SECANT DIMENSIONS OF MINIMAL ORBITS: COMPUTATIONS AND CONJECTURES

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ABSTRACT

We present an algorithm for computing the dimensions of higher secant varieties of minimal orbits. Experiments with this algorithm lead to many conjectures on secant dimensions, especially of Grassmannians and Segre products. For these two classes of minimal orbits, we also point out a relation between the existence of certain codes and non-defectiveness of certain higher secant varieties.

1. Introduction

A generic polynomial of degree d in $\mathbb{C}[x]$ can be written as a sum of $\lfloor (d+1)/2 \rfloor$ powers $(ax+b)^d,\ a,b \in \mathbb{C}$. A generic $n \times n$-matrix of rank k and trace 0 is the sum of k matrices of rank 1 and trace 0; in fact, this is true for *any* trace zero matrix of rank k, though that doesn't matter here. But what is the generic rank of a tensor in $(\mathbb{C}^2)^{\otimes 10}$, i.e., if we want to write a generic element of this tensor power as a sum of decomposable tensors, then how many do we need?

These are instances of a general type of problem, which has been solved only in very few cases. In this note we do not solve many instances, either, but we do present a program for investigating small concrete instances. Also, we will boldly state some conjectures that our experiments with this program suggest. We hope that this note will be an incentive for people working in this field, either to prove or disprove our conjectures, or to use our program and experiment for themselves.

To be more concrete on the type of problem that our program can handle, let G be a reductive complex algebraic group and let V be a non-trivial irreducible module for G. The projective space $\mathbb{P}V$ contains a unique (Zariski-)closed orbit X, and we let $C \subseteq V$ be the affine cone over X. For any natural number k, we write kX for the Zariski closure of the union of all projective $(k-1)$-spaces spanned by k points on X; kX is called the $(k-1)$st *secant variety* of X. Often the term *secant variety* itself is used for the 1st secant variety, while those for $k > 2$ are referred to as *higher secant varieties*. More concretely, the affine cone over kX is the Zariski closure of

$$kC := \{v_1 + \ldots + v_k \mid v_i \in C\}.$$

In the examples above G is equal to SL_2 acting on the space V of binary forms of degree d, or to SL_n acting on its Lie algebra, or to SL_2^{10} acting on the space $(\mathbb{C}^2)^{\otimes 10}$ of 10-tensors, respectively. Accordingly, C is the set of pure d-th powers of linear forms, or the set of trace zero matrices of rank at most 1, or the set of all pure 10-tensors.

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One can ask many questions about the sets kX or kC. For instance: what are polynomial equations for the kX? For the matrix example we know them: kC is the set of trace zero matrices of rank at most k, and these are characterised (even scheme-theoretically, see [7]) by the vanishing of all $(k+1)$-minors. For the binary forms and the 10-tensors we do not know equations. Another question: are the sets kC closed? In the matrix case they are, for the 10-tensors we do not know, and for the binary forms they are not. One can show, in fact, that a polynomial with a zero of multiplicity m, $0 < m < d$, cannot be written as a sum of less than $m+1$ pure d-th powers, so that the sets kC with $\lfloor (d+1)/2 \rfloor \leq k < d$ cannot possibly be closed.

But by far the most modest property of kC to want to determine is its dimension $\dim kC := \dim \overline{kC} = \dim kX + 1$, and this is precisely what our algorithm does. As the addition map $\overline{(k-1)C} \times C \to \overline{kC}$ is dominant, $\dim kC$ is at most $\dim C + \dim(k-1)C$; we call the minimum of the latter number and $\dim V$ the *expected dimension* of kC. If kC has the expected dimension, then kC (and $\overline{kC}$ and kX) are called *non-defective*. Otherwise, $kC, \overline{kC}$, and kX are called *defective*. The difference $\min\{\dim V, \dim C + \dim(k-1)C\} - \dim kC$ is called the *k-defect*. If kC is not defective for any $k \geq 1$, then call C and X themselves non-defective; otherwise, we call them defective.

As we will see below, calculating $\dim kC$ in concrete cases boils down to straightforward linear algebra computations—at least if one allows for a small error probability—and only in rare concrete cases does kC *not* have the expected dimension. And yet it is very difficult to prove anything substantial in this direction.

