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PRESENTED BY: Prashant Shrivastava

ACCEPTED BY THE DEPARTMENT OF PHYSICS

Ira Rothstein 5/11/18

IRA ROTHSTEIN, CHAIR PROFESSOR DATE

Scott Dodelson 5/16/18

SCOTT DODELSON, DEPT HEAD DATE

APPROVED BY THE COLLEGE COUNCIL

Rebecca Doerge 5/17/18

REBECCA DOERGE, DEAN DATE

Spontaneously Broken Spacetime Symmetries and Fermi Liquid Theory

by

Prashant Shrivastava

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Advisor: Professor Ira Z. Rothstein

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Abstract

I discuss the implications of spontaneously broken spacetime symmetries in context of Condensed Matter Physics (CMP) and High Energy Physics (HEP). Starting with a minimal set of assumptions, a lot can be learned about complicated many body quantum systems using the ideas from *Coset Construction* and *Effective Field theories* (EFT). The fact that any symmetry of the full theory must be realized, linearly or non-linearly, in the infra red (IR) limit puts strong constraints on the low energy dynamics of any EFT. In this work, I have mainly focused on the *Fermi liquid theory* (FLT), which in some sense is the simplest CMP system. Even though the phenomenology of FLT has been worked out extensively, its analysis from the point of view of spacetime symmetry breaking has shed some new light on the theory and led to the development of a novel idea the so-called *Dynamical Inverse Higgs Mechanism*, which is a new way of non-linear realization of broken spacetime symmetries. This technique is developed in this work in great detail.

I also worked on a different, yet more intuitive method of understanding broken spacetime symmetries by using the spacetime symmetry algebra. This is related to a well known concept in EFTs, called the Reparametrization Invariance (RPI). I work out several examples of the constraints that arise in HEP EFTs from RPI and how they can also be derived from the symmetry algebra. Lastly in the appendix, I also provide some detailed calculations related to the main body of my thesis and a few other applications of Coset Construction besides Fermi liquids.

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Chapter 1

Introduction

Landau's Fermi liquid theory (FLT) is a theory of interacting fermions which describes the normal state of metals [1, 2, 3]. In modern parlance the ubiquity of Fermi liquid behavior is a consequence of the fact that under a certain set of generic assumptions the long distance or low energy behavior is governed by a universal fixed point. That is, the Fermi liquids fall into a universality class. The starting assumption of the theory is that the interacting system can be reached by beginning with a free theory and adiabatically turning on the interactions such that the free fermion states evolve into interacting quasi-particles with same charge and spin as an electron but not necessarily the same mass. Landau showed that, under these assumptions, the width or inverse lifetime (Γ) of these quasi-particles is suppressed due to Pauli blocking of the final states such that $\Gamma(E) \sim (E - E_F)^2$, where E is the energy of the quasi-particle and E_F is the Fermi Energy or chemical potential. This *a posteriori* justifies the notion of a quasi-particle. The signature of FLT behavior in the normal phase of metals has generic features such as a resistivity scaling as T^2, the existence of zero sound and long lived gapless excitations.

An Effective Field theory (EFT) description of the Fermi surfaces was developed in Ref. [4, 5, 6] which is based on an expansion around the Fermi surface and becomes exact in the low energy or infra red (IR) limit where $E/E_F \to 0$. This is the unique EFT that describes the universality class that is FLT. The founding assumption of the EFT is the existence of long lived quasi-particles with the quantum numbers of the electron. Once the theory has been defined, one shows that it does predict a quasi-particle width, $\Gamma(E) \sim (E - E_F)^2$, a self-consistency check of the EFT.

As is the case for all EFT's, the defining characteristics of the theory are the relevant symmetries and the power counting parameter. In particular if any symmetry of the underlying microscopic theory are broken spontaneously then they must be non-linearly realized in the IR. Naively, the spontaneous breaking of symmetry should, by Goldstone's theorem [7, 8], lead to the existence of a massless scalar bosons and hence, by understanding the symmetry breaking pattern we can identify the relevant degrees of freedom for the EFT.

The case of Fermi liquids in temperature range, $T_F >> T >> T_c$[1], is very interesting from the point view of symmetry breaking since the only spontaneously broken symmetry is the Galilean boost due to a finite chemical potential and so in addition to the quasi-particles, we should have a massless scalar boson called the framon [9] in the IR.

It is well known that, when space-time symmetries are spontaneously broken there need not be a one-to-one map between broken generators and Goldstone bosons (GB). This is usually explained as being due to an “Inverse Higgs Constraint” (IHC) [10, 11, 12]. These constraints arise as a consequence of the fact that it is often possible that only one GB is needed to assure invariance under multiple symmetry transformations [13]. In most condensed matter systems, spacetime translations are spontaneously broken due to the presence of an atomic lattice, which results in the existence of phonons and due to an IHC, the Galilean boost Ward identity can be satisfied without an independent framon. Another way of saying this is that phonons are sufficient to non-linearly realize the boost invariance in the IR and we don’t need an additional degree of freedom.

However in the case of Fermi liquids, like Helium-3 at low temperatures, we can assume translational invariance is unbroken since we work with definite momentum states and ignore the atomic lattice. Moreover we can also assume that the Fermi surface is rotational symmetric.[2]. Under these assumptions, one should expect that framon is a relevant degree of freedom since there are no IHC and we should treat the low energy EFT of the normal state of metals as an interacting theory of electron quasi-particles and framons. But it can be shown[3] that existence of massless scalar will violate the critical assumption of the FLT, which is that the width of the quasi-particles goes as $(E - E_F)^2$. It seems that from the point of view spacetime symmetry breaking FLT is inconsistent.

In Ref. [14] it has been shown that this in fact is not true and FLT does non-linearly realize the Galilean boost symmetry, even without a framon. This happens because of the so-called Dynamical Inverse Higgs Mechanism (DIHM) [14] where the symmetry enforces an operator constraint, which results in a relation between the Wilson coefficients of the theory. In the case of FLT, this operator constraint is the famous Landau effective mass relation [1], which relates the bare mass of an electron to its effective mass. DIHM can also help in resolving a long standing problem about the normal state of a Unitary Fermi Liquid [15].

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[1] T_F is the Fermi temperature and T_c is the critical temperature for phase transition into a generic superfluid state.

[2] The assumption of rotational invariance of Fermi surface can be relaxed and the theory of such systems is discussed in later chapters.

[3] This result is derived in the appendix.

Chapter 2

Dynamical Inverse Higgs Mechanism

When symmetries are broken spontaneously they are manifested non-linearly in the IR. The realization of the broken symmetry, in general, will include gapless Goldstone bosons (GBs). When space-time symmetries are unbroken, Goldstone bosons are derivatively coupled and are irrelevant in the IR. If there are other gapless modes in the spectrum, not associated with symmetry breaking, the Goldstones may be ignored to first approximation. The canonical example of such a scenario is the Fermi liquid theory of metals where phonons do not play a role at leading order[1]. Of course, if there are no other gapless modes, or if the Goldstones couple to sources, then they are of primary importance. An example of such a scenario is the QCD chiral Lagrangian.

When space-time symmetries are broken, GBs can be non-derivatively coupled. Two canonical examples being the relativistic dilaton and the Goldstone bosons of broken rotational invariance in Fermi liquids. Such non-derivative couplings lead to marginal or relevant interactions which can drastically affect the IR physics. For instance, when rotational invariance is broken in a Fermi liquid and translations are unbroken (nematic order), the quasi-particles decay into Goldstones [16, 17, 18] leading to a width which scale as $\Gamma \sim E^{\alpha}$ with $\alpha < 2$. While relativistic dilatons can generate long range forces and, as such, their couplings are highly constrained[19]. The necessary conditions for non-derivatively coupled Goldstones in non-relativistic theories were discussed by Vishwanath and Watanabe in [20].

For relativistic theories the breaking of internal symmetries leads to a one to one correspondence between Goldstone bosons and generators which shift the vacuum [7, 8]. Moreover, the Goldstone boson is manifested as a delta function in the spectral density. When space-time symmetries are broken[21] (this includes the

[1]Potential, off-shell phonons play an indirect role in that they contribute to the attractive piece of the four Fermi coupling once they have been integrated out.

aforementioned non-relativistic case) we are no longer assured about the existence of a Goldstone boson associated with a broken generator X. Suppose we have a order parameter ϕ such that

$$\langle\Omega \mid [X, \phi] \mid \Omega\rangle \neq 0, \tag{2.1}$$

where

$$X = \int d^{d-1}x j_0^X(x). \tag{2.2}$$

It follows that [2]

$$\sum_n \delta^{(d-1)}(\vec{p}_n) \left[\langle\Omega \mid j_0^X(0) \mid n\rangle\langle n \mid \phi(0) \mid \Omega\rangle e^{iE_nt} - \langle\Omega \mid \phi(0) \mid n\rangle\langle n \mid j_0^X(0) \mid \Omega\rangle e^{-iE_nt}\right] \neq 0. \tag{2.3}$$

We assume that the system preserves a discrete translational invariance, so that there exists some notion of a conserved momentum. Given that X is a conserved charge, we see that symmetry breaking implies the existence of a zero energy state when $\vec{p}_n \to 0$. However, we can not say anything about the associated spectral weight other than the fact that it has to non-zero. This state may be arbitrarily wide. Thus if we are to count Goldstone bosons when space-time symmetries are broken we must define what we mean by a Goldstone boson. For our purposes we will define a Goldstone mode as having to satisfy the definition of a quasi-particle, $\Gamma \leq \frac{E^2}{E_F}$ in the limit of vanishing energy. Also note that (2.3) does not preclude the possibility of having multiple gapless states.

For non-relativistic systems, there can be no symmetry breaking if the vacuum is trivial since pair creation is disallowed. Thus a non-relativistic system which manifests *any* symmetry breaking necessarily has a ground state which breaks at least boost invariance and one can not separate space-time from internal symmetry breaking. However, in the literature when internal symmetries are broken, the breaking of boost symmetry is usually ignored. We will come back to this important issue below.

Goldstone bosons may have various dispersion relations. Inequalities for counting rules for the type I ($E \sim p$) and type II ($E \sim p^2$) Goldstones[3] associated with internal symmetry breaking were first written down by Nielsen and Chadha [22]. Since then a series of papers ultimately led to the final result for the number of Goldstones (N) [23, 24, 25, 26, 27, 28] when the group G is broken to H[29]

$$N = \dim(\mathrm{G/H}) - \frac{1}{2}\mathrm{rank}[\rho] \tag{2.4}$$

[2] We assume here that there are no long range forces so that surface terms may be dropped and that generators have canonical translational properties. See [21] for a discussion.

[3] The dispersion relation need not be limited to these two choices. Higher order relations are possible in some systems, e.g. the vibrations of a stiff rod. Non-analytic dispersion relations such as those discussed in [29] are due to integrating out fields with analytic dispersion relations.

where $\rho_{ab} = -i\langle[X_a, j_b^a(0)]\rangle$ and X_a are the broken charges of the full group G and $j^b(0)$ are the associated charge densities. Furthermore counting rules for gapped Goldstones (with both calculable and incalculable gaps) have been developed [37, 36].

The analysis leading to the result (2.4) does not hold when space-time symmetries are broken. Consider the case of a canonical superfluid. This system breaks a $U(1)$ symmetry corresponding to particle number and the rank of ρ vanishes leading to a prediction of only one Goldstone boson. However, we must ask what justifies ignoring the ersatz GB arising from the breaking of boost invariance? The answer lies in whats known as the as the "Inverse Higgs Mechanism"(IHM) [10, 12] (see also [13]).

The counting of (gapless) Goldstones still follows once we have established the necessary criteria for the IHM. When two broken generators X, X' obey a relation of the form

$$[\bar{P}, X] \propto X' \tag{2.5}$$

where $\bar{P}$ are the unbroken translations and X and X' are not in the same H multiplet, it *may* be possible to eliminate the Goldstone associated with X. As emphasized in [37] the algebraic relation (2.5) may or may not be the signal of a redundancy. That depends upon the nature of the order parameter. In particular, given a set of broken generators X^a a redundancy exists when there is a non-trivial solution to the equation

$$\pi^a(x)X^a\langle O(x)\rangle = 0 \tag{2.6}$$

where $\langle O(x)\rangle$ is the order parameter. As an example consider the symmetry breaking pattern for a metal. The lattice breaks rotations, translations and boosts. The boost Goldstone η^i and rotation Goldstone θ^i can be easily seen to be redundant since

$$K^i = P^i t - Mx^i \tag{2.7}$$

and

$$J^i = \epsilon^{ijk}x^j P^k \tag{2.8}$$

thus, assuming that the mass M is unbroken, i.e. no condensation, we have

$$(\theta^k(x)\epsilon^{ijk}x^i(i\partial^j) + \eta^i(x)(it\partial^i) + \pi^a(x)(i\partial^a))\langle O(x)\rangle = 0 \tag{2.9}$$

so that both rotations and boosts can be compensated for by a Goldstone dependent translation [13].

In any case when condition (2.5) is satisfied, it is often possible to impose a constraint on the fields which is consistent with the symmetries. This constraint is called the Inverse Higgs Constraints (IHC) which is associated with the IHM.

2.0.1 The Missing Goldstones

As was pointed out in [9] there are cases for which there is no inverse Higgs constraints and yet the Goldstones still do not appear. If particle number is spontaneously broken, then due to the fact that $[P, K] \propto M$, there is an IHC which allows one to eliminate the boost Goldstone. But if there is no IHM involving the boost generator, one must include the boost Goldstone in the analysis. In [9] the authors considered two such symmetry breaking patterns called type-I and type-II "framids". The former is a system in which the only broken symmetry is boost invariance while the latter also breaks rotations. A cursory check of the Galilean algebra shows that none of the broken generators satisfy (2.5) in these cases and yet the Goldstones associated with boosts, dubbed the "framons", are nowhere to be seen in nature.

Another missing Goldstone boson arises in the case of non-relativistic dilatation invariance. The authors of [30] point out that given that the dilaton σ transforms under Galilean boosts as

$$\sigma(x,t) \to \sigma(x - vt, t) \tag{2.10}$$

there is no way to write down a boost invariant kinetic term[4] since the time derivative of the dilaton transforms non-trivially. As also pointed out in [30], if the $U(1)$ of particle number is broken then, as a consequence of the algebraic relation,

$$[P_i, K_j] = i\delta_{ij} M \tag{2.11}$$

boost invariance is also broken (assuming translations are unbroken). As such, there is an IHM at play and the Ward identities may be saturated without the need for a dilaton. This begs the question, can one write down a sensible dilaton kinetic term if there is no particle condensate? The answer, as will be discussed below, is yes as long as the framon is included in the action. So we see that the questions of the framon and the dilaton are intimately connected. Thus the puzzle of the non-relativistic dilaton remains, and its resolution is tied to the fate of the framon.

As will be discussed below a resolution of the framon puzzle is closely related to the fact that when space-time symmetries are broken, Goldstone bosons need not be derivatively coupled. To explore this possibility we utilize the coset construction which will allow us to generalize the criteria for non-derivatively coupled Goldstones given in [20][5] . Furthermore, we will use the coset methodology to construct the theory of Fermi liquids with the symmetry breaking pattern of type

[4] Matter fields transform as projective representations under boosts which allow for the canonical kinetic energy term.

[5] We generalize [20] in two ways. [20] states that there can be no Goldstone associated with boost invariance due the non-vanishing commutator between boosts and the Hamiltonian. Here we show the need for the boost Goldstone and show that it couples non-derivatively. The coset methodology also allows us to consider relativistic generalizations.

I/II framids as realized by a canonical Fermi liquid without/with nematic order. We will see that the resolution of the framon issue follows from the dynamics of the effective field theory. By treating the Goldstone boson as a Lagrange multiplier we will generate a set of constraints, that are generalizations of the Landau conditions in canonical Fermi liquids, which when imposed, lead to the proper symmetry realization. We dub this the "Dynamical Inverse Higgs Mechanism" (DIHM), because the Goldstones are absent but not for algebraic reasons. We will then go a step further and discuss a more general symmetry breaking pattern where both boost invariance and Schrodinger invariance are spontaneously broken, which we call a "type-III framid". In this case one might expect both a framon and a non-relativistic dilaton to arise. We will see that again, they do not, but their absence greatly constrains the form of the effective field theory. This analysis was recently used to prove that degenerate electrons interacting in the unitary limit can not behave like a Fermi liquid [15] in the unbroken phase.

