

# Minimax regret and strategic uncertainty\*

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## Abstract

This paper introduces a new solution concept, a minimax regret equilibrium, which allows for the possibility that players are uncertain about the rationality and conjectures of their opponents. We provide several applications of our concept. In particular, we consider price-setting environments and show that optimal pricing policy follows a non-degenerate distribution. The induced price dispersion is consistent with experimental and empirical observations (Baye and Morgan (2004)).

**Keywords:** minimax regret, rationality, conjectures, price dispersion, auction.

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# 1 Introduction

Strategic situations confront individuals with the delicate tasks of conjecturing other individuals' decisions, that is, they face *strategic uncertainty*. Naturally, individuals might rely on their prior information or knowledge in forming their conjectures. For instance, if an individual knows that his opponents are rational, then he can infer that they will not play strictly dominated strategies.<sup>1</sup> Furthermore, common knowledge in rationality leads to rationalizable conjectures (Bernheim (1984) and Pearce (1984)). Likewise, common knowledge of conjectures, mutual knowledge of rationality and payoffs, and existence of a common prior imply that conjectures form a Nash equilibrium when viewed as mixed strategies (Aumann and Brandenburger (1995)). The aim of this paper is to introduce a new solution concept, called a *minimax regret equilibrium*, which postulates neither mutual or common knowledge in rationality nor common knowledge of conjectures.

Now, if an individual is uncertain about the rationality of his opponents, which conjectures about his opponents' actions should he form? This is a very intricate issue as there is at best little to guide the individual. Admittedly, he can form a subjective probabilistic assessment and play a best response to his assessment. However, any subjective assessment is largely arbitrary, and there is no obvious reasons to favor one assessment over another. Bayesian theory is silent on how to form initial probabilistic assessments (Morris (1995)). Moreover, experimental evidence such as the Ellsberg's paradox suggests that individuals frequently experience difficulties in forming a unique assessment. In this paper, we postulate that "regret" guides individuals in forming probabilistic assessments and, ultimately, in making choices. More precisely, we use the model of *minimax regret with multiple priors*, recently axiomatized by Hayashi (2008) and Stoye (2007b), to represent the preferences of individuals. In essence, the minimax regret criterion captures the idea that individuals are concerned with foregone opportunities. Before proceeding, we wish to stress that the concern for minimizing maximal regret does not arise from any behavioral or emotional considerations. Rather, it is a consequence of relaxing some of the axioms of subjective

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<sup>1</sup>In this paper, "knowledge" refers to belief with probability 1 (certainty).

expected utility, in particular the axiom of independence to irrelevant alternatives. Furthermore, the behavior of an individual concerned with regret is indistinguishable from the behavior of an individual who has formed a unique probabilistic assessment (a Bayesian), provided that this assessment is one that leads to maximal regret. Therefore, we may say that minimax regret does indeed guide individuals in forming their probabilistic assessments.

While we include the case where an individual conjectures that any action profile of his opponents might be played, we allow, more generally, for conjectures to be constrained. For instance, conjectures might be constrained to be correct with some minimal probability (i.e., approximate common knowledge in conjectures) or consistent with almost mutual knowledge in rationality. We can now provide an informal definition of our solution concept. A profile of actions is a *minimax regret equilibrium* if the action of a player is optimal given his conjecture about his opponents' play. And his conjecture is consistent with the criterion of minimax regret and initial constraints on conjectures. A parametric variant of special interest is called an  $\varepsilon$ -*minimax regret equilibrium*. In an  $\varepsilon$ -minimax regret equilibrium, conjectures are directly related to the equilibrium actions as follows. With probability  $1 - \varepsilon$ , a player believes (or conjectures) that his opponents will play according to the equilibrium actions while, with probability  $\varepsilon$ , the player is completely uncertain about his opponents' play. The set of initial assessments is therefore the  $\varepsilon$ -contamination neighborhood around the equilibrium actions. It transpires that this parameterized version of a minimax regret equilibrium is extremely simple, tractable and insightful for economic applications.

We provide several applications of our solution concept. In particular, we consider price-setting environments *à la* Bertrand and characterize their  $\varepsilon$ -minimax regret equilibria. In such environments, firms face two sources of regret. First, a firm's price might turned out to be lower than the lowest price of its competitors. Had the firm posted a higher price, its profit would have been higher. Second, a firm's price might turned out to be higher than the lowest price of its competitors, and the regret arises from not serving the market at all. The exposure to these two sources of regret has important economic applications. In any  $\varepsilon$ -minimax regret equilibrium, firms price above marginal costs and make a positive profit. The intuition is simple. Since

a firm is concerned with foregone opportunities and, in particular, with the possibility that its competitors might price close to the monopoly price, its optimal pricing policy reflects these concerns and, consequently, the price posted is strictly above the marginal cost (in order to minimize (maximal) regret). Moreover, as the number of firms gets larger, the pricing policy converges to the competitive equilibrium. Furthermore, when there are at least three firms competing in the market or costs are heterogeneous, the equilibrium pricing policy exhibits a kink at a price close to the monopoly price. All these equilibrium predictions agree remarkably well with empirical and experimental observations, as documented by Baye and Morgan (2004).

Some related concepts have already appeared in the literature. The closest is Klibanoff's (1996) concept of equilibrium with uncertainty aversion. The essential difference between Klibanoff's concept and ours is that Klibanoff assumes that players conform with the maximin criterion (with multiple priors), whereas we assume that they conform with the minimax regret criterion. Neither we nor Klibanoff assume mutual knowledge in rationality. Consequently, equilibria with uncertainty aversion as well as minimax regret equilibria might not be rationalizable. While conceptually very similar, these two approaches might give very different predictions in games, as we will see. Another solution concept, which adopts the maximin criterion and which is called a belief equilibrium, is offered by Lo (1996). Lo's concept differs from Klibanoff's concept and ours in that it assumes common knowledge in rationality and, consequently, belief equilibria are rationalizable. It would be straightforward to adapt Lo's concept to the minimax regret criterion, but we did not choose to do it. Indeed, in a wide range of experiments on the iterated deletion of strictly dominated strategies, a vast majority of subjects seems to be uncertain about the rationality of others (see Camerer (2003, Chapter 5) for a survey). Furthermore, a slight doubt about the rationality of others can yield very interesting predictions in economic models e.g., price dispersion, the existence of large and speculative trade (Neeman (1996)), just to name a few. Another closely related concept is the concept of ambiguous equilibrium (Mukerji (1995)), which adopts the concept of Choquet expected utility and  $\varepsilon$ -ambiguous beliefs, a close relative to  $\varepsilon$ -contamination.

While all these approaches are largely complementary and, indeed, share similar axiomatic and epistemic foundations, we advocate in favor of the minimax regret equilibrium. Indeed, the maximin criterion often leads to unsatisfactory predictions in strategic situations. In the price-setting environments mentioned above, the maximin solution implies that sellers price at the marginal cost and make zero profit (the Bertrand-Nash predictions). These predictions sharply contrast not only with our predictions, but also with empirical evidences. Bergemann and Schlag (2005, 2008), Halpern and Pass (2008), Linhart (2001) and Linhart and Radner (1989) make similar observations in other settings such as monopoly pricing and bilateral bargaining. Ultimately, a solution concept should be judged according to its merits in economic applications. We have written this paper with this perspective in mind and hope that its user-friendly exposition will help applied theorists to apply our solution concept fruitfully in future research.

The paper is organized as follows. Section 2 summarizes the axiomatization of the minimax regret criterion and gives the definition of a minimax regret equilibrium. Section 3 offers some examples and properties of a minimax regret equilibrium, while Section 4 provides an economic application. Lastly, Section 5 concludes.

## 2 Minimax Regret Equilibrium

### 2.1 Regret in Decision Theory

This section provides a brief review of “regret-type” decision rules. We refer the reader to Savage (1951), Milnor (1954), Hayashi (2008), Puppe and Schlag (2007) and Stoye (2007a,b) for in-depth treatments.

Consider a finite set  $\Omega$  of states of the world and a finite set of outcomes  $A$ . For any finite set  $X$ , we denote  $\Delta(X)$  the set of all probabilities over  $X$ , that is,  $\Delta(X) := \{\sigma \in \mathbb{R}_+^{|X|} : \sum_{x \in X} \sigma(x) = 1\}$ . An *act*  $f$  is a mapping from  $\Omega$  to  $\Delta(A)$ , the set of lotteries over  $A$ , and we denote a *menu* of acts by  $\mathcal{F}$ . The primitive of the model is a preference relation  $\succeq_{\mathcal{F}}$  over acts belonging to the menu  $\mathcal{F}$ .

Minimax regret theory departs from subjective expected utility theory

(Anscombe and Aumann (1963)) in two important ways. First, it weakens the axiom of independence to irrelevant alternatives to the axiom of *independence to never-optimal alternatives*. In words, the axiom of independence to never-optimal alternatives states that the act  $f$  is preferred to the act  $g$  in the menu  $\mathcal{F}$  if and only if the act  $f$  is preferred to the act  $g$  in the menu  $\mathcal{F}'$  obtained by complementing  $\mathcal{F}$  with never-optimal acts.<sup>2</sup> Second, it imposes an axiom of *ambiguity aversion* as in Gilboa and Schmeidler (1989), that is, if an individual is indifferent between acts  $f$  and  $g$ , then he (weakly) prefers the mixing of  $f$  and  $g$  to either of them. An ambiguity averse individual prefers to hedge bets across states.<sup>3</sup> Weakening menu independence together with ambiguity aversion leads to the following numerical representation of  $\succeq_{\mathcal{F}}$ : there exists a function  $U : A \rightarrow \mathbb{R}$  such that for all  $f \in \mathcal{F}$ ,  $g \in \mathcal{F}$ ,  $f \succeq_{\mathcal{F}} g$  if and only if

$$\max_{\omega \in \Omega} [\max_{h \in \mathcal{F}} u(h, \omega) - u(f, \omega)] \leq \max_{\omega \in \Omega} [\max_{h \in \mathcal{F}} u(h, \omega) - u(g, \omega)], \quad (1)$$

with  $u(f, \omega)$  the expected payoff of the act  $f$  in state  $\omega$ , i.e.,  $u(f, \omega) := \sum_{a \in A} U(a) f(\omega)(a)$ . In Eq. (1), the term “ $\max_{h \in \mathcal{F}} u(h, \omega) - u(f, \omega)$ ” is the difference between the highest payoff an individual would have got had he known the state was  $\omega$ , and the payoff obtained by choosing  $f$ . We can thus *interpret* this term as the regret an individual might experience by choosing  $f$ . Consequently, the act  $f$  is chosen over the act  $g$  if it minimizes the maximal regret, hence the term “minimax regret.” However, it is important to bear in mind that the axiomatization of minimax regret does not rely on any regret-led behaviors; it is rather “as if” individuals wish to minimize their maximal regret. It is also worth noting that no prior beliefs explicitly appear in Eq. (1). Or, more precisely, an individual considers all prior beliefs  $\pi \in \Delta(\Omega)$  possible: there is complete uncertainty. Another well-known theory of complete uncertainty is *maximin*. Briefly, maximin differs from minimax regret in that it postulates the axiom of independence to

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<sup>2</sup>An act  $h$  added to menu  $\mathcal{F}$  is never-optimal if for all states  $\omega \in \Omega$ , there is some act  $f' \in \mathcal{F}$  such that  $f'(\omega)$  is preferred to  $h(\omega)$  where  $f'(\omega)$  and  $h(\omega)$  are identified with constant acts. (Stoye (2007a), p. 4.)