First, however, we list some important things that *are* known about these higher secant varieties. The standard reference for secant varieties, containing a wealth of classification results on varieties with constraints on their secant dimensions, is [19].

(1) Take $G = \mathrm{SL}_n$ and let V be the space of homogeneous polynomials in $x_1, \ldots, x_n$ of degree d. Then C is the set of d-th powers of linear forms. A simple duality shows that $\dim kC$ is the *codimension* of the space of homogeneous polynomials in $x_1, \ldots, x_n$ of degree d that vanish together with all their first partial derivatives on k fixed, generic points. This relates higher secant varieties to the problem of multivariate interpolation, which was solved in the series of papers [1, 2, 3].

(2) For G a simple algebraic group acting on its Lie algebra $\mathfrak{g}$, the set C consists of all "extremal elements", that is: elements $X \in \mathfrak{g}$ for which $\mathrm{ad}(X)^2\mathfrak{g} \subseteq \mathbb{C}X$. The first secant variety is known in this case ([14, 15]), and for classical G the higher secant varieties were completely determined in [4].

(3) For $G = \mathrm{SL}_n$ and V the d-fold exterior power of $\mathbb{C}^n$ the set C is the affine cone over the Grassmannian, in its Plücker embedding, of d-dimensional vector spaces in $\mathbb{C}^n$. The paper [8] lists some defective Grassmannians, and proves that for $d > 2$ and $kd \leq n$ the variety kC is not defective. In Subsection 4.1 we generalise this latter result, and conjecture that the list of defective Grassmannians in [8] is complete.

From here, we proceed as follows: in Section 2 we present our algorithm for computing $\dim kC$, Section 3 deals with some implementation issues, and Section

4 lists our conjectures based on experiments with that implementation. These conjectures concern Grassmannians and Segre products, as well as a general finiteness statement.

2. The algorithm

We retain the notation G, V, X, C from the Introduction.

2.1. **The highest weight orbit.** We reduce the computation of $\dim kC$ to straightforward linear algebra as follows. First, we recall that X is the orbit of highest weight vectors. More precisely, let B be a Borel subgroup of G and let T be a maximal torus of G in B. For standard notions in algebraic group theory such as these we refer to [5]. Let $v_0 \in V$ span the unique B-stable line $\mathbb{C}v_0$ in V. Then the cone C over the minimal orbit X equals $C = Gv_0 \cup \{0\}$—recall that V was assumed non-trivial, so that this really is a cone. Let $P \supseteq B$ be the parabolic subgroup of G stabilising $\mathbb{C}v_0$, and denote by $\mathfrak{g}, \mathfrak{p}$ the Lie algebras of G, P, respectively.

2.2. **A dense orbit under a unipotent subgroup.** Now let $\mathfrak{u}_-$ be the direct sum of all T-root spaces in $\mathfrak{g}$ that are not in $\mathfrak{p}$. Then $\mathfrak{u}_-$ is the Lie algebra of a unique connected (unipotent) subgroup U_- of G. Let $X_1, \ldots, X_r$ be a basis of $\mathfrak{u}_-$ consisting of T-root vectors. Then the following statements are well known:

(1) The map $\mathbb{C}^r \to U_-$ sending $(t_1, \ldots, t_r)$ to $\exp(t_1X_1)\cdots\exp(t_rX_r)$ is an isomorphism of varieties.
(2) The U_--orbit U_-v_0 is the intersection of C with the affine hyperplane where the v_0-coordinate is 1 (relative to a weight basis of V containing v_0).
(3) Hence the image of Uv_0 in $\mathbb{P}V$ is dense in X.

Our program works, in fact, with elements in Uv_0 rather than all of C.

2.3. **Terracini's lemma.** Consider the addition map $\pi : C^k \to \overline{kC},\ (v_1, \ldots, v_k) \mapsto \sum_i v_i$. By elementary algebraic geometry the map sending the k-tuple $\mathbf{v} = (v_1, \ldots, v_k)$ to the rank of $d_{\mathbf{v}}\pi$ is lower semi-continuous, and its generic value is $\dim(kC)$ by the dominance of π. On the other hand, the image of $d_{\mathbf{v}}\pi$ equals

$$\sum_{i=1}^{k} T_{v_i}C,$$

where $T_{v_i}C$ denotes the tangent space to C at v_i, regarded as a linear subspace of V. We conclude that the dimension of this latter space is always a lower bound for $\dim kC$, while it is equal to $\dim kC$ for generic tuples $\mathbf{v}$. This observation is, in fact, one of the first results in the theory of join and secant varieties, and due to Terracini [18].

