

On layered stable processes

CHRISTIAN HOUDRÉ* AND REIICHIRO KAWAI†

Abstract

Layered stable (multivariate) distributions and processes are defined and studied. A layered stable process combines stable trends of two different indexes, one of them possibly Gaussian. More precisely, over short intervals it is close to a stable process, while over long intervals it approximates another stable (possibly Gaussian) process. We also investigate the absolute continuity of a layered stable process with respect to its short range limiting stable process. A series representation of layered stable processes is derived, giving insights into both the structure of the sample paths and of the short and long range behaviors of the process. This series representation is further used for sample paths simulation.

Keywords: Lévy processes, stable distributions and processes, layered stable distributions and processes.

AMS Subject Classification (2000): 60G52, 60G51, 60E07, 60F05.

1 Introduction and preliminaries

Stable processes form one of the simplest class of Lévy processes without Gaussian component. They have been thoroughly studied by many authors and have been used in several fields of applications, such as statistical physics, queuing theory, and mathematical finance. A major reason for their attractiveness is due to the scaling property induced by the structure of the corresponding Lévy measure. Sato [11] and Samorodnitsky and Taqqu [9] contain many basic facts on stable distributions and processes. Recent generalizations of stable processes can also be found, for example, in Barndorff-Nielsen and Shepard [2] and in Rosiński [8]. These new classes which are of great interest in applications, have moreover motivated our study.

Published in *Bernoulli* (2007) **13**(1) 252-278.

*School of Mathematics, Georgia Institute of Technology, Atlanta, GA, 30332-0160, USA. E-mail: houdre@math.gatech.edu

†Financial Engineering, Fixed Income Department, Daiwa Securities SMBC Co.Ltd., 1-14-5, Eitai, Koto-ku, Tokyo, 135-0034, Japan. E-mail: reiichiro.kawai@ybb.ne.jp. The views expressed in this paper are those of the author and do not reflect those of Daiwa Securities SMBC Co.Ltd.

In the present paper, we introduce and study further generalizations which we call *layered stable*. They are defined in terms of the structure of their Lévy measure whose radial component behaves asymptotically as an inverse polynomial, of different orders, near zero and at infinity. *The inner and outer (stability) indexes* correspond respectively to these orders of polynomial decay. This simple layering leads to the following properties: The outer index determines the moment properties (Proposition 2.3), while the variational properties depend on the inner index (Proposition 2.5). On the other hand, the inner and outer indexes also correspond to the short and long range behavior of the sample paths. Over short intervals, a layered stable process behaves like a stable one with the corresponding inner index (Theorem 3.1). The long range behavior has two modes depending on the outer index. When the outer index is strictly smaller than two, a layered stable process is close to a stable process with this index, while it behaves like Brownian motion if the outer index is strictly greater than two (Theorem 3.2). In relation with the short time behavior, we also investigate the mutual absolute continuity of a layered stable process and of its short time limiting stable process (Theorem 4.1). A shot noise series representation reveals the nature of layering and also gives direct insights into the properties of layered stable processes. We present typical sample paths of a layered stable process, which are simulated via the series representation, for various combinations of stability indexes in order to cover all the possible short and long time behaviors.

Let us begin with some general notations which will be used throughout the text. \mathbb{R}^d is the d -dimensional Euclidean space with the norm $\|\cdot\|$, $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$, $\mathcal{B}(\mathbb{R}_0^d)$ is the Borel σ -field of \mathbb{R}_0^d , $\mathcal{S}^{d-1} := \{z \in \mathbb{R}^d : \|z\| = 1\}$. The symbol $'$ is used to denote transpose, so $z' = (z_1, \dots, z_d) \in \mathbb{R}^d$, while A' is the transpose of the matrix A . We denote by $\|\cdot\|_o$ the operator norm of a linear transformation, so if $A \in \mathbb{R}^{d \times d}$, then $\|A\|_o = \sup_{\|x\| \leq 1} \|Ax\|$. $f(x) \sim g(x)$ indicates that $f(x)/g(x) \rightarrow 1$, as $x \rightarrow x_0 \in [-\infty, \infty]$, while $f(x) \asymp g(x)$ is used to mean that there exist two positive constants c_1 and c_2 such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$, for all x in an appropriate set. $\mathcal{L}(X)$ is the law of the random vector X , while $\stackrel{\mathcal{L}}{=}$ and $\stackrel{\mathcal{L}}{\rightarrow}$ denote respectively equality and convergence in distribution, or of the finite dimensional distributions when random processes are considered. Moreover, \xrightarrow{d} is used for the weak convergence of random processes in the space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ of càdlàg functions from $[0, \infty)$ into \mathbb{R}^d equipped with the Skorohod topology, while \xrightarrow{v} denotes convergence in the vague topology. For any $r > 0$, T_r is a transformation of measures on \mathbb{R}^d given, for any positive measure ρ , by $(T_r \rho)(B) = \rho(r^{-1}B)$, $B \in \mathcal{B}(\mathbb{R}^d)$. $\mathbb{P}|_{\mathcal{F}_t}$ is the restriction of a probability measure \mathbb{P} to the σ -field \mathcal{F}_t , while ΔX_t denotes the jump of X at time t , that is, $\Delta X_t := X_t - X_{t-}$. Finally, and throughout, all the multivariate or matricial integrals are defined component-wise.

Recall that an infinitely divisible probability measure μ on \mathbb{R}^d , without Gaussian component, is called *stable* if its Lévy measure is given by

$$\nu(B) = \int_{\mathcal{S}^{d-1}} \sigma(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where $\alpha \in (0, 2)$ is the stability index and where σ is a finite positive measure on S^{d-1} . It is well known that the characteristic function of μ is given by

$$\begin{aligned}\widehat{\mu}(y) &= \exp \left[i\langle y, \eta \rangle + \int_{\mathbb{R}_0^d} (e^{i\langle y, z \rangle} - 1 - i\langle y, z \rangle \mathbb{1}_{(0,1]}(\|z\|)) \nu_\alpha(dz) \right] \\ &= \begin{cases} \exp \left[i\langle y, \tau_\alpha \rangle - c_\alpha \int_{S^{d-1}} |\langle y, \xi \rangle|^\alpha (1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn} \langle y, \xi \rangle) \sigma(d\xi) \right], & \text{if } \alpha \neq 1, \\ \exp \left[i\langle y, \tau_1 \rangle - c_1 \int_{S^{d-1}} (|\langle y, \xi \rangle| + i \frac{2}{\pi} \langle y, \xi \rangle \ln |\langle y, \xi \rangle|) \sigma(d\xi) \right], & \text{if } \alpha = 1, \end{cases}\end{aligned}\quad (1.1)$$

for some $\eta \in \mathbb{R}^d$, and where $c_\alpha = |\Gamma(-\alpha) \cos \frac{\pi\alpha}{2}|$ when $\alpha \neq 1$ while $c_1 = \pi/2$, with moreover $\tau_\alpha = \eta - \frac{1}{1-\alpha} \int_{S^{d-1}} \xi \sigma(d\xi)$ when $\alpha \neq 1$ and $\tau_1 = \eta - (1 - \gamma) \int_{S^{d-1}} \xi \sigma(d\xi)$, $\gamma (= 0.5772\dots)$ being the Euler constant. A Lévy process $\{X_t : t \geq 0\}$ such that $\mathcal{L}(X_1) \sim \mu$ is called a *stable process*. Stable processes enjoy the *self-similarity* property, i.e., for any $a > 0$,

$$\{X_{at} : t \geq 0\} \stackrel{\mathcal{L}}{=} \{a^{1/\alpha} X_t + bt : t \geq 0\},$$

for some $b \in \mathbb{R}^d$. Next, we recall a shot noise series representation of stable processes on a fixed finite horizon $[0, T]$, $T > 0$. Related results can be found, for example, in Theorem 1.4.5 of Samorodnitsky and Taqqu [9]. The centering constants given below are obtained in Proposition 5.5 of Rosiński [8].

Lemma 1.1. *Let $T > 0$. Let $\{T_i\}_{i \geq 1}$ be a sequence of iid uniform random variables on $[0, T]$, let $\{\Gamma_i\}_{i \geq 1}$ be an arrival times of a standard Poisson process, and let $\{V_i\}_{i \geq 1}$ a sequence of iid random vectors in S^{d-1} with common distribution $\sigma(d\xi)/\sigma(S^{d-1})$. Also let*

$$z_0 = \begin{cases} 0, & \text{if } \alpha \in (0, 1), \\ \int_{S^{d-1}} \xi \sigma(d\xi) / \sigma(S^{d-1}), & \text{if } \alpha \in [1, 2), \end{cases}$$

and

$$b_T = \begin{cases} 0, & \text{if } \alpha \in (0, 1), \\ \sigma(S^{d-1}) T (\gamma + \ln(\sigma(S^{d-1}) T)), & \text{if } \alpha = 1, \\ \left(\frac{\alpha}{\sigma(S^{d-1}) T} \right)^{-1/\alpha} \zeta(1/\alpha), & \text{if } \alpha \in (1, 2), \end{cases}$$

where ζ denotes the Riemann zeta function. Then, the stochastic process

$$\left\{ \sum_{i=1}^{\infty} \left[\left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1}) T} \right)^{-1/\alpha} V_i \mathbb{1}(T_i \leq t) - \left(\frac{\alpha i}{\sigma(S^{d-1}) T} \right)^{-1/\alpha} z_0 \frac{t}{T} \right] + b_T z_0 \frac{t}{T} : t \in [0, T] \right\},$$

converges almost surely uniformly in t to an α -stable process $\{X_t : t \in [0, T]\}$ satisfying $\mathbb{E}[e^{i\langle y, X_T \rangle}] = \widehat{\mu}(y)^T$, where $\widehat{\mu}$ given by (1.1) with

$$\eta = \begin{cases} \frac{1}{1-\alpha} \int_{S^{d-1}} \xi \sigma(d\xi), & \text{if } \alpha \neq 1, \\ 0, & \text{if } \alpha = 1. \end{cases}$$

2 Definition and basic properties

We first define a layered stable multivariate distribution by making precise the structure of its Lévy measure in polar coordinates.

Definition 2.1. On \mathbb{R}^d , let μ be an infinitely divisible probability measure without Gaussian component. Then, μ is called layered stable if its Lévy measure on \mathbb{R}_0^d is given by

$$v(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) q(r, \xi) dr, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \quad (2.1)$$

where σ is a finite positive measure on S^{d-1} , and q is a locally integrable function from $(0, \infty) \times S^{d-1}$ to $(0, \infty)$ and such that as $r \rightarrow 0$,

$$q(r, \xi) \sim c_1(\xi) r^{-\alpha-1}, \quad (2.2)$$

and as $r \rightarrow \infty$,

$$q(r, \xi) \sim c_2(\xi) r^{-\beta-1}, \quad (2.3)$$

for σ -a.e. $\xi \in S^{d-1}$, where c_1 and c_2 are positive, integrable (with respect to σ) functions on S^{d-1} , and where $(\alpha, \beta) \in (0, 2) \times (0, \infty)$.

