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ENVIRONMENTAL QUALITY, LIFE EXPECTANCY, AND SUSTAINABLE ECONOMIC GROWTH

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Environmental Quality, Life Expectancy, and Sustainable Economic Growth[*]

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Abstract

I construct a model of a growing economy with pollution. The analysis of the model shows that the interactions between capital accumulation, endogenous longevity and environmental quality determine both the long-run growth rate of the economy and the pattern of convergence (i.e., monotonic or cyclical) towards the balanced growth path. I argue that such interactions can provide a possible explanatory factor behind the, empirically observed, negative correlation of long-run growth with its short-term cycles. Furthermore, the model may capture the observed pattern whereby economic growth and mortality rates appear to be negatively related in the long-run, but positively related in the short-run.

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1 Introduction

Following the renewed interest on issues related to growing economies, during the late 1980s, some economists initiated a strand of literature in which elements of environmental quality were incorporated into otherwise standard models of economic growth (e.g., John and Peccherino, 1994; Bovenberg and Smulders, 1996; Smulders and Gradus, 1996; Stokey, 1998; Hartman and Kwon, 2005). These analyses addressed various issues such as the (economic/ecological) sustainability of balanced output

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growth, the impact of pollution abatement policies, the joint dynamics of the physical capital and pollution stocks etc.

Intuitive reasoning and actual data support the idea that any interactions between the quality of the environment and economic growth are by-directional. Apart from the obvious negative impact of aggregate economic activity on environmental quality, there are equally important positive effects flowing from the natural environment to the economy as a whole. A prominent candidate for such positive repercussions is related to the beneficial aspect of environmental quality for the health status of the wider population. People exposed to environments which are contaminated and eroded by various pollutants (e.g., chemicals, toxins, smoke, radioactive substances and litter) face a profoundly adverse impact to their overall health characteristics.[1] Quantitatively, this impact appears to be nothing less than staggering: Pimentel *et al.* (1998) estimate that, each year, roughly 40% of deaths worldwide can be attributed to factors related with environmental degradation. For these reasons, an improved environment may entail economic benefits – benefits that take the form of higher labour productivity and the promotion of capital formation due to the increase of the availability of funds derived from economy-wide saving. The latter aspect, in particular, is related to the idea that an increase in life expectancy (due to better health characteristics) reduces the effective rate of time preference and stimulates a person's motive to postpone consumption for later stages of his/her lifetime.

In addition to the apparent implications for the trend of output growth, these considerations may also direct our attention to issues related to the pattern of an economy's convergence towards the long-run growth equilibrium. The reason why such transitional dynamics may prove to be of considerable interest can be clarified with the following argument. Environmental quality is beneficial for growth which, however, results in more pollution that, subsequently, mitigates the quality of the environment endowed to future generations. The lower environmental quality will impede economic activity and cause a decrease of pollutant emissions that, subsequently, may allow nature to bestow an improved environment to future generations. If such interactions are strong enough, then convergence to the long-run equilibrium can be non-monotonic – in the sense that such linkages may be crucial for the emergence of cyclical growth during the transition.

[1] See, for example, Koshal (1976), Holget *et al.* (1999) and Grigg (2004).

These ideas motivate the current analysis. I construct an overlapping generations economy in which individuals face the probability of dying prematurely – an outcome that inhibits their incentive for saving when young. This probability is a decreasing function of environmental quality, which goes through certain dynamic adjustments over time. In particular, it degrades as a result of pollution (a side-effect of aggregate economic activity) and it improves with the existing stock of environmental quality (indicating that an improved environment is better equipped in absorbing the negative repercussions of persistent pollution). Although economic activity has a negative externality on environmental quality, this adverse effect is mitigated by the government's provision of abatement capital which reduces the negative effects of pollution. A crucial feature of the model's equilibrium is that the existing level of environmental quality generates both positive and negative effects on its future prospects. The former are related to the process of regeneration which is inherent in the natural environment. The latter emerge because an improved environment supports higher longevity. As a result, it induces more saving, it promotes capital accumulation and enhances output growth temporarily. Higher growth, however, is responsible for the emission of more pollutant substances which undermine environmental prospects.

Depending on which effect of the current on the future environmental stock is stronger near the steady state – something that, ceteris paribus, depends on how 'dirty' is the output production technology operated in the economy – the transitional dynamics indicate that the economy may experience either a monotonic or an oscillatory convergence towards the sustainable balanced growth path. In addition, the 'dirtiness' of the technology is an inhibiting factor for the long-run growth rate of output since it contributes to a lower steady state level for environmental quality, which is associated with lower life expectancy (and lower aggregate saving) in equilibrium. As a result, if emission rates surpass a certain threshold, the economy experiences a cyclical convergence towards a relatively low growth rate, as opposed to the case where pollutant emissions are below the threshold and the economy experiences a smooth (i.e., monotonic) transition towards a higher growth rate. One upshot of this argument is that pollution can provide an explanatory factor for the negative correlation between cycles and growth.[2] Another implication of this set-up is that it captures the scenario whereby –

[2] In terms of the linkages between economic and environmental phenomena, the possibility of endogenous fluctuations has been raised before by Zhang (1999) and Seegmuller and Verchère (2004). Both analyses find that endogenous cycles of period two may emerge if emission rates are sufficiently high. The main difference of my analysis is that I explicitly consider the (well-documented and significant) effects of environmental quality on life expectancy. Another difference is that I abstract from the possibility of limit

despite being positively correlated in the long-run – longevity and economic activity may actually be negatively related in the short-run.

The rest of the paper is organised as follows: In Section 2, I outline the fundamentals of the economy. Section 3 describes the economy's temporary and dynamic equilibrium and Section 4 derives the balanced growth path. In Section 5, I present the economy's transitional dynamics towards the sustainable balanced growth path. Section 6 discusses the model's implications for the correlation between (short-term) cycles and (long-term) growth as well as some possible policy implications. In Section 7, I conclude.

2 The Economy

Time is discrete and indexed by $t = 0,1,...\infty$. I consider an artificial economy which produces a single consumable commodity. The economy is inhabited by an infinitely lived government and a population of agents that belong to overlapping generations and face a potential lifetime of two periods. The two periods of a person's lifespan are 'youth' and 'old age'. For simplicity, I normalise the population of young individuals to unity. An individuals' lifespan is uncertain as she may die before reaching her old age. The probability of premature death is a decreasing function of environmental quality (i.e., the cleanliness of air, soil and water, the availability of natural resources such as forestry and other forms of plantation etc.) – an idea that manifests the beneficial impact of environmental quality on the health status of the population. The quality of the environment is inhibited as a consequence of pollution which is a by-product of aggregate economic activity. The government levies taxes from firms in order to finance the formation and provision of public abatement capital – a policy that preserves the quality of the environment and allows the sustainability of the economy's balanced growth.