2.0.2 The Paths to Symmetry Realization

We see that there are three paths to space-time symmetry realization: No inverse Higgs constraints are applied and the system retains one Goldstone for each broken generator. Some or all of the constraints are applied and we have a reduced number of Goldstones due to the existence of IHCs, or a Goldstone can be eliminated via the DIHM with or without the application of other inverse Higgs constraints. In this paper we will consider all three scenarios in the context of degenerate fermions. We will show two examples of DIHMs, one for boosts and the other for dilatations.

2.1 Review Coset Construction

A powerful method for generating actions with the appropriate non-linearly realized broken symmetries was developed for internal symmetries by CCWZ [31, 32] and later generalized to space-time symmetries by Volkov and Ogievetsky [11, 12]. We refer the reader to original literature for details and here only rapidly review the salient points of this coset construction. The method uses the fact that the Goldstones coordinatize the coset space G/H where G is the symmetry group of the microscopic action and H is the symmetry sub-group left unbroken by the vacuum. The vacuum manifold is parameterized by

$$U = e^{i\vec{\pi}\cdot\vec{X}} \tag{2.12}$$

where $\vec{\pi}$ are the Goldstone fields and $\vec{X}$ the corresponding broken generators. The unbroken generators will be denoted by $\vec{T}$. This parameterization will be generalized when we break space-time symmetries. As discussed below, we may use U

to write down the most-general action consistent with the symmetry breaking pattern, including terms where the Goldstone couples to other gapless (non-Goldstone) fields in the theory. Notice that the coset construction seems to imply that there must be at least one Goldstone boson. However, this need not be the case, as mentioned above. It could very well be that we can construct an invariant action without the need for a Goldstone, even without an inverse Higgs constraint. We will show that if this is indeed possible then the coset construction is a useful tool in determining this *non-Goldstone* action.

Once space-time symmetries are broken, the symmetry group is no longer compact. As such the structure constants can not necessarily be fully antisymmetric [6] and consistency requires that one generalize the vacuum parameterization to include the unbroken translations $(\bar{P}^\mu)$[7] such that

$$U = e^{i\bar{P}\cdot x} e^{i\pi\cdot X}. \tag{2.13}$$

The number of unbroken translations may be enhanced if there exist internal translational symmetries as in the case of solids or fluids [33]. In such cases the direct product of the internal and space-time translations are broken to the diagonal subgroup by the solid. In this work we will not be considering such cases as we are interested in zero temperature ground states with delocalized particles.

The Maurer-Cartan (MC) form decomposes into a set of well defined geometrical objects,

$$U^{-1}\partial_\mu U = E_\mu^A(\bar{P}_A + \nabla_A \pi^a X^a + A_A^b T^b). \tag{2.14}$$

The vierbein E relates the global frame to the transformed (acted upon by G/H) frame. In this way, the covariant derivatives on the matter fields in the local frame are written as

$$\nabla_A \psi \equiv (E_A^\mu \partial_\mu + iT^q A_A^q)\psi \tag{2.15}$$

such that under a boost

$$\nabla_A \psi \to e^{\frac{i}{2}mv^2 t - im\vec{v}\cdot\vec{x}} \nabla_A \psi. \tag{2.16}$$

From (2.14) we can extract the vierbein, the covariant derivative of Goldstone fields $(\nabla\pi)$ and the Gauge fields (A) and use these objects to construct our action which will be invariant under the full symmetry group G by forming H invariants. For a complete discussion of the coset construction and its application to broken spacetime symmetries in multiple contexts, we refer the reader to [34].

[6] A consequence of this fact is that it is not longer true that $[T, X] = X$.

[7] When translations are broken by localized semi-classical objects (i.e. defects) the coordinate is lifted to the status of a dynamical variable; see for instance [34, 38, 39].

2.2 Non-Derivatively Coupled (NDC) Goldstone Bosons

In [20] the criteria necessary to generate theories with non-derivatively coupled Goldstones is given by

$$[X_i, \vec{P}] \neq 0. \tag{2.17}$$

where X^i is a broken generator and $\vec{P}$ are the unbroken space-time translations. The authors argue that the forward scattering matrix elements of broken generators X formally diverge

$$\lim_{\vec{k}' \to \vec{k}} \langle \vec{k} \mid X \mid \vec{k}' \rangle \to \infty. \tag{2.18}$$

which compensates for the explicit factor of the Goldstone momentum in the coupling. One may be concerned with the fact that X is not a well defined operator at infinite volume, and that the limiting procedure is not well defined. However, we will see below that the coset construction supports the authors claims and allows us to, trivially, generalize their criteria to relativistic systems. (2.17) is a necessary but not a sufficient criteria for the existence of a non-derivatively coupled Goldstones since we must also ensure that it can not be removed via an IHM.

Within the coset formalism the search for non-derivative couplings starts with understanding how the Goldstones couple to generic matter fields. As such, we need to determine under what conditions a Goldstone arises in the vierbein or connection without any derivatives acting upon it. Thus a necessary condition for non-derivative coupling is the generalization of (2.17), i.e.

$$[\bar{P}^\mu, X] \neq 0. \tag{2.19}$$

Note the distinction between this criteria and (2.17). First (2.19) only involves the unbroken canonical spatial translations $\bar{P}$ which can differ from P, not only because of the zero component, but more generally if there are internal translational symmetries. This is however, a distinction without a difference because internal and space-time symmetries commute. But an important distinction between (2.17) and (2.19) is the fact that (2.19) allows for the non-commutation with the Hamiltonian as being a criteria for NDC Goldstones. As a matter of fact, this explains the NDC nature of the dilaton (both relativistic as well as non-relativistic[8]). Also we will see that whether or not G is the Poincare or Galilean group is of no consequence as far the the criteria for non-derivative coupling is concerned.

To see that (2.19) is a sufficient criteria for NDC, assuming the Goldstone boson associated with X is not removed by the inverse Higgs mechanism, we note that the veirbein will contribute to the measure via

$$S = \int d^4x \sqrt{E^2} \ldots.. \tag{2.20}$$

[8]The non-relativistic case being of particular importance below.

so that as long as the determinant of the vierbein contains a term linear in the Goldstone[9] , there will be a NDC to matter fields. From (2.14) we can see that if $[\bar{P}, X] \sim \bar{P}$ then the Goldstone associated with X will arise in E. However, the Goldstone will often be absent from the volume factor as in the case of broken boosts or rotations. Thus the first NDC will come from the covariantization of the derivatives $E_a^\mu(\pi)\partial_\mu$. Alternatively if $[\bar{P}, X] \sim T$, then the Goldstone will show up in the connection, in which case the NDC will arise from the covariant derivative acting on the matter fields.

Finally, note that if G is the Galilean group then due to relation the Eq. (2.11) if the $U(1)$ particle number is unbroken, then the boost Goldstone will be associated with the connection. Whereas if G is the Poincaré group then the boost will be in the vierbein. But in either case framid will be non-derivatively coupled.

2.3 Framids

2.3.1 Non-Relativistic Framids

The type I framid as defined in [9] is a system where boost invariance is spontaneously broken, but all other space time symmetries are intact. The coset construction only cares about the symmetry breaking pattern and not the definite choice of the order parameter. As was emphasized in [37] the choice of order parameters can affect how the symmetry is realized if there exist gapped Goldstones (assuming the gap size is hierarchically small compared to the cut-off). In particular the representation of the order parameter(s) will determine whether or not the inverse Higgs conditions (2.5) leads to a redundancy or a gap. However, here we are only interested in the truly gapless modes, so in this respect the order parameter will be irrelevant. Nonetheless, we are interested in a certain class of order parameter, i.e. those whose commutator with boost generators have a non-vanishing vacuum expectation value (e.g. the momentum density). This class of order parameters yield Goldstones which are collective excitations. Whereby a "collective excitation" we mean a quasi-particle pole (or resonance) which exists as a consequence of the fact that the vacuum is not annihilated by some conserved charge. Put another way, the modes are excitations of the material responsible for the breaking of boost invariance. This definition sets apart say the pion in QCD from the plasmon in a metal.

Cases where the framid are not collective modes correspond to speculative theories beyond the standard model of particle physics and Relativity, such as Einstein-Aether theory [41], where a four vector gets a time-like expectation value.

$$\langle A_\mu \rangle = n_\mu. \tag{2.21}$$

[9]That E contains term linear in the Goldstone follows from the fact that the Goldstone acts as the transformation parameter.

The resulting theory contains three Goldstone modes corresponding to the framons [42]. The lack of the evidence for a Goldstone arising in Einstein-Aether theory allows us to place bounds on the couplings (see for instance [43]). However, we know that condensed matter systems break boost, and if the symmetry breaking pattern is such that there are no IHC around to eliminate the framids from the spectrum it is incumbent upon us to determine their fate.

It is tempting to disregard boost Goldstones since the associated generator does not commute with the Hamiltonian and hence there is no flat direction. However, the existence of the relativistic dilaton immediately dispels this notion. Furthermore, the inclusion of the framid into the coset parameterization is *necessary* for consistency. Moreover, according to the criteria for NDC (2.19) we should expect the coupling to the framid to be at least marginal.

To manifest framids in the laboratory we need systems which break boosts yet whose ground state does not break any symmetry which would lead to an inverse Higgs constraint. Thus we may eliminate electrons moving in a crystal background as well as superfluids/superconductors from the list of possibilities. It would seem that we are relegated to degenerate electrons in the unbroken phase.

One might be concerned that the Kohn and Luttinger [2] effect ensures that *all* Fermi liquids superconduct, even if the coupling function is repulsive in all channels in the UV. However, all we really need to manifest a framon is for there to be a temperature window between the boost symmetry breaking scale (E_F), and the critical temperature T_c. For a Fermi liquid the critical temperature scales as

$$T_c \sim \Lambda_\star \ll E_F \tag{2.22}$$

where $\Lambda_\star$ is the strong coupling scale which is typically exponentially suppressed. Thus there is a range of temperatures where the framid should contribute to the heat capacity. This is as opposed to the bosonic case where the critical temperature is set by the number density

$$T_C \sim n^{-1/3}. \tag{2.23}$$

and the boost symmetry breaking scale is of the same order.

Thus we have narrowed our search for framons to degenerate Fermi gases whose phenomenology certainly shows no signs of non-derivatively coupled Goldstone. One might be tempted to interpret zero sound as the boost Goldstone, however, the interaction between electrons due to zero sound exchange vanishes in the forward scattering limit.

2.3.2 Coset Construction of Fermi Liquid EFT with Rotational Symmetry: Type I Framid

We begin our investigation by building the coset construction for type I framids (i.e. systems with broken boosts but unbroken rotations). We consider the case of broken Galilean invariance, as the relativistic case will follow in a similar manner.

The vacuum manifold is parameterized by

$$U = e^{iP\cdot x} e^{-i\vec{K}\cdot\vec{\eta}(x)} \tag{2.24}$$

Calculating the MC-form, we can extract the vierbein

$$E_0^0 = 1 \quad E_i^j = \delta_i^j, \quad E_0^i = \eta^i, E_i^0 = 0. \tag{2.25}$$

The gauge field is given by

$$A_i = -\eta_i, A_0 = -\frac{1}{2}\vec{\eta}^2 \tag{2.26}$$

and the covariant derivatives of the framids are (up to lowest order in fields and derivatives)

$$\nabla_0 \eta^i = \dot{\eta}^i \quad \nabla_i \eta^j = \partial_i \eta^j. \tag{2.27}$$

The free action for the Goldstone follows by writing down all terms which are invariant under the linearly realized H symmetry

$$S = \int d^d x dt \Big(\frac{1}{2}\dot{\eta}_i^2 - \frac{1}{2}u_T^2(\partial_i\eta_j)^2 - \frac{1}{2}u_L^2(\partial\cdot\eta)^2\Big) \tag{2.28}$$

Following eq. (2.15), the coupling for the Goldstone to matter fields via the covariant derivative is given by

$$S_0 \;=\; \int d^d x dt\ \psi^\dagger \left[i(\partial_0 + \eta^i\partial_i) + \frac{1}{2}m\vec{\eta}^2 + \varepsilon(i\partial_i + m\eta_i)\right]\psi \tag{2.29}$$

where ε is the unknown dispersion relation that is fixed by the dynamics. Due to the central extension of the Galilean algebra, the fermion under a boost transformation with velocity $\vec{v}$ transforms as

$$\psi(x,t) \to e^{\frac{i}{2}m\vec{v}^2 t - im\vec{v}\cdot\vec{x}}\psi(x,t). \tag{2.30}$$

while the Goldstone field η undergoes a shift

$$\vec{\eta} \to \vec{\eta} + \vec{v}. \tag{2.31}$$

The η^2 term will be sub-leading and not play role in the remainder of our discussion.

As in the standard EFT description of Fermi liquids [5, 6] the quasi-particle self interaction is most conveniently written in momentum space

$$S_{int} = \prod_{i,a} \int d^d k_i dt\ g(\vec{k}_i + m\vec{\eta})\psi^\dagger_{k_1}(t)\psi_{k_2}(t)\psi^\dagger_{k_3}(t)\psi_{k_4}(t)\delta^d\left(\sum_i k_i\right) \tag{2.32}$$

Higher order polynomials in the matter field ψ are technically irrelevant (see below). g is the coupling function which now formally depends upon the framon. The assumption of spherical symmetry implies g is a scalar. Notice that the η is non-derivatively coupled, as expected from our considerations of the algebra, which can lead to non Fermi liquid behavior. Given that He^3, e.g., is well described by Fermi liquid theory, the framid must somehow decouple, yet it must do so in such a way that the theory remains boost invariant.

2.3.3 Multiple Realizations Of Broken Symmetry

Before moving onto further discussion about the framids in Fermi liquids, we want to highlight a subtle point about non-linear realizations of broken symmetries, which is that the same symmetry breaking pattern can lead to contrasting physical theories with very different particle content. This usually happens when there are two different order parameters. However, below we show that even with same order parameter we can have two different realizations of the symmetry. An example of this is the case of a massive complex scalar particle (ϕ) coupled to gauge fields. To power count this theory it is useful to introduce the notion of a field label as was introduced in Heavy Quark Effective Theory (HQET) where one is interested in the dynamics of a massive source which interacts with light gauge fields carrying momenta much less than the quark mass. The label is introduced by defining a re-phased field

$$\phi(x) = \sum_v e^{imv\cdot x} h_v(x), \tag{2.33}$$

such that v defines a superselection sector[44]. Derivatives acting on $h_v(x)$ scale as "residual momenta" (k) which obey $k \ll m$. The vacuum of the system, labeled by v, breaks boost invariance and so we expect that framid should exist as an independent degree of freedom. Typically, the Goldstone modes are associated with collective excitations of a system which are clearly absent as the choice of vacuum is not dynamical. Nonetheless the boost invariance must be non-linearly realized.

Using the covariant derivatives derived in the previous section, we can write down the most general action for ϕ which is invariant under translations and rotations,

$$\begin{aligned}\mathcal{L}_\phi &= \frac{i}{2}\left(\phi^\dagger(t,\vec{x})(\partial_t + \vec{\eta}\cdot\vec{\partial} + \frac{i}{2}m\vec{\eta}^2)\phi(t,\vec{x}) - [(\partial_t + \vec{\eta}\cdot\vec{\partial} - \frac{i}{2}m\vec{\eta}^2)\phi^\dagger(t,\vec{x})]\phi(t,\vec{x})\right) \\ &+ \frac{c_1}{2m}((\vec{\partial} + im\vec{\eta})\phi^\dagger(t,\vec{x}))(\vec{\partial} - im\vec{\eta})\phi(t,\vec{x}).\end{aligned} \tag{2.34}$$

If we choose $c_1 = 1$ then η decouples from ϕ and we get the standard non-relativistic kinetic term for a free particle. Had we started with a theory without the η, then c_1 can be fixed by requiring the theory to obey Galilean algebra, in particular by satisfying the commutator $[H, K_i] = iP_i$. c_1 can equally well be fixed by Reparametrization Invariance (RPI) [46], which is related to the freedom in splitting the heavy quark momentum into a large and small piece (more on this below). However we can leave c_1 to be completely arbitrary and keep η in the spectrum and the theory will still respect all the symmetries. The two theories (with and without η) are completely different and we have no reason to believe they will lead to same physics in the IR and yet they have the same symmetry breaking pattern and the same order parameter (local momentum density). Thus there are multiple ways of realizing the boost symmetry. While it would seem that this is a rather trivial example, we note that only difference between the HQET ground state and that of a Fermi liquid lies in the change in the number density from one to Avogadro's number.

