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inefficiency of head taxes**

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Abstract

The literature on local public (school) finance has shown that the use of local head taxes to finance schools leads to an efficient allocation of households and pupils to districts (Tiebout, 1956; Hamilton, 1975; Calabrese et al., 2009). This paper revises this well established result, using a two-community model with a housing market that adds two layers of realism to the analysis: not every household receives direct benefits from schools (e.g. some do not have children at school age) and communities are vertically differentiated, in the sense that one of them is exogenously preferred to the other by every household. In such context, head taxation leads to an inefficient allocation of households to districts, even if local governments set local spending levels efficiently given their population. The inefficiency emerges because too many intermediate income "in-school" households reside in the rich district in equilibrium. Income taxation is inefficient as well but, in a counter-intuitive result, it may cause smaller efficiency losses than a lump-sum tax.

1 Introduction

In line with ideas first presented in Tiebout's (1956) classical article, head taxation has traditionally been considered efficient and non-distortionary in the local public finance literature. First, because tax-payers can never change their tax burden, head taxation is non-distortionary. Second, when it is used to finance the provision of *local public goods* that are equally available to all the residents in a jurisdiction, it constitutes an efficient benefit tax, as the tax burden is equally shared among the beneficiaries. Hence, it has been shown, head taxes induce an efficient allocation of households to districts, internalising location externalities that emerge under ability-to-pay taxes such as the property tax or an income tax. In a recent and important paper, Calabrese et al. (2009) demonstrate this in the context of a rich multi-community model with housing markets. To be precise, in their model, if local choices of head taxes are efficient (i.e. reflect residents' average preferences), then households' residential choices do not generate externalities, the reason being that head taxation is equivalent to marginal cost pricing.

This paper reevaluates this important question. The model considered introduces two main departures from their model. First, households not only differ by income and tastes but also in the number of children at school age or, in other words, in the level of spending they are entitled to receive from the local government. Second, adding an extra layer of realism, school districts are assumed to be vertically differentiated (as in, for example, de Bartolome and Ross, 2004) so that, apart from local school finance, there exist another exogenous source of income segregation. Furthermore, to present results in a most transparent way, housing markets are considerably simplified and preferences are assumed quasi-linear in private consumption.

The analysis shows that the efficiency properties of local head taxation do not hold in the setting considered. In particular, too many relatively poor households reside in the rich district in market equilibrium. Location externalities emerge because the valuation in-school households make of the urban area does not reflect the cost of their education but its tax price, which is inefficiently low. Moreover, while tax prices do not affect their private valuation relative to that of out-of-school households, given that the latter also pay taxes, the cost of their education affects their relative marginal social valuation, as out-of-school households do not receive education spending. Furthermore, the analysis also proves that, while income taxation is inefficient too, the size of the inefficiency emerging under such tax scheme may well be

smaller than that arising with local head taxes. This paper is thus related to de Bartolomé (1990), who uncovered the possibility that peer group effects rendered head taxes inefficient.

The remaining of the paper is organised as follows. The next Section introduces the model. Section 3 characterises market equilibrium under head taxation, which is compared to the optimal allocation in Section 4. The next one presents results on income taxation. Section 6 concludes.

2 The model

Consider a city divided in two separate and distinct areas. The areas are vertically differentiated, in the sense that, *ceteris paribus*, all citizens would prefer to live in the same area, due to some exogenous characteristic (distance from a business, shopping or cultural centre, landscape, altitude, climate, physical amenities such as parks, and so on) which makes it intrinsically more attractive.¹ With no loss of generality, and for the sake of definiteness, following the typical pattern of many cities in Europe, we call “urban” the desirable area and “suburbs” the other. The difference in the quality of local amenities is denoted by $\Delta_u > 0$. Districts will be indexed with j . Each area of the city constitutes a school district whose boundaries are fixed. School districts provide tuition-free public education to every child living within their catchment area. For simplicity sake, private schools are excluded from the analysis.²

A population of households with mass normalised to 1 lives in the city. Households differ along two dimensions: income and household type: a proportion $\gamma \in (0, 1)$ of households have one child at school age while the rest have none.³ Households with children at school age (*in-school households*) differ from those without (*out-of-school households*) in the trade-off between local (public) expenditure on education and the private consumption good. In-school households consist of a mother and a daughter, which implies a

¹Physical barriers often divide the city into separate areas with different desirability: many European cities are built along rivers, which separate the two sides: more recently railway lines and major roads have had the same effect.

²Note however that the presence of a proportion of children attending private schools in the rich district would also render head taxation inefficient.

³Among out-of-school households there could be some who have children in the private or home education sector irrespective of what public schools look like (e.g. children from very rich or very religious households).

ratio between the number of children and the population of households equal to γ . Household types are indexed with $k = i, o$. In the analysis that follows I will assume that $\gamma > 1/2$. Household income is denoted by $y \in D \equiv [\underline{y}, \bar{y}]$, and is distributed in the population according to $\Phi_k(y) \in [0, 1]$; $k = i, o$. Income distribution functions are continuous, strictly increasing on all their support and have densities $\phi_i(y) = \Phi'_i(y)$ and $\phi_o(y) = \Phi'_o(y)$. The total (and average) income is given by:

$$Y = \int_{\underline{y}}^{\bar{y}} y (\phi_i(y) + \phi_o(y)) dy \quad (1)$$

I assume, for simplicity, that the two income distribution functions are identical: $\Phi_o(y) = \gamma\Phi(y)$; and $\Phi_i(y) = (1 - \gamma)\Phi(y)$.

To present results in a transparent way, the model contains the simplest possible representation of housing markets: each district has a fixed supply of homogenous houses which, to avoid uninteresting complications, is assumed identical and equal to one half: $H_u = H_s = 1/2$.⁴ This implies the city has capacity to just house the population.⁵ Equilibrium in the housing market will entail the existence of a (weakly) positive housing rent premium in one of the two districts. That is to say, in equilibrium, either $r_u \geq r_s = c$, or $r_s \geq r_u = c$, will hold, where c stands for construction costs which are normalised away to zero. To save on notation, in what follows, I will let $r = r_u - r_s$ be the urban area rent premium, which will take a negative value whenever $r_s > r_u = 0$.

Preferences of out-of-school households are specified over private consumption (x) and the amenity, whereas those of in-school ones are defined, additionally, over the child's future income. Thus, out-of-school households do not derive any utility from local public education and only care for taxes and the amenity.⁶ To facilitate welfare analysis, and following de Bartolome

⁴A very close model of local housing markets is used, for instance, by Bénabou (1996).

⁵It also implies that the population of out-of-school households is not large enough to fill any of the communities, while that of in-school households cannot be accommodated in a single district. The analysis thus restricts attention to cases where the two districts always have a school, avoiding potential equilibria which are irrelevant to the results presented.

