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SA1. Boltzmann Kinetic Equation in Curvilinear Coordinates and Curved Spaces

In the absence of an external force, a particle has a constant velocity and moves along a straight line in Euclidean space according to the 1st law of Newton. That is, time derivatives of the particle position and the velocity vectors are described by

$$\dot{\mathbf{x}} = \mathbf{v}, \quad \dot{\mathbf{v}} = 0 \tag{A.1}$$

where $\mathbf{x}$ and $\mathbf{v}$ are the position and velocity vectors, respectively. If we express the velocity vector in a curvilinear coordinate system, we have

$$\mathbf{v} = v^i \mathbf{g}_i(q) \tag{A.2}$$

Therefore,

$$\dot{\mathbf{v}} = 0 \ \rightarrow \ \dot{v}^i \mathbf{g}_i + v^i \frac{\partial \mathbf{g}_i}{\partial q^j} v^j = 0 \tag{A.3}$$

where in the above we have used the definition $v^j = \dot{q}^j$.

Since

$$\frac{\partial \mathbf{g}_i}{\partial q^j} = \frac{\partial \mathbf{g}_i}{\partial q^j} \cdot \mathbf{g}^k \mathbf{g}_k \equiv \Gamma^k_{ij} \mathbf{g}_k, \tag{A.4}$$

we have, by rearranging dummy indices

$$\dot{\mathbf{v}} = 0 \;\to\; \dot{v}^i \mathbf{g}_i + v^j v^k \Gamma^i_{jk} \mathbf{g}_i = 0 \tag{A.5}$$

Therefore, there is an effective acceleration (inertial force) in the space of coordinates, namely

$$\dot{v}^i = -v^j v^k \Gamma^i_{jk} \tag{A.6}$$

Based on the properties above, we are ready to write the Boltzmann equation in curvilinear coordinates,

$$\partial_t N + \frac{\partial}{\partial q^i}(v^i N) + \frac{\partial}{\partial v^i}(\dot{v}^i N) = \Omega \tag{A.7}$$

where $N \equiv N(q, \bar{v}, t)$ denotes the number of particles inside a small pocket $((x^i, x^i + dx^i), (v^i, v^i + dv^i); i = 1, 2, 3)$ of fluid of volume $J(q)$. $\Omega = \Omega(q, \bar{v}, t)$ is the collision term as discussed in the text, **Eq. 14** satisfying local mass and momentum conservaton laws

$$\int d\bar{v}\Omega(q, \bar{v}, t) = 0, \quad \int d\bar{v} v^i \Omega(q, \bar{v}, t) = 0, \tag{A.8}$$

where the integral operator above is defined as $\int d\bar{v} \equiv \int dv^1 dv^2 dv^3$. Substituting (A.6) into (A.7), we get

$$\partial_t N + \frac{\partial}{\partial q^i}(v^i N) - \frac{\partial}{\partial v^i}(v^j v^k \Gamma^i_{jk} N) = \Omega \tag{A.9}$$

Define a particle density function

$$J(q) f(q, \bar{v}, t) \equiv N(q, \bar{v}, t) \tag{A.10}$$

then we have the hydrodynamic moments specified below

$$\begin{aligned} \int d\bar{v} N(q, \bar{v}, t) &= J(q) \int d\bar{v} f(q, \bar{v}, t) = J(q)\rho(q, t) \\ \int d\bar{v} v^i N(q, \bar{v}, t) &= J(q) \int d\bar{v} v^i f(q, \bar{v}, t) = J(q)\rho(q, t) u^i(q, t) \end{aligned} \tag{A.11}$$

Taking the moment integral of the Boltzmann equation (A.9), and using the collision properties of (A.8), we obtain the two continuity equations, corresponding to mass and momentum conservations respectively

$$\begin{aligned} \int d\bar{v} \{\partial_t N + \frac{\partial}{\partial q^i}(v^i N) - \frac{\partial}{\partial v^i}(v^j v^k \Gamma^i_{jk} N)\} = 0 \\ \int d\bar{v} v^i \{\partial_t N + \frac{\partial}{\partial q^j}(v^j N) - \frac{\partial}{\partial v^l}(v^j v^k \Gamma^l_{jk} N)\} = 0 \end{aligned} \tag{A.12}$$

The term $\int d\bar{v} \frac{\partial}{\partial v^i}(v^j v^k \Gamma^i_{jk} N) = 0$ via integration by parts, and using definition in (A.11) we get the mass continuity equation as follows

$$\partial_t (J\rho) + \frac{\partial}{\partial q^i}(J\rho u^i) = 0 \tag{A.13}$$

or in the more familiar form

$$\partial_t \rho + \frac{1}{J}\frac{\partial}{\partial q^i}(J\rho u^i) = 0 \tag{A.14}$$

since the volume J is not dependent on time t.