2.4. **The algorithm.** Now our algorithm, which we have implemented in `GAP` [9], is as follows:

Input: $(\mathfrak{g}, \lambda, k)$, where $\mathfrak{g}$ is a split rational semisimple Lie algebra with a distinguished split Cartan subalgebra $\mathfrak{h}$ and a distinguished Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$; $\lambda \in \mathfrak{h}^*$ is a $\mathfrak{b}$-dominant weight; and k is a natural number.
Output: a lower bound for $\dim kC$ which with high probability equals $\dim kC$.
Method:

(1) Construct the irreducible representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of highest weight λ.

(2) Denote by v_0 the highest weight vector of V, and compute representatives $(X_1, \ldots, X_r)$ of the negative $\mathfrak{h}$-root spaces in $\mathfrak{g}$ that do not vanish on v_0 (these span $\mathfrak{u}_-$).
(3) For $i \in \{1, \ldots, r\}$ compute those divided powers $\rho(X_i)^d/(d!)$ that are non-zero.
(4) Set $T := \{0\}$, the zero subspace of V.
(5) Compute $T_{v_0}C := Kv_0 + \mathfrak{u}_- v_0$.
(6) Repeat k times the following steps:
    (a) Choose rational numbers $t_1, \ldots, t_r$ at random.
    (b) Compute $u := \exp(t_1\rho(X_1)) \cdots \exp(t_r\rho(X_r))$ using the divided powers of the $\rho(X_i)$ for faster computation of the exponentials.
    (c) Set $T := T + uT_{v_0}C$.
(7) return $\dim T$.

3. Implementation

We have implemented the algorithm in the computer algebra system GAP4 ([9]), using the built-in functionality for semisimple Lie algebras and their representations. All steps are rather straightforward to implement. It turns out that when working over the field $\mathbb{Q}$, the main bottleneck of the algorithm is the computation of a basis of the space T in Step 6(c). For example the computation of the secant dimensions of the Grassmannian of 5-dimensional subspaces in an 11-dimensional vector space took 362 seconds, of which 309 were spent in the basis computation of Step 6(c).[1] This is due to the fact that the coefficients of the vectors grow very fast (probably because of their random nature). In the example mentioned before, the vectors in a triangularised basis were dense, and contained rational numbers of up to 70 digits in both denominator and numerator.

For this reason we have performed the computations modulo a prime p. This however presents a new problem: the coefficients of the matrices of the divided power in Step (3) may not be integral. We can get around this by computing an "admissible lattice" in the highest weight module V (cf. [12]). An algorithm for this purpose is not present in GAP4. We have implemented an algorithm for this based on the theory of crystal bases (cf. [13]). Roughly this works as follows. First we note that V is also a module for the quantum group $U_q(\mathfrak{g})$. Now from the crystal graph of V we get a set of elements F_i in the negative part of $U_q(\mathfrak{g})$, with the property that $\{F_i \cdot v_0\}$ spans an admissible lattice (for details we refer to [10], [16]). Each F_i can be mapped to an element F_i' of the negative part of the universal enveloping algebra $U(\mathfrak{g})$. Then $\{F_i' \cdot v_0\}$ spans an admissible lattice of V. This approach has the advantage that we do not need to check linear independence of the basis elements. The necessary algorithms for quantum groups are implemented in the GAP4 package QuaGroup ([11]). With this the computation in the example above took 71 seconds, with only 3 seconds spent in Step 6(c).

When computing modulo a prime the computed dimensions may be smaller than the ones over $\mathbb{Q}$. However, we have an upper bound for the dimension of kC (namely $k \dim C$) which "usually" gives the correct dimension. It rarely happens that this upper bound is not reached. However, if this happens to be the case, then we perform the computation modulo a bigger prime, and eventually over $\mathbb{Q}$. If we still

[1] The computations in this section were done on a 2GHz processor, with 500M RAM memory for GAP

do not attain the upper bound in that case, we conclude that we are in a defective situation with high probability.