2.3.4 Power Counting

To determine the possible symmetry realization in a Fermi liquid, we must first discuss the systematics of the relevant EFT whose action is given by (2.29). The matter fields (which we will call electrons from here on) are effectively expanded around the Fermi surface, by removing the large energy and momentum components via the redefinition

$$\psi(x) = \sum_{\theta} e^{i\varepsilon(k_F)t} e^{-i\vec{\mathbf{k}}(\theta)\cdot\vec{x}} \psi_{\vec{\mathbf{k}}(\theta)}(x), \tag{2.35}$$

the assumption of rotational invariance implying that the magnitude of $|\vec{\mathbf{k}}(\theta)| = k_F$. The field label $\vec{\mathbf{k}}(\theta)$ is the large momenta around which we expand. As opposed to the HQET case, here the bins are dynamical and there is no super-selection rule. This case is more akin to NRQCD [45] where the labels change due to Coulomb exchange. Notice that there is a sum over the labels as opposed to an integral, this illustrates the fact that we have effectively tessellated the Fermi surfaces into "bins". The size of each bin will scale as $\lambda \sim E/E_F$. The fact that theory should not depend upon the bin size imposes constraints on the action. That is, we should be able deform the momentum around any fixed value we wish, by an amount scaling as λ, and the theory should be invariant. This re-parameterization invariance (RPI) [46] implies that the action can only be a function of $\vec{k}_F + \vec{\partial}$. In general RPI generates relations between leading order and sub-leading Wilson coefficients. It is convenient for power counting purposes to introduce a label operator $\mathcal{P}$ [47] such that

$$\vec{\mathcal{P}}\psi_{\vec{\mathbf{k}}(\theta)}(x) = \vec{\mathbf{k}}(\theta)\psi_{\vec{\mathbf{k}}(\theta)}(x). \tag{2.36}$$

Full theory derivatives then decompose into the RPI invariant combination $\vec{\mathcal{P}} + i\vec{\partial}$. In this way we may drop the exponential factors as long as we assume label

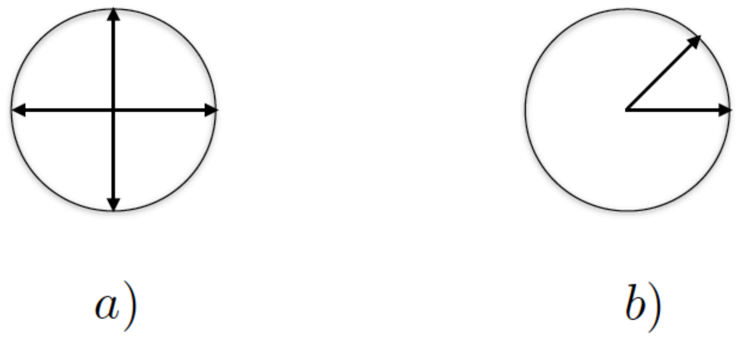

Figure 2.1: Allowed kinematic configuration for quasi-particle scattering. Diagram (a) is the BCS back to back configuration which leads to Cooper pair condensation. (b) Forward scattering, in which the final state momenta lie on top of the initial state momenta.

momentum conservation at each vertex. The action becomes

$$S_0 = \sum_{\vec{\mathbf{k}}(\theta)} \int d^d x dt \; \psi^\dagger_{\vec{\mathbf{k}}(\theta)}(x) \left(i\partial_0 - \vec{\eta}(x) \cdot (\vec{\mathcal{P}} + i\vec{\partial}) + \tfrac{1}{2} m \vec{\eta}(x)^2 \right) \psi_{\vec{\mathbf{k}}(\theta)}(x)$$
$$+ \int d^d x dt \; \psi^\dagger_{\vec{\mathbf{k}}(\theta)}(x) \left(\varepsilon(\vec{\mathcal{P}} + i\vec{\partial} + m\vec{\eta}(x) - \mu) \right) \psi_{\vec{\mathbf{k}}(\theta)}(x) \qquad (2.37)$$

Notice that the interaction with η does not change the quasi-particle label. The reasons for this will be discussed below once we have fixed the power counting systematics. Under a boost the labels are left invariant but the residual momentum shifts. Furthermore under a boost the time derivative transforms as

$$i\partial_0 \to i\partial_0 + \vec{v} \cdot \vec{\mathcal{P}}. \qquad (2.38)$$

Review of EFT of Fermi Liquids Scalings

We first review the EFT of Fermi liquids and its power counting (for details see [4, 5, 6]). In the EFT, the power counting is such that the momenta perpendicular to the Fermi surface ($k_\perp$) scale as $\lambda \sim E/\Lambda$, where the theories' breakdown scale is $\Lambda \sim E_F$. With this scaling, the most relevant terms in the action come from expanding the energy and coupling function around the Fermi surface and keeping the leading term in $k_\perp$.

The scaling of the electron field

$$\psi(\vec{k}, t) \sim \lambda^{-1/2}, \qquad (2.39)$$

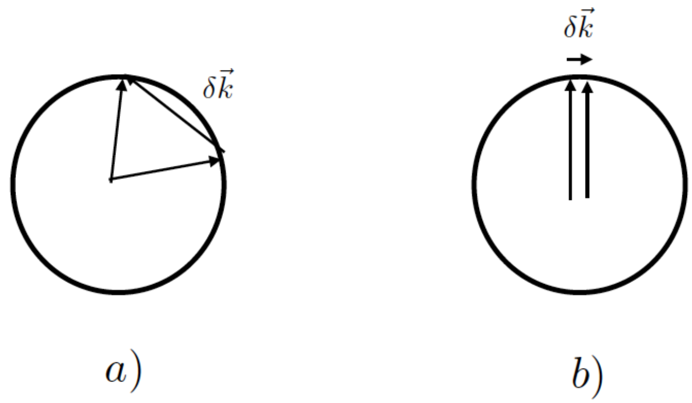

Figure 2.2: Allowed kinematic configuration for framid-quasi-particle scattering. Diagram (a) involves an off-shell framid, which can be integrated out. (b) shows the interaction with a soft framid leading to near forward scattering.

follows from the equal time commutator

$$\{\psi(\vec{k}, t=0), \psi^\dagger(\vec{p}, t=0)\} \sim \delta(k_\perp - p_\perp)\delta^{d-1}(k_\parallel - p_\parallel) \sim 1/\lambda \tag{2.40}$$

since $k_\parallel$ does not scale. Thus ignoring the framon for the moment, the leading order action is given by [10]

$$\begin{aligned} S_{FL} &= \sum_{\vec{\mathbf{k}}} \int d^d l dt\ \psi^\dagger_{\vec{\mathbf{k}}}(t,l)(i\partial_0 + \vec{l}_\perp \cdot \vec{v}_F)\psi_{-\vec{\mathbf{k}}}(t,-l) \\ &+ \sum_{\vec{\mathbf{k}}_i} \int d^d l_i dt\ \frac{g(\vec{\mathbf{k}}_i)}{2}\psi^\dagger_{\mathbf{k_1}}(t,l_1)\psi_{\mathbf{k_2}}(t,l_2)\psi^\dagger_{\mathbf{k_3}}(t,l_3)\psi_{\mathbf{k_4}}(t,l_4)\delta^d(\sum_i l_i) \end{aligned} \tag{2.41}$$

The Fermi velocity defined as $\vec{v}_F = \frac{\partial \varepsilon}{\partial k^i_\perp}|_{k_F}$ is constant on a spherically symmetric Fermi surface. In the last term there is a Kronecker delta for the label momenta that is implied. The residual momenta scale as $l_\perp \sim \lambda$ and $l_\parallel \sim 1$. The latter scaling might seem odd given that it is a residual momentum. However, $l_\parallel$ scales as the bin size, which does not play a role for Fermi surfaces which are featureless. Another way of saying this is that the $l_\parallel$ integral can be absorbed into the label sum.

Naively the interaction terms looks irrelevant because the delta function scale as λ^0 for generic kinematic configurations so that, once the scaling of the measure is taken into account ($\sim \lambda^3$), the operator will scale like λ. However, there are

[10] We are ignoring spin as it will not play a role in our discussion.

two configurations for which one of the delta function will scale as $1/\lambda$: The BCS configuration (back to back incoming momenta) and forward scattering. These two configurations are shown in figure (1), where it can be seen that these are the only two possible configurations that allow for momentum conservation that keep all momentum within λ of the Fermi surface.

It is convenient to decompose the BCS coupling into partial waves g_l. A one loop calculation (which is exact) of the beta function shows that g_l are either marginally relevant/irrelevant for attractive/repulsive UV initial data. The forward scattering coupling does not run, but plays an important role in the IR nonetheless. Interestingly, below we will show that Galilean invariance is sufficient to prove that the forward scattering and BCS kinematics are the only possible marginal/relevant interactions. This result follows without the need to consider the effects of the special kinematics on the power counting of the four Fermi operator.

Power Counting in the Coset Construction

Let us now derive the power counting from the coset construction.

We begin with the kinematics of the framid interactions. The two allowed scattering configurations are shown in figure (2). Figure (a) shows the interaction of a quasi-particle with a framid that is far off its mass shell in the sense that $E \ll k$, in the EFT language this would be called a "potential framid" and can be integrated out. Thus these interactions are swept, along with those of the phonon and screened electromagnetic interactions, into a non-local coupling. Note that the potential is effectively local because the labels on the incoming and outgoing quasiparticles can not be the same and hence it is analytic in (the small) residual momenta[11]. Figure (b) shows the interaction with an on-shell framon whose momentum is necessarily soft $k \sim E \ll E_F$. If we define our power counting parameter as $\lambda \sim E/E_F$, then we only know that $k \sim \lambda^n$, where n is yet to be determined. However, symmetries fix n as the covariant derivative must scale homogeneously in λ for the theory to be boost invariant. That is, η must scale in the same way as the residual momentum of order λ, so that $\eta \sim \lambda$. Given that $\partial \sim \lambda^n$ we can fix n by considering the canonical commutator

$$[\eta^i(x), \dot{\eta}^j(0)] \sim \lambda^{n+2} \sim \delta^d(x)\delta^{ij} \sim \lambda^{dn} \tag{2.42}$$

thus $n = \frac{2}{d-1}$. Thus we see that in two spatial dimensions $k \sim \lambda^2$ and the framons can not change the (residual) momentum of the quasi-particles and only their zero mode is relevant. This however is not the case in three dimensions where the

[11]In this sense it is better to think of the $1/k^2$ in the interaction as a Wilson coefficient.

framon carries off residual momentum $k \sim \lambda$. Expanding the action (2.37)

$$\begin{aligned} S_0 &= \sum_{\vec{\mathbf{k}}(\theta)} \int d^d x dt \ \psi^\dagger_{\vec{\mathbf{k}}(\theta)}(x) \left[i\partial_0 - \vec{\eta}(x) \cdot \vec{\mathbf{k}}(\theta) + (i\vec{\partial} + m\vec{\eta}(x)) \cdot \frac{\partial \varepsilon}{\partial k} \right] \psi_{\vec{\mathbf{k}}(\theta)}(x) + ... \\ &= \sum_{\vec{\mathbf{k}}(\theta)} \int d^d x dt \ \psi^\dagger_{\vec{\mathbf{k}}(\theta)}(x) \left[i\partial_0 - \vec{\eta}(x) \cdot \vec{v}_F(\theta)(m - m^\star) + i\vec{v}_F(\theta) \cdot \vec{\partial} \right] \psi_{\vec{\mathbf{k}}(\theta)}(x) + ... \end{aligned} \tag{2.43}$$

where $m^\star$ is the effective mass defined by $\frac{\partial \varepsilon}{\partial \vec{k}} = \vec{v}_F = \frac{\vec{k}_F}{m^\star}$. In two dimensions we must multipole expand the framon field to preserve manifest power counting [48], which leaves only the coupling to the framon zero mode. The leading order action is given by

$$S_0^{d=2} = \sum_{\vec{\mathbf{k}}(\theta)} \int d^2 x dt \ \psi^\dagger_{\vec{\mathbf{k}}(\theta)}(x) \left[i\partial_0 - \vec{\eta}(0) \cdot \vec{v}_F(\theta)(m - m^\star) + i\vec{v}_F(\theta) \cdot \vec{\partial} \right] \psi_{\vec{\mathbf{k}}(\theta)}(x) + ... \tag{2.44}$$

From here on to simplify the notation we will be dropping the label sum and the bold font for labels as all momenta unless stated otherwise will be labels.

Before we move on to determine the consequences of the multipole expansion let us pause to clarify this unusual scaling. Typically in an EFT the scaling of the fields follows from the scaling of the momenta not the other way around as in this case. Indeed, it would be useful to understand what happens to loops with momenta scaling as λ and not λ^2. However, symmetries forbid such contributions and it must be that if we do not multipole expand the framon interaction, that power counting and boost invariance are incompatible.

Thus we see that in two spatial dimensions, the symmetries can not be realized via a Goldstone as the framon equations of motion allow us to eliminate it from the theory, as will be discussed below. In three spatial dimensions this conclusion does not follow.

The Framid as Lagrange Multiplier and the Landau Relation

Let us consider the ramifications of the multipole expansion of the framon in two dimensions.[12] Since the kinetic piece of the framon action vanishes for the constant zero mode η plays the role of a Lagrange multiplier.

Expanding the action for the four-Fermi interaction term leads to the coupling

$$S_{int} = \prod_i \int d^d k_i dt \sum_j \frac{m}{2} \vec{\eta} \cdot \frac{\partial g(k_j)}{\partial \vec{k}_j} \psi^\dagger_{k_4}(t) \psi^\dagger_{k_3}(t) \psi_{k_2}(t) \psi_{k_1}(t) \delta^d \left(\sum_i k_i \right). \tag{2.45}$$

[12] Even though our arguments in this section are strictly valid only for d=2, we keep d arbitrary to generalize it latter to d=3.

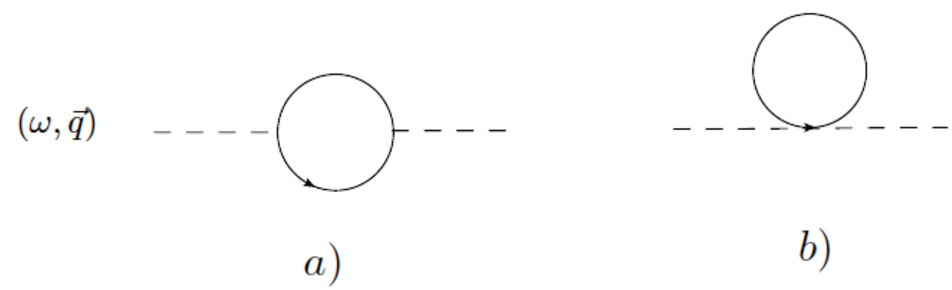

Figure 2.3: Diagram a) could contribute to wave function renormalization whereas both a) and b) could contribute to a mass. At zero external momentum the two diagrams cancel as dictated by boost invariance.

