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R-matrix and Mickelsson algebras for orthosymplectic quantum groups

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Let $\mathfrak{g}$ be a complex orthogonal or symplectic Lie algebra and $\mathfrak{g}' \subset \mathfrak{g}$ the Lie subalgebra of rank $\mathrm{rk}\,\mathfrak{g}' = \mathrm{rk}\,\mathfrak{g} - 1$ of the same type. We give an explicit construction of generators of the Mickelsson algebra $Z_q(\mathfrak{g}, \mathfrak{g}')$ in terms of Chevalley generators via the R-matrix of $U_q(\mathfrak{g})$. *© 2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4927582]

I. INTRODUCTION

In the mathematics literature, lowering and raising operators are known as generators of step algebras, which were originally introduced by Mickelsson[1] for reductive pairs of Lie algebras, $\mathfrak{g}' \subset \mathfrak{g}$. These algebras naturally act on $\mathfrak{g}'$-singular vectors in $U(\mathfrak{g})$-modules and are important in representation theory.[2,3]

The general theory of step algebras for classical universal enveloping algebras was developed in Refs. 2 and 4 and extended to the special linear and orthogonal quantum groups in Ref. 5. They admit a natural description in terms of extremal projectors,[4] introduced for classical groups in Refs. 6 and 7 and generalized to the quantum group case in Refs. 8 and 9. It is known that the step algebra $Z(\mathfrak{g}, \mathfrak{g}')$ is generated by the image of the orthogonal complement $\mathfrak{g} \ominus \mathfrak{g}'$ under the extremal projector of the $\mathfrak{g}'$. Another description of lowering/raising operators for classical groups was obtained in Refs. 10, 11, and 3 in an explicit form of polynomials in $\mathfrak{g}$.

A generalization of the results of Refs. 10 and 11 to quantum $\mathfrak{gl}(n)$ can be found in Ref. 12. In this special case, the lowering operators can be also conveniently expressed through "modified commutators" in the Chevalley generators of $U(\mathfrak{g})$ with coefficients in the field of fractions of $U(\mathfrak{h})$. Extending[3] to orthogonal and symplectic quantum groups is not straightforward, since there are no nilpotent triangular Lie subalgebras $\mathfrak{g}_\pm$ in $U_q(\mathfrak{g})$ but only their deformed associative envelope. We suggest such a generalization, where the lack of $\mathfrak{g}_\pm$ is compensated by the entries of the universal R-matrix with one leg projected to the natural representation. Those entries are nicely expressed through modified commutators in the Chevalley generators turning into elements of $\mathfrak{g}_\pm$ in the quasi-classical limit. Their commutation relation with the Chevalley generators modifies the classical commutation relations with $\mathfrak{g}_\pm$ in a tractable way. This enabled us to generalize the results of Refs. 10, 11, and 3 and construct generators of Mickelsson algebras for the non-exceptional quantum groups. Explicit form of these generators is useful in quantization of conjugacy classes, because they are related to singular vectors generating certain submodules involved.[13,14]

A. Quantized universal enveloping algebra

In this paper, $\mathfrak{g}$ is a complex simple Lie algebra of type B, C, or D. The case of $\mathfrak{gl}(n)$ can be easily derived from here due to the natural inclusion $U_q\big(\mathfrak{gl}(n)\big) \subset U_q(\mathfrak{g})$, so we do not pay special attention to it. We choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the inner product $(.,.)$ on $\mathfrak{h}^*$ normalized to the unit length of the highest weight of the natural representation. By R, we denote the root system of $\mathfrak{g}$ with a fixed subsystem of positive roots $\mathrm{R}^+ \subset \mathrm{R}$ and the basis of simple roots $\Pi^+ \subset \mathrm{R}^+$. For

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every $\lambda \in \mathfrak{h}^*$, we denote by h_λ its image under the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$, that is $(\lambda, \beta) = \beta(h_\lambda)$ for all $\beta \in \mathfrak{h}^*$. We put $\rho = \frac{1}{2}\sum_{\alpha \in \mathrm{R}^+} \alpha$ for the Weyl vector.