⁶This is not essential for any of the results in the paper. The only requirement is that households with children have a stronger taste for education than childless ones. The empirical evidence in Rubinfeld (1977), Baldson and Brunner (2004) shows that the strength of preferences for education decline with age; that offered by Fletcher and Kenny (2008) lends specific support to a view of the political process where childless households

and Ross (2004), a quasi-linear, money-metric utility function represents households' preferences in the model.⁷ Formally, the utility function of a type k household with income y is given by:

$$U_k(x, e, \Delta_u; y) = x + \delta_c^k h(e, y) + \delta_u y \Delta_u \quad (2)$$

where: $\delta_c^k, \delta_u \in \{0, 1\}$ indicate whether or not a household has a child ($\delta_c^i = 1, \delta_c^o = 0$) and if they live in the urban area of the city $\delta_u = 1$. Hence, indirect utility is given by:

$$v_j^k(T_j, e_j, r_j, \Delta_u; y) = y - T_j - r_j + \delta_c^k h(e, y) + \delta_u y \Delta_u \quad (3)$$

In-school households care for school quality because it enhances the child's future income. The offspring experience at school and its impact on her future income depends on her household socioeconomic background. Hence, in equation (2), offspring future income is a function of two inputs: school quality and home inputs. Units of school quality are normalised to be measured in units of spending, while home inputs are measured by household income (or parental human capital): $h(e, y)$. The human capital production function is monotonically increasing in the both of its arguments and strictly concave; moreover, home and school inputs are assumed complements:⁸ i.e. $h''_{ey} > 0$. The latter assumption is justified by the well documented impact of households' socioeconomic status on school performance, and implies a positive income elasticity of demand for education.⁹ Finally, note it is assumed that richer households derive greater benefits from urban amenities.¹⁰

To focus the analysis on location externalities, I assume that spending per pupil is chosen efficiently in every district, given the population of pupils.

ally with the poor in voting for low spending. The theoretical model proposed by Levy (2005) adopts the same assumption.

⁷The choice of a money-metric utility function completely separates efficiency and equity considerations in the analysis: because the marginal utility of consumption is constant and equal to 1 (i.e. households are risk neutral), given an allocation of households to districts, aggregate welfare is invariant to the distribution of private consumption and taxes in the population. Therefore, the only inefficiency source here is the possibly suboptimal allocation of households to districts.

⁸Subscripts indicate derivatives.

⁹Complementarity of home and school inputs is not controversial: parental inputs, home resources at early ages and during K-12 education, or social networking are all important to school achievement and improve with households socioeconomic status.

¹⁰This representation of utility implies households are willing to pay a share Δ_u of their income in order to enjoy the better amenities located in the urban area.

That is to say, levels of local spending satisfy the Samuelsonian condition for publicly provided public goods, i.e. are such that the average (across the local population of in-school households) marginal rate of substitution between education spending and numeraire consumption equals the marginal rate of transformation (which recall is normalised to 1). I then study the location equilibrium of the city in order to compare the outcomes that emerge under head to the efficient allocation.¹¹ Of course, local governments must meet their budget constraint, given by:

$$e_j n_j = T_j N_j \tag{4}$$

under per capita taxation.

Timing is as follows. Households make a single strategic choice: where to buy (or rent) a house, or which community to live in. This takes place at the first stage and determines the value of the rent premium in the price of a house in the urban district. When taking that decision, households anticipate the equilibrium vector of local levels of spending and taxation. At the second stage, spending and taxes are determined by the Samuelsonian efficiency condition, given the allocation of households to districts. Finally, at the third stage, households send their kids to school and consume all available income.

3 Head-Tax Equilibrium

I first study market equilibrium under head taxation. The economy is in market equilibrium if (i) local governments provide the efficient level of provision, given their population, and balance their budget; (ii) households make optimal consumption choices conditional on district choice and cannot obtain higher utility by moving to the other district; and (iii) housing markets are in equilibrium. An interior equilibrium will be any in which the two types of households are present in the two districts. Two types of segregated equilibrium may exist, as the better school may be located in the exogenously better area or not. The analysis focuses on equilibria where the urban area provides

¹¹Note that in this simple setting, with homogenous quality units of housing, head taxation is equivalent to property taxation.

better schooling (i.e. $e_u > e_s$).¹² I will name these *pushed-to-suburbia* equilibrium.

Let us define two bid-rent functions that provide the maximum amount of the numeraire a household of income y and type $k = i, o$ is willing to pay as rent premium for a house in the urban area, given a vector of local policies (y, e_u, e_s, T_s, T_u) and given the amenity quality gap Δ_u . These are given by the value of the rent that makes that household indifferent among the two areas:

$$r_i(y, e_u, e_s, T_s, T_u, \Delta_u) = h(e_u, y) - h(e_s, y) + (T_s - T_u) + y\Delta_u \quad (5)$$

$$r_o(y, T_s, T_u, \Delta_u) = (T_s - T_u) + y\Delta_u \quad (6)$$

The next lemma presents two basic properties of bid-rent functions that will be used to find the market equilibrium allocation of households to districts.

Lemma 1 *Suppose local governments use head taxation and that $e_u > e_s$, then (a) The bid rent functions r_i, r_o are increasing in income; and (b) $r_i(y) > r_o(y), \forall y > 0$.*

Proof. These results derive in a straightforward manner from assumptions made on preferences and technology. (a) The relevant partial derivatives are:

$$\begin{aligned} r_y^{i'}(\cdot) &= h'_y(e_u, y) - h'_y(e_s, y) + \Delta_u > 0 \\ r_y^{o'}(\cdot) &= \Delta_u > 0 \end{aligned}$$

Then, $\Delta_u > 0$ and $h''_{ey}(e, y) > 0$ guarantee that r_i is increasing in income, while $\Delta_u > 0$ implies the same for out-of-school households. (b) Complementarity between home and school inputs in the production of human capital also ensures that in-school households are willing to pay more for a house in the district providing better schooling than out-of-school ones with identical income. ■

¹²While there may exist equilibria in which the good school is located at the suburbs, efficiency requires the rich living in the good district because of them willing to pay more from the amenity (note that an equivalent argument could be made if the utility function were concave in private consumption and every household received the same benefits from the amenity). On the other hand, the symmetric equilibrium that exists in typical community models cannot emerge in this model due to the amenity quality gap.

Remark 1 Strict single-crossing conditions. *Part (a) of the lemma implies the following single-crossing conditions:*

$$v_i^s(e_s, T_s; y_i^H) = v_i^u(e_u, T_u, \Delta_u; y_i^H) \Rightarrow v_i^s(e_s, T_s; y) \left\{ \begin{array}{l} < \\ > \end{array} \right\} v_i^u(e_u, T_u, \Delta_u; y); \forall y \left\{ \begin{array}{l} > \\ < \end{array} \right\} y_i^H \quad (7)$$

$$v_o^s(T_s; y_o^H) = v_o^u(T_u, \Delta_u; y_o^H) \Rightarrow v_o^s(T_s; y) \left\{ \begin{array}{l} < \\ > \end{array} \right\} v_o^u(T_u, \Delta_u; y); \forall y \left\{ \begin{array}{l} > \\ < \end{array} \right\} y_o^H \quad (8)$$

the former holds provided $e_u > e_s$, while the latter requires $\Delta_u > 0$. In words, if households of type k and income y_k^H are indifferent among districts, then all households of the same type and higher (lower) income strictly prefer the urban area (the suburbs). The single-crossing conditions imply equilibrium will be characterised by within-type income segregation if households of a given type can be found in the two districts. Moreover, by continuity, any 'interior' equilibrium will have two border incomes, denoted y_k^H , that make households of the corresponding type indifferent between districts.¹³

Remark 2 Income mixing ($y_i^H < y_o^H$). *Part (b) of the lemma, in turn, entails that income mixing¹⁴ will characterise any pushed-to-suburbia equilibrium. In particular, some relatively poor in-school households outbid relatively rich out-of-school ones from the central city, as they are willing to give up more private consumption than their identical income counterparts in order to get access to the better school, because the quality of schooling is not valued by an out-of-school household.¹⁵*

Given the size of districts, housing markets clearance requires the two border incomes to satisfy a housing market constraint, which will be denoted

¹³By an "interior" equilibrium I mean one that has households of each type living in the two districts. In a non-interior equilibrium one of the types of households concentrates in one of the districts and in general that type will not have a 'border' income.

