Pricing of Discretely Sampled Asian Options under Lévy Processes

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Abstract. We develop a new method for pricing options on discretely sampled arithmetic average in exponential Lévy models. The main idea is the reduction to a backward induction procedure for the difference W_n between the Asian option with averaging over n sampling periods and the price of the European option with maturity one period. This allows for an efficient truncation of the state space. At each step of backward induction, W_n is calculated accurately and fast using a piece-wise interpolation or splines, fast convolution and either flat iFT and (refined) iFFT or the parabolic iFT. Numerical results demonstrate the advantages of the method.

Keywords: Option pricing, flat iFT method, parabolic iFT method, FFT, refined and enhanced FFT, Lévy processes, KoBoL, CGMY, BM, Asian options.

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CHAPTER 1

Introduction

An Asian option is an option whose terminal payoff depends on the average values of the underlying asset during some period of the option's lifetime. Thus, an Asian option is path-dependent. Asian-style derivatives constitute an important family of derivative securities with a wide variety of applications in financial markets. There are a number of economic reasons to write a contract with the an averaging feature. For example, in the foreign exchange market, Asian options allow one to reduce risk of exchange risk fluctuations, and such an option is typically cheaper than European options. Another reason is that the price manipulation becomes more difficult, even in the thinly traded asset market near option's maturity.

There are Asian options both of the European-style and the American-style, the averaging can be either arithmetic or geometric, and the sampling can be either continuous or discrete. The payoffs of continuously sampled arithmetic average Asian options are of the form

$$\frac{1}{T - T_0} \int_{T_0}^{T} w_u S_u du,$$

where w_t are weights; if the sampling is discrete, the payoffs are of the form

$$\frac{1}{N+1} \sum_{j=0}^{N} w_{T_j} S_{T_j},$$

where $0 \leq T_0 < T_1 \ldots < T_N = T$ are the sampling dates. It is easy to see that the joint distribution of the arithmetic average is quite complicated to characterize analytically, and derivation of efficient explicit formulas for prices of Asian option is very involved, with some exceptions.

1. Pricing in the exponential BM model

In the exponential Brownian motion model (Black-Scholes model), pricing of the continuously sampled geometric average options is easy and quite straightforward (and the same holds in Lévy models and a number of other popular models). However, in the same exponential Brownian motion model, pricing the arithmetic average Asians options is far from trivial. In early studies of continuously sampled Asian option, such options were approximated by the geometric average Asian options. This approach significantly underprices the option (see the discussion in Musiela and Rutkowski [**56**] and the references therein). To overcome this deficiency, a number of authors have suggested various analytical approximations for the distribution of the arithmetic average. For example, Turnbull and Wakeman [**62**] use the lognormal approximation with matched first and second moments, Milevsky and Posner [**55**] use the reciprocal gamma approximation. The problem with approximations of this sort is that no reliable error estimates are available.

One of the most celebrated analytical tools is to use the Laplace transform of the Asian option price, introduced by Geman and Yor [**38**]. Later, the approach has been extensively studied, for example, by Fu *et al.* [**35**], Carr and Schröder [**24, 25**], Shaw [**59**]. Linetsky [**45**] takes a new direction, and derives analytical formulas using spectral expansion for the value of the continuously sampled arithmetic Asian option.

While the methods above are tailored to continuously sampled Asian option, other popular numerical approaches, such as the PDE-approach (see, e.g., Večeř [**63, 64**], Rogers and Shi [**57**], Alziary *et al.* [**5**], Andreasen [**6**], Lipton [**46**], Zhang [**66, 67**], Dewynne and Shaw [**31**] and the bibliography therein) and the Monte Carlo method (MC) (see, e.g., Kemna and Vorst [**39**], Boyle *et al.* [**19**]) can be applied to both continuously and discretely sampled Asian options.

For the discretely sampled Asian options, another approach is to numerically evaluate the density of the sum of random variables as the convolution of individual densities. The second step of this method involves numerical integration of the option's payoff function with respect to this density function. The method is initiated by Carverhill and Clewlow [**26**], who rely on the use of the fast Fourier transform to evaluate the joint probability density function.

The pricing of Asian option in the Black-Scholes model has been dealt with by a host of researchers, and the list of papers above is by no means complete.

2. Pricing in exponential Lévy models

Lévy models provide a better fit to empirical asset price distributions that typically have fatter tails than Gaussian ones, and can reproduce volatility smile phenomena in option prices. For an introduction to applications of these models applied to finance, we refer the reader to S. Boyarchenko and Levendorskiǐ [**17**] and Cont and Tankov [**30**]. Pricing the continuously sampled geometric average options in exponential Lévy models is easy and quite straightforward (see S.Boyarchenko and Levendorskiǐ [**17**]) but pricing of arithmetic Asians presents serious mathematical and computational difficulties.

For continuously sampled Asian options, Bayraktar and Xing [**8**] derive analytical formulas which can be realized numerically fairly fast and accurately. Cai and Kou [**20**] obtain a closed-form solution for the double-Laplace transform of Asian options under hyper-exponential jump diffusion model, and suggest a numerical procedure for the realization of this closed form solution. The variant of the PDE-approach due to Večeř [**63, 64**] was extended to processes with jumps by Večeř and Xu [**65**].

For discretely sampled Asian options, Benhamou [**9**], Fusai and Meucci [**36**] enhance the method of approximating the joint probability density function of [**26**], and price Asians under Lévy processes. Since the transition density function of a Lévy process is generally unknown in the closed form, the accuracy of this approach crucially rely on the accuracy on an approximation of the transition density function. Černý and Kyriakou [**27**] reduce the pricing problem to a sequence of European options in the one-factor model, and use trapezoidal rule as the numerical realization of the (inverse) Fourier transform to approximate the prices of European options. More recently, Chen *et al.* [**28, 29**] developed a Monte Carlo algorithm for simulating Lévy processes, based on calculation of the pdf using the Fourier inversion, and applied this algorithm to pricing discretely sampled Asian option. In Albrecher et al. [**1, 2, 3, 4**], Lemmens *et al.* [**42**], pricing bounds for Asian options under Lévy processes are derived and hedging strategies analyzed.

3. General structure of our method

In the present thesis, we consider discretely averaged Asian options. As Černý and Kyriakou [**27**], we first reduce the pricing problem to a series of pricing of European options. However, the efficiency of an approach of this kind strongly depends on the type of the constructed sequence of European options, and the efficiency of the pricing the European options which inevitably have rather complicated payoffs. Therefore, it is necessary to employ additional tricks to enhance the accuracy and speed of calculations, and an efficient control of several sources of errors is of the paramount importance.

We use the reduction to a series of European options, whose terminal payoffs vanish at $-\infty$ and on $[0, \infty)$. Given the error tolerance, we choose $x_1 < x_M < 0$, and, for each option, replace the payoff on $(-\infty, x_1]$ with the leading term of asymptotics, which is of the form $c \cdot e^x$, $c \in \mathbb{R}$. This trick is efficient, and $|x_1|$ can be chosen moderately large in absolute value. On $[x_M, 0]$, we set the payoff to 0. On $[x_1, x_M]$, we approximate the payoff by a piece-wise polynomial function or use splines. We derive the bounds for the partial truncation and interpolation errors. The derivation is fairly non-trivial due to a rather complicated structure of the payoff. These bounds are used to give recommendations for the choice of the parameters of our numerical scheme.

At each step of the induction procedure, after the modification of the payoff have been made, we can calculate the price of the European option explicitly using the Fourier inversion. We rewrite the resulting formula as a sum of products of the values of the payoff function at the points of the chosen uniform grid, and values of two auxiliary functions, at points of a grid of the form $\ell\Delta$, $-M \le \ell \le M$. These two functions are the same at each step of the backward induction procedure (in the case of higher order interpolation, more than two functions are needed). The sum can be represented as two terms plus a discrete convolution, which can be realized very fast using the fast convolution algorithm (this idea goes back to Eydeland [**32**]). The values of the auxiliary functions can be calculated only once (and stored); various realizations of the inverse

Fourier transform can be used for this purpose. We use the simplified trapezoid rule, which is very efficient if the integrand is analytic in a strip, and allows for an efficient error control. The reason is that the disretization error of the infinite trapezoid rule decays exponentially as the mesh approaches 0 if the integrand is an analytic function in a strip around the line of integration. Crucial to this analysis are the fundamental properties of the Whittaker cardinal series (Sinc expansion) for functions that are analytic in a strip, introduced to finance in Feng and Linetsky [**34**]. The conformal change of variables introduced in S. Boyarchenko and Levendorskiĭ [**18**] (*parabolic iFT* method) allows one to greatly increase the rate of the decay of the integrand at infinity, and decrease the number of terms in the simplified trapezoid rule and the CPU time.

4. Organization of the thesis

In Chapter 2, we explain the main idea and outline the steps of our method. Error estimates and recommendations for the choices of the parameters of the scheme given the error tolerance are derived in Chapters 3 and 4. In Chapter 5, we present an explicit algorithm, and produce numerical examples to illustrate the relative performance of different methods. Chapter 6 concludes. In Appendix A, we give an overview of necessary facts of the theory of Lévy processes, and remind to the reader the general pricing formulas for the options of the European type. Technical details are relegated to Appendix B and Appendix C. Several possible directions in which our method can be developed further are outlined in Appendix D.

CHAPTER 2

Pricing Asian calls and puts

1. Option specification and the general scheme of calculation

We assume that the riskless rate $r \geq 0$ is constant, the market is arbitrage free, the underlying asset pays no dividends, and, under an *equivalent martingale measure* (EMM) $\mathbb{Q}$ chosen for pricing, the spot price process for the underlying is an exponential of a Lévy process X: $S_t = e^{X_t}$. Let

$$A(T; \{S_{T_j}\}_{j=0}^N) = \frac{1}{N+1} \sum_{j=0}^{N} S_{T_j}$$

be the arithmetic average of the asset price, where the dates $(0 =)T_0 < T_1 < \cdots < T_N (= T)$ are specified in the contract. The terminal payoffs of the Asian put and call options with strike K and maturity date T are

$$G(K, T; \{S_{T_j}\}_{j=0}^N) = (K - A(T; \{S_{T_j}\}_{j=0}^N))_+$$

and

$$G(K, T; \{S_{T_j}\}_{j=0}^N) = (A(T; \{S_{T_j}\}_{j=0}^N) - K)_+,$$

respectively; and the time-0 price of the corresponding option is given by

$$\mathcal{V}(K, T; S_0; \{T_j\}_{j=0}^N) = e^{-rT} \mathbb{E} \left[G(K, T; \{S_{T_j}\}_{j=0}^N) \mid S_0 \right], \qquad (2.1)$$

where $\mathbb{E}$ is the expectation operation under $\mathbb{Q}$, and S_0 is the current spot price of the underlying. We will consider the pricing of the Asian put option; pricing the Asian call option can be reduced to pricing the Asian put option via the put-call parity:

$$\mathcal{V}^c(K,T;\{S_{T_j}\}_{j=0}^N) = e^{-rT}\left(\mathbb{E}\left[A(T;\{S_{T_j}\}_{j=0}^N) \mid S_0\right] - K\right) \\ +\mathcal{V}(K,T;S_0;\{T_j\}_{j=0}^N), \tag{2.2}$$

where $\mathcal{V}$ is the price of the Asian put option. The first term on the RHS is the weighted sum of S_0 and the moment generating functions of $X_{T_j}|X_0$, $j = 1,2,\ldots,N$, hence, it can be easily calculated. It remains to calculate the price of the Asian put option.

Rewrite the terminal payoffs of the Asian put option G as

$$G(K,T;\{S_{T_j}\}_{j=0}^N) = \frac{1}{N+1}\left((N+1)K - S_0 - \sum_{j=1}^N S_{T_j}\right)_+ .$$

If $(N+1)K - S_0 \le 0$, then $G(K,T;\{S_{T_j}\}_{j=0}^N) = 0$, and the time-0 price of the option is 0. Below, we assume that

$$(N+1)K - S_0 > 0,$$

and set

$$x = \ln S_0 - \ln((N+1)K - S_0).$$

Assuming further that the sampling dates are equally spaced: $T_j = j\bar{\Delta}$, where $\bar{\Delta} = T/N$, and taking into account that X is a Lévy process, we rewrite (2.1) as

$$\mathcal{V}(K,T;S_0;\{T_j\}_{j=0}^N) = \frac{e^{-rT}}{N+1}((N+1)K - S_0)_+ V_N(x), \tag{2.3}$$

where

$$V_n(x) = \mathbb{E}^x\left[(1 - e^{X_{\bar{\Delta}}} - \cdots - e^{X_{n\bar{\Delta}}})_+\right], \quad n = 1,2,\ldots,N \tag{2.4}$$

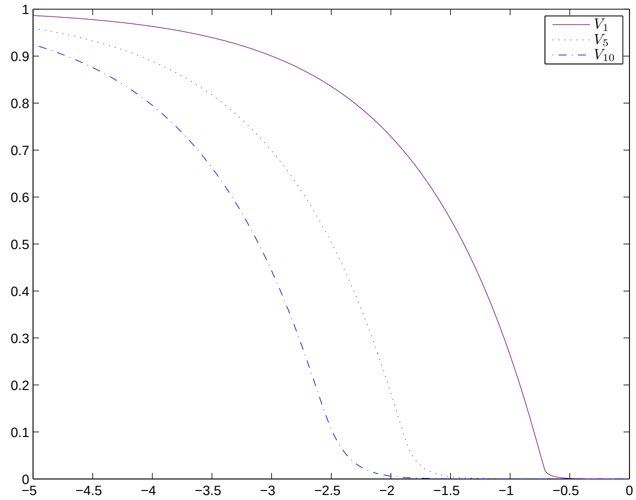

FIGURE 1. Typical curves of $(1-e^x)_+ \cdot V_n(x-\ln(1-e^x)_+)$. *KoBoL parameters:* $\nu = 0.2$, $c_+ = c_- = 1.1136$, $\lambda_+ = 3$, $\lambda_- = -10$, $\mu \approx 0.30403$. *Asian put option parameters:* $r = 0.04$, $T = 1$, $S = 100$, $N = 12$ $(\bar{\Delta} = T/N)$.

and $\mathbb{E}_t^x[\cdot] = \mathbb{E}[\cdot \mid X_t = x]$. Using the law of iterated expectations, we obtain

$$\begin{aligned} V_n(x) = \mathbb{E}^x \Big[(1-e^{X_{\bar{\Delta}}})_+ \cdot \mathbb{E}_{\bar{\Delta}}^{X_{\bar{\Delta}}} \Big[\Big(1 - e^{X_{2\bar{\Delta}} - \ln(1-e^{X_{\bar{\Delta}}})_+} \\ - \cdots - e^{X_{n\bar{\Delta}} - \ln(1-e^{X_{\bar{\Delta}}})_+}\Big)_+\Big]\Big] \\ = \mathbb{E}^x \left[(1-e^{X_{\bar{\Delta}}})_+ \cdot V_{n-1}\left(X_{\bar{\Delta}} - \ln(1-e^{X_{\bar{\Delta}}})_+\right)\right]. \end{aligned} \tag{2.5}$$

In Figure 1, we illustrate how the curve $(1-e^x)_+ \cdot V_n(x - \ln(1-e^x)_+)$ behaves.

To reduce the truncation error in the state space, we introduce

$$W_n(x) = V_n(x) - V_1(x).$$

For $n = 1$, $W_1(x) \equiv 0$; for $n = 2$,

$$W_2(x) = \mathbb{E}^x[(1-e^{X_{\bar{\Delta}}})_+ \cdot (V_1(X_{\bar{\Delta}} - \ln(1-e^{X_{\bar{\Delta}}})_+) - 1)]; \tag{2.6}$$

and for $n = 2, 3, \ldots, N-1$,

$$W_{n+1}(x) = \mathbb{E}^x[(1 - e^{X_{\bar{\Delta}}})_+ \cdot W_n(X_{\bar{\Delta}} - \ln(1 - e^{X_{\bar{\Delta}}})_+)] + W_2(x). \quad (2.7)$$

Introduce

$$f_1(x) \quad = \quad (1 - e^x)_+ \cdot (V_1(x - \ln(1 - e^x)_+) - 1), \quad (2.8)$$

and, for $n = 2, 3, \ldots, N-1$, define

$$f_n(x) \quad = \quad (1 - e^x)_+ \cdot W_n(x - \ln(1 - e^x)_+). \quad (2.9)$$

Formally, the calculation of the option price using (2.6) and (2.7) is quite straightforward:

1. calculate $V_1(x) = \mathbb{E}^x[(1 - e^{X_{\bar{\Delta}}})_+]$ (at the points of the chosen grid);
2. approximate f_1 in (2.8) with piece-wise polynomials on $[x_1, x_M]$ and by an exponential function on $(-\infty, x_1]$, substitute the result into (2.6) and calculate W_2;
3. in the cycle w.r.t $n = 2, \ldots, N-1$, approximate f_n in (2.9) by piece-wise polynomials on $[x_1, x_M]$ and by an exponential function on $(-\infty, x_1]$, substitute into (2.7) and calculate W_{n+1};
4. calculate $V_N(y) = W_N(y) + V_1(y)$ at $y = \ln S_0 - \ln((N+1)S_0 - K)$, and then the option value using (2.3).

At each step of calculation, there are several numerical parameters we need to choose, namely,

- mesh Δ and truncation parameters x_1, x_M in the state space domain, which define the x-grid; and
- line of integration ω, mesh ζ and truncation parameter Λ in the frequency domain, which define the dual grid.

(For a detailed numerical algorithm, see Chapter 5 Section 1.) Since V_1 is the price of the put option with strike 1, maturity $\bar{\Delta}$, in the model with zero riskless rate, V_1 can be easily calculated using the Fourier transform technique (see S. Boyarchenko and Levendorskiĭ [**18**] for the review). Since the Fourier transform of the piece-wise polynomial interpolant is a rational function, which can be easily calculated, W_n can be calculated almost as easily as V_1, but it is non-trivial to take the several errors involved into account accurately, and derive sufficiently simple and accurate recommendations for the choice of the parameters of the numerical scheme.

In the remaining part of this chapter, we list the main formulas of the numerical scheme; the derivation of error bounds, recommendations for the choice of the parameters of the numerical scheme, and proofs are relegated to the following chapters.

Remark 2.1. A more efficient approach is to use

$$W_n(x) = V_n(x) - V_{n-1}(x)$$

for $n \geq 2$. Then, W_2 is as in (2.6), and for $n = 2, 3, \ldots, N-1$,

$$\begin{aligned} V_n(x) &= V_{n-1}(x) - W_n(x) \\ W_{n+1}(x) &= \mathbb{E}^x[(1 - e^{X_{\bar{\Delta}}})_+ \cdot W_n(X_{\bar{\Delta}} - \ln(1 - e^{X_{\bar{\Delta}}})_+)]. \end{aligned}$$

Since this approach produce the exact same result for the choice of the truncation parameters given in Chapter 3 Section 1, we will use the definition $W_n = V_n - V_1$.

2. Classes of processes

We recall that every Lévy process $X = \{X_t\}_{t\geq 0}$ has a characteristic exponent, which is a continuous function $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ satisfying $\psi(0) = 0$

and

$$\mathbb{E}\big[e^{i\xi X_t}\big] = e^{-t\psi(\xi)} \qquad \forall\, \xi \in \mathbb{R},\ t \geq 0;$$

and, conversely, the law of a Lévy process is uniquely determined by its characteristic exponent ([**58**, Theorem 7.10, Proposition 2.5]). (See Appendix A for an overview of necessary facts of the theory of Lévy processes.)

Let $\lambda_- < -1 < 0 < \lambda_+$. We assume that $\psi(\xi)$ of X is of the form

$$\psi(\xi) = -i\mu\xi + \psi^0(\xi),$$

where $\psi^0(\xi)$ admits the analytic continuation into the complex plane with the cuts $i(-\infty, \lambda_-]$ and $i[\lambda_+, +\infty)$, and has the following asymptotics as $\rho \to +\infty$: for any $\phi \in (-\pi/2, \pi/2)$, $\omega_+ \in (\lambda_-, \lambda_+)$,

$$\psi^0(i\omega_+ + e^{i\phi}\rho) \quad \sim \quad d_+^0 e^{i\phi\nu}\rho^\nu(1 + O(\rho^{-1})) \tag{2.10}$$

$$\partial_\rho \operatorname{Re}\psi^0(i\omega_+ + e^{i\phi}\rho) \quad \sim \quad \nu d_+^0 \cos(\phi\nu)\rho^{\nu-1}(1 + o(1)) \tag{2.11}$$

where $d_+^0 > 0$. These properties are valid for wide classes of processes used in the theoretical and empirical studies of financial markets. See M. Boyarchenko *et al.* [**11**], Levendorskiĭ [**43**].

Examples 2.2. a) The main example is KoBoL (see S. Boyarchenko and Levendorskiĭ [**15, 16, 17**]) of order $\nu \in (0, 2), \nu \neq 1$, with the characteristic exponent

$$\psi^0(\xi) = \Gamma(-\nu)[c_+({\lambda_+}^\nu - (\lambda_+ + i\xi)^\nu) + c_-((-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu)], \tag{2.12}$$

where $c_\pm > 0$, $\lambda_- < 0 < \lambda_+$. In this thesis, we will make explicit calculations in the almost symmetric case

$$c_+ = c_- = c,$$

which is also known as CGMY model (see Carr *et al.* [**21**]). It is easily seen that, in this case, (2.10)–(2.11) hold with $d_+^0 = -2c\Gamma(-\nu)\cos(\nu\pi/2) > 0$.

b) Normal Inverse Gaussian processes (NIG) constructed by Barndorff-Nielsen [**7**] satisfy (2.10)–(2.11) with $\nu = 1$ and $d_+^0 = \delta$, where δ is the intensity parameter of NIG. Almost all processes of β-class (Kuznetsov [**41**]) satisfy (2.10) as well.

c) VG model introduced to finance by Madan and co-authors [**53, 52, 51**]

$$\psi^0(\xi) = c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)] \tag{2.13}$$

(we use this non-standard parametrization of VG model to make an analogy with KoBoL more transparent), hence, $\psi^0(\xi)$ stabilizes to $2c\ln|\xi|$ at infinity. This implies that the option price is rather irregular unless $\bar\Delta$ is fairly large. In the result, the justification of even the piece-wise linear interpolation is possible only for sufficiently large $\bar\Delta$, and a higher order interpolation is, essentially, impossible in the sense that it will lead to large discretization errors. If (2.10) holds with $\nu > 0$, the option price is of class C^∞; however, for small ν and/or $\bar\Delta$ the derivatives can be too large, and, therefore, a higher order interpolation has a larger interpolation error.

Remark 2.3. In (2.10)–(2.11), we only need the asymptotics for $\phi \in (-\pi/2, \pi/2)$, since for any $\phi \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$,

$$\psi^0(\rho e^{i\phi}) = \overline{\psi^0(\rho e^{i\phi'})}, \quad \phi' \in (-\pi/2, \pi/2).$$

Explicitly, let $a := \mathrm{Re}(\rho e^{i\phi}) < 0$, $b := \mathrm{Im}(\rho e^{i\phi}) \in \mathbb{R} \setminus \{0\}$, the following identities holds:

$$\begin{aligned} e^{-t\overline{\psi(a+ib)}} = \overline{e^{-t\psi(a+ib)}} &= \overline{\mathbb{E}[e^{i(a+ib)X_t}]} \\ &= \mathbb{E}[\overline{e^{i(a+ib)X_t}}] \\ &= \mathbb{E}[e^{i(-a+ib)X_t}] = e^{-t\psi(-a+ib)}, \quad t \geq 0 \end{aligned}$$

therefore, $\overline{\psi(a+ib)} = \psi(-a+ib)$, and

$$\overline{\psi^0(\rho e^{i\phi})} = \overline{\psi^0(a+ib)} = \psi^0(-a+ib) = \psi^0(\rho e^{i\phi'}),$$

where $\phi' \in (-\pi/2, \pi/2)$.

3. Calculation of V_1: flat iFT method

In this section, we calculate $V_1(x) = \mathbb{E}^x[(1-e^{X_{\bar{\Delta}}})_+]$ by using the flat iFT method. Take $\omega \in (0, \lambda_+)$, expand $G(x) = (1-e^x)_+$ into the Fourier integral

$$G(x) = (2\pi)^{-1} \int_{\mathrm{Im}\,\xi=\omega} e^{ix\xi} \hat{G}(\xi) d\xi \tag{2.14}$$

where $\hat{G}$ is the Fourier transform of G:

$$\hat{G}(\xi) = \int_{-\infty}^{0} e^{-ix\xi}(1-e^x)dx = -\frac{1}{\xi(\xi+i)}, \tag{2.15}$$

and substitute (2.14) into $V_1(x) = \mathbb{E}^x[G(X_{\bar{\Delta}})]$. Since $\mathrm{Re}\,\psi(\xi)$ is uniformly bounded from below on any horizontal line inside the strip of analyticity $\mathrm{Im}\,\xi \in (\lambda_-, \lambda_+)$ of $\psi(\xi)$, we can apply Fubini's theorem, and, using $\mathbb{E}^x[e^{i\xi X_{\bar{\Delta}}}] = e^{ix\xi - \bar{\Delta}\psi(\xi)}$, obtain

$$V_1(x) = \frac{1}{2\pi} \int_{\mathrm{Im}\,\xi=\omega} e^{ix\xi - \bar{\Delta}\psi(\xi)} \hat{G}(\xi) d\xi. \tag{2.16}$$

By making the change the variable $\xi = i\omega + \eta$, we obtain

$$V_1(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix(i\omega+\eta)-\bar{\Delta}\psi_(i\omega+\eta)}\hat{G}(i\omega+\eta)d\eta. \tag{2.17}$$

Taking into account that, for real η and ω,

- $\overline{e^{ix(i\omega+\eta)}} = e^{ix(i\omega-\eta)}$;
- $\overline{\hat{G}(i\omega+\eta)} = \hat{G}(i\omega-\eta)$;
- in particular, the characteristic function of a real-valued random variable, hence the characteristic exponent ψ, enjoy the same property:

$$\begin{aligned}\overline{e^{-\bar{\Delta}\psi(i\omega+\eta)}} &= \overline{\mathbb{E}\left[e^{i(i\omega+\eta)X_{\bar{\Delta}}}\right]} \\ &= \mathbb{E}\left[\overline{e^{i(i\omega+\eta)X_{\bar{\Delta}}}}\right] \\ &= \mathbb{E}[e^{i(i\omega-\eta)X_{\bar{\Delta}}}] = e^{-\bar{\Delta}\psi(i\omega-\eta)}.\end{aligned}$$

Applying the above properties to (2.17), we obtain

$$\begin{aligned} V_1(x) &= \frac{1}{2\pi}\left(\int_{-\infty}^{0}+\int_{0}^{\infty}\right) e^{ix(i\omega+\eta)-\bar{\Delta}\psi_(i\omega+\eta)}\hat{G}(i\omega+\eta)d\eta \\ &= \frac{1}{2\pi}\int_0^\infty \Big(e^{ix(i\omega-\eta)-\bar{\Delta}\psi_(i\omega-\eta)}\hat{G}(i\omega-\eta) \\ &\qquad\qquad + e^{ix(i\omega+\eta)-\bar{\Delta}\psi_(i\omega+\eta)}\hat{G}(i\omega+\eta)\Big)\, d\eta \\ &= \frac{1}{2\pi}\int_0^\infty \Big(\overline{e^{ix(i\omega+\eta)-\bar{\Delta}\psi_(i\omega+\eta)}\hat{G}(i\omega+\eta)} \\ &\qquad\qquad + e^{ix(i\omega+\eta)-\bar{\Delta}\psi_(i\omega+\eta)}\hat{G}(i\omega+\eta)\Big)\, d\eta \\ &= \frac{1}{\pi}\,\mathrm{Re}\left[\int_0^\infty e^{ix(i\omega+\eta)-\bar{\Delta}\psi_(i\omega+\eta)}\hat{G}(i\omega+\eta)d\eta\right]. \end{aligned} \tag{2.18}$$

If ν, the order of the process, and/or $\bar{\Delta}$, the interval between the sampling dates, are not small, then the integral (2.16) can be calculated very fast with the absolute error of order 10^{-7}–10^{-8} using the simplified trapezoid rule

$$V_1(x) \approx -\frac{\zeta}{\pi}\,\mathrm{Re}\left[\sum_{j=1}^{M_c} \frac{e^{ix\xi_j-\bar{\Delta}\psi(\xi_j)}}{\xi_j(\xi_j+i)}(1-\delta_{j1}/2)\right], \tag{2.19}$$

where δ_{jk} is Kronecker's delta, $\xi_j = i\omega + \zeta(j-1)$, $j = 1, 2, \dots, M_c$, are uniformly spaced points on the line of integration with a moderately large number of points M_c. If, in addition, the values $V_1(x_k)$ are needed at points of an equally spaced grid $x_j = x_1 + (j-1)\Delta$, $j = 1, 2, \dots, M$, then the iFFT can be used to increase the speed of calculations.

REMARKS 2.4. a) Equation (2.16) holds not only for $\omega \in (0, \lambda_+)$, for example, by setting $\omega \in (\lambda_-, -1)$, we obtain the price of the standard European call option with the same strike and expiry date.

b) For V_1 in (2.16), the integrand has simple poles at $-i$ and 0, and is analytic in the strips $\mathrm{Im}\,\xi \in (\lambda_-, -1)$, $\mathrm{Im}\,\xi \in (-1, 0)$ and $\mathrm{Im}\,\xi \in (0, \lambda_+)$. It follows from Theorem 4.1 that the wider the strip of analyticity is, the smaller the discretization error can be made by an appropriate choice of the line of integration inside the strip. Therefore, if the three strips above are of sizably different widths, it is advantageous to choose the widest strip. We can move from one strip to another using the residue theorem.