<sup>3</sup>The axiom of symmetry is also imposed. Loosely speaking, it states that “a preference ordering should not impose prior beliefs by implicitly assigning different likelihoods to different events.” (Stoye (2007a), p. 11.)

irrelevant alternatives, but relaxes the axiom of independence. The axiom of independence states that the act  $f$  is preferred to the act  $g$  in the menu  $\mathcal{F}$  if and only if any mixture of the acts  $f$  and  $h$  is preferred to the same mixture of the acts  $g$  and  $h$  in the menu composed of all the mixtures of acts in  $\mathcal{F}$  and  $\{h\}$ .<sup>4</sup>

To illustrate the above concepts, let us consider the example below with three states  $\{\omega_1, \omega_2, \omega_3\}$  and  $n > 4$ .

	$\omega_1$	$\omega_2$	$\omega_3$
$f$	0	$n$	$(n-1)/2$
$g$	$1/n$	1	$n/2$

For  $n$  large, both acts  $f$  and  $g$  are very similar in states  $\omega_1$  and  $\omega_3$  and, therefore, one might expect that  $f$  is preferred over  $g$  (since in state  $\omega_2$ ,  $f$  gives a disproportionally larger payoff than  $g$ ). Indeed,  $f$  is preferred over  $g$  according to the minimax regret theory. In contrast,  $g$  is preferred over  $f$  according to the maximin theory. The problem with maximin is the entire focus on the worst states of the world, rather than to the states in which the choice of an act is the most consequential as with minimax regret. With minimax regret, the choice between two acts might depend on the menus considered, however. To see this, consider the act  $h = (-n, -n, n)$ . We have that  $f$  is preferred over  $g$  in the menu  $\{f, g\}$ , but  $g$  is preferred over  $f$  in the menu  $\{f, g, h\}$ . We do not find this violation of the axiom of independence to irrelevant alternatives disturbing. Experimental evidences indeed suggest that the choice between two acts depends on the presence or absence of other options (see e.g., Simonson and Tversky (1992)).<sup>5</sup>

To capture the existence of partial prior information, we consider a variant of the minimax regret theory introduced by Hayashi (2008) (see also Stoye (2007b)), which allows for a restricted set of prior assessments. Relaxing the symmetry axiom, which captures the lack of prior information in the

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<sup>4</sup>The axiom of independence to irrelevant alternatives states that the act  $f$  is preferred to the act  $g$  in the menu  $\mathcal{F}$  if and only if  $f$  is preferred to  $g$  in the menu  $\mathcal{F}'$  for all menus  $\mathcal{F}$  and  $\mathcal{F}'$ .

<sup>5</sup>See Stoye (2007a) for more on this issue.

axiomatization of minimax regret given by (1), we get the existence of a closed and convex set  $\Pi \subseteq \Delta(\Omega)$  of prior beliefs such that  $f \succeq_{\mathcal{F}} g$  if and only if

$$\max_{\pi \in \Pi} \sum_{\omega \in \Omega} [\max_{h \in \mathcal{F}} u(h, \omega) - u(f, \omega)] \pi(\omega) \leq \max_{\pi \in \Pi} \sum_{\omega \in \Omega} [\max_{h \in \mathcal{F}} u(h, \omega) - u(g, \omega)] \pi(\omega).$$

Note that when  $\Pi = \Delta(\Omega)$ , the minimax regret theory with multiple priors reduces to the standard minimax regret theory à la Savage and when  $\Pi = \{\pi^*\}$ , it reduces to standard subjective expected utility. In the next section, the concept of minimax regret with multiple priors will be the cornerstone of our solution concept: minimax regret equilibrium.

## 2.2 Strategic-form Games

Let  $g := \langle N, (A_i, u_i)_{i \in N} \rangle$  be a strategic-form game with  $N := \{1, \dots, n\}$  the set of players,  $A_i$  the finite set of actions available to player  $i$ , and  $u_i : A := \times_i A_i \rightarrow \mathbb{R}$  the payoff function of player  $i$ . With a slight abuse of notation, we denote by  $G := \langle N, (\Sigma_i, u_i)_{i \in N} \rangle$  the mixed extension of  $g$ , that is,  $\Sigma_i = \Delta(A_i)$  is the set of mixed actions of player  $i$  and  $u_i : \Sigma := \times_i \Sigma_i \rightarrow \mathbb{R}$  is the payoff function.<sup>6</sup> Denote  $\Sigma_{-i} := \times_{j \in N \setminus \{i\}} \Sigma_j$  and  $\sigma_{-i}$  a generic element of  $\Sigma_{-i}$ . Similarly,  $a_{-i}$  denotes a generic element of  $A_{-i}$ . We say that the action  $\sigma_i^*$  *dominates* the action  $\sigma_i$  if  $u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$  for all  $\sigma_{-i} \in \Sigma_{-i}$ , and  $u_i(\sigma_i^*, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$  for some  $\sigma_{-i} \in \Sigma_{-i}$ . An action is *dominant* if it dominates all other actions. Similarly, we say that the action  $\sigma_i^*$  *strictly dominates* the action  $\sigma_i$  if  $u_i(\sigma_i^*, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$  for all  $\sigma_{-i} \in \Sigma_{-i}$ . An action is *strictly dominant* if it strictly dominates all other actions. A Nash equilibrium of the game  $G$  is a profile of (mixed) actions  $\sigma^*$  such that for all  $a_i^*$  in the support of  $\sigma_i^*$ ,  $u_i(a_i^*, \sigma_{-i}^*) \geq u_i(a_i, \sigma_{-i}^*)$  for all  $a_i \in A_i$ , for all  $i \in N$ . In words,  $\sigma_i^*$  is the common belief (conjecture) of player  $i$ 's opponents about the pure actions player  $i$  will play. And rational players best-reply to their conjectures. This paper proposes a new solution concept for games, which presupposes neither mutual knowledge of rationality nor common knowledge of conjectures. We call this solution concept a minimax regret equilibrium.

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<sup>6</sup>Precisely,  $u_i(\sigma_1, \dots, \sigma_n) = \sum_{a_1 \in A_1} \dots \sum_{a_n \in A_n} \sigma_1(a_1) \dots \sigma_n(a_n) u_i(a_1, \dots, a_n)$ .

We first define player  $i$ 's *ex-post regret* associated with any profile of **pure** actions  $(a_i, a_{-i})$  as

$$r_i(a_i, a_{-i}) := \sup_{\hat{a}_i \in A_i} u_i(\hat{a}_i, a_{-i}) - u_i(a_i, a_{-i}), \quad (2)$$

that is, this is the difference between the payoff player  $i$  obtains when the profile of actions  $(a_i, a_{-i})$  is played and the highest payoff he might have obtained had he known that his opponents were playing  $a_{-i}$ . The regret associated with the profile of mixed strategies  $(\sigma_i, \sigma_{-i})$  is then given by:

$$R_i(\sigma_i, \sigma_{-i}) := \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \sigma_i(a_i) \sigma_{-i}(a_{-i}) r_i(a_i, a_{-i}). \quad (3)$$

Before defining the concept of a minimax regret equilibrium, let us discuss in more details the concept of regret. Regret as axiomatized in decision theory (see above) is defined with respect to a set of states of the world. Extending this framework to choices in strategic situations, we identify the profile of actions  $a_{-i}$  chosen by the other players with such a state. The alternative of identifying an opponents' profile of mixed actions with a state of the world is problematic. It is indeed fundamental for the motivation of the axioms that the states are not related: changing an outcome in one state should not have an impact on the outcomes in other states. If states were identified with mixed actions, this would not be the case.

For each player  $i$ , let  $\Pi_i \subseteq \Delta(A_{-i})$  be some (compact and convex) set of player  $i$ 's beliefs (conjectures) about the play of his opponents. It is important to note that although conjectures are about opponents' mixed actions and, therefore, incorporate the fact that players play independently, player  $i$ 's belief  $\pi_i \in \Pi_i$  about opponents' pure actions might be correlated. To see this, suppose that players 2 and 3 have two actions each,  $a$  and  $b$ , and player 1 conjectures that they play the mixed action  $\sigma_2(a) = \sigma_3(a) = 1$  with probability  $1/4$  and the mixed action  $\sigma_2(a) = \sigma_2(b) = 1/3$  with probability  $3/4$ . Although player 1's conjecture puts strictly positive probability to independent mixing only, his belief  $\pi_1$  over the play of his opponents is given by  $\pi_1(a, a) = \pi_1(b, b) = 1/3$  and  $\pi_1(a, b) = \pi_1(b, a) = 1/6$ , a correlated distribution.<sup>7</sup> Thus, even though players play independently, conjectures might be

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<sup>7</sup>See Fudenberg and Levine (1993) for more on this.

correlated distributions. A special case is where player  $i$  cannot rule out any mixed action profile, i.e.,  $\Pi_i = \Delta(A_{-i})$ . In this case, we speak of *complete uncertainty*.