Using the equations of motion for η gives the operator constraint $O_i^B = 0$ where

$$\begin{aligned} O_i^B &= \int \frac{d^d p}{(2\pi)^d}\, \psi_p^\dagger(t) \left(p_i - m \frac{\partial \varepsilon_p}{\partial p_i} \right) \psi_p(t) \\ &- \frac{m}{2} \int \prod_{a=1}^{4} \frac{d^d p_a}{(2\pi)^d} \delta^{(d)} \left(\sum_i p_i \right) \left(\sum_i \frac{\partial g(p_a)}{\partial p_{i,a}} \right) \psi_{p_4}^\dagger(t) \psi_{p_3}^\dagger(t) \psi_{p_2}(t) \psi_{p_1}(t). \end{aligned} \tag{2.46}$$

This is a strong operator constraint both technically and colloquially. Notice that the constraint is non-local in the sense that it is integrated. This is crucial, as the constraint is a function of the Noether charges. Indeed, current algebra imposes this same constraint $O_i^B = 0$ as shown in the appendix where we also derive the relativistic generalization of this constraint.

The power counting of the terms in this constraint deserve attention. The first two terms scale as λ^0 while the last term naively scales as λ^2 since the measure scales as λ^4.

Recall that at this point we have not made any assumption about special kinematics so the delta function does not scale. We are trying to derive the fact that the only relevant couplings have these special kinematics. Thus we might naively think that we can drop the quartic term in the constraint. In general this is true, but there is an exception as we now explain. We begin by noticing that the quartic term is time dependent while the quadratic terms (being conserved charges) are not. Thus it would seem that the last term must vanish (to the order we are working). However, if we insert the quartic term in a two point function the time dependence will cancel.

Consider taking matrix of element of (2.46) in a 1-particle state with momentum $\vec{k}$ ($|\vec{k}| = k_F$)

$$k_i = m\frac{\partial\varepsilon(k)}{\partial k_i} + \frac{2m}{(2\pi)^d}\int d^dp\Big(\frac{\partial g(k,p,p,k)}{\partial p_i} + \frac{\partial g(k,p,p,k)}{\partial k_i}\Big)\theta(p_F - p) \tag{2.47}$$

We can now see why that the interaction term is enhanced because the radial integral, naively scaling as λ is actuality scaling as order one. This is a consequence of the power divergence of the integral. Such mixing of orders is commonplace in effective field theories when a cut-off regulator is used. However, here the cut-off (the radius of the Fermi surface) is physical[13].

We re-write this result in the form

$$k_i = m\frac{\partial\varepsilon_k}{\partial k_i} + 2m\frac{\partial}{\partial k_i}\int\frac{d^dp}{(2\pi)^d}\ \theta(p_F - p)g(p,k) + 2m\int\frac{d^dp}{(2\pi)^d}\ g(p,k)\delta(\varepsilon_F - \varepsilon)\frac{\partial\varepsilon_p}{\partial p_i}. \tag{2.48}$$

The second term on the RHS vanishes by spherical symmetry.

Next using the assumption of rotational invariance, and expanding the coupling function in Legendre polynomials, $g(\theta) = \sum_l g_l P_l(\cos\theta)$[14] we get

$$\frac{k_F}{m} = v_F + \frac{2p_F}{(2\pi)^2}\int d\theta\ \cos\theta\sum_l g_l P_l(\cos\theta). \tag{2.49}$$

Using this result we get the famous Landau relation [1] for a Fermi Liquid

$$\frac{m^\star}{m} = 1 + \frac{1}{3}\frac{2m^\star}{(2\pi)^2}g_1 \tag{2.50}$$

Notice that at this point it is not clear that this result will hold to all orders in perturbation theory.

It is interesting to ask whether or not more information can be extracted from the constraint by considering a two body state. However, as is seen by inspection the insertion of the constraint operator on external lines will will automatically be satisfied once the Landau condition is imposed, and furthermore the insertion of the quartic function into a four point amplitude will be suppressed since there is no power divergence that can enhance its scaling.

We can glean more information from the Landau criteria by utilizing the fact that the equation is RG invariant. Differentiating it with respect to the RG scale implies that the beta function vanishes. For generic momenta the four point one loop interaction diverges logarithmically. To avoid this conclusion we impose a

[13]In canonical EFT's one uses dimensional regularization exactly to avoid this mixing issue which complicates the power counting.

[14]We take d=2 for sake of simplicity but the results are valid for arbitrary d.

kinematic constraint to suppress the one loop result. If we consider the forward scattering interaction,

$$S_F = \int d^dxdt \sum_{\mathbf{k}(\theta_i)} g(\theta_1,\theta_2)\psi^\dagger_{\mathbf{k}(\theta_1)}(x,t)\psi_{\mathbf{k}(\theta_1)}(x,t)\psi^\dagger_{\mathbf{k}(\theta_2)}(x,t)\psi_{\mathbf{k}(\theta_2)}(x,t) \tag{2.51}$$

then the one loop result *vanishes* since the constraints imply that the loop involves no sum over the large label bins leading to a power suppressed result.

It would seem that we have ruled out the possibility of a BCS interaction which has a non-vanishing beta function at one loop. However, this is not the case as such an interaction would not contribute to the Landau relation since the tadpole diagram vanishes for the BCS interaction.

Thus we have reached the conclusion that the only allowed interactions are BCS and forward scattering. We are not claiming that this is a rigorous proof since we have assumed that the only sensible coupling with vanishing beta function is forward scattering. Furthermore, our argument regarding the acceptability of the BCS coupling is based on the fact that our arguments allow for any coupling which leads to a vanishing tadpole (with no associated counter-term). It is possible that there are other allowed kinematic configurations, however, assuming a featureless Fermi surface[15] we have not been able to find any sensible examples.

Recall at this point the result in the section only hold at one loop. However, now that we have restricted our interactions to BCS and forward scattering we know that that the Landau relation holds to all orders. This well known result follows from the fact that tadpole corrections to the one loop insertion of the constraint are pure counter-term and vanish.

Finally recall that this result assumed that the framon acts as a Lagrange multiplier. However, this was only forced upon us in two dimensions. In three dimensions, there is the logical possibility that the framon remains in the spectrum and there is no DIHM at play. This will be discussed below when we list the possible paths to symmetry realization in three dimensions.

2.4 Fermi Liquid with broken rotational invariance

Let us now consider the case where the rotational symmetry is broken by the Fermi surface (the typeII framid). We work in two spatial dimensions for the sake of simplicity. Again, to avoid an algebraic inverse Higgs constraints, we assume that the $U(1)$ particle number is unbroken. The vacuum will be parameterized by

$$U(\vec{\eta},\Theta,x) = e^{iP\cdot x}e^{-i\vec{K}\cdot\vec{\eta}(x)}e^{-iL\Theta(x)} \tag{2.52}$$

[15] In cases where the Fermi surface is singular there are other relevant interactions whose self contractions would vanish [50] algebraically.

The rotational Goldstone boson (Θ) is called the "angulon" has been studied in the context of electronic systems [16] as well as in neutron stars [49], although to our knowledge its non-linear self interactions have not been previously derived.

Calculating the MC-form we may extract the vierbein

$$E_0^0 = 1 \quad E_i^j = R_i^j(\Theta), \quad E_0^i = R^{ij}(\Theta)\eta^j, E_i^0 = 0. \tag{2.53}$$

where $R(\Theta)$ is the two dimensional rotation matrix. The gauge fields are

$$A_i = -R_{ij}(\Theta)\eta_j, \quad A_0 = -\frac{1}{2}\vec{\eta}^2. \tag{2.54}$$

the covariant derivatives of the angulons are

$$\nabla_0\Theta = \dot{\Theta}_i + \vec{\eta}\cdot\vec{\partial}\Theta, \quad \nabla_i\Theta = R_{ij}(\Theta)\partial_j\Theta = \partial_i\Theta + \epsilon_{ij}\Theta\partial_j\Theta + ... \tag{2.55}$$

The quadratic piece of the quasiparticle action is given by

$$S_\psi = \int d^dxdt\ \psi^\dagger(\vec{x},t)\left[i\left(\partial_0 + R_{ij}(\Theta)\eta_j\partial_i\right) + \frac{1}{2}m\vec{\eta}^2 + \varepsilon\left(R(\Theta)_{ij}(i\partial_j + m\eta_j)\right)\right]\psi(\vec{x},t) \tag{2.56}$$

The kinetic piece of the angulon Lagrangian consistent with time reversal and parity invariance is given by

$$L_{KE} = (\dot{\Theta})^2 + D^{ij}(\nabla_i\Theta)(\nabla_j\Theta), \tag{2.57}$$

so the angulon is a "type I" Goldstone, i.e. $E \sim p$. Unlike the framid, the angulon scaling is not fixed by symmetry and its momentum scaling is determined by the maximum momentum transfer consistent with the effective theory i.e. the scattering of an electron with an angulon should leave the electron near the Fermi surface to within λ thus the angulon momentum scales as λ and following the same arguments as above the field $\Theta(x) \sim \lambda^{1/2}$ in two spatial dimensions and as λ in three.

Expanding the action (2.56) and keeping on the leading order piece we have

$$S_\psi = \int d^dxdt\ \psi^\dagger\left[i\partial_0 + i\vec{\eta}\cdot\vec{\partial} + \vec{v}_F\cdot(i\vec{\partial} + m\vec{\eta}) + i\Theta v_F^i\partial^j\epsilon_{ij}\right]\psi. \tag{2.58}$$

We see that for $d = 2$, the interaction with the angulon is relevant and thus destroys Fermi liquid behavior. In $d = 3$ it is classically marginal and the fate of Fermi liquid behavior is determined by the sign of the beta function for this coupling.

Notice that the breaking of rotational symmetry does not effect the operator relation (2.50) imposed by the non-linearly realization of boost invariance. However, at least in two spatial dimensions, the Landau relation (2.50) is no longer justified, as the angulon coupling becomes strong in the IR and quasi-particle picture breaks down. In three dimensions it is possible that a perturbative result for the Landau relation could follow if the theory remains weakly coupled. In any case the operator constraint (2.46) must hold for the system to be boost invariant. However, in strong coupling it is not easy to deduce the physical ramifications. It would be interesting to utilize this constraint to generate new prediction in systems with broken rotational symmetry. In particular it is interesting to ask whether or not one can impose a DIHC to eliminate the angulon from the spectrum.

2.4.1 The Stability of Goldstone Boson Mass Under Renormalization

As can be seen from the actions (2.28) and (2.57), a Goldstone boson mass is forbidden despite the fact that the Goldstone boson need not be derivatively coupled. There are no gapped Goldstones as a consequence of the fact that there is no inverse Higgs mechanism for our chosen symmetry breaking pattern. If there is no anomaly then we should expect that this masslessness should persist to all orders in perturbation theory, indeed it should hold non-perturbatively. Vishwanath and Watanabe showed the cancellation of angulon mass correction at one loop [20] but they did not consider the framon. Given that we have constructed the full action, the all orders proof follows from the Ward identity. Nonetheless is it instructive to study the one loop case in order to distinguish the framon from the angulon. The Goldstone mass can be read off by considering the quadratic piece of the effective action generated by integrating out the electrons in a constant Goldstone background. For the angulon we find

$$\Gamma[\theta] = i\frac{\int d\omega d^d p Log\left[\omega - \varepsilon(R(\theta)\vec{p})\right]}{\int d\omega d^d p Log\left[\omega - \varepsilon(\vec{p})\right]} \tag{2.59}$$

which is independent of θ as a consequence rotational invariance of the measure. This result tells us that, at the level of the integrals, there must be an algebraic cancellation between the two diagrams which contribute to the mass at one loop shown in figure 3. Note this should NOT be expected for the framon, since boost symmetry breaking is sensitive to the UV scale E_F, whereas the angulon only knows about the shape of the Fermi surface and not its depth. Of course, boost invariance dictates the framon mass must vanish if we use a boost invariant regulator, i.e. not a cut-off. The situation is analogous to the case of the dilaton whose mass corrections vanish in dimensional regularization but necessitates counter-terms when using a cut-off. Such counter-terms should not be considered fine tuning.

2.5 Broken Conformal symmetry: Eliminating the non-Relativistic Dilaton

As mentioned in the introduction, consequences of spontaneous breaking of conformal invariance in non-relativistic systems is unique as the non-relativistic kinetic term for the dilaton appears to be in tension with boost invariance [30]. As such, we will study systems for which the broken symmetries are dilatations (D), special conformal transformations (C) and boosts (K_i). The relevant commutators of the Schrodinger group (the non-relativistic conformal group) are

$$[H, C] = iD \tag{2.60}$$

$$[P_i, C] = -iK_i \tag{2.61}$$

as these relations imply a reduction in the naive number of Goldstones. Furthermore, note that (2.60) implies that if dilatations are broken then so are the special conformal transformations. The vacuum is parameterized via

$$U = e^{iP\cdot x}e^{-i\vec{K}\cdot\vec{\eta}}e^{-iC\lambda}e^{-iD\phi}. \tag{2.62}$$

The algebra implies that both λ and ϕ are redundant degrees of freedom. The ensuing vierbein is given by

$$E_0^0 = e^{-2\phi} \quad E_0^i = \eta^i e^{-\phi} \quad E_i^j = \delta_i^j e^{-\phi}. \tag{2.63}$$

The gauge fields are

$$A_0 = -\frac{1}{2}\vec{\eta}^2 e^{2\phi} \quad \vec{A} = -\vec{\eta}e^{\phi} \tag{2.64}$$

and the covariant derivatives are given by

$$\begin{aligned}
\nabla_j \eta^i &= e^{2\phi}(\lambda\delta_j^i + \partial_j\eta^i) && (2.65)\\
\nabla_0 \eta^i &= -e^{3\phi}(\dot{\eta}^i + \vec{\eta}\cdot\vec{\partial}\eta^i) && (2.66)\\
\nabla_0 \phi &= -e^{2\phi}(\lambda + \dot{\phi} + \vec{\eta}\cdot\vec{\partial}\phi) && (2.67)\\
\nabla_i \phi &= e^{\phi}\partial_i\phi && (2.68)\\
\nabla_0 \lambda &= -e^{4\phi}(\dot{\lambda} + \vec{\eta}\cdot\vec{\partial}\lambda + \lambda^2) && (2.69)\\
\nabla_i \lambda &= e^{3\phi}\partial_i\lambda && (2.70)
\end{aligned}$$

The invariance of these objects under boosts, dilatations and special conformal transformations follows by first determining the non-linear transformation properties of the Goldstones via the relation

$$gU(\pi) = U(\pi', g)h(g, \pi), \tag{2.71}$$

where $g \in G$ and $h \in H$. Table 2.1 gives the resulting transformation properties of the Goldstones.

We see that there are two possible inverse Higgs constraints coming from setting the covariant derivatives in (2.65) and (2.67) to zero. Linearizing yields the two possible inverse Higgs relations from (2.65) and (2.67)

$$\begin{aligned} \lambda &= -\dot{\phi} + ... \\ \lambda &= \frac{1}{3}\vec{\partial} \cdot \vec{\eta} + ... \end{aligned} \tag{2.72}$$

Let us now address the question of the possible symmetry realizations. We will see that no matter what path is chosen, the systems will not behave like a canonical Fermi liquid [15]. We may choose not to eliminate any Goldstones, however note that in this case, the λ gets gapped as (2.69) is time reversal invariant and thus an allowed term in the action without squaring it. This realization includes two non-derivatively coupled Goldstones which would invalidate a Fermi liquid description [20]. If we use one IHC then again we will have the same spectrum and the same conclusion is reached. Finally we may consider using both constraints such that we equate

$$\dot{\phi} = \partial \cdot \eta \tag{2.73}$$

which would lead to a theory which appears non-local.[16] Thus, although we have two possible constraints we can only impose one while maintaining locality. This is a consequence of the fact we have

$$[H, C] = iD \qquad [C, P] = iK, \tag{2.74}$$

so that the two constraints are linked establishing the fact that the criteria for Goldstone elimination stated in (2.5) must be amended. If two of the relation involve the same generator on the LHS then there is one fewer allowable constraint. We know of no other cases where this happens. The final possibility is that we eliminate both η and ϕ using DIHMs as discussed in the next section.

2.5.1 Consequence of Broken Conformal Symmetry via the DIHM

To derive the relevant DIHCs we will again build the coset and treat both the dilaton and the framon as Lagrange multipliers. As in the previous cases, in two spatial dimensions this is not a choice as a consequence of power counting and symmetry. Notice that λ will not play a role as it shows up neither in the vierbein nor the connection. We have already written down the most general boost invariant

[16] This non-locality in EFT arises due to a poor choice of variables and is not in any sense fundamental since the underlying theory is local.