If we push the line of integration in (2.16) down and cross the pole at zero but remain above the second pole, we obtain

$$V_1(x) = 1 - (2\pi)^{-1} \int_{\mathrm{Im}\,\xi=\omega} \frac{e^{ix\xi-\bar{\Delta}\psi(\xi)}}{\xi(\xi+i)} d\xi, \tag{2.20}$$

for any $\omega \in (-1, 0)$. This is advantageous if $\lambda_+ < 1$ and $-\lambda_- - 1 < 1$. If $-\lambda_- - 1 > \max\{\lambda_+, 1\}$, we push the line of integration further down, thereby re-deriving the put-call parity relation

$$V_1(x) = 1 - e^{x-\bar{\Delta}\psi(-i)} - (2\pi)^{-1} \int_{\mathrm{Im}\,\xi=\omega} \frac{e^{ix\xi-\bar{\Delta}\psi(\xi)}}{\xi(\xi+i)} d\xi, \tag{2.21}$$

with $\omega \in (\lambda_-, -1)$.

A shift of this sort is especially useful if $x' := x + \mu\bar{\Delta}$ is negative and large in absolute value as it is quite often the case in the lower part of the x-grid.

c) If ν and/or $\bar{\Delta}$ are small then the integrand decays very slowly at infinity, and too many terms may be needed to satisfy the desired error tolerance. In these cases, the preliminary conformal deformation of the contour of integration in (2.16) with the subsequent change of the variables (parabolic iFT) can be used to greatly decrease the number of terms in the simplified trapezoid rule (see Chapter 4 and S. Boyarchenko and Levendorskiĭ [**18**]).

d) An alternative is the refined version of iFFT, which reduces the calculation of the sum in (2.19) to application of several copies of iFFT (see M. Boyarchenko and Levendorskiĭ [**12, 13**]), but if the integrand decays very slowly then parabolic iFT becomes indispensable.

4. Calculation of W_{n+1}, $n = 1, 2, \ldots, N-1$

4.1. Partial truncation of f_n. Let f_n, $n = 1, 2, \ldots, N-1$ be as in (2.8) and (2.9). The intervals of integration in (2.6) and (2.7) will be truncated from above at a point $x_M < 0$, equivalently, f_n will be set to 0. On $(-\infty, x_1]$, we replace each integrand in (2.6) and (2.7) with the leading term of the asymptotics as $x \to -\infty$. The bounds of the partial truncation errors and recommendations for the choice of x_1 and x_M (given the desired error tolerance ϵ at each step) will be given in Chapter 3 Section 1. The starting point is the following lemma.

LEMMA 2.5. *Let X be a Lévy process, whose characteristic exponent is analytic in a strip* $\operatorname{Im}\xi \in (\lambda_-, \lambda_+)$ *around the real axis. Then, for*

$n = 2, \dots, N$, *and any* $\omega \in (\lambda_-, 0]$, *as* $x \to -\infty$,

$$f_1(x) = -e^{x - \bar{\Delta}\psi(-i)} + O(e^{-\omega x - \bar{\Delta}\psi(i\omega)}) \tag{2.22}$$

$$f_n(x) = c_n e^x + O(e^{-\omega x - \bar{\Delta}\psi(i\omega)}), \qquad n = 2, 3, \dots, N \tag{2.23}$$

where

$$c_n = -e^{-2\bar{\Delta}\psi(-i)} \cdot \frac{1 - e^{(1-n)\bar{\Delta}\psi(-i)}}{1 - e^{-\bar{\Delta}\psi(-i)}}. \tag{2.24}$$

For the proof, see Appendix B. Note that (2.22) and (2.23) are useful only if we take $\omega \in (\lambda_-, -1)$.

4.2. Payoff modification in the state space. When we calculate W_{n+1}, we assume that $f_n(x_j)$, $j = 1, 2, \dots, M$, have been calculated. We set $u_M = 0$, $u_1 = c_n e^{x_1}$, where $c_1 = -e^{-\bar{\Delta}\psi(-i)}$, and c_n is given by (2.24) if $n \geq 2$, use $u_j = f_n(x_j)$, $j = 2, 3, \dots, M-1$, and approximate f_n by the function u defined by:

(1) $u(x) = c_n e^x$, $x \leq x_1$;
(2) $u(x) = 0$, $x_M \leq x < \infty$; and
(3)

$$u(x) = u_j + \frac{u_{j+1} - u_j}{\Delta}(x - x_j), \quad x_j \leq x \leq x_{j+1},$$

for $j = 2, 3, \dots, M-1$.

The last part defines the piece-wise linear interpolation on $[x_1, x_M]$. For the interpolation procedures of higher order and spline approximations, see Appendix C.

The Fourier transform $\hat{u}$ is easy to calculate. Fix $\omega \in (0, \lambda_+)$. For ξ on the line $\operatorname{Im}\xi = \omega$,

$$\begin{aligned}\hat{u}(\xi) &= c_n e^{x_1} \cdot \widehat{\mathcal{U}}_{tr}(\xi) - u_1 \cdot e^{-ix_1\xi}\widehat{\mathcal{U}}_M(\xi) \\ &\quad +(1/\Delta)\sum_{j=1}^{M} u_j \cdot e^{-ix_j\xi}\widehat{\mathcal{U}}(\xi),\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}\widehat{\mathcal{U}}_{tr}(\xi) &= (1 - i\xi)^{-1}e^{-i\xi x_1}, \\ \widehat{\mathcal{U}}_M(\xi) &= (i\xi\Delta)^{-2}\left(e^{i\xi\Delta} - i\xi\Delta - 1\right), \\ \widehat{\mathcal{U}}(\xi) &= (i\xi)^{-2}\left(e^{i\xi\Delta} + e^{-i\xi\Delta} - 2\right).\end{aligned}$$

Applying $u_1 = c_n e^{x_1}$ and rearranging the first two terms in (2.25), we obtain

$$\begin{aligned}\hat{u}(\xi) &= -u_1 \cdot e^{-ix_1\xi}\left(\hat{G}(\xi) + \widehat{\mathcal{U}}_1(\xi)/\Delta\right) \\ &\quad +(1/\Delta)\sum_{j=1}^{M} u_j \cdot e^{-ix_j\xi}\widehat{\mathcal{U}}(\xi),\end{aligned} \tag{2.26}$$

where $\hat{G}$ is as in (2.15), and

$$\widehat{\mathcal{U}}_1(\xi) = (i\xi)^{-2}\left(e^{i\xi\Delta} - 1\right).$$

For an integer $s \geq 2$, define

$$\mathcal{V}_s(x) = \frac{1}{2\pi}\int_{\operatorname{Im}\xi=\omega} \frac{e^{ix\xi - \bar{\Delta}\psi(\xi)}}{(i\xi)^s}d\xi. \tag{2.27}$$

(when the piece-wise linear interpolation is used, only $\mathcal{V}_2$ will be needed; $\mathcal{V}_s$, $s > 2$, appear if an interpolation procedure of a higher order is used).

Using (2.27) and (2.26), we obtain

$$\begin{aligned}
& \mathbb{E}^x \left[u(X_{\bar\Delta})\right] \\
= \;& (2\pi)^{-1} \int_{\mathrm{Im}\,\xi=\omega} e^{ix\xi-\bar\Delta\psi(\xi)}\hat{u}(\xi)d\xi \\
= \;& -u_1 \cdot \left(V_1(x-x_1) + \frac{\mathcal{V}_2(x-x_0)-\mathcal{V}_2(x-x_1)}{\Delta}\right) \\
& + \sum_{j=1}^{M} u_j \cdot \frac{\mathcal{V}_2(x-x_{j-1}) - 2\mathcal{V}_2(x-x_j) + \mathcal{V}_2(x-x_{j+1})}{\Delta}, \quad (2.28)
\end{aligned}$$

where $x_0 = x_1 - \Delta$ and $x_{M+1} = x_1 + \Delta$.

Functions $\mathcal{V}_s$ can be calculated similarly to V_1:

$$\mathcal{V}_s(x) \approx \frac{\zeta}{\pi}\,\mathrm{Re}\left[\sum_{j=1}^{M_c} \frac{e^{ix\xi_j-\bar\Delta\psi(\xi_j)}}{(i\xi_j)^s}(1-\delta_{j1}/2)\right], \quad (2.29)$$

where δ_{jk} is Kronecker's delta, $\xi_j = i\omega + \zeta(j-1)$, $j = 1, 2, \dots, M_c$, are uniformly spaced points on the line of integration with a moderately large number of points M_c. For the analysis of the impact of the errors of the calculation of $\mathcal{V}_2$ on the errors of the numerical realization of (2.28), and recommendations for the choice of ω and mesh, see Chapter 4.

The following remark is similar to the one in Remarks 2.4.

Remark 2.6. The integrand in (2.27) has a pole of order s at zero, and is analytic in the strips $\mathrm{Im}\,\xi \in (\lambda_-, 0)$ and $\mathrm{Im}\,\xi \in (0, \lambda_+)$. It follows from Theorem 4.1 that the wider the strip of analyticity is, the smaller the discretization error can be made by an appropriate choice of the line of integration inside the strip. Therefore, if the two strips above are of sizably different widths, it is advantageous to choose the widest strip. We can move from one strip to another using the residue theorem.

If we push the line of integration in (2.27) down and cross the pole, we obtain

$$\mathcal{V}_s(x) = -\frac{i^{1-s}}{(s-1)!}\frac{d^{s-1}}{d\xi^{s-1}}e^{ix'\xi-\bar{\Delta}\psi^0(\xi)}\Big|_{\xi=0} + \frac{1}{2\pi}\int_{\mathrm{Im}\,\xi=\omega}\frac{e^{ix\xi-\bar{\Delta}\psi(\xi)}}{(i\xi)^s}d\xi, \quad (2.30)$$

for any $\omega \in (\lambda_-, 0)$.

4.3. Efficient realizations of (2.28)**.** At each step of backward induction, we use (2.28) to calculate values of

$$\mathbb{E}^y[f_n(X_{\bar{\Delta}})] \approx \mathbb{E}^y[u(X_{\bar{\Delta}})]$$

at points $y = y_k := x_k - \ln(1-e^{x_k})_+$, $k = 1, 2, \dots, M$, as follows:

1. extend the grid $(x_j)_{j=1}^M$ to $(x_j)_{j=1}^{M_1}$, where $M_1 \ge M$ is the minimal integer such that $x_{M_1} \ge x_M - \ln(1-e^{x_M})_+$;
2. calculate $V_1(\ell\Delta)$ for $\ell = 0, 1, \dots, M_1 - 1$, and $\mathcal{V}_2(\ell\Delta)$ for $-M_1 \le \ell \le M_1$ using flat iFT and (refined) iFFT or parabolic iFT;
3. calculate $\mathbb{E}^{x_k}[u(X_{\bar{\Delta}})]$, $k = 1, 2, \dots, M_1$, using (2.28) and fast convolution;
4. calculate $\mathbb{E}^{y_j}[u(X_{\bar{\Delta}})]$, $j = 1, 2, \dots, M$, using an appropriate interpolation procedure.

The interpolation error at Step 4 admits a bound similar to the bound for the error of interpolation of f_n by u on (x_1, x_M) (see Chapter 3 Section 3), therefore, if we use the same interpolation procedure in both cases, we can take the second error into account by multiplying the bound for the first interpolation error by 2.

CHAPTER 3

Calculation and errors in the state space

1. Partial truncation error

LEMMA 3.1. *We have*

$$\|W_n\|_{L_\infty} \le 1, \quad n = 2, 3, \dots \tag{3.1}$$

PROOF. Since $0 < (1 - e^{x_1})_+ \le 1$ for all x_1, we have

$$0 < V_1(x) = \mathbb{E}^x[(1 - e^{X_{\bar\Delta}})_+] < 1 \quad \text{for all } x.$$

Similarly, for $n \ge 2$, applying

$$V_n(x) = \mathbb{E}^x\left[(1 - e^{X_{\bar\Delta}} - \cdots - e^{X_{n\bar\Delta}})_+\right]$$

and using the inequality

$$0 < (1 - e^{x_1} - \cdots - e^{x_n})_+ \le (1 - e^{x_1})_+ \le 1,$$

we obtain

$$0 < V_n(x) < V_1(x) < 1.$$

Therefore,

$$-1 < -V_1(x) < V_n(x) - V_1(x) < 0.$$

Using the definition: $W_n = V_n - V_1$, we obtain (3.1). □

LEMMA 3.2. *Let X be a Lévy process, whose characteristic exponent is analytic in a strip $\operatorname{Im}\xi \in (\lambda_-, \lambda_+)$ around the real axis, and f_n be as in*

(2.9). *Then, for $n = 2, 3, \ldots, N$, and any $\omega \in [0, \lambda_+)$, we have*

$$|f_n(x)| \le e^{-\bar{\Delta}\psi(i\omega)}(-x)^{1+\omega}(1 + o(1)) \quad as \quad x \uparrow 0. \tag{3.2}$$

Proof. W_n are expectations of functions that are bounded by 1 and vanish above 0. Hence, for any $\omega \in [0, \lambda_+)$,

$$|W_n(x)| \le \mathbb{E}^x[\ _{(-\infty,0)}(X_{\bar{\Delta}})] \le \mathbb{E}^x[e^{-\omega X_{\bar{\Delta}}}] = e^{-\omega x - \bar{\Delta}\psi(i\omega)}.$$

Let $y = x - \ln(1 - e^x)$, where $x < 0$. As $x \uparrow 0$, $y \to +\infty$, therefore, for $x \in (-\infty, 0)$:

$$|f_n(x)| = |(1 - e^x)_+ W_n(x - \ln(1 - e^x)_+)| \le e^{-\bar{\Delta}\psi(i\omega) - \omega x}(1 - e^x)^{1+\omega}.$$

It remains to apply the Taylor formula to $(1 - e^x)$ around $x = 0$. $\square$

To calculate W_{n+1} using (2.7), we need values of W_n and W_2. When we use an interpolation procedure to approximate $f_n(x)$ in (2.8)–(2.9), we need values of the latter function on $(-\infty, 0)$. We truncate $f_n(x)$ at $x = x_M < 0$, where x_M is found from the condition $|f_n(x)| \le \epsilon$, for all $x \in (x_M, 0)$, and $\epsilon > 0$ is the error tolerance. Let ϵ_1 be the truncation error at each step, and set $\epsilon = \epsilon_1/2$. Using (3.2), we find an approximate recommendation

$$x_M = -\left(\epsilon \cdot e^{\bar{\Delta}\cdot\psi(i\omega_+)}\right)^{1/(1+\omega_+)}, \tag{3.3}$$

where $\omega_+ \in (0, \lambda_+)$. If λ_+ is not very large, and $\psi(i\lambda_+) < \infty$ as it is the case with KoBoL and NIG, then we can use (3.3) with $\omega_+ = \lambda_+$. If λ_+ is not large and X is VG, or if λ_+ is very large, we can take ω_+ so that $\bar{\Delta}\psi(i\omega_+) = 0.1 \ln \epsilon$, and set

$$x_M = -\epsilon^{1.1/(1+\omega_+)}. \tag{3.4}$$

On the strength of Lemma 2.5, we choose the lowest point $x_1 \le \ln S_0 - \ln((N+1)K - S_0)$ of the grid so that, for all $x \in [x_1, x_{M_1}]$,

$$C \cdot \mathbb{E}^x \left[e^{-\omega_- X_{\bar{\Delta}} - \bar{\Delta}\psi(i\omega_-)} \quad {}_{(-\infty,x_1)}(X_{\bar{\Delta}}) \right] \le \epsilon,$$

where C is the constant in the O-terms in (2.22) and (2.23). Clearly, we can find x_1 from

$$x_1 = (\ln(\epsilon/C) + \bar{\Delta}\psi(i\omega_-))/(-\omega_-).$$

The constant C is unknown. If ϵ is very small, then $\ln(\epsilon/C) \approx \ln \epsilon$, and we can use an approximate recommendation

$$x_1 = (1.1 \ln \epsilon + \bar{\Delta}\psi(i\omega_-))/(-\omega_-). \tag{3.5}$$

If X is KoBoL or NIG and λ_- is not very large, then we may use (3.5) with $\omega_- = \lambda_-/2$. Otherwise, we can take ω_- so that $\bar{\Delta}\psi(i\omega_-) = 0.1 \ln \epsilon$.

Since

$$|\mathbb{E}^x[u(X_{\bar{\Delta}})]| \le \|u\|_{L_\infty},$$

the total truncation error is bounded by the sum of truncation errors at each step of backward induction. If ϵ_{tr} is the error tolerance for the total truncation error, and N is the number of steps, we choose $\epsilon_1 = \epsilon_{tr}/(N-1)$. When we derive estimates for the other types of errors at the current step, we may assume that there is no truncation at all. Similarly, we may assume that the calculation of the expectation that serves as an input at the next induction step contains no truncation error. The same argument will be used below to account for the impact of the other sources of errors. In the nutshell, each error estimate can be derived as if there have been no errors before.

2. Interpolation: preliminary error estimates

Denote by $p_{\bar{\Delta}}$ the transition density of X. In Appendix B, we prove

LEMMA 3.3. *a) Let X satisfy* (2.10) *and* (2.11).

Then, $\forall\ s \in \mathbb{Z}_+$, the following approximate bound holds

$$\|p_{\bar{\Delta}}^{(s)}\|_{L_1} \le \frac{2\Gamma(s/\nu)}{(d_+^0)^{s/\nu}\pi\nu D(s)}\bar{\Delta}^{-s/\nu}, \tag{3.6}$$

where

$$D(s) = \sup_{\phi\in(0,\min\{\pi/2,\pi/(2\nu)\}}(\cos(\phi\nu))^{s/\nu}\cos(\phi-\pi/2).$$

b) Let X be a VG, and let $s \in \mathbb{Z}_+$, $s < 2c\bar{\Delta}$, where $c > 0$ is the constant in (2.13).

Then the following approximate bound holds

$$\|p_{\bar{\Delta}}^{(s)}\|_{L_1} \le \frac{2}{\pi\cos(\phi-\pi/2)(2c\bar{\Delta}-s)}. \tag{3.7}$$

Lemma 3.3 gives a simple approximate upper bound for $\|p_{\bar{\Delta}}^{(s)}\|_{L_1}$ in the cases of KoBoL and VG models. For NIG, one can derive an estimate similar to the one for KoBoL.

If (2.10) holds with $\nu > 0$, the option price is of class C^∞; however, for small ν and/or $\bar{\Delta}$ the derivatives can be too large, and therefore, a higher order interpolation will have a larger interpolation error. In the VG model, the option price is rather irregular unless $\bar{\Delta}$ is fairly large. In the result, the justification of even the piece-wise linear interpolation used in the backward induction procedure is possible only for sufficiently large $\bar{\Delta}$, and a higher order interpolation is, essentially, impossible in the sense that it will lead to large discretization errors.

Let f_n be as in (2.8)–(2.9). In Appendix B, we prove

LEMMA 3.4. *For any integer $n \geq 1$ and $s \geq 2$ such that $\|p_{\bar{\Delta}}^{(s-1)}\|_{L_1} < \infty$,*

$$\left|f_n^{(s)}(x)\right| \leq 1 + (2(n+\delta_{n1}) - 3)\left(s + \sum_{j=2}^{s}\binom{s}{j}\|p_{\bar{\Delta}}^{(j-1)}\|_{L_1}\right) \tag{3.8}$$

In the case of the piece-wise linear interpolation, we need the simplest case of bound (3.8): for any $n \geq 1$,

$$\left|f_n^{(2)}(x)\right| \leq 1 + (2(n+\delta_{n1}) - 3)(\|p'_{\bar{\Delta}}\|_{L_1} + 2) \tag{3.9}$$

3. Interpolation error

A grid $\vec{x}$ on $[x_1, x_M]$ is chosen to interpolate functions f_n; the integral over $(-\infty, x_1]$ is calculated replacing $f_n(x)$ with $c_n e^x$, where c_n is given by (2.24) (this is the leading term of asymptotics in (2.22) (resp., (2.23)) if $n = 1$ (resp., if $n \geq 2$)). We use a uniformly spaced grid

$$x_j = x_l + (j-1)\Delta, \quad j = 1, 2, \ldots, M,$$

where $\Delta = (x_M - x_1)/(M-1)$. Assuming that a polynomial or spline interpolation procedure $f_{n;app}$ of order s is chosen, with the error estimate of the form

$$\|f_n - f_{n;app}\|_{L_\infty} \leq C_s\|f_n^{(s+1)}\|_{L_\infty}\Delta^{s+1}, \tag{3.10}$$

where C_s is a universal constant[1], we need to derive an estimate for the norm of the derivative on the RHS. Lemma 3.4 provides simple approximate upper bounds (3.8) for $\|f_n^{(s+1)}\|_{L_\infty}$, hence, for the discretization error. For instance, in the case of the piece-wise linear interpolation, using the error estimate (3.10) and the bounds (3.9), we find that the total sum

[1]Recall that in the case of the piece-wise linear interpolation, $C_2 = 1/8$, in the cases of cubic interpolation and cubic splines, $C_4 = 1/24$, and, in the case of cubic Hermite splines, $C_4 = 1/384$

of interpolation errors at all steps of the backward induction procedure, err_{int}, admits an approximate upper bound via

$$\mathrm{err}_{int} \le \frac{1}{8}\Delta^2 \left((3 + \|p'_{\bar{\Delta}}\|_{L_1}) \cdot N + \sum_{n=2}^{N} (1 + (2n-3)(\|p'_{\bar{\Delta}}\|_{L_1} + 2)) \right). \tag{3.11}$$

By simplifying the RHS, we have

$$\mathrm{err}_{int} \le \frac{1}{8}\Delta^2 \left((2N^2 + 1) + (N^2 - N + 1)\|p'_{\bar{\Delta}}\|_{L_1} \right).$$

For a process of order $\nu > 0$, applying (3.6), we obtain

$$\mathrm{err}_{int} \le \frac{1}{8}\Delta^2 \left((2N^2 + 1) + (N^2 - N + 1)\frac{2\Gamma(1/\nu)}{(d^0_+)^{1/\nu}\pi\nu D(1)}\bar{\Delta}^{-1/\nu} \right),$$

where

$$D(1) = \sup_{\phi\in(0,\min\{\pi/2,\pi/(2\nu)\}} (\cos(\phi\nu))^{1/\nu}\cos(\phi - \pi/2).$$

Similarly, one can obtain an error bound for the case of VG model.

Let ϵ_{int} be the error tolerance allocated for the total interpolation error. Lemmas 3.4 and 3.3 taken together allow us to choose Δ as a function of ϵ_{int}, $\bar{\Delta}$, parameters of the process, and s, the order of the interpolation procedure. We choose s with the maximal $\Delta = \Delta_s$; if several Δ_s are rather close, we choose the interpolation procedure with the smallest s.

If one uses an interpolation procedure to obtain values of V_1 and W_n at points

$$x = y_k := x_k - \ln(1 - e^{x_k})_+, \quad k = 1, 2, \ldots, M$$

(see Step 4 in Chapter 2 Section 4.3), one needs to take into account the errors of this interpolation. Suppose that a polynomial or spline interpolation procedure $V_{1;app}$ and $W_{n;app}$ of order s is chosen, with the error estimates similar to (3.10)

$$\|V_1 - V_{1;app}\|_{L_\infty} \leq C_s \|V_1^{(s+1)}\|_{L_\infty} \Delta^{s+1} \quad (3.12)$$

$$\|W_n - W_{n;app}\|_{L_\infty} \leq C_s \|W_n^{(s+1)}\|_{L_\infty} \Delta^{s+1}. \quad (3.13)$$

We will prove in Lemma B.3, that the constants in the RHSs of (3.12) and (3.13) are smaller than the constant in the bound in (3.10), hence, it suffices to choose the interpolation scheme and Δ as above, for $\epsilon_{int}/2$ instead of ϵ_{int}.

CHAPTER 4

Calculation and errors in the dual space

In this chapter, we consider the case of the piece-wise linear interpolation, and analyze the impact of errors of calculation of values of V_1 and $\mathcal{V}_2$ on the error of W_n calculated using (2.28). (The interpolation procedures of higher order can be analyzed similarly — see Appendix C.) We need to satisfy a rather large error tolerance for two sequences (of values of V_1 and $\mathcal{V}_2/\Delta$) in L_∞-norm when a very small factor u_1 is present and a fairly small error tolerance in L_1-norm of the sequence of values of $\mathcal{V}_2/\Delta$. In both cases, it is unnecessary to calculate the values with high accuracy near $x' = 0$, where it is especially difficult to satisfy a small error tolerance (see S. Boyarchenko and Levendorskiǐ [**18**]).

1. General estimates

Let $\epsilon_c > 0$ be the desired error tolerance for the error induced by calculation of V_1 and $\mathcal{V}_2$ at each step. Denote by $\epsilon(V, y)$ the absolute errors of the calculation of function $V = V_1$, $\mathcal{V}_s$ using flat iFT or parabolic iFT. Then the absolute error of the calculation of $\mathbb{E}^x[u(X_{\bar{\Delta}})]$ using (2.28) is bounded by

$$\begin{aligned}
&|u_1| \cdot \max_{0\le\ell\le M_1-1} \epsilon(V_1, \ell\Delta) \\
&+|u_1| \cdot (2/\Delta) \max_{0\le\ell\le M_1} \epsilon(\mathcal{V}_2, \ell\Delta) \\
&+(4/\Delta) \cdot \max_{1\le k\le M} \sum_{\ell} |u_k| \cdot \epsilon(\mathcal{V}_2, (k-\ell)\Delta). \qquad (4.1)
\end{aligned}$$

Since $|u_1|$ is small, we can make the first two term in (4.1) very small using a dual grid of a fairly large mesh ζ and rather small number of points M_c. For small Δ, the third term admits a fairly accurate bound via the L_1-norm of $\epsilon(\mathcal{V}_2, \cdot)$ times the L_∞-norm of u. The latter being bounded by 1, it suffices to satisfy the bounds

$$\max_{0\le\ell\le M_1-1} \epsilon(V_1, \ell\Delta) \le 0.05 \cdot \epsilon_c/|u_1| \tag{4.2}$$

$$\max_{0\le\ell\le M_1} \epsilon(\mathcal{V}_2, \ell\Delta) \le 0.05 \cdot \Delta \cdot \epsilon_c/(2|u_1|) \tag{4.3}$$

$$\|\epsilon(\mathcal{V}_2, \cdot)\|_{L_1} < 0.9 \cdot \Delta \cdot \epsilon_c/4. \tag{4.4}$$

Note that if $\bar{\Delta}$ and ν are not very small, and (4.4) is satisfied, then, typically, (4.2) and (4.3) are satisfied as well.

The error $\epsilon(V, \cdot)$, $V = V_1, \mathcal{V}_s$ consists of the discretization and truncation errors, which are denoted by $\epsilon_d(V, \cdot)$ and $\epsilon_{tr}(V, \cdot)$, respectively. In the remaining part of the chapter, we follow the study of Boyarchenko and Levendorskiĭ [**18**]. First, we derive error bounds $\epsilon_d(V, \cdot)$, $\epsilon_{tr}(V, \cdot)$, and $\|\epsilon(\mathcal{V}_2, \cdot)\|_{L_1}$ of the flat iFT method. These bounds are used to derive fairly accurate prescriptions for numerical parameters. Next, we apply parabolic iFT method to the calculation of V_1 and $\mathcal{V}_s$, and derive their error bounds and choice of numerical parameters.

We try to keep this chapter of the thesis as self-contained as possible. All the necessary definitions, facts and key ideas are summarized in Section 2.1 and 3.3. For more detailed exposition, see [**18**].

2. Flat iFT: error analysis

2.1. Preliminary.

2.1.1. *Discretization error.* As Feng and Linetsky [**34**], Boyarchenko and Levendorskiĭ [**18**], we start with Stenger's therem [**60**, Theorem 3.2.1], which we reformulate in accordance with our system of notation. Let

$$\mathcal{D}(\mu_-, \mu_+) := \{\xi \mid \operatorname{Im} \xi \in (\mu_-, \mu_+)\}$$

be the strip in the complex plane. Let $H^1(\mathcal{D}(\mu_-, \mu_+))$ denote the Hardy space of functions g satisfying

(1) for $\xi \in \mathcal{D}(\mu_-, \mu_+)$, $g(\xi)$ is analytic;

(2)
$$\int_{\mu_-}^{\mu_+} |g(\eta + i\omega)| d\omega \to 0 \quad \text{as} \quad \eta \to \pm\infty;$$

(3) the Hardy norm is finite:

$$\|g\|_{\mathcal{D}(\mu_-,\mu_+)} := \lim_{\omega \uparrow \mu_+} \int_{\mathbb{R}} |g(\eta + i\omega)| d\eta + \lim_{\omega \downarrow \mu_-} \int_{\mathbb{R}} |g(\eta + i\omega)| d\eta < \infty. \quad (4.5)$$

Take $\omega \in (\mu_-, \mu_+)$. We are interested in the error of the replacement of the integral

$$V = \int_{\operatorname{Im} \xi = \omega} g(\xi) d\xi$$

with the infinite sum:

$$\zeta \cdot \sum_{l \in \mathbb{Z}} g(\xi_l),$$

where $\xi_l = l\zeta + i\omega$, $l \in \mathbb{Z}$, $\zeta > 0$. Denote by $\epsilon_d(V; \omega, \zeta)$ the error

$$\epsilon_d(V; \omega, \zeta) = \int_{\operatorname{Im} \xi = \omega} g(\xi) d\xi - \zeta \cdot \sum_{l \in \mathbb{Z}} g(\xi_l).$$

For $\omega \in (\mu_-, \mu_+)$, set

$$d(\omega) = \min\{\omega - \mu_-, \mu_+ - \omega\}.$$

The discretization error ϵ_d admits an upper bound via

THEOREM 4.1. [**60**, Theorem 3.2.1]

$$|\epsilon_d(V;\omega,\zeta)| \le \frac{e^{-2\pi d(\omega)/\zeta}}{1-e^{-2\pi d(\omega)/\zeta}}\|g\|_{\mathcal{D}(\mu_-,\mu_+)}.$$

To derive fairly accurate prescriptions for an almost optimal choice of ζ for a given error tolerance, one needs to estimate the Hardy norm and apply the following corollary.