Let us call $q(\cdot, \cdot)$ the q -function of μ , or of its Lévy measure v . Clearly, v is well defined as a Lévy measure since it behaves like an α -stable Lévy measure near the origin while decaying like a β -Pareto density when sufficiently far away from the origin. Moreover, α and β are respectively called the inner and outer (stability) index of μ , or of v .

For convenience, we henceforth use the notations σ_1 and σ_2 for the finite positive measures on S^{d-1} defined respectively by

$$\sigma_1(B) := \int_B c_1(\xi) \sigma(d\xi), \quad B \in \mathcal{B}(S^{d-1}), \quad (2.4)$$

and

$$\sigma_2(B) := \int_B c_2(\xi) \sigma(d\xi), \quad B \in \mathcal{B}(S^{d-1}), \quad (2.5)$$

while v_σ^α is used for the positive measure on \mathbb{R}_0^d given by

$$v_\sigma^\alpha(B) := \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{dr}{r^{\alpha+1}}, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \quad (2.6)$$

where $\alpha \in (0, \infty)$ and where σ is a finite positive measure on S^{d-1} . Note that if $\alpha \in (0, 2)$, v_σ^α is simply an α -stable Lévy measure, while it is not well defined as a Lévy measure when $\alpha \geq 2$.

Example 2.2. The following example of layered stable Lévy measure is simple, yet interesting:

$$\begin{aligned} v(B) &= \int_B \mathbb{1}_{(0,1]}(\|z\|) v_\sigma^\alpha(dz) + \int_B \mathbb{1}_{(1,+\infty)}(\|z\|) v_\sigma^\beta(dz) \\ &= \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{dr}{r^{\alpha+1} \mathbb{1}_{(0,1]}(r) + r^{\beta+1} \mathbb{1}_{(1,+\infty)}(r)}, \quad B \in \mathcal{B}(\mathbb{R}_0^d). \end{aligned} \quad (2.7)$$

Its q -function is given by

$$q(r, \xi) = r^{-\alpha-1} \mathbb{1}_{(0,1]}(r) + r^{-\beta-1} \mathbb{1}_{(1,+\infty)}(r), \quad \xi \in S^{d-1},$$

which is independent of ξ . The measure ν consists of two disjoint domains of stability, and this construction results in two layers for the radial component associated with each respective stability index. The name “*layered stable*” originates from this special structure.

Recall that an infinitely divisible probability measure μ on \mathbb{R}^d is said to be of class L_0 , or self-decomposable if for any $b > 1$, there exists a probability measure ρ_b such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$. Equivalently, the Lévy measure of μ has the form

$$\int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) k_\xi(r) \frac{dr}{r}, \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where σ is a finite positive measure on S^{d-1} and where $k_\xi(r)$ is a nonnegative function measurable in $\xi \in S^{d-1}$ and decreasing in $r > 0$. Clearly, the Lévy measure (2.7) induces a self-decomposable measure. Moreover, the classes L_m , $m = 1, 2, \dots$, are defined recursively as follows: $\mu \in L_m$ if for every $b > 1$, there exists $\rho_b \in L_{m-1}$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$. It is then also clear that $L_0 \supset L_1 \supset L_2 \supset \dots$. Let $h_\xi(u) := k_\xi(e^{-u})$, be the so-called h -function of μ , or of its Lévy measure. Then, alternatively, $\mu \in L_0$ is shown to be in L_m if and only if $h_\xi(u) \in C^{m-1}$ and $h^{(j)} \geq 0$, for $j = 0, 1, \dots, m-1$. (See Sato [10] for more details.) The h -function of the Lévy measure (2.7) is given by

$$h_\xi(u) = e^{\alpha u} \mathbb{1}_{(0, \infty)}(u) + e^{\beta u} \mathbb{1}_{(-\infty, 0]}(u),$$

which is in C^0 but not in C^1 . Therefore, the infinitely divisible probability measure induced by (2.7) is in L_1 , but not in L_2 .

The following result asserts that layered stable distributions have the same probability tail behavior as β -Pareto distributions, or β -stable distributions if $\beta \in (0, 2)$.

Proposition 2.3. (Moments) *Let μ be a layered stable distribution with Lévy measure ν given by (2.1) and let also σ_2 be the measure given by (2.5). If $\sigma_2(S^{d-1}) \neq 0$, then*

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) \begin{cases} < +\infty, & p \in (0, \beta), \\ = +\infty, & p \in [\beta, \infty). \end{cases}$$

Moreover, $\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty$, $p \geq \beta$ and $\int_{\mathbb{R}^d} e^{\theta \|x\|} \mu(dx) < \infty$, $\theta > 0$ if and only if $\sigma_2(S^{d-1}) = 0$.

Proof. By Theorem 25.3 of Sato [11], it is enough to show that the restriction of ν to the set $\{z \in \mathbb{R}_0^d : \|z\| > 1\}$ has the corresponding moment properties. First, assume $\sigma_2(S^{d-1}) \neq 0$. Since $\int_{\|z\|>1} \|z\|^p \nu(dz) = \int_{S^{d-1}} \sigma(d\xi) \int_1^\infty r^p q(r, \xi) dr$, and then by (2.3), the right hand side is bounded

from above and below by constant multiples of $\sigma_2(S^{d-1}) \int_1^\infty r^p \frac{dr}{r^{\beta+1}}$, if $p \in (0, \beta)$, while it is otherwise clearly infinite. Next, assume $\sigma_2(S^{d-1}) = 0$ and let $p \in [\beta, \infty)$. Then, there exists $M > 0$ such that

$$\int_{\|z\|>1} \|z\|^p \mathbf{v}(dz) \asymp \int_{S^{d-1}} \sigma(d\xi) \int_1^M r^p q(r, \xi) dr,$$

and

$$\int_{\|z\|>1} e^{\theta\|z\|} \mathbf{v}(dz) \asymp \int_{S^{d-1}} \sigma(d\xi) \int_1^M e^{\theta r} q(r, \xi) dr.$$

Conversely, if $\sigma_2(S^{d-1}) \neq 0$ and $p \in [\beta, \infty)$, then $\int_{\|z\|>1} \|z\|^p \mathbf{v}(dz) = +\infty$ as already shown and, once more, by (2.3), $\int_{\|z\|>1} e^{\theta\|z\|} \mathbf{v}(dz) = \int_{S^{d-1}} \sigma(d\xi) \int_1^\infty e^{\theta r} q(r, \xi) dr = +\infty$. \square

Let us define the associated Lévy processes.

Definition 2.4. *A Lévy process, without Gaussian component, is called layered stable if its Lévy measure is given by (2.1).*

Henceforth, $\{X_t^{LS} : t \geq 0\}$ denotes a layered stable process in \mathbb{R}^d . Its characteristic function at time 1 is given by

$$\mathbb{E}[e^{i\langle y, X_1^{LS} \rangle}] = \exp \left[i\langle y, \eta \rangle + \int_{\mathbb{R}_0^d} (e^{i\langle y, z \rangle} - 1 - i\langle y, z \rangle \mathbb{1}_{(0,1]}(\|z\|)) \mathbf{v}(dz) \right], \quad (2.8)$$

where \mathbf{v} is the Lévy measure given by (2.1) and $\eta \in \mathbb{R}^d$. For convenience of notation, we write $\{X_t^{LS} : t \geq 0\} \sim LS_{\alpha, \beta}(\sigma, q; \eta)$ when (2.8) holds. Similarly, for $\alpha \in (0, 2)$, $\{X_t^{(\alpha)} : t \geq 0\}$ denotes an α -stable Lévy process. Its characteristic function at time 1 is given by

$$\mathbb{E}[e^{i\langle y, X_1^{(\alpha)} \rangle}] = \begin{cases} \exp \left[i\langle y, \eta \rangle + \int_{\mathbb{R}_0^d} (e^{i\langle y, z \rangle} - 1) \mathbf{v}_\sigma^\alpha(dz) \right], & \text{if } \alpha \in (0, 1), \\ \exp \left[i\langle y, \eta \rangle + \int_{\mathbb{R}_0^d} (e^{i\langle y, z \rangle} - 1 - i\langle y, z \rangle \mathbb{1}_{(0,1]}(\|z\|)) \mathbf{v}_\sigma^1(dz) \right], & \text{if } \alpha = 1, \\ \exp \left[i\langle y, \eta \rangle + \int_{\mathbb{R}_0^d} (e^{i\langle y, z \rangle} - 1 - i\langle y, z \rangle) \mathbf{v}_\sigma^\alpha(dz) \right], & \text{if } \alpha \in (1, 2), \end{cases} \quad (2.9)$$

where \mathbf{v}_σ^α is given by (2.6), and we write $\{X_t^{(\alpha)} : t \geq 0\} \sim S_\alpha(\sigma; \eta)$ when (2.9) holds.

Recall that a stochastic process $\{X_t : t \in [0, T]\}$ is said to have a.s. finite p -th variation if

$$\mathbb{P} \left(\sup_{\pi \in \mathcal{P}} \sum_{t_i \in \pi} \|X_{t_i} - X_{t_{i-1}}\|^p < \infty \right) = 1,$$

where \mathcal{P} is the set of all the finite partitions π of $[0, T]$, of the form $0 = t_0 \leq t_1 \leq \dots \leq t_{n-1} \leq t_n = T$, $n \geq 1$. A layered stable process shares the variational properties of a stable process with inner index α .

Proposition 2.5. (p -th variation) Let $X := \{X_t^{LS} : t \geq 0\} \sim LS_{\alpha,\beta}(\sigma, q; \eta)$.

(i) If $\sigma_1(S^{d-1}) > 0$, then X has a.s. finite first variation on every interval of positive length if and only if $\alpha \in (0, 1)$.

(ii) If $\sigma_1(S^{d-1}) > 0$, $(\alpha, \beta) \in [1, 2) \times (1, \infty)$ and $\eta = -\int_{S^{d-1}} \xi \sigma(d\xi) \int_1^\infty r q(r, \xi) dr$, then X has a.s. finite p -th variation on every interval of positive length if and only if $p > \alpha$.

(iii) If $\sigma_1(S^{d-1}) = 0$, then X has a.s. finite first variation on every interval of positive length.

Proof. (i) Recall that, near the origin, the radial component of the layered stable Lévy measure behaves like an α -stable Lévy measure. The first claim then follows immediately from Theorem 3 of Gikhman and Skorokhod [4].

(ii) Since X is now centered, Théorème III b of Bretagnolle [3] directly applies.