A convenient parametrization of the belief sets is the so-called  $\varepsilon$ -contamination neighborhood around some given profile  $\sigma_{-i}^*$ , with  $\varepsilon \in [0, 1]$ . In this case, with probability  $1 - \varepsilon$ , a player believes that his opponents will play  $\sigma_{-i}^* \in \Sigma_{-i}$  and, with probability  $\varepsilon$ , is completely uncertain about the play of his opponents. Formally, we have that  $\Pi_i := \Pi_{i\varepsilon}(\sigma_{-i}^*) = \{(1 - \varepsilon)\sigma_{-i}^* + \varepsilon\sigma_{-i}, \sigma_{-i} \in \Delta(A_{-i})\}$ . An alternative is to consider the Cartesian product of *independent*  $\varepsilon$ -contamination neighborhoods, that is,  $\Pi_i := \Pi_{i\varepsilon}(\sigma_{-i}^*) = \times_{j \neq i} \{\sigma_j : \sigma_j = (1 - \varepsilon)\sigma_j^* + \varepsilon\sigma_j' \text{ for all } \sigma_j' \in \Sigma_j\}$ . In that case, a player is completely certain that his opponents play  $\sigma_{-i}^*$  with probability  $(1 - \varepsilon)^{n-1}$ , is completely uncertain with probability  $\varepsilon^{n-1}$ , and partially certain otherwise. For instance, with probability  $\binom{n-1}{1}(1 - \varepsilon)\varepsilon^{n-2}$ , a player is certain that one of his opponent plays  $\sigma_j^*$  and is completely uncertain about the play of the other opponents. As argued above, even though conjectures are about mixed actions, beliefs about opponents' pure actions might be correlated, thus advocating in favor of the (correlated)  $\varepsilon$ -contamination neighborhoods. Moreover, it is not clear to us how a player can be certain that some of his opponents will play a given strategy profile while, simultaneously, being uncertain about the play of the other opponents. Correlated contamination is therefore our preferred formulation. Yet, as in the original definition of rationalizability (Berheim (1984) and Pearce (1984)), one might insist on the independence, in which case the product of independent  $\varepsilon$ -contaminations is the appropriate choice. As we will see later, both approaches might give different predictions in applications. To maintain focus and simplicity, we assume that  $\varepsilon$  is the same for each player; this can be easily relaxed.

**Definition 1** *A profile of strategies  $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$  is a minimax regret equilibrium relative to  $(\Pi_1, \dots, \Pi_N)$  if for each player  $i \in N$ ,  $\sigma_{-i}^* \in \Pi_i$ , and*

$$\max_{\sigma_{-i} \in \Pi_i} R_i(\sigma_i^*, \sigma_{-i}) \leq \max_{\sigma_{-i} \in \Pi_i} R_i(\sigma_i, \sigma_{-i}), \quad (4)$$

for all  $\sigma_i \in \Sigma_i$ .

Several remarks are worth making. First, a Nash equilibrium  $(\sigma_1^*, \dots, \sigma_n^*)$

is a minimax regret equilibrium relative to  $(\{\sigma_{-1}^*\}, \dots, \{\sigma_{-n}^*\})$ . Second, for given belief sets  $(\Pi_i)_{i \in N}$ , a minimax regret equilibrium might not exist. For instance, consider the prisoner dilemma game, below.

	$a$	$b$
$a$	3, 3	0, 4
$b$	4, 0	1, 1

$G0$

There is clearly no minimax regret equilibrium relative to  $(\{\delta_a\}, \{\delta_a\})$ , where  $\delta_a$  is the Dirac mass on  $a$ . Yet, if the belief sets are  $\varepsilon$ -contamination neighborhoods around the “equilibrium” strategies (hence beliefs are endogenously determined), existence of an equilibrium, called an  $\varepsilon$ -minimax regret equilibrium, is guaranteed. The formal definition of an  $\varepsilon$ -minimax regret equilibrium is below:<sup>8</sup>

**Definition 2** *An  $\varepsilon$ -minimax regret equilibrium is a minimax regret equilibrium  $(\sigma_i^*, \sigma_{-i}^*)$  relative to the  $\varepsilon$ -contamination neighborhoods  $(\Pi_{i\varepsilon}(\sigma_{-i}^*))_{i \in N}$ .*

Third, an alternative definition of a minimax regret equilibrium would include the belief sets  $(\Pi_i)_{i \in N}$  as part of the equilibrium. Since we do not assume explicit randomization, we do not find this alternative definition compelling. Indeed, it would amount to define two separate sets of beliefs for each player, which might be inconsistent with equilibrium reasoning. To see this, suppose there are only two players and let  $(\sigma_i^*, \Pi_i^*)_i$  be a minimax regret equilibrium (with the proposed alternative definition). Then for player  $i$ , not only  $\sigma_j^*$  represents his belief about player  $j$ 's play, but also  $\Pi_i^*$ . And these two sets might not coincide. How can player  $i$  endogenously entertain two different sets of beliefs? In other words, if equilibrium reasoning leads player  $i$  to the belief  $\sigma_j^*$ , why does the same equilibrium reasoning lead him to the beliefs  $\Pi_i^*$ , which might be different from  $\sigma_j^*$ ?

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<sup>8</sup>Note that in an  $\varepsilon$ -minimax regret equilibrium, the support of  $\sigma_{-i}^*$  is included in the support of  $\pi_i^*$  for all  $\pi_i^* \in \Pi_i^*(=:\Pi_{i\varepsilon}(\sigma_{-i}^*))$ , a requirement imposed by Marinacci (2000).

### 3 Properties and examples

This section presents some properties and examples of  $\varepsilon$ -minimax regret equilibria. Our first result, Theorem 1, gives a saddle-point interpretation of a minimax regret equilibrium, which will prove extremely useful in applications.

**Theorem 1 (Saddle point)** *The profile  $(\sigma_i^*, \sigma_{-i}^*)$  is a minimax regret equilibrium relative to  $(\Pi_i)_{i \in N}$  if and only if there exists  $\pi^* \in \times_{i \in N} \Pi_i$  such that  $R_i(\sigma_i^*, \pi_i^*) \geq R_i(\sigma_i^*, \pi_i)$  for all  $\pi_i \in \Pi_i$  and  $R_i(\sigma_i^*, \pi_i^*) \leq R_i(\sigma_i, \pi_i^*)$  for all  $\sigma_i \in \Sigma_i$  for all  $i \in N$ .*

**Proof** ( $\Leftarrow$ ). For any  $i \in N$ , let  $(\sigma_i^*, \pi_i^*) \in \Sigma_i \times \Pi_i$  be a saddle-point of  $R_i$  i.e., for all  $\sigma_i \in \Sigma_i$  and  $\pi_i \in \Pi_i$ :

$$R_i(\sigma_i, \pi_i^*) \geq R_i(\sigma_i^*, \pi_i^*) \geq R_i(\sigma_i^*, \pi_i).$$

It follows that

$$R_i(\sigma_i^*, \pi_i^*) = \max_{\pi_i \in \Pi_i} R_i(\sigma_i^*, \pi_i) \leq R_i(\sigma_i, \pi_i^*) \leq \max_{\pi_i \in \Pi_i} R_i(\sigma_i, \pi_i),$$

for all  $\sigma_i \in \Sigma_i$ . Henceforth,  $(\sigma_i^*)_{i \in N}$  is a minimax regret equilibrium relative to  $(\Pi_i)_i$ .

( $\Rightarrow$ ). Let  $(\sigma_i^*, \sigma_{-i}^*)$  be a minimax regret equilibrium relative to  $(\Pi_i)_i$ . In particular, this implies that  $\sigma_{-i}^* \in \Pi_i$  and there exists a  $\pi_i^* \in \Pi_i$  such that

$$R_i(\sigma_i^*, \pi_i^*) = \min_{\sigma_i \in \Sigma_i} \max_{\pi_i \in \Pi_i} R_i(\sigma_i, \pi_i).$$

Since the belief sets  $(\Pi_i)_{i \in N}$  are compact and convex and the regret functions are bilinear, it follows from the Minimax Theorem (Von Neumann (1928)) that

$$R(\sigma_i, \pi_i^*) \geq \min_{\sigma'_i \in \Sigma_i} R_i(\sigma'_i, \pi_i^*) = \max_{\pi'_i \in \Pi_i} \min_{\sigma'_i \in \Sigma_i} R_i(\sigma'_i, \pi'_i) \quad (5)$$

$$= \min_{\sigma'_i \in \Sigma_i} \max_{\pi'_i \in \Pi_i} R_i(\sigma'_i, \pi'_i) = \max_{\pi'_i \in \Pi_i} R_i(\sigma_i^*, \pi'_i) \geq R_i(\sigma_i^*, \pi_i), \quad (6)$$

for all  $\sigma_i \in \Sigma_i$  and  $\pi_i \in \Pi_i$ . Henceforth,  $(\sigma_i^*, \pi_i^*)$  is a saddle point, which completes the proof.  $\square$

It follows from Theorem 1 that finding a minimax regret equilibrium  $\sigma^*$  relative to belief sets  $(\Pi_i)_i$  is equivalent to checking whether  $(\sigma_i^*, \pi_i^*)$  is a Nash equilibrium of a two-player zero-sum game between player  $i$  and a fictitious player,  $i$ 's "Nature," in which player  $i$ 's action set is  $\Sigma_i$ , Nature's action set is  $\Pi_i$ , and the payoff function to Nature is  $R_i$ . Before presenting an example, three further remarks are worth making. First, the requirement in Theorem 1 that  $R_i(\sigma_i^*, \pi_i^*) \leq R_i(\sigma_i, \pi_i^*)$  for all  $\sigma_i \in \Sigma_i$  is equivalent to  $\sigma_i^*$  being player  $i$ 's best-reply to the conjecture  $\pi_i^*$  that is,  $u_i(\sigma_i^*, \pi_i^*) \geq u_i(\sigma_i, \pi_i^*)$  for all  $\sigma_i \in \Sigma_i$ . This is a rationality requirement. Thus, it is as if a player selects a belief about the play of his opponents according to the minimax regret criterion, and best replies to it. Furthermore, for  $\varepsilon$  small enough, conjectures are almost common knowledge and, consequently,  $\varepsilon$ -minimax regret equilibria are approximate Nash equilibria. Indeed, in an  $\varepsilon$ -minimax regret equilibrium  $\sigma^*$ , each player  $i$  assigns probability at least  $1 - \varepsilon$  to the mixed action  $\sigma_{-i}^*$  being played.<sup>9</sup> Second, any finite game admits an  $\varepsilon$ -minimax regret equilibrium.