Table 2.1: Infinitesimal variation of Goldstones under broken charges

	$\vec{K}(\vec{\beta})$	$D(\alpha)$	$C(\rho)$
$\vec{\eta}$	$-\vec{\beta}$	$\alpha\vec{\eta}$	$-\rho\vec{x} + t\rho\vec{\eta}$
ϕ	0	$-\alpha$	$-\rho t$
λ	0	$2\alpha\lambda$	$\rho - 2\rho t\lambda$
∂_t	-$\vec{\beta}\cdot\vec{\partial}$	$2\alpha\partial_t$	$2t\rho\partial_t + \rho\vec{x}\cdot\vec{\partial}$
∂_x	0	$\alpha\partial_x$	$-t\rho\partial_x$

interaction in (2.37) and (2.45), which we now amend using the new version of the vierbein and gauge field (2.63, 2.64). The invariant action for the quasi-particle is given by

$$S_0 = \int \frac{d^d p dt}{(2\pi)^d} e^{2\phi}\ \psi^\dagger_{\vec{\mathbf{p}}}(t)\left[(ie^{-2\phi}\partial_0 - e^{-2\phi}\vec{\eta}\cdot\vec{\mathbf{p}} + \tilde{\varepsilon}(e^{-\phi}(\vec{p}+m\vec{\eta})) + \mu_{\mathrm{F}}\right]\psi_{\vec{\mathbf{p}}}(t), \tag{2.75}$$

Here the energy functional $\tilde{\varepsilon}(p)$ is the energy of the quasi-particle measured from the Fermi surface since we have explicitly included the chemical potential μ_F in the action. For notational convenience, we will drop the explicit factor of μ_F and redefine the energy functional as $\varepsilon(p) = \mu_F + \tilde{\varepsilon}(p)$. As far as the interactions are concerned we have

$$\begin{aligned} S_{int} &= \frac{1}{2}\int \prod_{a=1}^{4} \frac{d^d p_a}{(2\pi)^d} dt \delta^{(d)}(p_1 + p_2 - p_3 - p_4) e^{(2-d)\phi}\ g(e^{-\phi}(\vec{\mathbf{p}}_i + m\vec{\eta}_i), e^{-\phi}\mu) \\ &\quad \psi^\dagger_{\vec{\mathbf{p}}_1}(t)\psi_{\vec{\mathbf{p}}_2}(t)\psi^\dagger_{\vec{\mathbf{p}}_3}(t)\psi_{\vec{\mathbf{p}}_4}(t). \end{aligned} \tag{2.76}$$

Here we have also introduced the renormalization scale μ in the coupling. The Landau relation (2.46) which ensures boost invariance remains unchanged but we generate a new constraint by setting η to zero and varying the action (2.75) and (2.76) with respect to ϕ.

Expanding (2.75) and (2.76) to leading order in ϕ,

$$\begin{aligned} S^\phi &= \sum_{\vec{\mathbf{k}}}\int d^d p dt\ \ \phi\ \psi^\dagger_{\vec{\mathbf{p}}}(t)\left[2\varepsilon(p) - p^i\frac{\partial\varepsilon}{\partial p_i}\right]\psi_{\vec{\mathbf{p}}}(t) \\ &+ \frac{1}{2}\prod_{a=1}^{4}\int d^d p_a dt\ \phi\left[(2-d)g(\vec{\mathbf{p}}_i,\mu) - \vec{\mathbf{p}}_i\cdot\frac{\partial g(\vec{\mathbf{p}}_i,\mu)}{\partial\vec{\mathbf{p}}_i} - \mu\frac{\partial g(\vec{\mathbf{p}}_i,\mu)}{\partial\mu}\right]\psi^\dagger_{\vec{\mathbf{p}}_1}(t)\psi_{\vec{\mathbf{p}}_2}(t)\psi^\dagger_{\vec{\mathbf{p}}_3}(t)\psi_{\vec{\mathbf{p}}_4}(t) \end{aligned} \tag{2.77}$$

The constraint follows from imposing $\frac{\delta S^\phi}{\delta \phi} = \mathcal{O}_\phi = 0$.

$$\begin{aligned}
\mathcal{O}_\phi &= \sum_{\vec{\mathbf{k}}} \int d^d p dt \;\; \psi^\dagger_{\vec{\mathbf{p}}}(t) \left[2\varepsilon(p) - p^i \frac{\partial \varepsilon}{\partial p_i} \right] \psi_{\vec{\mathbf{p}}}(t) \\
&+ \frac{1}{2} \prod_{a=1}^{4} \int d^d p_a dt \;\; \left[(2-d) g(\vec{\mathbf{p}}_i, \mu) - \vec{\mathbf{p}}_i \cdot \frac{\partial g(\vec{\mathbf{p}}_i, \mu)}{\partial \vec{\mathbf{p}}_i} - \mu \frac{\partial g(\vec{\mathbf{p}}_i, \mu)}{\partial \mu} \right] \psi^\dagger_{\vec{\mathbf{p}}_1}(t) \psi_{\vec{\mathbf{p}}_2}(t) \psi^\dagger_{\vec{\mathbf{p}}_3}(t) \psi_{\vec{\mathbf{p}}_4}(t)
\end{aligned} \tag{2.78}$$

Let us now see if a Fermi liquid description is consistent with these constraints. Given our assumption of rotational invariance and the notion of a well defined Fermi surface, the marginal coupling is only a function of the angles which are scale invariant. Thus the second term in the last line of (2.78) vanishes, and, as such, if we take the one particle matrix element we see that the quadratic and quartic terms must vanish separately since the quadratic term will depend upon the amplitude of the incoming external momentum and the quartic will not. In three dimensions we see that the coupling has power law running which is inconsistent with Fermi liquid theory, and in two dimensions the theory is free. Thus we conclude that: *fermions at unitarity are not properly described by Fermi liquid theory.*

We can also consider how these symmetry constraints can be utilized if we assume that the microscopic theory is defined via the action (2.41) (i.e. its is not an effective theory) as done in simulations. In this case since there is no restriction to forward scattering there is no mechanism by which the quadratic term can cancel with the quartic for all choices of states. Then taking the one particle matrix element of (2.78) we have the constraints

$$\boxed{\varepsilon = \frac{p^2}{2m^\star}} \tag{2.79}$$

and

$$\boxed{0 = (2-d) g(\vec{\mathbf{p}}_i, \mu) - \vec{\mathbf{p}}_i \cdot \frac{\partial g(\vec{\mathbf{p}}_i, \mu)}{\partial \vec{\mathbf{p}}_i} - \beta(g).} \tag{2.80}$$

For S-wave scattering ($g(p,\mu)$=$g(\mu)$), $m = m^\star$ due to the Landau condition in (2.50) and $(2-d)g(\mu) = \beta(g)$. For higher angular momentum channels, (2.80) gives us the beta function to all orders.

Chapter 3

Poincaré algebra and Re-parametrization Invariance

3.1 Introduction

In EFTs like Soft-Collinear Effective Theory (SCET) [52] and Heavy Quark Effective Theory (HQET) [53], the Lorentz symmetry is broken by choice of the directions n and $\overline{n}$ along which we quantize SCET or the choice of heavy quark velocity v in HQET. A part of this symmetry breaking is natural since the choice of n describes a collimated jet in SCET and the choice of v picks out a special frame in HQET. However a part of the symmetry is restored by the freedom in choosing n and v, which is known as Reparametrization Invariance (RPI) [46]. RPI is also related to the freedom in splitting the momentum into a large and residual piece as was discussed in Chapter 2. RPI connects operators at different order in power counting by relating their Wilson coefficients.

An alternate way of reproducing the same constraints as RPI is by using Poincaré algebra. We can write down the most general Lagrangian for SCET and HQET consistent with Gauge invariance and power counting. Since this action has to be invariant under Poincaré symmetry, it means that at every order in the power counting, symmetry algebra must also be satisfied as the symmetry is not explicitly broken but only non-linearly realized. If we construct the Noether charges for Poincaré algebra using effective Lagrangians with arbitrary Wilson coefficients and impose the constraint that they must satisfy the algebra, we should get the same results as from RPI.

In this chapter, I will elaborate in detail how this can be done not only for free part of the theory but also for interaction terms. I will begin with SCET and show how we can construct the Lagrangian for a collinear field by using Poincaré algebra. Then as an example of interacting theory, I will discuss the constraints on Heavy-Light decay currents in SCET and Heavy-Heavy decay currents in HQET.

3.2 Constraints on Soft-Collinear Lagrangian

QCD in light-cone coordinates can be described by choice of two light-like vectors n and $\overline{n}$ which satisfy

$$n^2 = 0, \qquad \overline{n}^2 = 0, \qquad n \cdot \overline{n} = 2, \tag{3.1}$$

where the last condition is a normalization choice. A standard choice for n and $\overline{n}$ is,

$$n^\mu = (1, 0, 0, 1), \qquad \overline{n}^\mu = (1, 0, 0, -1). \tag{3.2}$$

The physical intuition behind this choice is that we want to use SCET to describe jets produced in high energy collisions and this choice corresponds to production of two back to back jets say in e^+e^- collisions. However any other choice will work equally well. The quark momentum (p)[1] can be expanded in this light-cone basis

$$p^\mu = n^\mu \frac{\overline{n} \cdot p}{2} + \overline{n}^\mu \frac{n \cdot p}{2} + p_\perp^\mu, \tag{3.3}$$

where $\perp$ component is orthogonal to both n and $\overline{n}$ and we typically express p as

$$p = (p^+, p^-, \vec{p}_\perp). \tag{3.4}$$

A similar expression can be written down for the position vector (x) as well.

If a quark is moving along say n direction then a large component of its energy is concentrated along n direction with small momentum fluctuations in $\perp$ and $\overline{n}$ direction. If we assume $p^- \sim Q$ where Q is the hard scale in the problem and define $\lambda << 1$[2] as a power counting parameter describing small momentum fluctuations in other directions, then the on-shell condition for quark $p^+p^- = \vec{p}_\perp^{\,2}$, requires us to scale,

$$p^+ \sim Q\lambda^2, \qquad p_\perp \sim Q\lambda. \tag{3.5}$$

This is the standard scaling for a collinear quark in SCET. We can also define two projection operators,

$$P_n = \frac{\not{n}\not{\overline{n}}}{4}, \qquad P_{\overline{n}} = \frac{\not{\overline{n}}\not{n}}{4}, \tag{3.6}$$

where $\not{n} = n^\mu \gamma_\mu$ and $\not{\overline{n}} = \overline{n}^\mu \gamma_\mu$. These operators also satisfy $P_n + P_{\overline{n}} = 1$. Using these two operators, we can write the QCD quark field $\psi(x)$ as

$$\psi = P_n \psi + P_{\overline{n}} \psi = \xi_n + \phi_{\overline{n}}. \tag{3.7}$$

The fields ξ_n and $\phi_{\overline{n}}$ satisfy the following spin relations,

$$\not{n}\xi_n = 0, \qquad P_n \xi_n = \xi_n, \qquad \not{\overline{n}}\phi_{\overline{n}} = 0, \qquad P_{\overline{n}}\phi_{\overline{n}} = \phi_{\overline{n}}. \tag{3.8}$$

[1] We assume $p^2 = 0$ i.e quarks are massless.

[2] We can take λ to be of the order of $\frac{\Lambda_{QCD}}{Q}$.

Similar spin relations can be derived for quark spinors. Substituting Eq.(3.7) into the massless QCD action, $\mathcal{L}_{QCD} = \overline{\psi}(x) i\partial\!\!\!/ \psi(x)$ and using Eq.(3.8), we get

$$\mathcal{L}_{QCD} = \overline{\xi}_n \frac{\bar{n}\!\!\!/}{2} in \cdot \partial \xi_n + \overline{\phi}_{\overline{n}} i\partial\!\!\!/_\perp \xi_n + \overline{\xi}_n i\partial\!\!\!/_\perp \phi_{\overline{n}} + \overline{\phi}_{\overline{n}} \frac{n\!\!\!/}{2} i\overline{n} \cdot \partial \phi_{\overline{n}}. \tag{3.9}$$

The field $\phi_{\overline{n}}$ is sub-leading in power counting as compared to ξ_n [52] and can integrated out by performing the path integral over quadratic terms in $\phi_{\overline{n}}$. This is equivalent to using the leading order equations of motion of $\phi_{\overline{n}}$ and substituting it back in the Eq.(3.9). Doing so we get the Lagrangian for a collinear quark (ξ_n) with momentum scaling $p_c = Q(\lambda^2, 1, \lambda)$,

$$\mathcal{L}_{\xi_n} = \overline{\xi}_n \Big(i\partial_+ + \partial\!\!\!/_\perp \frac{i}{\partial_-} \partial\!\!\!/_\perp \Big) \frac{\bar{n}\!\!\!/}{2} \xi_n. \tag{3.10}$$

Here the notation is $\partial_+ = n.\partial = \frac{\partial}{\partial x^-}$, $\overline{n}.\partial = \partial_- = \frac{\partial}{\partial x^+}$ and $\partial\!\!\!/_\perp = \gamma^\mu \partial_{\mu,\perp}$ and we have chosen x^- as our time. For a more detailed discussion on construction of SCET Lagrangian from QCD, see Ref.[52].

By requiring the leading order action to be $\mathcal{O}(1)$ in power counting, we see that $\xi_n \sim \lambda$ since the measure d^4x scales as λ^{-4} for a collinear particle. For canonical quantization, we need the conjugate momenta to this field,

$$\Pi_n = \frac{\partial L}{\partial(\partial_+ \xi_n)} = \overline{\xi}_n i \frac{\bar{n}\!\!\!/}{2}. \tag{3.11}$$

The anti-commutation relation used to quantize the theory is,

$$\{\xi_n(x), \overline{\xi}_n(x')\} = \frac{n\!\!\!/}{2} \delta^2(x_\perp - x'_\perp)\delta(x_+ - x'_+). \tag{3.12}$$

Since x^- is being treated as time variable so the Hamiltonian is given by p^+, which can be calculated from the energy-momentum tensor ($\Theta^{\mu\nu}$) constructed from Eq.(3.10).

$$p^+ = \int d^2x_\perp dx^+ \Theta^{-+} = -\frac{1}{2} \int \overline{\xi}_n \Big(\partial\!\!\!/_\perp \frac{i}{\partial_-} \partial\!\!\!/_\perp \Big) \frac{\bar{n}\!\!\!/}{2} \xi_n \tag{3.13}$$

If there are additional operators in $\mathcal{L}_{\xi_n}$ representing interactions of collinear quark with say the $SU(3)$ gauge field, then they just get added to this Hamiltonian.

The momentum operator along $x^\perp$ is,

$$p^\perp = \int \Theta^{-\perp} = \int \overline{\xi}_n \frac{\bar{n}\!\!\!/}{2} i\partial^\perp \xi_n. \tag{3.14}$$

$p^\perp$ does not change if no operators with x^- derivatives are added to $\mathcal{L}_{\xi_n}$. The boost operator in $x^\perp$ direction is $M^{-\perp}$,

$$M^{-\perp} = \int (x^- \Theta^{-\perp} - x^\perp \Theta^{--}) = \int \overline{\xi}_n \Big(-\frac{1}{2} x^\perp i\partial^- \frac{\bar{n}\!\!\!/}{2} + x^- i\partial^\perp \frac{\bar{n}\!\!\!/}{2} - \frac{\bar{n}\!\!\!/}{2} S^{-\perp} \Big) \xi_n \tag{3.15}$$

where $S^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. However,

$$\frac{\bar{\not{n}}}{2} S^{\perp -} = -\frac{i}{4}\frac{\bar{\not{n}}}{2}[\bar{\not{n}}, \gamma^\perp] = 0 \qquad (3.16)$$

since $\bar{\not{n}}\bar{\not{n}} = \bar{n}^2 = 0$, $\bar{\not{n}}\gamma^\perp = -\gamma^\perp\bar{\not{n}}$ and $\bar{\not{n}}\xi_n = 0$. So the spin dependent term in $M^{-\perp}$ can be ignored.