(iii) Letting ν be the Lévy measure of X , there exists $\varepsilon \in (0, 1)$ such that $\nu(\{z \in \mathbb{R}_0^d : \|z\| \leq \varepsilon\}) < +\infty$ and so $\int_{\|z\| \leq 1} \|z\|^p \nu(dz) < +\infty$, $p \geq 1$. As in (i), the result follows from Theorem 3 of [4]. \square

Let us now consider a series representation for a general layered stable process $\{X_t^{LS} : t \geq 0\} \sim LS_{\alpha,\beta}(\sigma, q; 0)$. Fix $T > 0$. Let $\{T_i\}_{i \geq 1}$ be a sequence of iid uniform random variables on $[0, T]$, let $\{\Gamma_i\}_{i \geq 1}$ be Poisson arrivals with rate 1, and let $\{V_i\}_{i \geq 1}$ be a sequence of iid random vectors in S^{d-1} with common distribution $\sigma(d\xi)/\sigma(S^{d-1})$. Assume moreover that the random sequences $\{T_i\}_{i \geq 1}$, $\{\Gamma_i\}_{i \geq 1}$, and $\{V_i\}_{i \geq 1}$ are all mutually independent. Also, let

$$\overleftarrow{q}(u, \xi) := \inf \left\{ r > 0 : \sigma(S^{d-1}) \int_r^\infty q(s, \xi) ds < u \right\},$$

and let $\{b_i\}_{i \geq 1}$ be a sequence of constants given by

$$b_i = \int_{i-1}^i \mathbb{E}[\overleftarrow{q}(s/T, V_1) V_1 \mathbb{1}(\overleftarrow{q}(s/T, V_1) \leq 1)] ds,$$

Then, by Theorem 5.1 of Rosiński [7], the stochastic process

$$\left\{ \sum_{i=1}^\infty \left[\overleftarrow{q}(\Gamma_i/T, V_i) V_i \mathbb{1}(T_i \leq t) - b_i \frac{t}{T} \right] : t \in [0, T] \right\}, \quad (2.10)$$

converges almost surely uniformly in t to a Lévy process whose marginal law at time 1 is identical to $LS_{\alpha,\beta}(\sigma, q; 0)$.

Example 2.6. The Lévy measure (2.7) leads to a very illustrative series representation. Indeed,

$$\overleftarrow{q}(r, \xi) = \left(\frac{\beta r}{\sigma(S^{d-1})} \right)^{-1/\beta} \mathbb{1}_{(0, \sigma(S^{d-1})/\beta]}(r) + \left(\frac{\alpha r}{\sigma(S^{d-1})} + 1 - \frac{\alpha}{\beta} \right)^{-1/\alpha} \mathbb{1}_{(\sigma(S^{d-1})/\beta, +\infty)}(r),$$

and so the stochastic process

$$\left\{ \sum_{i=1}^{\infty} \left[\left(\left(\frac{\beta \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\beta} \mathbb{1}_{(0, \sigma(S^{d-1})T/\beta]}(\Gamma_i) + \left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})T} + 1 - \frac{\alpha}{\beta} \right)^{-1/\alpha} \mathbb{1}_{(\sigma(S^{d-1})T/\beta, \infty)}(\Gamma_i) \right) V_i \mathbb{1}(T_i \leq t) - b_i z_0 \frac{t}{T} \right] : t \in [0, T] \right\}, \quad (2.11)$$

where

$$b_i = \left(\frac{\beta}{\sigma(S^{d-1})T} \right)^{-1/\beta} \frac{(i \wedge \sigma(S^{d-1})T/\beta)^{1-1/\beta} - ((i-1) \wedge \sigma(S^{d-1})T/\beta)^{1-1/\beta}}{1 - 1/\beta},$$

converges almost surely uniformly in t to a Lévy process whose marginal law at time 1 is identical to $LS_{\alpha, \beta}(\sigma, q; 0)$, with $z_0 = \int_{S^{d-1}} \xi \sigma(d\xi) / \sigma(S^{d-1})$. This series representation directly reveals the nature of the layering; all the jumps with absolute size greater than 1 are due to the β -stable shot noise $\left(\frac{\beta \Gamma_i}{\sigma(S^{d-1})} \right)^{-1/\beta} V_i$, while smaller jumps come from the terms $\left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})} + 1 - \frac{\alpha}{\beta} \right)^{-1/\alpha} V_i$, whose jump size is very close to the α -stable shot noise when Γ_i is sufficiently large.

3 Short and long range behavior

We now present the first main result of this section by giving the short range behavior of a layered stable process. The results of this section were motivated by Section 3 of Rosiński [8]. Recall that σ_1 and σ_2 are the finite positive measures respectively given in (2.4) and (2.5), and that for any $r > 0$, T_r transforms the positive measure ρ , via $(T_r \rho)(B) = \rho(r^{-1}B)$, $B \in \mathcal{B}(\mathbb{R}^d)$. For convenience, we will use, throughout this section, the notation $\nu_{\sigma, q}^{\alpha, \beta}$ for the Lévy measure of a layered stable process $LS_{\alpha, \beta}(\sigma, q; \eta)$.

Theorem 3.1. *Short range behavior:* Let $\{X_t^{LS} : t \geq 0\} \sim LS_{\alpha, \beta}(\sigma, q; 0)$, let

$$\eta_{\alpha, \beta} = \begin{cases} \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^1 r q(r, \xi) dr, & \text{if } \alpha \in (0, 1), \\ - \int_{S^{d-1}} \xi \sigma(d\xi) \int_1^\infty r q(r, \xi) dr, & \text{if } (\alpha, \beta) \in (1, 2) \times (1, \infty), \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$b_{\alpha, \beta} = \begin{cases} \frac{1}{\alpha-1} \int_{S^{d-1}} \xi \sigma_1(d\xi), & \text{if } (\alpha, \beta) \in (1, 2) \times (0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Then, as $h \rightarrow 0$, $h > 0$,

$$\{h^{-1/\alpha} (X_{ht}^{LS} + ht \eta_{\alpha, \beta}) - t b_{\alpha, \beta} : t \geq 0\} \xrightarrow{d} \{X_t^{(\alpha)} : t \geq 0\},$$

where $\{X_t^{(\alpha)} : t \geq 0\} \sim S_\alpha(\sigma_1; 0)$.

Proof. Since a layered stable process is a Lévy process, by a theorem of Skorohod (see Theorem 15.17 of Kallenberg [6]), it suffices to show the weak convergence of its marginals at time 1. To this end, we will show the proper convergence of the generating triplet of the infinitely divisible law, following Theorem 15.14 of Kallenberg [6].

For the convergence of the Lévy measure, we need to show that as $h \rightarrow 0$,

$$h(T_{h^{-1/\alpha}} \mathbf{v}_{\sigma, q}^{\alpha, \beta}) \xrightarrow{v} \mathbf{v}_{\sigma_1}^\alpha,$$

or equivalently that,

$$\lim_{h \rightarrow 0} \int_{\mathbb{R}_0^d} f(z) h(T_{h^{-1/\alpha}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) = \int_{\mathbb{R}_0^d} f(z) \mathbf{v}_{\sigma_1}^\alpha(dz),$$

for all bounded continuous function $f: \mathbb{R}_0^d \rightarrow \mathbb{R}$, vanishing in a neighborhood of the origin. Letting f be such a function with $|f| \leq C < \infty$ and $f(z) \equiv 0$ on $\{z \in \mathbb{R}_0^d : \|z\| \leq \varepsilon\}$, for some $\varepsilon > 0$, we get by (2.2),

$$\begin{aligned} \int_{\mathbb{R}_0^d} f(z) h(T_{h^{-1/\alpha}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) &= h \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty f(h^{-1/\alpha} r \xi) q(r, \xi) dr \\ &\rightarrow \int_{S^{d-1}} c_1(\xi) \sigma(d\xi) \int_0^\infty f(r \xi) \frac{dr}{r^{\alpha+1}}, \end{aligned}$$

as $h \rightarrow 0$, where the last convergence is justified as follows. For $h \in (0, 1)$, we have

$$\begin{aligned} h \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty f(h^{-1/\alpha} r \xi) q(r, \xi) dr \\ = h \int_{S^{d-1}} \sigma(d\xi) \int_{h^{1/\alpha} \varepsilon}^\varepsilon f(h^{-1/\alpha} r \xi) q(r, \xi) dr + h \int_{S^{d-1}} \sigma(d\xi) \int_\varepsilon^\infty f(h^{-1/\alpha} r \xi) q(r, \xi) dr. \end{aligned}$$

Since $|f| \leq C$, and since the function q is locally integrable, i.e., integrable over any compact subset of $(0, \infty) \times S^{d-1}$, the condition (2.3) ensures that the second iterated integral above is bounded independently of h . Hence, as $h \rightarrow 0$, the second term above goes to zero. For the first term, note that the conditions on q and f ensure that

$$\lim_{h \rightarrow 0} h \int_{h^{1/\alpha} \varepsilon}^\varepsilon f(h^{-1/\alpha} r \xi) q(r, \xi) dr = \int_\varepsilon^\infty f(r \xi) c_1(\xi) \frac{dr}{r^{\alpha+1}},$$

for σ -a.e. $\xi \in S^{d-1}$, and then dominated convergence allows to conclude.