2.1 Firms

Output is produced by perfectly competitive firms who combine capital, denoted k_t, and labour, denoted l_t, to produce y_t units of goods according to

$$y_t = Ak_t^a(h_t l_t)^{1-a}, \quad A > 0, \; a \in (0,1). \tag{1}$$

cycles; instead, I allow the economy to settle down, eventually, to its balanced growth path. As a result, I derive implications for the correlation between long-term growth and 'short-term' growth cycles.

The variable h_t is an economy-wide indicator of labour productivity. To guarantee an existence of an equilibrium with positive long-run growth, I assume that labour productivity is proportional to the economy's aggregate stock of capital, $\overline{k}_t$, according to

$$h_t = v\overline{k}_t, \quad v > 0. \tag{2}$$

This assumption follows Frankel (1962) and Romer (1986), and captures the idea of a learning-by-doing externality through which the investment process by firms advances their stock of knowledge which, subsequently, spreads over the whole economy in the manner of a public good.

The government imposes a marginal tax rate $\tau \in (0,1)$ on output production. As a result, firms will have net revenues of $(1-\tau)y_t$. Profit maximisation by firms requires that the per unit costs of productive inputs are equal to their respective marginal products. Denoting the payments to capital and labour by R_t and w_t respectively, the above arguments imply that

$$R_t = (1-\tau)aAk_t^{a-1}(h_t l_t)^{1-a} = (1-\tau)a\frac{y_t}{k_t}, \tag{3}$$

and

$$w_t = (1-\tau)(1-a)Ak_t^a(h_t l_t)^{-a}h_t = (1-\tau)(1-a)\frac{y_t}{l_t}. \tag{4}$$

I shall now turn my attention to the description of the underlying dynamics for environmental quality.

2.2 The Quality of the Environment

I treat the quality of the environment, denoted e_t, as a renewable resource that takes values on the interval $[0, E]$ and evolves according to

$$e_{t+1} = f(e_t) - D_{t+1}, \tag{5}$$

with $0 < f' < 1$ and $f'' \leq 0$. I also assume $f(0) > 0$, which guarantees the existence of a non-negative solution for environmental quality, and $f(E) = E$, which implies that, in the absence of environmental degradation, captured by the variable D_{t+1}, the steady state level for environmental quality would be at its maximum.[3]

[3] The use of these assumptions on the description of environmental dynamics is widespread in the literature of economic growth with environmental issues. See Bovenberg and Smulders (1996) and Jouvet *et al.* (2005) among others.

The main reason why the environment may degrade over time emerges from the various pollutants that are generated by aggregate economic activity. I denote pollutant emissions by P_{t+1} and I assume that one unit of output produced generates $p > 0$ units of pollution. Therefore, for an economy that produces y_{t+1} units of output, the degree of unabated pollution is

$$P_{t+1} = p y_{t+1} . \tag{6}$$

Following Harrington *et al.* (2005), I assume that the government can reduce the adverse impact of economic activity on the environment by utilising its revenues from taxation in order to mitigate pollution through the provision of abatement capital, denoted z_{t+1}. One may think that 'abatement' capital includes recycling facilities, wastewater management facilities, installation and operation of renewable energy techniques that prevent the emission of greenhouse gases and toxic pollutants (e.g., wind turbines, hydroelectric plants and solar photovoltaics) etc. The formation of public capital takes place according to

$$z_{t+1} = \tau y_t , \tag{7}$$

and it is assumed that, in the initial period of activity, the economy is endowed with abatement capital $z_0 > 0$.

Pollutant emissions and abatement capital determine the ultimate extent of environmental degradation due to pollution. Their impact is captured by the function

$$D_{t+1} = \tilde{D}(P_{t+1}, z_{t+1}) , \tag{8}$$

where $\tilde{D}_P(\cdot) > 0$, $\tilde{D}_{PP}(\cdot) \leq 0$, $\tilde{D}_z(\cdot) < 0$ and $\tilde{D}_{zz}(\cdot) > 0$. A specification that captures these assumptions and allows the possibility of a sustainable long-run growth rate, while maintaining analytical solutions, is one for which the function $\tilde{D}(\cdot)$ is homogeneous of degree zero. In particular, I consider the functional form

$$D_{t+1} = P_{t+1}^{\delta} z_{t+1}^{-\delta} , \quad \delta \in (0,1] . \tag{9}$$

For the remaining analysis, I shall restrict my attention to the simplifying scenario whereby $\delta = 1$ (Harrington *et al.*, 2005). In addition, I shall utilise a specific functional form for $f(e_t)$ according to which

$$f(e_t) = (1-\eta)E + \eta e_t , \quad \eta \in (0,1) . \tag{10}$$

Some discussion on the choice of this functional form is necessary here. In general, this specification considers the term $(1-\eta) \in (0,1)$ as an indicator of the environment's capacity to absorb pollution. If $\eta = 0$, the absorption capacity is perfect and (prior to any productive activity taking place at the beginning of a period) the economy is endowed with the maximum level of environmental quality, because nature has absorbed any negative impact of pollution from the preceding period. If $\eta = 1$, the absorption capacity is non-existent and (at the beginning of each period) environmental quality is just the one endowed from the preceding period. The case where $0 < \eta < 1$ is an intermediate scenario whereby the environment possesses some absorption capacity, albeit an imperfect one.

Further clarification of these arguments is possible if we explicitly consider the dynamics of the pollution stock. For the sake of the argument, suppose that the stock of pollution, denoted π_t, evolves according to $\pi_t = \eta\pi_{t-1} + D_t$ (with D_t being the flow of pollution) and that environmental quality is given by $e_t = E - \pi_t$. Naturally, the scenario with $\eta \neq 0$ illustrates the idea that pollution is persistent due to nature's imperfect absorption capacity. Using $e_t = E - \pi_t$ in the dynamics of the pollution stock yields $e_t = (1-\eta)E + \eta e_{t-1} - D_t$, which corresponds to the dynamics of environmental quality described by (5) and (10).

Of course, this analysis indicates that, during the initial period of activity $t = 0$, the corresponding level of environmental quality (e_0) is partially determined by the initial stocks of physical (k_0) and abatement (z_0) capital. This is because the economy does not begin with a positive pollution stock. Only after any activity takes place does the stock of pollution begins its evolution. Therefore, at $t = 0$, it is $\pi_{-1} = 0$ and $e_0 = E - \pi_0 = E - D_0$ or (after using the equilibrium conditions $l_0 = 1$ and $k_0 = \overline{k}_0$)

$$e_0 = E - \frac{pAv^{1-a}k_0}{z_0} \equiv \breve{e}(k_0, z_0). \tag{11}$$

Alternatively, one may think that, prior to any activity taking place at the very beginning of its existence, the economy is endowed with the maximum degree of environmental quality, i.e., $e_{-1} = E$. Obviously, equation (11) indicates that we need to restrict attention to scenarios where ecological capacity and initial conditions do not violate

Condition 1. *Given (11),* $\breve{e}(k_0, z_0) > 0$ *holds.*

This requirement makes sense: production taking place at $t = 0$ must not exhaust more than the nature's total available resources that determine environmental quality.