¹⁴I define the "amount" of income mixing as the mass of households with incomes such that households of the other type with the same income live in the other district.

¹⁵Perfect income segregation across districts cannot characterise a pushed to suburbia equilibrium. Interestingly, in a model with amenities, perfect income segregation could only characterise an equilibrium if $e_u = e_s$, e.g. with equalised central spending. Only in that case bid rent functions are identical across types. But in the context of the model consider here, perfect income segregation implies the population of in-school households has higher income in the urban area and, hence, that $e_u > e_s$.

as $y_o = z(y_i)$.¹⁶

Lemma 2 Housing market constraint. *Let y_i be the border income for in-school households; the housing market constraint $y_o = z(y_i)$ provides the border income of out-of-school households such that if all households of type k and income above y_k live in the urban area, and the rest reside in the suburbs, housing demand equals housing supply in both districts; $z(y_i)$ is implicitly defined by:*

$$(1 - \gamma)(1 - \Phi(z(y_i))) + \gamma(1 - \Phi(y_i)) = 1/2 \quad (9)$$

The housing market constraint (9) requires the sum of the mass of households of each type with income above the corresponding border income being equal to urban housing supply. Provided Φ is continuous, which is assumed by the model, z is a continuous function. Clearly, z is also strictly decreasing, that is, border incomes move in opposite directions; as y_i goes up, relatively low in-school households replace higher income out-of-school ones from the urban area.¹⁷ Finally, note that income mixing rises as y_i gets away from \tilde{y} such that $\tilde{y} = z(\tilde{y})$.

Every allocation of households to districts satisfying within-type income segregation and the housing market constraint can thus be characterised by a unique in-school border income y_i . Furthermore, to every in-school border income y_i corresponds a unique vector of local policy variables $(e_u(y^i), e_s(y^i), T_u(y^i), T_s(y^i))$. The Samuelsonian efficiency condition determines levels of local school spending which, in turn, and along the budget constraints, determine local tax bills. It is thus possible to define two *border-income bid rent functions*, which yield, for any in-school border income y_i , the maximum price premium each type's border household is willing to pay for a house in the urban area, when local policy variables are set at $(e_j(y_i), T_j(y_i))$. These are denoted with $R_k(y_i)$ and are given by:

$$R_i(y_i) = h(e_u(y_i), y_i) - h(e_s(y_i), y_i) + T_s(y_i) - T_u(y_i) + y_i \Delta_u \quad (10)$$

¹⁶Of course, this constraint must be satisfied by the optimal allocation and any equilibrium with income taxation as well. Hence, I denote border incomes without the superscript.

¹⁷The domain of $z(y)$ depends on the size of the central district relative to the proportion of each type of households. Given the simplifying assumption that $\gamma > H_u = H_s > 1 - \gamma$ the domain of z is $S \equiv [\underline{y}^i, \bar{y}^i]$ where \underline{y}^i is the minimum value in-school border income may reach (i.e. such that in-school households with income above \underline{y}^i have mass equal to 1/2); and \bar{y}^i is the minimum value in-school border income may reach so that $(1 - \gamma) + \gamma [1 - \Phi(\bar{y}^i)] = H_u$. Therefore, $z(\underline{y}^i) = \bar{y}$ and $z(\bar{y}^i) = \underline{y}$.

$$R_o(y_i) = T_s(y_i) - T_u(y_i) + z(y_i) \Delta_u \quad (11)$$

Note that because $z(y_i)$, $e_j(y_i)$ and $T_j(y_i)$ are continuous, *border-income bid rent functions* are continuous too.

The next proposition proves existence of a head-tax pushed-to-suburbia equilibrium.

Proposition 1 Existence. *Under head taxation, a pushed-to-suburbia equilibrium exists. Border incomes satisfy $y_o = z(y_i)$, and $y_i^H < y_o^H = z(y_i^H)$. Moreover:*

(a) In an interior equilibrium, *border incomes and the urban area rent premium y_i^H , y_o^H , r^H satisfy: $r^H = R_i(y_i^H) = R_o(y_i^H)$, or*

$$y_o^H = y_i^H + \frac{h(e_u(y_i^H), y_i^H) - h(e_s(y_i^H), y_i^H)}{\Delta_u}. \quad (12)$$

(b) In a corner equilibrium, y_i^H, y_o^H, r^H satisfy: $y_o^H = \bar{y}$, $y_i^H = \underline{y}^i$; $R_i(\underline{y}^i) \geq R_o(\underline{y}^i)$; and $r^H \in [R_o(\underline{y}^i), R_i(\underline{y}^i)]$.

Proof. Let \tilde{y} denote the level of income such that $z(\tilde{y}) = \tilde{y}$, that is, the (unique) border income implying perfect income segregation. Recalling that the domain of z is $[\underline{y}^i, \bar{y}^i]$, let us first examine the interval $(\underline{y}^i, \tilde{y})$:

An *interior equilibrium* with rent premium r^H exists if $r^H = R_i(y_i^H) = R_o(y_i^H)$ for some $y_i^H \in (\underline{y}^i, \tilde{y})$. In that case, the two border incomes are simultaneously indifferent with the equilibrium rent r^H and the vector of local policy variables $(e_u(y_i^H), e_s(y_i^H), T_u(y_i^H), T_s(y_i^H))$. The single-crossing conditions (28) and (29) then ensure all households maximise utility in their district of residence. Recalling that $R_i(y_i^H)$ and $R_o(y_i^H)$ are continuous functions which satisfy $R_i(\tilde{y}) > R_o(\tilde{y})$, then, if $R_i(\underline{y}^i) - R_o(\underline{y}^i) < 0$, there exists some $y_i^H \in (\underline{y}^i, \tilde{y})$ such that $R_i(y_i^H) = R_o(y_i^H)$.

If, on the contrary, $R_i(\underline{y}^i) - R_o(\underline{y}^i) \geq 0$ then a *corner equilibrium* exists in which in-school households with income above \underline{y}^i populate the urban centre, while the rest live in the suburbs. This is an equilibrium for any value of the rent premium in the interval $[R_o(\underline{y}^i), R_i(\underline{y}^i)]$. Clearly, the allocation of households to districts implied by $y_i = \tilde{y}$ cannot be an equilibrium, as $R_i(\tilde{y}) > R_o(\tilde{y})$. By continuity, that means in-school households with income just below \tilde{y} are willing to outbid out-of-school ones with income just above \tilde{y} from the rich district, something incompatible with equilibrium. Finally, no

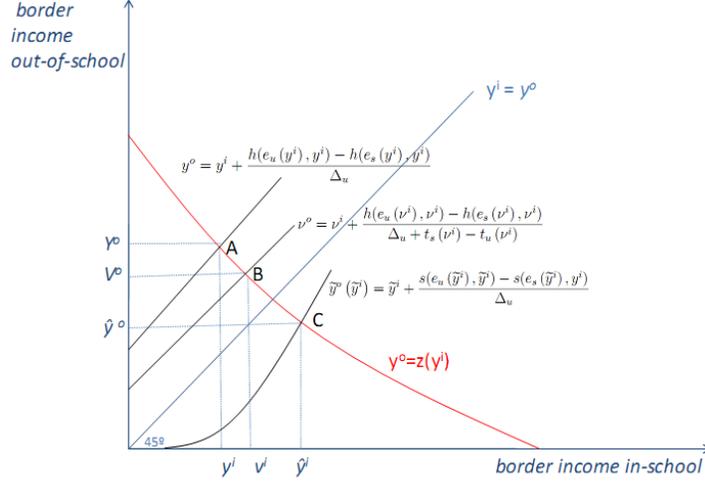


Figure 1
Equilibrium with income tax, head tax and efficient allocation

equilibrium may exist for any $y_i > \tilde{y}$ because, within that range, $y_i < z(y_i)$ and, hence, $e_u > e_s \Rightarrow R_i(y_i) > R_o(y_i)$. ■

Point A in Figure 1 corresponds to a pushed-to-suburbia equilibrium under head taxation.