**Theorem 2** *For any  $\varepsilon \in [0, 1]$ , there exists an  $\varepsilon$ -minimax regret equilibrium.*

The existence of an  $\varepsilon$ -minimax regret equilibrium  $(\sigma_i^*, \sigma_{-i}^*)$  follows from the fact that  $\sigma_{-i}^* \in \Pi_i(\sigma_{-i}^*)$  and standard fixed-point arguments. Third, our framework is easily generalized to games with infinite strategy spaces. For instance, if  $A_i$  is a compact Hausdorff space,  $\Delta(A_i)$  the set of (regular, countably additive) probability measures on the Borel subsets of  $A_i$ , then a sufficient condition for Theorem 1 to hold is that the zero-sum game between each player  $i$  and  $i$ 's Nature is payoff secure (see Reny (1999)).<sup>10</sup> The game of price competition studied in Section 4 is payoff secure.

Our first example carefully spells out all the steps necessary to find the  $\varepsilon$ -minimax regret equilibria of a game, and illustrates the importance of Theorem 1. Consider the mixed extension of the game  $G1$  below.

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<sup>9</sup>The solution concept shares similar epistemic foundations with the concept of  $\varepsilon$ -ambiguous equilibrium of Mukerji (1995).

<sup>10</sup>More precisely, Reny (1999) requires the mixed extension of the zero-sum game between player  $i$  and  $i$ 's Nature to be better-reply secure. Since the game is zero-sum, its mixed extension is reciprocally upper semi-continuous. Therefore, payoff security insures that the mixed extension is better-reply secure.

	$a$	$b$
$a$	3, 3	0, 5
$b$	5, 0	-3, -3

$G1$

The game  $G1$  has two pure Nash equilibria  $(a, b)$  and  $(b, a)$  and one mixed equilibrium  $((3/5, 2/5)(3/5, 2/5))$ . Let us first consider the case of complete uncertainty, so that  $\varepsilon = 1$ . To find the  $\varepsilon$ -minimax regret equilibria with  $\varepsilon = 1$ , we first construct the (ex-post) regret table for player 1 as follows:

	$a$	$b$
$a$	2	0
$b$	0	3

For instance, if both players play  $a$ , player 1 experiences an ex-post regret of 2 as the best action would have been  $b$ , had he known that player 2 was playing  $a$ . Second, we use Theorem 1 to search for an equilibrium of the zero-sum game between player 1 and 1's Nature, represented below.

	$a$	$b$
$a$	-2, 2	0, 0
$b$	0, 0	-3, 3

This game has a unique Nash equilibrium, in which player 1 chooses the mixed action  $(3/5, 2/5)$ . It guarantees a maximal regret of  $6/5$ . Similarly, for player 2. Therefore, the totally mixed Nash equilibrium  $((3/5, 2/5), (3/5, 2/5))$  is the unique  $\varepsilon$ -minimax regret equilibrium with  $\varepsilon = 1$ . We now show that it is an  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon$ .

Let  $(\sigma_1(a), \sigma_1(b)), (\sigma_2(a), \sigma_2(b))$  be a mixed action profile. In the zero-sum game between player 1 and 1's "Nature," the payoff to player 1 if he plays  $a_1 \in \{a, b\}$  and Nature plays  $a_2 \in \{a, b\}$  is

$$-(1 - \varepsilon)(\sigma_2(a)r_1(a_1, a) + \sigma_2(b)r_1(a_1, b)) - \varepsilon r_1(a_1, a_2).$$

Since  $((3/5, 2/5), (3/5, 2/5))$  is an  $\varepsilon$ -minimax regret equilibrium with  $\varepsilon = 1$ , we have that  $(3/5)r_1(a, a) + (2/5)r_1(a, b) = (3/5)r_1(b, a) + (2/5)r_1(b, b)$ . Thus, when player 1 conjectures that player 2 is playing the mixed action  $(\sigma_2(a), \sigma_2(b)) = (3/5, 2/5)$  with probability at least  $1 - \varepsilon$ , player 1's payoff in the zero-sum game is  $-(1 - \varepsilon)(6/5) - \varepsilon r_1(a_1, a_2)$ . The zero-sum game between player 1 and 1's "Nature" is therefore given by:

	$a$	$b$
$a$	$-(1 - \varepsilon)\frac{6}{5} - \varepsilon 2, (1 - \varepsilon)\frac{6}{5} + \varepsilon 2$	$-(1 - \varepsilon)\frac{6}{5}, (1 - \varepsilon)\frac{6}{5}$
$b$	$-(1 - \varepsilon)\frac{6}{5}, (1 - \varepsilon)\frac{6}{5}$	$-(1 - \varepsilon)\frac{6}{5} - \varepsilon 3, (1 - \varepsilon)\frac{6}{5} + \varepsilon 3$

Clearly,  $((3/5, 2/5)(3/5, 2/5))$  is an equilibrium of this game and, consequently, is an  $\varepsilon$ -minimax regret for all  $\varepsilon$ . The (symmetric) mixed Nash equilibrium survives any uncertainty from small to large. We can apply the exact same arguments to show that  $((3/5, 2/5)(3/5, 2/5))$  is the only non-degenerate  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon > 0$ .

Lastly, we can similarly check that  $(a, b)$  and  $(b, a)$  are  $\varepsilon$ -minimax regret equilibria for  $\varepsilon \leq 2/5$ . Note that  $(a, a)$  is not a minimax regret equilibrium for any  $\varepsilon$ .<sup>11</sup>

Our second example shows that the representation of belief sets as either the product of independent  $\varepsilon$ -contaminations or a (correlated)  $\varepsilon$ -contamination has important consequences for the equilibrium characterization. Consider the game  $G2$  below taken from van Damme (1991, p. 29).

	$a$	$b$		$a$	$b$
$a$	1, 1, 1	1, 0, 1	$a$	1, 1, 0	0, 0, 0
$b$	1, 1, 1	0, 0, 1	$b$	0, 1, 0	1, 0, 0
	$a$	$b$		$a$	$b$

Note that action  $a$  is strictly dominant for both players 2 and 3. The game has a continuum of Nash equilibria, in which player 1 randomizes between  $a$  and  $b$  with any probability and players 2 and 3 play  $a$ . For  $n$ -player

<sup>11</sup>It is, however, an  $\varepsilon$ -maximin equilibrium if  $\varepsilon$  is large enough.

games ( $n \geq 3$ ), the representation of belief sets as  $\varepsilon$ -contaminations entails a choice: either we model it as the product of independent  $\varepsilon$ -contaminations or as a (correlated)  $\varepsilon$ -contamination. First, consider the former. Let us check whether  $(a, a, a)$  is an  $\varepsilon$ -minimax regret equilibrium for some  $\varepsilon > 0$ . For players 2 and 3, since  $a$  is a strictly dominant action, it is clearly part of an  $\varepsilon$ -minimax regret equilibrium (see Proposition 1). Turning to player 1, we construct his regret table:

	$a, a$	$a, b$	$b, a$	$b, b$
$a$	0	0	0	1
$b$	0	1	1	0

and then consider the zero-sum game between player 1 and his “Nature”:

	$a, a$	$a, b$	$b, a$	$b, b$
$a$	0, 0	0, 0	0, 0	$-\varepsilon, \varepsilon$
$b$	0, 0	$-\varepsilon, \varepsilon$	$-\varepsilon, \varepsilon$	0, 0

Clearly,  $a$  is not part of any equilibrium of the above game. If player 1 considers playing  $a$ , 1’s Nature maximizes his regret by playing  $(b, b)$ , in which case a deviation to  $b$  is profitable for player 1. Henceforth,  $(a, a, a)$  is not an  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon > 0$ . Similarly, for  $(b, a, a)$ . In fact, the unique  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon > 0$  is  $((1/2, 1/2), (1, 0), (1, 0))$ .

Second, suppose that belief sets are represented by the product of independent contaminations. Let us check whether  $(a, a, a)$  is an  $\varepsilon$ -minimax regret equilibrium. The zero-sum game between player 1 and 1’s “Nature” is now:

	$a, a$	$a, b$	$b, a$	$b, b$
$a$	0, 0	0, 0	0, 0	$-\varepsilon^2, \varepsilon^2$
$b$	0, 0	$-\varepsilon, \varepsilon$	$-\varepsilon, \varepsilon$	$-2(\varepsilon - \varepsilon^2), 2(\varepsilon - \varepsilon^2)$

For instance, if 1's "Nature" plays  $(b, b)$ , player 1's regret of playing  $a$  is

$$(1 - \varepsilon)^2 r_1(a, a, a) + (1 - \varepsilon)\varepsilon r_1(a, a, b) + \varepsilon(1 - \varepsilon)r_1(a, b, a) + \varepsilon^2 r_1(a, b, b).$$

It follows that  $(a, a, a)$  is an  $\varepsilon$ -minimax regret equilibrium (with independent contaminations) if  $\varepsilon \leq 2/3$ . The intuition is the following. If player 1 considers playing  $a$ , 1's "Nature" maximizes player 1's regret by playing  $(b, b)$ , which occurs with probability  $\varepsilon^2$ . If  $(b, b)$  is played, player 1's regret is 1 and, consequently, his expected regret is  $\varepsilon^2$ . However, if player 1 considers deviating to  $b$ , his regret is 1 when the profile of actions  $(a, b)$  or  $(b, a)$  is played; this occurs with probability  $2\varepsilon(1 - \varepsilon)$ . Since player 1 is indifferent between  $a$  and  $b$  when  $(a, a)$  is played by his opponents, it follows that player 1 is better off playing  $a$  if  $\varepsilon^2 \leq 2\varepsilon(1 - \varepsilon)$  i.e., if  $\varepsilon \leq 2/3$ .

The next two propositions are about dominated actions, iteratively dominated actions and minimax regret equilibria. Before stating these propositions, our next example shows that a Nash equilibrium in weakly dominated actions can be an  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon$ . Consider the game  $G3$  below.

