

# A suggestion for constructing a large time-varying conditional covariance matrix<sup>\*</sup>

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## ABSTRACT

The construction of large conditional matrices has posed a problem in the empirical literature because the direct extension of the univariate GARCH model to a multivariate setting produces large numbers of parameters to be estimated as the number of equations rises. A number of procedures have previously aimed to simplify the model and restrict the number of parameters, but these procedures typically involve either invalid or undesirable restrictions. This paper suggests an alternative way forward, based on the GARCH approach, which allows conditional covariance matrices of unlimited size to be constructed. The procedure is computationally straightforward to implement. At no point in the procedure is it necessary to estimate anything other than a univariate GARCH model.

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## **1. Introduction**

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Finance theory provides many areas where the calculation of a large conditional covariance matrix is called for. Examples include optimal portfolio allocations of the Markowitz type, value at risk calculations for large numbers of assets and capital asset pricing calculations for a large number of assets. It is well-recognized that in empirical work these matrices should be measured by a *conditional* covariance matrix, since this is the true measure of risk, and that, in general, these conditional covariance matrices should be time-varying. The natural econometric analogue used to measure such a time-varying conditional covariance matrix is the GARCH family of models, developed from Engle (1982), and a direct multivariate extension of the GARCH model, developed by Bollerslev (1986). However, the empirical application of multivariate GARCH presents a problem of dimensionality because standard multivariate GARCH models quickly generate very large numbers of parameters as the number of variables in the system increases. For example, consider the following system:

$$\begin{aligned} Y_t &= \beta_t X + \varepsilon_t \quad \varepsilon_t \sim N(0, \Sigma_t) \\ \text{vech}(\Sigma_t) &= W + A \text{vech}(\varepsilon_{t-1} \varepsilon'_{t-1}) + B \text{vech}(\Sigma_{t-1}) \end{aligned} \quad (1)$$

where  $Y$  is a vector of  $n$  endogenous variable,  $X$  is a vector of suitable exogenous variables,  $\varepsilon_t$  is an innovation vector,  $\Sigma$  a conditional variance-covariance matrix, and we have limited the model to a system GARCH (1,1) specification. This model is a direct generalization of the standard univariate GARCH model, but it is intractable for anything other than a very small number of variables. For example, if the number of variables in the system were 5, the model would require estimation of 465 parameters in the  $W$ ,  $A$  and  $B$  matrices; the number of parameters grows exponentially with  $n$ , the number of variables in the model.

There are a number of ways to reduce this problem of dimensionality, but none of them is entirely satisfactory. It is possible to make the  $A$  and  $B$  matrices diagonal, but this effectively eliminates the interaction between the covariances and severely limits their time-variation. A popular model, based on the work of Boda, Engle, Kraft and Kroner (1990), known as the BEKK model, is given below:

$$\Sigma_t = V'V + A' \varepsilon_{t-1} \varepsilon'_{t-1} A + B' \Sigma_{t-1} B \quad (2)$$

where  $A$  and  $B$  are matrices of parameters and  $V$  is an upper triangular matrix of parameters.

This model allows fairly complex interactions between the covariances and also ensures that the covariance matrix is positive definite, but it nevertheless involves a large number of parameters as  $n$  rises. For a system where  $n=5$ , the model contains 75 parameters in the variance equation, and the number again grows rapidly as  $n$  rises.

Another alternative would be to use *factor* GARCH models; the assumption here is that there are only a small number of factors underlying the variables being modelled, allowing a much more parsimonious formulation of the model. However, factor GARCH models limit the amount of time-variation in the covariances; also, the assumption of a small number of factors may also be questioned. A further commonly-used approach is the constant conditional correlation model in which the time-varying conditional covariances are parameterized to be proportional to the product of the corresponding conditional standard deviations. This condition, however, restricts precisely the part of the model we are most interested in, and, where this has been tested, this assumption is almost invariably rejected.

There is, therefore, a need for a technique that will allow estimation of large conditional covariance matrices efficiently and without the restrictions imposed by the above methods. In Section 2, we provide such a method based on a variant of this technique recently used to derive a measure of systemic risk for the European banking system (Gibson, Hall and Tavlás, 2016). Section 3 presents a simple Monte Carlo illustration of the method and performs an example based on some real world data. Section 4 concludes.