Next, the convergence of the Gaussian component holds since for each $\kappa > 0$,

$$\begin{aligned} \int_{\|z\| \leq \kappa} z z' h(T_{h^{-1/\alpha}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) &= \int_{S^{d-1}} \xi \xi' \sigma(d\xi) \int_0^{h^{1/\alpha} \kappa} r^2 h^{1-2/\alpha} q(r, \xi) dr \\ &= \int_{S^{d-1}} \xi \xi' \sigma(d\xi) \int_0^\kappa r^2 h^{1+1/\alpha} q(h^{1/\alpha} r, \xi) dr \\ &\rightarrow \int_{S^{d-1}} \xi \xi' \sigma_1(d\xi) \int_0^\kappa r^2 \frac{dr}{r^{\alpha+1}} = \int_{\|z\| \leq \kappa} z z' \mathbf{v}_{\sigma_1}^\alpha(dz), \end{aligned}$$

as $h \rightarrow 0$. Above, the passage to the limit is justified, by dominated convergence, since the condition (2.2) ensures that

$$\lim_{h \rightarrow 0} \int_0^\kappa r^2 h^{1+1/\alpha} q(h^{1/\alpha} r, \xi) dr = c_1(\xi) \int_0^\kappa r^2 \frac{dr}{r^{\alpha+1}},$$

for σ -a.e. $\xi \in S^{d-1}$, and since, moreover, for sufficiently small h ,

$$\left\| \int_{S^{d-1}} \xi \xi' \sigma(d\xi) \int_0^\kappa r^2 h^{1+1/\alpha} q(h^{1/\alpha} r, \xi) dr \right\|_o \leq \frac{2\kappa^{2-\alpha}}{2-\alpha} \left\| \int_{S^{d-1}} \xi \xi' \sigma_1(d\xi) \right\|_o < +\infty.$$

For the convergence of the drift part, assume first that $(\alpha, \beta) \notin (1, 2) \times (0, 1]$. For a σ -finite positive measure ν on \mathbb{R}_0^d , let

$$C_\alpha(\nu) := \begin{cases} \int_{\|z\| \leq 1} z \nu(dz), & \text{if } \alpha \in (0, 1), \\ 0, & \text{if } \alpha = 1, \\ - \int_{\|z\| > 1} z \nu(dz), & \text{if } \alpha \in (1, 2). \end{cases} \quad (3.1)$$

Clearly, $\eta_{\alpha, \beta} = C_\alpha(\nu_{\sigma, q}^{\alpha, \beta})$ and we show that for each $\kappa > 0$, as $h \rightarrow 0$,

$$C_\alpha(h(T_{h^{-1/\alpha}} \nu_{\sigma, q}^{\alpha, \beta})) - \int_{\kappa < \|z\| \leq 1} zh(T_{h^{-1/\alpha}} \nu_{\sigma, q}^{\alpha, \beta})(dz) \rightarrow C_\alpha(\nu_{\sigma_1}^\alpha) - \int_{\kappa < \|z\| \leq 1} z \nu_{\sigma_1}^\alpha(dz),$$

where the integral $\int_{\kappa < \|z\| \leq 1}$ is understood to be $-\int_{1 < \|z\| \leq \kappa}$, when $\kappa > 1$. We have

$$\begin{aligned} C_\alpha(h(T_{h^{-1/\alpha}} \nu_{\sigma, q}^{\alpha, \beta})) - \int_{\kappa < \|z\| \leq 1} zh(T_{h^{-1/\alpha}} \nu_{\sigma, q}^{\alpha, \beta})(dz) \\ = \begin{cases} \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^{h^{1/\alpha} \kappa} r h^{1-1/\alpha} q(r, \xi) dr, & \text{if } \alpha \in (0, 1), \\ \int_{S^{d-1}} \xi \sigma(d\xi) \int_{h\kappa}^h r q(r, \xi) dr, & \text{if } \alpha = 1, \\ \int_{S^{d-1}} \xi \sigma(d\xi) \int_{h^{1/\alpha} \kappa}^\infty r h^{1-1/\alpha} q(r, \xi) dr, & \text{if } \alpha \in (1, 2), \end{cases} \end{aligned}$$

and it remains to show that as $h \rightarrow 0$, each term above converges, respectively, to

$$\begin{cases} \int_{S^{d-1}} \xi \sigma_1(d\xi) \int_0^\kappa r \frac{dr}{r^{\alpha+1}}, & \text{if } \alpha \in (0, 1), \\ \int_{S^{d-1}} \xi \sigma_1(d\xi) \int_\kappa^1 r \frac{dr}{r^2}, & \text{if } \alpha = 1, \\ \int_{S^{d-1}} \xi \sigma_1(d\xi) \int_\kappa^\infty r \frac{dr}{r^{\alpha+1}}, & \text{if } \alpha \in (1, 2). \end{cases}$$

First, for $\alpha \in (0, 1)$,

$$h^{1-1/\alpha} \int_0^{h^{1/\alpha} \kappa} r q(r, \xi) dr = \int_0^\kappa r h^{1+1/\alpha} q(h^{1/\alpha} r, \xi) dr \rightarrow c_1(\xi) \int_0^\kappa r \frac{dr}{r^{\alpha+1}},$$

while, for $\alpha = 1$,

$$\int_{h\kappa}^h r q(r, \xi) dr = \int_\kappa^1 r h^2 q(hr, \xi) dr \rightarrow c_1(\xi) \int_\kappa^1 r \frac{dr}{r^2},$$

making use of the conditions on q . Next, for $\alpha \in (1, 2)$, and for $h \in (0, 1)$,

$$h^{1-1/\alpha} \int_{h^{1/\alpha} \kappa}^{\infty} r q(r, \xi) dr = h^{1-1/\alpha} \int_{\kappa}^{\infty} r q(r, \xi) dr + h^{1-1/\alpha} \int_{h^{1/\alpha} \kappa}^{\kappa} r q(r, \xi) dr.$$

Above, on the right hand side, the first integral is bounded independently of h (by the conditions on q and since $\beta \in (1, \infty)$), and thus the corresponding first term converges to zero with h . For the second integral,

$$h^{1-1/\alpha} \int_{h^{1/\alpha} \kappa}^{\kappa} r q(r, \xi) dr = \int_{\kappa}^{h^{-1/\alpha} \kappa} r h^{1+1/\alpha} q(h^{1/\alpha} r, \xi) dr \rightarrow c_1(\xi) \int_{\kappa}^{\infty} r \frac{dr}{r^{\alpha+1}} < +\infty,$$

where the conditions on q justify the convergence. Finally, assume $(\alpha, \beta) \in (1, 2) \times (0, 1]$. Then, for each $\kappa > 0$, as $h \rightarrow 0$,

$$\begin{aligned} \int_{\kappa < \|z\| \leq 1} z h(T_{h^{-1/\alpha}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) \\ = \int_{S^{d-1}} \xi \sigma(d\xi) \int_{\kappa}^1 r h^{1+1/\alpha} q(h^{1/\alpha} r, \xi) dr \rightarrow \int_{S^{d-1}} \xi \sigma_1(d\xi) \int_{\kappa}^1 r \frac{dr}{r^{\alpha+1}}, \end{aligned}$$

where the convergence can be justified as before, and we thus get

$$-b_{\alpha, \beta} - \int_{\kappa < \|z\| \leq 1} z h(T_{h^{-1/\alpha}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) \rightarrow - \int_{\|z\| > \kappa} z \mathbf{v}_{\sigma_1}^{\alpha}(dz),$$

which completes the proof. \square

Our next result is also of importance. Unlike the short range behavior, the long range behavior of a layered stable process depends on its outer stability index β . This behavior is akin to a β -stable process if $\beta \in (0, 2)$, while akin to a Brownian motion whenever $\beta \in (2, \infty)$.

Theorem 3.2. *Long range behavior:* Let $\{X_t^{LS} : t \geq 0\} \sim LS_{\alpha, \beta}(\sigma, q; 0)$.

(i) Let $\beta \in (0, 2)$, let

$$\eta_{\alpha, \beta} = \begin{cases} \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^1 r q(r, \xi) dr, & \text{if } (\alpha, \beta) \in (0, 1) \times (0, 1), \\ - \int_{S^{d-1}} \xi \sigma(d\xi) \int_1^{\infty} r q(r, \xi) dr, & \text{if } \beta \in (1, 2), \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$b_{\alpha, \beta} = \begin{cases} \frac{1}{1-\beta} \int_{S^{d-1}} \xi \sigma_2(d\xi), & \text{if } (\alpha, \beta) \in [1, 2) \times (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Then, as $h \rightarrow +\infty$,

$$\{h^{-1/\beta} (X_{ht}^{LS} + ht \eta_{\alpha, \beta}) + t b_{\alpha, \beta} : t \geq 0\} \xrightarrow{d} \{X_t^{(\beta)} : t \geq 0\},$$

where $\{X_t^{(\beta)} : t \geq 0\} \sim S_\beta(\sigma_2; 0)$.

(ii) Let $\beta \in (2, \infty)$ and let

$$\eta = - \int_{S^{d-1}} \xi \sigma(d\xi) \int_1^\infty r q(r, \xi) dr. \quad (3.2)$$

Then, as $h \rightarrow +\infty$,

$$\{h^{-1/2}(X_{ht}^{LS} + ht\eta) : t \geq 0\} \xrightarrow{d} \{W_t : t \geq 0\}, \quad (3.3)$$

where $\{W_t : t \geq 0\}$ is a centered Brownian motion with covariance matrix $\int_{\mathbb{R}_0^d} zz' v_{\sigma, q}^{\alpha, \beta}(dz)$.

Proof. The claim (i) can be proved as (i) in Theorem 3.1. For the convergence of the Lévy measure, we will show that

$$\lim_{h \rightarrow \infty} \int_{\mathbb{R}_0^d} f(z) h(T_{h^{-1/\beta}} v_{\sigma, q}^{\alpha, \beta})(dz) = \int_{\mathbb{R}_0^d} f(z) v_{\sigma_2}^\beta(dz),$$

for all bounded continuous function $f : \mathbb{R}_0^d \rightarrow \mathbb{R}$ vanishing in a neighborhood of the origin. Letting f be such a function with $|f| \leq C < \infty$ and $f(z) \equiv 0$ on $\{z \in \mathbb{R}_0^d : \|z\| \leq \varepsilon\}$, for some $\varepsilon > 0$, we get by (2.3),

$$\begin{aligned} \int_{\mathbb{R}_0^d} f(z) h(T_{h^{-1/\beta}} v_{\sigma, q}^{\alpha, \beta})(dz) &= \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty f(r\xi) h^{1+1/\beta} q(h^{1/\beta} r, \xi) dr \\ &\rightarrow \int_{S^{d-1}} c_2(\xi) \sigma(d\xi) \int_0^\infty f(r\xi) \frac{dr}{r^{\beta+1}}, \end{aligned}$$

as $h \rightarrow \infty$, where the passage to the limit is, again, justified by dominated convergence with the conditions on q and f . For the convergence of the Gaussian component, we have as $h \rightarrow \infty$ and for each $\kappa > 0$,

$$\begin{aligned} \int_{\|z\| \leq \kappa} zz' h(T_{h^{-1/\beta}} v_{\sigma, q}^{\alpha, \beta})(dz) &= \int_{S^{d-1}} \xi \xi' \sigma(d\xi) \int_0^\kappa r^2 h^{1+1/\beta} q(h^{1/\beta} r, \xi) dr \\ &\rightarrow \int_{S^{d-1}} \xi \xi' \sigma_2(d\xi) \int_0^\kappa r^2 \frac{dr}{r^{\beta+1}} = \int_{\|z\| \leq \kappa} zz' v_{\sigma_2}^\beta(dz), \end{aligned}$$

where the limit is obtained as in Theorem 3.1, making use of the condition (2.3).