Another problem occurs when the dimension of V gets large (e.g., close to 1000). Then storing the matrices in Step (3) may lead to memory problems. To get around this we used an ad-hoc implementation of sparse matrices (only storing the nonzero entries). This greatly reduces the memory requirements, and for dimensions greater than roughly 500 leads to a speed-up for the matrix multiplications in Step 6 (b) as well.

n	d	total	module	basis	$\dim V(\lambda)$	k_{max}
11	5	71	60	3	462	14
12	6	222	165	23	924	24
13	6	973	686	160	1716	39
14	7	5456	3249	1380	3432	68

TABLE 1. Time (in seconds) for the computation of the secant dimensions of d-dimensional subspaces in n-space. The third column has the total time spent. The fourth and fifth have respectively the time used for the construction of the module along with the matrices in Step (3), and for the computation of the basis in Step 6(c). The sixth column displays the dimension of the $\mathfrak{g}$-module, and the last column has k_{max}, which means that $\dim kC$ has been computed for $1 \leq k \leq k_{max}$.

Table 1 contains some run-times of the algorithm, when computing the secant dimensions of the Grassmannian of d-dimensional subspaces of n-space. We see that the running times increase rather sharply, mainly because the same holds for the dimensions of the $\mathfrak{g}$-modules. Most of the time is spent on the construction of the module $V(\lambda)$ and the matrices of Step (3). For small n the time used in Step 6(c) is negligible, but as n increases the percentage of the time spent in that step also increases.

4. Conjectures

4.1. **Grassmannians.** For $G = \mathrm{SL}_n$ and $V = \bigwedge^d(\mathbb{C}^n)$ the minimal orbit X is the Grassmannian, in its Plücker embedding, of d-dimensional vector subspaces of $\mathbb{C}^n$, and the cone C over X is the set of (completely) decomposable wedge-products in V. For $d = 2$ the set kC equals is the set of all skew-symmetric matrices of (usual matrix-) rank at most $2k$ (see, e.g., [19] or [4]); we therefore exclude $d = 2$ in the following conjecture.

Conjecture 4.1. *Suppose that $d > 2$ and also that $2d \leq n$. Then C is defective in exactly the following cases:*

1. *$n = 7$ and $d = 3$, in which case $(\dim kC)_k$ equals $(13, 26, *34, 35)$;*
2. *$n = 8$ and $d = 4$, in which case $(\dim kC)_k$ equals $(17, 34, *50, *64, 70)$; or*
3. *$n = 9$ and $d = 3$, in which case $(\dim kC)_k$ equals $(19, 38, 57, *74, 84)$,*

where $*$ indicates the defective dimensions.

We have verified this conjecture with our program for all n up to 14. The defective Grassmannians in the list above were already found in [8]. In fact, it is somewhat tedious, but not hard, to prove the following proposition by hand.

Proposition 4.2. *In the setting of Conjecture 4.1, write $n = qd+r$ with $0 \leq r < d$. Then kC is not defective for $k = 1, \ldots, q$, and $(q+1)C$ is defective if and only if $(n, d, q+1) \in \{(7, 3, 3), (8, 4, 3), (9, 3, 4)\}$.*

Outline of proof. In fact, for these k the group SL_n has a dense orbit on the k-fold Cartesian product X^k of X: Choose a basis

$$\begin{matrix} e_{11} & \ldots & e_{1d} \\ e_{21} & \ldots & e_{2d} \\ \vdots & & \vdots \\ e_{q,1} & \ldots & e_{q,d} \\ e_{q+1,1} & \ldots & e_{q+1,r} \end{matrix} .$$

For $i = 1, \ldots, q$ let v_i be the wedge product of the i-th row of this array. For $j = r+1, \ldots, d$ let s_j be the sum of the j-th column in this array, and set

$$v_{q+1} := e_{q+1,1} \cdots e_{q+1,r} s_{r+1} \cdots s_d,$$

where the product is the wedge product in the Grassmann algebra. Then the orbit of $(\mathbb{C}v_1, \ldots, \mathbb{C}v_k)$ is dense in X^k for all $k \leq q+1$, so the dimension of kC equals the dimension of the sum

$$T_{v_1}C + \ldots + T_{v_k}C.$$

For $k < q+1$ this sum is readily seen to be direct, and we are done. For $k = q+1$ we find that kC is defective if and only if $T_{v_k}C$ contains a non-zero element of $\bigoplus_{i \leq q} T_{v_i}C$. Some easy, though tedious, combinatorics and linear algebra shows that this happens exactly in the cases listed above. □