Integrate by parts, and use $\frac{\partial v^i}{\partial v^j} = \delta^i_j$,

$$\int d\bar{v} v^i \frac{\partial}{\partial v^l}(v^j v^k \Gamma^l_{jk} N) = -\int d\bar{v} v^j v^k \Gamma^i_{jk} N$$

Hence, the second equation in (A.12) becomes

$$\partial_t(\rho u^i) + \frac{1}{J}\frac{\partial}{\partial q^j}(J\Pi^{ij}) + \Gamma^i_{jk}\Pi^{jk} = 0 \tag{A.15}$$

where the momentum flux tensor $\Pi^{ij} = \Pi^{ij}(q,t)$ is defined by

$$\Pi^{ij} \equiv \int d\bar{v} v^i v^j f(q,\bar{v},t) \tag{A.16}$$

Eqn.(A.15) is known as the Cauchy's transport equation.

We can separate the momentum flux tensor into two parts associated, respectively, to the equilibrium and the non-equilibrium parts of the distributions $f(q,\bar{v},t) = f^{eq}(q,\bar{v},t) + f^{neq}(q,\bar{v},t)$, so that $\Pi^{ij}(q,t) = \Pi^{ij,eq}(q,t) + \Pi^{ij,neq}(q,t)$. The equilibrium distribution is given by the Maxwell-Boltzmann form,

$$f^{eq} = \rho W\, exp[-\frac{\mathbf{U}^2}{2\theta}] \tag{A.17}$$

where $\mathbf{U} \equiv \mathbf{v} - \mathbf{u}$ and θ is the temperature. In terms of curvilinear coordinates,

$$\mathbf{U}^2 = \mathbf{U}\cdot\mathbf{U} = U^i\mathbf{g}_i \cdot U^j\mathbf{g}_j = U^i g_{ij} U^j$$

Hence, we can rewrite (A.17) in terms of curvilinear coordinates below

$$f^{eq} = \rho W\, exp[-\frac{g_{ij}U^iU^j}{2\theta}] \tag{A.18}$$

and the normalization factor $W = 1/\sqrt{(2\theta\pi)^3 det[g^{ij}]}$. Here, the inverse metric tensor $[g^{ij}]$ is defined such that $g^{ik}g_{kj} = \delta^i_j$ in differential geometry. $det[g^{ij}]$ is the determinant of $[g^{ij}]$. Using a few basic properties of Gaussian integral,

$$\int dUW\, exp[-\frac{g_{ij}U^iU^j}{2\theta}] = 1$$

$$\int dUW\, U^kU^l exp[-\frac{g_{ij}U^iU^j}{2\theta}] = g^{kl}\theta$$

$$\int dUW\, U^{k_1}U^{k_2}\cdots U^{k_n} exp[-\frac{g_{ij}U^iU^j}{2\theta}] = 0, \quad n = odd\ number \tag{A.19}$$

we immediately obtain from (A.16) and (A.18) that

$$\rho = \int d\bar{v} f^{eq}, \quad \rho u^i = \int d\bar{v} v^i f^{eq},$$

$$\Pi^{ij,eq} = \int d\bar{v} v^i v^j f^{eq} = g^{ij}\rho\theta + \rho u^i u^j \tag{A.20}$$

Therefore, at the Euler order, in which the momentum flux tensor only includes the equilibrium contribution, eqn.(A.15) reduces to

$$\partial_t(\rho u^i) + \frac{1}{J}\frac{\partial}{\partial q^j}(J\Pi^{ij,eq}) + \Gamma^i_{jk}\Pi^{jk,eq} = 0 \tag{A.21}$$