Substitution of equations (6)-(10) and $\delta = 1$ in (5) allows the explicit derivation of the dynamics of environmental quality as

$$e_{t+1} = (1-\eta)E + \eta e_t - \frac{p}{\tau} g_{t+1}, \tag{12}$$

where $g_{t+1} = y_{t+1} / y_t$ is the temporary growth rate of output. As long as the emission rate is positive (i.e., $p > 0$), the growth rate of output impedes the process of regeneration for environmental quality and does not allow it to settle at its maximum level. Current pollution is proportional to current production while abatement capital is formed by revenues (in the form of output) levied through taxation in the previous period. As a result, the greater is current production relative to past production (that is, the greater is the temporary growth rate of output), the greater is the extent of environmental degradation as well.

2.3 Consumers

Each period, a unit mass of young consumers comes into existence. A young consumer is endowed with one unit of labour which she supplies inelastically to firms in exchange for the market wage w_t. This represents her only source of income during her lifetime because, when old, she does not have any endowment of labour units. For this reason, if she desires to consume in the second period of her life, she needs to consume only a fraction of her income when young and save the remaining amount for retirement. I assume that a young worker will survive towards old age with probability $\theta_t \in [0,1)$, whereas with probability $1-\theta_t$ she dies prematurely and cannot consume when old. This is a source of uncertainty that will obviously impinge on her optimal saving decisions, as shall become clear later.

Longevity, which is captured by the probability of survival, is an increasing function of environmental quality, e_t, according to

$$\theta_t = \Theta(e_t), \tag{13}$$

where $\Theta' > 0$, $\Theta'' < 0$, $\Theta(0) = 0$ and $\Theta(E) = \tilde{\theta} < 1$. In addition, $\Theta'(0) = \widehat{\vartheta} < \infty$ and $\Theta'(E) = \underset{\smile}{\vartheta}$.[4] These assumptions capture the notion that a cleaner and more prosperous environment is a promoting factor for the health status and, therefore, the life

[4] The restriction $\Theta(0) = 0$ is not essential and the results are qualitatively identical even with $\Theta(0) > 0$, as long as $\Theta(0)$ is not sufficiently high. Otherwise, the non-negativity of e_{t+1} will be undermined.

expectancy of the wider population.[5] Evidence in support of this idea abounds. In many developing countries, drinking water is contaminated from untreated household and industrial wastes which may cause infectious diseases like cholera and diarrhoea. Chemicals, sulphur oxides and carbon oxides released into the air, mainly as a result of industrial activity, can provide a prominent cause of various diseases (e.g., those affecting the human respiratory system). Soil pollutants have a direct impact on the food chain through which they can inhibit the health status of many people through food poisoning, malnutrition and other (potentially terminal) diseases generated from the absorption of toxins and chemicals.

If, on the one hand, the mortality shock is favourable, i.e., with probability θ_t, the young person survives and is able to consume in both periods. Consequently her ex post utility is given by $(1-\chi)\ln c_t^t + \chi \ln c_{t+1}^t$, where c_i^j denotes consumption at period i of an agent born at period j, and $\chi \in (0,1)$ is the psychological weight on the utility derived from future consumption. If, on the other hand, the mortality shock is unfavourable, i.e., with probability $1-\theta_t$, the person passes away prematurely and her ex post utility is given by $(1-\chi)\ln c_t^t$. Consequently, an agent's ex ante (i.e., expected) lifetime utility is given by

$$u_t = (1-\chi)\ln c_t^t + \theta_t \chi \ln c_{t+1}^t . \tag{14}$$

A young consumer will maximise her expected lifetime utility, subject to the constraints for consumption during youth and old age. Denoting saving by s_t, these constraints are given by $c_t^t = w_t - s_t$ and $c_{t+1}^t = r_{t+1} s_t$ respectively. The variable r_{t+1} is the gross return that financial intermediaries provide on saving. I discuss the operational activities of financial intermediaries in the subsequent part of the paper.

2.4 Financial Intermediaries

Financial intermediaries accept deposits by young consumers and transform these funds into capital which they rent to firms at a cost of R_{t+1} per unit. They are perfectly competitive and provide a gross rate of return r_{t+1} to their depositors.[6]

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[5] Nevertheless, other exogenous factors (e.g., accidents) may still cause untimely death, that is why $\Theta(E) < 1$.

[6] At $t = 0$, the initial endowment of capital, k_0, belongs to the initial old generation who provides it directly to firms.

As a means of resolving the issue of saving decisions under uncertain lifetimes, I follow Chakraborty (2004) and appeal to the idea that financial intermediaries represent mutual funds that accept deposits in return for an annuity. Specifically, the mutual fund promises to provide retirement income, $r_{t+1}s_t$, contingent on the depositor's survival to old age. Otherwise, the income of those who die is equally shared among surviving members of the mutual fund.[7]

Given the above, there are two conditions describing the equilibrium in the financial market. The first one relates to the flow of funds and, in particular, is described by the equality between aggregate saving and aggregate investment. That is

$$k_{t+1} = s_t . \tag{15}$$

The second condition relates to the fact that financial intermediaries operate under perfect competition when they channel capital from households to firms. Therefore, these intermediaries derive zero economic profits from their activities. Equivalently, the costs per unit of funds deposited must be equal to the revenues per unit of funds provided in the form of capital. Combined with the idea that the financial market offers annuities contingent on the depositor's survival, the above imply that[8]

$$\theta_t r_{t+1} = \mathrm{R}_{t+1} . \tag{16}$$

With these considerations, I have completed the description of the fundamental characteristics of the economy. I now turn to the analysis of its equilibrium.

3 Equilibrium

The economy's fundamentals can be utilised for the derivation of its temporary equilibrium. I describe this through

Definition 1. *The temporary equilibrium of the economy is a set of quantities* $\left\{c_t^{t-1}, c_t^t, c_{t+1}^t, s_t, l_t, y_t, e_t, \theta_t, h_t, k_t, k_{t+1}, z_{t+1}\right\}$ *and prices* $\left\{w_t, \mathrm{R}_t, \mathrm{R}_{t+1}, r_{t+1}\right\}$ *such that:*

(i) *Given w_t, θ_t and r_{t+1}, the quantities c_t^t, c_{t+1}^t and s_t solve the optimisation problem of an agent born at time t;*

[7] The assumption of perfect annuity markets is made for analytical convenience. An alternative scenario would be to consider such markets as absent, in which case accidental bequests could accrue to the young as a result of their parents' untimely death. With a constant survival probability, such an assumption would not have caused any analytical inconvenience. Nevertheless, with time varying survival probability (as in the present analysis), the analytical complication would be insurmountable and clear-cut solutions impossible.

[8] I assume that the use of capital in production results in full depreciation of its (productive) value.

