \frac{1}{2} \int_{\Gamma_D} v^2\, d\mathcal{H}^1 - \frac{1}{8} \sum_{i=1}^{2} c_i^2 = \mathcal{K}_1(u_0, v),
\end{aligned}$$

where in the last step we have used the fact that $v_{n_k} \rightharpoonup v$, $\xi^i_{n_k} \rightharpoonup 0$ in $L^2(\Gamma_D)$, and so

$$\liminf_{k\to\infty} \int_{\Gamma_D \setminus \bigcup_i B_{\varepsilon_{n_k}}(\boldsymbol{x}_i)} \left(v_{n_k} - \xi^1_{n_k} - \xi^2_n \right)^2 d\mathcal{H}^1 \geq \int_{\Gamma_D} v^2\, d\mathcal{H}^1.$$

Step 3: (*Limsup inequality*) Let $u = u_0$ and $v \in L^2(\Gamma_D)$. We extend v to zero in $\partial\Omega \setminus \Gamma_D$ and assume first that $v \in H^{1/2}(\partial\Omega)$ (in what follows, although with a slight abuse of notation, we identify v with its extension). Then there exists $v^* \in H^1(\Omega)$ such that $v^* = v$ on $\partial\Omega$ in the sense of traces (see Theorem 18.40 in [70]). Set

$$u_n := u_0 + \varepsilon_n \sqrt{|\log \varepsilon_n|}(z_n + v^*),$$

where z_n is defined as in (4.3.18). As one can check (see (4.3.24) and (4.3.26)), $\{(u_n, z_n + v^*)\}_n$ is a recovery sequence for (u_0, v).

If $v \in L^2(\partial\Omega) \setminus H^{1/2}(\partial\Omega)$ we consider a sequence $\{v_n\}_n$ of functions in $H^{1/2}(\partial\Omega)$ such that

$$\|v_n - v\|_{L^2(\partial\Omega)} \to 0 \quad \text{as } n \to \infty, \tag{4.4.2}$$

and for every $n \in \mathbb{N}$ we let $v^*_n \in H^1(\Omega)$ be such that $v^*_n = v_n$ on $\partial\Omega$ and

$$\|v^*_n\|_{H^1(\Omega)} \leq c\|v_n\|_{H^{1/2}(\partial\Omega)}, \tag{4.4.3}$$

where $c > 0$ is independent of n (see Theorem 18.40 in [70]). Furthermore, notice that by a standard mollification argument we can also assume that

$$\|\varepsilon_n^{1/2} v_n\|_{H^{1/2}(\partial\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{4.4.4}$$

Set

$$u_n := u_0 + \varepsilon_n \sqrt{|\log \varepsilon_n|}(z_n + v_n^*)$$

and notice that by (4.4.3) and (4.4.4), $\|\varepsilon_n^{1/2} \nabla(z_n + v_n^*)\|_{L^2(\Omega;\mathbb{R}^2)} \to 0$ as $n \to \infty$. Thus, we can proceed as in (4.3.24) and (4.3.26). $\square$

Theorem 4.4.2. *Under the assumptions of Theorem 1.2.6, let*

$$\mathcal{K}_\varepsilon^{(2)} \colon L^2(\Omega) \times L^2_{\text{loc}}(\mathbb{R}^2_+) \times L^2_{\text{loc}}(\mathbb{R}^2_+) \times L^2_{\text{loc}}(\Gamma_D) \to \overline{\mathbb{R}}$$

be defined via

$$\mathcal{K}_\varepsilon^{(2)}(u, v_1, v_2, w) := \mathcal{F}_\varepsilon^{(2)}(u) \tag{4.4.5}$$

if

$$\begin{cases} u - u_0 - \varepsilon u_1 = \sqrt{\varepsilon} V_{i,\varepsilon} & \text{in } \Omega \cap B_\rho(\boldsymbol{x}_i), \\ u - u_0 - \varepsilon u_1 = \varepsilon w & \text{on } \Gamma_D \setminus B_\varepsilon(\boldsymbol{x}_i), \end{cases} \tag{4.4.6}$$

where the functions $V_{i,\varepsilon}$ are defined in polar coordinates by

$$\bar{V}_{i,\varepsilon}(r_i, \theta_i) := \bar{v}_i\left(\frac{r_i}{\varepsilon}, \theta_i\right), \tag{4.4.7}$$

