BRAUER CONFIGURATION ALGEBRAS: A GENERALIZATION OF BRAUER GRAPH ALGEBRAS

EDWARD L. GREEN AND SIBYLLE SCHROLL

Abstract. In this paper we introduce a generalization of a Brauer graph algebra which we call a Brauer configuration algebra. As with Brauer graphs and Brauer graph algebras, to each Brauer configuration, there is an associated Brauer configuration algebra. We show that Brauer configuration algebras are finite dimensional symmetric algebras. After studying and analysing structural properties of Brauer configurations and Brauer configuration algebras, we show that a Brauer configuration algebra is multiserial; that is, its Jacobson radical is a sum of uniserial modules whose pairwise intersection is either zero or a simple module. The paper ends with a detailed study of the relationship between radical cubed zero Brauer configuration algebras, symmetric matrices with non-negative integer entries, finite graphs and associated symmetric radical cubed zero algebras.

Introduction

The classification of algebras into finite, tame and wild representation type has led to many structural insights in the representation theory of finite dimensional algebras. Algebras have finite representation type if there are only finitely many isomorphism classes of indecomposable modules. In this case, the representation theory is usually very well understood and these algebras often serve as first examples or test cases for new ideas and conjectures. To cite but a few examples in this direction see for example [D1, D2, Ma] or [LY, Ri].

All other algebras have infinite representation type, and by [Dr] they are either tame or wild. If an algebra has tame representation type its representation theory usually still exhibits a certain regularity in its structure making calculations and establishing and proving conjectures often possible. One example of this is the class of Brauer graph algebras, which are, depending on their presentation, also known as symmetric special biserial algebras [R, S]. Brauer graph algebras are tame algebras and much of their representation theory is well-understood, see for example [A, GR, Ri, K] for a classification of derived equivalence classes, [ESk1] for their Auslander-Reiten quiver, [GSS] for their covering theory or [AG, GSST] for results on their Ext algebra. Recently there has been renewed interest in Brauer graph algebras, stemming from their connection with cluster theory on the one side [L, MS, S] and with mutation and derived equivalences on the other side [A, AAC, Z1, Z2]. Furthermore, Brauer graph algebras naturally appear in the derived equivalence classification of self-injective algebras of finite and tame representation type, see [Sk] and the references within. One reason that Brauer graph algebras are so well-studied and understood is that the combinatorial data of the underlying Brauer graph encodes much of the

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representation theory of a Brauer graph algebra such as for example, projective resolutions [G, R] or information on the structure of the Auslander-Reiten quiver [BS, Du].

In the general setting of wild algebras, however, the picture is very different and much less is known or understood. In the cases where we do have some understanding, the wild algebras under consideration are often endowed with some additional structure such as for example in [E] where wild blocks of group algebras are considered, or in [EKS], or in [KSY] where the algebras have other additional properties.

In this paper we introduce a new class of mostly wild algebras, called *Brauer configuration algebras*. These algebras have additional structure arising from combinatorial data, called a *Brauer configuration*. Brauer configuration algebras are a generalization of Brauer graph algebras, in the sense that every Brauer graph is a Brauer configuration and every Brauer graph algebra is a Brauer configuration algebra. However, unlike Brauer graph algebras, Brauer configuration algebras are in general of wild representation type. But just as the Brauer graph encodes the representation theory of Brauer graph algebras, the expectation is that the Brauer configuration will encode the representation theory of Brauer configuration algebras. As a first step in this direction, we show that the Brauer configuration yields the Loewy structure of the indecomposable projective modules of a Brauer configuration algebra and that the dimension of the algebra can be directly read off the Brauer configuration (Proposition 3.13). In fact, surprisingly, one can show that the radical of every finitely generated module over a Brauer configuration algebra, more generally, a special multiserial algebra, is the sum of uniserial modules, the intersection of any two being (0) or a simple module [GS]. We note that multiserial and special multiserial algebras were first introduced in [VHW]. In [KY] multiserial algebras have been studied with a focus on hereditary multiserial rings, and with a slightly differing definition multiserial algebras have been studied in [BM, J, M]. Furthermore, there are a number of other results valid for Brauer graph algebras that generalize to Brauer configuration algebras, see [GS], suggesting that the study of these rings and their representations could lead to interesting results.

Brauer configuration algebras contain another class of well-studied algebras that also have a combinatorial presentation in the form of a finite graph, namely that of symmetric algebras with radical cube zero [GS]. Symmetric algebras with radical cube zero have been well-studied, see for example, [B, ES1]. The representation type of symmetric algebras with radical cube zero has been classified in terms of the underlying graph in [B] and for all but a finite number of families they are of wild representation type. In the case of infinite representation type, these algebras are Koszul and thus we have some understanding of their representation theory and their cohomology in particular [ES1]. In this paper we classify the Brauer configurations such that the associated Brauer configuration algebras are *canonical* symmetric algebras with radical cube zero.

Brauer configuration algebras arise naturally in yet another context (in a different presentation). Namely, almost all representatives of derived equivalence classes of symmetric algebras of finite representation type in the classification by Skowronski et al., see [Sk] and the references within, are in fact Brauer configuration algebras. Those that are not Brauer configuration algebras are in fact deformations of Brauer graph algebras.

In Brauer configuraton algebras, similarly to the symmetric special biserial algebras, a path (of length at least 2) is non-zero if and only if it lies in a *special cycle*. It is a consequence of the existence of the special cycles, that the projective indecomposable modules are such that their heart, given by the quotient of the radical modulo the socle, is a direct sum of uniserial modules (Theorem 3.10).

We will now sumarize the most important results in this paper.

A *Brauer configuration* is a combinatorial data $\Gamma = \{\Gamma_0, \Gamma_1, \mu, \mathfrak{o}\}$, where Γ_0 is the set of vertices of Γ, Γ_1 is a set of multisets of elements of Γ_0, μ is a function called the *multiplicity* and $\mathfrak{o}$ is called the *orientation* of Γ (see section 1 for the precise definition of a Brauer configuration). Note that by a slight abuse of notation we call the elements in Γ_1 *polygons*. We denote by Λ_Γ the Brauer configuration algebra associated to Γ.

First we have some structural results about Brauer configuration algebras.

Theorem A.

(1) *A Brauer configuration algebra is a finite dimensional symmetric algebra.*

(2) *Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$ is a decomposition of Γ into two disconnected Brauer configurations Γ_1 and Γ_2. Then there is an algebra isomorphism $\Lambda_\Gamma \simeq \Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2}$ between the associated Brauer configuration algebras.*

(3) *The Brauer configuration algebra associated to a connected Brauer configuration is an indecomposable algebra.*

(4) *A Brauer graph algebra is a Brauer configuration algebra.*

In [VHW] multiserial algebras have been defined. They are a generalisation of biserial algebras as defined in [T, F]. A *multiserial algebra* is defined to be a finite dimensional algebra A such that, for every indecomposable projective left and right A-module P, $\mathrm{rad}(P)$ is a sum of uniserial submodules, say $\mathrm{rad}(P) = \sum_{i=1}^{n} U_i$, for some n and uniserial submodules U_i of $\mathrm{rad}(P)$ with the property that, if $i \neq j$, $U_i \cap U_j$ is either 0 or a simple A-module. The following result explores the structure of the indecomposable projective modules of a Brauer configuration algebra.

Theorem B. *Let Λ be a Brauer configuration algebra with Brauer configuration Γ.*

(1) There is a bijective correspondence between the set of projective indecomposable Λ-modules and the polygons in Γ.

(2) If P is a projective indecomposable Λ-module corresponding to a polygon V in Γ. Then rad(P) is a sum of r indecomposable uniserial modules, where r is the number of (nontruncated) vertices of V and where the intersection of any two of the uniserial modules is a simple Λ-module.

Since Brauer configuration algebras are symmetric, it then directly follows from the definition of a multiserial algebra that Brauer configuration algebras are multiserial.

Corollary C. *A Brauer configuration algebra is a multiserial algebra.*

Finally we study a bijective correspondence and its consequences between finite graphs, symmetric matrices and (ordered) Brauer configurations. For this we will restrict ourselves to the case where the Brauer configuration has no self-folded polygons, or equivalently to the case where there is no repetition of vertices in the polygons. We classify the Brauer configuration algebras with Jacobson radical cubed zero in this case: they correspond exactly to the *canonical* symmetric algebras with Jacobson radical cubed zero.

The paper is outlined as follows. In section 1 we define Brauer configurations and in section 2 Brauer configuration algebras. Both sections contain examples to illustrate the newly defined concepts. Section 3 starts out with the basic properties of Brauer configuration algebras, we then define special cycles and use these to show that the projective-injective modules of a Brauer configuration algebra are multiserial. In section 4 we define canonical symmetric algebras with radical cube zero and relate them to Brauer configuration algebras with radical cube zero and show that they correspond to exactly those Brauer configuration algebras whose Brauer configurations consist of polygons that have no self-foldings.

1. Brauer Configurations

In this section we define Brauer configurations which are generalizations of Brauer graphs.

For the readers benefit we briefly provide a definition of Brauer graph algebras. Let K be a field. A Brauer graph algebra is constructed from the combinatorial information contained in a *Brauer graph*, which is a 4-tuple $(\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ where (Γ_0, Γ_1) is a finite (undirected) graph with vertex set Γ_0 and edge set Γ_1. The graph may contain loops and multiple edges. Next, μ is a set map from $\Gamma_0 \to \mathbb{N}$ where $\mathbb{N}$ denotes the positive integers. A vertex $\alpha \in \Gamma_0$ is called *truncated* if $\mu(\alpha) = 1$ and α is the endpoint of one and only one edge. Last, $\mathfrak{o}$ is an orientation of (Γ_0, Γ_1); that is, at each vertex α in Γ_0, $\mathfrak{o}$ a cyclic ordering of the edges having v as one of its endpoints. The formal definition of the orientation and the construction a Brauer graph algebra from a Brauer graph is a special case of the construction of a Brauer configuration algebra from a Brauer configuration, which we give in detail below.

1.1. Definition of Brauer configurations.

We begin with a tuple $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 is a (finite) set of *vertices* of Γ and Γ_1 is a finite collection of labeled finite sets of vertices where repetitions are allowed. That is, Γ_1 is a finite collection of finite labeled multisets whose elements are in Γ_0.

Example 1.1. We provide two examples which we will continue to use throughout the paper to help clarify both definitions and concepts.

Example 1: $\Gamma = (\Gamma_0, \Gamma_1)$ with $\Gamma_0 = \{1,2,3,4,5,6,7,8\}$ and $\Gamma_1 = \{V_1 = \{1,2,3,4,7\}, V_2 = \{1,2,3,8\}, V_3 = \{4,5\}, V_4 = \{4,6\}, V_5 = \{1,4\}\}$.

Example 2: $\Delta = \{\Delta_0, \Delta_1\}$, with $\Delta_0 = \{1,2,3,4\}$ and $\Delta_1 = \{V_1 = \{1,1,1,2\}, V_2 = \{1,1,3\}, V_3 = \{1,2,3,4\}\}$.

We abuse notation and call the elements of Γ_1 *polygons*. The use of the term 'polygon' will become clear when we discuss realizations of configurations in section 1.2. We call the elements of a polygon V the *vertices of* V. If V is a polygon in Γ_1 and α is a vertex in Γ_0, define $\mathrm{occ}(\alpha, V)$ to be the number of times α occurs as a vertex in V and define the *valence* of α, $\mathrm{val}(\alpha)$, to be $\sum_{V \in \Gamma_1} \mathrm{occ}(\alpha, V)$.

Example 1.2. In the first example above, we have $\mathrm{occ}(1, V_1) = 1, \mathrm{occ}(1, V_2) = 1$, $\mathrm{occ}(1, V_3) = 0$, etc and $\mathrm{val}(1) = 3$ and $\mathrm{val}(2) = 2$, etc.
In the second example, we have $\mathrm{occ}(1, V_1) = 3, \mathrm{occ}(1, V_2) = 2, \mathrm{occ}(1, V_3) = 1$, etc. and $\mathrm{val}(1) = 6$ and $\mathrm{val}(2) = 2$, etc.