(ii) *Given* w_t *and* R_t*, firms choose quantities for* l_t *and* k_t *to maximise profits;*

(iii) *The labour market clears, i.e.,* $l_t = 1$*;*

(iv) *The goods market clears, i.e.,* $y_t = c_t^t + \theta_{t-1} c_t^{t-1} + k_{t+1} + z_{t+1}$*;*

(v) *The financial market clears, i.e.,* $k_{t+1} = s_t$ *and* $\theta_t r_{t+1} = \mathrm{R}_{t+1}$*;*

(vi) *The government's budget is balanced, i.e.,* $z_{t+1} = \tau y_t$*.*

The optimisation problem of a young person requires $\partial u_t / \partial s_t = 0$. This leads to a solution for saving given by

$$s_t = \frac{\chi \theta_t}{1 - \chi + \chi \theta_t} w_t . \tag{17}$$

Equation (17) indicates that the agent's saving constitutes a fraction of her labour income. As expected, the saving rate is increasing in the probability of survival. Had survival been certain (i.e., if $\theta_t = 1$), the agent would have saved a fraction equal to the weight she assigns to the utility accrued from second period consumption. However, the possibility of premature death induces the agent to devote a lower amount for retirement income and increase her consumption during youth. This is because a low θ_t reduces the incremental utility benefit of consuming when old. To restore the equilibrium, individuals must reduce the incremental utility cost of postponing consumption while young – effectively, they can achieve this by increasing the level of consumption during their youth. Obviously, the variation in saving behaviour in response to variations in life expectancy, apparent in (17), captures this idea.[9]

Using the equilibrium condition $l_t = 1$ and substituting (4) and (17) in (15) yields

$$k_{t+1} = \frac{(1-\tau)(1-a)\chi\theta_t}{1 - \chi + \chi\theta_t} y_t . \tag{18}$$

The aggregate investment rate varies with θ_t indicating how life expectancy affects the availability of funds through saving behaviour.

[9] The results are consistent with the economy-wide resource constraint. To see this, recall that, towards the end of a period T, only θ_T young agents will survive to maturity. With this in mind, use the per-period budget constraints and equations (7), (15) and (16) to write $\theta_{t-1} c_t^{t-1} + c_t^t + k_{t+1} + z_{t+1} = \theta_{t-1} r_t s_{t-1} + w_t - s_t + k_{t+1} + \tau y_t = \mathrm{R}_t k_t + w_t + \tau y_t$. Labour market clearing requires $l_t = 1$, therefore $\mathrm{R}_t k_t + w_t + \tau y_t = \mathrm{R}_t k_t + w_t l_t + \tau y_t$. Using (3) and (4) we have $\mathrm{R}_t k_t + w_t l_t + \tau y_t = a(1-\tau) y_t + (1-a)(1-\tau) y_t + \tau y_t = y_t$.

Now, substitute (2) in (1), use $l_t = 1$ and $\overline{k}_t = k_t$, and write the resulting expression in terms of $t+1$. It yields

$$y_{t+1} = Av^{1-a}k_{t+1}. \tag{19}$$

Substituting (13) and (18) in (19) and, subsequently, dividing both sides by y_t yields the temporary growth rate

$$g_{t+1} = \frac{y_{t+1}}{y_t} = \gamma\psi(e_t), \tag{20}$$

where $\gamma = Av^{1-a}(1-\tau)(1-a)$ and $\psi(e_t) = \chi\Theta(e_t)/[1-\chi+\chi\Theta(e_t)]$. Finally, we can substitute (19) in equation (12) to write the dynamics of environmental quality as[10]

$$e_{t+1} = (1-\eta)E + \eta e_t - \frac{p\gamma\psi(e_t)}{\tau} = \Phi(e_t), \tag{21}$$

where $\Phi(0) = (1-\eta)E > 0$, since $\Theta(0) = \psi(0) = 0$, $\Phi'(e_t) = \eta - \frac{p\gamma\psi'(e_t)}{\tau} \gtrless 0$ and $\Phi''(e_t) = -\frac{p\gamma\psi''(e_t)}{\tau} > 0$ since $\psi''(e_t) < 0$.

The dynamics expressed in equations (20) and (21) depict the inter-temporal behaviour of the economy. This may be formally described as

Definition 2. *Given $k_0, z_0 > 0$ and Condition 1, the dynamic equilibrium is a sequence of temporary equilibria that satisfy*

(i) $\quad g_{t+1} = \frac{y_{t+1}}{y_t} = \gamma\psi(e_t)$;

(ii) $\quad e_{t+1} = \Phi(e_t)$.

The dynamics of growth and environmental quality determine the transitional behaviour of the economy towards its long-run (steady state) equilibrium. Although the production function can be reduced to an 'AK' type (as it has become apparent from (19)), the economy does not settle to its long-run equilibrium automatically, as in other models of this sort. Instead, it displays some transitional dynamics towards the balanced growth path. The reason, of course, lies on the fact that environmental quality, which affects the growth rate through its implications for life expectancy and saving behaviour,

[10] Formally, (21) should be $e_{t+1} = \max[\Phi(e_t), 0]$. Later, however, I shall impose a restriction (see Footnote 11) that guarantees $\Phi(e_t) > 0 \quad \forall e_t \in [0, E]$.

undergoes a gradual adjustment towards its steady state level – an adjustment traced from the dynamics of equation (21). The derivation of this steady state equilibrium is the issue to which I now turn, while the analysis of the transitional behaviour of the economy shall be considered in a subsequent part of the paper.

4 The Sustainable Balanced Growth Path

A description of the steady state is provided by

Definition 3. *The steady state equilibrium is a sustainable balanced growth path in which output, capital intensity and consumption grow at a net rate* $\hat{g}-1$ *while environmental quality obtains a stationary level* $\hat{e} \in (0, E)$.

For now, I shall focus my attention to the steady state with the purpose of deriving the outcomes that transpire in the long-run. The equilibrium is obtained via

Proposition 1. *If there is a stationary solution for environmental quality then this solution is unique. Consequently, there exists a unique sustainable balanced growth path.*

Proof. A stationary solution for environmental quality is one for which $e_{t+1} = e_t = e \ \forall t$. In terms of equation (21), we need to find an interior $\hat{e} \in (0, E)$ such that $\hat{e} = \Phi(\hat{e})$. Substituting $e_{t+1} = e_t = e$ in (21) and rearranging yields $M(e) = E$ where $M(e) = e + \dfrac{p\gamma\psi(e)}{(1-\eta)\tau}$. Obviously, $\Theta(0) = 0$ implies $\psi(0) = \chi\Theta(0)/[1-\chi+\chi\Theta(0)] = 0$ therefore $M(0) = 0$, while $M(E) = E + \dfrac{p\gamma\chi\tilde{\theta}}{(1-\eta)\tau(1-\chi+\chi\tilde{\theta})} > E$. In addition, $M' = 1 + \dfrac{p\gamma\psi'}{(1-\eta)\tau} > 0$ because $\psi' = (1-\chi)\chi\Theta'/[1-\chi+\chi\Theta(\cdot)]^2 > 0$ given that Θ' is positive by assumption. As a result, there exists a unique $\hat{e} \in (0, E)$ satisfying $M(\hat{e}) = E \Leftrightarrow \hat{e} = \Phi(\hat{e})$ and leading to a unique long-run (gross) growth rate $\hat{g} = \gamma\psi(\hat{e})$. ■