4 The Inefficiency of Head Taxation

I adopt a utilitarian approach and define the Social Welfare Function (SWF) as the unweighted sum of utility in the economy. The SWF thus includes the utility of absentee landowners, assumed, as well, linear in the private good. Although Pareto-optimal allocations could be found in which $e_s > e_u$, implying that higher income in-school households reside in the suburbs, aggregate welfare would still be smaller than in a pushed-to-suburbia efficient equilibrium. The reason is that higher income households derive greater benefits from the urban amenities. The exposition will thus restrict attention to cases where the urban school is better.

Because utility is linear in x and the SWF weights equally household and landowners' utility, the distribution of the (net) tax burden across households as well as the transfer of housing rents from households to landowners do not affect aggregate welfare. Thus, for simplicity, and without loss of generality,

I assume school spending is financed locally through head taxes, that households do not pay any price for their houses, and that there are no government transfers to households or landowners.¹⁸

Following Calabrese et al. (2009), let $a_u^k(y)$ be the proportion of households of type k and income y that are allocated to the central district. Hence, $a_s^k(y) = 1 - a_u^k(y)$. Using (2), the unweighted SWF is simply defined as:

$$\begin{aligned}
W &= \int_D (y - T_u + h(e_u, y) + y\Delta_u) a_u^i(y) \gamma \phi(y) dy + \\
&\quad + \int_D (y - T_s + h(e_s, y)) (1 - a_u^i(y)) \gamma \phi(y) dy + \\
&\quad + \int_D (y - T_u + y\Delta_u) a_u^o(y) (1 - \gamma) \phi(y) dy + \\
&\quad + \int_D (y - T_s) (1 - a_u^o(y)) (1 - \gamma) \phi(y) dy
\end{aligned} \tag{13}$$

The SWF is maximised with respect to $e_u, e_s, a_u^i(y), a_u^o(y), T_u$ and T_s , subject to eight constraints, which include four nonnegativity constraints ($e_u \geq 0, e_s \geq 0, T_u \geq 0$ and $T_s \geq 0$), the housing market constraint, with associated Lagrange multiplier λ_h , and which can now be rewritten as:

$$\int_D a_u^i(y) \gamma \phi(y) + \int_D a_u^o(y) (1 - \gamma) \phi(y) = H_u, \tag{14}$$

two local budget equality constraints (with associated multipliers λ_j):

$$e_u \int_D a_u^i(y) \gamma \phi(y) = T_u \left[\int_D a_u^i(y) \gamma \phi(y) + \int_D a_u^o(y) (1 - \gamma) \phi(y) \right] \tag{15}$$

and

$$e_s \int_D (1 - a_u^i(y)) \gamma \phi(y) = T_s \left[\int_D (1 - a_u^i(y)) \gamma \phi(y) + \int_D (1 - a_u^o(y)) (1 - \gamma) \phi(y) \right], \tag{16}$$

and two demographic feasibility restrictions:

$$a_u^k(y) \in [0, 1]; k = i, o. \tag{17}$$

¹⁸Because of the absence of risk aversion in households and landowners preferences, any level of the rent would be efficient. It is thus without loss of generality that I impose $r = 0$ from the outset.

The solution to this problem can be found in the Appendix.

In the efficient allocation, districts provide the level of spending that maximises the future income of the local population of children. This is the level satisfying the usual samuelsonian condition for publicly provided private goods. Given the quasi-linear specification of utility, this is attained when the average (across in-school households) marginal productivity of education spending in the generation of future income equals its marginal cost, equal to 1:

$$\frac{\int_D h'_e(e_u^*, y) a_u^i(y) \gamma \phi(y) dy}{\int_D a_u^i(y) \gamma \phi(y) dy} = 1 \quad (18)$$

$$\frac{\int_D h'_e(e_s^*, y) (1 - a_u^i(y)) \gamma \phi(y) dy}{\int_D (1 - a_u^i(y)) \gamma \phi(y) dy} = 1 \quad (19)$$

The solution to the Social Planner's problem yields two *Marginal Social Value* functions, which I denote $MSV_u^k(y)$, $k = i, o$, providing the marginal rise in aggregate welfare derived from marginally increasing the proportion $a_u^k(y)$ of households of type k and income y assigned to the urban area instead of to the suburbs. These are obtained by dividing the FOCs corresponding to $a_u^k(y) \gamma \phi(y)$, for $k = i$, and $(1 - \gamma) \phi(y)$, for $k = o$.

$$MSV_u^i(y) = [h(e_u, y) - e_u] - [h(e_s, y) - e_s] + \Delta_u y \quad (20)$$

$$MSV_u^o(y) = \Delta_u y \quad (21)$$

Next, let $s(e, y)$ be the *education surplus* of a child who receives education spending e and whose household has income y ; this is given by the continuous function $s(e, y) \equiv h(e, y) - e$. Then, (20) can be rewritten as:

$$MSV_u^i(y) = [s(e_u, y) - s(e_s, y)] + \Delta_u y \quad (22)$$

Lemma 3 *Given $e_u > e_s$: (a) $MSV_u^k(y)$, $k = i, o$, are increasing in income; (b) $MSV_u^i(y) > MSV_u^o(y) \forall y < \tilde{y}$, $MSV_u^i(y) < MSV_u^o(y) \forall y > \tilde{y}$.*

Proof. (a) $MSV_u^{o'}(y) = \Delta_u > 0$; $MSV_u^{i'}(y) = h'_y(e_u, y) - h'_y(e_s, y) + \Delta_u > 0$.

(b) Note first that $MSV_u^i(y) \left\{ \begin{smallmatrix} \geq \\ < \end{smallmatrix} \right\} = MSV_u^o(y)$ if and only if $s(e_u, y) - s(e_s, y) \left\{ \begin{smallmatrix} \geq \\ < \end{smallmatrix} \right\} 0$. Given $h''_{ee} < 0$, households preferences for education spending are single-peaked. Moreover, $h''_{ey} > 0$ implies that the peak shifts to the right as income rises. Given $e_u > e_s$, then, for any y such that $e^*(y) > e_u > e_s \Rightarrow s(e_u, y) - s(e_s, y) > 0$, whilst for any y such that $e^*(y) < e_s < e_u \Rightarrow s(e_u, y) - s(e_s, y) < 0$.