For $\Gamma = (\Gamma_0, \Gamma_1)$ to be a Brauer configuration, we need two more pieces of information. One is a *multiplicity function* $\mu \colon \Gamma_0 \to \mathbb{N}$, where $\mathbb{N}$ denotes the strictly positive integers. The other is an *orientation* $\mathfrak{o}$ for Γ.

Example 1.3. In the first example, we choose $\mu(3) = 3$, $\mu(5) = 2$, and $\mu(i) = 1$ for all other vertices. In the second example, we choose $\mu(i) = 1$ for all vertices i. Of course, there are many other choices for the multiplicity function than the arbitrary ones given here.

An *orientation for* Γ is a choice, for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which α occurs as a vertex, including repetitions. More precisely, for each vertex $\alpha \in \Gamma_0$, let $V_1, \dots, V_t$ be the list of polygons in which α occurs as a vertex, with a polygon V occuring $\mathrm{occ}(\alpha, V)$ times in the list, that is V occurs the number of times α occurs as a vertex in V. The cyclic order at vertex α is obtained by linearly ordering the list, say $V_{i_1} < \cdots < V_{i_t}$ and by adding $V_{i_t} < V_{i_1}$. Finally, note that if $V_1 < \cdots < V_t$ is the chosen cyclic ordering at vertex α, then the same ordering can be represented by any cyclic permutation such as $V_2 < V_3 < \cdots < V_t < V_1$ or $V_3 < V_4 < \cdots < V_t < V_1 < V_2$, etc.

Example 1.4. In the first example, the list of polygons ocuring at vertex 1 is V_1, V_5, V_2, and V_1, V_2 at vertex 2, V_1, V_2 at vertex 3, V_1, V_3, V_4, V_5 at vertex 4, V_3 at vertex 5, V_4 at vertex 6, V_1 at vertex 7, and V_2 at vertex 8.
In the second example, the list of polygons ocuring at vertex 1 is $V_1^{(1)}, V_1^{(2)}, V_1^{(3)}, V_2^{(1)}$, $V_2^{(2)}, V_3$, where $V_1^{(1)}, V_1^{(2)}, V_1^{(3)}$ are the three ocurrences of vertex 1 in V_1, etc.

An orientation is then given by cyclically ordering the lists of polygons at each vertex. Thus, for the first example, one orientation would be: $V_1 < V_5 < V_2$ at vertex 1, $V_1 < V_2$ at vertex 2, $V_1 < V_2$ at vertex 3, $V_1 < V_4 < V_3 < V_5$ at vertex 4, V_3 at vertex 5, V_4 at vertex 6, V_1 at vertex 7, and V_2 at vertex 8. Note that for vertices 2,4,5,6,7,8 there is only one choice for cyclic ordering. On the other hand, there are 2 choices for the cyclic ordering at vertex 1 and 3! choices at vertex 4. For later use, we call the orientation given above $\mathfrak{o}_1(\Gamma)$ and let $\mathfrak{o}_2(\Gamma)$ be the orientation with orderings $V_1 < V_2 < V_5$ at vertex 1 and $V_1 < V_5 < V_3 < V_4$ at vertex 4. For the remainder of the paper, unless otherwise stated, we will use the orientation $\mathfrak{o}_1(\Gamma)$ when referring to example 1.

In the second example, let $\mathfrak{o}(\Delta)$ be the orientation given by the orderings $V_1^{(1)} < V_1^{(2)} < V_1^{(3)} < V_2^{(1)} < V_2^{(2)} < V_3$ at vertex 1, $V_1 < V_3$ at vertex 2, and $V_2 < V_3$ at vertex 3. There are many other choices of orientations for this example and they are typically associated to non-isomorphic Brauer configuration algebras.

Definition 1.5. A *Brauer configuration* is a tuple $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$, where Γ_0 is a set of vertices, Γ_1 is a set of polygons, μ is a multiplicity function, and $\mathfrak{o}$ is an orientation, such that the following conditions hold.

C1. Every vertex in Γ_0 is a vertex in at least one polygon in Γ_1.

C2. Every polygon in Γ_1 has at least two vertices.

C3. Every polygon in Γ_1 has at least one vertex α such that $\mathrm{val}(\alpha)\mu(\alpha) > 1$.

Note that if V and V' are two distinct polygons in Γ_1, it is possible that V and V' have identical sets of vertices. We distinguish between the two using their labels V and V'. Also note that $(\{1,2\}, \{V = \{1,2\}\}, \mu, \mathfrak{o})$ with $\mu(1) = \mu(2) = 1$ violates C3 and hence is not a Brauer configuration. However, by convention this algebra usually is nevertheless consider to be a Brauer graph algebra and is isomorphic to the truncated polynomial algebra $k[x]/(x^2)$.

Our next goal is to show that a Brauer configuration is the union of connected Brauer configurations. More precisely, let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ be a Brauer configuration. We say that Γ is *disconnected* if there are two Brauer configurations $\Gamma' = (\Gamma_0', \Gamma_1', \mu', \mathfrak{o}')$ and $\Gamma'' = (\Gamma_0'', \Gamma_1'', \mu'', \mathfrak{o}'')$ such that

(1) $\{\Gamma_0', \Gamma_0''\}$ is a partition of Γ_0,

(2) for every polygon $V \in \Gamma_1$, the vertices of V are either all in Γ_0' or are all in Γ_0'',

(3) $\{\Gamma_1', \Gamma_1''\}$ constitutes a partition of Γ_1,

(4) μ' (resp. μ'') is a restriction of μ to Γ_0' (resp. Γ_0''), and

(5) the orientations $\mathfrak{o}'$ and $\mathfrak{o}''$ are induced by $\mathfrak{o}$.

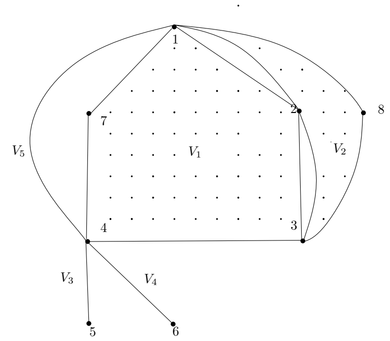
In this case, we write $\Gamma = \Gamma' \cup \Gamma''$. We say Γ *is connected* if it is not disconnected. It is clear that any Brauer configuration can be uniquely written as a union of connected Brauer configurations. We call these connected Brauer configurations the *connected components of* Γ.

We say a polygon in a Brauer configuration is *self-folded* if there is at least one vertex which occurs more than once in V. See the section below on realizations for a justification of the terminology.

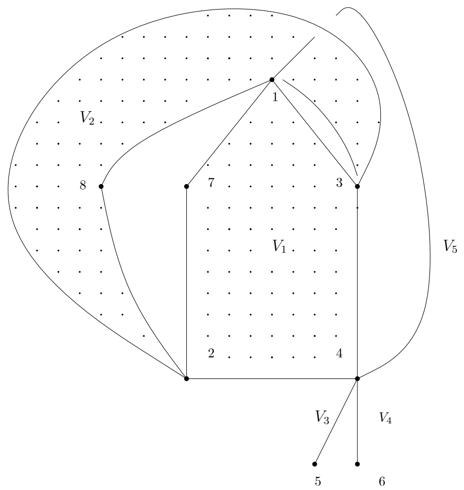
Remark 1.6. *A connected Brauer configuration, all of whose polygons are 2-gons is a Brauer graph and a self-folded 2-gon is a loop. For the definition of Brauer graphs and Brauer graph algebras, see, for example,* [B]*, for a definition in terms of ribbon graphs, see* [MS] *or for a presentation more closely related to the present paper, see* [GSS]*.*

1.2. **Realizations of Brauer configurations.** It is very useful to visualize a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$. For this, one represents each polygon in Γ_1 by an actual polygon. That is given a polygon (or multiset) $V = \{\alpha_1, \ldots, \alpha_d\}$ in Γ_1, V is visualized by an actual d-gon with vertices labeled by the α_i. Although there usually are many ways to perform the vertex labeling, just one is chosen. In particular, the order in which one labels the vertices of the actual polygon is not important. If a vertex $\alpha \in \Gamma_0$ occurs more than once in V, that is, V is self-folded, we identify all vertices labeled α in the actual polygon V. Finally, we identify vertices of different polygons if they correspond to the same vertex in Γ_0. We call such a choice, a *realization of the configuration* Γ. The actual theory and proofs in this paper never refer to or use any realization, however in terms of visualizing and understanding the results and proofs, realizations of configurations are a useful tool.

Example 1.7. We provide two realizations of the first example given above. Here is the first realization.

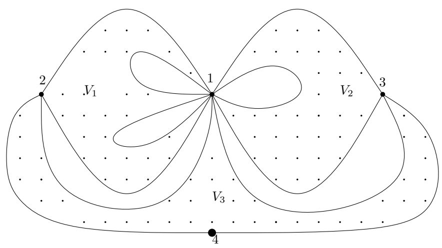


Note that the vertices in the polygon V_1 are 'ordered' 1,2,3,4,7 and in the 5-gon representing V_1, 1 is adjacent to 2, 2 is adjacent to 3, etc. If we change the order in which vertices occur, we will, in general, change the realization. For example, in the first example, if instead, we ordered the vertices 1,3,4,2,7 in the realization, we would obtain the following Brauer configuration.



We note that these are two different realizations of the same Brauer configuration and that the first realization is embeddable in the plane but the second is not.

Below is a realization of Δ, the second example, in which there are a number of self-foldings.



1.3. **Truncated vertices and reduced Brauer configurations.** We now define the crucial concept of a truncated vertex in a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$. We say a vertex $\alpha \in \Gamma_0$ is *truncated* if $\mathrm{val}(\alpha)\mu(\alpha) = 1$; that is, α occurs exactly once in exactly one $V \in \Gamma_1$ and $\mu(\alpha) = 1$. A vertex that is not truncated is called a *nontruncated vertex*.

Example 1.8. In the first example, vertices 6,7, and 8 are truncated. Note that vertex 5 is not truncated, even though $\mathrm{val}(5) = 1$, since $\mu(5) = 2 > 1$. In the second example, vertex 4 is truncated.

We will see in what follows that a truncated vertex only plays a role if it is one of the two vertices of a 2-gon. Namely, we introduce a reduction procedure which removes truncated vertices from polygons with 3 or more vertices. This will not affect the associated Brauer configuration algebra defined in Section 2, see Proposition 2.7.

The reduction procedure for removing a truncated vertex occurring in a d-gon, $d \geq 3$ is defined as follows. Suppose that $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ is a Brauer configuration and that $\alpha \in \Gamma_0$ is a truncated vertex in a d-gon $V \in \Gamma_1$ with $d \geq 3$. Note that since α is truncated, $\mathrm{val}(\alpha) = 1$ and this implies that V is the unique polygon in Γ_1 having α as a vertex. After reordering the vertices in V, we may assume that V is the d-gon $(\alpha_1, \ldots, \alpha_d)$, with $\alpha = \alpha_d$. Let $\Gamma' = (\Gamma'_0, \Gamma'_1, \mu', \mathfrak{o}')$, where

(1) $\Gamma'_0 = \Gamma_0 \setminus \{\alpha\}$,

(2) V' be the $(d-1)$-gon $\{\alpha_1, \ldots, \alpha_{d-1}\}$,

(3) $\Gamma_1' = (\Gamma_1 \setminus \{V\}) \cup \{V'\}$,

(4) $\mu' = \mu_{|\Gamma_0'}$.

(5) $\mathfrak{o}'$ is the orientation induced from the orientation $\mathfrak{o}$.

We see that Γ' is simply obtained from Γ by "removing" the truncated vertex α from V. Note that the number of polygons in Γ and Γ' are the same and only one polygon in Γ' has one less vertex.