	$a$	$b$	$c$
$a$	0, 0	6, 6	0, 6
$b$	6, 6	0, 0	0, 0
$c$	6, 0	0, 0	1, 1

$G3$

The profile of actions  $((1/2, 1/2, 0), (1/2, 1/2, 0))$  is a Nash equilibrium in weakly dominated actions.<sup>12</sup> To show that it is an  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon$ , we construct the table representing the zero-sum game between player 1 and 1's "Nature" (the table is similar for player 2):

	$a$	$b$	$c$
$a$	$(1 - \varepsilon)3 + \varepsilon 6$	$(1 - \varepsilon)3 + \varepsilon 0$	$(1 - \varepsilon)3 + \varepsilon 1$
$b$	$(1 - \varepsilon)3 + \varepsilon 0$	$(1 - \varepsilon)3 + \varepsilon 6$	$(1 - \varepsilon)3 + \varepsilon 1$
$c$	$(1 - \varepsilon)3 + \varepsilon 0$	$(1 - \varepsilon)3 + \varepsilon 6$	$(1 - \varepsilon)3 + \varepsilon 0$

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<sup>12</sup>For instance,  $(1/2, 1/2, 0)$  is weakly dominated by  $(1/2, 1/4, 1/4)$ .

Note that the payoff in each cell is the payoff of 1's Nature. It is then easy to check that for any  $\varepsilon$ ,  $((1/2, 1/2, 0), (1/2, 1/2, 0))$  is a Nash equilibrium of this zero-sum game. A similar argument for player 2 shows that  $((1/2, 1/2, 0), (1/2, 1/2, 0))$ , albeit weakly dominated, is indeed an  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon$ .

**Proposition 1** *Let  $\sigma^*$  be a minimax regret equilibrium relative to  $(\Pi_i)_{i \in N}$ . (i) If  $\sigma_i^{**}$  is a dominant action for player  $i$ , then  $\sigma_i^* = \sigma_i^{**}$ . (ii) If  $a_i$  is a strictly dominated action, then  $\sigma_i^*(a_i) = 0$ .*

The proof directly follows from the definition of a minimax regret equilibrium and dominated/dominant actions and is left to the reader.<sup>13</sup> Proposition 1 implies that Nash equilibria in dominant actions are robust to the introduction of uncertainty about the rationality and conjectures of opponents. This result is not surprising since players are assumed to be rational in our model. Part (ii) also shows that players do not play strictly dominated strategies in an  $\varepsilon$ -minimax regret equilibrium. In fact, for  $\varepsilon$  small enough, any pure action in the support of an  $\varepsilon$ -minimax regret equilibrium survives iterated deletion of strictly dominated pure actions.

**Proposition 2** *There exists  $\varepsilon^* > 0$  such that for any  $\varepsilon < \varepsilon^*$ , if  $\sigma^*$  is an  $\varepsilon$ -minimax regret equilibrium and  $a_i$  does not survive iterated deletion of strictly dominated pure actions, then  $\sigma_i^*(a_i) = 0$ .*

**Proof** We present the proof for the case of correlated  $\varepsilon$ -contaminations. The case of independent  $\varepsilon$ -contaminations is similar. Let  $(\varepsilon^m)_{m \in \mathbb{N}}$  be any sequence converging to 0 and, for each  $m \in \mathbb{N}$ , let  $\sigma^m$  be an  $\varepsilon^m$ -minimax regret equilibrium of  $G$ . Set  $A_i^0 := A_i$  and define recursively:

$$A_i^k := \{a_i \in A_i^{k-1} : \text{there is no } a'_i \in A_i^{k-1} \text{ such that } u_i(a'_i, a_{-i}) > u_i(a_i, a_{-i}) \forall a_{-i} \in A_{-i}^{k-1}\},$$

that is,  $A_i^k$  is the set of player  $i$ 's pure strategies that survives  $k$  rounds of iterated deletion of strictly dominated pure strategies. We want to show that there exists an  $M^*$  such that for all  $m > M^*$ , the support of  $\sigma^m$  is included in  $\bigcap_{k=0}^{\infty} A_i^k$ .

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<sup>13</sup>The proof can be found in the working paper version, Renou and Schlag (2008).

First, we show that the support of  $\sigma_i^m$  is in  $A_i^1$  for each player  $i \in N$ , for each  $m \in \mathbb{N}$ . By contradiction, suppose that  $a_i^*$  is in the support of  $\sigma_i^m$ , but  $a_i^* \notin A_i^1$  for some player  $i$ , for some  $m$ . This implies that there exists an action  $\bar{a}_i$  such that  $r_i(\bar{a}_i, a_{-i}) < r_i(a_i^*, a_{-i})$  for all  $a_{-i} \in A_{-i}$ . Consider the mixed strategy  $\bar{\sigma}_i$  with  $\bar{\sigma}_i(a_i^*) = 0$ ,  $\bar{\sigma}_i(\bar{a}_i) = \sigma_i^m(a_i^*) + \sigma_i^m(\bar{a}_i)$ , and  $\bar{\sigma}_i(a_i) = \sigma_i^m(a_i)$  for all  $a_i \neq a_i^*, \bar{a}_i$ . It follows that  $R_i(\bar{\sigma}_i, a_{-i}) < R_i(\sigma_i^m, a_{-i})$  for all  $a_{-i} \in A_{-i}$  since

$$R_i(\bar{\sigma}_i, a_{-i}) - R_i(\sigma_i^m, a_{-i}) = \sigma_i^m(a_i^*)(r_i(\bar{a}_i, a_{-i}) - r_i(a_i^*, a_{-i})).$$

Consequently,  $\sigma_i^m$  cannot be an  $\varepsilon^m$  minimax regret equilibrium, a contradiction. Hence, the support of  $\sigma_i^m$  is included in  $A_i^1$  for each player  $i \in N$ , for any  $m \in \mathbb{N}$ .

Second, we show that there exists an  $M^*$  such that the support of  $\sigma_i^m$  is included  $A_i^2$  for each player  $i \in N$ , for all  $m > M^*$ . By contradiction, suppose that for any  $M$ , there exists an action  $a_i^m$  in the support of  $\sigma_i^m$  with  $a_i^m \notin A_i^2$  for some player  $i$ , for some  $m > M$  (hence,  $A_i^2 \subset A_i^1$ ). Without loss of generality, assume it is for all  $m > M$ . This implies that there exists an action  $\bar{a}_i^m$  such that  $r_i(\bar{a}_i^m, a_{-i}) < r_i(a_i^m, a_{-i})$  for all  $a_{-i} \in A_{-i}^1$ , for all  $m > M$ . For each  $m > M$ , construct the mixed strategy  $\bar{\sigma}_i^m$  as  $\bar{\sigma}_i$  above.

Since  $\sigma^m$  is an  $\varepsilon^m$ -minimax regret equilibrium and the expected regret is bi-linear, we must have that:

$$(1 - \varepsilon^m)\sigma_i^m(a_i^m) \sum_{a_{-i}} (r_i(\bar{a}_i^m, a_{-i}) - r_i(a_i^m, a_{-i}))\sigma_{-i}^m(a_{-i}) + \varepsilon^m \sigma_i^m(a_i^m) [\max_{a_{-i}} r_i(\bar{a}_i^m, a_{-i}) - \max_{a_{-i}} r_i(a_i^m, a_{-i})] \geq 0.$$

By the preceding arguments, the first term is strictly negative since the support of  $\sigma_{-i}^m$  is in  $A_{-i}^1$ . As for the second term, it is bounded from above by:

$$K := \max_{(a'_i, a''_i) \in (A_i^1 \setminus A_i^2) \times A_i^2} [\max_{a_{-i}} r_i(a'_i, a_{-i}) - \max_{a_{-i}} r_i(a''_i, a_{-i})].$$

Note that  $K$  is well-defined by finiteness of the action spaces. Furthermore, there exists a  $\bar{M}$  such for all  $m > \bar{M}$ ,

$$(1 - \varepsilon^m)\sigma_i^m(a_i^m) \sum_{a_{-i}} (r_i(\bar{a}_i^m, a_{-i}) - r_i(a_i^m, a_{-i}))\sigma_{-i}^m(a_{-i}) + \varepsilon^m K < 0.$$

Consequently, there exists a  $M^{(2)}$  such that  $\sigma^m$  cannot simultaneously be an  $\varepsilon^m$ -minimax regret equilibrium for  $m > M^{(2)}$  and the support of  $\sigma_i^m$  not included in  $A_i^2$  for some player  $i$ . By induction, we can find such a  $M^{(k)}$  for any  $k > 2$ .

Lastly, since we consider finite strategic-form games, there exists a  $K$  such that the iterated deletion of strictly dominated actions stops after  $K$  rounds, and set  $M^*$  equal to the minimum of the  $M^{(k)}$  for  $k = 1, \dots, K$ .  $\square$

Proposition 2 thus states that for  $\varepsilon$  small enough, the pure actions in the support of an  $\varepsilon$ -minimax regret equilibrium survive iterated deletion of strictly dominated *pure* actions. Of course, for  $\varepsilon$  large enough, an  $\varepsilon$ -minimax regret equilibrium might not be rationalizable. This is not surprising as rationalizability relies on common knowledge in rationality, while minimax regret equilibrium does not even assume mutual knowledge in rationality.<sup>14</sup> Furthermore, we hasten to stress that an  $\varepsilon$ -minimax regret equilibrium might *not* survive iterated deletion of strictly dominated (mixed and pure) actions even for small  $\varepsilon > 0$ , as the following example illustrates.

	$a$	$b$	$c$
$a$	-1, 1	1, -1	4, -2
$b$	1, -1	-1, 1	0, -2

$G4$

Observe that the pure action  $c$  is strictly dominated for player 2. Deleting  $c$ , the game is a game of matching pennies with  $((1/2, 1/2)(1/2, 1/2))$  as the unique  $\varepsilon$ -minimax regret equilibrium for all  $\varepsilon$  (with a regret of 1 to each player). However, it is not an  $\varepsilon$ -minimax regret equilibrium of the game with action  $c$ . The reason is simple. If player 1 uniformly randomizes between  $a$  and  $b$ , 1's "Nature" maximizes player 1's regret by playing  $c$ , which gives a maximal regret of  $1 + 3\varepsilon > 1$ . However, if player 1 faces the mixed strategy  $(1 - \varepsilon)(1/2, 1/2, 0) + \varepsilon(0, 0, 1)$ ,  $a$  is the unique best-reply. The unique  $\varepsilon$ -minimax regret equilibrium is  $((2/3, 1/3)(1/2, 1/2, 0))$  for any  $\varepsilon > 0$ .