For income levels y such that $e(y) \in (e_u, e_s)$, $s(e_u, y) - s(e_s, y) < 0$ when $e^*(y)$ is closer enough to e_s than to e_u , and viceversa. Hence, as $e(y)$ is continuous and increasing, and $s(e, y)$ is continuous and single-peaked, for every pair $e_u > e_s$ there exists a unique income level, \tilde{y} , with $e_u > e(\tilde{y}) > e_s$ for which $s(e_u, y) - s(e_s, y) = 0$; above which is positive and below which is negative. ■

Remark 3 *Because both marginal social value functions are increasing in income, the efficient allocation requires within-types income segregation. Therefore, as in the analysis of market equilibrium, the allocation of households to districts can be characterised by two border incomes, which will be linked by the housing market constraint (14).*

Remark 4 *Efficient income mixing ($\tilde{y}_i^H \left\{ \begin{smallmatrix} \geq \\ < \end{smallmatrix} \right\} \tilde{y}_o^H$). $MSV_u^i(y)$ is smaller than $MSV_u^o(y)$ when the education surplus of households with income y is greater in the suburbs than in the urban area. This will occur in cases where the most preferred spending level of a household with income y is sufficiently closer to e_s than to e_u (i.e. if the income of the optimal in-school border household is sufficiently closer to the average income of households of this type that live in the suburbs than to the average of those residing in the centre). Contrary to what happens in a pushed-to-suburbia market equilibrium, border out-of-school households in an efficient outcome may have lower income than their in-school counterparts (Figure 2).*

Next, let $\mu_u^i(y)$ be defined as the *border marginal social value function* of in-school households. It yields the marginal social value of admitting an in-school household with income y to the urban area when that is the in-school border income.¹⁹ It is given by:

$$\mu_u^i(y) = [s(e_u(y), y) - s(e_s(y), y)] + \Delta_u y \quad (23)$$

¹⁹The border marginal social value function $\mu_u^i(y)$ of in-school households is not neces-

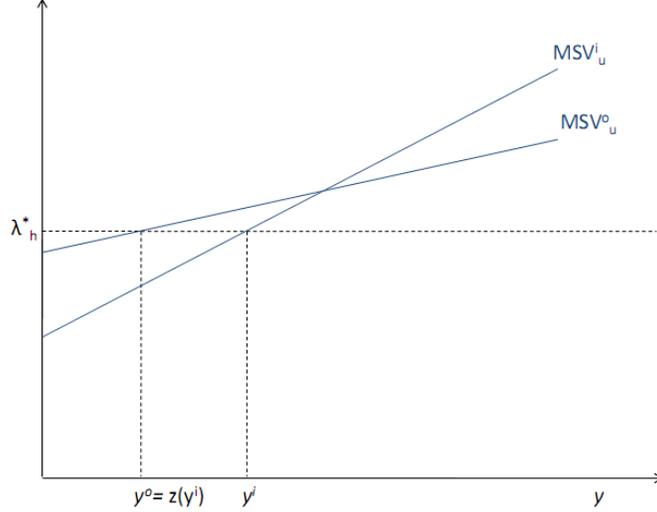


Figure 2
Optimal border incomes with $y^i > y^o = z(y^i)$

In order to find the optimal allocation, the marginal social value of admitting in-school households with border income y to the centre needs to be compared with that of admitting out-of-school households of income $z(y)$. The latter will be given by $\mu_u^o(y)$, which can be written:

$$\mu_u^o(y) = \Delta_u z(y).$$

The Social Planner's problem always has a solution. The optimal outcome may be of one of three types, depending on whether out-of-school households are all assigned to the centre, to the suburbs, or can be found in the two districts. This is studied in proposition 2 and proved in the Appendix.

Proposition 2 *In an optimal allocation:*

(a) *Corner solution, with all out-of-school households assigned to the centre: the distribution of in-school households across districts satisfies within-type income segregation; border incomes satisfy $\hat{y}^i = \bar{y}^i$ and $\hat{y}^o = \underline{y}$, and*

sarily increasing in income, as its derivative is equal to:

$$\frac{\partial \mu(y)}{\partial y} = [h'_y(e_u(y), y) - h'_y(e_s(y), y)] + \frac{de_u(y)}{dy} [h'_e(e_u(y), y) - 1] - \frac{de_s(y)}{dy} [h'_e(e_s(y), y) - 1]$$

where the last two terms are negative as $e_u(y) > e(y) > e_s(y)$.

$MSV_u^i(\bar{y}^i) \leq MSV_u^o(\underline{y}) > 0$; the optimal allocation rule is:

$$\begin{aligned} a_u^i(y) &= 0 & \forall y < \tilde{y}^i \\ a_u^i(y) &\in [0, 1] & y = \tilde{y}^i \\ a_u^i(y) &= 1 & \forall y > \tilde{y}^i \\ a_u^o(y) &= 1 & \forall y \end{aligned}$$

(b) Interior solution, with households of each type assigned to the two districts: the distribution of households across districts satisfies within-type income segregation; the two border incomes satisfy:

$$\tilde{y}^k = y / \mu_u^k(y) = \lambda_h^* > 0; k = i, o$$

. The optimal allocation rule for households of type k is:

$$\begin{aligned} a_u^k(y) &= 0 & \forall y < \tilde{y}^k \\ a_u^k(y) &\in [0, 1] & y = \tilde{y}^k \\ a_u^k(y) &= 1 & \forall y > \tilde{y}^k \end{aligned}$$

(c) Corner solution, with all out-of-school households assigned to the suburbs: the distribution of in-school households across districts satisfies within-type income segregation; border incomes, satisfy $\tilde{y}^i = \underline{y}^i$ and $\tilde{y}^o = \bar{y}$, and $\lambda_h^* = MSV_u^i(\underline{y}^i) \geq MSV_u^o(\bar{y}) > 0$; the optimal allocation rule is:

$$\begin{aligned} a_u^i(y) &= 0 & \forall y < \tilde{y}^i \\ a_u^i(y) &\in [0, 1] & y = \tilde{y}^i \\ a_u^i(y) &= 1 & \forall y > \tilde{y}^i \\ a_u^o(y) &= 0 & \forall y \end{aligned}$$

Proof. See Appendix. ■

The next proposition proves head taxation is inefficient, except when efficiency requires only in-school households to live in the urban centre. Point C in figure 1 represents an efficient allocation.

Proposition 3 *Under head taxation, a pushed-to-suburbia equilibrium is inefficient except in cases where it is efficient to have only in-school households in the centre.*

Proof. In a corner solution of type (a) in proposition 2 the efficient border income of in-school households is the maximum in its domain \bar{y}^i while all out-of-school households reside in the centre, so that $\tilde{y}^i = \bar{y}^i > \tilde{y}^o = \underline{y}^i$. In market equilibrium with head taxation, however, by proposition 1, $y_i^H < y_o^H$.