COROLLARY 4.2. [**18**, Corollary 2.2] *Let the error tolerance $\epsilon > 0$ for the discretization error be small so that $\epsilon_g := \epsilon/\|g\|_{\mathcal{D}(\mu_-,\mu_+)} < 1$. If*

$$\zeta \le 2\pi d(\omega)/\ln(1+1/\epsilon_g), \tag{4.6}$$

then $\epsilon_d(V;\omega,\zeta) \le \epsilon$.

We first consider the case of V_1. Apply Theorem 4.1 with $g(\xi)$ the integrand in (2.16) multiplied by $(2\pi)^{-1}$:

$$g(\xi) = -(2\pi)^{-1}\frac{e^{ix\xi-\bar{\Delta}\psi(\xi)}}{\xi(\xi+i)}.$$

Function g is analytic in the strips

- $[\mu_-,\mu_+] \subset (0,\lambda_+)$,
- $[\mu_-,\mu_+] \subset (-1,0)$,
- $[\mu_-,\mu_+] \subset (\lambda_-,-1)$.

Using the notation of ϵ_d, we can write the error of the replacement of the exact option price (2.16) with the infinite sum:

$$\epsilon_d(V_1;\omega,\zeta) = \int_{\operatorname{Im}\xi=\omega} g(\xi)d\xi - \zeta\cdot\sum_{l\in\mathbb{Z}} g(\xi_l),$$

where $\xi_l = l\zeta + i\omega$, $l\in\mathbb{Z}$, $\zeta>0$; and, for $\omega\in(\mu_-,\mu_+)$.

Proposition 4.3. [**18**, Proposition 2.3] *For any* $[\mu_-, \mu_+] \subset (0, \lambda_+)$, *or* $[\mu_-, \mu_+] \subset (-1, 0)$, *or* $[\mu_-, \mu_+] \subset (-\lambda_-, -1)$, *the error* ϵ_d *admits an upper bound via*

$$|\epsilon_d(V_1; \omega, \zeta)| \le \frac{e^{-2\pi d(\omega)/\zeta}}{1 - e^{-2\pi d(\omega)/\zeta}} \|g\|_{\mathcal{D}(\mu_-,\mu_+)}, \tag{4.7}$$

where $\omega \in (\mu_-, \mu_+)$, $d(\omega) = \min\{\omega - \mu_-, \mu_+ - \omega\}$. *Moreover, the Hardy norm* $\|g\|_{\mathcal{D}(\mu_-,\mu_+)}$ *given by* (4.5) *admits an estimate via*

a) For any $[\mu_-, \mu_+] \subset (0, \lambda_+)$ *or* $[\mu_-, \mu_+] \subset (-1, 0)$, $\|g\|_{\mathcal{D}(\mu_-,\mu_+)}$ *admits an estimate via*

$$\|g\|_{\mathcal{D}(\mu_-,\mu_+)} \le \sum_{\gamma=\{\mu_-,\mu_+\}} \frac{e^{-\gamma x - \bar{\Delta}\psi(i\gamma)}}{2|\gamma|}.$$

b) For any $[\mu_-, \mu_+] \subset (\lambda_-, -1)$, $\|g\|_{\mathcal{D}(\mu_-,\mu_+)}$ *admits an estimate via*

$$\|g\|_{\mathcal{D}(\mu_-,\mu_+)} \le \sum_{\gamma=\{\mu_-,\mu_+\}} \frac{e^{-\gamma x - \bar{\Delta}\psi(i\gamma)}}{2|-\gamma - 1|}.$$

(Note that the result is slightly different from the one in [**18**] due to the typo in the latter result.)

Below, we consider the error control of the numerical scheme for $\mathcal{V}_s$ with $s \ge 2$. Denote by $g_s(\xi)$ the integrand in (2.27) multiplied by $(2\pi)^{-1}$:

$$g_s(\xi) = \frac{1}{2\pi} \frac{e^{i\xi x - \bar{\Delta}\psi(\xi)}}{(i\xi)^s}.$$

Function g_s is analytic in two strips $\operatorname{Im}\xi \in (0, \lambda_+)$ and $\operatorname{Im}\xi \in (\lambda_-, 0)$ (note the slight difference with the case of the function g above). We take $0 < \mu_- < \omega < \mu_+ < \lambda_+$; if we push the line of integration below the real line, we will have to consider $\lambda_- < \mu_- < \omega < \mu_+ < 0$.

Setting $\xi_l = l\zeta + i\omega$, $l \in \mathbb{Z}$, $\zeta > 0$, and replacing the integral in (2.27) with infinite sum, we obtain

$$\mathcal{V}_s(x) \approx \zeta \cdot \sum_{l \in \mathbb{Z}} g_s(\xi_l). \tag{4.8}$$

The discretization error

$$\epsilon_d(\mathcal{V}_s; \omega, \zeta) = \int_{\mathrm{Im}\,\xi=\omega} g_s(\xi) d\xi - \zeta \cdot \sum_{l \in \mathbb{Z}} g_s(\xi_l)$$

admits the following bound.

PROPOSITION 4.4. *For any* $[\mu_-, \mu_+] \subset (0, \lambda_+)$ *or* $[\mu_-, \mu_+] \subset (\lambda_-, 0)$, *the error of the replacement of the exact integral* (2.27) *with the infinite sum* (4.8) *admits an upper bound via*

$$|\epsilon_d(V_s; \omega, \zeta)| \le \frac{e^{-2\pi d(\omega)/\zeta}}{1 - e^{-2\pi d(\omega)/\zeta}} \|g_s\|_{\mathcal{D}(\mu_-, \mu_+)}, \tag{4.9}$$

where $\omega \in (\mu_-, \mu_+)$, $d(\omega) = \min\{\omega - \mu_-, \mu_+ - \omega\}$. *Moreover, the Hardy norm* $\|g_s\|_{\mathcal{D}(\mu_-, \mu_+)}$ *given by* (4.5) *admits an estimate via*

$$\|g_s\|_{\mathcal{D}(\mu_-, \mu_+)} \le \sum_{\gamma = \{\mu_-, \mu_+\}} \frac{e^{-\gamma x - \bar{\Delta}\psi(i\gamma)}}{\pi |\gamma|^{s-1}} \cdot D_s, \tag{4.10}$$

where

$$D_s = \begin{cases} \frac{\pi}{2} & \text{if } s = 2 \\ 1 & \text{if } s = 3 \\ \frac{\pi}{2} \times \frac{1}{2} \times \frac{3}{4} \times \cdots \times \frac{s-3}{s-2} & \text{if } s = 4, 6, \dots \\ \frac{2}{3} \times \frac{4}{5} \times \cdots \times \frac{s-3}{s-2} & \text{if } s = 5, 7, \dots \end{cases} \tag{4.11}$$

PROOF. (4.9) follows from Theorem 4.1. To estimate $\|g_s\|_{\mathcal{D}(\mu_-, \mu_+)}$, we consider

$$I = \int_{\mathbb{R}} |g_s(\eta + i\omega)| d\eta = \frac{e^{-\omega x}}{2\pi} \int_{\mathbb{R}} |e^{-\bar{\Delta}\psi(\eta + i\omega)}| \cdot |i(\eta + i\omega)|^{-s} d\eta.$$

Since

$$\begin{aligned} \left|e^{-\bar{\Delta}\psi(\eta+i\omega)}\right| = \left|\mathbb{E}[e^{i(\eta+i\omega)X_{\bar{\Delta}}}]\right| \le \mathbb{E}\left[\left|e^{i(\eta+i\omega)X_{\bar{\Delta}}}\right|\right] &= \mathbb{E}[e^{-\omega X_{\bar{\Delta}}}] \\ &= e^{-\bar{\Delta}\psi(i\omega)}, \end{aligned}$$

and $|\eta+i\omega|^{-s} = (\eta^2+\omega^2)^{-s/2}$,

$$I \le \pi^{-1}e^{-\omega x - \bar{\Delta}\psi(i\omega)}\int_0^\infty (\eta^2+\omega^2)^{-s/2}d\eta.$$

The change of variables $\eta \mapsto \omega \cdot \tan\theta$ transforms the integral into

$$|\omega|^{1-s}\int_0^{\pi/2}(\cos\theta)^{s-2}d\theta,$$

and the statement of the proposition follows. □

2.1.2. *Truncation error.* The integrand in the Fourier inversion formula which defines V_1 decays at infinity as the integrand in the formula for $\mathcal{V}_2$, therefore, it suffices to consider the truncation error of the infinite sum in the formula for $\mathcal{V}_s$, $s \ge 2$. Making the change of variable $\xi \mapsto \eta + i\omega$, we rewrite (2.27) as

$$\mathcal{V}_s(x) = \int_{\mathbb{R}} g_s(\eta+i\omega)d\eta = \frac{e^{-\omega x}}{2\pi}\int_{\mathbb{R}} \frac{e^{i\eta x - \bar{\Delta}\psi(\eta+i\omega)}}{(i(\eta+i\omega))^s}d\eta. \tag{4.12}$$

If ζ is small, the truncation error of the replacement of the infinite sum (4.8) with the finite sum:

$$\zeta \cdot \sum_{l\in\mathbb{Z}} g_s(\xi_l) - \zeta \cdot \sum_{-M_c}^{M_c} g_s(\xi_l)$$

approximately equals to the truncation error of the replacement of the improper integral in (4.12) with the definite integral:

$$\int_{\mathbb{R}} g_s(\eta+i\omega)d\eta - \int_{-\Lambda}^{\Lambda} g_s(\eta+i\omega)d\eta,$$

where $\Lambda \approx M_c\zeta$. Denote by ϵ_{tr} the error of this replacement.

PROPOSITION 4.5. *In the case of KoBoL of order $\nu \in (0,2)$, $\nu \neq 1$,*

$$|\epsilon_{tr}(x;\Lambda)| \le \frac{e^{-\omega(x+\mu\bar{\Delta})}}{\pi\nu} \cdot C_2 \cdot (\bar{\Delta}C_\infty)^{(s-1)/\nu} \int_A^\infty e^{-t} t^{(1-s)/\nu-1} dt, \quad (4.13)$$

where $A = A(\Lambda) := \bar{\Delta}C_\infty\Lambda^\nu$,

$$C_2 = \exp(-\bar{\Delta}c\Gamma(-\nu)({\lambda_+}^\nu + (-\lambda_-)^\nu)),$$

$$C_\infty = C_\infty(\Lambda) := -c\Gamma(-\nu)\,\mathrm{Re}\left\{(-i)^\nu + i^\nu + \nu(\lambda_+ i^{\nu-1} - \lambda_-(-i)^{\nu-1})/\Lambda\right\}.$$

PROOF. It is sufficient to show that

$$|\epsilon_{tr}(x;\Lambda)| \le \frac{e^{-\omega(x+\mu\bar{\Delta})}}{\pi} \int_\Lambda^\infty |e^{-\bar{\Delta}\psi^0(\eta+i\omega)}|\eta^{-s} d\eta,$$

where $\psi^0(\xi) := \psi(\xi) + i\mu\xi$. As $\Lambda \to +\infty$,

$$|e^{-\bar{\Delta}\psi^0(\eta+i\omega)}| \le C_2(1+o(1))e^{-\bar{\Delta}C_\infty|\eta|^\nu}.$$

Therefore, we have an approximate upper bound

$$|\epsilon_{tr}(x;\Lambda)| \le \pi^{-1}e^{-\omega(x+\mu\bar{\Delta})}C_2 \int_\Lambda^\infty e^{-\bar{\Delta}C_\infty\eta^\nu}\eta^{-s}d\eta.$$

Making the change of variable: $\bar{\Delta}C_\infty\eta^\nu \mapsto t$, we obtain the bound (4.13). $\square$

Note that the integral

$$\int_x^\infty e^{-t}t^{a-1}dt$$

is the upper incomplete gamma function (if $a > 0$) or an exponential integral (if $a = 0$) or reducible to the upper incomplete gamma function via integration by parts (if $a < 0$).

2.2. Choice of ω, ζ, M_c. We consider the case when both $|\lambda_-|$ and λ_+ are not small. To calculate V_1, we take $\omega \in (0, \lambda_+)$ if $x' = x + \mu\bar{\Delta} \geq 0$; otherwise, we push the line of integration down, and use (2.21) with $\omega \in (\lambda_-, -1)$. Similarly, when calculating $\mathcal{V}_s$, we take $\omega \in (0, \lambda_+)$ if $x' \geq 0$; otherwise, we use (2.30) with $\omega \in (\lambda_-, 0)$.

Given a small desired error tolerance $\epsilon > 0$ for the discretization error, we choose ω and ζ using Corollary 4.2 and Proposition 4.4. If $\|g_s\|_{\mathcal{D}(\mu_-,\mu_+)}/\epsilon \gg 1$, it suffices to choose

$$\zeta = 2\pi d(\omega)/(\ln(\|g_s\|_{\mathcal{D}(\mu_-,\mu_+)}) - \ln \epsilon). \tag{4.14}$$

We choose (μ_-, μ_+) and $\omega \in (\mu_-, \mu_+)$ to maximize the RHS in (4.14). First, set $\omega = (\mu_+ - \mu_-)/2$, then $d(\omega) = (\mu_+ - \mu_-)/2$. Next, simplifying the bound (4.10),

$$\|g_s\|_{\mathcal{D}(\mu_-,\mu_+)} \leq 2 \cdot \max_{\gamma=\{\mu_-,\mu_+\}} \frac{e^{-\gamma x - \bar{\Delta}\psi(i\gamma)}}{\pi|\gamma|^{s-1}} \cdot D_s,$$

where D_s is the constant as in (4.11), we find that

$$\zeta = 2\pi \cdot d(\omega) \cdot \left(\ln(2\pi^{-1}D_s/\epsilon) + \min_{\gamma=\mu_\pm}\left(-\gamma x - \bar{\Delta}\psi(i\gamma) + (1-s)\ln|\gamma|\right)\right)^{-1}$$

implies (4.14). Finally, for a small error tolerance, we can simplify the choice following the prescription in [**18**]. Set $\epsilon_1 = \epsilon\pi/(2D_s)$.

(I) If $x' = x + \mu\bar{\Delta} \geq 0$, then
  1. if $-0.1 \cdot \ln \epsilon_1 \geq -\bar{\Delta}\psi^0(i\lambda_+ - 0) + (1-s)\ln \lambda_+$ or $\nu < 1$, set $\mu_+ = \lambda_+$, otherwise, find μ_+ as the unique positive solution of equation $-0.1 \cdot \ln \epsilon_1 = -\bar{\Delta}\psi^0(i\mu_+) + (1-s)\ln \mu_+$;
  2. set $\omega_+ = \mu_+/2$, and $\zeta_+ = -\pi\mu_+/(1.1 \ln \epsilon)$.

(II) If $x' < 0$, then, for the calculation of $V_1(x)$ (resp., $\mathcal{V}_s(x)$),

1. if $-0.1 \cdot \ln \epsilon_1 \geq -\bar{\Delta}\psi^0(i\lambda_- + 0) + (1-s)\ln(-\lambda_-)$ or $\nu < 1$, set $\mu_- = \lambda_-$, otherwise, find μ_- as the unique positive solution of equation $-0.1 \cdot \ln \epsilon = -\bar{\Delta}\psi^0(i\mu_-) + (1-s)\ln(-\mu_-)$;
2. set $\omega_- = (\mu_- - 1)/2$ (resp., $\omega_- = \mu_-/2$), and $\zeta_- = \pi(-\mu_- - 1)/(1.1 \ln \epsilon)$ (resp., $\zeta_- = \pi\mu_-/(1.1 \ln \epsilon)$).

For the choice of M_c for $x' \geq 0$ (resp., $x' < 0$), one can first apply (4.13) to find Λ, then use ζ above to choose the positive integer M_c such that $M_c\zeta \geq \Lambda$.

2.3. Error estimate of $\|\epsilon(\mathcal{V}_2, \cdot)\|_{L_1}$ and choices of ω, ζ, M_c. Consider the case when flat iFT method is used to calculate $\mathcal{V}_2(x')$ ($x' = x + \mu\bar{\Delta}$). Denote by $\epsilon_d(\mathcal{V}_2, x')$ and $\epsilon_{tr}(\mathcal{V}_2, x')$ the discretization and truncation errors of the calculation of $\mathcal{V}_2(x')$. Since

$$\|\epsilon(\mathcal{V}_2, \cdot)\|_{L_1} = \|\epsilon_d(\mathcal{V}_2, \cdot)\|_{L_1} + \|\epsilon_{tr}(\mathcal{V}_2, \cdot)\|_{L_1},$$

it suffices to consider each term on the RHS above separately.

Let

$$L_d(\gamma, \mu_-, \mu_+) = \frac{e^{-2\pi d(\omega)/\zeta}}{1 - e^{-2\pi d(\omega)/\zeta}} \cdot \frac{e^{-\bar{\Delta}\psi(i\gamma)}}{\pi|\gamma|} \cdot D_2, \tag{4.15}$$

where ω, $d(\omega)$ and D_2 are as in Proposition 4.4, and L_t denote the RHS in the upper bounds (4.13) when $x = x' - \mu\bar{\Delta} = -\mu\bar{\Delta}$.

Lemma 4.6.

$$\begin{aligned} \Delta \cdot \|\epsilon_d(\mathcal{V}_2, \cdot)\|_{L_1} \quad &\leq \quad \sum_{\gamma=\{\mu_-^-, \mu_+^-\}} |\gamma_-|^{-1} L_d(\gamma_-, \mu_-^-, \mu_+^-) \\ &\qquad + \sum_{\gamma=\{\mu_-^+, \mu_+^+\}} |\gamma_+|^{-1} L_d(\gamma_+, \mu_-^+, \mu_+^+) \qquad (4.16) \\ \Delta \cdot \|\epsilon_{tr}(\mathcal{V}_2, \cdot)\|_{L_1} \quad &\leq \quad |\omega_-|^{-1} L_t + |\omega_+|^{-1} L_t, \qquad (4.17) \end{aligned}$$

where $[\mu_-^-, \mu_+^-] \subset (\lambda_-, 0)$, $[\mu_-^+, \mu_-^+] \subset (0, \lambda_+)$, $\omega_- \in (\lambda_-, 0)$, $\omega_+ \in (0, \lambda_+)$.

PROOF. Since

$$\|\epsilon_d(\mathcal{V}_2, \cdot)\|_{L_1} = \|\epsilon_d(\mathcal{V}_2 \;\; {}_{(-\infty,0)}, \cdot)\|_{L_1} + \|\epsilon_d(\mathcal{V}_2 \;\; {}_{[0,\infty)}, \cdot)\|_{L_1}$$

$$\|\epsilon_{tr}(\mathcal{V}_2, \cdot)\|_{L_1} = \|\epsilon_{tr}(\mathcal{V}_2 \;\; {}_{(-\infty,0)}, \cdot)\|_{L_1} + \|\epsilon_{tr}(\mathcal{V}_2 \;\; {}_{[0,\infty)}, \cdot)\|_{L_1},$$

it suffices to consider each term on the RHS above separately.

When calculating $\mathcal{V}_2$, assume that we take the line of integration $\omega_+ \in [\mu_-^+, \mu_-^+] \subset (0, \lambda_+)$ if $x' \geq 0$; otherwise, we use (2.30) with $\omega_- \in [\mu_-^-, \mu_+^-] \subset (\lambda_-, 0)$.

We consider only the case $x' \geq 0$, the case $x' \leq 0$ are proved similarly. By Proposition 4.4, the discretization error admits an bound via

$$|\epsilon_d(\mathcal{V}_2 \;\; {}_{[0,\infty)}, x')| \leq \sum_{\gamma = \{\mu_-^+, \mu_+^+\}} e^{-\gamma_+ x'} \cdot L_d(\gamma_+, \mu_-^+, \mu_+^+),$$

where $L_d(\gamma_+, \mu_-^+, \mu_+^+)$ is as in (4.15). If ω_+, μ_-^+, μ_+^+ are fixed, then, $|\epsilon_d(\mathcal{V}_2 \;\; {}_{[0,\infty)}, x')|$ has the largest upper bounds at $x' = 0$. It follows that

$$\begin{aligned}
\Delta \cdot \|\epsilon_d(\mathcal{V}_2 \;\; {}_{[0,\infty)}, \cdot)\|_{L_1} \quad &\leq \quad \sum_{\gamma = \{\mu_-^+, \mu_+^+\}} \int_0^{+\infty} e^{-\gamma_+ x'} L_d(\gamma_+, \mu_-^+, \mu_+^+) dx' \\
&\leq \quad \sum_{\gamma = \{\mu_-^+, \mu_+^+\}} L_d(\gamma_+, \mu_-^+, \mu_+^+) \cdot |\gamma_+|^{-1}.
\end{aligned}$$

To derive bound for $\|\epsilon_{tr}(\mathcal{V}_2 \;\; {}_{[0,\infty)}, \cdot)\|_{L_1}$, we use Proposition 4.5. The truncation error admits an bound via

$$|\epsilon_{tr}(\mathcal{V}_2 \;\; {}_{[0,\infty)}, x')| \leq e^{-\omega_+ x'} \cdot L_t,$$

where L_t is the upper bounds (4.13) when $x = x' - \mu\bar{\Delta} = -\mu\bar{\Delta}$. Integrating $e^{-\omega_+ x'}$ w.r.t. x', we obtain

$$\Delta \cdot \|\epsilon_d(\mathcal{V}_2 \;\; {}_{[0,\infty)}, \cdot)\|_{L_1} \leq |\omega_+|^{-1} \cdot L_t.$$

□

Using (4.16), together with (4.4), a trivial modification of the prescription in Section 2.2 gives the choices of ω, ζ. Set $\epsilon_1 = 0.9\cdot\Delta^2\cdot\epsilon_c/4\cdot\pi/(2D_2)$.

(I) If $x' = x + \mu\bar{\Delta} \geq 0$, then

1. if $-0.1\cdot\ln\epsilon_1 \geq -\bar{\Delta}\psi^0(i\lambda_+ - 0) - s\ln\lambda_+$ or $\nu < 1$, set $\mu_+ = \lambda_+$, otherwise, find μ_+ as the unique positive solution of equation $-0.1\cdot\ln\epsilon = -\bar{\Delta}\psi^0(i\mu_+) - s\ln\mu_+$;
2. set $\omega_+ = \mu_+/2$, and $\zeta_+ = -\pi\mu_+/(1.1\ln\epsilon)$.

(II) If $x' < 0$, then

1. if $-0.1\cdot\ln\epsilon_1 \geq -\bar{\Delta}\psi^0(i\lambda_- + 0) - s\ln(-\lambda_-)$ or $\nu < 1$, set $\mu_- = \lambda_-$, otherwise, find μ_- as the unique positive solution of equation $-0.1\cdot\ln\epsilon = -\bar{\Delta}\psi^0(i\mu_-) - s\ln(-\mu_-)$;
2. set $\omega_- = \mu_-/2$, and $\zeta_- = \pi\mu_-/(1.1\ln\epsilon)$.

For the choice of M_c for $x' \geq 0$ (resp., $x' < 0$), one first multiply the RHS of (4.13) by $|\omega_+|^{-1}$ (resp., $|\omega_-|^{-1}$) and set to be ϵ_1; next, find Λ; then, use ζ above to choose the positive integer M_c such that $M_c\zeta \geq \Lambda$.

3. Parabolic iFT: formulas and error analysis

In this section, we first consider the conformal change of variable in (2.16) and (2.27). For the case of KoBoL model, we derive the asymptotics of the integrand. These asymptotics can be used to

- justify the transformation,
- compare the rate of decay of the integrand at infinity with the one in flat iFT method, and,
- derive bound for the truncation error.

3.1. General remarks. Parabolic iFT method allows us to obtain an integral with a better rate of convergence. For example, for V_1 in

(2.16), as $|\eta| \to +\infty$ along the lines $\operatorname{Im} \eta \in (0, \lambda_+)$, the integrand in the parabolic iFT method decays as

$$|\eta|^{-1-\alpha} \cdot e^{A \cdot x' |\eta|^{\alpha} + B \cdot \bar{\Delta} |\eta|^{\alpha\nu}}, \quad A,\, B < 0,$$

while the integrand in the flat iFT method decays as

$$|\xi|^{-2} \cdot e^{x'|\eta| - \bar{\Delta}|\xi|^{\nu}}$$

for ξ in the same strip. (See Section 3.2.)

The faster convergence of the resulting integrand in the parabolic iFT method amount to a spectacular improvement, especially for processes of order $\nu < 1$ (finite variation case) and close to maturity. One can decrease the number of terms in the simplified trapezoid rule by 10–100 times.

3.2. Main formulas. We have to evaluate V_1 in (2.16) and $\mathcal{V}_2$ in (2.27). If ν or $\bar{\Delta}$ are very small, as it may be the case, and if the calculations must be very accurate (error tolerance of order 10^{-9} and smaller), as it is necessary if the number of sampling dates is large, then accurate calculations using (2.19) and (2.29) (flat iFT) become extremely difficult because M_c of order of dozens of million may be needed. Both problems can be solved if we make an appropriate conformal deformation of the contour of integration in (2.16) (resp., (2.27)) with the following conformal change of variable to greatly increase the rate of decay of the integrand and decrease the number of terms in the simplified trapezoid rule.

First, we take $\alpha \in [1, \alpha_0)$, where $\alpha_0 > 1$ depends on x' and the order of the process, and, will be given while we derive the asymptotics of the resulting integrand. The method is labeled *parabolic iFT* of order α. Next, the process of the transformation depend on the following cases.

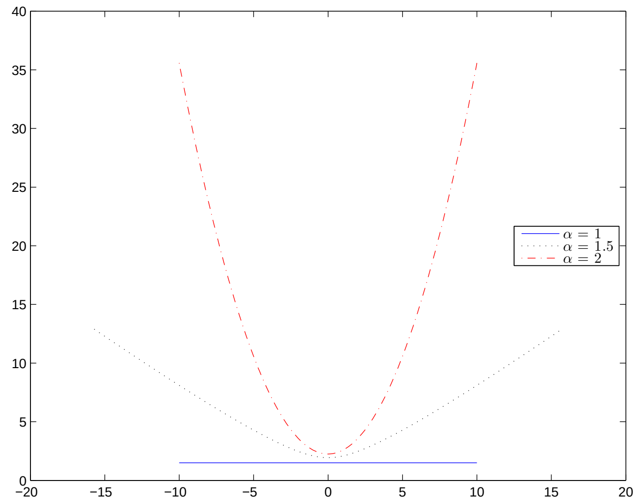

FIGURE 1. Typical curves of $\chi_\alpha^+(\eta) = i\lambda_+ - i{\lambda_+}^{1-\alpha}(\lambda_+ + i\eta)^\alpha$, $\eta \in i\omega + \mathbb{R}$ ($\omega \in (0, \lambda_+)$).

3.2.1. *Transformation of* (2.16)*: case* $x' = x + \mu\bar{\Delta} \geq 0$. Let $\omega \in (0, \lambda_+)$, $\eta' \in \mathbb{R}$, and $\eta = i\omega + \eta'$. We make the change of variable

$$\xi = \chi_\alpha^+(\eta) = i\lambda_+ - i{\lambda_+}^{1-\alpha}(\lambda_+ + i\eta)^\alpha, \quad \eta \in i\omega + \mathbb{R};$$

in the result, we obtain

$$V_1(x) = -(2\pi)^{-1} \int_{\operatorname{Im}\eta=\omega} \frac{e^{ix'\chi_\alpha^+(\eta) - \bar{\Delta}\psi^0(\chi_\alpha^+(\eta))}}{\chi_\alpha^+(\eta)(\chi_\alpha^+(\eta)+i)} \alpha \left(\frac{\lambda_+ + i\eta}{\lambda_+}\right)^{\alpha-1} d\eta, \quad (4.18)$$

where $\psi^0(\xi) = i\mu\xi + \psi(\xi)$.

The conformal change of variable is equivalent to the conformal deformation of the contour of integration with the subsequent change of the variables. Figure 3.2.1 illustrates that the typical curves of the contour χ_α^+ with different values of α.

If $\alpha \in (1, 2)$, the image is the following obtuse angle

$$\{i\lambda_+ + z \mid z \neq 0, \arg z \notin [\pi/2 - \pi(1-\alpha/2), \pi/2 + \pi(1-\alpha/2)]\}.$$

For $\alpha = 2$, the image is the complex plane with the cut $i[\lambda_+, +\infty]$. For $\alpha \in [2, \alpha_0)$, the contour belongs to an appropriate Riemann surface. One can use the explicit parametrization $\xi = \chi_\alpha^+(\eta)$ with $\eta = i\omega + \eta'$ in (4.18), if we define

$$\begin{aligned}\left(\lambda_+ + i\chi_\alpha^+(\eta)\right)^\nu &= e^{\nu \ln\left(\lambda_+ + i\chi_\alpha^+(\eta)\right)}\\ \left(-\lambda_- - i\chi_\alpha^+(\eta)\right)^\nu &= e^{\nu \ln\left(-\lambda_- - i\chi_\alpha^+(\eta)\right)}.\end{aligned}$$

Then, for each $\alpha \in [1, \alpha_0)$, the integrand in (4.18) admits the analytic continuation w.r.t. η into the strip $\operatorname{Im}\eta \in (0, \lambda_+)$.