More explicitly,

$$\partial_t(\rho u^i) + \frac{1}{J}\frac{\partial}{\partial q^j}(J[g^{ij}\rho\theta + \rho u^i u^j]) + \Gamma^i_{jk}[g^{jk}\rho\theta + \rho u^j u^k] = 0 \tag{A.22}$$

However, since the underlying space is "flat" (Euclidean), the metric tensor obeys the following property

$$\frac{1}{J}\frac{\partial}{\partial q^j}(Jg^{ij}) + \Gamma^i_{jk}g^{jk} = 0 \tag{A.23}$$

Substituting (A.23) into eqn.(A.22), we arrive at a more standard form of the Euler equation,

$$\partial_t(\rho u^i) + \frac{1}{J}\frac{\partial}{\partial q^j}(J\rho u^i u^j) + \Gamma^i_{jk}\rho u^j u^k = -g^{ij}\frac{\partial p}{\partial q^j} \tag{A.24}$$

with the pressure defined by an ideal gas equation of state, $p = \rho\theta$.

SA2. Derivation of the Navier-Stokes Equations in General Coordinates

To derive the Navier-Stokes hydrodynamics up to the viscous order, we use the Chapman-Enskog expansion procedure **Eq. 28**,

$$\partial_t = \epsilon\partial_{t_0} + \epsilon^2\partial_{t_1}; \quad \frac{\partial}{\partial q^i} = \epsilon\frac{\partial}{\partial q^i}; \quad \frac{\partial}{\partial v^i} = \epsilon\frac{\partial}{\partial v^i}$$

and

$$N = N^{eq} + \epsilon N^{(1)} + \epsilon^2 N^{(2)} + \cdots$$

Here ϵ ($<< 1$) denotes a small number. Thus, the Boltzmann equation (A.9) leads to the following two equations,

$$\partial_{t_0}N^{eq} + \frac{\partial}{\partial q^i}(v^i N^{eq}) - \frac{\partial}{\partial v^i}(v^j v^k \Gamma^i_{jk} N^{eq}) = -\frac{1}{\tau}N^{(1)} \tag{B.1}$$

and

$$\partial_{t_1}N^{eq} + \partial_{t_0}N^{(1)} + \frac{\partial}{\partial q^i}(v^i N^{(1)}) - \frac{\partial}{\partial v^i}(v^j v^k \Gamma^i_{jk} N^{(1)}) = -\frac{1}{\tau}N^{(2)} \tag{B.2}$$

where, for simplicity, in the above we have used the BGK collision operator form $\Omega = -(N - N^{eq})/\tau$ (**Eq.24**). Taking the mass and momentum moments over (B.1), and use the properties in (A.20) as well as conservation of mass and momentum by the collision in (A.8), we immediately obtain the leading (Euler) hydrodynamics as given by (A.14) and (A.24),

$$\partial_{t_0}\rho + \frac{1}{J}\frac{\partial}{\partial q^i}(J\rho u^i) = 0$$
$$\partial_{t_0}(\rho u^i) + \frac{1}{J}\frac{\partial}{\partial q^j}(J\rho u^i u^j) + \Gamma^i_{jk}\rho u^j u^k = -g^{ij}\frac{\partial p}{\partial q^j} \tag{B.3}$$

with an ideal gas equation of state, $p \equiv \rho\theta$.

Eqn.(B.1) can be inverted to give,

$$N^{(1)} = -\tau[\partial_{t_0}N^{eq} + \frac{\partial}{\partial q^i}(v^i N^{eq}) - \frac{\partial}{\partial v^i}(v^j v^k \Gamma^i_{jk} N^{eq})] \tag{B.4}$$

With the definition of particle density distribution function (A.10), $N(q,\bar{v},t) = J(q)f(q,\bar{v},t)$, and J depends on q only, hence eqn.(B.4) is equivalent to

$$f^{(1)} = -\tau[\partial_{t_0}f^{eq} + \frac{1}{J}\frac{\partial}{\partial q^i}(v^i J f^{eq}) - \frac{\partial}{\partial v^i}(v^j v^k \Gamma^i_{jk} f^{eq})] \tag{B.5}$$