## **2. A Suggested Approach**

Our suggestion rests on a simple relationship between the variance of two variables and the variance of the sum of those two variables. Considering two variables  $x$  and  $y$ , then

$$E(x_t + y_t)^2 = E[(x_t + y_t)(x_t + y_t)] = E(x_t^2) + E(y_t^2) + 2E(x_t y_t) \quad (3)$$

or in conditional variance terms:

$$\text{Var}(x_t + y_t | \Omega_t) = \text{Var}(x_t | \Omega_t) + \text{Var}(y_t | \Omega_t) + 2\text{Cov}(x_t y_t | \Omega_t) \quad (4)$$

To estimate the conditional covariance between any two variables, we may estimate a univariate GARCH model for each of the variables and then estimate a further univariate GARCH model for the sum of the variables. Then, we simply calculate the covariance as:

$$\text{Cov}(x_t y_t | \Omega_t) = [\text{Var}(x_t + y_t | \Omega_t) - \text{Var}(x_t | \Omega_t) - \text{Var}(y_t | \Omega_t)] / 2 \quad (5)$$

Specifically, our approach consists of the following steps. (1) We add  $x$  and  $y$  and we run a GARCH on that sum, obtaining the variance of the sum of  $x$  and  $y$ . (2) We run a GARCH on  $x$  and  $y$  separately. Those two steps provide all three terms on the right-hand-side of equation (5) so that we have worked-out the covariance matrix. Note that this procedure allows us to derive all of the covariance elements of a large covariance matrix.

If each of the conditional variance estimates is consistent, then our estimate of the covariance will also be a consistent estimate. The theoretical properties of standard GARCH models has been extensively discussed in the literature and the conditions under which they are consistent is well established -- see Weiss (1986). The application in (5) is, however, a little unusual as it involves the two variables  $x$  and  $y$ , which are conventional GARCH processes and then the sum of these two variables ( $x+y$ ). This unusual case has been studied by Nijman and Sentana (1996) and Komunjer (2001), who term the individual models for  $x$  and  $y$  as 'strong' GARCH processes and the summation variable ( $x+y$ ) as a 'weak' GARCH process. Komunjer shows that the conditions for consistency of a weak GARCH process are not the same as a strong one; details of the required assumptions for consistency are given in Komunjer (2001). An important point to note here, however, is that the main focuses of these proofs is on the parameters of the GARCH process being consistently estimated. If they are consistent then the estimate of the conditional variance is also consistent. Even in the case when the parameters are not consistent, it is still possible that the conditional variance is consistently estimated as long as sufficient lags are included in the GARCH specification; this follows from the Wold decomposition.

### **3. Empirical application**

### 3.1 Monte Carlo experiments

In order to test the validity of the proposed procedure, we have conducted a number of Monte Carlo experiments. All experiments are carried out for a sample of 1,000 observations and with 1,000 replications. In the first set of experiments, we consider the case of two variables that are normally distributed  $N(0,1)$  with constant covariance ranging from -0.95 to 1. In the second case we consider two variables that have a changing covariance of different forms; first, we consider a covariance which changes according to a linear trend from -0.8 to +0.8; second we consider a covariance which follows a sine wave pattern, varying between -0.8 and +0.8 every 50 periods. Finally, we consider a model in which the covariance is generated by a GARCH like process  $cov_t = 0.6 cov_{t-1} + 0.1 e_t$  where  $e_t$  is distributed  $N(0,1)$ .

Table 1: Monte Carlo results for a constant covariance

True value	-0.95	-.75	-.5	-.25	0	0.25	0.5	0.75	1
estimated	-0.98	-.751	-.5	-.26	-0.00005	0.258	0.52	0.8	1