Finally, we prove the convergence of the drift part. Assume first that $(\alpha, \beta) \notin [1, 2) \times (0, 1)$. Let $C_\beta(v)$ be the defined as in (3.1), but with β replacing α . Clearly, $\eta_{\alpha, \beta} = C_\beta(v_{\sigma, q}^{\alpha, \beta})$. We then show that for each $\kappa > 0$, as $h \rightarrow \infty$,

$$C_\beta(h(T_{h^{-1/\beta}} v_{\sigma, q}^{\alpha, \beta})) - \int_{\kappa < \|z\| \leq 1} zh(T_{h^{-1/\beta}} v_{\sigma, q}^{\alpha, \beta})(dz) \rightarrow C_\beta(v_{\sigma_2}^\beta) - \int_{\kappa < \|z\| \leq 1} zv_{\sigma_2}^\beta(dz),$$

where the integral $\int_{\kappa < \|z\| \leq 1}$ is again understood to be $-\int_{1 < \|z\| \leq \kappa}$, when $\kappa > 1$. As in Theorem 3.1, we get, as $h \rightarrow \infty$,

$$C_\beta(h(T_{h^{-1/\beta}} v_{\sigma, q}^{\alpha, \beta})) - \int_{\kappa < \|z\| \leq 1} zh(T_{h^{-1/\beta}} v_{\sigma, q}^{\alpha, \beta})(dz) \rightarrow \begin{cases} \int_{S^{d-1}} \xi \sigma_2(d\xi) \int_0^\kappa r \frac{dr}{r^{\beta+1}}, & \text{if } \beta \in (0, 1), \\ \int_{S^{d-1}} \xi \sigma_2(d\xi) \int_\kappa^1 r \frac{dr}{r^2}, & \text{if } \beta = 1, \\ \int_{S^{d-1}} \xi \sigma_2(d\xi) \int_\kappa^\infty r \frac{dr}{r^{\beta+1}}, & \text{if } \beta \in (1, 2). \end{cases}$$

Next, let $(\alpha, \beta) \in [1, 2) \times (0, 1)$. Then, observe that for each $\kappa > 0$, and as $h \rightarrow \infty$,

$$-b_{\alpha, \beta} - \int_{\kappa < \|z\| \leq 1} zh(T_{h^{-1/\alpha}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) \rightarrow - \int_{\|z\| > \kappa} z \mathbf{v}_{\sigma_1}^{\alpha}(dz),$$

where the convergence holds true as before. This completes the proof of (i).

(ii) The random vector $h^{-1/2} X_h^{LS}$ is infinitely divisible with generating triplet

$$\left(- \int_{\|z\| \geq 1} zh(T_{h^{-1/2}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz), 0, h(T_{h^{-1/2}} \mathbf{v}_{\sigma, q}^{\alpha, \beta}) \right).$$

We first prove the vague convergence, to zero, of the Lévy measure $h(T_{h^{-1/2}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})$. Let f be a bounded continuous function from \mathbb{R}_0^d to \mathbb{R} such that $|f| \leq C < \infty$ and $f(z) \equiv 0$ on $\{z \in \mathbb{R}^d : \|z\| \leq \varepsilon\}$, for some $\varepsilon > 0$. Then, as $h \rightarrow \infty$,

$$\begin{aligned} h^{\beta/2-1} \int_{\mathbb{R}_0^d} f(z) h(T_{h^{-1/2}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) &= \int_{S^{d-1}} \sigma(d\xi) \int_{\varepsilon}^{\infty} f(r\xi) h^{\beta/2+1/2} q(h^{1/2}r, \xi) dr \\ &\rightarrow \int_{S^{d-1}} \sigma(d\xi) \int_{\varepsilon}^{\infty} f(r\xi) \frac{dr}{r^{\beta+1}}, \end{aligned}$$

where the conditions on q and f ensure the passage to the limit. Since $\beta > 2$, we conclude that, as $h \rightarrow \infty$, $h(T_{h^{-1/2}} \mathbf{v}_{\sigma, q}^{\alpha, \beta}) \xrightarrow{v} 0$.

For the convergence of the Gaussian component, we have as $h \rightarrow +\infty$ and for each $\kappa > 0$,

$$\int_{\|z\| \leq \kappa} zz' h(T_{h^{-1/2}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) = \int_{\|z\| \leq h^{1/2}\kappa} zz' \mathbf{v}_{\sigma, q}^{\alpha, \beta}(dz) \rightarrow \int_{\mathbb{R}_0^d} zz' \mathbf{v}_{\sigma, q}^{\alpha, \beta}(dz), \quad (3.4)$$

which is clearly well defined since $\int_{\mathbb{R}_0^d} \|z\|^2 \mathbf{v}_{\sigma, q}^{\alpha, \beta}(dz) < \infty$. Finally, write

$$\int_{\|z\| > \kappa} zh(T_{h^{-1/2}} \mathbf{v}_{\sigma, q}^{\alpha, \beta})(dz) = \int_{S^{d-1}} \xi \sigma(d\xi) \int_{\kappa}^{\infty} rh^{3/2} q(h^{1/2}r, \xi) dr, \quad (3.5)$$

and with the conditions imposed on q , the dominated convergence theorem ensures that, as $h \rightarrow \infty$,

$$h^{\beta/2-1} \int_{\kappa}^{\infty} rh^{3/2} q(h^{1/2}r, \xi) dr \rightarrow c_2(\xi) \int_{\kappa}^{\infty} r \frac{dr}{r^{\beta+1}}. \quad (3.6)$$

Since $\beta > 2$, the convergence of the drift term is proved. \square

For $\beta = 2$, layered stable processes do not seem to possess any nice long range property, and this can be seen from the improper convergence of the Lévy measure, i.e., as $h \rightarrow \infty$, $h(T_{h^{-1/2}} \mathbf{v}_{\sigma, q}^{\alpha, 2})$ converges vaguely to

$$\int_{S^{d-1}} \sigma_2(d\xi) \int_0^{\infty} \mathbb{1}_B(r\xi) \frac{dr}{r^{2+1}}, \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

which is not well defined as a Lévy measure. However, additional assumptions on σ_2 lead to the weak convergence towards a Brownian motion as β approaches 2.

Proposition 3.3. Let $\{X_t^{LS} : t \geq 0\} \sim LS_{\alpha,\beta}(\sigma, q; 0)$ in \mathbb{R}^d .

(i) Let $\beta \in (1, 2)$ and let $\eta = -\int_{S^{d-1}} \xi \sigma(d\xi) \int_1^\infty r q(r, \xi) dr$. If σ_2 is uniform on S^{d-1} such that $\sigma_2(S^{d-1}) = d(2 - \beta)$, then

$$\{h^{-1/\beta}(X_{ht}^{LS} + ht\eta) : t \geq 0\} \xrightarrow{d} \{W_t : t \geq 0\}, \quad \text{as } h \rightarrow \infty, \beta \uparrow 2,$$

where $\{W_t : t \geq 0\}$ is a d -dimensional (centered) standard Brownian motion. (Above, the limit is taken over $h \rightarrow \infty$ first.)

(ii) Let $\beta \in (2, \infty)$ and let η be the constant (3.2). If σ_2 is symmetric such that $\sigma_2(S^{d-1}) = \beta - 2$, then

$$\{h^{-1/2}(X_{ht}^{LS} + ht\eta) : t \geq 0\} \xrightarrow{d} \{W_t : t \geq 0\}, \quad \text{as } h \rightarrow \infty, \beta \downarrow 2,$$

where $\{W_t : t \geq 0\}$ is a centered Brownian motion with covariance matrix $\int_{\mathbb{R}_0^d} zz' \nu_{\sigma,q}^{\alpha,2}(dz)$. (Above, the limit can be taken either over $h \rightarrow \infty$ or over $\beta \downarrow 2$ first.)

Proof. (i) By Theorem 3.1 (i), $h^{-1/\beta}(X_h^{LS} + h\eta) \xrightarrow{\mathcal{L}} X_1^{(\beta)}$, as $h \rightarrow \infty$, where $\{X_t^{(\beta)} : t \geq 0\} \sim S_\beta(\sigma_2; 0)$. Then, by E.18.8-18.9 of Sato [11], we get $\mathbb{E}[e^{i\langle y, X_1^{(\beta)} \rangle}] = \exp[-c_{\beta,d} \|y\|^\beta]$, where

$$c_{\beta,d} = \frac{\Gamma(d/2)\Gamma((2-\beta)/2)}{2^\beta \beta \Gamma((\beta+d)/2)} \sigma_2(S^{d-1}).$$

Taking $\beta \uparrow 2$ and since $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we get the result.

(ii) The vague convergence, to zero, of the Lévy measure $h(T_{h^{-1/2}} \nu_{\sigma,q}^{\alpha,\beta})$ can be proved just as in Theorem 3.2 (ii). Next, in view of (3.5)-(3.6), we have

$$\lim_{h \rightarrow \infty} h^{\beta/2-1} \left\| \int_{\|z\| > \kappa} zh(T_{h^{-1/2}} \nu_{\sigma,q}^{\alpha,\beta})(dz) \right\| = \frac{\kappa^{1-\beta}}{\beta-1} \left\| \int_{S^{d-1}} \xi \sigma_2(d\xi) \right\| = 0,$$

where the last equality holds by the symmetry of σ_2 . Since $\beta > 2$, we get

$$\lim_{h \rightarrow \infty} \left\| \int_{\|z\| > \kappa} zh(T_{h^{-1/2}} \nu_{\sigma,q}^{\alpha,\beta})(dz) \right\| = 0,$$

which proves the convergence of the drift component. Finally, in view of (3.4), we have

$$\int_{\mathbb{R}_0^d} \|z\|^2 \nu_{\sigma,q}^{\alpha,\beta}(dz) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty r^2 q(r, \xi) dr < \infty,$$

using the conditions on q and since $\beta > 2$. The proof is complete. \square

Remark 3.4. The short range behavior (Theorem 3.1) and the (non-Gaussian) long range behavior (Theorem 3.2 (i)) can also be inferred from the series representation (2.10). For simplicity, consider the symmetric case. Letting $X_t := \sum_{i=1}^\infty \overleftarrow{q}(\Gamma_i/T, V_i) V_i \mathbf{1}(T_i \leq t)$, we have

$$h^{-1/\alpha} X_{ht} = \sum_{i=1}^\infty h^{-1/\alpha} \overleftarrow{q}(\Gamma_i/(hT), V_i) V_i \mathbf{1}(hT_i \leq ht),$$

and so for each $u > 0$ and each $\xi \in S^{d-1}$ such that $c_1(\xi) \in [0, \infty)$, bounded convergence gives

$$\begin{aligned} h^{-1/\alpha} \overleftarrow{q}(h^{-1}u, \xi) &= h^{-1/\alpha} \inf \left\{ r > 0 : \sigma(S^{d-1}) \int_r^\infty q(s, \xi) ds < h^{-1}u \right\} \\ &= \inf \left\{ r > 0 : \sigma(S^{d-1}) \int_r^\infty h^{1+1/\alpha} q(h^{1/\alpha}s, \xi) ds < u \right\} \\ &\rightarrow \inf \left\{ r > 0 : c_1(\xi) \sigma(S^{d-1}) \int_r^\infty s^{-\alpha-1} ds < u \right\} = \left(\frac{\alpha u}{c_1(\xi) \sigma(S^{d-1})} \right)^{-1/\alpha}, \end{aligned}$$

as $h \rightarrow 0$, which is indeed an α -stable shot noise. The (non-Gaussian) long range behavior can be similarly inferred.