In an interior solution, $\tilde{y}^o = z(\tilde{y}^i)$, and $\mu_u^i(\tilde{y}^i) = \mu_u^o(\tilde{y}^o) = \lambda_h^*$, or:

$$\tilde{y}^o(\tilde{y}^i) = \tilde{y}^i + \frac{s(e_u(\tilde{y}^i), \tilde{y}^i) - s(e_s(\tilde{y}^i), \tilde{y}^i)}{\Delta_u} \quad (24)$$

First, note that the numerator in the second addend of the RHS of (24) may be positive, negative or equal to zero and, thus, that $\tilde{y}^o \leq \tilde{y}^i$. If $\tilde{y}^o \leq \tilde{y}^i$ then market equilibrium with head taxation is inefficient as, again, by proposition 1, $y_i^H < y_o^H$. As in the previous case, too many relatively low income in-school households live in the centre in market equilibrium. Consider now the case where $\tilde{y}^o > \tilde{y}^i$. Given the definition of the education surplus, $s(e_u, y)$, and $e_u > e_s$, the graph of (24) is always below that of (12) in (y^i, y^o) space (figure 1) and therefore cuts $z(y)$ at a point where $\tilde{y}^i > y_i^H$ and hence $\tilde{y}^o < y_o^H$. That is, the equilibrium again has too many relatively low income in-school households living in the urban area.

Lastly, if the SPP has a corner solution of type (c) in proposition 2, then the market equilibrium will be efficient, as in that case $\mu_u^o(\bar{y}) \leq \mu_u^i(\underline{y}^i)$, id est,

$$\Delta_u \bar{y} \leq \Delta_u \underline{y}^i + s(e_u(\underline{y}^i), \underline{y}^i) - s(e_s(\underline{y}^i), \underline{y}^i) < \Delta_u \underline{y}^i + h(e_u(\underline{y}^i), \underline{y}^i) - h(e_s(\underline{y}^i), \underline{y}^i)$$

where the latter inequality implies the existence of a corner equilibrium with head taxation. ■

Proposition 3 shows that head taxes cannot induce an efficient allocation. Attaining efficiency requires, on the one hand, households with children to internalise the full cost of their education, and, on the other hand, the choices of households without children not to be affected by local school funding. Instead, they should only reflect their private valuation of the amenity quality gap. However, the private valuation of the urban area by the two types of households differs from the marginal social value of them being assigned to that district. In the case of in-school households, because not every tax payer increases the cost of local education, head taxes do not reflect the full per-pupil cost of local schools. Hence, the tax surplus they obtain in district j ($h(e_j, y) - T_j$) is always greater than their education surplus ($h(e_j, y) - e_j$).

Moreover, as the proportion of children is in general different in the two districts, the tax differential does not reflect true spending differences across districts. In the case of out-of-school households, their location choices *are* affected by local school finance unless the tax bill is the same in the two districts. Therefore, residential choices are inefficient because the relative willingness to pay for a house in the centre of in-school households is inefficiently high. In other words, under head taxation, the allocation with border income equal to the optimal border income \tilde{y}^i cannot be an equilibrium because in-school households of income $\tilde{y}^i - \varepsilon$; $\varepsilon > 0$ would be outbid out-of-school households with income $\tilde{y}^o = z(\tilde{y}^i)$ from the urban centre. Inefficiency thus emerges because too many relatively poor in-school households live in the urban area in market equilibrium. In cases where it is efficient to fill the urban area just with in-school households, clearly, this locational externality does not emerge.

5 Head taxation can be more distortionary than income taxation

This Section derives the inefficiency generated by income taxation and compares it to that induced by lump-sum taxation. Interestingly, the analysis reveals that income taxation may actually lead to lower efficiency losses than head taxation. Under proportional income taxation, the local budget constraints of district j is:

$$e_j n_j = t_j Y_j, \quad (25)$$

while the bid rent functions are given by:

$$\rho_i(y, e_u, e_s, t_s, t_u, \Delta_u) = \delta h(e_u, y) - \delta h(e_s, y) + y(t_s - t_u) + y\Delta_u \quad (26)$$

$$\rho_o(y, t_s, t_u, \Delta_u) = y(t_s - t_u) + y\Delta_u \quad (27)$$

which satisfy part (b) in lemma 1 and require two standard single-crossing conditions to fulfill part (a) as well. These are:

$$\rho_y^{ii}(\cdot) = [h_y'(e_u, y) - t_u] - [h_y'(e_s, y) - t_s] + \Delta_u > 0 \quad (28)$$

and

$$\rho_y^{oo}(\cdot) = t_s - t_u + \Delta_u > 0. \quad (29)$$

Focusing again on pushed to suburbia equilibrium, then, the vector of local policies can be expressed as a function of the border income of in-school households $(e_u(y^i), e_s(y^i), t_s(y^i), t_u(y^i))$.²⁰ It is then possible to define two *border bid rent functions*, analogous to (5) and (6), which will be denoted by $P_k(y^i)$ and can be written as follows:

$$P_i(y^i) = h(e_u(y^i), y^i) - h(e_s(y^i), y^i) + y^i (t_s(y^i) - t_u(y^i)) + y^i \Delta_u$$

$$P_o(y^i) = z(y^i) (t_s(y^i) - t_u(y^i)) + z(y^i) \Delta_u$$

Proposition 4 *Suppose the single-crossing conditions (28) and (29) are satisfied. Under proportional income taxation, a pushed-to-suburbia equilibrium exists. Border incomes ν^i, ν^o satisfy $\nu^i < \nu^o$, and $\nu^o = z(\nu^i)$. Moreover:*

(a) *In an interior equilibrium, border incomes and the urban area rent premium ν^i, ν^o, r^I satisfy: $r^I = P_i(\nu^i) = P_o(\nu^i)$, which implies:*

$$\nu^o = \nu^i + \frac{h(e_u(\nu^i), \nu^i) - h(e_s(\nu^i), \nu^i)}{\Delta_u + t_s(\nu^i) - t_u(\nu^i)} \quad (30)$$

(b) *In a corner equilibrium, ν^i, ν^o, r^I satisfy: $\nu^o = \bar{y}$, $\nu^i = \underline{y}^i$; $P_i(\nu^i) \geq P_o(\nu^i)$; and $r^I \in [P_o(\nu^i), P_i(\nu^i)]$.*

Proof. The proof of existence is analogous to the proof to proposition 1 and is omitted for the sake of brevity. ■

As in the case of head-tax equilibria, the comparison between (30) and (24) shows that the allocation of households to districts will, in general, be inefficient under income taxation. Nevertheless, the next and final result shows that for any economy in which a pushed-to-suburbia income-tax equilibrium with larger tax rates in the poor district exists, another pushed-to-suburbia head-tax equilibrium inducing larger efficiency losses exists as well. Point B in figure 1 represents an income-tax pushed-to-suburbia equilibrium inducing smaller efficiency losses than the head-tax equilibrium (point A).

²⁰When the specified single-crossing conditions hold, the distribution of in-school households across districts are determined by the border income y^i , along with local efficient levels of spending. Given the corresponding border income of out-of-school households y^o and the local governments' budget constraints, local income tax rates are also uniquely determined.