Table 1 shows the Monte Carlo results for the case of a constant covariance where the true covariance varies between -0.95 and 1. The results show that in every case the average estimate of the covariance over the 1,000 replications is virtually the same as the true covariance. In Table 2, we show the results for the three cases of a time-varying covariance, that is, the linearly increasing covariance, the deterministic cyclical covariance and the GARCH style stochastic covariance. It is worth pointing out that the GARCH model is actually misspecified in the case of the first two cases. In the trended case a GARCH process can approximately capture this; in the cyclical case the GARCH model is seriously misspecified as it cannot generate cycles of this kind. In the final case of a GARCH-style changing covariance, the GARCH model is, of course, well suited to the DGP. The numbers reported in Table 2 are the mean discrepancy between the DGP covariance and the covariance, actually estimated by our procedure for every period and then averaged over the 1,000 replications. The numbers are average differences in order to give some idea of the scale. We can think of these numbers as percentages of the true variance for the two variables, as this is set to one. The linear trend covariance has an average error of 2.8% of the variance while the cyclical case has an average error of -4%. As we would expect, the

best performing case is where the GARCH model is well specified and in this case the average error is only 0.05% of the variance. The estimation process seems to perform well even when the model is misspecified, but it performs very well in the case of the GARCH style DGP.

Table 2: Monte Carlo results for a changing covariance of different structures.

Linear trend covariance	Cyclical covariance	GARCH covariance
-0.028	-0.04	0.0005

### ***3.2 An example using UK banks' stock market valuation***

Here we illustrate the proposed technique by estimating the covariance matrix for the total stock market valuation of five large UK banks using daily data over the period March 1, 2000 to November 4, 2016<sup>1</sup>. We are then able to calculate the full conditional covariance matrix among these five banks. As discussed above, the technique may be easily extended to any number of banks. Figures 1 and 2 show the conditional variances and covariances for the five banks.

These two figures illustrate that the technique works but of course they do not show that it provides the correct answer. To cast further light on the technique, we estimate a bivariate GARCH model for the first two banks based on the general GARCH structure (1) above. This is a very small system. Nevertheless, it took over 100 iterations to converge and around 30 seconds of computer time. Larger systems become impossible to estimate very quickly. Figure 3 shows the conditional covariance of the first two banks derived from the system GARCH compared with the results from our new technique. GARCH\_01\_02 is the estimate of the conditional covariance derived from the system estimation, VUK12 is the estimate derived by our suggested approach. As shown in Figure 3, the results are very similar, and the correlation coefficient between the two is 0.9446. Figure 4 shows an analysis of the percentage difference between the two estimates. The average percentage difference is around 7 per cent, the distribution is quite symmetrical. The main difference

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<sup>1</sup> The data are from Datastream (Thomson Reuters). The banks are HSBC, Lloyds Banking Group, Barclays, Royal Bank of Scotland and Standard Chartered.

between the two measures seems to come at the points where the covariance peaks, and here our approach seems to produce larger peaks than the bivariate GARCH technique.

#### **4. Conclusions**

We have proposed a new way of using the GARCH family of models to estimate large conditional covariance structures. This approach has no natural limitation to the size of the covariance matrix that can be estimated. We have provided two simple Monte Carlo experiments that confirm the performance of the technique. We have also compared the results with those from a standard small multivariate model and found the results to be essentially identical.

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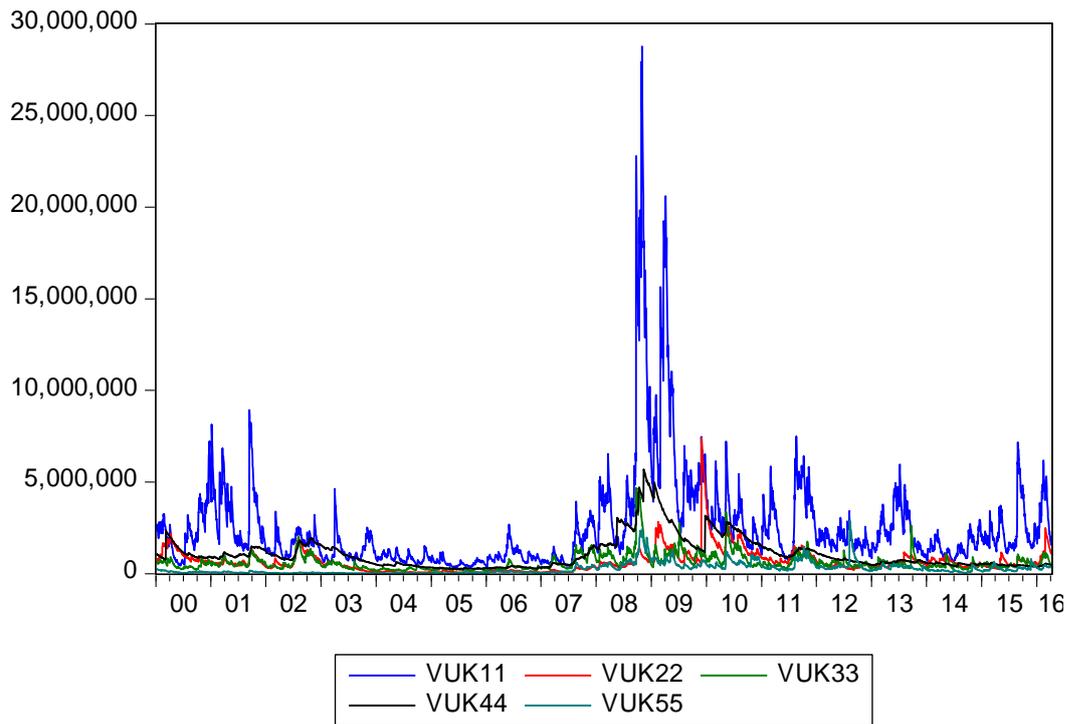