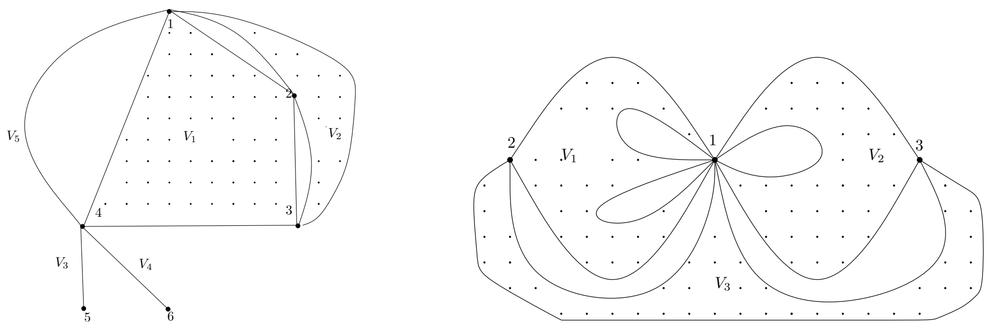
If Γ' also has a truncated vertex in a d'-gon, with $d' \geq 3$, we can remove it and obtain a Brauer configuration Γ'', where Γ'' has 2 less vertices than Γ. Continuing in this fashion, we arrive at a Brauer configuration Γ^* in which, if α is a truncated vertex, it occurs in a 2-gon and hence there are no more reductions that can be performed in Γ^*. We call Γ^* a *reduced Brauer configuration associated to* Γ. If $\Gamma = \Gamma^*$ we say that Γ is a *reduced* Brauer configuration. We leave the proof of the next result to the reader.

Lemma 1.9. *Let Γ be a Brauer configuration and suppose that Γ^* and Γ^{**} are two reduced configurations associated to Γ. Then we may choose a relabeling of the vertices of Γ^{**} so that $\Gamma^* = \Gamma^{**}$.* □

The above lemma allows us to talk about 'the' reduced configuration associated to a Brauer configuration.

Example 1.10. In the first example, the reduced configuration is obtained by removing vertices 7 and 8. Note that although vertex 6 is truncated, it is in the 2-gon V_4. In the second example, the reduced configuration is obtained by removing vertex 4.

Realizations of the reduced Brauer configurations for these two examples are given below.



Given a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$, we note that Γ is reduced if and only if every polygon V in Γ_1 satisfies one of the following conditions:

(1) V contains no truncated vertices.

(2) V is a 2-gon with one truncated vertex.

2. Brauer configuration algebras

In this section we define Brauer configuration algebras. As described in the introduction, Brauer configuration algebras are generalizations of Brauer graph algebras.

Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ be a Brauer configuration. For each nontruncated vertex $\alpha \in \Gamma_0$, consider the list of polygons V containing α such that V occurs in this list $\operatorname{occ}(\alpha, V)$ times. As described in Section 1, the orientation $\mathfrak{o}$ provides a cyclic ordering of this list. We call such a cyclically ordered list the *successor sequence at* α. Suppose that $V_1 < \cdots < V_t$ is the successor sequence at some nontruncated vertex α (with $\operatorname{val}(\alpha) = t$). Then we say that V_{i+1} is the *successor of* V_i *at* α, for $1 \leq i \leq t$, where $V_{t+1} = V_1$. Note that if $\operatorname{val}(\alpha) = 1$, $\mu(\alpha) > 1$ and α is vertex in polygon V, then the successor sequence at α is just V.

Example 2.1. The successor sequences for our two examples are already given by the orientations in Example 1.4. For instance, for the first example (with orientation $\mathfrak{o}_1(\Gamma)$), the successor sequence of vertex 4 is $V_1 < V_4 < V_3 < V_5$ (or $V_4 < V_3 < V_5 < V_1$ etc.) For the second example, the successor sequence for vertex 1 is $V_1^{(1)} < V_1^{(2)} < V_1^{(3)} < V_2^{(1)} < V_2^{(2)} < V_3$.

A Brauer configuration algebra Λ_Γ associated to a Brauer configuration Γ is defined by giving Λ_Γ as a path algebra of a quiver modulo an ideal of relations. Fix a field K and let $\Gamma = (\Gamma_0, \Gamma_1, \mathfrak{o}, \mu)$ be a Brauer configuration, with $\Gamma_1 = \{V_1, \ldots, V_m\}$.

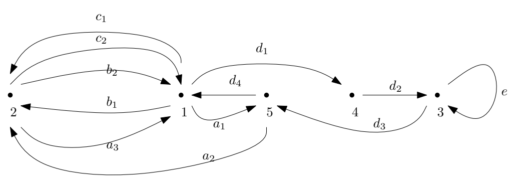
2.1. **The quiver of a Brauer configuration algebra.** We define the quiver $\mathcal{Q}_\Gamma$ as follows. The vertex set $\{v_1, \ldots, v_m\}$ of $\mathcal{Q}_\Gamma$ is in correspondence with the set of polygons $\{V_1, \ldots, V_m\}$ in Γ_1, noting that there is one vertex in $\mathcal{Q}_\Gamma$ for every polygon in Γ_1. We call v_i (respectively, V_i) the vertex (resp. polygon) *associated* to V_i (resp. v_i). In order to define the arrows in $\mathcal{Q}_\Gamma$, we use the successor sequences. For each nontruncated vertex $\alpha \in \Gamma_0$, and each successor V' of V at α, there is an arrow from v to v' in $\mathcal{Q}_\Gamma$, where v and v' are the vertices in $\mathcal{Q}_\Gamma$ associated to the polygons V and V' in Γ_1, respectively.

Note that V' can be the successor of V more than once at a given vertex of Γ_0, and also that V' can be the successor of V at more than one vertex of Γ_0. For each such occurrence there is an arrow from v to v'. In particular, $\mathcal{Q}_\Gamma$ may have multiple arrows from v to v'.

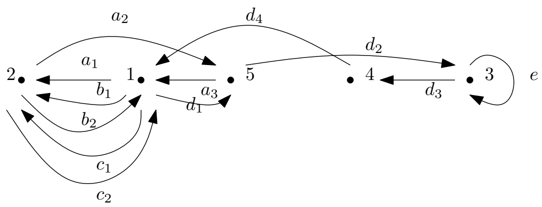
Thus, every arrow in $\mathcal{Q}_\Gamma$ is associated to a vertex $\alpha \in \Gamma_0$ and two polygons V and V' in Γ_1 such that V' is the successor of V at α. Conversely, associated to two polygons V, V', such that V' is the successor of V at some vertex $\alpha \in \Gamma_0$, there is an arrow from v to v' in $\mathcal{Q}_\Gamma$.

Example 2.2. For the first example, recall that we took $V_1 < V_5 < V_2$ as as the ordered list of polygons at vertex 1 for the orientation $\mathfrak{o}_1(\Gamma)$. It is the successor sequence at vertex 1 of Γ that yields the arrows a_1, a_2, and a_3 in the quiver below. For example, V_5 is the successor of V_1 at vertex 1 yielding the arrow a_1. The successor sequence $V_1 < V_2$ at vertex 2 yields the arrows b_1 and b_2. The successor sequence at vertex 3 yields c_1 and c_2, that at vertex 4 yields $d_1, \ldots, d_4$, and that at vertex 5 yields e.

The quiver associated to $(\Gamma_0, \Gamma_1, \mu, \mathfrak{o}_1(\Gamma))$ is

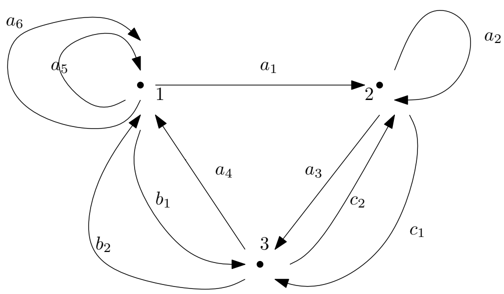


The quiver associated to $(\Gamma_0, \Gamma_1, \mu, \mathfrak{o}_2(\Gamma))$ is



We note that the two orientations $\mathfrak{o}_1(\Gamma)$ and $\mathfrak{o}_2(\Gamma)$ give rise to two non-isomorphic quivers; that is, they are non-isomorphic as oriented graphs. More precisely, the number of arrows going into a vertex, the number of arrows going out of a vertex and the fact that vertex 3 has a loop show that if there were an isomorphism of quivers, vertex 3 would be sent to vertex 3 and vertex 5 would be sent to vertex 5. But the arrow between vertices 3 and 5 are in opposite directions in the two quivers. Hence the quivers are not isomorphic.

The quiver associated to $\Delta = (\Delta_0, \Delta_1, \mu, \mathfrak{o}(\Delta))$ is



Remark 2.3. When we apply the above construction to Brauer configuration algebras where all polygons are 2-gons, we recover the usual quiver of a Brauer graph algebra.

2.2. **Ideal of relations and definition of a Brauer configuration algebra.** We define a set of elements ρ_Γ in $K\mathcal{Q}_\Gamma$ which will generate the ideal of relations I_Γ of the Brauer configuration algebra associated to the Brauer configuration Γ. There are three types of relations in ρ_Γ. For this we need the following definitions.

For each nontruncated vertex $\alpha \in \Gamma_0$ with successor sequence $V_1 < V_2 < \ldots < V_{\mathrm{val}(\alpha)}$, let $C_j = a_j a_{j+1} \cdots a_{\mathrm{val}(\alpha)} a_1 \cdots a_{j-1}$ be the cycle in $\mathcal{Q}_\Gamma$, for $j = 1, \ldots, \mathrm{val}(\alpha)$, where the arrow a_r corresponds to the polygon V_{r+1} being the successor of the polygon V_r at the vertex α. Now fix a polygon V in Γ_1 and suppose that $\mathrm{occ}(\alpha, V) = t \geq 1$. Then there are t indices $i_1, \ldots, i_t$ such that $V = V_{i_j}$. We define the *special α-cycles at v* to be the cycles $C_{i_1}, \ldots, C_{i_t}$, where v is the vertex in the quiver of $\mathcal{Q}_\Gamma$ associated to the polygon V. Note that each C_{i_j} is a cycle in $\mathcal{Q}_\Gamma$, beginning and ending at the vertex v and if α occurs only once in V and $\mu(\alpha) = 1$, then there is only one special α-cycle at v. Furthermore, if V is a polygon containing n vertices, counting repetitions, then there are a total of n different special α-cycles at v, one for each $\alpha \in V$ where each repetition of α in V gives rise to a different special α-cycle at v. Note however that the special cycles, corresponding to repetitions of the same vertex α in V, are cyclic permutations of each other.

We will sometimes say C is a special α-cycle if v is understood or just a special cycle if both v and α are understood.

Example 2.4. In the first example with orientation $\mathfrak{o}_1(\Gamma)$, the special 1-cycle at v_1 is $a_1a_2a_3$, the special 2-cycle at v_1 is b_1b_2, etc. The special 1-cycle at v_2 is the cyclic permutation $a_3a_1a_2$ of the special 1-cycle at v_1. Similarly, the special 2-cycle at v_2 is b_2b_1. There are no special j-cycles for $j = 6, 7, 8$ since they are truncated vertices. Note that e^2 is the unique 5-cycle at v_3 since $\mu(5) = 2$.

In the second example, there are three special 1-cycles at v_1, $a_1a_2\cdots a_6$, $a_6a_1a_2\cdots a_5$, and $a_5a_6a_1a_2\cdots a_4$. There are two special 1-cycles at v_2 $a_2a_3\cdots a_6a_1$ and $a_3a_4\cdots a_6a_1a_2$. There is only one special 1-cycle at v_3. Since vertex 4 of Γ is truncated, there are no special 4-cycles. Since 2 is not a vertex in V_2, there are no special 2-cycles at v_2 and similarly, since 3 is not in V_1 there are no special 3-cycles at v_1.

We now define the three types of relations by setting:

Relations of type one. For each polygon $V = \{\alpha_1, \dots, \alpha_m\} \in \Gamma_1$ and each pair of nontruncated vertices α_i and α_j in V, ρ_Γ contains all relations of the form $C^{\mu(\alpha_i)} - (C')^{\mu(\alpha_j)}$ or $(C')^{\mu(\alpha_j)} - C^{\mu(\alpha_i)}$ where C is a special α_i-cycle at v and C' is a special α_j-cycle at v.