To ensure the integrand in (4.18) is of class L_1 on the line $\operatorname{Im}\eta = \omega$, we justify the transformation by deriving the asymptotic behavior of each factor of the integrand in (4.18) as $\eta' := \operatorname{Re}\eta \to \pm\infty$ along the line $\operatorname{Im}\eta = \omega$:

(I)
$$\left|\frac{1}{\chi_\alpha^+(\eta)\cdot(\chi_\alpha^+(\eta)+i)}\alpha\cdot\left(\frac{\lambda_+ + i\eta}{\lambda_+}\right)^{\alpha-1}\right|,$$

(II)
$$\left|e^{ix'\chi_\alpha^+(\eta)}\right| = e^{\operatorname{Re}\left(ix'\chi_\alpha^+(\eta)\right)},$$

(III)
$$\left|e^{-\bar{\Delta}\psi^0(\chi_\alpha^+(\eta))}\right| = e^{-\bar{\Delta}\cdot\operatorname{Re}\left(\psi^0(\chi_\alpha^+(\eta))\right)}.$$

(I). For $s \geq 2$, and as $\eta' \to \pm\infty$, we have $(\chi_\alpha^+(\eta) + i) \sim \chi_\alpha^+(\eta)$, and

$$\left|\chi_\alpha^+(\eta)\right|^{-s} \sim {\lambda_+}^{-(1-\alpha)s}\rho^{-\alpha s} \tag{4.19}$$

$$\left|\alpha{\lambda_+}^{1-\alpha}(\lambda_+ + i\eta)^{\alpha-1}\right| \sim \alpha{\lambda_+}^{1-\alpha}\rho^{\alpha-1}, \tag{4.20}$$

where

$$\rho = \sqrt{(\lambda_+ - \omega)^2 + \eta'^2}. \tag{4.21}$$

Therefore, for $\alpha \geq 1$, the factor

$$\left| \frac{1}{\chi_\alpha^+(\eta) \cdot (\chi_\alpha^+(\eta) + i)} \alpha \cdot \left(\frac{\lambda_+ + i\eta}{\lambda_+} \right)^{\alpha-1} \right| \sim \alpha {\lambda_+}^{\alpha-1} \rho^{-\alpha-1}, \tag{4.22}$$

that is, decays as $\rho^{-\alpha-1}$ at infinity. It remains to consider the exponent factor in (4.18).

(II). For $x' \geq 0$, we have

$$\begin{aligned} \mathrm{Re}\left(ix'\chi_\alpha^+(\eta)\right) &= \mathrm{Re}\left(ix'(i\lambda_+ - i{\lambda_+}^{1-\alpha}(\lambda_+ + i\eta)^\alpha)\right) \\ &= -x'\lambda_+ + x'{\lambda_+}^{1-\alpha}\,\mathrm{Re}(\lambda_+ - \omega + i\eta')^\alpha \\ &= -x'\lambda_+ + x'{\lambda_+}^{1-\alpha}\,\mathrm{Re}\left(\rho^\alpha e^{i\alpha\phi}\right) \\ &= -x'\lambda_+ + x'{\lambda_+}^{1-\alpha}\rho^\alpha \cos(\alpha\phi), \end{aligned}$$

where ρ is as in (4.21), and

$$\phi = \arctan\left(\frac{\eta'}{\lambda_+ - \omega} \right). \tag{4.23}$$

Therefore, as $\eta' \to \pm\infty$,

$$\mathrm{Re}\left(ix'\chi_\alpha^+(\eta)\right) \sim -x'\lambda_+ + x'{\lambda_+}^{1-\alpha}\rho^\alpha \cos(\pm\alpha\pi/2). \tag{4.24}$$

If $\alpha \in (1,3)$, $\cos(\pm\alpha\pi/2) < 0$, and $\mathrm{Re}\,(ix'\chi_\alpha^+(\eta)) \to -\infty$.

(III). For a KoBoL process of order $\nu \in (0,2)$, $\nu \neq 1$, by (2.12) ($c = c_+ = c_-$), $\psi^0(\chi_\alpha^+(\eta))$ can be written explicitly as

$$\psi^0(\chi_\alpha^+(\eta)) = c\Gamma(-\nu)\left[{\lambda_+}^\nu + \psi_+^0(\eta) + (-\lambda_-)^\nu + \psi_-^0(\eta)\right], \tag{4.25}$$

where

$$\begin{aligned}
\psi_+^0(\eta) &= -(\lambda_+ + i(i\lambda_+ - i{\lambda_+}^{1-\alpha}(\lambda_+ + i\eta)^\alpha))^\nu \\
&= -{\lambda_+}^{(1-\alpha)\nu}(\lambda_+ + i\eta)^{\alpha\nu} \\
&= -{\lambda_+}^{(1-\alpha)\nu} \cdot e^{\alpha\nu \ln(\lambda_+ + i\eta)} \\
\psi_-^0(\eta) &= -(-\lambda_- - i(i\lambda_+ - i{\lambda_+}^{1-\alpha}(\lambda_+ + i\eta)^\alpha))^\nu \\
&= -\left(-\lambda_- + \lambda_+ - {\lambda_+}^{1-\alpha}(\lambda_+ + i\eta)^\alpha\right)^\nu \\
&= -\left(-\lambda_- + \lambda_+ - {\lambda_+}^{1-\alpha} \cdot e^{\alpha \ln(\lambda_+ + i\eta)}\right)^\nu .
\end{aligned}$$

We have $\lambda_- < 0$, $0 < \omega < \lambda_+$, therefore, it can be shown that for any $\alpha \in [1, 4)$, curve

$$\mathbb{R} \ni \eta' \mapsto -\lambda_- + \lambda_+ - {\lambda_+}^{1-\alpha} \cdot e^{\alpha \ln(\lambda_+ + i\eta)} \in \mathbb{C}$$

does not cross $(-\infty, 0]$, and (4.25) is well-defined. We can write the terms $\mathrm{Re}\left(-\bar{\Delta} c \Gamma(-\nu)\psi_\pm^0(\eta)\right)$ in the form

$$\begin{aligned}
\mathrm{Re}\left(-\bar{\Delta} c \Gamma(-\nu)\psi_+^0(\eta)\right) &= \bar{\Delta} c \Gamma(-\nu){\lambda_+}^{(1-\alpha)\nu}\, \mathrm{Re}\left(\rho^{\alpha\nu} e^{i\alpha\nu\phi}\right) \\
&= \bar{\Delta} c \Gamma(-\nu){\lambda_+}^{(1-\alpha)\nu} \rho^{\alpha\nu} \cos(\alpha\nu\phi)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}\left(-\bar{\Delta} c \Gamma(-\nu)\psi_-^0(\eta)\right) &= \bar{\Delta} c \Gamma(-\nu){\lambda_+}^{(1-\alpha)\nu} \rho^{\alpha\nu}\, \mathrm{Re}\left(a - e^{i\alpha\phi}\right)^\nu \\
&= \bar{\Delta} c \Gamma(-\nu){\lambda_+}^{(1-\alpha)\nu} \rho^{\alpha\nu}\, \mathrm{Re}\left(a + e^{i(\alpha\phi + \pi)}\right)^\nu ,
\end{aligned}$$

where ρ and ϕ are as in (4.21) and (4.23), respectively, and

$$a = \frac{-\lambda_- + \lambda_+}{{\lambda_+}^{1-\alpha} \rho^\alpha}.$$

As $\eta' \to \pm\infty$, we have $\rho \to \infty$, $\phi \to \pm\pi/2$, $a \to 0$, therefore,

$$\begin{aligned}-\bar{\Delta}\,\mathrm{Re}\left(\psi^0(\chi_\alpha^+(\eta))\right) \quad &\sim \quad -\bar{\Delta}c\Gamma(-\nu)[{\lambda_+}^\nu + (-\lambda_-)^\nu] \\ &\quad\; +\bar{\Delta}c\Gamma(-\nu){\lambda_+}^{(1-\alpha)\nu}\rho^{\alpha\nu} \\ &\quad\; \times(\cos(\alpha\nu\pi/2) + \cos(\alpha\nu\pi/2 + \nu\pi)),\end{aligned}$$

applying the trigonometric identity

$$\cos\theta + \cos\vartheta = 2\cos\left(\frac{\theta+\vartheta}{2}\right)\cos\left(\frac{\theta-\vartheta}{2}\right),$$

and obtain

$$\begin{aligned}&-\bar{\Delta}\,\mathrm{Re}\left(\psi^0(\chi_\alpha^+(\eta))\right) \\ \sim\; &-\bar{\Delta}c\Gamma(-\nu)[{\lambda_+}^\nu + (-\lambda_-)^\nu] \\ &+\bar{\Delta}c\Gamma(-\nu){\lambda_+}^{(1-\alpha)\nu}\rho^{\alpha\nu}2\cos(\nu\pi/2)\cos((1-\alpha)\nu\pi/2). \quad (4.26)\end{aligned}$$

For $\nu \in (0, 2)$, $\nu \neq 1$, product $\Gamma(-\nu)\cos(\nu\pi/2) < 0$, should we wish that

$$-\bar{\Delta}\,\mathrm{Re}\left(\psi^0(\chi_\alpha^+(\eta))\right) \to -\infty, \quad \text{as } \eta' \to \pm\infty,$$

we must have $\cos((1-\alpha)\nu\pi/2) > 0$, equivalently,

$$(1-\alpha)\nu\pi/2 > -\pi/2, \quad \text{for } \alpha \geq 1.$$

Thus, $1 < \alpha < \min\{4, 1 + 1/\nu\}$.

If $\nu > 1$, the latter condition implies that $\alpha < 2$, hence, the real part of both terms in the exponent in the pricing formula (4.18) tend to $-\infty$ as $\eta' \to \pm\infty$. If $\nu \in [0+, 1)$, it is possible that $1 + 1/\nu < \alpha < 3$; the the real part of the first term tends to $-\infty$, as ρ^α (see (4.24)), and the second term tends to $+\infty$ as $\rho^{\alpha\nu}$ (see (4.26)). The real part of the sum tends to

$-\infty$ but the behavior can be rather irregular. Hence, in these cases, we would use $\alpha < \min\{3, 1 + 1/\nu\}$, as it was recommended in [**18**].

We conclude from the asymptotics (4.22), (4.24) and (4.26) that the integrand in (4.18) decays as

$$\rho^{-1-\alpha} \cdot e^{A\cdot|\eta|^\alpha + B\cdot\bar{\Delta}\rho^{\alpha\nu}}, \quad A,\, B < 0,$$

or equivalently,

$$|\eta|^{-1-\alpha} \cdot e^{A\cdot x'|\eta|^\alpha + B\cdot\bar{\Delta}|\eta|^{\alpha\nu}}, \quad A,\, B < 0,$$

as $|\eta| \to +\infty$ along the lines $\operatorname{Im}\eta \in (0, \lambda_+)$. The rate of decay is generally faster than the one in the flat iFT method

$$|\xi|^{-2} \cdot e^{x'|\eta| - \bar{\Delta}|\xi|^\nu}$$

with ξ in the same strip $\operatorname{Im}\xi \in (0, \lambda_+)$. The asymptotics can be easily derived from the asymptotic of ψ^0 (2.10) and the integrand of (2.16):

$$-\frac{1}{2\pi} \cdot \frac{e^{ix\xi - \bar{\Delta}\psi(\xi)}}{\xi(\xi + i)}.$$

3.2.2. *Transformation of* (2.16)*: case* $x' = x + \mu\bar{\Delta} < 0$. First, we use put-call parity, then, take $\omega \in (-\lambda_-, -1)$ and make the change of the variable

$$\xi = \chi^-_\alpha(\eta) = i\lambda_- + i(-\lambda_- - 1)^{1-\alpha}(-\lambda_- - i\eta)^\alpha, \quad \eta \in i\omega + \mathbb{R};$$

in the result, we obtain

$$V_1(x) = 1 - e^{x - \bar{\Delta}\psi(-i)} + V_1^c(x),$$

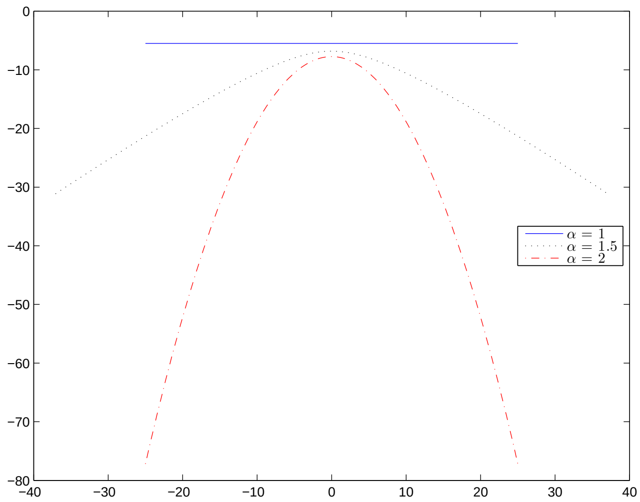

FIGURE 2. Typical curves of $\chi_\alpha^-(\eta) = i\lambda_- + i(-\lambda_- - 1)^{1-\alpha}(-\lambda_- - i\eta)^\alpha$, $\eta \in i\omega + \mathbb{R}$ ($\omega \in (\lambda_-, -1)$).

where

$$V_1^c(x) = -(2\pi)^{-1} \int_{\mathrm{Im}\,\eta=\omega} \frac{e^{ix'\chi_\alpha^-(\eta) - \bar{\Delta}\psi^0(\chi_\alpha^-(\eta))}}{\chi_\alpha^-(\eta)(\chi_\alpha^-(\eta)+i)} \alpha \left(\frac{-\lambda_- - i\eta}{-\lambda_- - 1}\right)^{\alpha-1} d\eta. \tag{4.27}$$

The conformal change of variable is equivalent to the conformal deformation of the contour of integration with the subsequent change of the variables. Figure 3.2.2 illustrates that the typical curves of the contour χ_α^+ with different values of α.

If $\alpha \in (1,2)$, the image is the following obtuse angle

$$\{i\lambda_- + z \mid z \neq 0, \arg z \notin [-\pi/2 - \pi(1-\alpha/2), -\pi/2 + \pi(1-\alpha/2)]\}.$$

For $\alpha = 2$, the image is the complex plane with the cut $i(-\infty, \lambda_-]$. For $\alpha \in [2, \alpha_0)$ the contour belongs to an appropriate Riemann surface. One can use the explicit parametrization $\xi = \chi_\alpha^-(\eta)$ with $\eta = i\omega + \eta'$ in (4.27), if we define

$$\begin{aligned}\left(\lambda_+ + i\chi^+_\alpha(\eta)\right)^\nu &= e^{\nu\ln\left(\lambda_+ + i\chi^+_\alpha(\eta)\right)}\\ \left(-\lambda_- - i\chi^-_\alpha(\eta)\right)^\nu &= e^{\nu\ln\left(-\lambda_- - i\chi^-_\alpha(\eta)\right)}\end{aligned}$$

Then, for each $\alpha \in [1, \alpha_0)$, the integrand in (4.27) admits the analytic continuation w.r.t. η into the strip $\operatorname{Im}\eta \in (\lambda_-, -1)$.

To ensure the integrand in (4.27) is of class L_1 on the line $\operatorname{Im}\eta = \omega$, the justification of the transformation can be made similarly to the case $x' \geq 0$, we list the key formulas, and leave the detailed derivation to the reader.

Let

$$\rho = \sqrt{(-\lambda_- + \omega)^2 + \eta'^2}, \quad \phi = \arctan\left(\frac{-\eta'}{-\lambda_- + \omega}\right).$$

For $\alpha \geq 1$, as $\eta' \to \pm\infty$, the factor of the integrand in (4.27)

$$\left|\frac{1}{\chi^-_\alpha(\eta)\cdot(\chi^-_\alpha(\eta)+i)}\alpha\cdot\left(\frac{-\lambda_- - i\eta}{-\lambda_- - 1}\right)^{\alpha-1}\right| \sim \alpha(-\lambda_- - 1)^{\alpha-1}\rho^{-\alpha-1}, \quad (4.28)$$

that is, decays as $\rho^{-\alpha-1}$ at infinity. For $x' \leq 0$,

$$\operatorname{Re}\left(ix'\chi^-_\alpha(\eta)\right) \sim -x'\lambda_- - x'(-\lambda_- - 1)^{1-\alpha}\rho^\alpha\cos(\pm\alpha\pi/2) \qquad (4.29)$$

as $\eta' \to \pm\infty$. If $\alpha \in (1, 3)$, then, $\cos(\pm\alpha\pi/2) < 0$, and $\operatorname{Re}(ix'\chi^-_\alpha(\eta)) \to -\infty$. For a KoBoL process of order $\nu \in (0, 2)$, $\nu \neq 1$,

$$\begin{aligned}-\bar{\Delta}\operatorname{Re}\left(\psi^0(\chi^-_\alpha(\eta))\right) \sim\ & -\bar{\Delta}c\Gamma(-\nu)[\lambda_+{}^\nu + (-\lambda_-)^\nu]\\ & +\bar{\Delta}c\Gamma(-\nu)(-\lambda_- - 1)^{(1-\alpha)\nu}\rho^{\alpha\nu}\\ & \times 2\cos(\nu\pi/2)\cos((1-\alpha)\nu\pi/2). \quad (4.30)\end{aligned}$$

as $\eta' \to \pm\infty$. We have $\Gamma(-\nu)\cos(\nu\pi/2) < 0$, therefore, if $1 < \alpha < \min\{4, 1+1/\nu\}$, $\cos((1-\alpha)\nu\pi/2) > 0$, and

$$-\bar{\Delta}\,\mathrm{Re}\left(\psi^0(\chi^-_\alpha(\eta))\right) \to -\infty, \quad \text{as } \eta' \to \pm\infty.$$

Finally, we would use $\alpha < \min\{3, 1+1/\nu\}$ if $\nu \in [0+, 1)$, as it was recommended in [**18**].

3.2.3. *Value of* α_0. We conclude that the integrand in (4.18) (resp., (4.27)) is of class L_1 on the line $\mathrm{Im}\,\eta = \omega$, uniformly in $\alpha \in [1, \alpha_0)$, where $\alpha_0 > 1$ depends on x' and the order of the process:

- if $x' > 0$ (resp., $x' < 0$) and $\nu \in [0+, 1)$, then $\alpha_0 = \min\{3, 1+1/\nu\}$;
- if $x' = 0$ and $\nu \in [0+, 1)$, then $\alpha_0 = \min\{4, 1+1/\nu\}$;
- if $x' \geq 0$ (resp., $x' \leq 0$) and $\nu \in [1, 2]$, then $\alpha_0 = 1 + 1/\nu$.

Therefore, (4.18) and (4.27) can be applied for any $\alpha \in [1, \alpha_0)$. The error estimates and recommendations discussed later are valid for these α.

3.2.4. *Numerical realization.* If $x' \geq 0$, by using (4.18), and taking into account that, for real η' and $\omega \in (0, \lambda_+)$,

- $\overline{i(i\omega + \eta')} = i(i\omega - \eta')$,
- $\overline{i\chi^+_\alpha(i\omega + \eta')} = i\chi^+_\alpha(i\omega - \eta')$
- $\overline{\psi^0(\chi^+_\alpha(i\omega + \eta'))} = \psi^0(\chi^+_\alpha(i\omega - \eta'))$,

and, similarly to (2.18), we obtain

$$V_1(x) = -\frac{1}{\pi}\,\mathrm{Re}\left[\int_{i\omega}^{i\omega+\infty} \frac{e^{ix'\chi^+_\alpha(\eta) - \bar{\Delta}\psi^0(\chi^+_\alpha(\eta))}}{\chi^+_\alpha(\eta)(\chi^+_\alpha(\eta)+i)}\alpha\left(\frac{\lambda_+ + i\eta}{\lambda_+}\right)^{\alpha-1} d\eta\right]. \quad (4.31)$$

We calculate the integrals in (4.31) using the simplified trapezoid rule:

$$V_1(x) \approx -\frac{\zeta}{\pi}\,\mathrm{Re}\sum_{j=1}^{M_c} \frac{e^{ix'\chi^+_\alpha(\eta_j) - \bar{\Delta}\psi^0(\chi^+_\alpha(\eta_j))}}{\chi^+_\alpha(\eta_j)(\chi^+_\alpha(\eta_j)+i)}\alpha\left(\frac{\lambda_+ + i\eta_j}{\lambda_+}\right)^{\alpha-1}(1 - \delta_{j1}/2), \quad (4.32)$$

where δ_{jk} is Kronecker's delta, $\eta_j = i\omega + (j-1)\zeta$, $j = 1, 2, \dots, M_c$, and ω is chosen appropriately. Similarly, if $x' < 0$, by using $\mathcal{V}_s^c$ in (4.27) and taking into account that, for real η' and $\omega \in (\lambda_-, -1)$,

- $\overline{i(i\omega + \eta')} = i(i\omega - \eta')$,
- $\overline{i\chi_\alpha^-(i\omega + \eta')} = i\chi_\alpha^-(i\omega - \eta')$
- $\overline{\psi^0(\chi_\alpha^-(i\omega + \eta'))} = \psi^0(\chi_\alpha^-(i\omega - \eta'))$,

we obtain

$$V_1^c(x) = -\frac{1}{\pi}\,\mathrm{Re}\left[\int_{i\omega}^{i\omega+\infty} \frac{e^{ix'\chi_\alpha^-(\eta) - \bar\Delta\psi^0(\chi_\alpha^-(\eta))}}{\chi_\alpha^-(\eta)(\chi_\alpha^-(\eta)+i)}\alpha\left(\frac{-\lambda_- - i\eta}{-\lambda_- - 1}\right)^{\alpha-1} d\eta\right]. \tag{4.33}$$

The simplified trapezoid rule reads

$$V_1^c(x) \approx -\frac{\zeta}{\pi}\,\mathrm{Re}\sum_{j=1}^{M_c} \frac{e^{ix'\chi_\alpha^-(\eta_j) - \bar\Delta\psi^0(\chi_\alpha^-(\eta_j))}}{\chi_\alpha^-(\eta_j)(\chi_\alpha^-(\eta_j)+i)}\alpha\left(\frac{-\lambda_- - i\eta_j}{-\lambda_- - 1}\right)^{\alpha-1}(1-\delta_{j1}/2). \tag{4.34}$$

The method allows one to satisfy the error tolerance of order 10^{-10} quite easily unless $x' = 0$, $\bar\Delta$ and ν are very close to 0.

3.2.5. *Transformation of* (2.27). First, we take $\alpha \in (1, \alpha_0)$, where α_0 is specified in Section 3.2.3. Next, if $x' = x + \mu\bar\Delta \ge 0$, we choose $\omega \in (0, \lambda_+)$, and let $\eta = i\omega + \mathbb{R}$. By making the change of the variable

$$\xi = \chi_\alpha^+(\eta) = i\lambda_+ - i{\lambda_+}^{1-\alpha}(\lambda_+ + i\eta)^\alpha, \quad \eta \in i\omega + \mathbb{R},$$

we obtain

$$\mathcal{V}_s(x) = (2\pi)^{-1}\int_{\mathrm{Im}\,\eta=\omega} \frac{e^{ix'\chi_\alpha^+(\eta) - \bar\Delta\psi^0(\chi_\alpha^+(\eta))}}{(i\chi_\alpha^+(\eta))^s}\alpha\left(\frac{\lambda_+ + i\eta}{\lambda_+}\right)^{\alpha-1} d\eta. \tag{4.35}$$

If $x' \le 0$, we push the line of integration in (2.27) down. In the process of the transformation, the contour crosses the pole at $\xi = 0$, which is of order s. Using the residue theorem, we obtain

$$\mathcal{V}_s(x') = -\frac{i^{1-s}}{(s-1)!}\,\frac{d^{s-1}}{d\xi^{s-1}}e^{ix'\xi-\bar{\Delta}\psi^0(\xi)}\Big|_{\xi=0} + \mathcal{V}_s^c(x'),$$

where

$$\mathcal{V}_s^c(x') = (2\pi)^{-1}\int_{\mathrm{Im}\,\eta=\omega}\frac{e^{ix'\xi-\bar{\Delta}\psi^0(\xi)}}{(i\xi)^s}d\xi, \quad \omega\in(\lambda_-,0).$$

We choose $\omega\in(\lambda_-,0)$, and let $\eta = i\omega+\mathbb{R}$. By making the change of the variable

$$\xi = \chi_\alpha^-(\eta) = i\lambda_- + i(-\lambda_-)^{1-\alpha}(-\lambda_- - i\eta)^\alpha, \quad \eta\in i\omega+\mathbb{R},$$

we obtain

$$\mathcal{V}_s^c(x) = (2\pi)^{-1}\int_{\mathrm{Im}\,\eta=\omega}\frac{e^{ix'\chi_\alpha^-(\eta)-\bar{\Delta}\psi^0(\chi_\alpha^-(\eta))}}{(i\chi_\alpha^-(\eta))^s}\alpha\left(\frac{-\lambda_- - i\eta}{-\lambda_-}\right)^{\alpha-1}d\eta. \quad (4.36)$$

Formulas (4.35) and (4.36) can be calculated similarly to V_1 in (4.18) and V_1^c in (4.27). First, we reduce the integral in (4.35) (resp., (4.36)) to an integral over $i\omega+\mathbb{R}^+$, $\omega\in(0,\lambda_+)$ (resp., $\omega\in(\lambda_-,0)$), and then apply trapezoid rule:

$$\begin{aligned}\mathcal{V}_s(x') &\approx \frac{\zeta}{\pi}\,\mathrm{Re}\sum_{j=1}^{M_c}\frac{e^{ix'\chi_\alpha^+(\eta_j)-\bar{\Delta}\psi^0(\chi_\alpha^+(\eta_j))}}{i\chi_\alpha^+(\eta_j)}\alpha\left(\frac{\lambda_+ + i\eta_j}{\lambda_+}\right)^{\alpha-1}(1-\delta_{j1}/2)\\ \mathcal{V}_s^c(x') &\approx \frac{\zeta}{\pi}\,\mathrm{Re}\sum_{j=1}^{M_c}\frac{e^{ix'\chi_\alpha^-(\eta_j)-\bar{\Delta}\psi^0(\chi_\alpha^-(\eta_j))}}{i\chi_\alpha^-(\eta_j)}\alpha\left(\frac{-\lambda_- - i\eta_j}{-\lambda_-}\right)^{\alpha-1}(1-\delta_{j1}/2),\end{aligned}$$

where δ_{jk} is Kronecker's delta, $\eta_j = i\omega+(j-1)\zeta$, $j=1,2,\dots,M_c$, and ω is chosen appropriately.

3.3. Errors: preliminary.

3.3.1. *Discretization error.* In the flat iFT method in Section 2.1, we defined the strip $\mathrm{Im}\,\xi\in[\mu_-,\mu_+]\subset(0,\lambda_+]$ in order to find the line of integration $\mathrm{Im}\,\xi = (\mu_- + \mu_+)/2$ and constant $d = (\mu_+ - \mu_-)/2$, which is needed to estimate the discretization error and find a sufficiently fine mesh ζ. Typically, μ_- is close to 0, hence, we use $\mu_- = 0$ as a fairly good approximation. In the parabolic iFT methods, the key idea to define the strip $\mathrm{Im}\,\eta \in [\sigma_-, \sigma_+]$ is to select the strip has the same intersection with the imaginary axis as $[\mu_-, \mu_+]$. Observe that the domain of analyticity outside the imaginary axis extends to infinity as $\mathrm{Im}\,\xi \to \pm\infty$, hence, one can expect that the main contribution to the discretization error comes from the bottleneck between the two cuts.