4 Absolute continuity with respect to short range limiting stable process

Two Lévy processes, which are mutually absolutely continuous, share any almost sure local behavior. The next theorem confirms this fact in relation with the short range behavior result obtained in Theorem 3.1. Indeed, given any layered stable process with respect to some probability measure, one can find a probability measure under which the layered stable process is identical in law to its short range limiting stable process. This result should be compared with Section 4 of Rosiński[8].

Recall that c_1 and c_2 are the integrable (with respect to σ) functions on S^{d-1} appearing in (2.2) and (2.3), while σ_1 and σ_2 are the finite positive measures (2.4) and (2.5), respectively. As before, we use the notation $\nu_{\sigma, q}^{\alpha, \beta}$ for the Lévy measure of a layered stable process $X := \{X_t : t \geq 0\} \sim LS_{\alpha, \beta}(\sigma, q; \eta)$, while ν_σ^α is the measure (2.6).

Theorem 4.1. *Let \mathbb{P} , \mathbb{Q} and \mathbb{T} be probability measures on (Ω, \mathcal{F}) such that under \mathbb{P} the canonical process $\{X_t : t \geq 0\}$ is a Lévy process in \mathbb{R}^d with $\mathcal{L}(X_1) \sim LS_{\alpha, \beta}(\sigma, q; k_0)$, while under \mathbb{Q} it is a Lévy process with $\mathcal{L}(X_1) \sim S_\alpha(\sigma_1; k_1)$, and let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of $\{X_t : t \geq 0\}$. Moreover, when $\beta \in (0, 2)$ and under \mathbb{T} , $\{X_t : t \geq 0\}$ is a Lévy process with $\mathcal{L}(X_1) \sim S_\beta(\sigma_2; \eta)$, for some $\eta \in \mathbb{R}^d$. Then,*

(i) $\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}|_{\mathcal{F}_t}$ are mutually absolutely continuous, for every $t > 0$, if and only if

$$k_0 - k_1 = \begin{cases} \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^1 r q(r, \xi) dr, & \text{if } \alpha \in (0, 1), \\ \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^1 r (q(r, \xi) - c_1(\xi) r^{-\alpha-1}) dr, & \text{if } \alpha = 1, \\ \frac{1}{\alpha-1} \int_{S^{d-1}} \xi \sigma_1(d\xi) + \int_{S^{d-1}} \xi \sigma(d\xi) \int_0^1 r (q(r, \xi) - c_1(\xi) r^{-\alpha-1}) dr, & \text{if } \alpha \in (1, 2). \end{cases}$$

(ii) If $\alpha \neq \beta$, then for any choice of $\eta \in \mathbb{R}^d$, $\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{T}|_{\mathcal{F}_t}$ are singular, for every $t > 0$.

(iii) For every $t > 0$,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{U_t},$$

where $\{U_t : t \geq 0\}$ is a Lévy process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$U_t := \lim_{\varepsilon \downarrow 0} \sum_{\{s \in (0, t] : \|\Delta X_s\| > \varepsilon\}} \left[\ln \left(\frac{q(\|\Delta X_s\|, \Delta X_s / \|\Delta X_s\|)}{c_1(\Delta X_s / \|\Delta X_s\|) \|\Delta X_s\|^{-\alpha-1}} \right) - t(v_{\sigma, q}^{\alpha, \beta} - v_{\sigma_1}^{\alpha}) (\{z \in \mathbb{R}_0^d : \|z\| > \varepsilon\}) \right], \quad (4.1)$$

and where the convergence holds \mathbb{P} -a.s., uniformly in t on every interval of positive length.

Proof. (i) By Theorem 33.1 and Remark 33.3 of Sato [11], to prove the result it is necessary and sufficient to show that the following three conditions hold:

$$\int_{\{z : |\varphi(z)| \leq 1\}} \varphi(z)^2 v_{\sigma_1}^{\alpha}(dz) < \infty, \quad (4.2)$$

$$\int_{\{z : \varphi(z) > 1\}} e^{\varphi(z)} v_{\sigma_1}^{\alpha}(dz) < \infty, \quad (4.3)$$

$$\int_{\{z : \varphi(z) < -1\}} v_{\sigma_1}^{\alpha}(dz) < \infty, \quad (4.4)$$

where the function $\varphi : \mathbb{R}_0^d \rightarrow \mathbb{R}$ is defined by $(dv_{\sigma, q}^{\alpha, \beta} / dv_{\sigma_1}^{\alpha})(z) = e^{\varphi(z)}$, that is,

$$\varphi(z) = \ln \left(\frac{q(\|z\|, z / \|z\|)}{c_1(z / \|z\|) \|z\|^{-\alpha-1}} \right), \quad z \in \mathbb{R}_0^d.$$

Now, observe that

$$\lim_{\|z\| \rightarrow 0} \varphi(z) = \lim_{\|z\| \rightarrow 0} \ln \left(\frac{c_1(z / \|z\|) \|z\|^{-\alpha-1}}{c_1(z / \|z\|) \|z\|^{-\alpha-1}} \right) = 0, \quad (4.5)$$

and that as $\|z\| \rightarrow \infty$,

$$\varphi(z) \sim \ln \left(\frac{c_2(z / \|z\|) \|z\|^{-\beta-1}}{c_1(z / \|z\|) \|z\|^{-\alpha-1}} \right) = \ln \left(\frac{c_2(z / \|z\|)}{c_1(z / \|z\|)} \right) + (\alpha - \beta) \ln \|z\| \rightarrow \begin{cases} -\infty, & \text{if } \alpha < \beta, \\ +\infty, & \text{if } \alpha > \beta. \end{cases} \quad (4.6)$$

The conditions (4.2) and (4.4) are thus immediately satisfied, via (4.5) and (4.6) with $\alpha < \beta$. In view of (4.6) with $\alpha > \beta$, the condition (4.3) is also satisfied since $\int_{\{z : \varphi(z) > 1\}} e^{\varphi(z)} v_{\sigma_1}^{\alpha}(dz)$ is bounded from above and below by constant multiples of $\int_{\|z\| > 1} \frac{q(\|z\|, z / \|z\|)}{c_1(z / \|z\|) \|z\|^{-\alpha-1}} v_{\sigma_1}^{\alpha}(dz) = v_{\sigma, q}^{\alpha, \beta}(\{z \in \mathbb{R}_0^d : \|z\| > 1\})$. When $\alpha = \beta \in (0, 2)$, we have, by (4.5) and (4.6),

$$\begin{cases} \lim_{\|z\| \rightarrow 0} \varphi(z) = 0, \\ \lim_{\|z\| \rightarrow \infty} |\varphi(z)| = \lim_{\|z\| \rightarrow \infty} \left| \ln \left(\frac{c_2(z / \|z\|)}{c_1(z / \|z\|)} \right) \right| < +\infty. \end{cases}$$

The condition (4.2) is then satisfied since $\int_{\{z : |\varphi(z)| \leq 1\}} \varphi(z)^2 v_{\sigma_1}^{\alpha}(dz)$ is bounded from above and below by constant multiples of $\int_{\|z\| > 1} \varphi(z)^2 v_{\sigma_1}^{\alpha}(dz)$, which is further bounded by a constant multiple of $v_{\sigma_1}^{\alpha}(\{z \in \mathbb{R}_0^d : \|z\| > 1\})$. The conditions (4.3) and (4.4) are also satisfied since the domains $\{z \in \mathbb{R}_0^d : \varphi(z) > 1\}$ and $\{z \in \mathbb{R}_0^d : \varphi(z) < -1\}$ are contained in compact subsets of \mathbb{R}_0^d .

(ii) It suffices to show that either one of the following two conditions always fails;

$$\int_{\{z:\psi(z)>1\}} e^{\psi(z)} \nu_{\sigma_2}^\beta(dz) < +\infty, \quad (4.7)$$

$$\int_{\{z:\psi(z)<-1\}} \nu_{\sigma_2}^\beta(dz) < +\infty, \quad (4.8)$$

where the function $\psi : S^{d-1} \rightarrow \mathbb{R}$ is defined via $(d\nu_{\sigma,q}^{\alpha,\beta}/d\nu_{\sigma_2}^\beta)(z) = e^{\psi(z)}$, that is,

$$\psi(z) = \ln \left(\frac{q(\|z\|, z/\|z\|)}{c_2(z/\|z\|)\|z\|^{-\beta-1}} \right), \quad z \in \mathbb{R}_0^d.$$

As in the proof of (i), observe that

$$\lim_{\|z\| \rightarrow \infty} \psi(z) = \lim_{\|z\| \rightarrow \infty} \ln \left(\frac{c_2(z/\|z\|)\|z\|^{-\alpha-1}}{c_2(z/\|z\|)\|z\|^{-\alpha-1}} \right) = 0,$$

and that as $\|z\| \rightarrow 0$,

$$\psi(z) \sim \ln \left(\frac{c_1(z/\|z\|)\|z\|^{-\alpha-1}}{c_2(z/\|z\|)\|z\|^{-\beta-1}} \right) = \ln \left(\frac{c_1(z/\|z\|)}{c_2(z/\|z\|)} \right) + (\beta - \alpha) \ln \|z\| \rightarrow \begin{cases} +\infty, & \text{if } \alpha > \beta, \\ -\infty, & \text{if } \alpha < \beta. \end{cases}$$

Therefore, the condition (4.7) fails when $\alpha > \beta$ since

$$\int_{\{z:\psi(z)>1\}} e^{\psi(z)} \nu_{\sigma_2}^\beta(dz) = \nu_{\alpha,q}^{\alpha,\beta}(\{z \in \mathbb{R}_0^d : \varphi(z) > 1\}) = +\infty,$$

while (4.8) fails when $\alpha < \beta$ since $\nu_{\sigma_2}^\beta(\{z \in \mathbb{R}_0^d : \psi(z) < -1\}) = +\infty$.