The Poincaré algebra condition in the light cone coordinates is

$$[p^+, M^{-\perp}] = \frac{i}{2} p^\perp. \qquad (3.17)$$

From the operators calculated above, it can be shown that this condition is satisfied to leading order in λ. Also $M^{-\perp}$ does not receive extra contributions from $\mathcal{L}_{\xi_n}$ unless an operator with x^- derivative is added at leading order. So operators like,

$$\bar{\xi}_n i\partial^\mu_\perp \frac{1}{i\partial_-} i\partial_{\mu,\perp} \frac{\bar{\not{n}}}{2}\xi_n, \qquad (3.18)$$

cannot be added to $\mathcal{L}_{\xi_n}$, even though it is leading order in power counting because it would give a contribution to p^+ but won't change $p^\perp$ or $M^{-\perp}$ and hence spoil the algebra. This constraint is obtained in SCET by requiring the theory to be invariant under RPI II [52]. This uniquely fixes the leading order SCET Lagrangian.

Now we split momentum into a label and ultra-soft piece with $p_{us} \sim Q(\lambda^2, \lambda^2, \lambda^2)$ to describe small fluctuations in the collinear direction. To do this it is convenient to re-define the collinear field (ξ_n) as,

$$\xi_n(x) = \sum_{P \neq 0} e^{-ix\cdot P} \xi_{n,P}(x). \qquad (3.19)$$

and define the derivative acting on ξ_n as,

$$i\partial_\mu = P_\mu + i\partial^{\mathrm{us}}_\mu, \qquad (3.20)$$

where P_μ is defined as the label operator. The label momentum now is just an operator which gives a number (label momentum of fields) and not a derivative with respect to x coordinates. The SCET Lagrangian after this separation of momenta into a label and ultra-soft piece, to $\mathcal{O}(\lambda^5)$ in power counting is,

$$\mathcal{L}^{\lambda^5}_{\xi_n} = e^{-iP.x}\bar{\xi}_n \Big(i\partial_+ + \not{P}_\perp \frac{1}{\bar{P}} \not{P}_\perp + C_1\ \not{P}_\perp \frac{1}{\bar{P}} i\not{\partial}_\perp + C_2\ i\not{\partial}_\perp \frac{1}{\bar{P}} \not{P}_\perp \Big) \frac{\bar{\not{n}}}{2} \xi_n \qquad (3.21)$$

Making the substitution Eq.(3.19) into the Noether charges, we get[3]

$$\begin{aligned} p_\perp &= \int \bar{\xi}_n \frac{\bar{\not{n}}}{2}\Big(P_\perp + i\partial_\perp\Big)\xi_n \\ M^{-\perp} &= \int \bar{\xi}_n \frac{\bar{\not{n}}}{2}\Big(x^-(P_\perp + i\partial_\perp) - \frac{1}{2}x^\perp(\bar{P} + i\partial^-)\Big)\xi_n \end{aligned} \qquad (3.22)$$

[3]We ignore the exponential factor in label momentum by assuming conservation of label momentum but the measure for Noether charges ($d^2x_\perp dx^+$) still scales as λ^{-2}.

To satisfy the algebra, we need to consider a term of $\mathcal{O}(\lambda^6)$ otherwise we will miss the ultra-soft piece in $p_\perp$. At this order there are two additional operators,

$$\mathcal{L}^{\lambda^6}_{\xi_n} = \bar{\xi}_n \frac{\bar{\not{n}}}{2}\Big(C_3\ i\not{\partial}_\perp \frac{i}{\overline{\mathcal{P}}}\not{\partial}_\perp + C_4\ \not{\mathcal{P}}_\perp \frac{1}{\overline{\mathcal{P}}} i\partial^- \frac{1}{\overline{\mathcal{P}}}\not{\mathcal{P}}_\perp\Big)\xi_n \tag{3.23}$$

We can fix coefficients C_1, C_2 and C_3 from the commutator $[p^+, M^{-\perp}] = \frac{i}{2}p^\perp$. The commutator gives,

$$\frac{i}{4}\int \bar{\xi}_n \frac{\bar{\not{n}}}{2}\Big[\big(C_1\gamma^\perp \not{\mathcal{P}}_\perp + C_2 \not{\mathcal{P}}_\perp \gamma^\perp\big)\Big(1+\frac{i\partial^-}{\overline{\mathcal{P}}}\Big) + C_3\big(\gamma^\perp \not{\partial}_\perp + \not{\partial}_\perp \gamma^\perp\big)\Big(1+\frac{i\partial^-}{\overline{\mathcal{P}}}\Big)\Big]\xi_n. \tag{3.24}$$

To matching with $\frac{i}{2}p_\perp$ we should set [4],

$$C_1 = C_2 = C_3 = 1. \tag{3.25}$$

There is also a left over piece from the commutator,

$$\frac{i}{2}\int \bar{\xi}_n \frac{\bar{\not{n}}}{2}(P^\perp + i\partial^\perp)\frac{i\partial^-}{\overline{\mathcal{P}}}\xi_n. \tag{3.26}$$

To cancel this contribution we need a higher dimensional operator in p^+, which is to say we need operators at $\mathcal{O}(\lambda^7)$,

$$\mathcal{L}^{\lambda^7}_{\xi_n} = \bar{\xi}_n \frac{\bar{\not{n}}}{2}\Big(C_5\ \not{\partial}_\perp \frac{i}{\overline{\mathcal{P}}} i\partial^- \frac{1}{\overline{\mathcal{P}}}\not{\mathcal{P}}_\perp + C_6\ \not{\mathcal{P}}_\perp \frac{1}{\overline{\mathcal{P}}} i\partial^- \frac{1}{\overline{\mathcal{P}}} i\not{\partial}_\perp\Big)\xi_n. \tag{3.27}$$

The additional contribution from $\mathcal{L}^{\lambda^7}_{\xi_n}$ to $[p^+, M^{-\perp}]$ is,

$$\frac{i}{4}\int \bar{\xi}_n \frac{\bar{\not{n}}}{2}\Big[\big(C_5\gamma^\perp \not{\mathcal{P}}_\perp + C_6 \not{\mathcal{P}}_\perp \gamma^\perp\big)\frac{i\partial^-}{\overline{\mathcal{P}}} + \Big(\frac{i\partial^-}{\overline{\mathcal{P}}}\Big)^2\big(C_5\gamma^\perp \not{\mathcal{P}}_\perp + C_6 \not{\mathcal{P}}_\perp \gamma^\perp\big)\Big]\xi_n. \tag{3.28}$$

The first term has to cancel the term in Eq.(3.26),

$$C_5 = C_6 = -1. \tag{3.29}$$

The only remaining Wilson Coefficient at this order which is yet to be fixed is C_4. It can be calculated by satisfying the commutator $[p^+, M^{-+}] = \frac{i}{2}p^+$.

$$M^{-+} = \int (x^-\Theta^{-+} - x^+\Theta^{--}) = \frac{1}{2}\int \bar{\xi}_n \frac{\bar{\not{n}}}{2}(x^- i\partial^+ - i x^+\partial^-)\xi_n - \frac{1}{2}\int \bar{\xi}_n \frac{\bar{\not{n}}}{2}\xi_n \tag{3.30}$$

The first term in p^+ which gives a non-zero contribution to this commutator is,

$$p^+ = -\frac{C_4}{2}\int \bar{\xi}_n \frac{\bar{\not{n}}}{2}\left(\not{\mathcal{P}}_\perp \frac{1}{\overline{\mathcal{P}}} i\partial^- \frac{1}{\overline{\mathcal{P}}}\not{\mathcal{P}}_\perp\right)\xi_n. \tag{3.31}$$

[4]We have to use the identity $P_{\mu,\perp}\left(\gamma^\mu\gamma^\perp + \gamma^\perp\gamma^\mu\right) = 2P_{\mu,\perp}g^{\mu\perp} = 2P_\perp$.

The commutator of these two operators gives,

$$[p^+, M^{-+}] = i\frac{C_4}{4}\int \overline{\xi}_n \frac{\bar{n}\!\!\!/}{2}\left(P\!\!\!/_\perp \frac{1}{\overline{P}} P\!\!\!/_\perp\right)\xi_n + O(\lambda^4), \tag{3.32}$$

and matching with $\frac{i}{2}p^+$ requires,

$$C_4 = -1. \tag{3.33}$$

To $\mathcal{O}(\lambda^7)$, the unique SCET Lagrangian is,

$$\begin{aligned}
\mathcal{L}^{\lambda^4}_{\xi_n} &= \overline{\xi}_n\Big(i\partial_+ + P\!\!\!/_\perp \frac{1}{\overline{P}} P\!\!\!/_\perp\Big)\frac{\bar{n}\!\!\!/}{2}\xi_n \\
\mathcal{L}^{\lambda^5}_{\xi_n} &= \overline{\xi}_n\Big(i\partial\!\!\!/_\perp \frac{1}{\overline{P}} P\!\!\!/_\perp + P\!\!\!/_\perp \frac{1}{\overline{P}} i\partial\!\!\!/_\perp\Big)\frac{\bar{n}\!\!\!/}{2}\xi_n \\
\mathcal{L}^{\lambda^6}_{\xi_n} &= \overline{\xi}_n\Big(i\partial\!\!\!/_\perp \frac{1}{\overline{P}} \partial\!\!\!/_\perp - P\!\!\!/_\perp \frac{1}{\overline{P}} i\partial_- \frac{1}{\overline{P}} P\!\!\!/_\perp\Big)\frac{\bar{n}\!\!\!/}{2}\xi_n \\
\mathcal{L}^{\lambda^7}_{\xi_n} &= \overline{\xi}_n\Big(-i\partial\!\!\!/_\perp \frac{1}{\overline{P}} i\partial_- \frac{1}{\overline{P}} P\!\!\!/_\perp - P\!\!\!/_\perp \frac{1}{\overline{P}} i\partial_- \frac{i}{\overline{P}} \partial\!\!\!/_\perp\Big)\frac{\bar{n}\!\!\!/}{2}\xi_n.
\end{aligned} \tag{3.34}$$

3.3 Constraints on Heavy to Light Decay Current Operators

To give an example for the application of Poincaré algebra constraints in an interacting theory, I will now discuss the decay of a heavy quark moving with velocity v and described by HQET field h_v. The leading order HQET action is given by [53],

$$\mathcal{L}_{HQET} = \overline{h}_v(x) iv\cdot\partial h_v(x), \tag{3.35}$$

and the total quark momentum p is given by $p = mv + k$ where $k \sim \Lambda_{QCD}$. The residual momentum in the heavy quark can be considered to be of the same order as an ultra-soft momentum ($Q\lambda^2$) of a collinear quark ξ_n. Hence the decay process will receive quantum corrections due to exchange of ultra-soft momentum between heavy and light quark.

In full QCD this process is mediated by a weak decay but since we are considering momentum exchange of the $\mathcal{O}(\Lambda_{QCD})$ we can integrate out the weak boson and its effects are encoded in the matching Wilson coefficients. The external current operators will be considered up to Next to Leading order (NLO) in λ.

The QCD operator, which gives rise to this decay is,

$$J_{QCD} = \overline{q}\Gamma b, \tag{3.36}$$

where q denotes the light quark, b is the heavy quark and Γ denotes a generic Dirac matrix structure of the weak current. At leading order in SCET, this operator is matched onto

$$J^0_{SCET} = \bar{\xi}_n \Gamma h_v. \tag{3.37}$$

The sub-leading pieces are given by

$$J^1_{SCET} = \bar{\xi}_n \frac{\not{\bar{n}}}{2} i \overleftarrow{\not{\partial}^c_\perp} \frac{1}{\overline{P}^\dagger} \Gamma h_v \tag{3.38}$$

The SCET current operator for heavy to light decay upto NLO in λ is,

$$J = \sum_i C_i \bar{\xi}_n \Gamma_i h_v + \sum_i B_i \bar{\xi}_n \frac{\not{\bar{n}}}{2} \overleftarrow{\not{\partial}^c_\perp} \frac{i}{\overline{P}^\dagger} \Gamma_i h_v. \tag{3.39}$$

Here Γ_i stands for all the allowed Dirac structures allowed for this current i.e. $\{\gamma^\mu, v^\mu, n^\mu\}$ etc. The Wilson Coefficients are related by RPI-II [52],

$$C_i = B_i, \tag{3.40}$$

for $\Gamma = \{\gamma^\mu, v^\mu\}$ to all order in perturbation theory. For $\Gamma = n^\mu$, there is an additional piece in the sub-leading current whose coefficient is again related to leading order current.

We can derive these same constraints on Wilson coefficients by considering the transformation of the collinear field ξ_n under Lorentz group. However to get the correct transformation for ξ_n under Lorentz group, in particular $S^{+\perp}$, we first need to consider full QCD, because the action of this generator on ξ_n vanishes if we construct it using the leading order collinear Lagrangian in Eq.(3.10).

$$S^{+\perp} = -\frac{i}{2} \int \Pi_n \sigma^{+\perp} \xi_n = \frac{i}{4} \int \bar{\xi}_n \frac{\not{\bar{n}}}{2} [\not{n}, \gamma^\perp] \xi_n = \int \bar{\xi}_n \gamma^\perp \xi_n = \int \bar{\xi}_n P_n \gamma^\perp \xi_n = 0 \tag{3.41}$$

Here we have used $P_n \xi_n = \xi_n$, $\bar{\xi}_n P_{\bar{n}} = \bar{\xi}_n$ and $\bar{\xi}_n \not{n} = 0$. However if we first consider the full QCD operator, $S^{+\perp}$ and then expand the Dirac field, we see that $[S^{+\perp}, \xi_n] \neq 0$. This is because from Dirac equation,

$$S^{+\perp} = -\frac{i}{2} \int \overline{\psi} \gamma^0 \sigma^{+\perp} \psi. \tag{3.42}$$

and if we expand $\psi = \xi_n + \phi_{\bar{n}}$, then we get

$$S^{+\perp} = i \int \bar{\xi}_n \gamma^\perp \phi_{\bar{n}}. \tag{3.43}$$

Here we have used $\gamma^0 = \frac{\not{n} + \not{\bar{n}}}{2}$ to split the conjugate momentum into two parts $\Pi_n = \bar{\xi}_n \frac{\not{\bar{n}}}{2}$ and $\Pi_{\bar{n}} = \overline{\phi}_{\bar{n}} \frac{\not{n}}{2}$. The commutator of this Lorentz generator with ξ_n is

$$[S^{+\perp}, \xi_n] = -i \frac{\not{n}}{2} \gamma^\perp \phi_{\bar{n}} \tag{3.44}$$

Since,

$$\phi_{\overline{n}} = \frac{i}{\overline{P}} \not{\partial}^c_\perp \frac{\overline{n}}{2} \xi_n, \tag{3.45}$$

which can be derived by using equations of motion for $\phi_{\overline{n}}$ from Eq.(3.10), we can see that Lorentz transformation of ξ_n under $S^{+\perp}$ mixes the large (ξ_n) and small components ($\phi_{\overline{n}}$) of the Dirac field and so every operator involving ξ_n should also include this small sub-leading piece when we are concerned with sub-leading operators in SCET. Replacing ξ_n by $\xi_n + \phi_{\overline{n}}$ in Eq.(3.37) and comparing with Eq.(3.39), we see that $B_i = C_i$

If we also include operators in the SCET current operator J, which are sub-leading due to splitting the momentum into collinear and ultra-soft piece, then Wilson coefficients of these operators can be fixed by requiring $[M^{-\perp}, J] = 0$ where $M^{-\perp}$ as was derived in the previous section.