Explicitly, for the case of V_1 in (4.18), in [**18**], the authors find $\sigma_\pm$ such that $\chi^+_\alpha(i\sigma_\pm) = i\mu_\pm$, set $\omega = (\sigma_- + \sigma_+)/2$, $d = (\sigma_+ - \sigma_-)/2$, and use ω to define the line of integration $\mathrm{Im}\,\eta = \omega$. We can find $\sigma_\pm$ solving equation

$$\frac{\lambda_+ - \mu_\pm}{\lambda_+} = \left(\frac{\lambda_+ - \sigma_\pm}{\lambda_+}\right)^\alpha,$$

equivalently,

$$\frac{\lambda_+ - \sigma_\pm}{\lambda_+} = \left(\frac{\lambda_+ - \mu_\pm}{\lambda_+}\right)^{1/\alpha},$$

and, finally,

$$\sigma_\pm = \lambda_+ - \lambda_+ \left(\frac{\lambda_+ - \mu_\pm}{\lambda_+}\right)^{1/\alpha}.$$

We conclude that

$$d_+ = \frac{\lambda_+}{2}\left[\left(\frac{\lambda_+ - \mu_-}{\lambda_+}\right)^{1/\alpha} - \left(\frac{\lambda_+ - \mu_+}{\lambda_+}\right)^{1/\alpha}\right], \tag{4.37}$$

$$\omega_+ = \lambda_+ - \frac{\lambda_+}{2}\left[\left(\frac{\lambda_+ - \mu_-}{\lambda_+}\right)^{1/\alpha} + \left(\frac{\lambda_+ - \mu_+}{\lambda_+}\right)^{1/\alpha}\right]; \tag{4.38}$$

Similarly, for the case of V_1^c in (4.27), we have

$$d_- = -\frac{(\lambda_- + 1)}{2}\left[\left(\frac{-\lambda_- + \mu_+}{-\lambda_- - 1}\right)^{1/\alpha} - \left(\frac{-\lambda_- + \mu_-}{-\lambda_- - 1}\right)^{1/\alpha}\right], \tag{4.39}$$

$$\omega_- = \lambda_- - \frac{(\lambda_- + 1)}{2}\left[\left(\frac{-\lambda_- + \mu_+}{-\lambda_- - 1}\right)^{1/\alpha} + \left(\frac{-\lambda_- + \mu_-}{-\lambda_- - 1}\right)^{1/\alpha}\right]. \tag{4.40}$$

The choice of ω and the constant d for the calculation of $\mathcal{V}_s$ in (4.35) and $\mathcal{V}_s^c$ in (4.36) are chosen in the same fashion. For $\mathcal{V}_s$, ω and d are the same as in (4.37) and (4.38), respectively. For $\mathcal{V}_s^c$, we have

$$d_- = -\frac{(\lambda_- + 1)}{2}\left[\left(\frac{-\lambda_- + \mu_+}{-\lambda_-}\right)^{1/\alpha} - \left(\frac{-\lambda_- + \mu_-}{-\lambda_-}\right)^{1/\alpha}\right], \qquad (4.41)$$

$$\omega_- = \lambda_- - \frac{(\lambda_- + 1)}{2}\left[\left(\frac{-\lambda_- + \mu_+}{-\lambda_-}\right)^{1/\alpha} + \left(\frac{-\lambda_- + \mu_-}{-\lambda_-}\right)^{1/\alpha}\right]. (4.42)$$

3.3.2. *Truncation error.* We consider the case of KoBoL of order $\nu \in (0,2)$, $\nu \neq 1$. The upper bounds for the truncation error and a procedure for the choice of $\Lambda = M_c\zeta$ are given in Propositions 5.2 and 5.8 in [**18**]. We reformulate the proposition in accordance with our asymptotics derived in Section 3.2.

Proposition 4.7. [**18**, Proposition 5.2] *Let $\nu \in (0,2), \nu \neq 1$, and assume that either $x' > 0$ and $\alpha \in [1, \min\{1 + 1/\nu, 3\})$, or $x' = 0$ and $\alpha \in [1, \min\{1 + 1/\nu, 4\})$. Then*

a) As $\rho := |\lambda_+ + i\eta| \to +\infty$, the integrand in (4.18) *is bounded in modulus by*

$$(2\pi)^{-1}\alpha{\lambda_+}^{\alpha-1}\rho^{-1-\alpha} \cdot e^{-x'\lambda_+ + A\cdot x'\rho^\alpha + B\cdot\bar{\Delta}\rho^{\alpha\nu} - c\bar{\Delta}\Gamma(-\nu)({\lambda_+}^\nu + (-\lambda_-)^\nu)}$$

where

$$A = {\lambda_+}^{1-\alpha} \cdot \cos(\alpha\pi/2) < 0, \qquad (4.43)$$

$$B = 2c\Gamma(-\nu){\lambda_+}^{(1-\alpha)\nu}\cos(\nu\pi/2) \cdot \cos(\nu(\alpha-1)\pi/2) < 0. \qquad (4.44)$$

b) Given an error tolerance ϵ for the truncation error, $\Lambda = M_c\zeta$ can be chosen as a number, which satisfies

$$\frac{e^{A\cdot x'\Lambda^\alpha + B\cdot\bar{\Delta}\Lambda^{\alpha\nu}}}{\Lambda^\alpha} < \epsilon\pi{\lambda_+}^{1-\alpha}e^{c\bar{\Delta}\Gamma(-\nu)({\lambda_+}^\nu + (-\lambda_-)^\nu) + x'\lambda_+}. \qquad (4.45)$$

(This recommendation is applicable only in the region of sufficiently large Λ, where the LHS is monotone or very close to a monotone function.)

PROOF. a) We use the asymptotics: (4.22), (4.24) and (4.26).

b) For a decreasing positive function f, it is evident that,

$$\int_\Lambda^\infty f(x)\cdot \alpha x^{-1-\alpha}dx \le f(\Lambda)\int_\Lambda^\infty \alpha x^{-1-\alpha}dx = f(\Lambda)\cdot \Lambda^{-\alpha}.$$

$\square$

PROPOSITION 4.8. [**18**, Proposition 5.8] *Let $\nu \in (0,2), \nu \neq 1$, and assume that either $x' < 0$ and $\alpha \in [1, \min\{1+1/\nu, 3\})$, or $x' = 0$ and $\alpha \in [1, \min\{1+1/\nu, 4\})$. Then*

a) As $\rho := |-\lambda_- - i\eta| \to +\infty$, the integrand in (4.27) is bounded in modulus by

$$(2\pi)^{-1}\alpha(-\lambda_- - 1)^{\alpha-1}\rho^{-1-\alpha}\cdot e^{-x'\lambda_- - A\cdot x'\rho^\alpha + B\cdot\bar\Delta\rho^{\alpha\nu} - c\bar\Delta\Gamma(-\nu)(\lambda_+{}^\nu + (-\lambda_-)^\nu)}$$

where

$$A = (-\lambda_- - 1)^{1-\alpha}\cdot\cos(\alpha\pi/2) < 0,$$
$$B = 2c\Gamma(-\nu)(-\lambda_- - 1)^{(1-\alpha)\nu}\cos(\nu\pi/2)\cdot\cos(\nu(\alpha-1)\pi/2) < 0.$$

b) Given an error tolerance ϵ for the truncation error, $\Lambda = M_c\zeta$ can be chosen as a number, which satisfies

$$\frac{e^{-A\cdot x'\Lambda^\alpha + B\cdot\bar\Delta\Lambda^{\alpha\nu}}}{\Lambda^\alpha} < \epsilon\pi(-\lambda_- - 1)^{1-\alpha}e^{c\bar\Delta\Gamma(-\nu)(\lambda_+{}^\nu + (-\lambda_-)^\nu) + x'\lambda_-}. \tag{4.46}$$

(This recommendation is applicable only in the region of sufficiently large Λ, where the LHS is monotone or very close to a monotone function.)

PROOF. a) We use the asymptotics: (4.28), (4.29) and (4.30).

b) For a decreasing positive function f, it is evident that,

$$\int_{\Lambda}^{\infty} f(x) \cdot \alpha x^{-1-\alpha} dx \leq f(\Lambda) \int_{\Lambda}^{\infty} \alpha x^{-1-\alpha} dx = f(\Lambda) \cdot \Lambda^{-\alpha}.$$

□

Below, we consider the error control of the numerical scheme of $\mathcal{V}_s$. The upper bounds for the truncation error and a procedure for the choice of $\Lambda = M_c\zeta$ are given in the following propositions, which are analogues of Proposition 5.2 and 5.8 in [**18**].

Proposition 4.9. *Let* $\nu \in (0, 2)$, $\nu \neq 1$, *and assume that either* $x' > 0$ *and* $\alpha \in [1, \min\{1+1/\nu, 3\})$, *or* $x' = 0$ *and* $\alpha \in [1, \min\{1+1/\nu, 4\})$. *Then*

a) As $\rho := |\lambda_+ + i\eta| \to +\infty$, *the integrand in* (4.35) *is bounded in modulus by*

$$(2\pi)^{-1}\alpha{\lambda_+}^{(\alpha-1)(s-1)}|\rho|^{-\alpha(s-1)-1} \cdot e^{-x'\lambda_+ + A\cdot x'\rho^{\alpha} + B\cdot\bar{\Delta}\rho^{\alpha\nu} - c\bar{\Delta}\Gamma(-\nu)({\lambda_+}^{\nu} + (-\lambda_-)^{\nu})}$$

where A *and* B *are as in* (4.43) *and* (4.44), *respectively.*

b) Given an error tolerance ϵ *for the truncation error,* $\Lambda = M_c\zeta$ *can be chosen as a number, which satisfies*

$$\frac{e^{A\cdot x'\Lambda^{\alpha} + B\cdot\bar{\Delta}\Lambda^{\alpha\nu}}}{(s-1)\Lambda^{\alpha(s-1)}} < \epsilon\pi{\lambda_+}^{-(\alpha-1)(s-1)} e^{c\bar{\Delta}\Gamma(-\nu)({\lambda_+}^{\nu} + (-\lambda_-)^{\nu})}. \tag{4.47}$$

(This recommendation is applicable only in the region of sufficiently large Λ, *where the LHS is monotone or very close to a monotone function.)*

Proof. a) In Section 3.2, we derived the asymptotics of the factors of the integrand in (4.35) as $\eta' := \mathrm{Re}\,\eta \to \pm\infty$ along the line $\mathrm{Im}\,\eta = \omega$. Here we summarize the results.

For $s \geq 2$, on the strength of (4.19) and (4.20), we have

$$\left| \frac{1}{(i\chi_\alpha^+(\eta))^s}\alpha\left(\frac{\lambda_+ + i\eta}{\lambda_+}\right)^{\alpha-1} \right| \sim \alpha{\lambda_+}^{(\alpha-1)(s-1)}\rho^{-\alpha(s-1)-1},$$

where $\rho := |\lambda_+ + i\eta|$. If $\alpha \geq 1$, the factor decays as $\rho^{-\alpha(s-1)-1}$ at infinity. The remaining exponent factor is showed on (4.24) and (4.26).

b) For a decreasing positive function f, it is evident that,

$$\int_\Lambda^\infty f(x)\cdot\alpha\cdot x^{-\alpha(s-1)-1}dx \leq \frac{f(\Lambda)}{(s-1)\Lambda^\alpha}.$$

$\square$

PROPOSITION 4.10. *Let $\nu \in (0,2), \nu \neq 1$, and assume that either $x' < 0$ and $\alpha \in [1, \min\{1+1/\nu, 3\})$, or $x' = 0$ and $\alpha \in [1, \min\{1+1/\nu, 4\})$. Then*

a) As $\rho := |-\lambda_- - i\eta| \to +\infty$, the integrand in (4.36) is bounded in modulus by

$$(2\pi)^{-1}\alpha(-\lambda_-)^{(\alpha-1)(s-1)}\rho^{-\alpha(s-1)-1}e^{-x'\lambda_- - A\cdot x'\rho^\alpha + B\cdot\bar{\Delta}\rho^{\alpha\nu} - c\bar{\Delta}\Gamma(-\nu)({\lambda_+}^\nu + (-\lambda_-)^\nu)}$$

where

$$A = \cos(\alpha\pi/2)(-\lambda_-)^{1-\alpha} < 0 \tag{4.48}$$

$$B = 2c\Gamma(-\nu)(-\lambda_-)^{(1-\alpha)\nu}\cos(\nu\pi/2)\cos(\nu(\alpha-1)\pi/2) < 0. \tag{4.49}$$

b) Given an error tolerance ϵ for the truncation error, $\Lambda = M_c\zeta$ can be chosen as a number, which satisfies

$$\frac{e^{-x'A(\Lambda)\Lambda^\alpha + \bar{\Delta}B(\Lambda)\Lambda^{\alpha\nu}}}{(s-1)\Lambda^{(s-1)\alpha}} < \epsilon\pi(-\lambda_-)^{1-\alpha}e^{c\bar{\Delta}\Gamma(-\nu)({\lambda_+}^\nu + (-\lambda_-)^\nu)}. \tag{4.50}$$

(This recommendation is applicable only in the region of sufficiently large Λ, where the LHS is monotone or very close to a monotone function.)

PROOF. We consider the conformal mapping:

$$\chi_\alpha^-(\eta) = i\lambda_- + i(-\lambda_-)^{1-\alpha}(-\lambda_- - i\eta)^\alpha,$$

where $\omega \in (\lambda_-, -1)$ and $\eta \in i\omega + \mathbb{R}$.

a) For $s \geq 2$, as $\eta' := \operatorname{Re}\eta \to \pm\infty$ along the line $\operatorname{Im}\eta = \omega$, we have

$$\begin{aligned} \left|\chi_\alpha^-(\eta)\right|^{-s} &\sim (-\lambda_-)^{-(1-\alpha)s}\rho^{-\alpha s} \\ \left|\alpha(-\lambda_-)^{1-\alpha}(-\lambda_- - i\eta)^{\alpha-1}\right| &\sim \alpha(\lambda_-)^{1-\alpha}\rho^{\alpha-1}, \end{aligned}$$

where $\rho := |-\lambda_- - i\eta|$, and hence,

$$\left|\frac{1}{(i\chi_\alpha^-(\eta))^s}\alpha\left(\frac{-\lambda_- - i\eta}{-\lambda_-}\right)^{\alpha-1}\right| \sim \alpha(-\lambda_-)^{(\alpha-1)(s-1)}\rho^{-\alpha(s-1)-1}.$$

If $\alpha \geq 1$, the factor decays as $\rho^{-\alpha(s-1)-1}$ at infinity. The asymptotics of the remaining exponent factor in (4.36) are similar to (4.29) and (4.30), and are derived as follows. For $x' \leq 0$,

$$\operatorname{Re}\left(ix'\chi_\alpha^-(\eta)\right) \sim -x'\lambda_- - x'(-\lambda_-)^{1-\alpha}\rho^\alpha\cos(\alpha\pi/2)$$

as $\eta' \to \pm\infty$. If $\alpha \in (1,3)$, then, $\cos(\alpha\pi/2) < 0$, and $\operatorname{Re}(ix'\chi_\alpha^-(\eta)) \to -\infty$. For a KoBoL process of order $\nu \in (0,2)$, $\nu \neq 1$,

$$\begin{aligned} -\bar{\Delta}\operatorname{Re}\left(\psi^0(\chi_\alpha^-(\eta))\right) &\sim -\bar{\Delta}c\Gamma(-\nu)[\lambda_+{}^\nu + (-\lambda_-)^\nu] \\ &\quad +\bar{\Delta}c\Gamma(-\nu)(-\lambda_-)^{(1-\alpha)\nu}\rho^{\alpha\nu} \\ &\quad \times 2\cos(\nu\pi/2)\cos((1-\alpha)\nu\pi/2). \end{aligned}$$

as $\eta' \to \pm\infty$. We have $\Gamma(-\nu)\cos(\nu\pi/2) < 0$, therefore, if $1 < \alpha < \min\{4, 1+1/\nu\}$, $\cos((1-\alpha)\nu\pi/2) > 0$, and

$$-\bar{\Delta}\operatorname{Re}\left(\psi^0(\chi_\alpha^-(\eta))\right) \to -\infty, \quad \text{as } \eta' \to \pm\infty.$$

b) For a decreasing positive function f, it is evident that,

$$\int_{\Lambda}^{\infty} f(x) \cdot \alpha \cdot x^{-\alpha(s-1)-1} dx \leq \frac{f(\Lambda)}{(s-1)\Lambda^{\alpha}}.$$

□

3.4. Choice of ω, ζ, M_c. One may use the same length and mesh of the η-grid in order to apply the vectorization procedure in MATLAB. Since the discretization error decays exponentially, we expect that the C++ implementation with the x'-dependent truncation parameter will be more efficient.

We follow the prescription in [**18**], and set $\epsilon_1 = \epsilon\pi/(2D_s)$, where D_s is as in (4.11).

(I) If $x' = x + \mu\bar{\Delta} \geq 0$, then, for the calculation of $V_1(x)$ and $\mathcal{V}_s(x)$,

1. if $-0.1 \cdot \ln \epsilon_1 \geq -\bar{\Delta}\psi^0(i\lambda_+ - 0) + (1-s)\ln \lambda_+$ or $\nu < 1$, set $\mu_+ = \lambda_+$, otherwise, find μ_+ as the unique positive solution of equation $-0.1 \cdot \ln \epsilon = -\bar{\Delta}\psi^0(i\mu_+) + (1-s)\ln \mu_+$;
2. set $\mu_- = 0$, and d_+ and ω_+ as in (4.37) and (4.38), respectively;
3. set $\zeta_+ = -\pi d_+/(1.1 \ln \epsilon)$.

(II) If $x' < 0$, then for the calculation of $V_1(x)$ (resp., $\mathcal{V}_s(x)$),

1. if $-0.1 \cdot \ln \epsilon_1 \geq -\bar{\Delta}\psi^0(i\lambda_- + 0) + (1-s)\ln(-\lambda_-)$ or $\nu < 1$, set $\mu_- = \lambda_-$, otherwise, find μ_+ as the unique positive solution of equation $-0.1 \cdot \ln \epsilon = -\bar{\Delta}\psi^0(i\mu_-) + (1-s)\ln(-\mu_-)$;
2. set $\mu_+ = -1$ (resp., $\mu_+ = 0$), and d_- and ω_- as in (4.39) (resp., (4.41)) and (4.40) (resp., (4.42));
3. set $\zeta_- = -\pi d_-/(1.1 \ln \epsilon)$.

For the choice of M_c for $x' \geq 0$ (resp., $x' < 0$), one can first apply (4.45) (resp., (4.46), (4.47) and (4.50)) to find Λ, then use ζ above to choose the positive integer M_c such that $M_c\zeta \geq \Lambda$.

3.5. Error estimate of $\|\epsilon(\mathcal{V}_2, \cdot)\|_{L_1}$ and choices of ω, ζ, M_c. Consider the case when parabolic iFT method is used to calculate $\mathcal{V}_2(x')$ ($x' = x + \mu\bar{\Delta}$). By analogy with [**18**], we use the same bound for $\|\epsilon_d(\mathcal{V}_2, \cdot)\|_{L_1}$ as for flat iFT. One can derive the Hardy norm of the integrand in (4.35) and (4.36), hence, derive accurate error estimate for the discretization and truncation error. However, in our case, we only need simple recommendation for the choice of ω, ζ and Λ. Using (4.16), together with (4.4), a trivial modification of the algorithm in Section 3.4 gives the choices of ω, ζ. Set $\epsilon_1 = 0.9 \cdot \Delta^2 \cdot \epsilon_c/4 \cdot \pi/(2D_2)$.

(I) If $x' = x + \mu\bar{\Delta} \geq 0$, then

1. if $-0.1 \cdot \ln \epsilon_1 \geq -\bar{\Delta}\psi^0(i\lambda_+ - 0) - s\ln\lambda_+$ or $\nu < 1$, set $\mu_+ = \lambda_+$, otherwise, find μ_+ as the unique positive solution of equation $-0.1 \cdot \ln \epsilon = -\bar{\Delta}\psi^0(i\mu_+) - s\ln\mu_+$;
2. set $\mu_- = 0$, and d_+ and ω_+ as in (4.37) and (4.38), respectively;
3. set $\zeta_+ = \pi d_+/(1.1\ln\epsilon)$;

(II) If $x' < 0$, then

1. if $-0.1 \cdot \ln \epsilon_1 \geq -\bar{\Delta}\psi^0(i\lambda_- + 0) - s\ln(-\lambda_-)$ or $\nu < 1$, set $\mu_- = \lambda_-$, otherwise, find μ_- as the unique positive solution of equation $-0.1 \cdot \ln \epsilon = -\bar{\Delta}\psi^0(i\mu_-) - s\ln(-\mu_-)$;
2. set $\omega_+ = 0$, and d_- and ω_- as in (4.41) and (4.42), respectively;
3. set $\zeta_- = \pi d_-/(1.1\ln\epsilon)$.

Let

$$L_t^+ = \frac{B_+ e^{\bar{\Delta}\Lambda_+^{\alpha\nu}}}{(s-1)\Lambda_+^{(s-1)\alpha}} \cdot \frac{e^{-c\bar{\Delta}\Gamma(-\nu)(\lambda_+{}^{\nu}+(-\lambda_-)^{\nu})}}{\pi\lambda_+{}^{1-\alpha}} \tag{4.51}$$

$$L_t^- = \frac{e^{B_-\bar{\Delta}\Lambda_-^{\alpha\nu}}}{(s-1)\Lambda_-^{(s-1)\alpha}} \cdot \frac{e^{-c\bar{\Delta}\Gamma(-\nu)(\lambda_+{}^{\nu}+(-\lambda_-)^{\nu})}}{\pi(-\lambda_-)^{1-\alpha}} \tag{4.52}$$

where B_+ and B_- are as in (4.44) and (4.49), respectively.

LEMMA 4.11.

$$\|\epsilon_{tr}(\mathcal{V}_2,\cdot)\|_{L_1} \leq |A_+\Lambda_+^{\alpha}|^{-1}\cdot L_t^+ + |A_-\Lambda_-^{\alpha}|^{-1}\cdot L_t^-,$$

where A_+ *and* A_- *is as in* (4.43) *and* (4.48).

PROOF. Since

$$\|\epsilon_{tr}(\mathcal{V}_2,\cdot)\|_{L_1} = \|\epsilon_{tr}(\mathcal{V}_2\ {}_{(-\infty,0)},\cdot)\|_{L_1} + \|\epsilon_{tr}(\mathcal{V}_2\ {}_{[0,\infty)},\cdot)\|_{L_1},$$

it suffices to consider each term on the RHS above separately. We consider only the case $x' \geq 0$, the case $x' \leq 0$ are proved similarly. By Proposition 4.9, the truncation error admits an bound via

$$|\epsilon_{tr}(\mathcal{V}_2\ {}_{[0,\infty)},x')| \leq e^{x'A_+(\Lambda_+)\Lambda_+\alpha}\cdot L_t^+,$$

where A_+ and L_t^+ are as in (4.43) and (4.51). Integrating $e^{x'A_+(\Lambda_+)\Lambda_+\alpha}$ w.r.t. x', we obtain

$$\|\epsilon_{tr}(\mathcal{V}_2\ {}_{[0,\infty)},\cdot)\|_{L_1} \leq |A_+(\Lambda_+)\Lambda_+^{\alpha}|^{-1}\cdot L_t^+.$$

□

Using the above lemma, we find $\Lambda^{\pm}$ such that

$$\begin{aligned}
\frac{e^{A_+x'\Lambda_+^{\alpha}+B_+\bar{\Delta}\Lambda_+^{\alpha\nu}}}{A_+(s-1)\Lambda_+^{s\alpha}} &< \epsilon_1\pi{\lambda_+}^{1-\alpha}e^{c\bar{\Delta}\Gamma(-\nu)({\lambda_+}^{\nu}+(-\lambda_-)^{\nu})}\\
\frac{e^{-A_-x'\Lambda_-^{\alpha}+B_-\bar{\Delta}\Lambda_-^{\alpha\nu}}}{A_-(s-1)\Lambda_-^{s\alpha}} &< \epsilon_1\pi(-\lambda_-)^{1-\alpha}e^{c\bar{\Delta}\Gamma(-\nu)({\lambda_+}^{\nu}+(-\lambda_-)^{\nu})},
\end{aligned}$$

where $\epsilon_1 = 0.9\cdot\Delta^2\cdot\epsilon_c/4\cdot\pi/(2D_2)$. Positive integer M_c can be found from $M_c^{\pm}\zeta^{\pm} \geq \Lambda_{\pm}$.

CHAPTER 5

Numerical algorithm and examples

1. Algorithm

We present an explicit algorithm for computing the prices of Asian put and call option. The parabolic iFT method with the choice of numerical parameters described in Chapter 4 Section 3.2 and 3.4 can be used to replace the flat iFT and (refined) FFT method.

The inputs are the spot price S_0, the strike K, the maturity date T, the number of equally spaced sampling dates $N+1$, the interest rate r, the parameters of the process, and the error tolerance ϵ. If $(N+1)K-S_0 \le 0$, then the price of the Asian put option is 0. In the algorithm below, we assume $(N+1)K - S_0 > 0$, and let V_1 and $\mathcal{V}_2$ be as in (2.16) and (2.27), respectively.

Step I. Choose the error tolerances ϵ_{tr}, ϵ_{int} and ϵ_c that will control, respectively, the truncation error, interpolation error, and the impact of the errors of calculation of values of functions $\mathcal{V}_2$ and V_1.

Let $h = e^{-rT}((N+1)K - S_0)/(N+1)$, and set $\epsilon_{tr} = \epsilon_{tr}/h$, $\epsilon_{int} = \epsilon_{int}/h$ and $\epsilon_c = \epsilon_c/h$ as the error tolerance for the calculation of V_N. Set $\bar{\Delta} = T/N$.

Step II. Set $\epsilon_1 = \epsilon_{tr}/(N-1)$. Using (3.4) and (3.5), find $x_1 < x_M < 0$ so that the errors of the truncation above x_M and partial truncation below x_1 do not exceed ϵ_1.

Step III. Choose the order of the interpolation procedure, which (approximately) maximizes mesh Δ of the grid in the state space given $\epsilon_{int}/2$ (recall that we do the interpolation at each step twice), $\bar{\Delta}$ and parameters of the process. Below, we assume that the piece-wise linear interpolation is used. One can choose Δ so that the RHS of (3.11) does not exceed $\epsilon_{int}/2$.

Step IV. Choose a smallest integer M such that $(M-1)\Delta > x_M - x_1$, and re-define $x_1 = x_M - (M-1)\Delta$. Choose a smallest integer M_1 such that $(M_1 - 1)\Delta > x_M - \ln(1 - e^{X_M})$, and construct grids:

$$\vec{x} = (x_j)_{j=1}^{M_1}, \quad \vec{y} = (y_j)_{j=1}^{M}, \quad \text{and} \quad \vec{z} = (z_\ell)_{\ell=-M_1}^{M_1},$$

where

$$x_j = x_1 + (j-1)\Delta, \quad y_j = x_j - \ln(1 - e^{x_j}), \quad \text{and} \quad z_\ell = \ell\Delta.$$

Step V. Set $\epsilon_1 = \epsilon_c/N$. Calculate $\vec{V_x} \approx V_1(\vec{x})$ using flat iFT and (refined) iFFT with error tolerance ϵ_1. See Chapter (4) Section 2.2 for the choice of ξ-grid in the frequency domain.

Use piece-wise linear interpolation to find values: $\vec{V_y} \approx V_1(\vec{y})$, then store these values. After that, set $\vec{u} = (1 - e^{\vec{x}})(\vec{V_y} - 1)$, and re-set $u_1 = e^{x_1 - \bar{\Delta}\psi(-i)}$ and $u_M = 0$.

Step VI. Calculate and store arrays $\vec{V_1} \approx V_1((z_l)_{l=0}^{M_1-1})$ and $\vec{\mathcal{V}_2} \approx \mathcal{V}_2(\vec{z})$ using flat iFT and (refined) iFFT with error tolerance $0.1 \cdot \epsilon_1/|u_1|$ and $0.9 \cdot \Delta^2 \cdot \epsilon_1/4$. See Chapter 4 Section 2.2 for the choice of ξ-grid in the frequency domain.

Step VII. Calculate $\vec{U} \approx W_2(\vec{x})$ using (2.28) and fast convolution algorithm. The inputs are $\vec{u}$, $\vec{V_1}$, and $\vec{\mathcal{V}_2}$. Let $\vec{W} = \vec{U}$.

Step VIII. In the cycle w.r.t. $n = 2, 3, \dots, N-1$,

- calculate and store array $\vec{V}_y \approx W_n(\vec{y})$ using $\vec{W}$ as the input and applying the piece-wise linear interpolation procedure;
- set $\vec{u} := (1 - e^{\vec{x}})\vec{V}_y$ and re-set $u_1 = c_n e^{x_1}$ and $u_M = 0$, where c_n is given by (2.24);
- calculate $\vec{W} \approx W_{n+1}(\vec{x})$ using (2.28), inputs: $\vec{u}$, $\vec{\mathcal{V}}_2$ and $\vec{V}_1$, and fast convolution algorithm;
- set $\vec{W} = \vec{W} + \vec{U}$.

Step IX. Set $\vec{V}_N = \vec{W} + \vec{V}_x$. Calculate $V_N(x)$ at $x = \ln(S_0/(S_0 - (N+1)K))$ using $\vec{V}_N$, piecewise linear interpolation procedure, and then the Asian put option value using (2.3) and the Asian call option value using (2.2).

2. Numerical examples

The results presented below were performed in MATLAB® 7.11.0 (R2010b), on a laptop with characteristics Intel® Celeron® Processor T1600 (1MB Cache, 1.66GHz, 667MHz FSB), under the Genuine Windows Vista™ Home Basic with Service Pack 2 (32-bit) operating system.

In all the examples, the benchmark prices are calculated using our method with very long and fine grid, both in the state space and the frequency domain. The set of numerical parameters guarantee a small error not larger than 10^{-10}.

2.1. KoBoL: example taken from [36]. In this subsection , we compare the performance of our algorithm with the performance of MC method and the method developed by Fusai and Meucci [**36**], for calculating the prices of discretely monitored Asian call options. As in [**36**], we assume that under a chosen EMM, the log-spot price, $X_t = \ln S_t$, of the underlying follows a KoBoL process (see (2.12)) with parameters $\nu = 1.2945$, $c = 0.0244$, $\lambda_+ = 0.0765$, and $\lambda_- = -7.5515$. Assume the interest rate is $r = 0.0367$, which allows us to find the remaining parameter $\mu \approx 0.138736$ from the EMM condition $r + \psi(-i) = 0$ (where $\psi(\xi)$ is the characteristic exponent of $\{X_t\}$). The process is of infinite variation since order $\nu = 1.2945 \geq 1$.

We calculate the prices of a discretely sampled Asian call option on the stock $S_t = e^{X_t}$, with spot price $S_0 = 100$, maturity date $T = 1$ year and the number of sampling dates $N = 12$, 50 and 250, respectively.