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<sup>14</sup>It is worth noting that the concept of conjectural equilibrium also shares this feature (Battigalli (1987)). Both concepts do not coincide, however. For an example, see game  $G2$  in the working paper.

The next proposition states that  $\varepsilon$ -minimax regret equilibria are  $\varepsilon'$ -Nash equilibria for  $\varepsilon'$  appropriately chosen. Intuitively, Theorem 1 implies that each player is best replying to a mixture between the mixed strategy of his opponents (with probability  $1-\varepsilon$ ) and a mixed strategy of his “Nature” (with probability  $\varepsilon$ ), and thus a player’s payoff cannot differ “too much” from his optimal payoff had he known the strategy of his opponents.

**Proposition 3** *Let  $\sigma^*$  be an  $\varepsilon$ -minimax regret equilibrium with  $\varepsilon < 1$ . There exists  $\gamma > 0$  such that  $\sigma^*$  is an  $\varepsilon'$ -Nash equilibrium with  $\varepsilon' \geq \gamma\varepsilon/(1-\varepsilon)$ .*

**Proof** Suppose that  $\sigma^*$  is an  $\varepsilon$ -minimax regret equilibrium (with correlated  $\varepsilon$ -contamination). If  $\varepsilon = 0$ , then  $\sigma^*$  is a Nash equilibrium, hence an  $\varepsilon'$ -Nash equilibrium with  $\varepsilon' = 0$ . Assume that  $\varepsilon > 0$ . Since  $\sigma^*$  is an  $\varepsilon$ -minimax regret equilibrium, we have

$$(1-\varepsilon)R_i(\sigma_i^*, \sigma_{-i}^*) + \varepsilon \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i^*, \hat{\sigma}_{-i}) \leq (1-\varepsilon)R_i(\sigma_i, \sigma_{-i}^*) + \varepsilon \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i, \hat{\sigma}_{-i}),$$

for all  $\sigma_i \in \Sigma_i$ , for all  $i \in N$ . This is equivalent to

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) + \frac{\varepsilon}{1-\varepsilon} \left( \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i^*, \hat{\sigma}_{-i}) - \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i, \hat{\sigma}_{-i}) \right),$$

for all  $\sigma_i \in \Sigma_i$ , for all  $i \in N$ . Consider the set  $S_i(\sigma^*) := \{\sigma_i \in \Sigma_i : u_i(\sigma_i^*, \sigma_{-i}^*) \leq u_i(\sigma_i, \sigma_{-i}^*)\}$  of player  $i$ ’ strategies that improve upon  $u_i(\sigma^*)$ . We have

$$\gamma_i(\sigma^*) := \min_{\hat{\sigma}_i \in S_i(\sigma^*)} \left( \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\hat{\sigma}_i, \hat{\sigma}_{-i}) - \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i^*, \hat{\sigma}_{-i}) \right) \leq 0,$$

since  $\sigma^*$  is an  $\varepsilon$ -minimax regret equilibrium,  $S_i(\sigma^*)$  is compact, and  $R_i$  is bi-continuous. Let  $\varepsilon' := \varepsilon \max_{i \in N} |\gamma_i(\sigma^*)|/(1-\varepsilon)$ . Then, we have for all  $i \in N$ , for all  $\sigma_i \in \Sigma_i$ ,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) - \varepsilon',$$

which is the desired result. Note that  $\varepsilon'$  depends on  $\sigma^*$ . To get a uniform bound, consider

$$\bar{\gamma}_i = \min_{\hat{\sigma}_i} \max_{\hat{\sigma}_{-i}} R_i(\hat{\sigma}_i, \hat{\sigma}_{-i}) - \max_{\hat{\sigma}_i} \max_{\hat{\sigma}_{-i}} R_i(\hat{\sigma}_i, \hat{\sigma}_{-i}),$$

and let  $\varepsilon' := \varepsilon \max_{i \in N} |\bar{\gamma}_i|/(1-\varepsilon)$ . With the product of independent  $\varepsilon$ -contaminations, the proof is similar and left to the reader.  $\square$

In an  $\varepsilon$ -minimax regret equilibrium, the parameter  $\varepsilon$  relates to the uncertainty about rationality and conjectures and, consequently, to the (expected) ex-post regret. In contrast, the parameter  $\varepsilon'$  relates to the ex-ante regret in an  $\varepsilon'$ -Nash equilibrium. Proposition 3 links these two parameters: we may therefore think of an  $\varepsilon$ -minimax regret equilibrium as an  $\varepsilon'$ -Nash equilibrium, in which players might not maximize their payoff even though they have correct conjectures about their opponents' strategies. Furthermore,  $\varepsilon'$  goes to zero as  $\varepsilon$  goes to zero, and  $\varepsilon'$  is monotone increasing in  $\varepsilon$ . As a consequence, the set of  $\varepsilon$ -minimax regret equilibria converges to a subset of Nash equilibria as  $\varepsilon$  goes to zero. The converse of Proposition 3 does not hold, however. For instance, in example *G2*, not all Nash equilibria, albeit  $\varepsilon'$ -equilibrium with  $\varepsilon' = 0$ , are  $\varepsilon$ -minimax regret equilibria, even with infinitesimally small uncertainty. However, any strict Nash equilibrium is an  $\varepsilon$ -minimax regret equilibrium for  $\varepsilon$  small enough.

**Proposition 4** *Let  $\sigma^*$  be a strict Nash equilibrium. There exists  $\varepsilon^* > 0$  such that  $\sigma^*$  is an  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon < \varepsilon^*$ .*

The proof of Proposition 4 is similar to the proof of Proposition 3 and left to the reader. Naturally, not all  $\varepsilon$ -minimax regret equilibria are strict Nash equilibria, even with infinitesimally small uncertainty.

Lastly, the concept of an  $\varepsilon$ -minimax regret equilibrium might be the basis for equilibrium refinement. From Proposition 3, letting  $\varepsilon$  going to zero makes it possible to select among Nash equilibria. Equilibrium selection is not our primary aim, but the following offers such an equilibrium selection along with a short discussion for the interested readers.

**Definition 3** *A profile of strategy  $\sigma^*$  is a regret-perfect equilibrium if there exist some sequences  $(\varepsilon_k)_{k \in \mathbb{N}}$  and  $(\sigma_k^*)_{k \in \mathbb{N}}$  such that : (i) for all  $k \in \mathbb{N}$ ,  $\varepsilon_k > 0$  and  $\lim_{k \rightarrow +\infty} \varepsilon_k = 0$ , (ii) for all  $k \in \mathbb{N}$ ,  $\sigma_k^*$  is an  $\varepsilon_k$ -minimax regret equilibrium, and (iii)  $\lim_{k \rightarrow +\infty} \sigma_k^* = \sigma^*$ .*

It follows from the existence of an  $\varepsilon$ -minimax regret equilibrium for each  $\varepsilon$  and the compactness of  $\Sigma$  that a regret-perfect equilibrium exists. Moreover, we have seen in example *G2* that this concept helps to select even among undominated Nash equilibria. From example *G3*, we also have that dominated

Nash equilibria might be regret-perfect equilibria. The reader might find this property rather unsatisfactory, and we would have agreed before working on this project. Indeed, if players *cautiously* believe that all their opponents' actions can be played, say because of trembles, then it is not optimal to play dominated actions. However, players are also excessively cautious, albeit differently, in our formulation. They believe that the worst possible trembles would materialize, i.e., the trembles that maximize a player regret. There is no clear justification for one form of cautiousness over another and, therefore, we do not feel troubled with this feature of a regret-perfect equilibrium. Hence, a regret-perfect equilibrium might be neither perfect nor proper. The converse also holds true. For instance, consider the game  $G6$ .

	$a$	$b$	$c$
$a$	1, 1	0, 0	-1, -2
$b$	0, 0	0, 0	0, -2
$c$	-2, -1	-2, 0	-2, -2

$G5$

The action profile  $(b, b)$  is a perfect and proper equilibrium of  $G6$  (van Damme (1991, p. 15)), but is not a regret-perfect equilibrium. Consequently, we have the following proposition.

**Proposition 5** *The set of regret-perfect equilibria is neither a subset nor a superset of the set of perfect (or proper) equilibria.*

## 4 An economic application: price competition

Consider a market in which  $n$  firms compete in prices to sell a homogeneous product. Each firm  $i$  posts a price  $p_i$  and the firm posting the lowest price wins the entire market. In the event of a tie, each firm charging the lowest price has an equal chance of serving the entire market. The market demand is unitary if the lowest price is smaller than 1, the monopoly price, and zero otherwise. Each firm marginal cost of production is  $c_i$ . The profit to firm

$i$  if it posts the price  $p_i$  and its competitors post the price  $p_{-i}$  is denoted  $u_i(p_i, p_{-i})$ .