Naturally, the steady state outcomes for the growth rate of output and environmental quality depend on different realisations for the economy's structural parameters. For subsequent purposes, it shall prove constructive to identify how pollutant emissions, captured by the ratio of pollution per unit of output produced (i.e., the emission rate p) affects the equilibrium solutions. To this purpose, a useful result takes the form of

Proposition 2. *A higher emission rate (i.e., greater p) results in a sustainable balanced growth path with lower environmental quality and lower output growth.*

Proof. In the steady state we have $M(\hat{e}) = E$. Revisiting the Proof of Proposition 1, we can see that $dM(\cdot)/dp > 0$. Given that $M(\cdot)$ is monotonically increasing in $\hat{e}$ as well, following an increase in p the equilibrium can be restored only at a lower value for environmental quality. Given that the steady state growth rate, $\hat{g}$, is also monotonically increasing in $\hat{e}$, a greater value for p will have an inhibiting effect on output growth. ■

Essentially, Proposition 2 implies that

$$\hat{e} = \varepsilon(p)\,;\ \hat{g} = \gamma\psi[\varepsilon(p)] = G(p)\,, \tag{22}$$

such that $\varepsilon' < 0$, therefore $G' = \partial\hat{g}/\partial p = \gamma\psi'\varepsilon' < 0$. This is a quite intuitive result: more pollution, for given levels of output, implies greater environmental degradation. The latter has an adverse impact on the health status of the population and causes a reduction in life expectancy – effectively, reducing aggregate saving and, therefore, aggregate investment. As a result, the inhibiting effect on the process of capital accumulation leads to a reduction of output growth in the long-run.

The notion of the steady state is meaningful – i.e., it can facilitate our understanding on how alterations in the economy's structure may affect its equilibrium outcomes – as long as we can establish that such equilibrium is stable. In terms of this model, we can ensure that the balanced growth path is a meaningful equilibrium notion once we guarantee the stability of the solution derived from the dynamics of environmental quality, as they are described in equation (21). Consequently, given (22), the balanced growth path (represented by $\hat{g}$) will be sustainable.

Since equation (21) represents a non-linear, first-order difference equation, the stability of the solution $\hat{e}$ is guaranteed as long as $|\Phi'(\hat{e})| < 1$ holds. With this in mind, I impose

Condition 2. *$p < \overline{p}$, where $\overline{p}$ is defined from* $\dfrac{\overline{p}\gamma\chi(1-\chi)\Theta'[\varepsilon(\overline{p})]}{\tau\{1-\chi+\chi\Theta[\varepsilon(\overline{p})]\}^2} = 1+\eta$.

Now we can derive

Lemma 1. *Given Condition 2, the steady state solution* $\hat{e}$ *is stable. Therefore, for* $k_0, z_0 > 0$ *and Condition 1,* $e_\infty = \hat{e}$ *and* $g_\infty = \hat{g}$.

Proof. See the Appendix. ■

The argument from Lemma 1 can be clarified by revisiting equation (21) and using a Taylor series approximation to linearise it around the steady state. That is

$$e_{t+1} = \Phi(\hat{e}) + \Phi'(\hat{e})(e_t - \hat{e}) . \tag{23}$$

Next, substitute $\hat{e} = \Phi(\hat{e})$ and denote $\Phi'(\hat{e}) = \beta$ in (23) to get

$$e_{t+1} = \hat{e} + \beta(e_t - \hat{e}) = \hat{e}(1-\beta) + \beta e_t . \tag{24}$$

Substituting recursively in (24) yields

$$e_t = \hat{e} + \beta^t (e_0 - \hat{e}) . \tag{25}$$

Finally, substitution of (25) in (20) yields

$$g_{t+1} = \gamma\psi[\hat{e} + \beta^t (e_0 - \hat{e})] . \tag{26}$$

Given Lemma 1, it is $|\beta| < 1$, hence $\lim_{t\to\infty} \beta^t = 0$. Applying this result is (25) yields

$$\lim_{t\to\infty} e_t = e_\infty = \lim_{t\to\infty} [\hat{e} + \beta^t (e_0 - \hat{e})] = \hat{e} .$$

Similarly, application of the above in (26) leads to

$$\lim_{t\to\infty} g_{t+1} = g_\infty = \lim_{t\to\infty} \{\gamma\psi[\hat{e} + \beta^t (e_0 - \hat{e})]\} = \gamma\psi(\hat{e}) = \hat{g} .$$

With Lemma 1, I guaranteed the stability of the balanced growth path.[11] In the next Section, I consider the economy's transitional dynamics and pattern of convergence towards the long-run equilibrium.

5 Transitional Dynamics

As I indicated in a preceding part of the paper, the economy's settlement towards its balanced growth path is not immediate, despite the fact that the production function is linear in the aggregate stock of capital. Insofar as the model incorporates stock variables

[11] I also use Condition 2 to impose $(1-\eta)E + \eta e_t - \frac{\overline{p}\gamma\psi(e_t)}{\tau} > 0 \Leftrightarrow \Phi(e_t) > 0 \ \forall e_t \in [0, E]$.

evolving in different sectors of the economy – and affecting each other's evolution in a by-directional manner – the settlement to the balanced growth path requires a gradual adjustment over time.

Furthermore, the pattern of convergence may not be straightforward as well. Instead, it may depend on the economy's structural parameters. In particular, there are parameter configurations for which the transition towards the sustainable balanced growth path can eventually become either monotonic or cyclical. Such possibilities are summarised in

Lemma 2. *Consider some* p^* *such that* $\frac{p^*\gamma\chi(1-\chi)\Theta'[\varepsilon(p^*)]}{\tau\{1-\chi+\chi\Theta[\varepsilon(p^*)]\}^2}=\eta$. *Then if* $p<p^*$ *the convergence towards the balanced growth path becomes (eventually) monotonic whereas if* $\overline{p}>p>p^*$ *the convergence towards the balanced growth path becomes (eventually) cyclical.*

Proof. See the Appendix. ■

There are two conflicting effects of the current level of environmental quality on the future one. On the one hand, there is a beneficial effect resulting from the natural process of environmental regeneration. On the other hand, the existing level of environmental quality promotes current growth because it raises life expectancy – an effect that, ultimately, impedes the future level of environmental quality because it exacerbates the extent of pollution emerging from economic activity.