Proposition 5 *Suppose the single-crossing conditions (28) and (29) are satisfied and that an interior income-tax pushed-to-suburbia equilibrium exists with border incomes $v^i < v^o$. Then, the size of the inefficiency induced by income taxation will be smaller than that arising under head taxation if $t_s(v^i) > t_u(v^i)$.*

Proof. In that case, $P_i(v^i) = P_o(v^i)$, that is

$$v^o = v^i + \frac{h(e_u(\nu^i), \nu^i) - h(e_s(\nu^i), \nu^i)}{\Delta_u + t_s(\nu^i) - t_u(\nu^i)} < v^i + \frac{h(e_u(\nu^i), \nu^i) - h(e_s(\nu^i), \nu^i)}{\Delta_u}$$

where the latter inequality follows from $t_s(v^i) > t_u(v^i)$ and implies that $R_o(v^i) < R_i(v^i)$. Then, adapting the proof to proposition 1 by substituting \tilde{y} for v^i demonstrates that a head-tax pushed-to-suburbia equilibrium exists as well with in-school border income $y_i^H < v^i$. ■

This result clashes with the view of local head taxes as efficiency-enhancing benefit taxes. In the model considered in this paper, not only head taxation induces an inefficient distribution of households to districts. Imposing lump-sum taxation may also be more distortionary than taxing an ability-to-pay measure as household income. The explanation lies on the competition between relatively low income in-school households and richer out-of-school ones to reside in the urban area in pushed to suburbia equilibrium. Under head taxation, tax bills do not change with income. Under income taxation, on the contrary, they do, making relative tax rates relevant. When the income tax rate is larger in the suburbs, the utility derived from residing in the urban centre, and hence bid rent functions, increase faster with income. As a consequence, in-school households with income, say, ν^i is able to outbid lower income out-of-school households than those they outbid under a system of head taxation. In other words, when the income tax rate is greater in the poor district, income taxation generates a stronger motive for higher income out-of-school households to buy a house in the centre than they have with lump-sum taxes. This effect reduces the mass of too low income in-school households that reside in the urban area in equilibrium.

The possibility that the income tax rate is higher in the poor district than in the rich one is not far-fetched. While efficient spending is lower in the suburbs, average income is smaller there as well. Relative income tax rates are also affected by the proportion of children to the population in each district. This proportion is larger in the urban area whenever $\nu^i <$

$z(\nu^i) = \nu^o$ but it is identical when the two types' border incomes are the same. Moreover, the effect of the proportion of children in local populations will be small for γ close to 1. Thus, this is not an element that could per se discard the possibility under consideration. As a matter of fact, income taxation data from Swiss communities show frequent cases of poor districts imposing higher income tax rates than richer ones. For instance, this was the case, in 1997, of four communities that belong to the canton of Basel-Land (city of Basel): Birsfelden, Mönchestein, Bottmingen, Binningen and Biel-Benken, whose data on mean, median income and mid income and high income tax rates are reported in Table 1:

	Mean income	Median income	Mid income tax rate	High income tax rate
Birsfelden	52,351	52,033	8.51	23.19
Mönchestein	58,962	54,567	8.13	22.26
Allschwill	69,302	63,138	7.94	21.77
Binningen	73,405	60,106	7.88	21.80
Biel-Benken	88,610	72,350	7.64	20.87

Table 1: Income taxation. Swiss communities (1997)
Source: Schmedheiny (2003)

6 Concluding Remarks

Head or lump-sum taxation has traditionally been considered efficient and non-distortionary. On the one hand, it is non-distortionary because taxpayers cannot alter their tax bill by changing their economic behaviour in any way. On the other hand, when used to fund the provision of *locally provided goods* equally available to all the residents in a jurisdiction, they act as an efficient benefit tax, in the sense that the tax burden is equally shared among the beneficiaries. Furthermore, in a multi-community context, head taxes have been shown to induce an efficient allocation of households to districts, controlling for location externalities and avoiding the sometimes called "musical-suburbs" problem, that is, the poor chasing the rich to benefit from tax induced redistribution, and the rich running away from the poor.

Adding two layers of realism (the presence of households without children at school-age and the exogenous vertical differentiation of districts) to an otherwise standard two-community model, this paper offers new insights

into the nature of inefficiencies arising in community models. The analysis showed that head taxes cannot perform the role of efficient prices and are unable to internalise location externalities. Head taxation leads to an inefficient allocation of households to districts, even though local governments choose spending levels efficiently, given their population. This result follows the contribution of de Bartolomé (1990), who uncovered the possibility that peer group effects rendered head taxes inefficient. The analysis went one step forward by showing, in a counter-intuitive result, that the size of the inefficiency may be larger than that arising under an ability-to-pay tax scheme such as an income tax.

The analysis is relevant for the debate on the practice of zoning policies by local governments. The use of zoning regulations has been justified on the grounds that they make property taxation closer to a head tax, inducing welfare gains (Hamilton, 1975; Calabrese et al., 2007). Results presented in this paper suggest the need to re-evaluate that conclusion.

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Appendix

Proof of proposition 2

First, note the problem always has a solution. Given the housing market constraint, lemma 3 and the requirement that local governments provide the locally efficient level of spending and balance their budget, the problem can be defined in a single choice variable, the in-school border income \tilde{y}^i .²¹ That is, it can be seen as that of choosing the in-school border income \tilde{y}^i . Existence of a solution is then assured by the compactness of the choice set, i.e. the domain $[\underline{y}^i, \bar{y}^i]$, and by the continuity of the objective function. Following the approach in Calabrese et al. (2009), the Lagrangian auxiliary function is:²²

$$\begin{aligned}
\mathcal{L} = & \int_D (y - T_u + h(e_u, y) + y\Delta_u) a_u^i(y) \gamma \phi(y) dy \\
& + \int_D (y - T_s + h(e_s, y)) (1 - a_u^i(y)) \gamma \phi(y) dy \\
& + \int_D (y - T_u + y\Delta_u) a_u^o(y) (1 - \gamma) \phi(y) dy \\
& + \int_D (y - T_s) (1 - a_u^o(y)) (1 - \gamma) \phi(y) dy \\
& - \lambda_h [n_u^i + n_u^o - H_u] \\
& - \lambda_u [e_u n_u^i - T_u [n_u^i + n_u^o]] \\
& - \lambda_s [e_s n_s^i - T_s [n_s^i + n_s^o]] \\
& - \lambda_{i1} [a_u^i(y) - 1] \\
& - \lambda_{i0} [-a_u^i(y) - 0] \\
& - \lambda_{o1} [a_u^o(y) - 1] \\
& - \lambda_{o0} [-a_u^o(y) - 0]
\end{aligned}$$

²¹The border income of in-school households \tilde{y}^i determines the population of children in the two districts and, along the housing market constraint, the border income of out-of-school households. Hence, local spending levels and tax bills are uniquely determined by \tilde{y}^i .

²²The Lagrangian function is written ignoring the nonnegativity constraints imposed on e_u , e_s , T_u and T_s . These are embedded in the complementary slackness conditions.

where:

$$\begin{aligned}
n_u^i &= \int_D a_u^i(y) \gamma \phi(y) dy; \\
n_u^o &= \int_D a_u^o(y) (1 - \gamma) \phi(y) dy; \\
n_s^i &= \int_D (1 - a_u^i(y)) \gamma \phi(y) dy \\
n_s^o &= \int_D (1 - a_u^o(y)) (1 - \gamma) \phi(y) dy
\end{aligned}$$