Figure 1: Conditional Variances

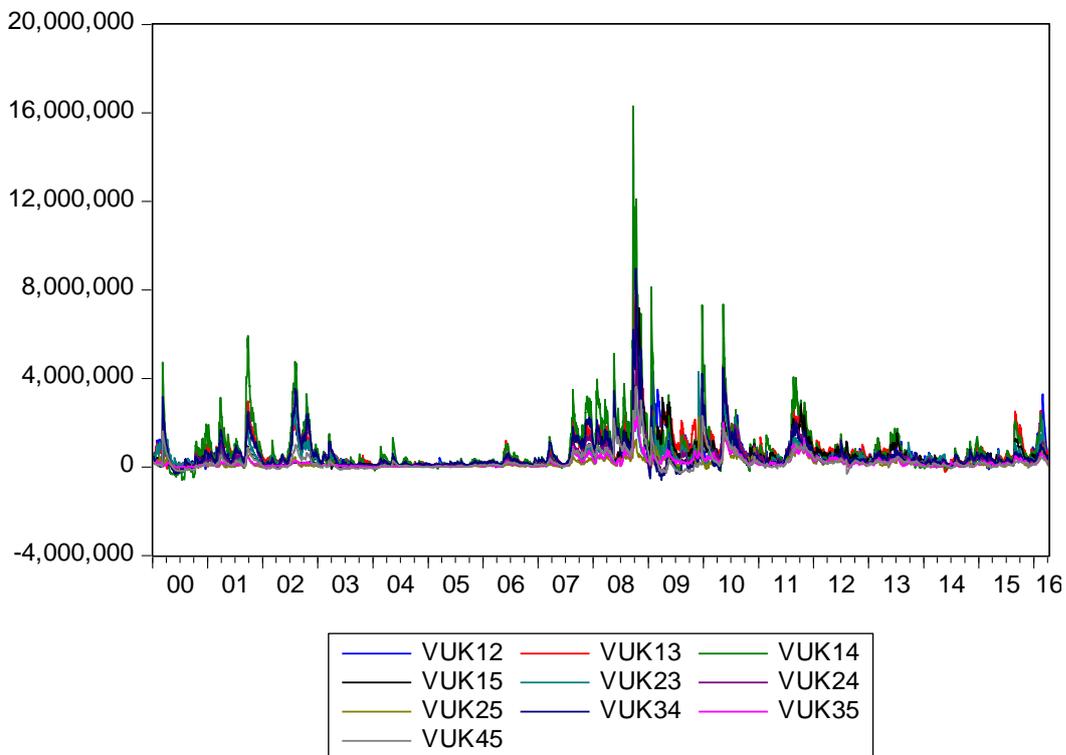


Figure 2: Conditional Covariances

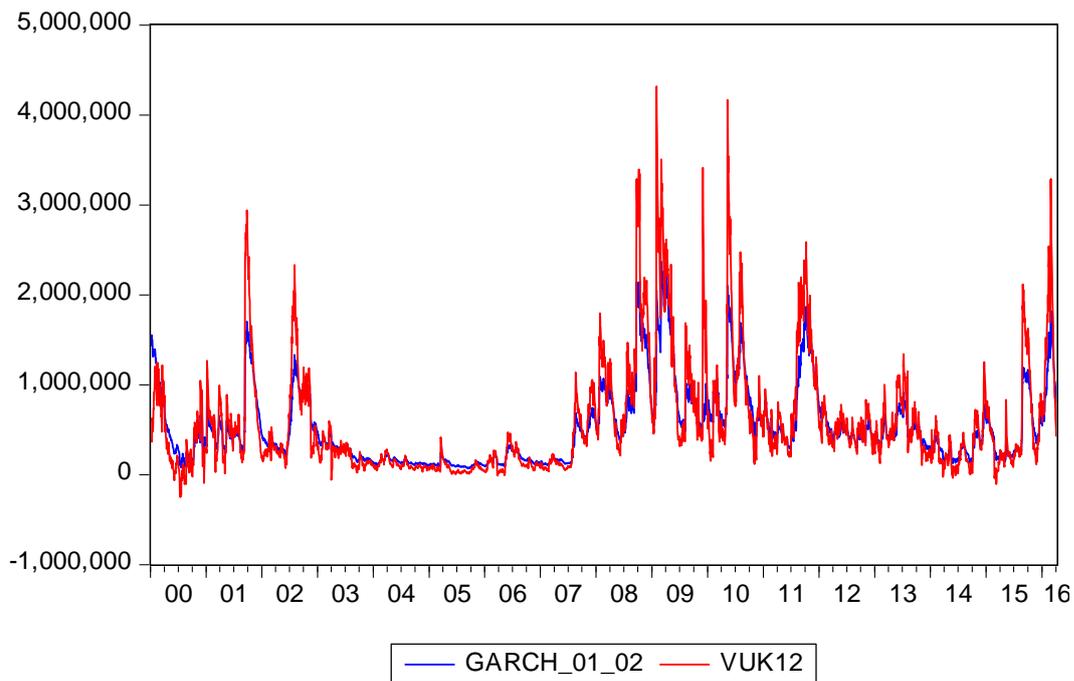


Figure 3: A comparison between a