With (B.3), it is easily checked via straightforward algebra that $N^{(1)}$ (and $f^{(1)}$) gives vanishing mass and momentum moments, namely

$$\int d\bar{v} N^{(1)}(q,\bar{v},t) = 0, \quad \int d\bar{v} v^i N^{(1)}(q,\bar{v},t) = 0$$

On the other hand, taking the momentum flux moment, we have from (B.4) the following,

$$\begin{aligned} J\Pi^{ij,neq} &\equiv \int d\bar{v} v^i v^j N^{(1)} \\ &= -\tau \int d\bar{v} v^i v^j [\partial_{t_0}N^{eq} + \frac{\partial}{\partial q^k}(v^k N^{eq}) - \frac{\partial}{\partial v^k}(v^l v^m \Gamma^k_{lm} N^{eq})] \end{aligned} \tag{B.6}$$

The last term in (B.6) above can be further simplified below,

$$\begin{aligned} \tau \int d\bar{v} v^i v^j \frac{\partial}{\partial v^k}(v^l v^m \Gamma^k_{lm} N^{eq}) &= -\tau \int d\bar{v} v^l v^m \Gamma^k_{lm} N^{eq} \frac{\partial}{\partial v^k}(v^i v^j) \\ &= -\tau \int d\bar{v} v^l v^m (\Gamma^i_{lm} v^j + \Gamma^j_{lm} v^i) N^{eq} \end{aligned} \tag{B.7}$$

Substituting (B.7) into (B.6), we obtain

$$\begin{aligned} J\Pi^{ij,neq} = &- \tau\{\int d\bar{v} v^i v^j [\partial_{t_0}N^{eq} + \frac{\partial}{\partial q^k}(v^k N^{eq})] \\ &+ \int d\bar{v} v^l v^m (\Gamma^i_{lm} v^j + \Gamma^j_{lm} v^i) N^{eq}\} \end{aligned} \tag{B.8}$$

Or equivalently, we have

$$J\Pi^{ij,neq} = -\tau\{J\partial_{t_0}\Pi^{ij,eq} + \frac{\partial}{\partial q^k}(JQ^{ijk,eq}) + J(\Gamma^i_{kl}Q^{jkl,eq} + \Gamma^j_{kl}Q^{ikl,eq})\} \tag{B.9}$$

where the equilibrium heat flux $Q^{ijk,eq}$ is defined as

$$Q^{ijk,eq} = \int d\bar{v} v^i v^j v^k f^{eq} \tag{B.10}$$

The integration (B.10) is straightforward to perform with f^{eq} (given by (A.18)), and it yields

$$Q^{ijk,eq} = \rho\theta(g^{ij}u^k + g^{jk}u^i + g^{ki}u^j) + \rho u^i u^j u^k \tag{B.11}$$

Therefore, together with (A.20), eqn.(B.9) becomes

$$\begin{aligned} J\Pi^{ij,neq} = & -\tau\{J[g^{ij}\partial_{t_0}(\rho\theta) + \partial_{t_0}(\rho u^i u^j)] \\ & + \frac{\partial}{\partial q^k}[J\rho\theta(g^{ij}u^k + g^{jk}u^i + g^{ki}u^j) + J\rho u^i u^j u^k] \\ & + \Gamma^i_{kl}[J\rho\theta(g^{jk}u^l + g^{kl}u^j + g^{lj}u^k) + J\rho u^j u^k u^l] \\ & + \Gamma^j_{kl}[J\rho\theta(g^{ik}u^l + g^{kl}u^i + g^{li}u^k) + J\rho u^i u^k u^l]\} \end{aligned} \tag{B.12}$$

Next, let us take into account the properties of the metric tensor in the flat space, these are

$$\begin{aligned} \frac{\partial}{\partial q^k} g^{ij} + \Gamma^i_{kl} g^{jl} + \Gamma^j_{kl} g^{il} = 0 \\ \frac{\partial}{\partial q^k}(J g^{jk}) + J\Gamma^j_{kl} g^{kl} = 0 \\ \frac{\partial}{\partial q^k}(J g^{ki}) + J\Gamma^i_{kl} g^{kl} = 0 \end{aligned} \tag{B.13}$$