A firm faces two sources of regret. First, firm  $i$ 's price might turn out to be lower than the lowest price of its competitors. Had firm  $i$  posted a slightly higher price, it would also have served the entire market and made a higher profit. Second, firm  $i$ 's price might turn out to be higher than the lowest price of its competitors, and the regret arises from not serving the market at all. The exposure to these two sources of regret has important economic implications, as we will see. Formally, the regret to firm  $i$  is:

$$r_i(p_i, p_{-i}) = (\min(\{(p_j)_{j \in N}\} \cup \{1\}) - c_i) - u_i(p_i, p_{-i}).$$

**Two firms and identical costs.** For simplicity, we start by considering the case of two firms and identical marginal costs, normalized to zero. We show that each firm charging a price  $p_i \in [\varepsilon, 1]$  according to the distribution

$$G(p_i) = \frac{1}{1 - \varepsilon}(1 - \varepsilon p_i^{-1})$$

constitutes an  $\varepsilon$ -minimax regret equilibrium. Let us conjecture that firm  $i$ 's regret is maximized at the monopoly price, i.e.,  $p = 1$ . Accordingly, with probability  $(1 - \varepsilon)$ , firm  $i$ 's competitor follows the pricing strategy  $G$  while, with probability  $\varepsilon$ , firm  $i$  faces an “irrational” competitor, and conjectures that it prices at the monopoly price. Firm  $i$ 's regret of posting any price  $p_i \in [\varepsilon, 1]$  is:

$$(1 - \varepsilon) \left( \int_{\varepsilon}^{p_i} p_j dG(p_j) + \int_{p_i}^1 (p_j - p_i) dG(p_j) \right) + \varepsilon(1 - p_i) = -\varepsilon \ln \varepsilon.$$

For almost any price  $p_i \in [\varepsilon, 1]$ , firm  $i$ 's regret is therefore constant and equal to  $-\varepsilon \ln \varepsilon$ .<sup>15</sup> Moreover, if firm  $i$  prices below  $\varepsilon$ , its regret is  $-\varepsilon \ln \varepsilon + \varepsilon - p_i$ , a non-profitable deviation. The intuition is simple: if firm  $i$  prices below  $\varepsilon$ , it is sure to serve the entire market. However, its exposure to potential regret is substantial: both the “rational” and “irrational” incarnations of its competitor might price all the way up to the monopoly price. Similarly, if firm

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<sup>15</sup>Due to the tie-breaking rule, there is a discontinuity at 1:  $i$ 's regret if it posts the monopoly price is  $-\varepsilon \ln \varepsilon + 0.5\varepsilon$ , which is higher than  $-\varepsilon \ln \varepsilon$ .

$i$  prices above 1, his regret is  $-\varepsilon \ln \varepsilon + \varepsilon$ , a non-profitable deviation. Consequently, the mixed strategy  $G$  satisfies all the requirements to be a mixed Nash equilibrium in the zero-sum game between firm  $i$  and  $i$ 's "Nature."

Let us now turn to our conjecture that firm  $i$ 's "Nature" maximizes  $i$ 's regret at the monopoly price, that is, we have to check that  $\hat{p} \mapsto R_i(G, \hat{p})$  is maximized at  $\hat{p} = 1$ . Clearly, firm  $i$ 's "Nature" will price neither strictly below  $\varepsilon$  nor strictly above 1. Firm  $i$ 's regret if "Nature" prices at  $\hat{p} \in [\varepsilon, 1]$  is

$$R_i(G, \hat{p}) = \hat{p} - \int_{\varepsilon}^{\hat{p}} p_i dG(p_i),$$

which is strictly convex in  $\hat{p}$ . So, all we have to check is that  $R_i(G, \varepsilon) \leq R_i(G, 1)$ , which is satisfied if

$$\varepsilon \leq 1 - \int_{\varepsilon}^1 p \frac{\varepsilon}{1 - \varepsilon} \frac{1}{p^2} dp = 1 + \frac{\varepsilon}{1 - \varepsilon} \ln \varepsilon.$$

This inequality holds if  $\varepsilon < 0.39423$ . From Theorem 1, it follows that for small values of  $\varepsilon$ , posting prices according to the randomized strategy  $G$  in an  $\varepsilon$ -minimax regret equilibrium.

In equilibrium, the expected profit to each firm is strictly positive and each firm prices strictly above the marginal cost with probability one. The average price is  $-(\varepsilon \ln \varepsilon) / (1 - \varepsilon)$ . For instance, when  $\varepsilon = 0.1$ , the average price is about 0.25. While the distribution  $G$  converges to  $\delta_{\{p \geq 0\}}$  in distribution as  $\varepsilon$  goes to zero, the marginal increase of the average price at  $\varepsilon = 0$  is equal to  $+\infty$ . Only small uncertainty regarding the strategy of one's competitor causes a dramatic increase in prices. Our predictions not only differ with the Bertrand-Nash predictions, but also with the "maxmin" predictions. Indeed, the worst a firm can face is that its competitor prices at the marginal cost, hence  $(0, 0)$  is the unique  $\varepsilon$ -maximin equilibrium for any  $\varepsilon$ . We can equivalently show that the Nash equilibrium  $(0, 0)$  is not an  $\varepsilon$ -minimax regret equilibrium for any  $\varepsilon > 0$ . Lastly, the  $\varepsilon$ -minimax regret equilibrium is almost surely unique, i.e., the pricing strategies are unique up to how prices are set on a set of measure 0.<sup>16</sup>

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<sup>16</sup>The proof can be found in the working paper version.

**Two firms and different marginal costs.** We now assume that the two firms have different marginal costs  $c_1$  and  $c_2$  with  $0 \leq c_1 < c_2 < 1$ . Firm 1 is the most efficient firm. For small  $\varepsilon$ , we can expect the efficient firm to undercut the inefficient firm, thus focusing on the event that its conjecture is correct (with probability  $1 - \varepsilon$ ). Consequently, let us conjecture that firm 1 prices according to the distribution  $G_1$  on  $[a, b]$ , firm 2 prices according to  $G_2$  on  $[a, 1]$ , and  $G_2$  first-order stochastically dominates  $G_1$ . What might maximize a firm's regret? Since we expect firm 1 to try to undercut firm 2, we can conjecture that firm 1's regret is maximized at the monopoly price (the highest price consistent with a positive demand). Thus, we expect firm 1 to face the distribution  $F_2 = \delta_{\{p_2: p_2 \geq 1\}}$  with probability  $\varepsilon$ . Regarding firm 2, the inefficient firm, we expect it to be mostly concerned with foregone profits when, with probability  $\varepsilon$ , its conjecture about firm 1 is incorrect. Since foregone profits are higher, the higher firm 1's price, we expect firm 2's regret to be maximized when it faces the distribution  $F_1$  with support  $[b, 1]$ . Let us show that we can indeed construct such an  $\varepsilon$ -minimax regret equilibrium.

First, we consider the indifference conditions. To be indifferent between almost all prices in the support of  $G_i$ , the following equality for firm  $i$  has to be satisfied

$$(p - c_i) (1 - (1 - \varepsilon) G_{-i}(p) - \varepsilon F_{-i}(p)) = a - c_i, \quad (7)$$

since  $G_{-i}(a) = F_{-i}(a) = 0$ . Moreover, since  $F_{-i}(b) = 0$ , we obtain that

$$G_{-i}(p) = \frac{1}{1 - \varepsilon} \left( 1 - \frac{a - c_i}{p - c_i} \right),$$

for all  $p \in [a, b]$ . Let us now focus on firm 1. Since we conjecture an equilibrium with  $G_1(b) = 1$ , we have

$$\frac{1}{1 - \varepsilon} \left( 1 - \frac{a - c_2}{b - c_2} \right) = 1,$$

and, consequently, the parameters  $a$  and  $b$  satisfy  $a = c_2 + \varepsilon(b - c_2)$ . Furthermore, since  $G_1(p) = 1$  for all  $p \geq b$ , firm 2's indifference condition (7) implies

$$F_1(p) = 1 - \frac{a - c_2}{\varepsilon(p - c_2)},$$

for (almost) all  $p \in [b, 1]$ . Let us now turn to firm 2.

For any  $p \in [b, 1]$ , the support of  $F_1$ , the indifference condition for 2's "Nature" (i.e., the "irrational" incarnation of firm 1) implies that the regret of firm 2 when facing price  $p$  is constant, that is,

$$R_2(G_2, p) = (p - c_2) - \int_a^p (x - c_2) g_2(x) dx$$

is constant in  $p$ .<sup>17</sup> Hence,  $g_2(p) = 1/(p - c_2)$  for all  $p \in [b, 1]$ , and a simple integration gives

$$G_2(p) = \frac{1}{1 - \varepsilon} \left( 1 - \frac{a - c_1}{b - c_1} \right) + \ln \frac{p - c_2}{b - c_2},$$

for all  $p \in [b, 1]$ . In particular, we need that  $G_2(1) = 1$  which implies that

$$a = c_1 + \left( 1 - (1 - \varepsilon) \left( 1 - \ln \frac{1 - c_2}{b - c_2} \right) \right) (b - c_1).$$

Together with the above expression for  $a$ , we obtain

$$c_2 - c_1 = (b - c_1) \ln \frac{1 - c_2}{b - c_2}. \quad (8)$$

Note that the parameter  $b$  does not depend on  $\varepsilon$ . Clearly, the above equation has a solution with  $b \geq c_2$ . Furthermore, as  $\varepsilon$  goes to zero, the parameter  $a$  goes to  $c_2$ , firm 1 prices at  $c_2$  and firm 2 prices according to the distribution  $\bar{G}_2(p) = 1 - \frac{c_2 - c_1}{p - c_1}$  for  $p \in [c_2, b]$ . By construction, firm 1 facing  $\bar{G}_2$  is indifferent over all prices in  $[c_2, b]$  and, therefore, these limit strategies constitute a mixed Nash equilibrium of the Bertrand game.

Second, we have to verify that no player has an incentive to deviate. Nature replacing firm 2 must be maximizing firm 1's regret at the monopoly price  $p = 1$  (since we assume  $F_2 = \delta_{\{p_2: p_2 \geq 1\}}$ ). As in the case with homogeneous cost, the regret is strictly convex in  $p$ , and a necessary and sufficient condition for  $R_1(G_1, a) \leq R_1(G_1, 1)$  is given by:

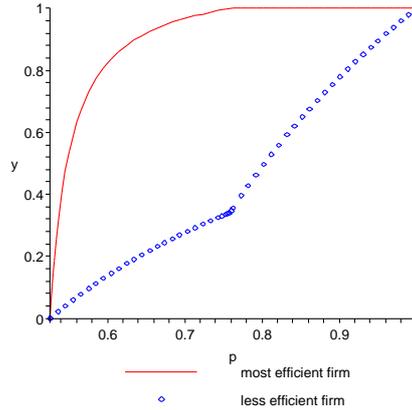
$$(1 - c_1) - \frac{a - c_2}{1 - \varepsilon} \int_a^b \frac{p - c_1}{(p - c_2)^2} dp \geq a - c_1. \quad (9)$$

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<sup>17</sup>Since  $R_2$  is the payoff function of 2's "Nature" in the zero-sum game between firm 2 and 2's "Nature".