When $p<p^*$, the former effect dominates around the steady state. Eventually, the economy will experience a monotonic convergence towards the balanced growth path (i.e., $\hat{e}$ and $\hat{g}=\gamma\psi(\hat{e})$) – a transition during which the quality of the environment and the growth rate of output will be either declining or increasing monotonically over time. When $p>p^*$, the latter effect dominates around the steady state. The dynamic transition towards the balanced growth path becomes more complex because improved environmental quality implies higher growth, which, subsequently, causes a deterioration of environmental quality. The latter implies lower growth (due to high mortality) as a result of which economic activity generates fewer pollutant emissions – an effect that improves environmental quality, and so on. The dynamic transition towards steady state growth will eventually become oscillatory (i.e., cyclical).[12]

[12] Cipriani and Makris (2007) find that the presence of endogenous longevity, coupled with intergenerational transfers, may be crucial for the emergence of local indeterminacies that indicate the

Note that, even though growth and longevity are always positively related in the long-run, there are scenarios in which the model generates a negative correlation between life expectancy and output growth in the short-run – mainly, scenarios whereby environmental phenomena lead to oscillatory patterns for economic activity (for which the growth rate of output is a strong proxy) and life expectancy. Although this possibility is in contrast to existing theoretical analyses (e.g., Chakraborty, 2004), in which growth and (endogenous) longevity are positively related both in the short- and in the long-run, it is actually supported by existing empirical studies. Tapia Granados (2005) provides evidence on short-term oscillations in mortality rates that are significantly correlated with fluctuations in economic activity – with mortality declining more strongly during recessions and, some times, increasing during expansions. Chay and Greenstone (2003) present evidence and argue that the positive correlation between the phase of economic activity and mortality is significantly related to the fact that recessions are associated with reductions in pollutant emissions that, subsequently, lead to an improvement for the prospects of infant survival due to better environmental conditions.[13]

Returning to the analysis of the model, the local behaviour (i.e., for a neighbourhood close enough to $\hat{e}$) of the economy towards its balanced growth path is easily traced from equations (25) and (26). When $0 < \Phi'(\hat{e}) < 1$, then for e_0 close enough to $\hat{e}$, environmental quality and, consequently, output growth will monotonically increase or decrease over time depending on whether (given Condition 1) e_0 is below or above $\hat{e}$ respectively. When $-1 < \Phi'(\hat{e}) < 0$, then for e_0 close enough to $\hat{e}$, output growth will display a pattern of alternating values above and below $\hat{g}$ over time – following, of course, the movements of e_t above and below $\hat{e}$ as time progresses.

presence of endogenously driven business cycles. However, they do not consider environmental variables and how these may be important determinants of life expectancy.

[13] There is also evidence (e.g., Mayer, 1999) to support the view that some environment-related variables may display oscillatory patterns over time.

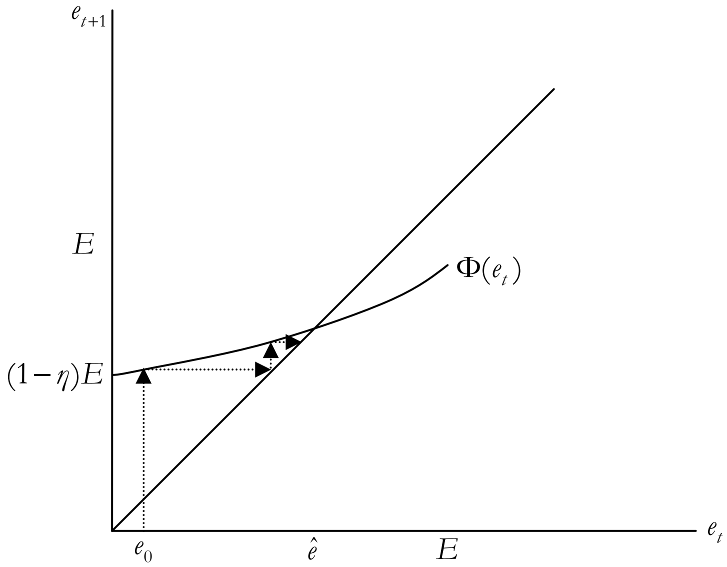

Figure 1: $0 < \Phi'(\hat{e}) < 1$

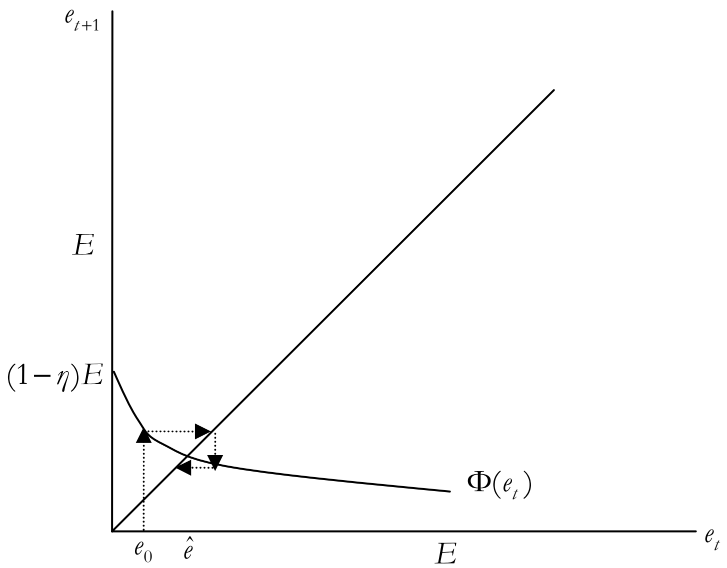

Figure 2: $-1 < \Phi'(\hat{e}) < 0$

Naturally, the above arguments apply at a 'global' level as well. Notice that the shape of the convex function $\Phi(\cdot)$ is determined by the relative strength of the parameter p. In particular, for p low (high) enough, the function $\Phi(\cdot)$ can be monotonically increasing (decreasing) – cases that are illustrated in Figures 1 and 2. For intermediate values of p, the function $\Phi(\cdot)$ displays a U-shaped graph. In Figure 3 we have $\Phi'(\hat{e}) \in (0,1)$ and convergence towards $\hat{e}$ becomes eventually monotonic while for $\Phi'(\hat{e}) \in (-1,0)$

convergence will eventually take place through oscillations (of reduced magnitude) around $\hat{e}$ - a scenario depicted in Figure 4.