Then we can further simply (B.12) to

$$\begin{aligned} J\Pi^{ij,neq} = & -\tau\{J[g^{ij}\partial_{t_0}(\rho\theta) + \partial_{t_0}(\rho u^i u^j)] \\ & + g^{ij}\frac{\partial}{\partial q^k}(J\rho\theta u^k) + J g^{jk}\frac{\partial}{\partial q^k}(\rho\theta u^i) + J g^{ki}\frac{\partial}{\partial q^k}(\rho\theta u^j) \\ & + \frac{\partial}{\partial q^k}(J\rho u^i u^j u^k) + \Gamma^i_{kl} J\rho\theta g^{jk} u^l + \Gamma^i_{kl} J\rho u^j u^k u^l \\ & + \Gamma^j_{kl} J\rho\theta g^{ki} u^l + \Gamma^j_{kl} J\rho u^i u^k u^l\} \end{aligned} \tag{B.14}$$

For the purpose of deriving hydrodynamics for isothermal fluid, we set $\theta = const$, in addition we substitute the Euler order relations from (B.3) for the terms with ∂_{t_0}, then after some straightforward algebra, eqn.(B.14) becomes

$$\Pi^{ij,neq} = -\tau\rho\theta\{g^{ik}(\frac{\partial u^j}{\partial q^k} + \Gamma^j_{kl}u^l) + g^{jk}(\frac{\partial u^i}{\partial q^k} + \Gamma^i_{kl}u^l)\} \tag{B.15}$$

Using the standard definition of covariant derivative in differential geometry,

$$u^i|_k \equiv \frac{\partial u^i}{\partial q^k} + \Gamma^i_{kl}u^l$$

together with the rate of strain definition as the symmetric form of velocity derivative,

$$S^{ij} = S^{ji} \equiv \frac{1}{2}(g^{jk}u^i|_k + g^{ik}u^j|_k)$$

then eqn.(B.15) further simplies to

$$\Pi^{ij,neq} = -2\mu S^{ij} \tag{B.16}$$

where $\mu \equiv \tau\rho\theta$ is the dynamic viscosity.

Taking mass and momentum moments of eqn.(B.2), and the vanishing right-hand side, we obtain

$$\begin{aligned} \partial_{t_1}\rho &= 0 \\ \partial_{t_1}(\rho u^i) &= \frac{1}{J}\frac{\partial}{\partial q^j}(2J\mu S^{ij}) + 2\Gamma^i_{jk}\mu S^{jk} \end{aligned} \tag{B.17}$$

Combining the first order hydrodynamics of (B.17) with that of the Euler order (B.3), we finally arrive at the full Navier-Stokes hydrodynamics in a curvilinear coordinate system

$$\begin{aligned} \partial_t\rho + \frac{1}{J}\frac{\partial}{\partial q^i}(J\rho u^i) &= 0 \\ \partial_t(\rho u^i) + \frac{1}{J}\frac{\partial}{\partial q^j}(J\rho u^i u^j) + \Gamma^i_{jk}\rho u^j u^k &= -g^{ij}\frac{\partial p}{\partial q^j} \\ &+ \frac{1}{J}\frac{\partial}{\partial q^j}(2J\mu S^{ij}) + 2\Gamma^i_{jk}\mu S^{jk} \end{aligned} \tag{B.18}$$

with an ideal gas equation of state, $p \equiv \rho\theta$. Eqns.(B.18) are in fact the mass continuity and the Navier-Stokes equation in coordinate-free operator forms

$$\begin{aligned} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) &= 0 \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) &= -\nabla p + \nabla\cdot(2\mu\mathbf{S}) \end{aligned} \tag{B.19}$$

SA3. Incompressibility in Curved Space with Underlying Euclidean Metrics

The size of a volume element in a 3^2 single particle phase-space is denoted as $J_p dq d\bar{v}$, where $dq \equiv dq^1dq^2dq^3$ and $d\bar{v} \equiv dv^1dv^2dv^3$. Clearly, for a curvilinear coordinate system, $J_p = J^2 = g$, with $g \equiv det[g_{ij}]$ being the determinant of the metric tensor, and $J = J(q)$ is the Jacobian of the curvilinear coordinates.