The First Order and Complementary Slackness Conditions are:²³

$$\frac{\partial \mathcal{L}}{\partial a_u^i(y)} = h(e_u, y) - h(e_s, y) + \Delta_u y - \lambda_h - \lambda_u e_u + \lambda_s e_s - \lambda_{i1} + \lambda_{i0} \leq 0; a_u^i(y) \in [0, 1]; \quad (31)$$

$$a_u^i(y) [h(e_u, y) - h(e_s, y) + \Delta_u y - \lambda_h - \lambda_u e_u + \lambda_s e_s - \lambda_{i1} + \lambda_{i0}] = 0 \quad (32)$$

$$-a_u^i(y) \leq 0; \lambda_{i0} \geq 0$$

$$\lambda_{i0} [-a_u^i(y)] = 0 \quad (33)$$

$$a_u^i(y) - 1 \leq 0; \lambda_{i1} \geq 0$$

$$\lambda_{i1} [a_u^i(y) - 1] = 0 \quad (34)$$

$$\frac{\partial \mathcal{L}}{\partial a_u^o(y)} = \Delta_u y - \lambda_h - \lambda_{o1} + \lambda_{o0} \leq 0; a_u^o(y) \in [0, 1]; \quad (35)$$

$$a_u^o(y) [\Delta_u y - \lambda_h - \lambda_{o1} + \lambda_{o0}] = 0 \quad (36)$$

$$-a_u^o(y) \leq 0; \lambda_{o0} \geq 0$$

$$\lambda_{o0} [-a_u^o(y)] = 0 \quad (37)$$

$$a_u^o(y) - 1 \leq 0; \lambda_{o1} \geq 0$$

$$\lambda_{o1} [a_u^o(y) - 1] = 0 \quad (38)$$

$$\frac{\partial \mathcal{L}}{\partial T_u} = -n_u^i(y) - n_u^o(y) + \lambda_u [n_u^i(y) + n_u^o(y)] = 0 \quad (39)$$

$$\frac{\partial \mathcal{L}}{\partial T_s} = -n_s^i(y) - n_s^o(y) + \lambda_s [n_s^i(y) + n_s^o(y)] = 0 \quad (40)$$

²³The inequalities (31) and (32) are divided by $\gamma \phi(y)$, while (35) (36) are divided by $(1 - \gamma) \phi(y)$.

$$\frac{\partial \mathcal{L}}{\partial e_u} = \int_D h'_e(e_u^*, y) a_u^i(y) \gamma \phi(y) dy - \lambda_u n_u^i(y) = 0 \quad (41)$$

$$\begin{aligned} \int_D h'_e(e_u^*, y) a_u^i(y) \gamma \phi(y) dy - \lambda_u n_u^i(y) &\leq 0; e_u \geq 0; \\ e_u \left[\int_D h'_e(e_u^*, y) a_u^i(y) \gamma \phi(y) dy - \lambda_u n_u^i(y) \right] &= 0 \\ \frac{\partial \mathcal{L}}{\partial e_s} = \int_D h'_e(e_s^*, y) (1 - a_u^i(y)) \gamma \phi(y) dy - \lambda_s n_s^i(y) &= 0 \quad (42) \end{aligned}$$

$$\begin{aligned} \int_D h'_e(e_s^*, y) (1 - a_u^i(y)) \gamma \phi(y) dy - \lambda_s n_s^i(y) &\leq 0; e_s \geq 0; \\ e_s \left[\int_D h'_e(e_s^*, y) (1 - a_u^i(y)) \gamma \phi(y) dy - \lambda_s n_s^i(y) \right] &= 0 \end{aligned}$$

From (39) and (40) $\lambda_u = \lambda_s = 1$. Then, (41) and (42) reduce to (18) and (19) respectively, setting locally efficient levels of spending. Moreover, (31) and (32) simplify to:

$$\begin{aligned} [h(e_u, y) - e_u] - [h(e_s, y) - e_s] + \Delta_u y - \lambda_h - \lambda_{i1} + \lambda_{i0} &\leq 0; a_u^i(y) \in [0, 1] \\ a_u^i(y) [[h(e_u, y) - e_u] - [h(e_s, y) - e_s] + \Delta_u y - \lambda_h - \lambda_{i1} + \lambda_{i0}] &= 0 \quad (43) \end{aligned}$$

Given the definition of $MSV_u^i(y)$ by the complementary slackness conditions (33), (34) and (43):

- If $a_u^i(y) \in (0, 1)$, then $\lambda_{i1} = \lambda_{i0} = 0$: (43) implies $[h(e_u, y) - e_u] - [h(e_s, y) - e_s] + \Delta_u y - \lambda_h = 0$, or $MSV_u^i(y) = \lambda_h$.
- If $a_u^i(y) = 0$, then $\lambda_{i1} = 0$, $\lambda_{i0} \geq 0$ and (43) imply $[h(e_u, y) - e_u] - [h(e_s, y) - e_s] + \Delta_u y - \lambda_h \leq 0$, or $MSV_u^i(y) \leq \lambda_h$.
- If $a_u^i(y) = 1$, then $\lambda_{i1} \geq 0$, $\lambda_{i0} = 0$ and (43) imply $[h(e_u, y) - e_u] - [h(e_s, y) - e_s] + \Delta_u y - \lambda_h \geq 0$, or $MSV_u^i(y) \geq \lambda_h$.

On the other hand, (35) and (36) become:

$$\begin{aligned}\Delta_u y - \lambda_h - \lambda_{o1} + \lambda_{o0} &\leq 0; a_u^o(y) \geq 0 \in [0, 1] \\ a_u^o(y) [\Delta_u y - \lambda_h - \lambda_{o1} + \lambda_{o0}] &= 0\end{aligned}\tag{44}$$

Hence, using the definition $MSV_u^o(y)$, and by the complementary slackness conditions (37), (38) and (44):

- If $a_u^o(y) \in (0, 1)$, then $\lambda_{o1} = \lambda_{o0} = 0$, and $\Delta_u y - \lambda_h = 0$ or $MSV_u^o(y) = \lambda_h$.
- If $a_u^o(y) = 0$, then $\lambda_{o1} = 0$, $\lambda_{o0} \geq 0$, and $\Delta_u y - \lambda_h \leq 0$, or $MSV_u^o(y) \leq \lambda_h$.
- If $a_u^o(y) = 1$, then $\lambda_{o1} \geq 0$, $\lambda_{o0} = 0$, and $\Delta_u y - \lambda_h \geq 0$, or $MSV_u^o(y) \geq \lambda_h$.

The value of the Lagrange multiplier corresponding to (14) at the solution (λ_h^*) provides the marginal increase in aggregate utility derived from assigning households with income equal to their type's border income to the urban area. By lemma 3 and the results above, households with income higher than their type's border level have $MSV_u^k(y) > \lambda_h^*$ and are all assigned to the centre; those with income satisfying $\forall y < \tilde{y}^k$ have $MSV_u^k(y) < \lambda_h^*$, and all reside in the suburbs in an efficient allocation; while households with income satisfying $y = \tilde{y}^k$ have $MSV_u^k(y) = \lambda_h^*$ and may be assigned to any district.

Noting that border MSV functions are continuous and that their domain is the compact set $[\underline{y}^i, \bar{y}^i]$, a corner solution with every out-of-school household residing in the centre emerges if and only if $\mu_u^i(\underline{y}^i) \geq \mu_u^o(\underline{y}^i) = \Delta_u \bar{y}$. If the latter inequality does not hold, i.e. if $\mu_u^i(\underline{y}^i) < \mu_u^o(\underline{y}^i)$ then it may be that these two functions cross for some $y \in (\underline{y}^i, \bar{y}^i)$ in which case an interior solution arises with $\mu_u^i(y) = \mu_u^o(y)$. Finally, if they do not cross in that interval, that is, if $\mu_u^i(\bar{y}^i) < \mu_u^o(\bar{y}^i) = \Delta_u \underline{y}$, then a corner solution with every out-of-school household living in the centre emerges.