The results of our calculation are summarized in Table 1. The numerical parameters of our algorithm are specified in the caption to the table. (We use the acronym "MC", "LX(f)" and "FM" to label the results obtained using our method (implemented with flat (refined) iFFT) and the method in [**36**].)

It is well known that the convergence of the MC estimator is very slow, therefore, we used $500,000$ paths. For simulating trajectories of KoBoL processes, we implemented the code[1] of Poirot and Tankov [**50**]. It is also well known that for processes of infinite variation, the accuracy of the results significantly affected by the simulation bias arise from truncating the small jumps, therefore, we truncate the jump with size less than 1×10^{-9}. From the table, we observe that MC produce results with relative error of order up to 10^{-3}.

For our results in column "LX(f)", we fix a moderately small mesh in the frequency domain, and change the mesh in the state space to see the change of the relative error. Due to the Nyquist relation $M\Delta\zeta = 2\pi$, halving Δ increases the length of the grid in the frequency domain. One

[1]The code is available at `http://www.math.jussieu.fr/~tankov/`.

can see that our method produce more accurate results than the method of "MC" and "FM", and faster. The initial length of the grid in the frequency domain depends on the number N of the sampling dates, therefore, we have to choose different M_2 in the refined FFT algorithm for different N. The reason is that as N increases, a longer grid is needed to ensure that the truncation error in the frequency domain is small. We observe that as Δ is halved, the error decreases by a factor of 10 and more. The exception is the case $K = 110$, $\bar{\Delta} = 1/50$ and $\Delta \approx 0.027781$, when the error decreases by a factor of 2. This is due to the fact that the truncation error in the frequency domain is rather large.

2.2. KoBoL: the examples taken from [27]. In this subsection , we compare the performance of our algorithm with the performance of the method developed by Černý and Kyriakou [**27**]. We use three parameter sets for KoBoL model (see (2.12)) considered in [**27**]:

$$\begin{aligned}
&\text{A:} && c = 0.2703,\ \lambda_- = -54.82,\ \lambda_+ = 17.56,\\
& && \nu = 0.8,\ \mu \approx 0.17753 \quad (m_2 \approx 0.01)\\
&\text{B:} && c = 0.6509,\ \lambda_- = -18.27,\ \lambda_+ = 5.853,\\
& && \nu = 0.8,\ \mu \approx 0.42432 \quad (m_2 \approx 0.09)\\
&\text{C:} && c = 0.9795,\ \lambda_- = -10.96,\ \lambda_+ = 3.512,\\
& && \nu = 0.8,\ \mu \approx 0.63587 \quad (m_2 \approx 0.25)
\end{aligned}$$

where m_2 is the second central moment of the KoBoL process. (Parameters μ are obtained from the EMM condition $r + \psi(-i) = 0$, where $\psi(\xi)$ is the characteristic exponent of $\{X_t\}$.) We list m_2 in order to facilitate the comparison with the Brownian motion model.

For these parameter sets, we calculate the prices of a discretely sampled Asian call option on the stock $S_t = e^{X_t}$, with spot price $S_0 = 100$, maturity date $T = 1$ year and the number of sampling dates $N = 50$; the riskless rate $r = 0.04$. The results are summarized in Table 4. The numerical parameters of our algorithm are specified in the caption to the table. (We use the acronym "LX" and "CK" to label the results obtained using our method and the method in [**27**], respectively.) We observe that the prices increase as m_2 increases as it is to be expected.

Our method takes much less CPU time than CK method to achieve the same level of accuracy. The CPU time in [**27**] was recorded on a relatively better computer than ours: Dell Latitude 620 Intel® Core™ 2 Duo Processor T7200 (4MB Cache, 2.00 GHz, 667 MHz FSB) and 2 GB RAM with MATLAB© R15.

For these sets of model and option parameters, the implementation of our approach with flat (refined) iFFT is faster than parabolic iFT. However, if we consider processes of small ν and/or $\bar{\Delta}$, the advantage of parabolic version is significant.

2.3. KoBoL: example with small ν. In this subsection, we compare the performance of flat (refined) iFFT with the performance of parabolic iFT. We take a KoBoL process with parameters $\nu = 0.2$, $c = 1.1136$, $\lambda_+ = 3$, $\lambda_- = -10$ from [**18**]. Assume the interest rate is $r = 0.04$, which allows us to find the remaining parameter $\mu \approx 0.30403$ from the EMM condition $r + \psi(-i) = 0$. The process is of finite variation since order $\nu = 0.2 \leq 1$.

For these parameter sets, we calculate the prices of a discretely sampled Asian call option on the stock $S_t = e^{X_t}$, with spot price $S_0 = 100$, maturity date $T = 1$ year and the number of sampling dates $N = 12$.

The results are summarized in Table 7. The numerical parameters of our algorithm are specified in the caption to the table. (We use the acronym "LX(f)" and "LX(p)" to label the results obtained using flat (refined) iFFT and parabolic iFT, respectively.)

For this process, even in the case with $\bar{\Delta}$ is not small, parabolic iFT perform much better than flat (refined) iFFT.

2.4. BM. In this subsection, we compare the performance of our method with the performance of the method developed in [**36**] and [**27**], for pricing discretely monitored Asian call option under the Brownian motion. The results are summarized in Table 8 and 11.

Since the probability density function of the increment behaves fairly regularly and has very thin tails, one does not need the FFT algorithm to enhance the efficiency and use a uniform grid. We first reduce the expectation (2.4) to the integration on the real line:

$$V_{n+1}(y) = \int_{-\infty}^{0} (1 - e^z)V_n(z - \ln(1 - e^z))p(z - y)dz, \tag{5.1}$$

where p is the normal probability density of the increments $X_{\bar{\Delta}}$ with the mean[2] $\mu\bar{\Delta}$ and variance $\sigma^2\bar{\Delta}$, and then, use Gaussian quadrature to evaluate the integral at $y = x - \ln(1 - e^x)$.

From our results, one can observe that Gaussian quadrature converge to the benchmark price very fast.

[2]The drift μ is determined by EMM condition: $\mu = r - \sigma^2/2$.

Table 1. Prices of Asian call options in the KoBoL model. Example 1: comparison with the results of Monte Carlo method (MC), Fusai and Meucci [**36**] (FM).

A: $N = 12$ ($\bar{\Delta} = T/N = 1/12$)

K	Benchmark	MC		LX(f)			FM		
		rel.err.	s.d.	$\Delta \approx$			$M =$		
				0.027958	0.013979	0.0069894	10000	5000	1000
90	12.7066281	1.0E-03	0.01171	-4.4E-06	-3.1E-07	-2.0E-08	-3.0E-05	-2.9E-05	-8.3E-05
100	5.0349805	3.3E-03	0.00863	1.3E-05	1.3E-06	8.8E-08	-1.2E-05	-2.4E-05	3.0E-04
110	1.0211530	7.5E-03	0.00453	1.1E-04	3.0E-06	2.9E-07	-2.9E-06	-5.2E-05	9.7E-04
CPU (sec.)				0.18532	0.33182	0.63647	N/A		

Column 2 contains the benchmark prices obtained using our method. Column labeled "MC" contain the relative difference between the benchmark prices and the results obtained using Monte Carlo method. Column labeled "LX(f)" contain the relative difference between the benchmark prices and the results obtained using our method. Column labeled "FM" contains the relative difference between the benchmark prices and the results obtained using the method of Fusai and Meucci [**36**]. The results are taken from the tables in *op. cit.*.

The example is taken from [**36**].
Asian call option parameters: $r = 0.0367$, $T = 1$, $S = 100$, N (number of sampling dates).
KoBoL parameters: $\nu = 1.2945$, $c = 0.0244$, $\lambda_+ = 0.0765$, $\lambda_- = -7.5515$, $\mu \approx 0.138736$.

Numerical parameters for benchmark prices: enhancement — cubic spline, $x_1 = -15$, $x_M = -0.0001$, $\Delta \approx 0.0002$, $\omega_+ = \lambda_+/2$, $\omega_- = -2$, $\zeta_1 \approx 0.008$ ($M_3 = 32$), and $\Lambda^\pm \approx 28766.5$ ($= M_2 \cdot 2\pi/\Delta$, $M_2 = 1$).

Numerical parameters of LX(f): enhancement — cubic spline, $x_1 = -8$, $x_M = -0.0001$, $\omega_+ = \lambda_+/2$, $\omega_- = -2$, $\zeta_1 \approx 0.045$ ($M_3 = 8$), and $\Lambda^\pm = M_2 \cdot 2\pi/\Delta$, $M_2 = 2$.

Numerical parameters for MC: number of trajectories — 5×10^5, truncation parameter — 1×10^{-9}, simulator for KoBoL — Tankov's code (`http://www.math.jussieu.fr/~tankov/`).

M — the number of points in M-point Gaussian quadrature;
Λ^+ (Λ^-) — the length of the truncated line of integration in upper (lower) half-plane of the frequency domain.

Table 2. Prices of Asian call options in the KoBoL model. Example 1: comparison with the results of Monte Carlo method (MC), Fusai and Meucci [**36**] (FM).

B: $N = 50$ ($\bar{\Delta} = T/N = 1/50$)

K	Benchmark	MC		LX(f)			FM		
				$\Delta \approx$			$M =$		
		rel.err.	s.d.	0.027781	0.013979	0.0069894	10000	5000	1000
90	12.7400351	-9.9E-04	0.01176	-1.3E-05	-1.2E-06	-9.4E-08	-1.2E-04	-1.2E-04	2.3E-05
100	5.0761189	-1.1E-03	0.00867	8.5E-05	7.1E-06	2.8E-07	-8.3E-05	-6.9E-05	-7.6E-04
110	1.0467955	-2.4E-03	0.00457	5.3E-05	2.6E-05	1.1E-06	-5.3E-05	-1.0E-04	2.6E-03
CPU (sec.)				0.42915	0.86946	1.6236	N/A		

Column 2 contains the benchmark prices obtained using our method. Column labeled "MC" contain the relative difference between the benchmark prices and the results obtained using Monte Carlo method. Column labeled "LX(f)" contain the relative difference between the benchmark prices and the results obtained using our method. Column labeled "FM" contains the relative difference between the benchmark prices and the results obtained using the method of Fusai and Meucci [**36**]. The results are taken from the tables in *op. cit.*.

The example is taken from [**36**].
Asian call option parameters: $r = 0.0367$, $T = 1$, $S = 100$, N (number of sampling dates).
KoBoL parameters: $\nu = 1.2945$, $c = 0.0244$, $\lambda_+ = 0.0765$, $\lambda_- = -7.5515$, $\mu \approx 0.138736$.

Numerical parameters for benchmark prices: enhancement — cubic spline, $x_1 = -15$, $x_M = -0.0001$, $\Delta \approx 0.0002$, $\omega_+ = \lambda_+/2$, $\omega_- = -2$, $\zeta_1 \approx 0.008$ ($M_3 = 32$), and $\Lambda^{\pm} \approx 28766.5$ ($= M_2 \cdot 2\pi/\Delta$, $M_2 = 1$).

Numerical parameters LX(f): enhancement — cubic spline, $x_1 = -8$, $x_M = -0.0001$, $\omega_+ = \lambda_+/2$, $\omega_- = -2$, $\zeta_1 \approx 0.045$ ($M_3 = 8$), and $\Lambda^{\pm} = M_2 \cdot 2\pi/\Delta$ $M_2 = 4$.

Numerical parameters for MC: number of trajectories — 5×10^5, truncation parameter — 1×10^{-9}, simulator for KoBoL — Tankov's code (`http://www.math.jussieu.fr/~tankov/`).

M — the number of points in M-point Gaussian quadrature;
Λ^+ (Λ^-) — the length of the truncated line of integration in upper (lower) half-plane of the frequency domain.

TABLE 3. Prices of Asian call options in the KoBoL model. Example 1: comparison with the results of Monte Carlo method (MC), Fusai and Meucci [**36**] (FM).

C: $N = 250$ ($\bar{\Delta} = T/N = 1/250$)

K	Benchmark	MC		LX(f)			FM		
				$\Delta \approx$			$M =$		
		rel.err.	s.d.	0.013868	0.0069342	0.0034671	10000	5000	1000
90	12.7491229	-1.4E-03	0.01178	-2.8E-05	-3.9E-07	-3.6E-08	-1.4E-04	-1.7E-03	
100	5.0874701	-2.8E-03	0.00870	-4.0E-05	2.3E-06	1.1E-07	-1.0E-04	-1.6E-03	N/A
110	1.0539774	-4.1E-03	0.00461	-1.5E-04	6.9E-06	4.8E-07	-8.3E-05	-1.4E-03	
CPU (sec.)				1.9053	3.9343	8.1125	N/A		

Column 2 contains the benchmark prices obtained using our method. Column labeled "MC" contain the relative difference between the benchmark prices and the results obtained using Monte Carlo method. Column labeled "LX(f)" contain the relative difference between the benchmark prices and the results obtained using our method. Column labeled "FM" contains the relative difference between the benchmark prices and the results obtained using the method of Fusai and Meucci [**36**]. The results are taken from the tables in *op. cit.*.

The example is taken from [**36**].
Asian call option parameters: $r = 0.0367$, $T = 1$, $S = 100$, N (number of sampling dates).
KoBoL parameters: $\nu = 1.2945$, $c = 0.0244$, $\lambda_+ = 0.0765$, $\lambda_- = -7.5515$, $\mu \approx 0.138736$.

Numerical parameters for benchmark prices: enhancement — cubic spline, $x_1 = -15$, $x_M = -0.0001$, $\Delta \approx 0.0002$, $\omega_+ = \lambda_+/2$, $\omega_- = -2$, $\zeta_1 \approx 0.008$ ($M_3 = 32$), and $\Lambda^\pm \approx 28766.5$ ($= M_2 \cdot 2\pi/\Delta$, $M_2 = 1$).

Numerical parameters LX(f): enhancement — cubic spline, $x_1 = -8$, $x_M = -0.0001$, $\omega_+ = \lambda_+/2$, $\omega_- = -2$, $\zeta_1 \approx 0.045$ ($M_3 = 8$), and $\Lambda^\pm = M_2 \cdot 2\pi/\Delta$, $M_2 = 4$.

Numerical parameters for MC: number of trajectories — 5×10^5, truncation parameter — 1×10^{-9}, simulator for KoBoL — Tankov's code (`http://www.math.jussieu.fr/~tankov/`).

M — the number of points in M-point Gaussian quadrature;
Λ^+ (Λ^-) — the length of the truncated line of integration in upper (lower) half-plane of the frequency domain.

TABLE 4. Prices of Asian call options in the KoBoL model. Example 2: comparison with the results of Černý and Kyriakou [**27**] (CK).

A: $c = 0.2703$, $\lambda_- = -54.82$, $\lambda_+ = 17.56$, $\nu = 0.8$, $\mu \approx 0.17753$ ($m_2 \approx 0.01$)

K	Benchmark	LX(f)			LX(p)			CK
		$\Delta \approx$			$\Delta \approx$			
		0.013898	0.0069492	0.0034746	0.013898	0.0069492	0.0034746	
90	11.6398812	-2.9E-06	-1.9E-07	-2.7E-09	-2.9E-06	-1.9E-07	-2.8E-09	-1.0E-07
100	3.3245835	3.9E-05	1.6E-06	1.4E-07	3.9E-05	1.6E-06	1.4E-07	-1.1E-06
110	0.1578768	-1.4E-03	-6.7E-05	-4.3E-06	-1.4E-03	-6.7E-05	-4.3E-06	-4.3E-05
CPU (sec.)		0.43966	0.61824	1.1756	0.78197	1.5812	3.8461	8.5*

Column 2 contains the benchmark prices obtained using our method. Columns labeled "LX(f)" and "LX(p)" contain the relative difference between the benchmark prices and the results obtained using our method, implemented with flat (refined) iFFT and parabolic iFT, respectively. Column labeled "CK" contains the relative difference between the benchmark prices and the results obtained using the method of Černý and Kyriakou [**27**]. The results are the same with our benchmark prices, they are taken from the tables in *op. cit.*.

The calculations of "LX(f)" and "LX(p)" presented were performed in MATLAB © 7.11.0 (R2010b), on a laptop with characteristics Intel® Celeron® Processor T1600 (1MB Cache, 1.66GHz, 667MHz FSB) and 1 GB RAM, under the Genuine Windows Vista ™ Home Basic with Service Pack 2 (32-bit) operating system. The calculation of "CK" presented were performed in MATLAB© R15, on a Dell Latitude 620 Intel® Core™ 2 Duo Processor T7200 (4MB Cache, 2.00 GHz, 667 MHz FSB) and 2 GB RAM.

The example is taken from [**27**].
Asian call option parameters: $r = 0.04$, $T = 1$, $S = 100$, $N = 50$ (number of sampling dates), $\bar{\Delta} = T/N = 0.02$.

Numerical parameters for benchmark prices: enhancement — cubic spline, $x_1 = -15$, $x_M = -0.0001$, $\Delta \approx 0.0002$, $\omega_+ = \min\{\lambda_+/2, 2\}$, $\omega_- = \max\{-2, (\lambda_- - 1)/2\}$, $\zeta_1 \approx 0.008$ ($M_3 = 32$), and $\Lambda^{\pm} \approx 28933$ ($= M_2 \cdot 2\pi/\Delta$, $M_2 = 1$).

Numerical parameters of LX(f) & LX(p): enhancement — cubic spline, $x_1 = -7.6025$, $x_M = -0.000046$, parameters of the numerical scheme in the dual space is chosen using the recommendations in the thesis with $\epsilon = 10^{-12}$.

Λ^+ (Λ^-) — the length of the truncated line of integration in upper (lower) half-plane of the frequency domain. m_2 — the second central moment of the KoBoL process.

Table 5. Prices of Asian call options in the KoBoL model. Example 2: comparison with the results of Černý and Kyriakou [**27**] (CK).

B: $c = 0.6509$, $\lambda_- = -18.27$, $\lambda_+ = 5.853$, $\nu = 0.8$, $\mu \approx 0.42432$ ($m_2 \approx 0.09$)

K	Benchmark	LX(f) $\Delta \approx$ 0.027896	LX(f) $\Delta \approx$ 0.013948	LX(f) $\Delta \approx$ 0.006974	LX(p) $\Delta \approx$ 0.027896	LX(p) $\Delta \approx$ 0.013948	LX(p) $\Delta \approx$ 0.006974	CK
90	13.7016037	7.1E-07	1.1E-07	2.1E-08	7.1E-07	1.1E-07	2.1E-08	-2.7E-07
100	7.3474239	1.8E-05	1.6E-06	9.3E-08	1.8E-05	1.6E-06	9.3E-08	-5.2E-07
110	3.2830822	4.7E-05	3.9E-06	2.2E-07	4.7E-05	3.9E-06	2.2E-07	-6.6E-07
CPU (sec.)		0.27935	0.36186	0.62633	0.55654	1.0373	2.3099	4.1*

Column 2 contains the benchmark prices obtained using our method. Columns labeled "LX(f)" and "LX(p)" contain the relative difference between the benchmark prices and the results obtained using our method, implemented with flat (refined) iFFT and parabolic iFT, respectively. Column labeled "CK" contains the relative difference between the benchmark prices and the results obtained using the method of Černý and Kyriakou [**27**]. The results are the same with our benchmark prices, they are taken from the tables in *op. cit.*.

The calculations of "LX(f)" and "LX(p)" presented were performed in MATLAB © 7.11.0 (R2010b), on a laptop with characteristics Intel® Celeron® Processor T1600 (1MB Cache, 1.66GHz, 667MHz FSB) and 1 GB RAM, under the Genuine Windows Vista ™ Home Basic with Service Pack 2 (32-bit) operating system. The calculation of "CK" presented were performed in MATLAB© R15, on a Dell Latitude 620 Intel® Core™ 2 Duo Processor T7200 (4MB Cache, 2.00 GHz, 667 MHz FSB) and 2 GB RAM.

The example is taken from [**27**].
Asian call option parameters: $r = 0.04$, $T = 1$, $S = 100$, $N = 50$ (number of sampling dates), $\bar{\Delta} = T/N = 0.02$.

Numerical parameters for benchmark prices: enhancement — cubic spline, $x_1 = -15$, $x_M = -0.0001$, $\Delta \approx 0.0002$, $\omega_+ = \min\{\lambda_+/2, 2\}$, $\omega_- = \max\{-2, (\lambda_- - 1)/2\}$, $\zeta_1 \approx 0.008$ ($M_3 = 32$), and $\Lambda^{\pm} \approx 28933$ ($= M_2 \cdot 2\pi/\Delta$, $M_2 = 1$).

Numerical parameters of LX(f) & LX(p): enhancement — cubic spline, $x_1 = -7.6156$, $x_M = -0.000046$, parameters of the numerical scheme in the dual space is chosen using the recommendations in the thesis with $\epsilon = 10^{-12}$.

Λ^+ (Λ^-) — the length of the truncated line of integration in upper (lower) half-plane of the frequency domain. m_2 — the second central moment of the KoBoL process.

TABLE 6. Prices of Asian call options in the KoBoL model. Example 2: comparison with the results of Černý and Kyriakou [**27**] (CK).

C: $c = 0.9795$, $\lambda_- = -10.96$, $\lambda_+ = 3.512$, $\nu = 0.8$, $\mu \approx 0.63587$ ($m_2 \approx 0.25$)

K	Benchmark	LX(f)			LX(p)			CK
		$\Delta \approx$			$\Delta \approx$			
		0.027981	0.013991	0.0069953	0.027981	0.013991	0.0069953	
90	16.7683558	8.0E-07	8.4E-08	1.0E-08	8.0E-07	8.4E-08	1.1E-08	-3.5E-07
100	11.2442404	2.7E-06	2.5E-07	2.0E-08	2.7E-06	2.5E-07	2.0E-08	-3.9E-08
110	7.1762405	4.6E-06	4.1E-07	2.9E-08	4.6E-06	4.1E-07	3.0E-08	-7.2E-08
CPU (sec.)		0.28205	0.43715	0.67224	0.68402	1.2826	2.6453	2.1*

Column 2 contains the benchmark prices obtained using our method. Columns labeled "LX(f)" and "LX(p)" contain the relative difference between the benchmark prices and the results obtained using our method, implemented with flat (refined) iFFT and parabolic iFT, respectively. Column labeled "CK" contains the relative difference between the benchmark prices and the results obtained using the method of Černý and Kyriakou [**27**]. The results are the same with our benchmark prices, they are taken from the tables in *op. cit.*.

The calculations of "LX(f)" and "LX(p)" presented were performed in MATLAB © 7.11.0 (R2010b), on a laptop with characteristics Intel® Celeron® Processor T1600 (1MB Cache, 1.66GHz, 667MHz FSB) and 1 GB RAM, under the Genuine Windows Vista ™ Home Basic with Service Pack 2 (32-bit) operating system. The calculation of "CK" presented were performed in MATLAB© R15, on a Dell Latitude 620 Intel® Core™ 2 Duo Processor T7200 (4MB Cache, 2.00 GHz, 667 MHz FSB) and 2 GB RAM.

The example is taken from [**27**].

Asian call option parameters: $r = 0.04$, $T = 1$, $S = 100$, $N = 50$ (number of sampling dates), $\bar{\Delta} = T/N = 0.02$.

Numerical parameters for benchmark prices: enhancement — cubic spline, $x_1 = -15$, $x_M = -0.0001$, $\Delta \approx 0.0002$, $\omega_+ = \min\{\lambda_+/2, 2\}$, $\omega_- = \max\{-2, (\lambda_- - 1)/2\}$, $\zeta_1 \approx 0.008$ ($M_3 = 32$), and $\Lambda^\pm \approx 28933$ ($= M_2 \cdot 2\pi/\Delta$, $M_2 = 1$).

Numerical parameters of LX(f) & LX(p): enhancement — cubic spline, $x_1 = -7.8067$, $x_M = -0.000019$, parameters of the numerical scheme in the dual space is chosen using the recommendations in the thesis with $\epsilon = 10^{-12}$.

Λ^+ (Λ^-) — the length of the truncated line of integration in upper (lower) half-plane of the frequency domain. m_2 — the second central moment of the KoBoL process.

Table 7. Prices of Asian call options in the KoBoL model. Example 3: flat (refined) iFFT vs. parabolic iFT.

$c = 1.1136,\ \lambda_- = -10,\ \lambda_+ = 3,\ \nu = 0.2,\ \mu \approx 0.30403\ (m_2 \approx 0.16)$

K	Benchmark	LX(f)			LX(p)		
		$\Delta \approx$			$\Delta \approx$		
		0.02836	0.014118	0.007059	0.02836	0.014118	0.007059
90	14.7955309.	-2.9E-06	-2.1E-07	1.9E-08	-2.9E-06	-2.1E-07	1.9E-08
100	8.2812183.	-9.6E-06	-7.8E-07	3.5E-08	-9.6E-06	-7.8E-07	3.5E-08
110	3.7180942.	-2.0E-05	-1.7E-06	1.5E-07	-2.0E-05	-1.8E-06	1.5E-07
CPU (sec.)		27.2752	27.77	27.8486	0.38203	0.79295	1.6773

Column 2 contains the benchmark prices obtained using our method. Columns labeled “LX(f)” and “LX(p)” contain the relative difference between the benchmark prices and the results obtained using our method, implemented with flat (refined) iFFT and parabolic iFT, respectively.

The example is taken from [**18**].

Asian call option parameters: $r = 0.04$, $T = 1$, $S = 100$, $N = 12$ (number of sampling dates), $\bar{\Delta} = T/N = 1/12$.

Numerical parameters for benchmark prices: enhancement — cubic spline, $x_1 = -15$, $x_M = -0.0001$, $\Delta \approx 0.0004$, $\omega_+ = \min\{\lambda_+/2, 2\}$, $\omega_- = \max\{-2, (\lambda_- - 1)/2\}$, $\zeta_1 \approx 0.16935$ ($M_3 = 2$), and $\Lambda^{\pm} \approx 455732$ ($= M_2 \cdot 2\pi/\Delta$, $M_2 = 32$).

Numerical parameters of LX(f) & LX(p): enhancement — cubic spline, $x_1 = -7.186$, $x_M = -0.00001$, parameters of the numerical scheme in the dual space is chosen using the recommendations in the thesis with $\epsilon = 10^{-10}$.

Λ^+ (Λ^-) — the length of the truncated line of integration in upper (lower) half-plane of the frequency domain. m_2 — the second central moment of the KoBoL process.

Table 8. Prices of Asian call options under BM. Example 1: comparison with the results of Fusai and Meucci [**36**] (FM).

A: $N = 12$ ($\bar{\Delta} = T/N = 1/12$)

K	Benchmark	LX			FM		
		$A = 4.0236$			$(u, l) =$ N/A		
		M=58	M=114	M=226	M=1000	M=5000	M=10000
90	11.90491575	-2.5E-03	-1.0E-08	0.0E+00	-5.3E-05	5.4E-06	4.6E-06
100	4.88196162	3.8E-03	-1.9E-07	-2.0E-15	5.8E-06	3.2E-05	2.8E-05
110	1.36303795	2.9E-01	-1.9E-07	0.0E+00	4.9E-04	7.5E-05	7.5E-05
CPU (sec.)		0.002395	0.005547	0.015265	N/A		

Column 2 contains the benchmark prices obtained using our method with the numerical parameters as follows. Column labeled "LX" contain the relative difference between the benchmark prices and the results obtained using our method. Column labeled "FM" contains the relative difference between the benchmark prices and the results obtained using the method of Fusai and Meucci [**36**]. The results are taken from the tables in *op. cit.*.

The example is taken from [**36**].
Asian call option parameters: $r = 0.0367$, $T = 1$, $S = 100$, N — number of sampling dates.
BM parameters: $\sigma = .17801$, $\mu \approx 0.020856$.

Numerical parameters for benchmark prices: $A = 15.6327$, $M = 3604$.

A — truncation parameter;
M — the number of points in M-point Gaussian quadrature.

TABLE 9. Prices of Asian call options under BM. Example 1: comparison with the results of Fusai and Meucci [**36**] (FM).

B: $N = 50$ ($\bar{\Delta} = T/N = 1/50$)

K	Benchmark	LX			FM		
		$A = 5.7548$			$(u, l) = \mathrm{N/A}$		
		M=332	M=662	M=1322	M=1000	M=5000	M=10000
90	11.93293820	-2.4E-09	0.0E+00	0.0E+00	3.8E-05	4.3E-06	6.5E-04
100	4.93720281	9.5E-08	1.8E-14	-2.2E-14	-1.9E-05	3.6E-05	3.1E-03
110	1.40251551	1.1E-06	6.4E-14	-1.5E-13	-3.7E-04	7.5E-05	7.8E-03
CPU (sec.)		0.040542	0.173280	0.616170	N/A		

Column 2 contains the benchmark prices obtained using our method with the numerical parameters as follows. Column labeled "LX" contain the relative difference between the benchmark prices and the results obtained using our method. Column labeled "FM" contains the relative difference between the benchmark prices and the results obtained using the method of Fusai and Meucci [**36**]. The results are taken from the tables in *op. cit.*.

The example is taken from [**36**].
Asian call option parameters: $r = 0.0367$, $T = 1$, $S = 100$, N — number of sampling dates.
BM parameters: $\sigma = .17801$, $\mu \approx 0.020856$.

Numerical parameters for benchmark prices: $A = 11.4577$, $M = 5288$.

A — truncation parameter;
M — the number of points in M-point Gaussian quadrature.

TABLE 10. Prices of Asian call options under BM. Example 1: comparison with the results of Fusai and Meucci [**36**] (FM).