(iii) This is a direct consequence of (i) with the help of Theorem 33.2 of Sato [11]. \square

Remark 4.2. As in Remark 2.2, let

$$q(r, \xi) = r^{-\alpha-1} \mathbb{1}_{(0,1]}(r) + r^{-\beta-1} \mathbb{1}_{(1,+\infty)}(r), \quad \xi \in S^{d-1}.$$

Then, the Lévy process $\{U_t : t \geq 0\}$ given in (4.1) becomes

$$U_t = (\alpha - \beta) \sum_{\{s \in (0,t] : \|\Delta X_s\| > 1\}} \ln(\|\Delta X_s\|) - t \left(\frac{1}{\beta} - \frac{1}{\alpha} \right) \sigma(S^{d-1}).$$

Intuitively speaking, the density transformation $(d\mathbb{Q}/d\mathbb{P})|_{\mathcal{F}_t}$ replaces all the β -stable jumps of a layered stable process up to time t (i.e., the jumps with absolute size greater than 1) by the corresponding α -stable jumps with the same jump direction. Moreover, when $\alpha < \beta$, the Lévy measure ν of $\mathcal{L}(U_1)$ is concentrated on $(-\infty, 0)$ and is given by

$$\nu(-\infty, y) = \alpha^{-1} \sigma(S^{d-1}) e^{\frac{\alpha}{\beta-\alpha}y}, \quad y < 0,$$

while when $\alpha > \beta$, it is concentrated on $(0, \infty)$ and is given by

$$v(y, \infty) = \alpha^{-1} \sigma(S^{d-1}) e^{\frac{\alpha}{\beta-\alpha} y}, \quad y > 0.$$

Let us next restate the absolute continuity result (Theorem 4.1) based on the fact that a series representation generates sample paths of a Lévy process directly by generating every single jump. For simplicity, we consider the symmetric case. Let $\{Y_t : t \geq 0\}$ be an α -stable process with $\mathcal{L}(Y_1) \sim S_\alpha(\sigma; k_1)$. By Lemma 1.1, there exists a version of $\{Y_t : t \in [0, T]\}$ given by

$$Y'_t = \sum_{i=1}^{\infty} \left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\alpha} V_i \mathbb{1}(T_i \leq t) + k_1 t.$$

Also, let $\{X_t : t \geq 0\}$ be a layered stable process with $\mathcal{L}(X_1) \sim LS_{\alpha, \beta}(\sigma, q; k_0)$. In view of the series representation (2.11), there exists a version of $\{X_t : t \in [0, T]\}$ given by

$$X'_t = \sum_{i=1}^{\infty} \left[\left(\frac{\beta \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\beta} \mathbb{1}_{(0, \sigma(S^{d-1})T/\beta)}(\Gamma_i) + \left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})T} + 1 - \frac{\alpha}{\beta} \right)^{-1/\alpha} \mathbb{1}_{(\sigma(S^{d-1})T/\beta, +\infty)}(\Gamma_i) \right] \times V_i \mathbb{1}(T_i \leq t) + k_0 t,$$

where all the random sequences are the same as those appearing in $\{Y'_t : t \in [0, T]\}$ above. By Theorem 4.1, they are mutually absolutely continuous if and only if

$$k_0 - k_1 = \begin{cases} \frac{1}{\alpha-1} \int_{S^{d-1}} \xi \sigma_1(d\xi), & \text{if } \alpha \in (0, 1) \cup (1, 2), \\ 0, & \text{if } \alpha = 1. \end{cases}$$

We infer that the Lévy process $\{U_t : t \in [0, T]\}$ in the Radon-Nikodym derivative of Theorem 4.1 (iii), that is,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = e^{U_t},$$

has a version given by

$$U'_t = -\frac{\alpha - \beta}{\alpha} \sum_{i=1}^{\infty} \ln \left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})T} \right) \mathbb{1}_{(0, \sigma(S^{d-1})T/\alpha]}(\Gamma_i) \mathbb{1}(T_i \leq t) - t \left(\frac{1}{\beta} - \frac{1}{\alpha} \right) \sigma(S^{d-1}).$$

As a direct consequence, we have

$$\mathbb{P}(X \in B) = \mathbb{E}_{\mathbb{P}}[e^{U'_T} \mathbb{1}_B(Y')], \quad B \in \mathcal{B}(\mathbb{D}([0, T], \mathbb{R}^d)).$$

Moreover, in view of Theorem 33.2 of Sato [11],

$$\frac{d\mathbb{P}}{d\mathbb{Q}} \Big|_{\mathcal{F}_t} = e^{-U_t},$$

and so we can derive a version of $\{U_t : t \in [0, T]\}$ in terms of the jumps of the layered stable process as follows:

$$U_t'' = -\frac{\alpha - \beta}{\beta} \sum_{i=1}^{\infty} \ln \left(\frac{\beta \Gamma_i}{\sigma(S^{d-1})T} \right) \mathbb{1}_{(0, \sigma(S^{d-1})T/\beta]}(\Gamma_i) \mathbb{1}(T_i \leq t) - t \left(\frac{1}{\beta} - \frac{1}{\alpha} \right) \sigma(S^{d-1}).$$

Similarly, we have

$$\mathbb{Q}(Y \in B) = \mathbb{E}_{\mathbb{Q}}[e^{-U_T''} \mathbb{1}_B(X')], \quad B \in \mathcal{B}(\mathbb{D}([0, T], \mathbb{R}^d)).$$

5 Concluding remarks

• The weak convergence towards a Brownian motion, proved in Proposition 3.3 (i), is interesting in the sense that a stable process with uniformly dependent components converges in law to a standard Brownian motion. It is also interesting to see how a stable process with independent components can converge towards a Brownian motion. To this end, for $i = 1, \dots, d$, let $a_i \in [0, \infty)$, let

$$b'_{i+} := (0, \dots, 0, +1, 0, \dots, 0), \quad b'_{i-} := (0, \dots, 0, -1, 0, \dots, 0),$$

where $+1$ and -1 are located at the i -th component, and set

$$\sigma(d\xi) := \sum_{i=1}^d \frac{2 - \alpha}{2} a_i (\delta_{b'_{i+}}(d\xi) + \delta_{b'_{i-}}(d\xi)), \quad \xi \in S^{d-1},$$

where δ is the Dirac measure. Clearly, σ is a symmetric finite positive measure on S^{d-1} . Also, let $\{X_t^{(\alpha)} : t \geq 0\} \sim S_{\alpha}(\sigma; 0)$. Then, if y_i is the i -th component of y , we have, by E.18.8-18.9 of Sato [11] and using $\Gamma(x+1) = x\Gamma(x)$, $x > 0$,

$$\begin{aligned} \mathbb{E}[e^{i\langle y, X_1^{(\alpha)} \rangle}] &= \exp \left[-\frac{\Gamma(1/2)\Gamma((2-\alpha)/2)}{2^{\alpha}\alpha\Gamma((1+\alpha)/2)} \int_{S^{d-1}} |\langle y, \xi \rangle|^{\alpha} \sigma(d\xi) \right] \\ &= \exp \left[-\frac{1}{2} \sum_{i=1}^d \frac{\Gamma(1/2)\Gamma(1+(2-\alpha)/2)}{2^{\alpha-2}\alpha\Gamma((1+\alpha)/2)} a_i |y_i|^{\alpha} \right] \\ &\rightarrow \exp \left[-\frac{1}{2} \sum_{i=1}^d a_i |y_i|^2 \right], \quad \text{as } \alpha \uparrow 2. \end{aligned}$$

Therefore, as $\alpha \uparrow 2$, we get $\{X_t^{(\alpha)} : t \geq 0\} \xrightarrow{d} \{W_t : t \geq 0\}$, where $\{W_t : t \geq 0\}$ is a Brownian motion with covariance matrix

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_d \end{pmatrix}.$$

- By making use of the absolute continuity of Lévy measures, we can derive two more forms of the series representation of a layered stable process corresponding to the Lévy measure (2.7), with $\alpha < \beta$. With the notations of Theorem 4.1, we get for $z \in \mathbb{R}_0^d$,

$$\frac{d\nu_{\sigma,q}^{\alpha,\beta}}{d\nu_{\sigma}^{\alpha}}(z) = \mathbb{1}_{(0,1]}(\|z\|) + \|z\|^{\alpha-\beta} \mathbb{1}_{(1,+\infty)}(\|z\|) \leq 1,$$

and

$$\frac{d\nu_{\sigma,q}^{\alpha,\beta}}{d\nu_{\sigma}^{\beta}}(z) = \|z\|^{\beta-\alpha} \mathbb{1}_{(0,1]}(\|z\|) + \mathbb{1}_{(1,+\infty)}(\|z\|) \leq 1.$$

Then, by the rejection method of Rosiński [7], the summands $\{\overleftarrow{q}(\Gamma_i/T, V_i)V_i\}_{i \geq 1}$ in (2.11) can be respectively replaced by

$$\left\{ \left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\alpha} \mathbb{1} \left(\frac{d\nu_{\sigma,q}^{\alpha,\beta}}{d\nu_{\sigma}^{\alpha}} \left(\left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\alpha} V_i \right) \geq U_i \right) V_i \right\}_{i \geq 1},$$

and

$$\left\{ \left(\frac{\beta \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\beta} \mathbb{1} \left(\frac{d\nu_{\sigma,q}^{\alpha,\beta}}{d\nu_{\sigma}^{\beta}} \left(\left(\frac{\beta \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\beta} V_i \right) \geq U_i \right) V_i \right\}_{i \geq 1},$$

where $\{U_i\}_{i \geq 1}$ is a sequence of iid uniform random variables on $[0, 1]$, independent of all the other random sequences.

- In complete similarity to the work presented in [5], it is possible to define a notion of fractional layered stable motion (fLSm). Then as in [5], over short intervals, fLSm will be close to fractional stable motion (with inner index α) while over long intervals it will be close to either fractional Brownian motion (if $\beta > 2$) or to fractional stable motion (with index $\beta < 2$).

- Let us observe some sample paths of a layered stable process, generated via the series representation (2.11). By Theorem 3.1 and 3.2, the entire situation is exhausted by the following three cases;

- (i) $\alpha < \beta < 2$,
- (ii) $\beta \in (2, \infty)$,
- (iii) $\alpha > \beta$ with $\beta \in (0, 2)$.

Figure 1 corresponds to the case (i) and typical sample paths of a symmetric layered stable process with $(\alpha, \beta) = (1.3, 1.9)$ are drawn in short, regular, and long range settings. For better comparison, we also drew its corresponding 1.3-stable and 1.9-stable processes. All these sample paths are generated via the series representation (2.11) for a layered stable process, or the one given in Lemma 1.1 for stable processes. Three sample paths within each figure are generated on

a common probability space in the sense that a common set of random sequences $\{\Gamma_i\}_{i \geq 1}$, $\{V_i\}_{i \geq 1}$ and $\{T_i\}_{i \geq 1}$ are used. The desired short and long range behaviors are apparent.

For the case (ii), we drew in Figure 2 typical sample paths of a symmetric layered stable process with $(\alpha, \beta) = (1.1, 2.5)$, along with its corresponding 1.1-stable process and a Brownian motion with a suitable variance. The layered stable process and the 1.1-stable process are generated as before, while the Brownian motion is independent of the others. As expected, the long range Gaussian-type behavior (Theorem 3.2 (ii)) is clearly apparent. These stable-type short range and Gaussian-type long range behaviors have long been considered to be very appealing in applications. In the left figure, the layered stable process and its short range limiting stable process are almost indistinguishable in a graphical sense (of course, not probabilistically).