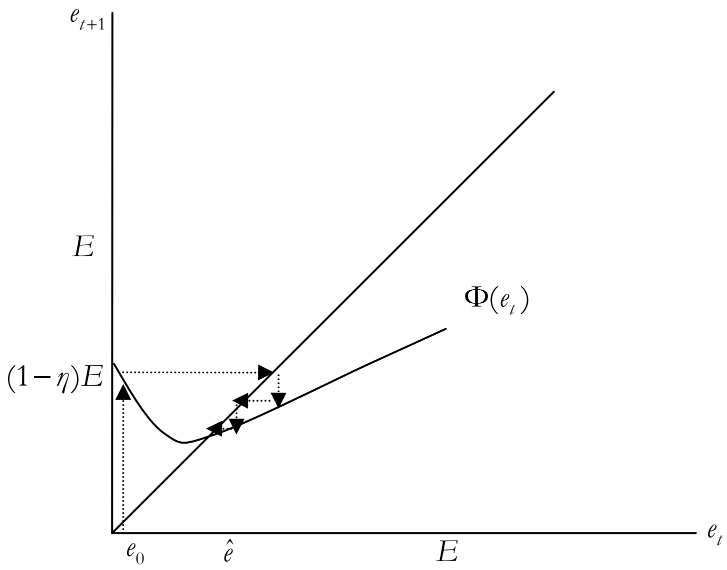

Figure 3: $0 < \Phi'(\hat{e}) < 1$

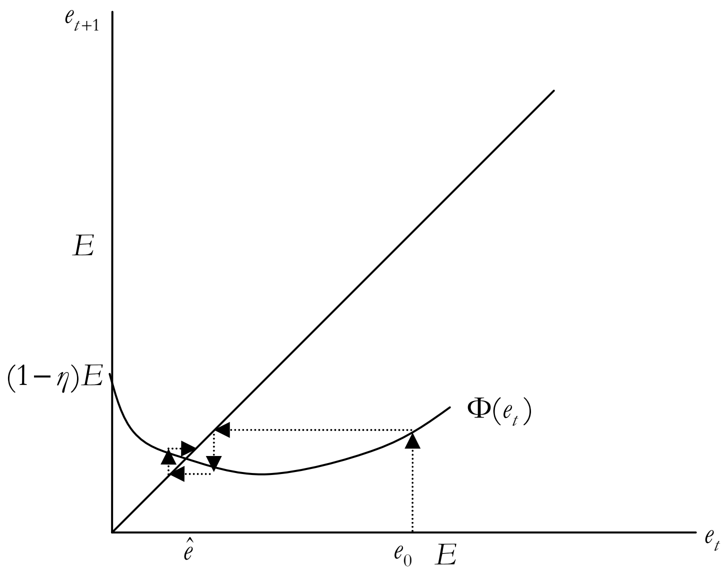

Figure 4: $-1 < \Phi'(\hat{e}) < 0$

6 Long-Run Growth and Convergence Patterns

So far, the analysis of the model has shown that the extent of pollutant emissions, captured by the parameter p, can provide an important determinant on whether the economy converges to its balanced growth path smoothly or whether it experiences growth cycles during the transition. In addition, a previous part of the analysis established that p impinges on the equilibrium growth rate of output, as pollution degrades environmental quality, reduces longevity and hurts the process of capital accumulation by inhibiting the aggregate saving behaviour of agents.

The aforementioned ideas bring forth another important facet of the interactions between the natural environment and economic activity. This can be summarised in

Proposition 3. *Consider two economies,* A *and* B*, which are otherwise identical apart from their emission rates. Specifically, suppose that the only difference in the structure of these economies derives from* $p^{\mathrm{A}} < p^* < p^{\mathrm{B}}$ *. Then, for* $k_0, z_0 > 0$ *and Condition 1, economy* A *will experience a monotonic convergence towards a growth rate* $\hat{g}^{\mathrm{A}}$ *whereas economy* B *will experience a cyclical convergence towards a lower growth rate* $\hat{g}^{\mathrm{B}} < \hat{g}^{\mathrm{A}}$ *.*

Proof. It follows directly from the results established in Proposition 2 and Lemma 2. ■

The upshot from Proposition 3 can be clarified with the help of the following

Corollary. *The pollutant emission rate can be an important determinant of the negative correlation between (long-run) output growth and its (short-run) cycles.*

According to this model, economies with technologies that emit pollutants above a certain threshold, will display a non-monotonic (oscillatory) transition to the long-run equilibrium and achieve a lower trend in terms of output growth. This idea provides a possible new dimension to an issue that has preoccupied researchers for many years – that is, the issue of the correlation between the trend of output growth and its cyclical volatility. Empirically, there exist a variety of analyses, the majority of which tend to conclude that, on average, growth rates are inversely correlated with proxies of their cycles (e.g., Ramey and Ramey, 1995; Turnovsky and Chattopadhyay, 2003). A relatively recent line of theoretical research, explores the analytics of this issue by employing stochastic endogenous growth models in which random shocks impinge on growth rates in a non-linear manner – meaning that mean-preserving spreads cause alterations in trend growth (e.g., Canton, 2002). In this respect, all these models provide a clear message concerning the causality of the relationship, since they predict that the degree of cyclical volatility causes a change in the growth rate of output.

This model provides a different view which is conceptually closer in spirit with the literature examining the emergence of endogenously-driven growth cycles (e.g., Matsuyama, 1999). In this analysis there are no exogenous shocks causing fluctuations in major variables; rather, it is the structure of the economy that determines both its equilibrium growth rate and the possibility that, during the transition, temporary growth rates may behave cyclically. Thus, long-run growth and its 'short-term' behaviour of alternating values above and below the steady state are natural economic phenomena that are inherently linked and driven by fundamentals – in this case, the interactions between capital accumulation, environmental quality and endogenous longevity: an economy that converges to its long-run equilibrium through oscillations, reaches a relatively low growth rate compared with an economy whose transition is smoother. [14]

Of course, specific attention has to be directed to the fact that this model abstracts from the possibility of limit cycles. Here, deterministic growth cycles – that is, oscillations through which growth takes alternating values above and below its trend – decline over time until the economy settles down to its long-run equilibrium. Nevertheless, given recent empirical evidence, this is not necessarily an undesirable feature: indeed, there is ample empirical support (e.g., Sensier and van Dijk, 2004; Stock and Watson, 2005) to suggest that many industrialised economies experience a reduction in their aggregate volatility – the observation that has been commonly labelled as 'the great moderation'.

6.1 Some Policy Implications

The economic framework presented in this paper, can provide clear policy implications – particularly, implications concerning the economic effects of environmental policies (e.g., pollution abatement). We can trace such effects by altering the government's policy instrument, i.e., the tax rate τ .

Clearly, a higher tax rate has both positive (due to improvements in environmental quality) and negative (due to the crowding-out impact on private investment) effects on growth. Effectively, there is a Laffer-type relationship between long-term growth and the marginal tax rate. It is straightforward to check that, ceteris paribus, an increase in the

[14] Of course, the labelling 'short-term' attached to cyclical volatility should be put into the context of what represents a period within a discrete overlapping generations setting. Thus, in terms of duration, the cycle here represent something of a Kondratiev-type wave.

marginal tax rate is more likely to be conducive for economic growth, the higher are the structural parameters determining the longevity factor for given levels of environmental quality. Therefore, pollution abatement policies may benefit the growth rate of output as long as the health sector of the economy is relatively advanced.

Naturally, there are implications for stabilisation policies in any framework that links long-term growth with short-term cycles. In terms of this model, the government could use its policy instrument to increase the critical emission rate above which the economy displays growth oscillations during the transition. Given the preceding arguments, however, such a policy would have implications for the equilibrium growth rate: depending on whether the tax rate required to eradicate growth oscillations is below or above the tax rate that maximises growth, then this type of 'stabilisation' (accruing from the government's policy of pollution abatement) would either increase or decrease the equilibrium growth rate in the long-run.