Suppose that $q \in \mathbb{C}$ is not a root of unity. Denote by $U_q(\mathfrak{g}_\pm)$ the $\mathbb{C}$-algebra generated by $e_{\pm\alpha}$, $\alpha \in \Pi^+$, subject to the q-Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_{\alpha_i}} e_{\pm\alpha_i}^{1-a_{ij}-k} e_{\pm\alpha_j} e_{\pm\alpha_i}^k = 0,$$

where $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, $i,j = 1,\dots,n = \mathrm{rk}\,\mathfrak{g}$, is the Cartan matrix, $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$, and

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]_q!}{[k]_q![m-k]_q!}, \quad [m]_q! = [1]_q \cdot [2]_q \cdots [m]_q.$$

Here, and further on, $[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$ whenever $q^{\pm z}$ make sense.

Denote by $U_q(\mathfrak{h})$ the commutative $\mathbb{C}$-algebra generated by $q^{\pm h_\alpha}$, $\alpha \in \Pi^+$. The quantum group $U_q(\mathfrak{g})$ is a $\mathbb{C}$-algebra generated by $U_q(\mathfrak{g}_\pm)$ and $U_q(\mathfrak{h})$ subject to the relations

$$q^{h_\alpha} e_{\pm\beta} q^{-h_\alpha} = q^{\pm(\alpha,\beta)} e_{\pm\beta}, \quad [e_\alpha, e_{-\beta}] = \delta_{\alpha,\beta} \frac{q^{h_\alpha} - q^{-h_\alpha}}{q_\alpha - q_\alpha^{-1}}.$$

Although $\mathfrak{h}$ is not contained in $U_q(\mathfrak{g})$, still it is convenient for us to keep reference to $\mathfrak{h}$.

Fix the comultiplication in $U_q(\mathfrak{g})$ as in Ref. 15,

$$\Delta(e_\alpha) = e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha, \quad \Delta(e_{-\alpha}) = e_{-\alpha} \otimes 1 + q^{-h_\alpha} \otimes e_{-\alpha},$$
$$\Delta(q^{\pm h_\alpha}) = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha},$$

for all $\alpha \in \Pi^+$.

The subalgebras $U_q(\mathfrak{b}_\pm) \subset U_q(\mathfrak{g})$ generated by $U_q(\mathfrak{g}_\pm)$ over $U_q(\mathfrak{h})$ are quantized universal enveloping algebras of the Borel subalgebras $\mathfrak{b}_\pm = \mathfrak{h} + \mathfrak{g}_\pm \subset \mathfrak{g}$.

The Chevalley generators e_α can be extended to a set of composite root vectors e_β for all $\beta \in \mathrm{R}$. A normally ordered set of root vectors generate a Poincaré-Birkhoff-Witt (PBW) basis of $U_q(\mathfrak{g})$ over $U_q(\mathfrak{h})$.[15] We will use $\mathfrak{g}_\pm$ to denote the vector space spanned by $\{e_{\pm\beta}\}_{\beta\in\mathrm{R}^+}$.

The universal R-matrix is an element of a certain extension of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. We heavily use the intertwining relation

$$\mathcal{R}\Delta(x) = \Delta^{op}(x)\mathcal{R}, \tag{1.1}$$

between the coproduct and its opposite for all $x \in U_q(\mathfrak{g})$. Let $\{\varepsilon_i\}_{i=1}^n \subset \mathfrak{h}^*$ be the standard orthonormal basis and $\{h_{\varepsilon_i}\}_{i=1}^n$ the corresponding dual basis in $\mathfrak{h}$. The exact expression for $\mathcal{R}$ can be extracted from Ref. 15, Theorem 8.3.9, as the ordered product

$$\mathcal{R} = q^{\sum_{i=1}^n h_{\varepsilon_i} \otimes h_{\varepsilon_i}} \prod_\beta \exp_{q_\beta}\{(1 - q_\beta^{-2})(e_\beta \otimes e_{-\beta})\} \in U_q(\mathfrak{b}_+)\hat{\otimes} U_q(\mathfrak{b}_-), \tag{1.2}$$

where $\exp_q(x) = \sum_{k=0}^\infty q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!}$.