Finally, it is easy to check that firm 1 has no incentive to price either above  $b$  or below  $a$ . Similarly, for firm 2 and 2's "Nature". Thus, if Eq. (9) holds,  $(G_1, G_2)$  is an  $\varepsilon$ -minimax regret equilibrium.

To illustrate our findings, let us consider a numerical example. Assume that  $c_1 = 0$ ,  $c_2 = 0.5$  and  $\varepsilon = 0.1$ . We have that  $a = 0.52$ ,  $b = 0.76$ , and the pricing policies are illustrated in the figure below.



**Several firms and identical costs.** We now consider the case of  $n \geq 3$  firms in order to illustrate, within an economic example, the differences between the two formulations of belief sets: correlated contaminations or product of independent contaminations. Costs are identical and normalized to zero. We first investigate the situation where the belief sets are product of independent contaminations. Each firm believes that each opponent independently chooses according to the distribution  $G$  with probability  $1 - \varepsilon$  and is uncertain about the opponents' play, otherwise. Let us look for an  $\varepsilon$ -minimax regret equilibrium in which all firms price according to the distribution  $G$  on  $[\varepsilon^{n-1}, 1]$  and each respective adversarial Nature maximizes regret at the monopoly price.

Let  $G_\varepsilon = (1 - \varepsilon)G + \varepsilon\delta_{\{p \geq 1\}}$ . For all  $p \in [\varepsilon^{n-1}, 1)$ , the profit to firm  $i$  is  $p(1 - G_\varepsilon(p))^{n-1}$  and together with the fact that it goes to  $\varepsilon^{n-1}$  as  $p$  goes to one, it follows that the distribution  $G$  is given by:<sup>18</sup>

$$G(p) = \frac{1}{1 - \varepsilon} (1 - \varepsilon p^{-1/(n-1)}).$$

<sup>18</sup>Let  $\tilde{G}_\varepsilon$  be the distribution of the order statistics  $\min(p_{-i})$ . The regret to firm  $i$  of

Denote  $g$  the density of  $G$ . Now, we show that each adversarial Nature indeed maximizes regret at the monopoly price. For this, we derive firm  $i$ 's regret by considering it as a sum of independent events. Consider the event in which  $n - m$  firms out of  $n - 1$  choose according to  $G$  while an adversarial Nature charges the prices of the remaining  $m - 1$  firms. This event occurs with probability  $(1 - \varepsilon)^{n-m} \varepsilon^{m-1} \binom{n-1}{n-m}$ . Let  $q$  be the lowest price charged by Nature. Conditional on this event, firm  $i$ 's regret is

$$\int_0^q pg(p)(n-m)(1-G(p))^{n-m-1} dp + q(1-G(q))^{n-m} - \int_0^q p(1-G(p))^{n-m} g(p) dp.$$

Differentiating this expression with respect to  $q$ , we obtain after simplifications:

$$(1-G(q))^{n-m}(1-qq(q)).$$

Lastly, note that

$$1-qq(q) = 1 - \frac{\varepsilon}{(1-\varepsilon)(n-1)} q^{-\frac{1}{n-1}},$$

and is increasing in  $q$ . Moreover, at  $q = \varepsilon^{n-1}$ , the above expression is equal to  $1 - \frac{1}{(1-\varepsilon)(n-1)}$ , which is strictly positive if  $\varepsilon < 1 - 1/(n-1)$ . Therefore, for  $\varepsilon < 1 - 1/(n-1)$ , the derivative of the regret with respect to  $q$  is strictly positive (except at the point  $p = 1$ ), hence the regret is maximized at the monopoly price.

The expected price is equal to  $\varepsilon(1-\varepsilon^{n-2})/[(1-\varepsilon)(n-2)]$  and is increasing in the degree of confidences about opponents' conjectures,  $\varepsilon$ , as in the case with two firms. However, unlike the two-firm case, the marginal increase in the expected price at  $\varepsilon = 0$  is now finite (equal to  $1/(n-2)$ ). Furthermore, the expected price is decreasing in  $n$  i.e., the more intense the competition,

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posting the price  $p_i$  is

$$\int_{\varepsilon^{n-1}}^{p_i} \min(p_{-i}) d\tilde{G}_\varepsilon(\min(p_{-i})) + \int_{p_i}^1 (\min(p_{-i}) - p_i) d\tilde{G}_\varepsilon(\min(p_{-i})),$$

and together with the boundary conditions give the distribution  $G$ .

the lower the expected price. As  $n \rightarrow +\infty$ , the  $\varepsilon$ -minimax regret equilibrium converges to the competitive equilibrium.

Let us now turn to the situation where a firm expects with probability  $1 - \varepsilon$  that *all* other firms independently price according to  $G$ , and is completely uncertain otherwise, that is, we consider the case of “correlated” contaminations, our preferred formulation. In that situation, adversarial Natures have a larger impact on the pricing policy of firms. Following the same logic as before, we have the following. Firms price according to the pricing distribution  $G$  with:

$$G(p) = \begin{cases} 1 - \left(\frac{\varepsilon(1-p)}{(1-\varepsilon)p}\right)^{1/(n-1)} & \text{for } p \in [\varepsilon, b] \\ 1 - \left(\frac{\varepsilon(1-b)}{(1-\varepsilon)b}\right)^{1/(n-1)} + \ln \frac{p}{b} & \text{for } p \in [b, 1], \end{cases}$$

with  $b$  satisfying the equality

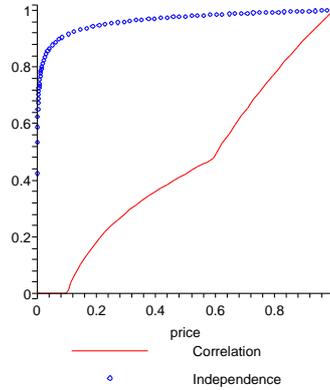
$$-\ln b = \left(\frac{\varepsilon(1-b)}{(1-\varepsilon)b}\right)^{1/(n-1)}.$$

Such a  $b$  exists for  $\varepsilon < 1/e$  where  $b \in (1/e, 1)$ . For completeness, let us give the strategy (distribution)  $F$  followed by each adversarial Nature:

$$F(p) = 1 - \frac{1}{p} + \frac{1-\varepsilon}{\varepsilon} \left( \left(\frac{\varepsilon(1-b)}{(1-\varepsilon)b}\right)^{1/(n-1)} - \ln \frac{p}{b} \right)^{n-1}$$

for  $p \in [b, 1]$ . As in the preceding cases, the reader can check that this is indeed an  $\varepsilon$ -minimax regret equilibrium for  $\varepsilon$  small enough i.e.,  $\varepsilon < (n-2)/(n-1)$ .

A simple numerical example helps to illustrate the differences between both formulations. With 5 firms and  $\varepsilon = 0.1$ , the pricing policies in the case of independent and correlated contaminations are represented in the graph below ( $b = 0.59$ ).



Both policies are strikingly different. With independent contaminations, a firm is mainly concerned with facing “rational” competitors who price close to the marginal cost; the likelihood to face only “irrational” firms is extremely small ( $10^{-4}$ ). Consequently, firms price very close to the marginal cost (see the curve in dots). In contrast, with correlated contaminations, the likelihood to face only “irrational” firms is disproportionately higher  $10^{-1}$  and, therefore, firms are more concerned with the possibility of foregoing opportunities. Their pricing policy reflects this concern and, accordingly, prices are more likely to be substantially above marginal costs.

To summarize, all our model share three distinctive features: 1) price dispersion, 2) firms make positive expected profit, and 3) firms price above marginal costs. Moreover, as the number of firms increases, the average price decreases (converging towards perfect competition), but price dispersion persists. Furthermore, with several firms and correlated contaminations, the pricing policy exhibits a kink at  $b$ , which is close to the monopoly price for  $\varepsilon$  small enough.

All these findings agree remarkably well with experimental and field data. For instance, using experimental and field data, Baye and Morgan (2004) document price dispersion, positive profit and pricing above marginal costs in price-setting environments. Furthermore, using data from experiments, Bayes and Morgan note the average price is decreasing in the number of firms. Moreover, the price distributions exhibit kinks at prices close to the

monopoly price.<sup>19</sup> All these qualitative observations are consistent with our theoretical findings. A thorough statistical analysis awaits future research.

## 5 Concluding Remarks

In this paper, we have proposed a solution concept, *minimax regret equilibrium*, where players are uncertain about the conjectures of their opponents and rationality. A parametric variant of our solution concept, an  $\varepsilon$ -minimax regret equilibrium, where uncertainty is modeled as  $\varepsilon$ -contaminations, is particularly appealing and intuitive. In an  $\varepsilon$ -minimax regret equilibrium, players have, with probability  $1 - \varepsilon$ , correct conjectures about their opponents' play. This specification greatly simplifies the computation of equilibria and most importantly for applications, one can study how the equilibrium predictions change as the degree of uncertainty varies. For instance, in the model of price competition with two identical firms, we show that even the slightest shadow of doubt about the behavior of a competitor creates a dramatic increase in equilibrium prices. Moreover, our prediction is essentially unique. Relaxing the assumptions of common knowledge in conjectures and mutual knowledge in rationality does not necessarily imply a loss of predictive power. We therefore believe that relaxing these assumptions might prove particularly important in explaining economic and social phenomena. Although further research is still required, our model explains price dispersion in price-setting environments, which is qualitatively consistent with empirical and experimental observations (Baye and Morgan (2004)).

In games of incomplete information, another source of strategic uncertainty is the private information about types. It is not difficult to see that the concept of minimax regret equilibrium readily extends to this class of games. We present such an extension in the working paper version (see also Hyafil and Boutilier (2004) and Hayashi (2007a)).<sup>20</sup>

Finally, we like to mention two issues, which we believe deserves further

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<sup>19</sup>Note the quantal response equilibrium studied in Baye and Morgan is inconsistent with the presence of kinks.

<sup>20</sup>Note, however, that if the type-space is a singleton, the concept of Hyafil and Boutilier (2004) reduces to the concept of Nash equilibrium (since there is no strategic uncertainty).

research. First, it would be nice to extend the concept of minimax regret equilibrium to extensive-form games. Hayashi (2007b) seems to be a good starting point. Second, we like to have a theory of learning of minimax regret equilibrium.

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