and $\mathcal{K}_\varepsilon^{(2)}(u, v_1, v_2, w) := +\infty$ otherwise. Then the family $\{\mathcal{K}_\varepsilon^{(2)}\}_\varepsilon$ Γ-converges with respect to the topology of $L^2(\Omega) \times L^2_{\text{loc}}(\mathbb{R}^2_+) \times L^2_{\text{loc}}(\mathbb{R}^2_+) \times L^2_{\text{loc}}(\Gamma_D)$ to the functional

$$\mathcal{K}_2(u, v_1, v_2, w) := \sum_{i=1}^2 \left[\frac{1}{2}\mathcal{J}_i(v_i) + B_i c_i + C_\varphi c_i^2\right] + \frac{1}{2}\int_{\Gamma_D} \left[\left(w - \sum_{i=1}^2 c_i \psi_i\right)^2 - \left(\partial_\nu u^0_{\text{reg}}\right)^2\right] d\mathcal{H}^1$$

if $u = u_0$, $v_1, v_2 \in H$, $w - \sum_{i=1}^2 c_i \psi_i \in L^2(\Gamma_D)$, and $\mathcal{K}_2(u, v_1, v_2, w) := +\infty$ otherwise, where B_i and C_φ are defined as in (1.2.18) *and* (1.2.19)*, respectively.*

Proof. **Step 1:** (*Liminf inequality*) Let $\varepsilon_n \to 0^+$ and $\{(u_n, v_{1,n}, v_{2,n}, w_n)\}_n$ be a sequence in $L^2(\Omega) \times L^2_{\text{loc}}(\mathbb{R}^2_+) \times L^2_{\text{loc}}(\mathbb{R}^2_+) \times L^2_{\text{loc}}(\Gamma_D)$ such that $(u_n, v_{1,n}, v_{2,n}, w_n) \to (u, v_1, v_2, w)$. Let $\boldsymbol{u}_n := (u_n, v_{1,n}, v_{2,n}, w_n)$. Reasoning as in the proof of Theorem 1.2.3, we can assume without loss of generality that

$$\liminf_{n\to\infty} \mathcal{K}_{\varepsilon_n}^{(2)}(\boldsymbol{u}_n) = \lim_{n\to\infty} \mathcal{K}_{\varepsilon_n}^{(2)}(\boldsymbol{u}_n) < \infty.$$

In particular, $\mathcal{K}_{\varepsilon_n}^{(2)}(\boldsymbol{u}_n) < \infty$ for every n sufficiently large. Let $\{u_{n_k}\}_k$ be the subsequence of $\{u_n\}_n$ given as in Theorem 1.2.6. By (4.3.30) (with w_n replaced by u_{n_k}), (4.4.5), (4.4.6), and (4.4.7) it follows that for every $\varepsilon_{n_k} < \delta < \rho$,

$$\mathcal{K}_{\varepsilon_{n_k}}^{(2)}(\boldsymbol{u}_{n_k}) = \sum_{i=1}^2 \left\{\frac{1}{2}\int_{\varepsilon_{n_k}}^\delta \left(\frac{\bar{v}_{i,n_k}(r_i/\varepsilon_{n_k}, 0)}{\sqrt{\varepsilon_{n_k}}} - \frac{c_i}{2}\bar{\varphi}(r_i) r_i^{-1/2}\right)^2 dr_i + B_{i,n_k} c_i + C_\varphi c_i^2\right.$$

$$
+\frac{1}{2}\int_0^{\varepsilon_{n_k}}\left(\frac{\bar{v}_{i,n_k}(r_i/\varepsilon_{n_k},0)^2}{\varepsilon_{n_k}}-c_i r_i^{-1/2}\frac{\bar{v}_{i,n_k}(r_i/\varepsilon_{n_k},0)}{\sqrt{\varepsilon_{n_k}}}\right)dr_i\Bigg\}
$$

$$
+\frac{1}{2}\int_{\Gamma_D\setminus\bigcup_i B_\delta(\boldsymbol{x}_i)}\left(w_{n_k}-\sum_{i=1}^2 c_i\psi_i\right)^2 d\mathcal{H}^1-\frac{1}{2}\int_{\Gamma_D}\left(\partial_\nu u^0_{\mathrm{reg}}\right)^2 d\mathcal{H}^1
$$