Another way to see this is to define a new collinear quark field $\hat{\xi}_n$, which should transform co-variantly under Lorentz transformations. So we require that this field obeys,

$$\sigma^{+\perp}\hat{\xi} \neq 0. \tag{3.46}$$

In terms of the old collinear field ξ_n, we can a general form for the new field $\hat{\xi}_n$ as,

$$\hat{\xi}_n = (1 + f(\partial)\Gamma)\xi_n. \tag{3.47}$$

where the only allowed Dirac matrix (Γ) which will give a non-vanishing result is $\Gamma = \not{\overline{n}}$. Also since the extra piece should be sub-leading in λ,

$$f(\partial) = A(i\not{\partial}_\perp) + ... \tag{3.48}$$

and to get the correct dimensions without spoiling the power counting, we should set $A = \frac{1}{\overline{P}}$. So our new Collinear field is

$$\hat{\xi}_n(x) = \left(1 + \frac{1}{\overline{P}} i\not{\partial}_\perp \frac{\not{\overline{n}}}{2}\right) \xi_n(x) + ... \tag{3.49}$$

where additional sub-leading terms will arise by expanding the momentum in collinear and ultra-soft piece. This new field is just the full Dirac field with small component expressed in terms of the ξ field. It is not surprising that Lorentz transformation requires us to bring back the small piece of the Dirac field, which we had integrated out at leading order in λ.

3.4 Constraints on Heavy to Heavy decay currents in HQET

Next we consider the Heavy to Heavy decay process ($b \to c$) in HQET and derive RPI constraints [53] using Poincaré algebra. From Dirac equation, the free

Hamiltonian for this system is,

$$H_0 = \int (m_b \bar{b}_v b_v + m_c \bar{c}_{v'} c_{v'} - \bar{b}_v i\vec{\gamma} \cdot \vec{\partial} b_v - \bar{c}_{v'} i\vec{\gamma} \cdot \vec{\partial} c_{v'}), \tag{3.50}$$

where b_v is a b-quark field of mass m_b moving with velocity v and $c_{v'}$ is the charm quark field of mass m_c with velocity v'. For a consistent power counting in m, we have included the mass term in the Hamiltonian. The Lorentz boost operator k_i derived from this Hamiltonian is given by,

$$k_i = tP_i - \int x_i \left(m_b \bar{b}_v b_v + m_c \bar{c}_v c_v - \bar{b}_v i\vec{\gamma} \cdot \vec{\partial} b_v - \bar{c}_v i\vec{\gamma} \cdot \vec{\partial} c_v \right) - \frac{i}{2} \int (\bar{b}_v \gamma_i b_v + \bar{c}_v \gamma_i c_v), \tag{3.51}$$

where P_i is the sum of momentum operator for b_v and $c_{v'}$. We quantize the theory by using the following anti-commutation relation for fields,

$$\{\psi_v(x), \overline{\psi}_{v'}(x')\}_{\alpha\beta} = \gamma^0_{\alpha\beta} \delta^3(x - x') \delta_{vv'}. \tag{3.52}$$

The external current operator for $b \to ce\overline{\nu}_e$ decay adds a term V to the Hamiltonian,

$$V = \int \left(C_1 \bar{c}_{v'} \gamma^\mu b_v + \frac{B_1}{2m_b} \bar{c}_{v'} \gamma^\mu i \partial\!\!\!/ b_v + \frac{B_8}{2m_c} \bar{c}_{v'} i \overleftarrow{\partial\!\!\!/} \gamma^\mu b_v \right) l_\mu + h.c \tag{3.53}$$

Here l_μ is just a four-vector independent of x representing the leptonic current while C_1, B_1 and B_8 are the unknown Wilson coefficients. A similar term can be added to Hamiltonian for the axial part of the current. Due to the presence of this additional term in the Hamiltonina, the boost operator gets a correction term W_i,

$$W_i = \int x_i \left(C_1 \bar{c}_{v'} \gamma^\mu b_v + \frac{B_1}{2m_b} \bar{c}_{v'} \gamma^\mu i \partial\!\!\!/ b_v + \frac{B_8}{2m_c} \bar{c}_{v'} i \overleftarrow{\partial\!\!\!/} \gamma^\mu b_v \right) l_\mu + h.c \tag{3.54}$$

This is because the Lorentz boost generator is given by $K_i = tP_i - \int x_i T^{00}$ where $T^{\mu\nu}$ is the energy momentum tensor, and T^{00} is the sum of free and the interacting Hamiltonian density. So we should split the boost operator into $K_i = k_i + W_i$, where k_i is from free part of the Hamiltonian density and W_i is due to interactions. The Poincarè algebra condition now becomes

$$[H_0, k_i] + [H_0, W_i] + [V, k_i] + [V, W_i] = iP_i \tag{3.55}$$

Imposing the algebra constraint we get,

$$i \int l_\mu \bar{c}_{v'} \left[\gamma^\mu \left(\frac{B_1 \gamma^i \gamma^0 + C_1 \gamma^0 \gamma^i}{2} \right) + \left(\frac{C_1 \gamma^i \gamma^0 - B_8 \gamma^0 \gamma^i}{2} \right) \gamma^\mu \right] b_v = 0. \tag{3.56}$$

And so by setting

$$C_1 = B_1 = -B_8, \tag{3.57}$$

we can satisfy the algebra. Hence to $\mathcal{O}(1/m)$ in the quark masses, the external current for $b \to c$ decay in HQET can be written as,

$$J^\mu = C_1 \bar{c}_{v'} \left[\Gamma^\mu + \frac{1}{2m_b} \Gamma^\mu i \not{\partial} - \frac{1}{2m_c} i \overleftarrow{\not{\partial}} \Gamma^\mu \right] b_v \tag{3.58}$$

where $\Gamma^\mu = \gamma^\mu (1 - \gamma_5)$.

Appendix A

Lifetime of Quasi-Particles

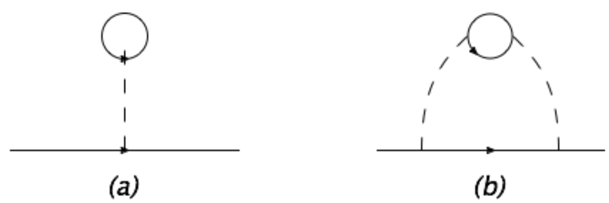

Figure A.1: Diagram (a) results in a shift in the chemical potential while (b) is the first self-energy correction for quasi-particles.

A.1 Without Goldstone Bosons

A critical assumption for the validity of FLT is the existence of well defined quasi-particle states. This is a self-consistency check on the theory. To prove that the width of the quasi-particles does indeed go as $\Gamma(E) \sim (E - E_F)^2$ close to the fermi surface, we consider the self energy diagrams shown in Fig. (A) which follow from the FLT action in Eq.(2.41). Since the diagram (a) does not have an imaginary part, the first correction to fermion self-energy comes from the two loop sunrise diagram. Define the fermion bubble as $i\Pi$ and assume that the coupling function of FLT $g(p_i) = g$ =constant,

$$i\Pi(\omega, \vec{q}) = g^2 \int \frac{d^d k}{(2\pi)^d} \int \frac{dk_0}{2\pi} \; G(k_0, \vec{k}) \; G(k_0 + \omega, \vec{k} + \vec{q}) \tag{A.1}$$

where,

$$G(k_0, \vec{k}) = \frac{1}{\omega - \varepsilon(k) + i\delta\ \mathrm{sgn}(\varepsilon(k))}, \tag{A.2}$$

is the free fermion propagator and $\mathrm{sgn}(x) = 1$ or -1 for $x > 0$ and $x < 0$ respectively. Performing the k_0 integral by contours,

$$i\Pi(\omega, \vec{q}) = ig^2 \int \frac{d^d k}{(2\pi)^d} \frac{f(\varepsilon(\vec{k}+\vec{q})) - f(\varepsilon(\vec{k}))}{\omega + \varepsilon(\vec{k}) - \varepsilon(\vec{k}+\vec{q}) + i\delta} \tag{A.3}$$

where $f(\varepsilon(\vec{k}+\vec{q})) = \theta(\varepsilon(\vec{k}+\vec{q}))\theta(-\varepsilon(\vec{k}))$. Expanding the function f in the limit $q << k$, we can estimate the imaginary part of Π,

$$\mathrm{Im}\Pi(\omega, \vec{q}) = \gamma \frac{\omega}{|\vec{q}|} \tag{A.4}$$

where,

$$\gamma = -\frac{g^2\pi}{v_F} \int\ dk\ k^{d-1}\delta(\varepsilon(\vec{k})) \int \frac{d\Omega_{d-1}}{(2\pi)^d}. \tag{A.5}$$

Now we can compute the imaginary part of the self-energy of the quasi-particle.

$$i\Sigma(p_0, \vec{p}) = ig^2 \int \frac{d^d q}{(2\pi)^d} \int \frac{d\omega}{2\pi} \mathrm{Im}\Pi(\omega, \vec{q}) G(\omega + p_0, \vec{p} + \vec{q}) \tag{A.6}$$

Once again perform the energy integral by contours and using a cutoff regulator Λ for the momentum integral,

$$\mathrm{Im}\Sigma(p_0, \vec{p}) \sim \gamma\ g^2\ \Lambda^{d-2} v_F(\Lambda^2 - p_\perp^2) \sim \beta p_0^2 \tag{A.7}$$

where β is a constant, the first term is a power divergence canceled by a counter term and we used the on-shell condition for the quasi-particle $p_0 = v_F p_\perp$. p_0 is the energy measured from the Fermi surface. Hence, we have shown that lifetime of quasi-particle indeed satisfies the Landau criterion and they are well defined states close to the Fermi surface.

A.2 With Goldstone Bosons

Using the calculation of previous section for the fermion bubble, we can immediately deduce that if at all any Goldstones are interacting with quasi-particles of FLT, then they will be overdamped since the one-loop correction to boson propagator in fig.(A.2) leads to an imaginary part resulting in a pole at $\omega \sim -i|\vec{q}|^3$. With this one-loop corrected boson propagator, now the quasi-particle self energy is, (after performing the energy integral)

$$\mathrm{Im}\Sigma(p_0, \vec{p}) \sim \int d^{d-1}k_{||} \int dk_\perp \frac{k_{||}}{|k_{||}|^3 + \gamma\ p_0 - v_F\gamma(p_\perp + k_\perp)} \tag{A.8}$$

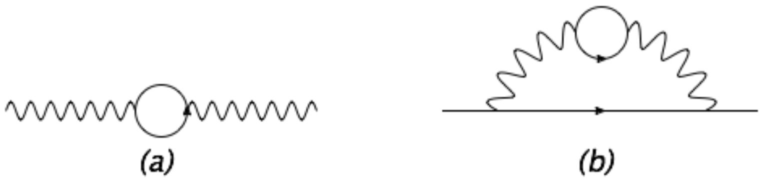

Figure A.2: Diagram (a) is the one loop correction to GB propagator resulting in over-damped Goldstones which feedbacks into the lifetime of quasi-particles through diagram (b) invalidating the Landau criterion.

where the momentum integral is split into parallel ($k_{||}$) and perpendicular ($k_{\perp}$) components to Fermi surface and γ was defined in previous section. Performing the $k_{\perp}$ integral with a cut-off Λ and ignoring a $\log(\Lambda^3)$ divergent term, we get

$$\mathrm{Im}\Sigma(p_0, \vec{p}) \sim \int d^d k_{||} \ln(k_{||}^3 + \gamma\, p_0) \tag{A.9}$$

Defining $k_{||} = x(\gamma\, p_0)^{1/3}$,

$$\mathrm{Im}\Sigma(p_0) \sim \alpha (p_0)^{d/3} \int dx \ln(1+x) x^{d-1} \sim \alpha' p_0^{d/3}. \tag{A.10}$$

So in $d = 2$ the quasi-particle lifetime goes to zero faster than its energy and hence its not a well defined quantum state while in $d = 3$ we will get a Marginal Fermi liquid.

Appendix B

Landau Relation from Symmetry algebra

B.1 Galilean algebra

The Landau relation can also be derived (similar to Landau's original derivation) by demanding that the Fermi Liquid action, without including the boost Goldstone, should be Galilean boost invariant. This is equivalent to satisfying the Galilean algebra by using the Noether charges constructed from the Fermi Liquid action. The only commutator of the Galilean algebra we need to satisfy is $[H, G_i] = iP_i$ where G_i is the generator of Galilean boost, H is the Hamiltonian and P_i is the momentum operator. In terms of the quasi-particle fields, these operators are given by

$$\begin{aligned} H =& \int d^dp\ \psi_p^\dagger \varepsilon(p)\psi_p + \prod_i \int d^dk_i\ \frac{g(k_i)}{2} \psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3}\psi_{k_4}\delta^{(d)}(k_1 + k_2 - k_3 - k_4) \\ G_i =& t\int d^dp\ \psi_p^\dagger p_i \psi_p - im\int d^3p\ \psi_p^\dagger \partial_i \psi_p \\ P_i =& \int d^dp\ \psi_p^\dagger p_i \psi_p \end{aligned} \tag{B.1}$$

Using anti-commutation relation $\{\psi_p, \psi_{p'}^\dagger\} = \delta^d(p - p')$ and satisfying $[H, G_i] = iP_i$, we get back the operator relation in (2.46).

B.2 Poincaré algebra

Here we derive the Landau relation for a relativistic Fermi liquid from current algebra. The derivation [3] is a little more involved in comparison to the Galilean algebra case. The commutator we need to satisfy is still $[H, K_i] = iP_i$ where K_i

is the generator of the Lorentz boost's but the Noether charges are different from their Galilean counterparts. Denoting H_0 as the free Hamiltonian and V as the interaction

$$
\begin{aligned}
H_0 &= \int d^d p \ \psi_p^\dagger \varepsilon(p) \psi_p \\
V &= \prod_i \int d^d k_i \ \frac{g(k_i)}{2} \psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3} \psi_{k_4} \delta^{(d)}(k_1 + k_2 - k_3 - k_4) \\
k_i &= t \int d^d p \ \psi_p^\dagger p_i \psi_p - i \int d^d p \partial_i \psi_p^\dagger \varepsilon(p) \psi_p \\
W_i &= \prod_i \int d^d k_i \ g(k_i) \partial_i \psi_{k_1}^\dagger \psi_{k_2}^\dagger \psi_{k_3} \psi_{k_4} \delta^{(d)}(k_1 + k_2 - k_3 - k_4) \\
P_i &= \int d^d p \ \psi_p^\dagger p_i \psi_p
\end{aligned}
\tag{B.2}
$$

Assuming weak interactions between quasi-particles and neglecting terms of $\mathcal{O}(g^2)$, (taking one particle matrix elements for a state with external momentum, k)

$$
\varepsilon(k) \frac{\partial}{\partial k_i} \left(\varepsilon(k) + \varepsilon(k) \ \int d^d p \langle 0 | \psi_p^\dagger \psi_p | 0 \rangle g(p,k) \right) + \int d^d p \langle 0 | \psi_p^\dagger \psi_p | 0 \rangle \frac{\partial}{\partial p_i} (\varepsilon(p) g(p,k)) = k_i
\tag{B.3}
$$

For forward scattering $g(p,k) = g(\cos\theta)$ where θ is angle between $\vec{p}$ and $\vec{k}$ and $\langle 0|\psi_p^\dagger \psi_p|0\rangle = \Theta(p_F - p)$ to leading order in g. Using $\varepsilon(k_F) = \mu$, where μ is the chemical potential and the definitions of effective mass, m^* and the density of states at the Fermi surface, $D(\mu)$,

$$
m^* = \mu \left(1 + \frac{1}{3} G_1 \right)
\tag{B.4}
$$

where we assumed $d = 2$ and expanded the coupling function $g(\theta)$ in Legendre polynomials , $g(\theta) = \sum_l g_l P_l(\cos\theta)$ and defined $G_l = D(\mu) g_l$.