Relations of type two. The type two relations are all paths of the form $C^{\mu(\alpha)}a$ where C is a special α-cycle and a is the first arrow in C.

Relations of type three. These relations are quadratic monomial relations of the form ab in $K\mathcal{Q}_\Gamma$ where ab is not a subpath of any special cycle.

Definition 2.5. Let K be a field and Γ a Brauer configuration. The *Brauer configuration algebra* Λ_Γ *associated to* Γ is defined to be $K\mathcal{Q}_\Gamma/I_\Gamma$, where $\mathcal{Q}_\Gamma$ is the quiver associated to Γ and I_Γ is the ideal in $K\mathcal{Q}_\Gamma$ generated by the set of relations ρ_Γ of type one, two and three.

We note that the set of relations ρ_Γ generating I_Γ is not necessarily minimal and usually contains redundant relations.

Example 2.6. For the first example, we list some of the relations of type one: $a_1a_2a_3 - b_1b_2, a_1a_2a_3 - (c_1c_2)^3, a_1a_2a_3 - d_1d_2d_3d_4, a_3a_1a_2 - b_2b_1, a_3a_1a_2 - (c_2c_1)^3, d_3d_4d_1d_2 - e^2$, etc. Many of the type one relations are redundant, for example, $b_1b_2 - (c_1c_2)^3$ and $(c_1c_2)^3 - d_1d_2d_3d_4$ follow from the above. Some of the type two relations are $a_1a_2a_3a_1, a_2a_3a_1a_2, a_3a_1a_2a_3, d_4d_1d_2d_3d_4$, $d_3d_4d_1d_2d_3$, e^3, $(c_1c_2)^3c_1$, $(c_2c_1)^3c_2$, etc. The type three relations are any of the paths of length two of the form a_ib_j, b_ja_i, a_ic_j, c_ja_i, a_id_j, d_ja_i, ea_i, and a_ie for all possible combinations of i, j. This gives a partial list of the type one, two, and three relations, and it includes many relations that are consequences of others.

For the second example, some of the type one relations are $a_1a_2\cdots a_6 - a_6a_1a_2\cdots a_5$, $a_1a_2\cdots a_6 - a_5a_6a_1a_2\cdots a_4, a_2a_3\cdots a_6a_1 - a_3a_4\cdots a_6a_1a_2$, and $b_2b_1 - c_2c_1$. Some type two relations are $a_6a_1\cdots a_6, a_1\cdots a_6a_1$, or $b_2b_1b_2$. Any $a_ib_j, b_ja_i, a_ic_j, c_ja_i, b_ic_j$ are type three relations. Some other relations of type three are a_6a_5, a_5^2, a_2^2 and a_5a_1.

Next we show that the reduction procedure for removing truncated vertices from a d-gon, $d \geq 3$, does not change the Brauer configuration algebra.

Proposition 2.7. *Let Γ be a Brauer configuration with associated Brauer configuration algebra Λ_Γ. Suppose $\alpha \in \Gamma_0$ is a truncated vertex in a polygon $V \in \Gamma_1$ and V is a d-gon,*

$d \geq 3$. Let Γ' be the Brauer configuration algebra obtained by removing the vertex α as in Section 1.3. Then the Brauer configuration algebra $\Lambda_{\Gamma'}$ associated to Γ' is isomorphic to Λ_Γ.

Proof. Since α is truncated, there is a unique polygon, say $V \in \Gamma_1$ in which α is a vertex. Again since α is truncated there are no special α-cycles. Thus, no arrows are created in the quiver of Λ by α, and the quivers of Λ and Λ' are the same. Similarly, the ideals of relations are seen to be the same and the result follows. □

2.3. **Special cycles.** In this section we investigate properties of special cycles and use these properties to obtain results about the muliplicative structure of a Brauer configuration algebra.

Let $\Lambda = K\mathcal{Q}/I$ be the Brauer configuration algebra associated to a reduced Brauer configuration Γ. Denote by $\pi\colon K\mathcal{Q} \to \Lambda$ the canonical surjection. Then if no confusion can arise we denote $\pi(x)$ by $\bar{x}$, for $x \in K\mathcal{Q}$.

We begin with a list of facts about successors and successor sequences translated into facts about special cycles. The proofs of these facts are immediate consequences of the definitions and are left to the reader. We denote by (F) facts relating to successor sequences and by (F$'$) the analogous facts expressed in terms of special cycles.

- (F1) If V' is a successor to V in the successor sequence at the (nontruncated) vertex $\alpha \in \Gamma_0$ then there is a unique arrow in $\mathcal{Q}$ from v to v' associated to V' being the successor of V.
- (F$'$1) If $a\colon v \to v'$ is an arrow in $\mathcal{Q}$ then up to cyclic permutation there is a unique special cycle C in which a occurs. If, in particular, C is a special α-cycle, for some $\alpha \in \Gamma_0$ then a is associated to one occurence of V' being a successor of V in the successor sequence at α.
- (F2) For each nontruncated vertex $\alpha \in \Gamma_0$ there is a unique successor sequence at α, up to cyclic permutation.
- (F$'$2) For each nontruncated vertex $\alpha \in \Gamma_0$, there is a unique special α-cycle, up to cyclic permutation.
- (F3) If V' is the successor of V at $\alpha \in \Gamma_0$, then, after cyclically reordering the successor sequence at α, the sequence begins with the chosen $V < V'$.
- (F$'$3) If a is an arrow in $\mathcal{Q}$, then there is a unique nontruncated $\alpha \in \Gamma_0$ and a unique special α-cyce C such that a is the first arrow in C. In particular, there are no repeated arrows in a special cycle.
- (F$'$4) If there is an arrow that occurs in two special cycles C and C', then there is a nontruncated vertex $\alpha \in \Gamma_0$ such that both C and C' are special α-cycles and C' is a cyclic permutation of C.
- (F5) The number of polygons in the successor sequence at a vertex $\alpha \in \Gamma_0$ is $\sum_{V \in \Gamma_1} \mathrm{occ}(\alpha, V)$.
- (F$'$5) The number of arrows in a special α-cycle is $\sum_{V \in \Gamma_1} \mathrm{occ}(\alpha, V)$.

The next result and its corollary show that there is a very tight multiplicative structure in Λ.

Proposition 2.8. *Let Γ be a Brauer configuration with associated Brauer configuration algebra $\Lambda = K\mathcal{Q}/I$ and let $V \in \Gamma_1$, $\alpha \in \Gamma_0$ a nontruncated vertex in Γ_0 that occurs in V. Let $C = a_1 a_2 \dots a_{\mathrm{val}(\alpha)}$ be a special α-cycle at v and C' the cyclic permutation $a_{j+1} \dots a_{\mathrm{val}(\alpha)} a_1 \dots a_j$. Let $p = a_1 a_2 \dots a_j$, for some $1 \le j \le \mathrm{val}(\alpha) - 1$ and set $x = C^s p$ and $y = pC'^s$, for some $0 \le s < \mu(\alpha)$. Then*

(1) $a_i \neq a_j$, for $i \neq j$.

(2) $\bar{x} \neq 0$.

(3) If a is an arrow in $\mathcal{Q}$, then $\overline{xa} \neq 0$ if and only if $a = a_{j+1}$.

(4) $\bar{y} \neq 0$.

(5) If a is an arrow in $\mathcal{Q}$, then $\overline{ay} \neq 0$ if and only if $a = a_{\mathrm{val}(\alpha)}$.

Proof. Part (1) follows from (F′3). The type two and three relations are monomial paths and the type one relations are differences $D^{\mu(\alpha)} - D'^{\mu(\alpha)}$ where D and D' are special α-cycles for some $\alpha \in \Gamma_0$. Since x and y have no subpaths that are type two relations or type three relations, we see (2) and (4) hold. Similarly, if $a \neq a_{j+1}$, then $a_j a$ is a type three relation and hence (3) holds. Finally, if $a \neq a_{\mathrm{val}(\alpha)}$ then aa_1 is a type three relation and hence (5) holds. □

Proposition 2.8 has the following consequence which plays an important role in [GS].

Proposition 2.9. *Let Γ be an indecomposable reduced Brauer configuration with associated Brauer configuration algebra $\Lambda = K\mathcal{Q}/I$ and assume* $\mathrm{rad}^2(\Lambda) \neq 0$. *Let a be an arrow in $\mathcal{Q}$. Then*

(1) there is a unique arrow b such that $\overline{ab} \neq 0$, and

(2) there is a unique arrow c such that $\overline{ca} \neq 0$.

Proof. First note that if P is an indecomposable projective Λ-module with $P\,\mathrm{rad}^2(\Lambda) = 0$, then by indecomposability and symmetry and the definition of the relations, $P/P\,\mathrm{rad}(\Lambda)$ and $P\,\mathrm{rad}(\Lambda)$ are isomorphic simple Λ-modules, and hence Λ would be isomorphic to $K[x]/(x)^2$, contradicting the assumption that $\mathrm{rad}^2(\Lambda) \neq 0$.

From the definition of the type one, two, and three relations, if x and y are arrows in $\mathcal{Q}$ then $\overline{xy} \neq 0$ if and only if there is a special cycle in which the arrows x and y occur and y immediately follows x. By (F′3), suppose that a is the first arrow in the special α-cycle $C = a_1 \cdots a_{\mathrm{val}(\alpha)}$ for some nontruncated vertex $\alpha \in \Gamma_0$. It follows that $b = a_2$ and $c = a_{\mathrm{val}(\alpha)}$ and we are done. □

From this result we obtain the following surprising consequence which shows that there is a tight connection between arrows in $\mathcal{Q}$ and paths of length 2 in $\mathcal{Q}$ that are not in the ideal I, the ideal generated by the relations of types one, two, and three. We introduce the following notation for this result. Set

$$\Pi = \{ab \mid a, b \text{ arrows in } \mathcal{Q} \text{ and } ab \notin I\}$$

that is Π is the set of paths of length 2 in $\mathcal{Q}$ that are not in I. The following corollary follows directly from 2.9.

Corollary 2.10. *Let Γ be an indecomposable reduced Brauer configuration with associated Brauer configuration algebra $\Lambda = K\mathcal{Q}/I$ and assume that* $\mathrm{rad}^2(\Lambda) \neq 0$. *Let Π be as defined above and define $f\colon \Pi \to \mathcal{Q}_1$ by $f(ab) = a$ and $g\colon \Pi \to \mathcal{Q}_1$ by $g(ab) = b$, where $a, b \in \mathcal{Q}_1$ with $ab \notin I$. Then the maps f and g are bijections.*

We call two special cycles in A *equivalent* if one is a cyclic permutation of the other. Suppose that there are t equivalence classes and let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be a full set of equivalence class representatives.

Proposition 2.11. *Let Γ be an indecomposable reduced Brauer configuration with associated Brauer configuration algebra $\Lambda = K\mathcal{Q}/I$ and assume that* $\mathrm{rad}^2(\Lambda) \neq 0$. *Let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be a complete set of representatives of special cycles. The following statements hold.*

(1) Any arrow of $\mathcal{Q}$ occurs once in exactly one of the special cycles in $\mathcal{C}$.

(2) The cardinality of $\mathcal{Q}_1$ is
$$|\mathcal{Q}_1| = \sum_{C_i \in \mathcal{C}} |C_i| = \sum_{\substack{\alpha \in \Gamma_0, \\ \alpha \text{ nontruncated}}} \sum_{V \in \Gamma_1} \mathrm{occ}(\alpha, V)$$
where $|C_i|$ denotes the number of arrows in the cycle C_i.

Proof. Given an arrow $a \in \mathcal{Q}_1$, there is a unique nontruncated $\alpha \in \Gamma_0$ such that a is in a special α-cycle by (F$'$1). Hence a is in every special α-cycle since any two special α-cycles are cyclic permutations of one another by (F$'$3) and (F$'$4). Thus, we can assume that a is in exactly one of the special cycles in $\mathcal{C}$. Since special cycles have no repeated arrows by (F$'$3), part (1) follows. Part (2) follows from part (1) and we are done. □

3. Properties of Brauer configuration algebras

In this section we prove some basic properties of Brauer configuration algebras. Assume that $\Lambda = K\mathcal{Q}/I$ where Λ is a Brauer configuration algebra associated to a Brauer configuration and $\mathcal{Q}$ is the quiver of Λ. We assume from now on that all Brauer configurations are reduced. We show that I is an admissible ideal and that Λ is a symmetric algebra. We also show that a Brauer configuration algebra is indecomposable if and only if its Brauer configuration is connected. Finally, we show that Λ is a multiserial algebra; that is we show that the heart of Λ is a direct sum of uniserial modules.