7 Concluding Remarks

The majority of existing theories tend to focus on the impact of capital accumulation and economic growth on environmental degradation, while eluding any possible feedback that the quality of the environment may entail for saving, factor accumulation, productivity and growth. This is despite the well-documented, and quantitatively significant, impact of pollution and environmental degradation on human health and life expectancy – aspects that may indicate how environmental factors may impinge on the economic behaviour and actions of the population.

In the preceding analysis I have sought to fill this gap and consider the by-directional nature of the environment-growth nexus, within an analytically tractable model of sustainable growth. On the one hand, output growth generates pollution and hurts the environment; on the other hand, environmental quality supports longevity and, as a result, promotes saving behaviour and capital accumulation. The results suggest that the linkages between factor accumulation, environmental quality and (endogenous) life expectancy have implications for both the pattern of an economy's convergence towards its balanced growth equilibrium and the economy's growth rate of output itself. In particular, if technologies emit pollutants above a certain critical rate, then the economy experiences growth cycles of declining magnitude until it settles to a balanced growth rate in the long run – a growth rate which is low, however, relative to that of an economy whose emission rates are below the critical level and experiences a smooth (i.e., monotonic) transition towards its long-run equilibrium. This is exactly the point that seems to suggest an intuitive explanation behind the empirically observed, negative correlation between growth rates and their cycles. Furthermore, the emergence of oscillatory patterns for the model's major variables (i.e., the growth rate of output, the quality of the environment, and the rate of mortality) allows the model to identify the possibility that – although inversely related in the long-run – economic growth and mortality rates may actually be positively related in the short-run. This distinct correlation of the two phenomena over the short-term and the long-term, finds support from existing empirical evidence.

Of course, the need to keep the analysis tractable and tightly focused means that the present framework abstracts from some important issues which should provide additional and important insights on the implications of the growth/development process for the quality of the environment and its sustainability. Regardless of this, however, the model's tight focus on specific issues on the growth/environmental quality nexus allows it to benefit from analytical solutions that provide clarity of both intuition and of all the mechanisms involved. As a result, it is able to reproduce outcomes that relate and account for 'real world' observations while, at the same time, providing some possible and intuitive explanations for their occurrence.

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Appendix

Proof of Lemma 1. Given (21), it is

$$\Phi'(e_t) = \eta - \frac{p\gamma\psi'(e_t)}{\tau} = \eta - \frac{p\gamma}{\tau}\frac{\chi(1-\chi)\Theta'(e_t)}{[1-\chi+\chi\Theta(e_t)]^2},$$

and evaluating at $e_t = \hat{e}$

$$\Phi'(\hat{e}) = \eta - \frac{p\gamma}{\tau}\frac{\chi(1-\chi)\Theta'(\hat{e})}{[1-\chi+\chi\Theta(\hat{e})]^2}.$$

Stability requires $|\Phi'(\hat{e})| < 1 \Leftrightarrow -1 < \Phi'(\hat{e}) < 1$. Obviously, $\Phi'(\hat{e}) < 1$ is satisfied because $\eta \in (0,1)$ by assumption. It remains to show that $\Phi'(\hat{e}) > -1$. Now, consider the expression

$$\frac{p\gamma\chi(1-\chi)\Theta'[\varepsilon(p)]}{\tau\{1-\chi+\chi\Theta[\varepsilon(p)]\}^2},$$

which is increasing in p because $\varepsilon' < 0$, $\Theta' > 0$ and $\Theta'' < 0$. Given that there is some $\overline{p}$ for which

$$\frac{\overline{p}\gamma\chi(1-\chi)\Theta'[\varepsilon(\overline{p})]}{\tau\{1-\chi+\chi\Theta[\varepsilon(\overline{p})]\}^2} = 1+\eta,$$

then for $p < \overline{p}$, which is true by virtue of Condition 2, the above expression takes the form of the inequality

$$\frac{p\gamma\chi(1-\chi)\Theta'[\varepsilon(p)]}{\tau\{1-\chi+\chi\Theta[\varepsilon(p)]\}^2} < 1+\eta.$$

Substituting (22) and rearranging, the inequality is expressed as

$$\eta - \frac{p\gamma}{\tau}\frac{\chi(1-\chi)\Theta'(\hat{e})}{[1-\chi+\chi\Theta(\hat{e})]^2} > -1 \Rightarrow$$

$$\eta - \frac{p\gamma\psi'(\hat{e})}{\tau} > -1 \Rightarrow$$

$$\Phi'(\hat{e}) > -1,$$

which completes the Proof. ■

Proof of Lemma 2. Similarly to the Proof for Lemma 1, I begin with the expression

$$\frac{p\gamma\chi(1-\chi)\Theta'[\varepsilon(p)]}{\tau\{1-\chi+\chi\Theta[\varepsilon(p)]\}^2},$$

which is increasing in p. Since there is some p^* for which

$$\frac{p^*\gamma\chi(1-\chi)\Theta'[\varepsilon(p^*)]}{\tau\{1-\chi+\chi\Theta[\varepsilon(p^*)]\}^2} = \eta,$$

a first conclusion is that $p^* < \overline{p}$ by Lemma 1. Now, for $p < p^*$, the above expression takes the form of the inequality

$$\frac{p\gamma\chi(1-\chi)\Theta'[\varepsilon(p)]}{\tau\{1-\chi+\chi\Theta[\varepsilon(p)]\}^2} < \eta,$$

which, after substitution of (22), can be written as

$$\eta - \frac{p\gamma}{\tau}\frac{\chi(1-\chi)\Theta'(\hat{e})}{[1-\chi+\chi\Theta(\hat{e})]^2} > 0 \Rightarrow$$

$$\eta - \frac{p\gamma\psi'(\hat{e})}{\tau} > 0 \Rightarrow$$

$$1 > \Phi'(\hat{e}) > 0,$$

since $\eta \in (0,1)$. Therefore, when $p < p^*$, convergence towards the balanced growth path will become monotonic. Next, consider the case for which $p > p^*$. Then

$$\frac{p\gamma\chi(1-\chi)\Theta'[\varepsilon(p)]}{\tau\{1-\chi+\chi\Theta[\varepsilon(p)]\}^2} > \eta \Rightarrow$$

$$\eta - \frac{p\gamma}{\tau}\frac{\chi(1-\chi)\Theta'(\hat{e})}{[1-\chi+\chi\Theta(\hat{e})]^2} < 0 \Rightarrow$$

$$\eta - \frac{p\gamma\psi'(\hat{e})}{\tau} < 0 \Rightarrow$$

$$-1 < \Phi'(\hat{e}) < 0,$$

by Lemma 1 and $p^* < \bar{p}$. Thus, when $p > p^*$, convergence towards the balanced growth path will become cyclical. ■