system GARCH and the new suggestion

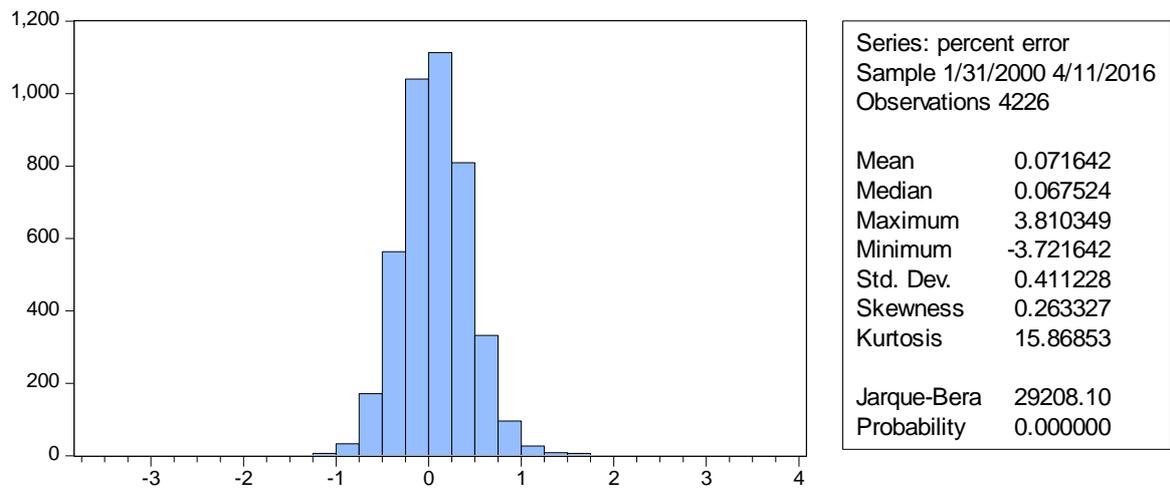


Figure 4: Analysis of the percentage difference between the system GARCH covariance and the new suggestion