3.1. **First properties of Brauer configuration algebras and a basis of Λ.** If p is a path in a quiver $\mathcal{Q}$, we let $\ell(p)$ denote the length of p.

Lemma 3.1. *Suppose that $\Lambda = K\mathcal{Q}/I$ is the Brauer configuration algebra associated to a Brauer configuration Γ. Let C be a special cycle and let p be a path of length ≥ 1 in $\mathcal{Q}$ such that the first arrow of p is the first arrow in C. Then $\bar{p} \neq 0$ if and only if p is a prefix of $C^{\mu(\alpha)}$.*

Proof. Since the first arrow of $p = a_1 \cdots a_m$ is the same as the first arrow of C, either p is a prefix of C^s for some s or there is an i such that a_i is an arrow in C but $a_i a_{i+1}$ is not in C. First assume $a_i a_{i+1}$ is not in C. Then $a_i a_{i+1}$ is not in any special cycle by Proposition 2.8(3). Hence $a_i a_{i+1}$ is a type three relation and $\bar{p} = 0$.

Assume that C is a special α-cycle for some nontruncated vertex $\alpha \in \Gamma_0$. Now suppose that p is a prefix of C^s. Then either $\ell(p) \leq \ell(C^{\mu(\alpha)})$ or $\ell(p) > \ell(C^{\mu(\alpha)})$.

First assume that $\ell(p) > \ell(C^{\mu(\alpha)})$. Then p contains $C^{\mu(\alpha)}a_1$ which is a type two relation. Hence, $\bar{p} = 0$.

Now suppose that $\ell(p) \leq \ell(C^{\mu(\alpha)})$. Then p contains no relations of type two or three. By the length assumption, type one relations do not affect p and we see that $\bar{p} \neq 0$. The proof is complete. $\square$

We let J denote the two sided ideal in $K\mathcal{Q}$ generated by the arrows in $\mathcal{Q}$. Recall that the ideal I in $K\mathcal{Q}$ is *admissible* if $J^N \subseteq I \subseteq J^2$, for some $N \geq 2$. Clearly, if I is admissible then Λ is finite dimensional.

Proposition 3.2. *Let $\Lambda = K\mathcal{Q}/I$ be the Brauer configuration algebra associated to a Brauer configuration Γ. Then I is admissible and Λ is a symmetric algebra.*

Proof. From the definition of the three types of relations, we see that I is contained in J^2. Consider the set

$$\mathcal{S} = \{C^{\mu(\alpha)} \mid \alpha \text{ is a nontruncated vertex and } C \text{ is a special } \alpha\text{-cycle }\}.$$

Let $N = \max_{C^{\mu(\alpha)} \in \mathcal{S}}(\ell(C^{\mu(\alpha)})) + 1$. If p is a path of length N, then p cannot be a prefix of any element in $\mathcal{S}$. Note that by (F$'$3), we see that every arrow in $\mathcal{Q}$ is the prefix of some cycle in $\mathcal{S}$. Using this observation and Lemma 3.1, we see that $\bar{p} = 0$; that is, $p \in I$. Thus $J^N \subseteq I$ and it follows that I is admissible.

Using Lemma 3.1, the reader may check that the two sided socle of Λ is generated by the elements of $\mathcal{S}$. In fact, for each $V \in \Gamma_1$, choose a nontruncated vertex α of V and one special α-cycle at v, C_V, in $\mathcal{Q}$. Then $\{\overline{C_V^{\mu(\alpha)}} \mid V \in \Gamma_1\}$ forms a K-basis of the two sided socle of Λ.

To show that Λ is symmetric, let $\phi\colon \Lambda \to K$ be the K-linear form defined as follows. Let p be a path in $\mathcal{Q}$. Then $\phi(\bar{p}) = 1$ if and only if $p \in \mathcal{S}$. If $p \notin \mathcal{S}$, let $\phi(\bar{p}) = 0$. It is easy to show that $\phi(ab) = \phi(ba)$. That $\operatorname{Ker}\phi$ contains no left or right ideals follows from the description of a K-basis of the two sided socle of Λ and that ϕ is 1 on elements of $\mathcal{S}$. It follows that Λ is a symmetric algebra. $\square$

The next result provides a useful K-basis of Λ.

Proposition 3.3. *Let Λ be the Brauer configuration algebra associated to the Brauer configuration Γ. For each $V \in \Gamma_1$, choose a nontruncated vertex α of V and exactly one special α-cycle C_V at v. Then keeping the above notation, we have*

$$\{\bar{p} \mid p \textit{ is a proper prefix of some } C^{\mu(\alpha)} \textit{ where } C \textit{ is a special } \alpha\textit{-cycle}\} \cup \{\overline{C_V^{\mu(\alpha)}} \mid V \in \Gamma_1\}$$

is a K-basis of Λ.

Proof. We have seen that $\{\overline{C_V^{\mu(\alpha)}} \mid V \in \Gamma_1\}$ is a K-basis of the socle of Λ. Using that every arrow is the start of a special cycle, Lemma 3.1 and that the only relations affecting proper subpaths of the special cycles are monomial relations (types two and three), the result follows. $\square$

3.2. **Decomposable and indecomposable Brauer configuration algebras.** We start by investigating disconnectedness of Brauer configurations.

Proposition 3.4. *Suppose the Brauer configuration* Γ *is disconnected and decomposes into Brauer configurations* $\Gamma' \cup \Gamma''$. *Then the associated Brauer configuration algebra* Λ_Γ *is isomorphic to the product* $\Lambda_{\Gamma'} \times \Lambda_{\Gamma''}$.

Proof. By definition of a disconnected Brauer configuration given in Section 1, there can be no arrows between vertices in $\mathcal{Q}_{\Gamma'}$ and $\mathcal{Q}_{\Gamma''}$ and the result follows.

□

We prove the converse.

Proposition 3.5. *If the Brauer configuration* Γ *is connected then the Brauer configuration algebra associated to* Γ *is indecomposable as an algebra.*

Proof. We show that if a Brauer configuration algebra is decomposable then the Brauer configuration is disconnected. Suppose that Γ is a Brauer configuration with associated Brauer configuration algebra Λ. Assume that Λ is not indecomposable and that $\Lambda \cong \Lambda' \times \Lambda''$. Let $\mathcal{Q}, \mathcal{Q}'$, and $\mathcal{Q}''$ be the quivers of Λ, Λ' and Λ'' respectively. Then $\mathcal{Q}$ is the disjoint union of $\mathcal{Q}'$ and $\mathcal{Q}''$. Let $A = \{V \in \Gamma_1 \mid \text{ the vertex in } \mathcal{Q} \text{ associated to } V \text{ is in } \mathcal{Q}'\}$ and $B = \{V \in \Gamma_1 \mid \text{ the vertex in } \mathcal{Q} \text{ associated to } V \text{ is in } \mathcal{Q}''\}$. Note that $A \cup B = \Gamma_1$ and $A \cap B = \emptyset$.

Let $\mathcal{A}$ be the set of vertices of the polygons in A and $\mathcal{B}$ be the set vertices of the polygons in B. Then we show that $\mathcal{A} \cap \mathcal{B} = \emptyset$. Indeed, suppose there is a vertex $\alpha \in \Gamma_0$ that is a vertex of a polygon $V \in A$ and of a polygon $V' \in B$. Then both V and V' occur in the successor sequence at α. Hence, if C is a special α-cycle, both v and v' occur as vertices in C. This contradicts the fact that there are no paths from v to v' in the quiver of Λ since v is a vertex in $\mathcal{Q}'$ and v' is a vertex in $\mathcal{Q}''$. Hence, by condition C1, we get a partition of $\Gamma_0 = \mathcal{A} \cup \mathcal{B}$.

In order to show that Γ is disconnected, assume for a contradiction that Γ is connected. Hence, since $\Gamma_1 = A \cup B$, $A \cap B = \emptyset$, $\Gamma_0 = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$, for Γ to be connected there must be some polygon V that has vertices from both $\mathcal{A}$ and $\mathcal{B}$; that is, there is a polygon $V \in \Gamma_1$ and $\alpha \in \mathcal{A}$, and $\beta \in \mathcal{B}$, such that α and β are vertices of V. This contradicts $A \cap B = \emptyset$, finishing the proof. □

3.3. **Gradings of Brauer configuration algebras.**

Proposition 3.6. *Let* Λ *be the Brauer configuration algebra associated to a connected Brauer configuration* Γ. *The algebra* Λ *has a length grading induced from the path algebra* $K\mathcal{Q}$ *if and only if there is an* $N \in \mathbb{Z}_{>0}$ *such that, for each nontruncated vertex* α *in* Γ_0, $\mathrm{val}(\alpha)\mu(\alpha) = N$.

Proof. If $\alpha \in \Gamma_0$ is a truncated vertex in the 2-gon V, then the projective indecomposable associated to V is uniserial and gives rise to a monomial relation of type two. Monomial relations are homogeneous under any grading.

Suppose Λ has a length grading. If C is a special α-cycle, then the length of $C^{\mu(\alpha)}$ is $\mathrm{val}(\alpha)\mu(\alpha)$. Since type two and three relations are monomials, the ideal of relations I is generated by length homogeneous relations if the type one relations are length homogeneous. Thus, for I to be generated by length homogeneous elements, all type one relations must be length homogeneous. But this implies that for a special α-cycle C at v and a special β-cycle C' at v, the relation $C^{\mu(\alpha)} - (C')^{\mu(\beta)}$ must be length homogeneous. Thus for all vertices α in a polygon, the $\mathrm{val}(\alpha)\mu(\alpha)$ must all be equal. Using connectedness, we see that if α is a vertex in polygon V and β is a vertex in polygon V', then there is a sequence of polygons $V_1, \dots, V_k$ and vertices α_i, β_i in V_i, such that $\alpha = \alpha_1$, $\beta_i = \alpha_{i+1}$ for $i = 1, \dots, k-1$ and $\beta = \beta_k$. The result now follows.

The converse immediately follows from the fact the if for each nontruncated vertex α in Γ_0, $\mathrm{val}(\alpha)\mu(\alpha) = N$ then all relations are length homogeneous. $\square$

3.4. **Projective indecomposable modules and uniserial modules.** We now describe the projective-injective indecomposable modules and the non-projective uniserial modules over a Brauer configuration algebra.

Let Λ be a Brauer configuration algebra associated to a reduced Brauer configuration Γ. In what follows we adopt the following notation, if V is a polygon in Γ and v is the vertex in the quiver of Λ associated to V, then we let P_V be the projective Λ-module (resp. S_V the simple Λ-module) associated to v. Let V be a polygon in Γ_1, α a nontruncated vertex in V with $n = \mathrm{val}(\alpha)$, and $C = a_1a_2 \cdots a_n$ a special α-cycle at v. Let a_i be an arrow from $v_i \to v_{i+1}$ and let V_i be the polygon in Γ_1 associated to v_i, for $i = 1, \dots, n$. Note that $V_1 = V_{n+1} = V$. It follows from Proposition 2.9 and Proposition 3.3 that based on the choice of C, we can now define a chain of uniserial submodules in the following way.