$$
+\frac{1}{2\varepsilon_{n_k}}\int_\Omega|\nabla(u_{n_k}-u_0)|^2\,d\boldsymbol{x}, \tag{4.4.8}
$$

where B_{i,n_k} is defined as in (4.3.28). Arguing as in the first step of the proof of Theorem 1.2.7, we arrive at

$$
\liminf_{k\to\infty}\mathcal{K}^{(2)}_{\varepsilon_{n_k}}(\boldsymbol{u}_{n_k})\geq\sum_{i=1}^2\left[\frac{1}{2}\mathcal{J}_i(v_i)+B_ic_i+C_\varphi c_i^2\right]+\frac{1}{2}\int_{\Gamma_D\setminus\bigcup_i B_\delta(\boldsymbol{x}_i)}\left(w-\sum_{i=1}^2 c_i\psi_i\right)^2 d\mathcal{H}^1
$$

$$
-\frac{1}{2}\int_{\Gamma_D}\left(\partial_\nu u^0_{\mathrm{reg}}\right)^2 d\mathcal{H}^1.
$$

To conclude the proof of the liminf inequality it is enough to let $\delta\to 0^+$.
Step 2: (*Limsup inequality*) Let (u_0,v_1,v_2,w) be such that $\mathcal{K}_2(u_0,v_1,v_2,w)<\infty$. We assume first that there exists $0<\delta<\rho/2$ such that

$$
w\in H^{1/2}\left(\Gamma_D\setminus\bigcup_{i=1}^2\overline{B_{\delta/4}(\boldsymbol{x}_i)}\right), \tag{4.4.9}
$$

and we extend it to a function in $H^{1/2}(\partial\Omega)$ (in what follows, although with a slight abuse of notation, we identify w with its extension). Then there exists $w^*\in H^1(\Omega)$ such that $w^*=w$ on $\partial\Omega$ in the sense of traces (see Theorem 18.40 in [70]). Set

$$
u_n\coloneqq u_0+\varepsilon_n u_1+\sqrt{\varepsilon_n}Z_n,
$$

where Z_n is given in polar coordinate at $\boldsymbol{x}_i$ by

$$
\bar{Z}^{(i)}_n(r_i,\theta_i)\coloneqq\bar{\varphi}\left(\frac{\rho}{2\delta}r_i\right)\bar{v}_i\left(\frac{r_i}{\varepsilon_n},\theta_i\right)+\sqrt{\varepsilon_n}\left(1-\bar{\varphi}\left(\frac{\rho}{2\delta}r_i\right)\right)\overline{w^*}^{(i)}(r_i,\theta_i),
$$

and $Z_n\coloneqq\sqrt{\varepsilon_n}w^*$ in $\Omega\setminus\bigcup_{i=1}^2 B_\rho(\boldsymbol{x}_i)$. We claim that $\{\boldsymbol{u}_n\}_n$, defined from $\{u_n\}_n$ via (4.4.6) and (4.4.7), is a recovery sequence for (u_0,v_1,v_2,w). Using the fact that $\bar{\varphi}\left(\frac{\rho}{2\delta}r_i\right)=1$ for $r_i\leq\delta$ and the change of variables $\varepsilon_n s=r_i$ (see also (4.3.35), (4.3.36), and (4.3.37)), we get

$$
\mathcal{J}_i(v_i)\geq\limsup_{n\to\infty}\Bigg\{\int_{B_\delta(\boldsymbol{x}_i)}|\nabla Z_n|^2\,d\boldsymbol{x}+\int_0^{\varepsilon_n}\left(\frac{\bar{Z}^{(i)}_n(r_i,0)^2}{\varepsilon_n}-c_i r_i^{-1/2}\frac{\bar{Z}^{(i)}_n(r_i,0)}{\sqrt{\varepsilon_n}}\right)dr_i
$$

$$
+\int_{\varepsilon_n}^{\delta}\left(\frac{\bar{Z}^{(i)}_n(r_i,0)}{\sqrt{\varepsilon_n}}-\frac{c_i}{2}\bar{\varphi}(r_i)r_i^{-1/2}\right)^2 dr_i\Bigg\}.
$$