C: $N = 250$ ($\bar{\Delta} = T/N = 1/250$)

K	Benchmark	LX			FM		
		$A = 9.1847$			$(u, l) =$ N/A		
		M=1060	M=2120	M=4240	M=1000	M=5000	M=10000
90	11.94056316	-5.5E-06	8.5E-15	0.0E+00	6.8E-05	1.1E-05	9.8E-06
100	4.95215688	-1.0E-04	6.3E-14	-4.4E-14	-5.5E-04	4.7E-05	3.5E-05
110	1.41336703	-1.2E-03	5.2E-13	1.5E-13	-3.3E-04	9.4E-05	1.0E-04
CPU (sec.)		0.93958	3.6859	14.7639	N/A		

Column 2 contains the benchmark prices obtained using our method with the numerical parameters as follows. Column labeled "LX" contain the relative difference between the benchmark prices and the results obtained using our method. Column labeled "FM" contains the relative difference between the benchmark prices and the results obtained using the method of Fusai and Meucci [**36**]. The results are taken from the tables in *op. cit.*.

The example is taken from [**36**].
Asian call option parameters: $r = 0.0367$, $T = 1$, $S = 100$, N — number of sampling dates.
BM parameters: $\sigma = .17801$, $\mu \approx 0.020856$.

Numerical parameters for benchmark prices: $A = 18.363$, $M = 16954$.

A — truncation parameter;
M — the number of points in M-point Gaussian quadrature.

Table 11. Prices of Asian call options under BM. Example 2: comparison with the results of Černý and Kyriakou [**27**] (CK).

A: $\sigma = 0.1$, $\mu \approx 0.035$

K	Benchmark	LX			CK
		$A = 5.5479$			
		M=320	M=638	M=1274	
90	11.58113414	7.5E-06	-3.2E-12	0.0E+00	-3.6E-07
100	3.33861712	1.6E-02	-1.1E-10	-2.1E-14	-2.1E-06
110	0.27375877	1.2E+00	1.8E-09	-1.2E-12	-3.2E-05
CPU (sec.)		0.036130	0.158800	0.581260	1*

B: $\sigma = 0.3$, $\mu \approx -0.005$

K	Benchmark	LX			CK
		$A = 6.3072$			
		M=182	M=364	M=728	
90	13.66981573	-1.8E-05	-7.3E-15	0.0E+00	-4.2E-07
100	7.69859896	-1.2E-04	-3.6E-14	-2.5E-15	-1.2E-06
110	3.89639940	-5.6E-04	-1.1E-13	-2.3E-14	-2.4E-06
CPU (sec.)		0.014509	0.055880	0.231310	0.3*

C: $\sigma = 0.5$, $\mu \approx -0.085$

K	Benchmark	LX			CK
		$A = 8.0298$			
		M=116	M=230	M=458	
90	17.19239284	5.2E-03	1.0E-13	1.2E-14	-1.7E-07
100	12.09153558	1.3E-02	2.2E-13	2.5E-14	-4.6E-07
110	8.31441256	3.0E-02	4.0E-13	5.1E-14	-3.1E-07
CPU (sec.)		0.007235	0.021060	0.101140	0.3*

Column 2 contains the benchmark prices obtained using our method with the numerical parameters as follows. Column labeled "LX" contain the relative difference between the benchmark prices and the results obtained using our method. For the method of Černý and Kyriakou [**27**], the results are taken from the tables in *op. cit.* and are the same as our benchmark prices.

The calculations of "LX" presented were performed in MATLAB© 7.11.0 (R2010b), on a laptop with characteristics Intel® Celeron® Processor T1600 (1MB Cache, 1.66GHz, 667MHz FSB) and 1 GB RAM, under the Genuine Windows Vista™Home Basic with Service Pack 2 (32-bit) operating system. The calculation of "CK" presented were performed in MATLAB© R15, on a Dell Latitude 620 Intel® Core™ 2 Duo Processor T7200 (4MB Cache, 2.00 GHz, 667 MHz FSB) and 2 GB RAM. *Asian call option parameters:* $r = 0.0367$, $T = 1$, $S = 100$, $N = 50$ (number of sampling dates). (The example is taken from [**27**].)

Numerical parameters for benchmark prices: *Panel A:* $A = 11.0353$, $M = 5092$, *Panel B:* $A = 11.4577$, $M = 5288$. *Panel C:* $A = 11.4577$, $M = 5288$.

A — truncation parameter;
M — the number of points in M-point Gaussian quadrature.

CHAPTER 6

Conclusion

We introduced a new method for pricing discretely sample Asian options, and suggest efficient numerical realization. We calculated prices of call options for several sets of parameters in the KoBoL and Brownian motion models, and demonstrated that our method are both more accurate and efficient than the method developed by Fusai and Meucci [**36**], and the method developed by Černý and Kyriakou [**27**].

By comparison with the implementation of the method developed in [**36**] and [**27**], not only is our approach faster, but it is also inherently more accurate. The complete disentanglement of the dual grids that we achieved allows one to simultaneously and independently control the errors arise from truncating and discretizing the log-price domain, and the errors of numerical Fourier inversion. We derive bounds for all sources of errors, with prescriptions for parameter choices according to a desired error tolerance.

For the sets of model parameters in [**36**] and [**27**], with the implementation of flat (refined) iFFT, our approach allows one to achieve the absolute error of 10^{-8} within 1 seconds; and with the implementation of parabolic iFT, our approach needs 2 seconds. However, if the order of the process and/or the interval between two sampling dates are small, parabolic iFT is significant faster than flat (refined) iFFT, which is due to the fact that the integrand of the Fourier inversion in flat iFT decays very slowly at infinity, and too many terms may be needed to satisfy the desired error tolerance, while the conformal deformation of the contour of integration with the subsequent change of the variables (parabolic iFT) can be used to greatly decrease the number of terms in the simplified trapezoid rule. In general, parabolic iFT is much faster than flat iFT for point-wise calculation. Since flat iFT allows one to use (refined) iFFT algorithm, if the order of the process and/or the interval between two sampling dates are not small, flat (refined) iFFT is faster than parabolic iFT.

APPENDIX A

Pricing European options under Lévy process

1. Lévy processes: general definitions and basic facts

For an exposition of the general theory of Lévy processes and their applications to pricing derivative securities, we refer the reader to [**10, 58**] and [**17, 30, 61**], respectively.

Definition A.1. A *one-dimensional Lévy process* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $X = \{X_t\}_{t\geq 0}$ of $\mathbb{R}$-valued random variables on Ω satisfying the following properties:

(1) Given an integer $n \geq 1$ and a collection of times $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}$, $\ldots$ $X_{t_n} - X_{t_{n-1}}$ are independent.

(2) $X_0 = 0$ almost surely.

(3) For any $t \geq 0$, the distribution of $X_{s+1} - X_s$ is independent of $s \geq 0$.

(4) Stochastic continuity: given $t \geq 0$ and $\epsilon > 0$, we have

$$\lim_{s\to t} \mathbb{P}\left[|X_s - X_t| > \epsilon\right] = 0.$$

(5) There exists a subset $\Omega_0 \subset \Omega$ such that $\mathbb{P}[\Omega_0] = 1$ and for every $\omega \in \Omega_0$, the trajectory $t \mapsto X_t(\omega)$ is right continuous in $t \geq 0$, and has left limits for all $t > 0$.

Poisson process is an example of a pure jump Lévy processes.

DEFINITION A.2. A stochastic process X on $\mathbb{R}$ is a *Poisson process* with parameter $\lambda > 0$ if it is a Lévy process and, for $t > 0$, X_t has Poisson distribution with mean λt:

$$\mu_t\{k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \quad \text{for} \quad k = 0, 1, 2, \dots,$$

and $\mu_t(B) = 0$ for any B containing no nonnegative integer.

The Poisson process and Brownian motion are fundamental examples of Lévy processes. They can be thought of as building blocks of Lévy processes because every Lévy process is a superposition of a Brownian motion and a (infinite) number of independent Poisson processes. The Brownian motion is the only subclass of Lévy processes with continuous sample paths. Sample paths of any other Lévy process exhibit jumps.

The primary tool in the analysis of distributions of Lévy processes is characteristic functions of distribution. We give definitions, properties, and examples of characteristic function.

DEFINITION A.3. Denote by $\hat{\mathbb{P}}_X$ the *characteristic function* of the distribution $\mathbb{P}_X$ of a random variable X on $\mathbb{R}$. $\hat{\mathbb{P}}_X$ admits the representation

$$\hat{\mathbb{P}}_X(\xi) = \int_{\mathbb{R}} e^{ix\xi}\mathbb{P}_X(x)dx.$$

PROPOSITION A.4. [**58**, Proposition 2.5] *Let $\mathbb{P}_1$, $\mathbb{P}_2$ be distribution on $\mathbb{R}$. If $\hat{\mathbb{P}}_1(\xi) = \hat{\mathbb{P}}_2(\xi)$ for $\xi \in \mathbb{R}$, then $\mathbb{P}_1 = \mathbb{P}_2$.*

The characteristic function of the distribution of the Lévy process X_t admits the representation

$$\mathbb{E}\left[e^{i\xi X_t}\right] = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}, t \geq 0. \tag{A.1}$$

The function ψ is called the *characteristic exponent* of X. It characterizes X in the sense that two Lévy processes without the same characteristic exponent have the same probability distribution $\mathbb{P}_{X_t}$, for each t (see [**58**, Theorem 7.10]).

EXAMPLE A.5. a) Let X be the BM with drift μ and volatility σ. Then the characteristic function of X is given by (A.1) with the characteristic exponent

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi.$$

b) Let X be the Poisson process with intensity λ. Then the characteristic function of X is given by (A.1) with the characteristic exponent

$$\psi(\xi) = \lambda(1 - e^{i\xi}).$$

The Lévy-Khintchine formula below describes all possible characteristic exponents, hence, all Lévy processes.

THEOREM A.6. *a) Let X be a Lévy process on $\mathbb{R}$. Then its characteristic exponent admits the representation*

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi + \int_{\mathbb{R}\setminus 0} \left(1 + i\xi x \quad {}_{[-1,1]}(x) - e^{ix\xi}\right) F(dx), \tag{A.2}$$

where $\sigma \geq 0$, $b \in \mathbb{R}$, and F is a measure on $\mathbb{R} \setminus 0$ satisfying

$$\int_{\mathbb{R}\setminus 0} \min\{|x|^2, 1\} F(dx) < \infty. \tag{A.3}$$

b) The representation (A.2) *is unique.*

c) Conversely, if $\sigma \geq 0$, $b \in \mathbb{R}$, and F is a measure on $\mathbb{R}$ satisfying (A.3)*, then there exists a Lévy process X defined by* (A.1) *and* (A.2).

The triplet (σ, F, b) is called the *generating triplet*. The Lévy measure, F, can be interpreted as follows: the expected number of jumps per unit of time from 0 into a measurable set $U \subset \mathbb{R} \setminus \{0\}$ equals $\int_U F(dx)$. The term $ix\xi \ _{[-1,1]}(x)$ in (A.2) is needed to ensure the convergence of the integral, and hence other functions can be used instead of $c(x) := \ _{[-1,1]}(x)$, for instance, $c(x) = 1/(1 + |x|^2)$; the σ and F are independent of the choice of c but b does depend on the choice.

Example A.7. If $\sigma = 0$ and

$$F(dx) = c(x/|x|) \cdot |x|^{-\nu-1} dx,$$

where $\nu \in (0, 2)$ and c is a non-negative function, then X is a stable Lévy process of index ν.

Classes of Lévy process can be constructed in different ways. For example, Hyperbolic processes are obtained by constructing a probability distribution and showing that is infinitely divisible. KoBoL family can be constructed by taking appropriate Lévy measures and making explicitly calculations in (A.2). Once characteristic exponents of some Lévy processes are constructed, one can extend the list by using subordination and linear transformation. In the remaining part of this section, we describe several families of Lévy process, which is widely used in finance, and remind to the reader the general pricing formulas for the options of the European type.

2. Classes of Lévy processes of exponential type

To the best of our knowledge, Lévy process were first introduced to finance by Mandelbrot [**54**] in 1963. Since then a variety of models based on Lévy processes have been proposed as models for asset prices and tested on empirical data. One of the principle motivations for departing from Gaussian models in finance has been to take into account some of the observed empirical properties of asset returns which disagree with these models. (See the review of literature in [**17, 30**].)

Mandelbrot [**54**] used stable Lévy processes of index ν to model the stock dynamics (see Example A.7). Stable distributions model well the behavior of real stocks and indices in the center of distribution of returns. However, the tails of stable distributions are too fat: polynomially decaying, whereas many empirical studies suggest that the tails decay exponentially. Even more importantly, one cannot use the stable Lévy processes in exponential Lévy models because the expectation of the stock price $\mathbb{E}[e^{X_t}] = \infty$, which makes the model unsuitable for consistent pricing.

One can preserve the behavior typical for distributions of stable Lévy processes in the central part, but make tails decay exponentially at infinity. These distributions occupy the middle ground between Gaussian distribution, with super-exponentially decaying tail, and stable distributions, with heavy polynomially decaying tails. The resulting classes of processes are called Lévy processes of exponential type.

Some examples of Lévy processes that are commonly used in empirical studies of financial markets are listed in Chapter 2 Section 2. These examples are given in terms of characteristic exponent. In the thesis, we can use the characteristic exponent, without referring to the initial definition. Below, we give definitions of several class of Lévy processes in terms of the Lévy measure. The flexibility of choice of the Lévy measure allows us to calibrate the model to market prices of options and reproduce statistical features that have motivated their use, for example, implied volatility skews/smiles.

Definition A.8. [**17**, Definition 3.1] A Lévy process X is called a KoBoL process of order $\nu < 2$, if it is a purely discontinuous Lévy process with the Lévy measure of the form

$$F(dx) = c_+ e^{\lambda_- x} x^{-\nu-1} \ {}_{(0,+\infty)}(x)dx + c_- e^{\lambda_+ x}|x|^{-\nu-1} \ {}_{(-\infty,0)}(x)dx,$$

where $c_\pm > 0$, and $\lambda_- < 0 < \lambda_+$. Constants λ_+ and λ_- are called the steepness parameters of the process.

If $\nu < 0$, then density $F(dx) \in L_1$. For example, the following special case $\nu = -1$ is very convenient for computations and simulations.

Definition A.9. A double exponential jump-diffusion model is a Lévy process with the generating triplet (σ, F, b), where $\sigma \neq 0$ and the Lévy measure is given by

$$F(dx) = c_+ e^{\lambda_- x} \ {}_{(0,+\infty)}(x)dx + c_- e^{\lambda_+ x} \ {}_{(-\infty,0)}(x)dx.$$

The double exponential jump-diffusion model was introduced to finance by Kou [**40**], and independently, by Lipton [**47**].

With $\nu = 0$ and $\sigma = 0$, we obtain the Variance Gamma Process (VGP), which was introduced to Finance by Madan and co-authors [**53, 52, 51**].

Definition A.10. A Variance Gamma Process is a pure jump process with the Lévy density of the form

$$F(dx) = c_+ e^{\lambda_- x} x^{-1} \ {}_{(0,+\infty)}(x)dx + c_- e^{\lambda_+ x}|x|^{-1} \ {}_{(-\infty,0)}(x)dx.$$

The standard definition of a VGP is by subordination of a Brownian motion: $X_t = Y_{Z_t}$, where Y_t is a Brownian motion, and Z_t is a stochoastic process with non-decreasing trajectories (subordinator). In Finance, a subordinator is interpreted as business time [**37**].

3. Pricing European options under Lévy processes

We consider a model frictionless market consisting of a riskless bond yielding the riskless rate of return r, and a stock, which is modeled as an exponential Lévy process $S_t = e^{X_t}$, under a chosen equivalent martingale measure (EMM) $\mathbb{Q}$. We assume that an EMM $\mathbb{Q}$ has been fixed once and for all, and all expectation operators appearing in this text will be with respect to this measure. The characteristic exponent ψ of X is also under this $\mathbb{Q}$.

If the stock does not pay dividends, then S_t must be a martingale under $\mathbb{Q}$. In terms of the characteristic exponent of the log-price process X, the EMM-condition can be written as follows:

$$r + \psi(-i) = 0, \tag{A.4}$$

where we implicitly assume that $\psi(\xi)$ admits the analytic continuation into the closed strip $-1 \le \operatorname{Im}\xi \le 0$ (if this is not the case, then $\mathbb{E}[S_t] = \infty$ for all $t > 0$, i.e., the process $\{S_t\}$ cannot be priced; we exclude this situation from our consideration). If the stock pays dividends at constant rate δ, then (A.4) must be replaced with the EMM condition becomes

$$r + \psi(-i) = \delta. \tag{A.5}$$

Let $V(t, x)$ be the price of the European option with maturity T and payoff $G(X_T)$, at time t and $X_t = x$. We assume that, under a risk-neutral measure $\mathbb{Q}$ chosen for pricing of options on the underlying stock or index, X is a Lévy process of exponential type $[\lambda_-, \lambda_+]$, with the characteristic exponent $\psi(\xi)$. Assume that, for some $\omega \in (\lambda_-, \lambda_+)$, function $G_\omega(x) :=$

$e^{\omega x}G(x)$ belongs to $L_1(\mathbb{R})$. Then $V(T,x)$ is finite, and

$$V(t,x) = \mathbb{E}\left[e^{-r(T-t)}G(X_T) \mid X_t = x\right] \tag{A.6}$$

We decompose G into the Fourier integral

$$G(x) = (2\pi)^{-1}\int_{\operatorname{Im}\xi=\omega} e^{ix\xi}\hat{G}(\xi)d\xi, \tag{A.7}$$

substitute (A.7) into the pricing formula (A.6) and change the order of integration. The result is

$$V(t,x) = (2\pi)^{-1}\int_{\operatorname{Im}\xi=\omega} e^{ix\xi-\tau(r+\psi(\xi))}\hat{G}(\xi)d\xi, \tag{A.8}$$

where $\tau = T - t > 0$. Note that the Fubini theorem is applicable for standard payoffs. Indeed, for digital options, the Fourier transform $\hat{G}(\xi)$ of the payoff decays as $|\xi|^{-1}$ as $\xi \to \infty$, and for puts and calls, $\hat{G}(\xi)$ decays as $|\xi|^{-2}$. Furthermore, for a regular Lévy process of exponential type, of order $\nu > 0$ (see Chapter 2 Section 2 for the definition), there exists $c = c(\omega) > 0$ and R such that

$$\operatorname{Re}\psi(\xi) > c|\xi|^{\nu}, \tag{A.9}$$

for ξ s.t. $\operatorname{Im}\xi = \omega, |\xi| > R$. Therefore, the exponential function under the integral sign decays at infinity faster than $|\xi|^{-N}$, for any N. For VG model, this function decays as $|\xi|^{-c\tau}$, where $c, \tau > 0$. In both cases, the integrand decays faster than $|\xi|^{-1-\epsilon}$, for some $\epsilon > 0$.

Examples A.11. a) Consider a European call option with the strike price K and expiry date T. The terminal payoff is

$$G(X(T)) = (e^{X(T)} - K)_+,$$

and the integral in (A.7) is well-defined for ξ in the half-plane $\operatorname{Im}\xi < -1$. Hence, we assume that $\lambda_- < -1$ (one can also treat the case $\lambda_- = -1$ but this will lead to additional unnecessary technical complications), and derive, for any $\omega \in (\lambda_-, -1)$,

$$V_{\text{call}}(t,x) = -\frac{K}{2\pi}\int_{\operatorname{Im}\xi=\omega} \frac{e^{i(x+\mu\tau)\xi-\tau(r+\psi^0(\xi))}}{(\xi+i)\xi}d\xi. \tag{A.10}$$

b) For the put option with the same strike and expiry date, the terminal payoff is

$$G(X(T)) = (K - e^{X(T)})_+,$$

and the integral in (A.7) is well-defined for ξ in the half-plane $\operatorname{Im}\xi > 0$. Hence, we assume that $\lambda_+ > 0$ (one can also treat the case $\lambda_+ = 0$ but this will lead to additional unnecessary technical complications). Take any $\omega' \in (0, \lambda_+)$; then

$$V_{\text{put}}(t,x) = -\frac{K}{2\pi}\int_{\operatorname{Im}\xi=\omega'} \frac{e^{i(x+\mu\tau)\xi-\tau(r+\psi^0(\xi))}}{(\xi+i)\xi}d\xi. \tag{A.11}$$

c) For the digital call option with the strike price K and expiry date T, the terminal payoff is

$$G(X(T)) = {}_{[\ln K,\infty)}(X(T)).$$

The integral in (A.7) is well-defined for ξ in the half-plane $\operatorname{Im}\xi < 0$. Hence, we assume that $\lambda_- < 0$ (one can also treat the case $\lambda_- = 0$ but this will lead to additional unnecessary technical complications). Take any $\omega \in (\lambda_-, 0)$; then

$$V_{\text{d.call}}(t,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\xi=\omega} \frac{e^{i(x+\mu\tau)\xi-\tau(r+\psi^0(\xi))}}{i\xi}d\xi. \tag{A.12}$$

Notice that the pricing formula above is a special case of a general formula (A.8). In the case of digitals, the Fourier transform of the payoff decays slowly, which leads to additional computational difficulties.

The standard approach to the numerical calculation of Fourier transforms is trapezoid rule. The approach was first applied with the FFT to pricing European options by Carr and Madan [**22**], produces prices at many points fairly fast, but may lead to sizable computational errors, and the setup in which one uses these techniques is not flexible enough to allow one to control these errors. This observation was made for the first time in Section 12.3 of [**17**] in the context of pricing of European options, and a new fast accurate method, integration along the cut method (IAC method), for pricing OTM European options was suggested. Later, deficiencies of iFT techniques were analyzed in a number of papers, for example, [**49, 48, 14, 23, 33, 12, 13, 44**], and various improvements of iFT techniques were suggested. However, certain important points are not addressed and sufficiently accurate general recommendations for an approximately optimal choice of parameters of numerical schemes are missing. More recently, Boyarchenko and Levendorskiĭ [**18**] review several variations of iFT in applications to pricing European options, analyze relations among these variations and derive general estimates for the discretization and truncation errors of the trapezoid rule, which can be used to choose an (approximately) optimal mesh and number of terms in the trapezoid rule. More importantly, a simple conformal map is used to deform the contour of integration so that the integrands in (A.10)–(A.12) decays very fast along the contour.

APPENDIX B

Technicalities

1. Proof of Lemma 2.5

We have

$$\begin{aligned} V_1(x) - 1 &= \mathbb{E}^x\left[-e^{X_{\bar\Delta}} + (e^{X_{\bar\Delta}} - 1)_+\right] \\ &= -e^{x-\bar\Delta\psi(-i)} + \mathbb{E}^x\left[e^{-\omega X_{\bar\Delta}} e^{\omega X_{\bar\Delta}}(e^{X_{\bar\Delta}} - 1)_+\right] \\ &\le -e^{x-\bar\Delta\psi(-i)} + \sup_{y\in\mathbb{R}}\{e^{\omega y}(e^y - 1)_+\} \cdot e^{-\omega x - \bar\Delta\psi(i\omega)} \end{aligned}$$

for any $\omega \in (\lambda_-, -1)$, and we obtain

$$\begin{aligned} f_1(x) &= (1-e^x)_+(V_1(x - \ln(1-e^x)_+) - 1) \qquad \text{(B.1)} \\ &\le -e^{x-\bar\Delta\psi(-i)} \;{}_{(-\infty,0)}(x) \\ &\quad + (1-e^x)_+ \sup_{y\in\mathbb{R}}\{e^{\omega y}(e^y - 1)_+\} \cdot e^{-\omega x + \omega\ln(1-e^x)_+ - \bar\Delta\psi(i\omega)} \\ &= -e^{x-\bar\Delta\psi(-i)} \;{}_{(-\infty,0)}(x) \\ &\quad + \sup_{y\in\mathbb{R}}\{e^{\omega y}(e^y - 1)_+\} \cdot e^{-\omega x - \bar\Delta\psi(i\omega)} \cdot (1-e^x)_+^{1+\omega}. \end{aligned}$$

As $x \to -\infty$, $(1-e^x)_+^{1+\omega} \to 1$, therefore,

$$f_1(x) = -e^{x-\bar\Delta\psi(-i)} + O(e^{-\omega x - \bar\Delta\psi(i\omega)}), \quad x \to -\infty. \qquad \text{(B.2)}$$

To derive similar bounds for W_2, first, choose $A > 0$, by applying the definition: $W_2(x) = \mathbb{E}^x\left[f_1(X_{\bar\Delta})\right]$, where f_1 is as in (B.1), we have

$$W_2(x) = \mathbb{E}^x\left[f_1(X_{\bar\Delta}) \;{}_{(-\infty,-A)}(X_{\bar\Delta})\right] + \mathbb{E}^x\left[f_1(X_{\bar\Delta}) \;{}_{[-A,+\infty)}(X_{\bar\Delta})\right].$$

Substitute (B.2) in the first term above, we find

$$\begin{aligned} W_2(x) &= \mathbb{E}^x\left[-e^{X_{\bar\Delta}-\bar\Delta\psi(-i)}\ {}_{(-\infty,-A)}(X_{\bar\Delta})\right] \\ &\quad +O\left(e^{-\omega X_{\bar\Delta}-\bar\Delta\psi(i\omega)}\ {}_{(-\infty,-A)}(X_{\bar\Delta})\right) \\ &\quad +\mathbb{E}^x\left[f_1(X_{\bar\Delta})\ {}_{[-A,+\infty)}(X_{\bar\Delta})\right] \\ &= \mathbb{E}^x\left[-e^{X_{\bar\Delta}-\bar\Delta\psi(-i)}\right]+O\left(\mathbb{E}^x\left[e^{-\omega X_{\bar\Delta}-\bar\Delta\psi(i\omega)}\ {}_{(-\infty,-A)}(X_{\bar\Delta})\right]\right) \\ &\quad +\mathbb{E}^x\left[e^{X_{\bar\Delta}-\bar\Delta\psi(-i)}\ {}_{[-A,+\infty)}+f_1(X_{\bar\Delta})\ {}_{[-A,+\infty)}(X_{\bar\Delta})\right] \\ &= -e^{x-2\bar\Delta\psi(-i)}+O\left(e^{-\omega x-\bar\Delta\psi(i\omega)}\right), \end{aligned} \tag{B.3}$$

where $\omega\in(\lambda_-,-1)$. By the same arguments above as in the derivation of (B.2),

$$\begin{aligned} f_2(x) &= (1-e^x)_+\cdot W_2(x-\ln(1-e^x)_+) \\ &= -e^{x-2\bar\Delta\psi(-i)}\ {}_{(-\infty,0)}(x)+O\left(e^{-\omega x-\bar\Delta\psi(i\omega)}\cdot(1-e^x)_+^{1+\omega}\right). \end{aligned}$$

As $x\to-\infty$, $(1-e^x)_+^{1+\omega}\to 1$, therefore,

$$f_2(x)=-e^{x-\bar\Delta\psi(-i)}+O(e^{-\omega x-\bar\Delta\psi(i\omega)}),\quad x\to-\infty.$$

Therefore, (2.23) holds for $n=2$.

By induction, using the recurrence relation of W_n

$$W_{n+1}(x)=\mathbb{E}^x[(1-e^{X_{\bar\Delta}})_+W_n(X_{\bar\Delta}-\ln(1-e^{X_{\bar\Delta}})_+)]+W_2(x),$$

for $n=3,4,\ldots$, the same argument above shows that

$$W_n(x)=c_ne^x+O(e^{-\omega x-\psi(i\omega)}), \tag{B.4}$$

where $c_1 = 0$ and $c_{n+1} = (c_n - e^{-\bar{\Delta}\psi(-i)})e^{-\bar{\Delta}\psi(-i)}$, $n \geq 2$. Set $a = e^{-\bar{\Delta}\psi(-i)}$. We easily find

$$c_n = -(a^n + \cdots + a^2) = -e^{-2\bar{\Delta}\psi(-i)} \frac{1 - e^{(1-n)\bar{\Delta}\psi(-i)}}{1 - e^{-\bar{\Delta}\psi(-i)}}.$$

Equation

$$f_n(x) = c_n e^x + O(e^{-\omega x - \bar{\Delta}\psi(i\omega)}), \quad x \to -\infty,$$

follows from (B.3) and (B.4).