Set $U_{n\mu(\alpha)} = U_{n\mu(\alpha)}(C)$ to be isomorphic to the simple module associated to $v = v_{n+1}$. Note that $U_{n\mu(\alpha)}$ has K-basis $\{\overline{(a_1a_2 \dots a_n)^{\mu(\alpha)}}\} = \{\overline{C^{\mu(\alpha)}}\}$. Assuming $U_{j+1} = U_{j+1}(C)$ is defined and $j \geq 1$, let $U_j = U_j(C)$ be the uniserial Λ-module containing U_{j+1} and such that U_j/U_{j+1} is isomorphic to the simple module associated to v_ℓ, where $j = kn + \ell$ with $1 \leq \ell < n$ and $0 \leq k < \mu(\alpha)$. Note that U_j has K-basis $\{\overline{(C^k a_1a_2 \dots a_\ell)}, \overline{(C^k a_1a_2 \dots a_{\ell+1})}, \dots, \overline{(C^{\mu(\alpha)})}\}$. Thus, we obtain a chain of uniserial modules

$$(0) \subset U_{n\mu(\alpha)} \subset U_{(n\mu(\alpha))-1} \subset \cdots \subset U_2 \subset U_1.$$

If V is a 2-gon with vertices α and β, with β truncated (and hence α is nontruncated), then the indecomposable projective Λ-module P_V is uniserial with K-basis $\{e_v, \overline{a_1}, \overline{a_1a_2}, \dots, \overline{(a_1a_2 \dots a_n)^{\mu(\alpha)}}\}$, where e_v is the primitive idempotent in Λ at vertex v, and we obtain a chain of uniserial modules

$$(0) \subset U_{n\mu(\alpha)} \subset U_{(n\mu(\alpha))-1} \subset \cdots \subset U_2 \subset U_1 \subset P_V.$$

Lemma 3.7. *Let Λ be a Brauer configuration algebra with connected Brauer configuration Γ. Let U be a non-projective uniserial Λ-module. With the notation above we have the following.*

(1) There is a polygon V in Γ, a nontruncated vertex α in V, and a special α-cycle C at v such that U is isomorphic to $U_j(C)$ for some $1 \leq j \leq \mathrm{val}(\alpha)\mu(\alpha)$. Furthermore, if U is not a simple Λ-module then V, α, and j, are unique and C is unique up to cyclic permutation.

(2) *Suppose that U is a non-zero, and non-simple uniserial module isomorphic to $U_j(C)$ obtained from a special α-cycle C and let U' be a non-projective uniserial Λ-module isomorphic to $U'_{j'}(C')$ obtained from a special β-cycle C' at v', with $\alpha \neq \beta$. Then* $\dim_K \operatorname{Hom}_\Lambda(U, U') \leq 1$.

Proof. Let U be a uniserial Λ-module. If U is a simple Λ-module then (1) holds. Assume that U is a nonsimple uniserial Λ-module. Let the socle of U be isomorphic to a simple module S_V for some $V \in \Gamma_1$. Since Γ is reduced, V is either a 2-gon with one truncated vertex or V is a d-gon, $d \geq 2$, and all the vertices of V are nontruncated. We have that U maps monomorphically into P_V since P_V is an indecomposable injective module. Applying Proposition 2.8 and Propositiom 2.9, we see that U is isomorphic to $U_j(C)$, for some special α-cycle at v and some j, $1 \leq j \leq \operatorname{val}(\alpha)\mu(\alpha)$. The last part of (1) again follows from Proposition 2.8 and Proposition 2.9

Part (2) follows from (F$'$4) since, if $\alpha \neq \beta$ then C and C' can have no arrows in common. In particular, the only possible map would be from U to the socle of U'. □

Proposition 3.8. *Let Λ be a Brauer configuration algebra associated to a connected Brauer configuraton Γ. Let U be a uniserial Λ-module. With the notation above we have that U is projective (uniserial) if and only if U is isomorphic to the indecomposable projective Λ-module associated to a 2-gon V having a truncated vertex.*

Proof. If V is a 2-gon with vertices α and β and β is truncated, then there is only one special α-cycle at v and by the discussion at the beginning of this section, P_V is a uniserial projective module. Conversely, suppose that V is a d-gon, $d \geq 2$, such that each vertex in V is nontruncated. Let $V = \{\alpha_1, \ldots, \alpha_d\}$ and, for $1 \leq i \leq d$, let C_i be a special α_i-cycle at V. These cycles yield distinct uniserial submodules $U_1(C_i)$, all having the same socle. Hence P_V is not a uniserial module. The result follows. □

Example 3.9. Let Λ be the Brauer configuration algebra associated to the Brauer configuration of our first example (after reducing). If X is a set of elements in Λ, let $\langle X \rangle$ denote the right submodule of Λ generated by X. Then, the polygon $V_1 = \{1, 2, 3, 4\}$, yields the following sequences of uniserial submodules:

$$\langle \overline{a_1 a_2 a_3} \rangle \subset \langle \overline{a_1 a_2} \rangle \subset \langle \overline{a_1} \rangle,$$
$$\langle \overline{b_1 b_2} \rangle \subset \langle \overline{b_1} \rangle,$$
$$\langle (\overline{c_1 c_2})^3 \rangle \subset \langle (\overline{c_1 c_2})^2 \overline{c_1} \rangle \subset \cdots \subset \langle \overline{c_1} \rangle,$$
$$\langle \overline{d_1 d_2 d_3 d_4} \rangle \subset \langle \overline{d_1 d_2 d_3} \rangle \subset \cdots \subset \langle \overline{d_1} \rangle.$$

Note that vertex 5 in V_3 is truncated and hence we get a uniserial projective Λ-module generated by the idempotent e_{v_3}. The sequence of uniserial modules in $e_{v_3}\Lambda$ is

$$\langle \overline{d_2 d_3 d_4 d_1} \rangle \subset \langle \overline{d_2 d_3 d_4} \rangle \subset \cdots \langle \overline{d_2} \rangle \subset \langle e_{v_3} \rangle.$$

Since the Brauer configuration of our second example (after reducing) has no truncated vertices, the associated Brauer configuration algebra has no uniserial projective modules.

Next, we describe the indecomposable projective modules over a Brauer configuration algebra.

Theorem 3.10. *Let Λ be a Brauer configuration algebra associated to a reduced Brauer configuration Γ. Let P be an indecomposable projective Λ-module associated to a d-gon V. Define an integer r by setting $r = d$ if all vertices of V are nontruncated and $r = 1$ if V is a 2-gon with one truncated vertex. Then $rad(P)$ is the sum of the r uniserial Λ-modules $\sum_C U_1(C)$ where C runs over the special α-cycles at v for every nontruncated vertex α of V. If $r > 1$, then $U_i \cap U_j$ is the simple socle of P. Furthermore, the heart of P, $\mathrm{rad}(P)/\operatorname{soc}(P)$ is a direct sum of uniserial Λ-modules.*

Proof. Let

$$\mathcal{C}_v = \{C \mid C \text{ is a special } \alpha\text{-cycle at } v, \alpha \text{ a nontruncated vertex in } V\} = \{C_1, \dots, C_r\}.$$

For each $C_i \in \mathcal{C}_v$, the uniserial Λ-module $U_1(C_i)$ is generated by the first arrow, a_i, in C_i. Let M denote the submodule of P generated by $a_1, \dots, a_r$. Note that $a_1, \dots, a_r$ are precisely the arrows in the quiver of Λ that start at vertex v (where v corresponds to V) and that $P = e_v\Lambda$, where e_v is the primitive idempotent in Λ at v. Then it follows that $M = \mathrm{rad}(P)$ and that $M = \sum_{i=1}^r U_1(C_i)$.

For $i = 1, \dots r$, we have seen that $\overline{C_i^{\mu(\alpha)}}$ is a nonzero element in the socle of P and, considering the relations of type 1 they are all equal. Since the type 2 and type 3 relations are monomial relations, it follows that, if $i \neq j$, then $U_1(C_i) \cap U_1(C_j)$ is the socle of P which is a simple Λ-module.

Again from the relations of types one, two and three, we see that we have a short exact sequence of Λ-modules

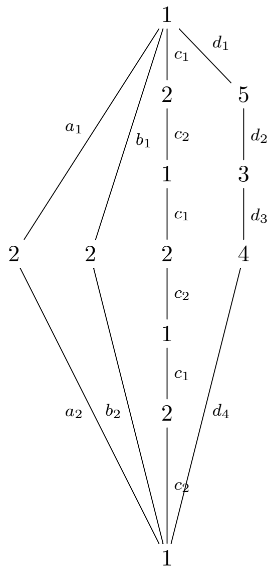
$$0 \to \oplus_{i=1}^{r-1} S_v \xrightarrow{f} \oplus_{i=1}^{r} U_1(C_i) \to M \to 0,$$

where $f(s_1, \dots, s_{r-1}) = (s_1, s_2 - s_1, s_3 - s_2, \dots, s_{r-1} - s_{r-2}, -s_{r-1})$. Factoring out the socles, we obtain an isomorphism.

$$\oplus_{i=1}^{r} U_1(C_i)/S_v \xrightarrow{\cong} M/\operatorname{soc}(P).$$

Noting that $M/\operatorname{soc}(P)$ is the heart of P, the proof is complete. □

Example 3.11. Let Λ be the Brauer configuration algebra corresponding to our first example and let P be the indecomposable projective Λ-module associated to the vertex 1 in the quiver of Λ which in turn corresponds to the polygon V_1 of the associated Brauer configuration. We give a schematic of P.



In [GS] the notion of a multiserial algebra is defined. Multiserial algebras are a direct generalization of biserial algebras. Namely, a K-algebra A is *multiserial* if the Jacobson radical of A e- as a left and right A-module is a direct sum of uniserial modules, the intersection of any two uniserial modules is either 0 or a simple A-module. It follows from Theorem 3.10 that a Brauer configuration algebra is multiserial and we also obtain the number of uniserial summands in the heart of each projective indecomposable:

Corollary 3.12. *Let Λ be a Brauer configuration algebra with Brauer configuration Γ.*

(1) *Λ is a multiserial algebra.*

(2) *The number of summands in the heart of a projective indecomposable Λ-module P such that $\mathrm{rad}^2(P) \neq 0$ equals the number of non-truncated vertices of the polygon in Γ corresponding to P counting repetitions.*

The next result uses Theorem 3.10 and shows that the dimension of a Brauer configuration algebra can easily be computed from its Brauer configuration.

1

Proposition 3.13. *Let Λ be a Brauer configuration algebra associated to the Brauer configuration Γ and let $\mathcal{C} = \{C_1, \dots, C_t\}$ be a full set of equivalence class representatives of special cycles. Assume that, for $i = 1, \dots, t$, C_i is a special α_i-cycle where α_i is a nontruncated* *vertex in* Γ*. Then*

$$\dim_K \Lambda = 2|Q_0| + \sum_{C_i \in \mathcal{C}} |C_i|(n_i|C_i| - 1),$$

where $|Q_0|$ *denotes the number of vertices of* Q*,* $|C_i|$ *denotes the number of arrows in the* α_i*-cycle* C_i *and* $n_i = \mu(\alpha_i)$*.*

Proof. Since $\dim_K \Lambda = \dim_K(\Lambda/\operatorname{rad}(\Lambda)) + \dim_K \operatorname{rad}(\Lambda)$ and $\dim_K(\Lambda/\operatorname{rad}(\Lambda)) = |Q_0|$ we must show that $\dim_K \operatorname{rad}(\Lambda) = |Q_0| + \sum_{i=1} |C_i||C_i - 1|^{\mu(\alpha_i)}$. Now $\dim \operatorname{rad}(\Lambda) = \dim_K \operatorname{soc}(\operatorname{rad}(\Lambda)) + \dim_K(\operatorname{rad}(\Lambda)/\operatorname{soc}(\operatorname{rad}(\Lambda))) = |Q_0| + \dim_K(\operatorname{rad}(\Lambda)/\operatorname{soc}(\operatorname{rad}(\Lambda)))$. We must show that $\operatorname{rad}(A)/\operatorname{soc}(\operatorname{rad}(A)) = \sum_{i=1} |C_i||C_i - 1|^{\mu(\alpha_i)}$. Using Theorem 3.10, we see that $\operatorname{rad}(\Lambda)/\operatorname{soc}(\Lambda)$ is isomorphic to the direct sum of the uniserial modules $a\Lambda/\operatorname{soc}(a\Lambda)$, that is, $\operatorname{rad}(\Lambda)/\operatorname{soc}(\Lambda) \simeq \oplus_{a \in Q_1}(a\Lambda/\operatorname{soc}(a\Lambda)$. But if $C = a_1 \cdots a_s$ is the special α-cycle with first arrow $a = a_1$ associated to the uniserial module $a\Lambda$, then $\dim_K(a\Lambda) = \mu(\alpha)s$. Hence, $\dim_K(a\Lambda/\operatorname{soc}(a\Lambda)) = \mu(\alpha)|C| - 1$. Noting there are $|C|$ special cycles equivalent to C, the result now follows. □

4. Radical cubed zero Brauer configuration algebras

In this section we classify the radical cubed zero Brauer configuration algebras such that the associated Brauer configuration has no self-foldings; that is, each polygon in the Brauer configuration contains no repeated vertices. More precisely, we associate Brauer configuration algebras to finite graphs and compare them to the symmetric radical cube zero algebras associated to the same graphs. In particular, given a symmetric radical cubed zero algebra associated to a graph, we provide a construction of a Brauer configuration, such that the associated Brauer configuration algebra is isomorphic to the given symmetric radical cubed zero algebra.