We conclude with an interesting link between secant dimensions of Grassmannians and coding theory: a *binary code* of length n and constant weight d is a subset of $\{0, 1\}^n$ where every element has exactly d entries equal to 1. The *(Hamming) distance* between two elements of $\{0, 1\}^n$ is the number of coordinates where they differ.

Theorem 4.3. *Retain the setting of Conjecture 4.1, and let B be a binary code of length n and constant weight d with $|B| = k$. Then the following holds.*

1. *If the distance between any two distinct elements of B is at least 6, then kC is not defective.*
2. *If every word in $\{0, 1\}^n$ of weight d has an element of B at distance at most 2, then $kC = V$.*

The second observation is also stated, in a slightly different form, in [17].

Proof. Let $e_1, \ldots, e_n$ be a basis of $\mathbb{C}^n$. To every word w in $\{0, 1\}^n$ of weight d we associate an element of V as follows: if $i_1 < \ldots < i_d$ are the coordinates i where $b_i = 1$, then we set

$$e_w := e_{i_1} \cdots e_{i_d} \in V.$$

Now

$$T_{e_w}C = \sum_{j=1}^{d} e_{i_1} \cdots e_{i_{j-1}} \mathbb{C}^n e_{i_{j+1}} \cdots e_{i_d}$$

is precisely the span of all e_u where u is a word of weight d at distance at most 2 from w. Hence the dimension of $\sum_{b \in B} T_{e_b} C$ (and therefore that of kC) is at least the cardinality of the set of all weight-d words in $\{0,1\}^n$ at distance at most 2 from B. This implies both statements. □

This observation generalises [8, Theorem 2.1.ii].

4.2. **Segre powers.** For $G = \mathrm{SL}_n^d$ and $V = (\mathbb{C}^n)^{\otimes d}$ the minimal orbit X is $(\mathbb{P}^{n-1})^d$ in its Segre embedding, and the affine cone C over X is the set of decomposable tensors in V. For $d = 2$ the set kC corresponds to the set of $n \times n$-matrices of rank $\leq k$, so we leave out this well-understood case from our study. Our conjecture is as follows.

Conjecture 4.4. *Suppose that $d \neq 2$. The variety C is defective if and only if*

1. *$n = 2$ and $d = 4$, in which case $(\dim kC)_k$ equals $(5, 10, *14, 16)$; or*
2. *$n = 3$ and $d = 3$, in which case $(\dim kC)_k$ equals $(7, 14, 21, *26, 27)$,*

where $*$ indicates the defective secant dimensions.

We have verified this conjecture with our program for $d + n \leq 8$ as well as for $n = 2$ and $d = 9, 10$ and for $d = 3$ and $n \leq 9$. Again, it is not hard but tedious to prove the following proposition.

Proposition 4.5. *In the setting of Conjecture 4.4, kC is not defective for $k \leq n$, and $(n + 1)C$ is defective if and only if $(n, d) \in \{(2, 4), (3, 3)\}$.*

The proof is completely analogous to that of Proposition 4.2: here $(\mathrm{SL}_n)^d$ has a dense orbit on X^k, hence the rank of the differential of the summation map $C^k \to kC$ need only be computed in a point of C^k over this orbit. We omit the details, but do report funny numeric coincidence: the same computation that shows that $3C$ is defective for the Grassmannian of 4-dimensional subspaces of an 8-dimensional space, also shows that $3C$ is defective for the 4th Segre power of $\mathbb{PC}^2$. The same (numeric) connection exists between the defect in $4C$ for the Grassmannian of 3-dimensional subspaces of a 9-dimensional space and the defect in $4C$ for the 3rd Segre power of $\mathbb{PC}^3$.

Again, we conclude with a link to coding theory.