2. Proof of Lemma 3.3

Since the transition operator is translation-invariant, we may assume that $\mu = 0$ and $\psi = \psi^0$. If X is the process of order $\nu > 0$ or VG and $\bar{\Delta}$ is large enough, we can use Fubini's theorem to justify the following equality

$$p^{(s)}_{\bar{\Delta}}(x) = (2\pi)^{-1} \int_{\operatorname{Im}\xi = \omega} e^{-ix\xi - \bar{\Delta}\psi^0(\xi)} (-i\xi)^s d\xi, \tag{B.5}$$

for any $\omega \in (\lambda_-, \lambda_+)$. We can use different ω depending on the sign of x. If $x < 0$, we use $\omega = \omega_+ \in (0, \lambda_+)$, and if $x \geq 0$, then we take $\omega = \omega_- \in (\lambda_-, 0)$. Next, we take $\phi \in (0, \min\{\pi/2, \pi/(2\nu)\})$, and introduce two contours

$$\mathcal{L}^+_{\omega_+,\phi} = i\omega_+ + (e^{i\phi}\mathbb{R}_+ \cup e^{i(\pi-\phi)}\mathbb{R}_+)$$

and

$$\mathcal{L}^-_{\omega_-,\phi} = i\omega_- + (e^{-i\phi}\mathbb{R}_+ \cup e^{i(-\pi+\phi)}\mathbb{R}_+).$$

For $x < 0$ (resp., $x \geq 0$), we deform the contour of integration in (B.5) into $\mathcal{L}^+_{\omega_+,\phi}$ (resp., $\mathcal{L}^-_{\omega_-,\phi}$). Since

$$\|p^{(s)}\|_{L_1} = \|p^{(s)}\ {}_{(-\infty,0)}\|_{L_1} + \|p^{(s)}\ {}_{(0,+\infty)}\|_{L_1},$$

it suffices to consider each term on the RHS above separately. The estimates being similar, we consider the first term. For $x < 0$, we have

$$p^{(s)}_{\bar{\Delta}}(x) = \frac{1}{\pi} \operatorname{Re} \int_{\mathcal{L}^+_{\omega_+,\phi}} e^{-ix\xi - \bar{\Delta}\psi^0(\xi)} (-i\xi)^s d\xi.$$

Changing the variable $\xi = i\omega_+ + e^{i\phi}\rho, \rho > 0$, we obtain

$$\begin{aligned} |p^{(s)}_{\bar{\Delta}}(x)| &\leq \frac{1}{\pi} \int_0^{+\infty} e^{(\omega_+ + \cos(\phi - \pi/2)\rho)x - \bar{\Delta} \operatorname{Re} \psi^0(i\omega_+ + e^{i\phi}\rho)} |i\omega_+ + e^{i\phi}\rho|^s d\rho \\ &= \frac{1}{\pi} \int_0^{+\infty} e^{(\omega_+ + \cos(\phi - \pi/2)\rho)x - \bar{\Delta} \operatorname{Re} \psi^0(i\omega_+ + e^{i\phi}\rho)} \\ &\qquad \times ((\omega_+ + \cos\phi \cdot \rho)^2 + (\sin\phi \cdot \rho)^2)^{s/2} d\rho. \end{aligned}$$

Since $\omega_+ > 0$ and $\cos(\phi - \pi/2) > 0$, we can integrate over the half-line $x < 0$ and obtain

$$\begin{aligned} &\|p^{(s)}_{\bar{\Delta}} \quad {}_{(-\infty,0)}\|_{L_1} \\ \leq\ & \frac{1}{\pi} \int_0^{+\infty} \frac{e^{-\operatorname{Re} \bar{\Delta}\psi^0(i\omega_+ + e^{i\phi}\rho)}}{\omega_+ + \cos(\phi - \pi/2)\rho} \\ &\qquad \times ((\omega_+ + \cos\phi \cdot \rho)^2) + (\sin\phi \cdot \rho)^2)^{s/2} d\rho. \end{aligned} \tag{B.6}$$

For simplicity, we assume that as $\rho \to +\infty$, for any $\phi \in (-\pi/2, \pi/2)$,

$$\psi^0(i\omega_+ + e^{i\phi}\rho) \sim d^0_+ e^{i\phi\nu} \rho^\nu (1 + O(\rho^{-1})) \tag{B.7}$$

$$\partial_\rho \operatorname{Re} \psi^0(i\omega_+ + e^{i\phi}\rho) \sim \nu d^0_+ \cos(\phi\nu) \rho^{\nu-1} (1 + o(1)) \tag{B.8}$$

where $d^0_+ > 0$; otherwise, in the following considerations, we need to impose an additional condition on ϕ:

$$(\operatorname{Re} d^0_+ e^{i\phi\nu}) > 0,$$

and replace $d^0_+ \cos(\phi\nu)$ with $\operatorname{Re}(d^0_+ e^{i\phi\nu})$. Then, using (B.8), we conclude that there exists $C > 0$ such that $\operatorname{Re} \psi^0(i\omega_+ + e^{i\phi}\rho)$ is monotone on

$(C, +\infty)$. One can easily verify that C is fairly moderate for typical parameters of processes under consideration. Hence, if $s \le 3$, which are of interest to us, and ω_+ is not too small (since we have the freedom of the choice of $\omega_+ \in (0, \lambda_+)$, the essential condition is $\lambda_+ > 0$ is not too small), we can replace the integral in (B.6) with the integral over $(C, +\infty)$. Making the change of variables $y = \bar\Delta \operatorname{Re} \psi^0(i\omega_+ + e^{i\phi}\rho)$, and taking (B.7) and (B.8) into account, we obtain

$$\begin{aligned}\rho &= \left(\frac{y}{\bar\Delta d^0_+ \cos(\phi\nu)}\right)^{1/\nu} (1 + O(y^{-1/\nu}))\\ \frac{d\rho}{dy} &\sim \nu^{-1}(\bar\Delta d^0_+ \cos(\phi\nu)))^{-1/\nu} y^{1/\nu - 1}\end{aligned}$$

and

$$\begin{aligned}&\frac{((\omega_+ + \cos\phi\cdot\rho)^2 + (\sin\phi\cdot\rho)^2)^{s/2}}{\omega_+ + \cos(\phi - \pi/2)\rho}\\ \sim\ &\frac{\rho^{s-1}}{\cos(\phi - \pi/2)}\\ \sim\ &(\cos(\phi - \pi/2))^{-1}(\bar\Delta d^0_+ \cos(\phi\nu)))^{(1-s)/\nu} y^{(s-1)/\nu}.\end{aligned}$$

Substituting into (B.6), we derive an approximate bound

$$\|p^{(s)}_{\bar\Delta}\ {}_{x<0}\|_{L_1} \le \frac{(\bar\Delta d^0_+)^{-s/\nu}}{\pi\nu D(s, \phi)} \int_{C^\nu \bar\Delta d^0_+ \cos(\phi\nu)}^{+\infty} e^{-y} y^{s/\nu - 1} d\rho, \tag{B.9}$$

where

$$D(s, \phi) = (\cos(\phi\nu))^{s/\nu} \cos(\phi - \pi/2).$$

Denote by $D(s)$ the supremum of $D(s, \phi)$ over $\phi \in (0, \min\{\pi/2, \pi/(2\nu)\})$. Then, using (B.9) and the same bound for $\|p^{(s)}_{\bar\Delta}\ {}_{x>0}\|_{L_1}$, we obtain (3.6).

For VG, assuming that $s < 2c\bar\Delta$, where c is the intensity parameter in

$$\psi^0(\xi) = c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)],$$

we deduce from (B.9) an approximate bound:

$$\begin{aligned}
\|p_{\bar{\Delta}}^{(s)}\|_{L_1} &\leq \frac{2}{\pi\cos(\phi-\pi/2)}\int_C^{+\infty}\rho^{-2c\bar{\Delta}+s-1}d\rho \\
&\leq \frac{2}{\pi\cos(\phi-\pi/2)(2c\bar{\Delta}-s)}.
\end{aligned}$$

3. Proof of Lemma 3.4

We first estimate the first derivative of V_n and W_n, $n = 1, 2, \ldots$.

Lemma B.1. *For all x and $n = 1, 2, \ldots$,*

$$|V_n'(x)| \leq n, \tag{B.10}$$

$$|W_{n+1}'(x)| \leq 2n-1. \tag{B.11}$$

Proof. Since $0 \leq (1-e^{x_1}-\cdots-e^{x_n})_+ \leq (1-e^{x_1})+ \leq 1$, we have $0 \leq V_n(x) \leq V_1(x) \leq 1$ for all x. Moreover, $W_n = V_n - V_1$, we have

$$-1 \leq W_n(x) \leq 0 \quad \text{for all} \quad x. \tag{B.12}$$

We consider classes of Lévy processes $X = \{X_t\}_{t\geq 0}$, which satisfy (ACT)-condition, that is, the transition measure P_{X_t} are absolutely continuous for all $t > 0$. This implies, in particular, that if g is continuous and g' is measurable and bounded, then

$$\frac{d}{dx}\mathbb{E}^x[g(X_{\bar{\Delta}})] = \mathbb{E}^x[g'(X_{\bar{\Delta}})]. \tag{B.13}$$

Applying (B.13) with $g(x) = (1-e^x)_+$, we find $-1 \leq V_1'(x) \leq 0$ for all x. For $n = 1, 2, \ldots$, let $G_n(x) = (1-e^x)_+ \cdot V_n(x-\ln(1-e^x)_+)$. We find that

$$G_n'(x) = -e^x \quad_{(-\infty,0)}(x)V_n(x-\ln(1-e^x)_+) + V_n'(x-\ln(1-e^x)_+),$$

therefore, $-2 \le G_1'(x) \le 0$. By definition (2.5):

$$V_n(x) = \mathbb{E}^x \left[(1 - e^{X_{\bar{\Delta}}})_+ V_{n-1} \left(X_{\bar{\Delta}} - \ln(1 - e^{X_{\bar{\Delta}}})_+ \right) \right],$$

we have

$$-2 \le V_2'(x) \le 0.$$

By induction, for $n = 3, 4, \ldots$, we easily find that

$$-n \le G_{n-1}'(x) \le 0 \quad \text{and} \quad -n \le V_n'(x) \le 0, \quad \text{for all } x.$$

Therefore, (B.10) holds.

Next, let f_1 be as in (2.8):

$$f_1(x) = (1 - e^x)_+ \cdot (V_1(x - \ln(1 - e^x)_+) - 1),$$

then

$$f_1'(x) = e^x \quad {}_{(-\infty,0)}(x)(V_1(x - \ln(1 - e^x)_+) - 1) + V_1'(x - \ln(1 - e^x)_+).$$

From (B.12) and (B.10), we find

$$-1 \le f_1'(x) \le 1.$$

Applying (B.13) with $g(x) = f_1(x)$ and using the definition of $W_2 = \mathbb{E}^x[f_1(X_{\bar{\Delta}})]$, we prove (B.11) for $|W_2'(x)| \le 1$. For $n = 2, 3, \ldots$, let

$$f_n(x) = (1 - e^x)_+ \cdot W_n(x - \ln(1 - e^x)_+).$$

then

$$f_n'(x) = -e^x \quad {}_{(-\infty,0)}(x) W_n(x - \ln(1 - e^x)_+) + W_n'(x - \ln(1 - e^x)_+).$$

Using $|W_2'(x)| \leq 1$ and (B.12), we find

$$-1 \leq f_2'(x) \leq 2.$$

Applying (B.13) with $g(x) = f_2(x)$ to

$$W_{n+1}'(x) = \frac{d}{dx}\mathbb{E}^x\left[f_n(X_{\bar{\Delta}})\right] + W_2'(x),$$

we obtain (B.11) with $n = 2$. By induction, for $n = 3, 4, \ldots$,

$$-n + 1 \leq f_n' \leq 2n - 2, \quad \text{and} \quad -n \leq W_{n+1}' \leq 2n - 1.$$

□

The bounds for the derivatives in Lemma B.1 are simple, convenient, and, if the number of the sampling dates is not too large, the constants in the bounds are moderate. However, even if we use the piece-wise linear interpolation, we need estimates for the second derivatives of V_1 and W_n. If we use the piece-wise cubic interpolation or cubic splines, we need bounds for the derivatives of order 4. We can derive bounds for the derivatives of V_1 and W_n of order $s+1 > 1$ in terms of the L_1-norm of the derivatives $p_{\bar{\Delta}}^{(s)}$ of the transition kernel.

Lemma B.2. *Let g be continuous with a measurable bounded derivative. Then*

$$\left|\frac{d^{s+1}}{dx^{s+1}}\mathbb{E}^x[g(X_{\bar{\Delta}})]\right| \leq \|p_{\bar{\Delta}}^{(s)}\|_{L_1} \cdot \|g'\|_{L_\infty}. \tag{B.14}$$

PROOF. For processes under consideration, the transition kernel is infinitely differentiable on $\mathbb{R}$, with the exception of 0 for VG, therefore,

$$\begin{aligned}\left|\frac{d^{s+1}}{dx^{s+1}}\mathbb{E}^x[g(X_{\bar\Delta})]\right| &= \left|\frac{d^s}{dx^s}\mathbb{E}^x[g'(X_{\bar\Delta})]\right| \\ &= \left|\int_{\mathbb{R}} p_{\bar\Delta}^{(s)}(y-x)(-1)^s g'(y)dy\right| \\ &\le \int_{\mathbb{R}} |p_{\bar\Delta}^{(s)}(y-x)|\cdot|g'(y)|dy \\ &\le \|g'\|_{L_\infty}\int_{\mathbb{R}}|p_{\bar\Delta}^{(s)}(y-x)|dy = \|p_{\bar\Delta}^{(s)}\|_{L_1}\cdot\|g'\|_{L_\infty}.\end{aligned}$$

□

LEMMA B.3. *For all s such that $\|p_{\bar\Delta}^{(s)}\|_{L_1}<\infty$, and all $n\ge 1$,*

$$\|V_n^{(s+1)}(x)\|_{L_\infty} \le n\cdot\|p_{\bar\Delta}^{(s)}\|_{L_1} \tag{B.15}$$

$$\|W_{n+1}^{(s+1)}(x)\|_{L_\infty} \le (2n-1)\cdot\|p_{\bar\Delta}^{(s)}\|_{L_1}. \tag{B.16}$$

PROOF. By applying Lemma B.2, and (B.10) in Lemma B.1, we obtain (B.15). Similarly, (B.16) follows from (B.11). □

LEMMA B.4. *Let $s\ge 1$ be an integer and g be $s-1$ times continuously differentiable on $(-\infty,0)$, with the derivative of order s measurable and bounded. Then, for $s=1,2,3,4$,*

$$\left|\frac{d^s}{dx^s}(1-e^x)_+ g(x-\ln(1-e^x)_+)\right| \le \sum_{j=0}^{s}\binom{s}{j}\|g^{(j)}\|_{L_\infty}. \tag{B.17}$$

PROOF. Let $x<0$. Change the variable $y=x-\ln(1-e^x)$. Then we have the following sequence of equivalent equalities

$$1-e^x=(1+e^y)^{-1},\quad \frac{dy}{dx}=(1-e^x)^{-1}=1+e^y,$$

and therefore,

$$\begin{aligned}\frac{d}{dx}(1-e^x)\cdot g(x-\ln(1-e^x)) &= \frac{dy}{dx}\frac{d}{dy}(1+e^y)^{-1}\cdot g(y)\\ &= u(y)\cdot g(y)+g'(y),\end{aligned}$$

where $u(y)=-e^y/(1+e^y)$. Direct calculations show that

$$\frac{d^s}{dx^s}(1-e^x)g(x-\ln(1-e^x))=\sum_{j=0}^{s} f_{js}(y)g^{(j)}(y),$$

where f_{js} are rational functions of e^y, and satisfies $f_{js}(y)\to 0$ as $y\to-\infty$, $|f_{js}(y)|\to\binom{s}{j}$ as $y\to+\infty$, and, at points of local minima of $|f_{js}|$ (if such points exist), $|f_{js}(y)|<\binom{s}{j}$ (since we consider $s\leq 4$, the verification can be easily done on the case-by-case basis). Hence, (B.17) holds. $\square$

By applying Lemma B.4 to $g=V_1$ and $g=W_n$ and taking into account Lemma B.3, we obtain Lemma 3.4.

APPENDIX C

Interpolation of higher order: cubic spline

We want to evaluate the action of Fourier transform to a measurable function f on $\mathbb{R}$:

$$(\mathcal{F}_{x\to\xi} f)(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx. \tag{C.1}$$

Given the function $e^{\operatorname{Im}\xi\cdot x} f(x) \in L_1(\mathbb{R})$, the right hand side of (C.1) converge absolutely. Typically, the function f has a kink or point of discontinuity, in such cases, it is advantageous to take into account this kink, and consider two sets of piecewise smooth polynomial approximation.

In the evaluation of (C.1), both for a smoother approximation and for a more efficient approximation, one has to go to piecewise polynomial approximations with higher order pieces. In this section, we describe the scheme for the piecewise cubic spline interpolation.

We assume that we are given the values of f on a uniformly spaced grid $\vec{x} = (x_k)_{k=1}^M$, where $x_k = x_1 + (k-1)\Delta$ and $\Delta > 0$ is fixed. Write $u_k = f(x_k)$ for all k. If $f(x)$ has a kink $x = h$, we choose a grid such that $x_h = h$; then we interpolate f on $[x_1, x_h]$ and $[x_h, x_M]$. For simplicity, below, we assume that f is smooth on $[x_1, x_M]$

First, we choose the interpolant u to f to consist of cubic pieces:

$$u(x) = S_k(x) = a_k + b_k x + c_k x^2 + d_k x^3, \quad x \in [x_k, x_{k+1}],\ k \geq 0. \tag{C.2}$$

The coefficients a_k, b_k, c_k and d_k are determined from the following continuity conditions:

$$S_k(x_k) = u_k \qquad S_k(x_{k+1}) = u_{k+1}$$

$$S_k'(x_{k+1}) = S_{k+1}'(x_{k+1}) \qquad S_k''(x_{k+1}) = S_{k+1}''(x_{k+1})$$

and two initial conditions:

$$S_1'(x_1) = s_1', \quad S_M'(x_M) = s_M'.$$

Let us write

$$S_k'(x_k) = s_k', \quad k = 1, \ldots, M.$$

Next, we replace the function $f(x)$ that appears in the integrand in (C.1) with the cubic spline functions we just described. This replaces $(\mathcal{F}_{x\to\xi} f)(\xi)$ with a sum of integrals of the form

$$\int_{x_k}^{x_{k+1}} e^{-i\xi x}(a_k + b_k x + c_k x^2 + d_k x^3)dx, \quad 1 \le k \le M.$$

A direct calculation ultimately leads to the following approximation:

$$\begin{aligned}(\mathcal{F}_{x\to\xi} f)(\xi) \approx e^{-i\xi x_1} &\left(\Delta u_1 \widehat{\mathcal{U}}_1(\xi) + \Delta^2 s_1' \widehat{\mathcal{U}}_1^d(\xi)\right) \\ &+ e^{-i\xi x_M}\left(\Delta u_M \widehat{\mathcal{U}}_M(\xi) + \Delta^2 s_M' \widehat{\mathcal{U}}_M^d\right), \\ &+ \sum_{k=2}^{M-1} e^{-i\xi x_k}\left(\Delta u_k \widehat{\mathcal{U}}(\xi) + \Delta^2 s_k' \widehat{\mathcal{U}}^d(\xi)\right) \qquad \text{(C.3)}\end{aligned}$$

where

$$\widehat{\mathcal{U}}_1(\xi) = (i\xi\Delta)^{-4}\left(12 - 6(i\xi\Delta) + (i\xi\Delta)^3 - 12e^{-i\xi\Delta} - 6(i\xi\Delta)e^{-i\xi\Delta}\right)$$

$$\widehat{\mathcal{U}}_M(\xi) = (i\xi\Delta)^{-4}\left(12 + 6(i\xi\Delta) - (i\xi\Delta)^3 - 12e^{i\xi\Delta} + 6(i\xi\Delta)e^{i\xi\Delta}\right)$$

$$\begin{aligned}
\widehat{\mathcal{U}}_1^d(\xi) &= (i\xi\Delta)^{-4}\left(6 - 4(i\xi\Delta) + (i\xi\Delta)^2 - 6e^{-i\xi\Delta} - 2(i\xi\Delta)e^{-i\xi\Delta}\right)\\
\widehat{\mathcal{U}}_M^d(\xi) &= (i\xi\Delta)^{-4}\left(-6 - 4(i\xi\Delta) - (i\xi\Delta)^2 + 6e^{i\xi\Delta} - 2(i\xi\Delta)e^{i\xi\Delta}\right)\\
\widehat{\mathcal{U}}(\xi) &= \widehat{\mathcal{U}}_1(\xi) + \widehat{\mathcal{U}}_M(\xi)\\
\widehat{\mathcal{U}}^d(\xi) &= \widehat{\mathcal{U}}_1^d(\xi) + \widehat{\mathcal{U}}_M^d(\xi).
\end{aligned}$$

By providing the slops at the end points, the slopes s'_k, $k = 2, \dots, M-1$ can be obtained by solving a tridiagonal linear system of equations:

$$s'_{k-1} + 4s'_k + s'_{k+1} = 3(-u_{k-1} + u_{k+1})/\Delta, \qquad 2 \le k \le M-1.$$

Following the setup in Chapter 2 Section 4.2, we set $u_M = 0$, $u_1 = c_n e^{x_1}$, where $c_1 = -e^{-\bar{\Delta}\psi(-i)}$, and c_n is given by (2.24) if $n \ge 2$, $u_j = f_n(x_j)$, $j = 2, 3, \dots, M-1$, and approximate f_n by the function u defined by:

(1) $u(x) = c_n e^x$, $x \le x_1$;
(2) $u(x) = 0$, $x_M \le x < \infty$; and
(3) use cubic spline (C.2) with $s'_M = 0$, $x_j \le x \le x_{j+1}$, $j = 2, 3, \dots, M-1$.

Using (C.3), we obtain an analogue of (2.28):

$$\mathbb{E}^x\left[u(X_{\bar{\Delta}})\right] = \mathcal{U}_1(x) + \mathcal{U}(x) + \mathcal{U}_1^d(x) + \mathcal{U}^d(x),$$

where

$$\begin{aligned}
\mathcal{U}_1(x) = -u_1\Bigg(V_1(x-x_1) &+ 12\cdot\frac{\mathcal{V}_4(x-x_1) - \mathcal{V}_4(x-x_0)}{\Delta^3}\\
&+6\cdot\frac{\mathcal{V}_3(x-x_1) - \mathcal{V}_3(x-x_0)}{\Delta^2}\Bigg)
\end{aligned}$$

$$\mathcal{U}(x) = \sum_{j=1}^{M} u_j \left(-12 \cdot \frac{\mathcal{V}_4(x - x_{j-1}) - 2\mathcal{V}_4(x - x_j) + \mathcal{V}_4(x - x_{j+1})}{\Delta^3} \right.$$
$$\left. +6 \cdot \frac{\mathcal{V}_3(x - x_{j-1}) - \mathcal{V}_3(x - x_{j+1})}{\Delta^2} \right)$$

$$\mathcal{U}_1^d(x) = -s_1' \left(6 \cdot \frac{-\mathcal{V}_4(x - x_1) + \mathcal{V}_4(x - x_0)}{\Delta^2} \right.$$
$$\left. -2 \cdot \frac{2\mathcal{V}_3(x - x_1) + \mathcal{V}_3(x - x_0)}{\Delta} - \mathcal{V}_2(x - x_1) \right)$$

$$\mathcal{U}^d(x) = \sum_{j=1}^{M} s_j' \left(6 \cdot \frac{\mathcal{V}_4(x - x_{j-1}) - \mathcal{V}_4(x - x_{j+1})}{\Delta^2} \right.$$
$$\left. -2 \cdot \frac{\mathcal{V}_3(x - x_{j-1}) + 4\mathcal{V}_3(x - x_j) + \mathcal{V}_3(x - x_{j+1})}{\Delta} \right),$$

$x_0 = x_1 - \Delta$ and $x_{M+1} = x_1 + \Delta$, V_1 is as in (2.16), and $\mathcal{V}_s$ is as in (2.27).

APPENDIX D

Alternative calculations

We outline several possible directions in which the method of Chapter 2 can be developed further. In the following setup, since (i)FFT and fast convolution algorithm are not applicable, and parabolic iFT is more efficient and accurate than flat iFT for point-wise calculation, we would recommend to use parabolic iFT when the calculations in the dual space are needed. For the choice of parameters of parabolic iFT, see Chapter 4 Section 3.4.

1. Approach 1

In (2.6)–(2.7), we have to evaluate V_1 in (2.16) and W_n in (2.28) at points of the form $y_k := x_k - \ln(1 - e^{x_k})_+$, where x_k are points of an equally spaced grid. By taking into account this feature, we re-arrange the algorithm in Chapter 2 Section 4.3 as follows:

1. calculate $V_1\left(\ell\Delta - \ln(1 - e^{x_{\ell+1}})\right)$ for $\ell = 0, 1, \ldots, M-1$, and $\mathcal{V}_2$ on a $M \times M$ matrix, with the entries $(i, j) = (1-i)\Delta - \ln(1 - e^{x_j})$, using parabolic iFT;
2. calculate $\mathbb{E}^{y_k}[u(X_{\bar{\Delta}})]$, $k = 1, 2, \ldots, M_1$, using (2.28) and matrix multiplication.

Moreover, one can work with non-uniformly spaced grid of $x_k = x_1 + \sum_{j=1}^{k-1} \Delta_j$. Then, (2.28) becomes

$$\begin{aligned}\mathbb{E}^x\left[u(X_{\bar{\Delta}})\right] = -u_1 \cdot \left(V_1(x - x_1) + \frac{\mathcal{V}_2(x - x_1) - \mathcal{V}_2(x - x_2)}{\Delta_1}\right) \\ + \sum_{j=2}^{M-1} u_j \cdot \left(\frac{\mathcal{V}_2(x - x_{j-1}) - \mathcal{V}_2(x - x_j)}{\Delta_{j-1}}\right. \\ \left. - \frac{\mathcal{V}_2(x - x_j) - \mathcal{V}_2(x - x_{j+1})}{\Delta_j}\right),\end{aligned} \tag{D.1}$$

where $\mathcal{V}_2$ is as in (2.27).

2. Approach 2

First, write the expectations (2.6) and (2.7) as the integrations on the real line:

$$W_2(y) = \int_{-\infty}^{0} f_1(z)p(z - y)dz, \tag{D.2}$$

and, for $n \geq 2$,

$$W_{n+1}(y) = \int_{-\infty}^{0} f_n(z)p(z - y)dz + W_2(y), \tag{D.3}$$

where f_1 and f_n are as in (2.8) and (2.9), respectively, and p is the the probability density of the increments $X_{\bar{\Delta}}$. Then, any standard quadrature can be used to evaluate the truncated integral at $y = z - \ln(1 - e^z)_+$, provided one can calculate f_n and p on chosen grids.

Explicitly, the integrals are calculated as follows. First, as in Chapter 2, we truncate the intervals of integrations in (D.2) and (D.3) from above at a point $z_M < 0$, and use a partial truncation from below, that is, on $(-\infty, z_1]$, we replace $f_n(z)$ with $c_n e^z$, where $c_1 = -e^{-\bar{\Delta}\psi(-i)}$, and c_n is given by (2.24) if $n \geq 2$. Next, we choose a quadrature, construct a grid of points $\vec{z}$ in $[z_1, z_M]$, and evaluate f_n on $\vec{z}$. Set $\vec{y} = \vec{z} - \ln(1 - e^{\vec{z}})$, and use parabolic iFT to accurately evaluate $p(z_i - y_j)$. Finally, apply the quadrature to approximate the integral:

$$W_{n+1}(y_j) \approx c_n e^{z_1} V(y_j - z_1) + \sum_{\ell=1}^{M} f_n(z_\ell) \cdot p(z_\ell - y_j) \cdot w_\ell + W_2(y_j) \cdot (1 - \delta_{n1}),$$

where w_ℓ are the weight of the chosen quadrature, and

$$V(x) = (2\pi)^{-1} \int_{\operatorname{Im}\xi=\omega} \frac{e^{ix\xi - \bar{\Delta}\psi(\xi)}}{(1 - i\xi)} d\xi, \tag{D.4}$$

$\omega \in (0, \lambda_+)$ if $x' = x + \mu\bar{\Delta} \geq 0$, and $\omega \in (\lambda_-, -1)$ otherwise.

This scheme is especially useful in cases when the probability density function of the increment behave relatively regular and its tail is relatively thin (for numerical examples, see Chapter 5 Section 2.4).

2.1. Algorithm. The following algorithm calculates $V_N(\gamma)$.

1. Choose truncation parameters z_1 and z_M.
2. Choose a quadrature method, and construct grids: $\vec{z} = (z_j)_{j=1}^{M}$, and set $\vec{y} = \vec{z} - \ln(1 - \exp(\vec{z}))$.
3. Calculate $\vec{V}_1 \approx V_1(\vec{y})$; and set $\vec{W}_1 = \vec{V}_1 - 1$.
4. Calculate $\vec{V} \approx V(\vec{y} - z_1)$, where V is as in (D.4); and for $j = 1, \dots, M$, calculate $p(\vec{z} - y_j)$.
5. Calculate $V_{1,\gamma} \approx V_1(\gamma)$, $V_\gamma \approx V(\gamma - z_1)$ and $p(\vec{z} - \gamma)$.
6. In the cycle w.r.t $k = 1, 2, \dots, N-2$,
   - If $k = 1$, set $c_k = -e^{-\bar{\Delta}\psi(-i)}$; otherwise, c_k is as in (2.24).
   - calculate

     $$\vec{W}_{k+1} = c_n e^{z_1} \vec{V} + \sum_{\ell=1}^{M} (1 - e^{z_\ell}) \cdot W_k(y_\ell) \cdot p(z_\ell - \vec{y}) \cdot w_\ell,$$

     where w_ℓ are the weight of the chosen quadrature;

- for $k = 1$, store $\vec{U} = \vec{W}_2$ and

$$W_\gamma = \sum_{\ell=1}^{M} (1 - e^{z_\ell}) \cdot W_k(y_\ell) \cdot p(z_\ell - \gamma) \cdot w_\ell;$$

- for $k = 2, 3, \ldots, N-2$, set $\vec{W}_{k+1} = \vec{W}_{k+1} + \vec{U}$.

7. Let c_{N-1} be as in (2.24), and calculate

$$V_N(\gamma) \approx c_{N-1} e^{z_1} V_\gamma + \sum_{\ell=1}^{M} (1 - e^{z_\ell}) \cdot W_{N-1}(y_\ell) \cdot p(z_\ell - \gamma) w_\ell + \vec{W}_\gamma + V_{1,\gamma}.$$

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