We begin with a well-known one-to-one correspondence. Fix a positive integer n and consider the set of finite graphs $G = (G_0, G_1)$ where $G_0 = \{1, 2, \ldots, n\}$ is the set of vertices of G and G_1 is the set of edges of G. We further suppose that G has no isolated vertices; in particular, the valency of each vertex is at least equal to one. We allow multiple edges and loops. We say two such graphs $G = (G_0, G_1)$ and $G' = (G_0, G_1')$ if there is a set isomorphism $\delta\colon G_1 \to G_1'$ such that, for each $e \in G_1$, the endpoints of e and $\delta(e)$ are the same. Effectively, two graphs are equivalent if they differ only in the names of the edges.

Let $\mathcal{G}_n$ denote the set of equivalence classes of finite graphs having n vertices labelled 1 to n and such that there are no vertices of valency zero.

Let $\mathcal{M}_n$ denote the set of symmetric $n \times n$ matrices with entries in the nonnegative integers such that no row only has 0 entries. Equivalently, no column has only 0 entries. Although the next result is well-known we include a short proof for completeness.

Lemma 4.1. *There is a one-to-one correspondence between* $\mathcal{G}_n$ *and* $\mathcal{M}_n$*.*

Proof. If G represents an equivalence class in $\mathcal{G}_n$, define $E(G)$ to be the $n \times n$ matrix with (i,j)-entry being the number of edges with endpoints the vertices i and j, if $1 \le i \ne j \le n$, and the (i,i)-entry of $E(G)$ is the number of loops at vertex i.

If $E = (e_{i,j}) \in \mathcal{M}_n$ let $G(E)$ be the equivalence class of a graph with vertex set $\{1, \ldots, n\}$ and $e_{i,j}$ edges with endpoints i and j.

We have $G(E(G)) = G$ and $E(G(E)) = E$. Finally, G having no isolated vertices corresponds to E having no zero row and no zero column and we are done. □

Our goal is to extend the one-to-one correspondence given in the above Lemma to include Brauer configurations. For this we need to introduce ordered Brauer configurations. But before we do so, we would like to motivate why this is necessary. Consider the following two (unequal) symmetric matrices

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The graphs associated to these matrices are both 4 cycles respectively given by

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[d, dash] \\ 4 \arrow[r, dash] & 3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] & 3 \arrow[d, dash] \\ 4 \arrow[r, dash] & 2. \end{tikzcd}$$

As the vertices of the graph will correspond to the polygons in a Brauer configuration, to obtain the desired one-to-one correspondence, one needs to distinguish two Brauer configurations that differ only in the labelling of the polygons. This is taken care of by 'ordering' the polygons, which is formally defined below.

Before giving the definition of ordering, we have the following result.

Lemma 4.2. *Let Λ be a indecomposable Brauer configuration algebra with Brauer configuration Γ. Then $\mathrm{rad}^3(\Lambda) = 0$ and $\mathrm{rad}^2(\Lambda) \neq 0$ if and only if $\mathrm{val}(\alpha)\mu(\alpha) = 2$, for each nontruncated vertex $\alpha \in \Gamma_0$.*

Proof. If α is a nontruncated vertex, then $\mathrm{val}(\alpha)\mu(\alpha) \geq 2$. If, for some $\alpha \in \Gamma_0$, $\mathrm{val}(\alpha)\mu(\alpha) \geq 3$ and C is a special α-cycle, then $C^{\mu(\alpha)}$ has length $\mathrm{val}(\alpha)\mu(\alpha)$ and is a nonzero element in $\mathrm{soc}(\Lambda)$. Thus $\mathrm{rad}^3(\Lambda) \neq 0$. Conversely, if $\mathrm{rad}^3(\Lambda) \neq 0$ then there must be some vertex α and a special α-cycle C such that the length of $C^{\mu(\alpha)}$ which equals $\mathrm{val}(\alpha)\mu(\alpha)$ is ≥ 3. To see this, if every $C^{\mu(\alpha)}$ is of length 2 then every path of length 3 must be in the ideal of relations. In particular, $\mathrm{rad}^3(\Lambda) = 0$, a contradiction. □

Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ be a reduced Brauer configuration such that $|\Gamma_1| = n$. That is, there are exactly n polygons in Γ_1, and suppose further that $1 \leq \mathrm{val}(\alpha)\mu(\alpha) \leq 2$, for all $\alpha \in \Gamma_0$. Recall that by our assumptions on Brauer configurations and the definition of reduced, if $\mathrm{val}(\alpha)\mu(\alpha) = 1$, then α is a truncated vertex in a 2-gon and the other vertex is not truncated.

We say Γ is *ordered by f* if $f\colon \{1, 2, 3, \ldots, n\} \to \Gamma_1$ is an isomorphism. Note that f 'orders' the polygons in Γ_1 with $f(i)$ being the i^{th} polygon. To simplify notation, we will usually omit f and use the terminology 'the i^{th} polygon' for $f(i)$ and say Γ is *ordered*. We remark that the assumption that $\mathrm{val}(\alpha)\mu(\alpha) \leq 2$ for all α, implies there is only one choice for $\mathfrak{o}$ and μ is determined by the truncated vertices. Namely, if α is a vertex with $\mathrm{val}(\alpha) = 2$ and $\mu(\alpha) = 1$ and if α is in the polygons V and V', with $V \neq V'$, then the successor sequence at α must be $V < V'$ which is the same as $V' < V$. If α is a vertex with $\mathrm{val}(\alpha) = 1$, α is either truncated or not. If α is truncated, $\mu(\alpha) = 1$ and if not, $\mu(\alpha) = 2$, and $\alpha \in V$, then the successor sequence for α is given by V. If α is truncated then α has no successor sequence.

We say an ordered reduced Brauer configuration Γ has *no self-foldings* if, for each polygon V in Γ_1, there are no repeated vertices in V.

Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ and $\Gamma' = (\Gamma'_0, \Gamma'_1, \mu', \mathfrak{o}')$ be two ordered reduced Brauer configurations with no self-foldings such that, for each $\alpha \in \Gamma_0$ or $\alpha \in \Gamma'_0$, we have $1 \leq \mathrm{val}(\alpha)\mu(\alpha) \leq 2$. We say Γ is *equivalent* to Γ' if there is a set isomorphism $\epsilon\colon \Gamma_0 \to \Gamma'_0$ such that if $\{\alpha_1, \ldots, \alpha_r\}$ is the i^{th} polygon in Γ_1, then $\{\epsilon(\alpha_1), \ldots, \epsilon(\alpha_r)\}$ is the i^{th} polygon in Γ'_1. Effectively, two ordered reduced Brauer configurations are equivalent if they differ only in the names of the vertices.

It easily follows from the definition, that if Γ and Γ' are equivalent Brauer configurations then the associated Brauer configuration algebras are isomorphic. This holds since the only difference between Γ and Γ' is the labeling of the vertices.

Let $\mathcal{B}_n$ be the set of equivalence classes of ordered reduced Brauer configurations Γ satisfying

(1) Γ has no self-foldings,

(2) Γ has exactly n polygons, and

(3) $1 \leq \mathrm{val}(\alpha)\mu(\alpha) \leq 2$, for all $\alpha \in \Gamma_0$.

Before stating our next result, we introduce some terminology. Let $G = (G_0, G_1)$ be a representative of an element of $\mathcal{G}_n$. A vertex in G is called a *leaf* if it has valency 1. If i is a leaf and i is an endpoint of the edge e, we say e is the *leaf edge associated to i*.

Proposition 4.3. *The set $\mathcal{B}_n$ is in one-to-one correspondence with the sets $\mathcal{G}_n$ and $\mathcal{M}_n$, for all strictly positive integers n.*

Proof. We show that $\mathcal{B}_n$ is in one-to-one correspondence with $\mathcal{G}_n$.

Let $\Gamma = (\Gamma_0, \Gamma_1, \mathfrak{o}, \mu)$ be an ordered, reduced Brauer configuration which represents an equivalence class in $\mathcal{B}_n$. We construct a graph $G(\Gamma)$ which represents an equivalence class in $\mathcal{G}_n$ as follows. For $i = 1, \ldots n$, let V_i be the i^{th} polygon in Γ_1. The vertex set of $G(\Gamma)$ is given by $\{1, \ldots, n\}$ with the polygon V_i in Γ corresponding to the vertex i in $G(\Gamma)$. The number of edges between vertices i and j in $G(\Gamma)$, where $i \neq j$, is equal to the number of vertices $\alpha \in \Gamma_0$ such that $\alpha \in V_i \cap V_j$. The number of loops at vertex i in $G(\Gamma)$ is equal to the number of vertices $\alpha \in \Gamma_0$ such that $\alpha \in V_i$, and $\mu(\alpha) = 2$. From Γ, we have constructed a graph $G(\Gamma)$. The reader may check that if Γ' is another Brauer configuration in the same equivalence class of $\mathcal{B}_n$ as Γ, then $G(\Gamma)$ and $G(\Gamma')$ are in the same equivalence class in $\mathcal{G}_n$. Thus, we have a well-defined map from $\mathcal{B}_n$ to $\mathcal{G}_n$.

We will construct the inverse map from $\mathcal{G}_n$ to $\mathcal{B}_n$. We begin with a special case. Let G be the graph with one vertex $\{1\}$ and one loop at 1. Set $\Gamma(G) = (\{\alpha, \beta\}, V = \{\alpha, \beta\}, \mu, \mathfrak{o})$ where $\mu(\alpha) = 1$ and $\mu(\beta) = 2$. Note that there is only one orientation since there is just one polygon. The vertex α is truncated. Then we send the equivalence class of G to the equivalence class of $\Gamma(G)$. It is clear that $G(\Gamma(G))$ is in the same equivalence class as G in $\mathcal{G}_n$.

Now let $G = (G_0, G_1)$ be a graph in an equivalence class in $\mathcal{G}_n$. Using the special case above, we may assume that G has no connected components consisting of a vertex and a loop. Define a Brauer configuration $\Gamma(G) = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ representing an equivalence class in $\mathcal{B}_n$ as follows. The vertex set of $\Gamma(G)$ is

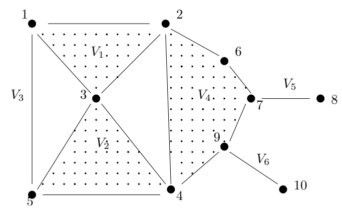
$$\Gamma(G)_0 = G_1 \cup \{(e, i) \mid i \text{ is a leaf and } e \text{ is the leaf-edge associated to } i\}.$$

The set of polygons $\{V_1, \ldots, V_n\}$ is in bijection with the set G_0 of vertices of G, where $V_i = \{e \in G_1 \mid i \text{ is an endpoint of } e\}$ if i is not a leaf and $V_i = \{e, (e,i)\}$ where i is a leaf and e is the leaf edge associated to i. If e is a loop at vertex i in G, then V_i contains e as an element once (since we are constructing Brauer configurations with no self-foldings; as an aside, Example 4.7 shows that if we allow self-foldings, there could be more than one Brauer configuration associated to a graph G). Note that the valency of a vertex in $\Gamma(G)$ is at most 2 since an edge in G has at most two endpoints. Therefore there are at most two polygons at any vertex of $\Gamma(G)$ and there is a unique cyclic order $\mathfrak{o}$. To clarify notation, if e is an edge in G then we will write $\bar{e}$ instead of e for the vertex in Γ. We define the multiplicity function as follows. If e is an edge of G and e is not a loop, then $\mu(\bar{e}) = 1$. If e is a loop, $\mu(\bar{e}) = 2$. Finally set $\mu(e,i) = 1$. We see that the vertices of $\Gamma(G)$ of the form (e,i) are precisely the truncated vertices of $\Gamma(G)$. Moreover, it is clear that $\Gamma(G)$ is ordered and reduced. Thus we have constructed a Brauer configuration $\Gamma(G)$. The reader may check that sending the equivalence class of G to the equivalence class of $\Gamma(G)$ is a well-defined map and inverse to the first map. $\square$

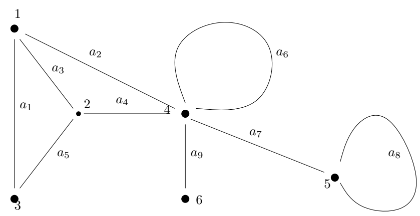
For a given graph G in an equivalence class of $\mathcal{G}_n$, call the Brauer configurations constructed in the proof of Proposition 4.3 above, the *Brauer configuration associated to the graph G* and denote it by $\Gamma(G)$.