Theorem 4.6. *In the setting of Conjecture 4.4 let B be a subset of $\{1, \ldots, n\}^d$ of size k. Then the following holds.*

1. *If the Hamming distance between any two elements of B is at least 3, then kC is not defective.*
2. *If every element of $\{1, \ldots, n\}^d$ is at distance at most 1 from an element of B, then $kC = V$.*

Again, the second part is also present in [17].

Proof. Let $e_1, \ldots, e_n$ be a basis of V. To an element $w = (i_1, \ldots, i_d)$ of $\{1, \ldots, n\}^d$ we associate the element $e_w := e_{i_1} \cdots e_{i_d}$ in V, where the product is taken in the tensor algebra over $\mathbb{C}^n$. Then $T_{e_w} C$ is precisely the span of all e_u where u is at distance ≤ 1 from w. Hence $\dim(kC)$ is at least the cardinality of the union of all balls of radius 1 around words in B. This implies both statements. □

4.3. **A finiteness question.** The experiments with our program suggest the following question: Fix the complex semisimple group G. Is it true that the set of all irreducible representations of G whose minimal orbit is defective is *finite*?

Though we dare not formulate this as a conjecture at this stage, the question is a very natural one. For instance, from the fundamental work of Alexander and Hirschowitz [1, 2, 3], we know that for each n only finitely many symmetric powers of the natural representation of SL_n have defective minimal orbits.

We give a conjecturally complete list of "defective highest weights" for some small groups; note that for $G = \mathrm{SL}_2$ the minimal orbit in *no* irreducible representation is defective.

Conjecture 4.7. *The only irreducible representations of G for which the minimal orbit is defective are those with the following highest weights (in the labelling of* [6]*):*

(1) *for G of type A_2:*
   (a) $2\omega_1$ *and* $2\omega_2$ *with secant dimensions* $(3, *5, 6)$,
   (b) $4\omega_1$ *and* $4\omega_2$ *with secant dimensions* $(3, 6, 9, 12, *14, 15)$,
   (c) $\omega_1 + \omega_2$ *with secant dimensions* $(4, *7, 8)$, *and*
   (d) $2\omega_1 + 2\omega_2$ *with secant dimensions* $(4, 8, 12, 16, 20, 24, *26, 27)$.

(2) *for G of type A_3:*
   (a) $\omega_1 + \omega_2$ *and* $\omega_2 + \omega_3$ *with secant dimensions* $(6, 12, *17, 20)$,
   (b) $2\omega_1$ *and* $2\omega_3$ *with secant dimensions* $(4, *7, *9, 10)$.
   (c) $\omega_1 + \omega_3$ *with secant dimensions* $(6, *11, *14, 15)$,
   (d) $2\omega_2$ *with secant dimensions* $(5, 10, *14, *17, *19, 20)$,
   (e) $4\omega_1$ *and* $4\omega_3$ *with secant dimensions* $(4, 8, 12, 16, 20, 24, 28, 32, *34, 35)$.

(3) *for G of type B_2:*
   (a) $2\omega_1$ *with secant dimensions* $(4, 8, *11, *13, 14)$,
   (b) $2\omega_2$ *with secant dimensions* $(4, *7, *9, 10)$,
   (c) $\omega_1 + \omega_2$ *with secant dimensions* $(5, 10, *14, 16)$.
   (d) $4\omega_2$ *with secant dimensions* $(4, 8, \ldots, 28, 32, *34, 35)$, *and*

(4) *for G of type G_2:*
   (a) $2\omega_1$ *with secant dimensions* $(6, 12, *17, *21, 24)$,
   (b) ω_2 *with secant dimensions* $(6, *11, 14)$, *and*
   (c) $2\omega_2$ *with secant dimensions* $(6, \ldots, 72, *76, 77)$.

The conjecture for A_2 and B_2 has been verified for weights $i\omega_1 + j\omega_2$ with $i+j \leq 6$. For A_3 the conjecture has been checked for highest weights $i\omega_1 + j\omega_2 + k\omega_3$ with $i + j + k \leq 4$, and for G_2 the conjecture has been verified for all highest weights $i\omega_1 + j\omega_2$ with $i + j \leq 4$.

To attack the question posed in this section, one would need completely new techniques, far beyond our easy algorithm. But we hope that the challenges boldly posed in this paper as conjectures will be taken up by some of our readers!

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