Finally, for the case (iii), we give in Figure 3 typical sample paths of a symmetric layered stable process with $(\alpha, \beta) = (1.9, 1.3)$, along with its corresponding 1.9-stable and 1.3-stable processes. Unlike the sample path behaviors observed in Figure 1, the path of the layered stable processes behaves more continuously (like a 1.9-stable) over short intervals, while more discontinuously over long intervals (like a 1.3-stable).

• To finish this study, we briefly introduce another generalization of stable processes. Again, on \mathbb{R}^d , let μ be an infinitely divisible probability measure without Gaussian component. Then, μ is *mixed stable* if its Lévy measure is given by

$$\nu(B) = \int_{(0,2)} \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{dr}{r^{\alpha+1}} \varphi(d\alpha), \quad B \in \mathcal{B}(\mathbb{R}_0^d), \quad (5.1)$$

where φ is a probability measure on $(0, 2)$ such that

$$\int_{(0,2)} \frac{1}{\alpha(2-\alpha)} \varphi(d\alpha) < \infty.$$

The simplest example of a mixed stable distribution is formed by convolution of stable distributions of different orders. This can be constructed by setting $\varphi(d\alpha) = \sum_k c_k \delta_{\alpha_k}(d\alpha)$, where for all k , $\alpha_k \in (0, 2)$, $c_k \geq 0$, and $\sum_k c_k = 1$.

Recall that in Example 2.2 we defined the classes L_m , $m = 0, 1, \dots$. Let also $L_\infty := \bigcap_{m=0}^\infty L_m$. It is proved in Theorem 3.4 of Sato [10] that an infinitely divisible probability measure without Gaussian component is in L_∞ if and only if its Lévy measure has the form (5.1), and that its characteristic function is given by

$$\begin{aligned} \widehat{\mu}(y) = \exp \left[i \langle y, \eta \rangle - \int_{(0,2)} c_\alpha \int_{S^{d-1}} |\langle y, \xi \rangle|^\alpha (1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn} \langle y, \xi \rangle) \sigma(d\xi) \varphi(d\alpha) \right. \\ \left. - \varphi(\{1\}) c_1 \int_{S^{d-1}} (|\langle y, \xi \rangle| + i \frac{2}{\pi} \langle y, \xi \rangle \ln |\langle y, \xi \rangle|) \sigma(d\xi) \right], \end{aligned}$$

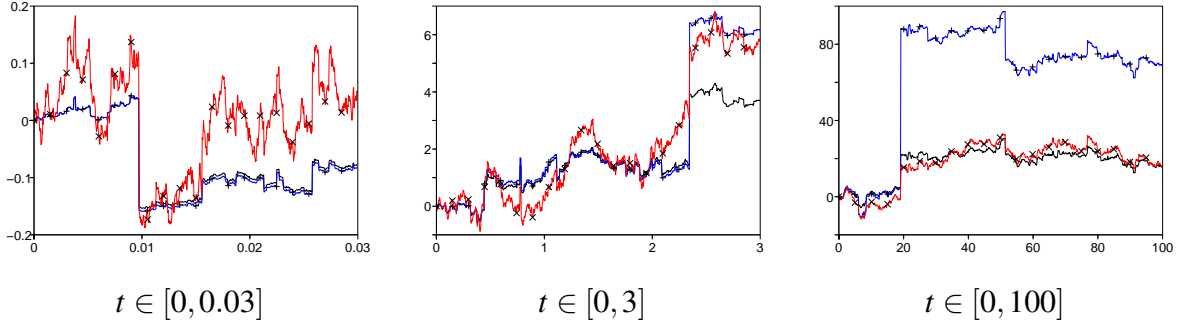


Figure 1: Typical sample paths of layered stable process (—) with $(\alpha, \beta) = (1.3, 1.9)$, 1.3-stable process (-+-), and 1.9-stable process (-x-)

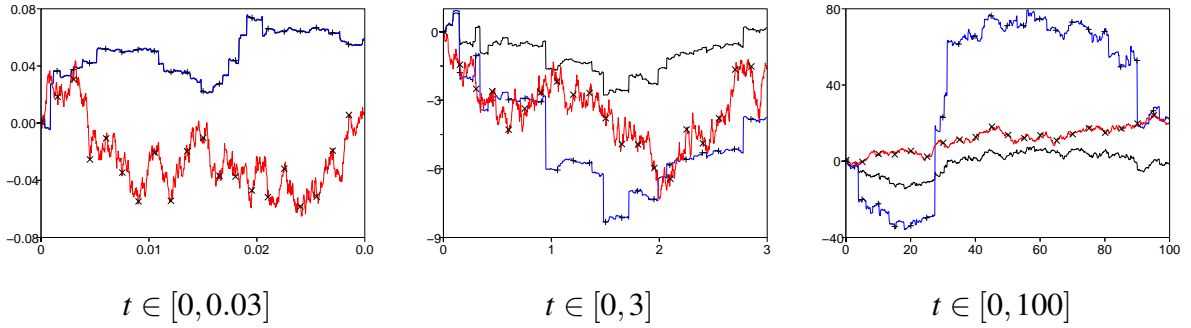


Figure 2: Typical sample paths of layered stable process (—) with $(\alpha, \beta) = (1.1, 2.5)$, 1.1-stable process (-+-), and a Brownian motion (-x-)

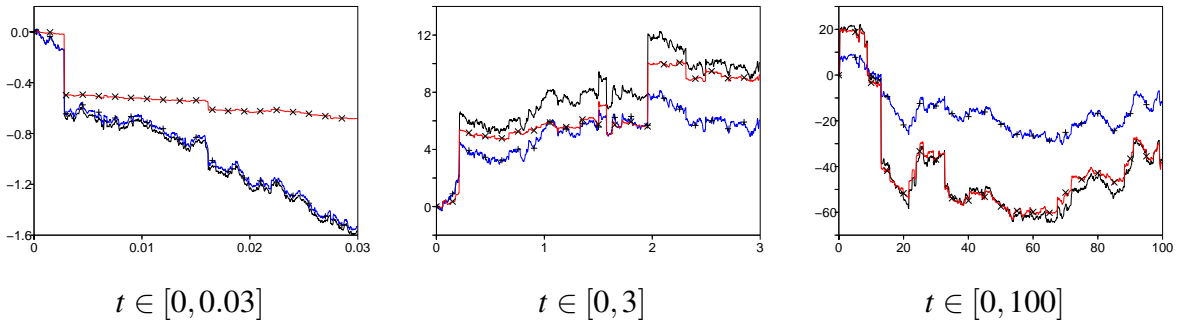


Figure 3: Typical sample paths of layered stable process (—) with $(\alpha, \beta) = (1.9, 1.3)$, 1.9-stable process (-+-), and 1.3-stable process (-x-)

for some $\eta \in \mathbb{R}^d$, and where $c_\alpha = |\Gamma(-\alpha) \cos \frac{\pi\alpha}{2}|$ when $\alpha \neq 1$ while $c_1 = \pi/2$. We have seen in Example 2.2 that an infinitely divisible probability measure is in L_0 if and only if the corresponding Lévy measure has the form

$$\int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) k_\xi(r) \frac{dr}{r}, \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where σ is a finite positive measure on S^{d-1} and where $k_\xi(r)$ is a nonnegative function measurable in $\xi \in S^{d-1}$ and decreasing in $r > 0$. Recently, Barndorff-Nielsen et al.[1] defined a new class of infinitely divisible distributions by further requiring that the function $k_\xi(r)$ be completely monotone in r for σ -a.e. ξ . Mixed stable distributions are indeed in this class since $\int_{(0,2)} r^{-\alpha} \varphi(d\alpha)$ is completely monotone.

Finally, note that the associated Lévy process that we call a *mixed stable process* possesses an interesting series representation. For simplicity, assume that σ in (5.1) is symmetric. Let $\{\Gamma_i\}_{i \geq 1}$, $\{T_i\}_{i \geq 1}$ and $\{V_i\}_{i \geq 1}$ be random sequences defined as before. In addition, let $\{\alpha_i\}_{i \geq 1}$ be a sequence of iid random variables with common distribution φ . Assume moreover that all these random sequences are mutually independent. Then, with the help of the generalized shot noise method of Rosiński [7], it can be shown that the stochastic process

$$\left\{ \sum_{i=1}^{\infty} \left(\frac{\alpha_i \Gamma_i}{\sigma(S^{d-1})T} \right)^{-1/\alpha_i} V_i \mathbb{1}(T_i \leq t) : t \in [0, T] \right\},$$

converges almost surely uniformly in t to a mixed stable process whose marginal law at time 1 is mixed stable with the Lévy measure (5.1). Comparing this result with the series representation of a stable process given in Lemma 1.1, a mixed stable process can be thought of as a stable process with each of its jumps obeying a randomly chosen stability index. This jump structure also implies that unlike the layered stable process, the mixed stable process does not alter its behavior in terms of the time range.

References

- [1] Barndorff-Nielsen, O.E., Maejima, M., Sato, K. (2006) Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations, *Bernoulli*, **12**, 1-33.
- [2] Barndorff-Nielsen, O.E., Shepard, N. (2002) Normal modified stable processes, *Theory Probab. Math. Statist.* **65**, 1-20.
- [3] Bretagnolle, J. (1972) p -variation de fonctions aléatoires, In: *Séminaire de Probabilités VI*, Lect. Notes in Math. **258**, Springer, 51-71.

- [4] Gikhman, I.I., Skorokhod, A.V. (1969) *Introduction to the theory of random processes*, W.B. Saunders.
- [5] Houdré, C., Kawai, R. (2006) On fractional tempered stable motion, *Stochastic Processes Appl.*, **116**, 1161-1184.
- [6] Kallenberg, O. (2001) *Foundations of Modern Probability* (2nd ed.), Springer.
- [7] Rosiński, J. (2001) Series representations of Lévy processes from the perspective of point processes, In: *Lévy Processes - Theory and Applications*, Eds. Barndorff-Nielsen, O.-E., Mikosch, T., Resnick, S.I., Birkhäuser, 401-415.
- [8] Rosiński, J. (2007) Tempering stable processes, To appear in *Stochastic Processes and their Applications*.
- [9] Samorodnitsky, G. Taqqu, M.S. (1994) *Stable non-Gaussian random processes*, Chapman & Hall, New York.
- [10] Sato, K. (1980) Class L of multivariate distributions and its subclasses, *J. Multivariate Anal.* **10**, 207-232.
- [11] Sato, K. (1999) *Lévy processes and infinitely divisible distributions*, Cambridge: Cambridge University Press.