The transport of a particle phase-space density function $\phi = \phi(q, \bar{v}, t)$, without collision, follows a continuity equation in phase space,

$$\partial_t(J_p\phi) + \frac{\partial}{\partial q^i}(J_p\phi v^i) + \frac{\partial}{\partial v^i}(J_p\phi\dot{v}^i) = 0 \tag{C.1}$$

Since a particle moves along a straight path (geodesics) for the underlying Euclidean space according to the 1st law of Newton, we have (see Appendix A)

$$\dot{v}^i = -\Gamma^i_{jk}v^j v^k \tag{C.2}$$

We can show that the motion of particles in phase-space is incompressible. Define the velocity field in phase-space as,

$$\dot{\mathbf{X}} \equiv (\dot{q}, \dot{\bar{v}}) = (\bar{v}, \dot{\bar{v}}) \tag{C.3}$$

then, the divergence of the velocity field in phase-space is given by

$$\nabla \cdot \dot{\mathbf{X}} \equiv \frac{\partial}{\partial q^i}(J_p v^i) + \frac{\partial}{\partial v^i}(J_p \dot{v}^i) \tag{C.4}$$

Substitute (C.2) in eqn.(C.4), we get

$$\nabla \cdot \dot{\mathbf{X}} = \frac{\partial}{\partial q^i}(J_p v^i) - \frac{\partial}{\partial v^i}(J_p \Gamma^i_{jk} v^j v^k) \tag{C.5}$$

But since both J_p $(= J^2)$ and Γ^i_{jk} are functions of q only, the second term in (C.5) becomes

$$\begin{aligned} \frac{\partial}{\partial v^i}(J_p \Gamma^i_{jk} v^j v^k) &= J_p \Gamma^i_{jk} \frac{\partial}{\partial v^i}(v^j v^k) = J_p \Gamma^i_{jk}[v^j \frac{\partial v^k}{\partial v^i} + v^k \frac{\partial v^j}{\partial v^i}] \\ &= J_p[\Gamma^i_{ji} v^j + \Gamma^i_{ik} v^k] = 2 J_p \Gamma^i_{ij} v^j \end{aligned} \tag{C.6}$$

where the last equality in the above is due to the symmetry of the Christoffel symbol, $\Gamma^i_{jk} = \Gamma^i_{kj}$. Furthermore, using the fundamental property

$$\Gamma^i_{ij} = \frac{1}{J} \frac{\partial J}{\partial q^j}$$

then (C.6) becomes,

$$\begin{aligned} \frac{\partial}{\partial v^i}(J_p \Gamma^i_{jk} v^j v^k) = 2 J_p \Gamma^i_{ij} v^j &= 2J \frac{\partial J}{\partial q^j} v^j \\ &= \frac{\partial J_p}{\partial q^j} v^j = \frac{\partial}{\partial q^j}(J_p v^j) \end{aligned} \tag{C.7}$$

the last equality is because $\frac{\partial}{\partial q^j} v^j = 0$. Plug the result of (C.7) into (C.5), hence we have proved the incompressibility property of the velocity field in the phase-space. That is,

$$\nabla \cdot \dot{\mathbf{X}} = \frac{\partial}{\partial q^i}(J_p v^i) + \frac{\partial}{\partial v^i}(J_p \dot{v}^i) = 0 \tag{C.8}$$

With the incompressibility property of the phase space velocity field, the continuity equation (C.1) takes on a form of the Vlasov equation,

$$\partial_t \phi + v^i \frac{\partial \phi}{\partial q^i} + \dot{v}^i \frac{\partial \phi}{\partial v^i} = 0 \tag{C.9}$$

It is worth to point out, the incompressibility property of the phase-space velocity field is an intrinsic property of the particle motion via Newtonian mechanics in Euclidean space. Hence this is true in any coordinate systems whether Cartesian or curvilinear. Nonetheless, it is useful to directly show that such a property is preserved in a curvilinear coordinate system.

Lastly, it is also convenient to define a density function below,

$$f(q, \bar{v}, t) \equiv J(q)\phi(q, \bar{v}, t)$$

so that hydrodynamic moments are given by simple integrations of $\bar{v}$, for instance

$$\rho = \int d\bar{v} f = \int dv^1 dv^2 dv^3 f; \quad \rho u^i = \int d\bar{v} v^i f = \int dv^1 dv^2 dv^3 v^i f$$