Appendix C

Coset Construction for Spacetime Symmetry Breaking

C.1 Free Particle Action from Coset Construction

A simple application of coset construction for spontaneously broken spacetime symmetries is illustrated in this section where we construct the action for a free massive and massless relativistic particle [34]. It is also straightforward to include gravity and gauge symmetries however here we will focus only on the free part of the action. The little group for a massive particle in three dimensions is the rotation group $SO(3)$, so the coset space is given by $G/H = ISO(3,1)/SO(3)$ where $ISO(3,1)$ is the Poincaré Group in 3+1 dimensions. A massive particle spontaneously breaks three spatial translations (P_i) and three boosts(K_i) while time translations (H) and rotations (J_i) are unbroken. We can use an Inverse Higgs constraints, $[H, K_i] = iP_i$, to remove the Goldstone modes for translations (π) in favor of those associated with boosts (η). The Coset Element is,

$$\Omega = e^{-iHt}e^{i\vec{\pi}\cdot\vec{P}}e^{i\vec{\eta}\cdot\vec{K}_i} = e^{-iHt}e^{i\vec{\pi}\cdot\vec{P}}\tilde{\Omega}, \tag{C.1}$$

and the Maurer-Cartan 1-form can be calculated as,

$$\Omega^{-1}\partial_\mu\Omega = i(\Lambda^\nu_\mu P_\nu + J^{\alpha\beta}(\Lambda^{-1})^\gamma_\alpha\partial_\mu\Lambda_{\gamma\beta}). \tag{C.2}$$

Here $\tilde{\Omega^{-1}}P_\mu\tilde{\Omega} = \Lambda^\nu_\mu P_\nu$ acts a Lorentz boost on the momentum generator and $J^{\alpha\beta}$ are generators of Lorentz transformation. Projecting this onto the particle world-line, we can calculate the Vierbein (E), gauge field (A_{ij}) and the covariant derivatives for the Goldstone fields,

$$\dot{x}^\mu\Omega^{-1}\partial_\mu\Omega = iE(P_0 + \nabla\pi^i P_i + \nabla\eta^i K_i + A^{ij}J_{ij}), \tag{C.3}$$

where $x^\mu = t, \vec{x}$ are the spacetime coordinates of the particle and $\dot{x}^\mu = \frac{dx^\mu}{dt}$. Comparing the two sides we get,

$$\begin{aligned} E &= \dot{x}^\mu \Lambda_\mu^0 \\ \nabla\pi^i &= E^{-1}\dot{x}^\mu \Lambda_\mu^i \\ \nabla\eta^i &= E^{-1}(\Lambda^{-1})_0^\nu \dot{\Lambda}_\nu^i \\ A_{ij} &= E^{-1}(\Lambda^{-1})_i^\nu \dot{\Lambda}_{\nu j} \end{aligned} \tag{C.4}$$

Using the Inverse Higgs constraint, we can set $\nabla\pi^i = E^{-1}\dot{x}^\mu\Lambda_\mu^i = 0$ which gives

$$\dot{x}^\mu = \Lambda_0^\mu, \tag{C.5}$$

due to orthogonality of Lorentz transformations $\Lambda_\mu^i \Lambda_0^\mu = 0$. We can express the vierbein as, $|E| = \sqrt{E^2} = \sqrt{\dot{x}^\mu \dot{x}^\nu \Lambda_\mu^\gamma \Lambda_{\gamma\nu}}$. Since Λ is a Lorentz transformation, it satisfies $\eta_{\mu\nu} = \Lambda_\mu^\alpha \eta_{\alpha\beta} \Lambda_\nu^\beta = \Lambda_\mu^\alpha \Lambda_{\alpha\nu}$. The leading order action for a massive particle is then given by,

$$S = -m\int dt|E| = -m\int\ dt\ \sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = -m\int\ ds \tag{C.6}$$

where m is the mass of the particle. Sub-leading terms in the action should be given by $\nabla\eta^i$ but we can eliminate them by using leading order equation of motion. This is because the velocity in co-moving frame of the particle is $u^\mu = \dot{x}^\mu = \Lambda_0^\mu$. Hence,

$$E\nabla\eta^i = \dot{\Lambda}_\nu^i \Lambda_0^\nu = -\Lambda_\nu^i \dot{\Lambda}_0^\nu = -\Lambda_\nu^i \dot{u}^\nu, \tag{C.7}$$

and by leading order equation of motion $\dot{u}^\nu = 0$. [1] So Eq.(C.6) is the exact action for a massive free particle to all orders. Notice that this action is invariant under $t \to t'(t)$ since the Jacobian $\frac{dt}{dt'}$ is canceled by the transformation of term inside the square-root,

$$\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \to \frac{dt'}{dt}\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{dt'}\frac{dx^\nu}{dt'}} \tag{C.8}$$

The free action for a massless particle can be derived from Eq.(C.6) but its not obvious how to take the massless limit of this action. To do so we introduce an auxiliary variable, e and rewrite Eq.(C.6) as,

$$S = -\frac{1}{2}\int\ d\lambda\ \left(\frac{\dot{x}^2}{e} + m^2 e\right) \tag{C.9}$$

Here λ is an affine parameter and dot represents derivative wrt λ. Since $e(\lambda)$ does not have a kinetic term, it is a non-dynamical variable and we can always eliminate it by using its equations of motion,

$$\frac{\delta S}{\delta e} = -\frac{\dot{x}^2}{e^2} + m^2 = 0 \tag{C.10}$$

[1] Acceleration of the particle in absence of any external force vanishes.

and get back Eq.(C.6). Hence adding this new variable does not change the physics. Now we can take $m \to 0$ limit in Eq.(C.11), to find the action for a massless particle.

$$S = -\int d\lambda \, \frac{\dot{x}^2}{2e} \tag{C.11}$$

This is similar to the Polyakov action for strings with an induced metric on 1-dimensional particle world line.

C.2 Lagrangian for Free HQET and SCET

Using Eq.(C.6) and Eq.(C.11), we can derive the classical free action for HQET and SCET respectively. For HQET, we write the four velocity of the heavy quark as $u^\mu = v^\mu + \frac{k^\mu}{m}$ where $v^2 = 1$ and k^μ are fluctuations of the order Λ_{QCD}. Substituting this expression for u^μ in Eq.(C.6) we get,

$$S = -m \int dt \left(1 + \frac{2v \cdot k}{m} + \frac{k^2}{m^2}\right)^{1/2} = -\int dt \left(m + v \cdot k + \frac{k^2}{2m} + ...\right) \tag{C.12}$$

Ignoring the first term, which is equivalent of integrating out the mass, gives the Lagrangian for HQET

$$\mathcal{L}_{HQET} = v \cdot k + \frac{k^2}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right) \tag{C.13}$$

For SCET, it is convenient to use the light cone coordinates defined in chapter 3. In the massless limit, the appropriate action would be Eq.(C.11). Replacing m^2 by $p^2 = p^+p^- - \vec{p}_\perp^2$ in Eq.(C.10) and by choosing $e = \frac{2}{P^-}$, we get

$$S_{SCET} = \int d\lambda \left(p^+ - \frac{\vec{p}_\perp^2}{p^-}\right) \tag{C.14}$$

This is the leading order action for a collinear particle in momentum space where p^- is much larger than other two momenta.

C.3 Massive spinning particle coupled to Electromagnetism

For a spinning particle coupled to electromagnetic (EM) vector potential A_μ, we consider all the generators of Poincaré group to be broken except time translations. We parametrize our Coset element as

$$\Omega = e^{iHt} e^{-i\vec{\pi}(x)\cdot\vec{P}} e^{i\alpha^{\mu\nu}(x) J_{\mu\nu}}. \tag{C.15}$$

where $\pi(x)$ and $\alpha^{\mu\nu}(x)$ are Goldstone bosons. The Maurer-Cartan 1-form is similar to that of the massive free particle without the gauge field A_{ij} and we get the same results for the covariant derivatives of GBs and vierbein as in that case except two key differences. First we have to gauge EM so replace $\partial_\mu \to (\partial_\mu + igA_\mu) = D_\mu$. Also leading order equations of motion don't vanish due to presence of an external force, $\nabla\eta^i = E^{-1}\Lambda^\nu_0\dot{\Lambda}^i_\nu \neq 0$. However this term is sub-leading in derivative expansion so can be ignored.

Also instead of separating Goldstone modes for boosts and rotations, we combined them into $\alpha^{\mu\nu}$ and the covariant derivative of $\alpha^{\mu\nu}$ is given by

$$\nabla\alpha^{\mu\nu} = \dot{x} \cdot D\alpha^{\mu\nu} = E^{-1}\Lambda^\mu_\gamma\dot{\Lambda}^{\gamma\nu} = E^{-1}\omega^{\mu\nu}, \tag{C.16}$$

which we can think of as angular velocity of the particle. This is convenient since we can easily include contributions of spin and orbital angular momentum in the same parameter. Since $\omega^{\mu\nu} = -\omega^{\nu\mu}$ because $\Lambda^\mu_\gamma\dot{\Lambda}^{\gamma\nu} = -\dot{\Lambda}^\mu_\gamma\Lambda^{\gamma\nu}$ so it can also correspond to $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ for a spin 1/2 fermion. However we are only interested in a classical picture here so this term is not required.

The most general Lagrangian should be constructed with E, D_μ and $\omega^{\mu\nu}$. The simplest terms involving ω, which can be written down are ω^2 and $\omega \cdot F$ where $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$ is the EM field tensor. The ω^2 will represent the rotational kinetic energy of the particle but since we are interested in point particle we can ignore it. Hence the lowest order non trivial term other than the standard kinetic terms for a massive particle is $\omega \cdot F$.

$$\mathcal{L}_{HQET} = v \cdot D + \frac{D^2}{2m} + C\frac{\omega^{\mu\nu}F_{\mu\nu}}{m} \tag{C.17}$$

This can be easily generalized to non-abelian gauge theories. Here C is an unknown Wilson coefficient which needs to be fixed at each order in perturbation theory by a matching calculation to the UV theory.

C.4 Crystals

A crystal has a symmetry breaking pattern similar to a massive particle i.e the broken symmetries are spatial translations ($\vec{P}$), boosts ($\vec{K}$) and rotations ($\vec{J}$). However a crystal also has an internal symmetry group, in particular internal translations ($\vec{Q}$) and rotations ($\vec{S}$), which in combination with spacetime symmetries give rise to a diagonal subgroup with unbroken translations ($\overline{P} = \vec{P} + \vec{Q}$) and rotations ($\overline{J} = \vec{J} + \vec{S}$). The broken and unbroken generators are summarized below:

$$\begin{array}{ll} \text{Unbroken} & H, M, \overline{P}, \overline{J} \\ \text{Broken} & \vec{P}, \vec{Q}, \vec{J}, \vec{S}, \vec{K} \end{array} \tag{C.18}$$

Since $P+Q$ and $J+S$ is unbroken, we only need to include Goldstone bosons for one set of translations and rotations i.e either of (P, J) or (Q, S). It is straightforward to derive the vierbein (E), covariant derivatives of Goldstones and the Gauge field (A) from the Maurer-Cartan 1-form. For calculation ease, we restrict ourselves to two spatial dimensions and choose to include GBs for (Q, S).

$$\begin{aligned}
E_0^0 &= 1, \quad E_0^i = \eta^i, \quad E_i^0 = 0, \quad E_i^j = \delta_i^j \\
A_0 &= -\frac{1}{2}\vec{\eta}^2, \quad A^i = -\eta^i \\
\nabla_0 \eta^i &= -\partial_0 \eta^i + \vec{\eta} \cdot \vec{\partial} \eta^i \\
\nabla_0 \theta &= -\partial_0 \theta + \vec{\eta} \cdot \vec{\partial} \theta \\
\nabla_0 \pi^i &= -R_j^i \partial_0 \pi^j + \eta^j R_j^k \partial_k \pi^i + \eta^i \\
\nabla_i \eta^j &= \partial_i \eta^j \\
\nabla_i \theta &= \partial_i \theta \\
\nabla_i \pi^j &= R_i^k (\partial_k \pi^j + \delta_k^j) - \delta_i^j
\end{aligned} \tag{C.19}$$

Here π is the phonon, η is the framon, θ is the angulon and $R_i^j = R_i^j(\theta)$ is the two-dimensional rotation matrix. The inverse Higgs constraints arise because

$$[K_i, H] = iP_i = i(\overline{P}_i - Q_i), \quad [S, \overline{P}_i] = i\epsilon_{ij} Q_j, \tag{C.20}$$

so we can set $\nabla_0 \pi^i = 0$ and $\epsilon_{ij} \nabla_i \pi^j = 0$. To leading order in the derivatives and fields, these two conditions give an algebraic solution for framon and angulon in terms of the phonon.

$$\begin{aligned}
\eta^i &= \partial_0 \pi^i + ... \\
\theta &= \frac{1}{2} \epsilon_{ij} \partial_i \pi^j + ...
\end{aligned} \tag{C.21}$$

Hence, the only relevant Goldstone boson at low energies for a crystal is the phonon.

C.5 Superfluids at Unitarity

In a normal superfluid phase $U(1)$ particle number invariance (Q) is broken along with Galilean boosts (K). If we consider superfluids at unitarity, we get two additional broken generators namely the dilatations (D) and special conformal transformations (C). We can assume that the spacetime translations are unbroken. The Coset space for this symmetry breaking pattern can be parametrized as,

$$\Omega = e^{iHt} e^{-i\vec{P}\cdot\vec{x}} e^{-i\vec{K}\cdot\vec{\eta}} e^{-iC\lambda} e^{-iD\phi} e^{-iQ\pi}. \tag{C.22}$$

Here η is the framon, π is the superfluid phonon, ϕ is the dilaton and λ is the GB for special conformal transformations. Again we can calculate the Maurer-Cartan 1-form and the extract covariant derivatives for the Goldstones and the vierbein (E).

$$
\begin{aligned}
E_0^0 &= e^{2\phi}, \quad E_0^i = e^{\phi}\eta^i, \quad E_i^0 = 0, \quad E_i^j = e^{\phi}\delta_i^j \\
\nabla_0\pi &= -e^{-2\phi}(\partial_0\pi - \vec{\eta}\cdot\vec{\partial}\pi + \frac{1}{2}m\vec{\eta}^2) \\
\nabla_i\pi &= e^{-\phi}(\partial_i\pi - m\eta_i) \\
\nabla_0\phi &= -e^{-2\phi}(\partial_0\phi - \vec{\eta}\cdot\vec{\partial}\phi - \lambda) \\
\nabla_i\phi &= e^{-\phi}\partial_i\phi \\
\nabla_0\lambda &= -e^{-4\phi}(\partial_0\lambda - \vec{\eta}\vec{\partial}\lambda - \lambda^2) \\
\nabla_i\lambda &= e^{-3\phi}\partial_i\lambda \\
\nabla_0\eta^i &= -e^{-3\phi}(\partial_0\eta^i - \vec{\eta}\cdot\vec{\partial}\eta^i) \\
\nabla_i\eta^j &= e^{-2\phi}(\partial_i\eta^j - \lambda\delta_i^j)
\end{aligned}
\tag{C.23}
$$

We again impose Inverse Higgs constraints due to following commutation relation between generators, which allow us to solve for η and λ in terms of π.

$$
[K_i, P_j] = i\delta_{ij}mQ, \qquad [C, P_i] = iK_i \tag{C.24}
$$

This allows us to set $\nabla_i\pi = 0$ and $\vec{\nabla}\cdot\vec{\eta} = 0$ from which we get,

$$
\begin{aligned}
\vec{\eta} &= \frac{1}{m}\vec{\partial}\pi \\
\lambda &= \frac{1}{3}\vec{\partial}\cdot\vec{\eta} = \frac{1}{3m}\vec{\partial}^2\pi.
\end{aligned}
\tag{C.25}
$$

Substituting these in $\nabla_0\pi$ and $\nabla_0\phi$ gives,

$$
\begin{aligned}
\nabla_0\pi &= -e^{-2\phi}\left(\partial_0\pi - \frac{1}{2m}(\partial_i\pi)^2\right) \\
\nabla_0\phi &= -e^{-2\phi}\left(\partial_0\phi - \frac{1}{m}\vec{\partial}\pi\cdot\vec{\partial}\phi - \frac{1}{3m}\vec{\partial}^2\pi\right)
\end{aligned}
\tag{C.26}
$$

The first of these conditions is the Witten-Wilzcek relation for superfluids while the second condition was derived in [30] as a way of writing down a boost invariant action for the dilaton, which according to Ref.[30] was only possible in the superfluid phase. However if the framon exists then a boost invariant action can be written down for the dilaton with unbroken $U(1)$ as well.

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