Example 4.4. To help clarify the above proof we provide examples of both constructions. Namely in the first instance, given a Brauer configuration we construct a representative of the associated graph equivalence class. Secondly, given a graph, we construct a representative of the corresponding Brauer configuration.

a) Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ be a Brauer configuration where $\Gamma_0 = \{1, 2, \ldots, 10\}$, $\Gamma_1 = \{V_1, \ldots, V_6\}$, with $V_1 = \{1,2,3\}, V_2 = \{3,4,5\}, V_3 = \{1,5\}, V_4 = \{2,4,6,7,9\}, V_5 = \{7,8\}$, and $V_6 = \{9,10\}$, $\mu(8) = 2 = \mu(6)$, and $\mu(i) = 1$, for $i \neq 6, 8$. Note that since at each vertex of Γ the valency is ≤ 2, there is a unique orientation $\mathfrak{o}$. We further remark that neither vertices 6 nor 8 are truncated since their multiplicities are greater than 1, however, vertex 10 is truncated. A realization of Γ is given by

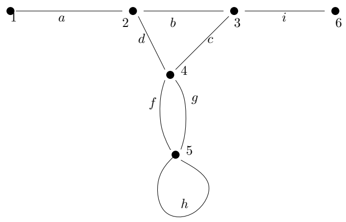


The reader may check that the equivalence class of the graph $G(\Gamma)$ as constructed in the theorem can be represented by

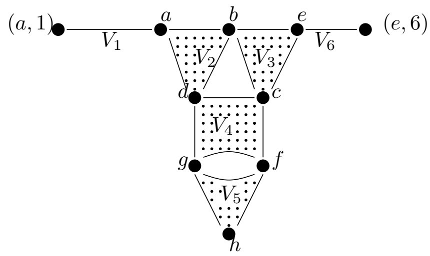


Note that the edges are labelled a_i, where i is the vertex in Γ yeilding a_i. Thus, since vertex 1 in Γ is in $V_1 \cap V_3$, we have an edge a_1 between 1 and 3. Similarly, for example, since vertices 6 and 8 in Γ has multiplicity 2, we get the loops a_6 and a_8. Note that since 10 is truncated in Γ, there is no a_{10} and vertex 6 of $G(\Gamma)$ is a leaf.

As a second example, we start with a graph G representing an equivalence class in $\mathcal{G}_6$



and construct the associated Brauer configuration, which is



Note that the two leaves at vertices 1 and 6 in G, result in two truncated vertices in $\Gamma(G)$; namely $(a, 1)$ and $(e, 6)$, respectively.

Given a graph G representing an equivalence class in $\mathcal{G}_n$, there is a standard construction of a radical cubed zero symmetric algebra which we now recall. We start by constructing a quiver, $\mathcal{Q}_G$, associated to G as follows. The vertex set of $\mathcal{Q}_G$ consists of n vertices labeled $\{1, \ldots, n\}$ and, for each edge e in G_1 that is not a loop, and such that e has endpoint vertices i and j in G_0, there are two arrows a and a' in $\mathcal{Q}_G$ such that a is an arrow from i to j and a' an an arrow from j to i. For each loop e at vertex i in G, the quiver $\mathcal{Q}_G$ has a loop b at vertex i.

We call aa', $a'a$, and b^2 *distinguished paths* in $\mathcal{Q}_G$.

Let I_G be the ideal in $K\mathcal{Q}_G$ generated by

(1) all $p - q$ where p and q are distinguished path with the same start and end vertices.
(2) all aba, for all arrows a, b in $\mathcal{Q}$ such that ab is a distinguished path.
(3) all non-distinguished paths of length 2.

Note that if G' is a graph equivalent to G then $\mathcal{Q}_G = \mathcal{Q}_{G'}$ and $I_G = I_{G'}$.

Keeping the notation above, we have the following result.

Lemma 4.5. *Let G be a graph representing an equivalence class in $\mathcal{G}_n$. Then $K\mathcal{Q}_G/I_G$ is a symmetric K-algebra with radical cubed zero.*

Proof. Let $\pi\colon K\mathcal{Q}_G \to K\mathcal{Q}_G/I_G$ be the canonical surjection. It is routine to check that $K\mathcal{Q}_G/I_G$ is a selfinjective algebra with radical cubed zero.

To see that $K\mathcal{Q}_G/I_G$ is symmetric, define $f\colon K\mathcal{Q}_G/I_G \to K$ as follows. If p is a path in $K\mathcal{Q}_G$, let

$$f(\pi(p)) = \begin{cases} 1, & \text{if } p \text{ is a distinguished path,} \\ 0, & \text{in all other cases.} \end{cases}$$

It is straightforward to show that the kernel of f contains no two-sided ideals and that $f(xy) = f(yx)$, for all $x, y \in K\mathcal{Q}_G/I_G$. $\square$

We call $A_G = K\mathcal{Q}_G/I_G$ the *canonical symmetric algebra with radical cubed zero associated to* G.

We now present the main result of this section.

Theorem 4.6. *Let G be a representative of an equivalence class in $\mathcal{G}_n$, let $\Gamma(G)$ be the Brauer configuration associated to G with corresponding Brauer configuration algebra $\Lambda_{\Gamma(G)}$ and let A_G be the canonical symmetric algebra with radical cubed zero associated to G. Then $\Lambda_{\Gamma(G)}$ and A_G are isomorphic as K-algebras.*

Proof. For ease of notation set $\Gamma = \Gamma(G)$, $\Lambda = \Lambda_{\Gamma(G)}$ and $A = A_G$. The quiver $\mathcal{Q}_\Lambda$ of Λ has vertex set $\{v_1, \ldots, v_n\}$ corresponding to the set of polygons $\{V_1, \ldots, V_n\}$. Recall that the vertex set Γ_0 of Γ is given by the edge set G_1 of G together with the truncated vertices (e, i), where i is a leaf and e is the leaf edge associated to i.

For the special case where G has one vertex and a loop at that vertex, both Λ and A are isomorphic to $K[x]/(x^3)$ since the Brauer configuration associated to G is given in the proof of Proposition 4.3 is a 2-gon with one vertex truncated and the other has multiplicity 2.

Thus we may assume that G has no connected component consisting of one vertex and a loop. Let $e \in \Gamma_0$ where e is an edge in G. To avoid confusion, we will write $\bar{e}$ when e is a vertex in Γ and e when e is an edge in G. Then, as an edge in G, e either is a loop or not. If e is a loop in G, then let i be the vertex in G_0 that is the endpoint of e. From our construction it then follows that, as a vertex in Γ_0, $\bar{e}$ has valence 1 in Γ, that $\bar{e} \in V_i$ and $\mu(\bar{e}) = 2$. But then in Γ, the successor sequence at $\bar{e}$ is V_i and it gives rise to a loop at v_i in $\mathcal{Q}_\Lambda$. On the other hand, if e is not a loop in G, then e has two endpoints i and j, with $1 \leq i \neq j \leq n$. It follows from our construction of Γ, that $\bar{e}$ as a vertex in Γ_0 is such that $\mathrm{val}(\bar{e}) = 2$ with $\bar{e} \in V_i$ and $\bar{e} \in V_j$. Furthermore, the successor sequence at $\bar{e}$ is $V_i < V_j$. It follows that $\bar{e}$ gives rise to two arrows in $\mathcal{Q}_\Lambda$, namely one arrow from i to j and one from j to i. Recall that the truncated vertices of Γ do not influence $\mathcal{Q}_\Lambda$.

Let $\mathcal{Q}_A$ be the quiver of the canonical symmetric algebra with radical cube zero A associated to G. It clearly follows from the construction of $\mathcal{Q}_\Lambda$ and $\mathcal{Q}_G$ that by sending v_i to i and an arrow from v_i to v_j to an arrow from i to j, we obtain an isomorphism $\mathcal{Q}_\Lambda \to \mathcal{Q}_G$. Let I_Γ be the ideal of relations of the Brauer configuration algebra associated to Γ and I_G be the ideal of relations of A. It easy to see that, under the isomorphism $\mathcal{Q}_\Lambda \to \mathcal{Q}_A$, the type one, two and three relations obtained from the Brauer configuration Γ map isomorphically to the relations (1), (2), and (3) for the canonical symmetric algebra with radical cube zero associated to G given above. This completes the proof.

□

One might ask if every radical cubed zero symmetric algebra is the canonical symmetric algebra with radical cube zero associated to a graph G. The answer is no as shown by the following example of a graph to which we can associate two non-isomorphic symmetric algebras with radical cube zero.

Example 4.7. We give an example of a graph G in $\mathcal{G}_1$ and two symmetric algebras associated to this graph, one of them being the canonical symmetric algebra with radical cube zero associated to G, the other one being a Brauer configuration algebra where the Brauer configuration has a self-folding. These two algebras are isomorphic if K contains an element i such that $i^2 = -1$ and they are non-isomorphic otherwise.

Let G be the graph with one vertex and two loops.

The canonical symmetric algebra associated to G is $A = K[x,y]/(xy, x^2 - y^2)$, where K is a field and $K[x,y]$ is the commutative polynomial ring in two variables. A Brauer configuration such that the associated Brauer configuration algebra is isomorphic to A is $\Gamma = (\{\alpha,\beta\}, V = \{\alpha,\beta\}, \mu, \mathfrak{o})$ with $\mu(\alpha) = \mu(\beta) = 2$. In fact, this corresponds to the Brauer graph algebra associated to the Brauer graph given by a single edge where both vertices have multiplicity two.

Now consider the algebra $A' = K[x,y]/(x^2, y^2)$. Then A' is a symmetric radical cubed zero algebra. The algebra A' is isomorphic to the Brauer configuration algebra associated to the Brauer configuration $\Gamma' = (\{\alpha\}, V = \{\alpha,\alpha\}, \mu, \mathfrak{o})$ with $\mu(\alpha) = 1$. Remark that Γ' has a self-folding. The algebra A' also corresponds to a Brauer graph algebra. Its Brauer graph is given by a loop with multiplicity one at the only vertex.

Note that the matrix associated to both A and A' is the 1×1 matrix (2).

Now suppose that $i \in K$ with $i^2 = -1$. Then the map defined by $x \mapsto x + iy, y \mapsto x - iy$ induces an isomorphism from A' to A. If, however, K does not contain an element squaring to -1 then the algebras A and A' defined above are not isomorphic.

Example 4.7 leads to the question: What can one say about an indecomposable symmetric radical cubed zero algebra of the form $K\mathcal{Q}/I$. This question is addressed in [GS] where it is shown that every such algebra is a Brauer configuration algebra, when one allows self-foldings in the Brauer configuration.

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Edward L. Green, Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA

E-mail address: green@math.vt.edu

Sibylle Schroll, Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom

E-mail address: schroll@le.ac.uk