In turn, it follows from (4.4.8) that

$$\limsup_{n\to\infty}\mathcal{K}_\varepsilon^{(2)}(\boldsymbol{u}_n) \leq \sum_{i=1}^{2}\left\{\frac{\mathcal{J}_i(v_i)}{2}+B_ic_i+C_\varphi c_i^2\right\}-\frac{1}{2}\int_{\Gamma_D}\left(\partial_\nu u^0_{\mathrm{reg}}\right)^2\,d\mathcal{H}^1$$

$$+\limsup_{n\to\infty}\frac{1}{2}\int_{\Gamma_D\setminus\bigcup_i B_\delta(\boldsymbol{x}_i)}\left(\frac{Z_n}{\sqrt{\varepsilon_n}}-\sum_{i=1}^{2}c_i\psi_i\right)^2\,d\mathcal{H}^1$$

$$+\limsup_{n\to\infty}\frac{1}{2}\int_{\Omega\setminus\bigcup_i B_\delta(\boldsymbol{x}_i)}|\nabla(Z_n+\sqrt{\varepsilon_n}u_1)|^2\,d\boldsymbol{x}. \tag{4.4.10}$$

By the convexity of the square function we have

$$\int_\delta^{2\delta}\left(\frac{\bar{Z}_n^{(i)}(r_i,0)}{\sqrt{\varepsilon_n}}-\frac{c_i}{2}\bar{\varphi}(r_i)r_i^{-1/2}\right)^2\,dr_i \leq \int_\delta^{2\delta}\bar{\varphi}\left(\frac{\rho}{2\delta}r_i\right)\left(\bar{v}_i(r_i/\varepsilon_n,0)-\frac{c_i}{2}\bar{\varphi}(r_i)r_i^{-1/2}\right)^2\,dr_i$$

$$+\int_\delta^{2\delta}\left(1-\bar{\varphi}\left(\frac{\rho}{2\delta}r_i\right)\right)\left(w-\frac{c_i}{2}\bar{\varphi}(r_i)r^{-1/2}\right)^2\,dr_i,$$

and therefore, since $\mathcal{J}_i(v_i)<\infty$,

$$\limsup_{n\to\infty}\int_\delta^{2\delta}\left(\frac{\bar{Z}_n^{(i)}(r_i,0)}{\sqrt{\varepsilon_n}}-\frac{c_i}{2}\bar{\varphi}(r_i)r_i^{-1/2}\right)^2\,dr_i \leq \int_\delta^{2\delta}\left(w-\frac{c_i}{2}\bar{\varphi}(r_i)r^{-1/2}\right)^2\,dr_i.$$

In addition, using the fact that $\bar{\varphi}\left(\frac{\rho}{2\delta}r_i\right)=0$ for $r_i\geq 2\delta$, we obtain

$$\int_{\Gamma_D\setminus\bigcup_i B_{2\delta}(\boldsymbol{x}_i)}\left(\frac{Z_n}{\sqrt{\varepsilon_n}}-\sum_{i=1}^{2}c_i\psi_i\right)^2\,d\mathcal{H}^1=\int_{\Gamma_D\setminus\bigcup_i B_{2\delta}(\boldsymbol{x}_i)}\left(w-\sum_{i=1}^{2}c_i\psi_i\right)^2\,d\mathcal{H}^1.$$

We now observe that the result of Lemma 4.3.5 straightforwardly extends to every $v_i\in H$ such that $\mathcal{J}_i(v_i)<\infty$. Consequently, we can argue as in the second step of the proof of Theorem 1.2.7 to deduce that

$$\limsup_{n\to\infty}\frac{1}{2}\int_{\Omega\setminus\bigcup_i B_\delta(\boldsymbol{x}_i)}|\nabla(Z_n+\sqrt{\varepsilon_n}u_1)|^2\,d\boldsymbol{x}=0.$$

This concludes the proof of the limsup inequality under the assumption that (4.4.9) is satisfied.

If on the other hand

$$w\notin H^{1/2}\left(\Gamma_D\setminus\bigcup_{i=1}^{2}\overline{B_{\delta/4}(\boldsymbol{x}_i)}\right)$$

for any $\delta>0$, we reproduce the mollification argument in (4.4.2) - (4.4.4) and proceed as before. □

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