We use the notation $e_i = e_{\alpha_i}$ and $f_i = e_{-\alpha_i}$ for $\alpha_i \in \Pi^+$, in all cases apart from $i = n$, $\mathfrak{g} = \mathfrak{so}(2n+1)$, where we set $f_n = [\frac{1}{2}]_q e_{-\alpha_n}$. The reason for this is twofold. First, the natural representation can be defined through the classical assignment on the generators, as given below. Second, we get rid of $q_{\alpha_n} = q^{\frac{1}{2}}$ and can work over $\mathbb{C}[q]$, as the relations involved turn into

$$[e_n, f_n] = \frac{q^{h_{\alpha_n}} - q^{-h_{\alpha_n}}}{q - q^{-1}},$$
$$f_n^3 f_{n-1} - (q + 1 + q^{-1}) f_n^2 f_{n-1} f_n + (q + 1 + q^{-1}) f_n f_{n-1} f_n^2 - f_{n-1} f_n^3 = 0.$$

It is easy to see that the square root of q disappears from the corresponding factor in presentation (1.2).

In what follows, we regard $\mathfrak{gl}(n) \subset \mathfrak{g}$ to be the Lie subalgebra with the simple roots $\{\alpha_i\}_{i=1}^{n-1}$ and $U_q\big(\mathfrak{gl}(n)\big)$ the corresponding quantum subgroup in $U_q(\mathfrak{g})$.

Consider the natural representation of $\mathfrak{g}$ in the vector space $\mathbb{C}^N$. We use the notation $i' = N + 1 - i$ for all integers $i = 1, \dots, N$. The assignment

$$\pi(e_i) = e_{i,i+1} \pm e_{i'-1,i'}, \quad \pi(f_i) = e_{i+1,i} \pm e_{i',i'-1}, \quad \pi(h_{\alpha_i}) = e_{ii} - e_{i+1,i+1} + e_{i'-1,i'-1} - e_{i'i'},$$

for $i = 1, \dots, n-1$, defines a direct sum of two representations of $\mathfrak{gl}(n)$ for each sign. It extends to the natural representation of the whole $\mathfrak{g}$ by

$$\pi(e_n) = e_{n,n+1} \pm e_{n'-1,n'}, \quad \pi(f_n) = e_{n+1,n} \pm e_{n',n'-1}, \quad \pi(h_{\alpha_n}) = e_{nn} - e_{n'n'},$$
$$\pi(e_n) = e_{nn'}, \quad \pi(f_n) = e_{n'n}, \quad \pi(h_{\alpha_n}) = 2e_{nn} - 2e_{n'n'},$$
$$\pi(e_n) = e_{n-1,n'} \pm e_{n,n'+1}, \quad \pi(f_n) = e_{n',n-1} \pm e_{n'+1,n}, \quad \pi(h_{\alpha_n}) = e_{n-1,n-1} + e_{nn} - e_{n'n'} - e_{n'+1,n'+1},$$

respectively, for $\mathfrak{g} = \mathfrak{so}(2n+1)$, $\mathfrak{g} = \mathfrak{sp}(2n)$, and $\mathfrak{g} = \mathfrak{so}(2n)$.

Two values of the sign give equivalent representations. The choice of minus corresponds to the standard representation that preserves the bilinear form with entries $C_{ij} = \delta_{i'j}$, for $\mathfrak{g} = \mathfrak{so}(N)$, and $C_{ij} = \mathrm{sign}(\frac{N+1}{2} - i)\delta_{i'j}$, for $\mathfrak{g} = \mathfrak{sp}(N)$. However, we fix the sign to + in order to simplify calculations. The above assignment also defines representations of $U_q(\mathfrak{g})$.

II. *R*-MATRIX OF NON-EXCEPTIONAL QUANTUM GROUPS

Define $\check{\mathcal{R}} = q^{-\sum_{i=1}^n h_{\varepsilon_i} \otimes h_{\varepsilon_i}} \mathcal{R}$. Denote by $\check{R}^- = (\pi \otimes \mathrm{id})(\check{\mathcal{R}}) \in \mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{g}_-)$ and by $\check{R}^+ = (\pi \otimes \mathrm{id})(\check{\mathcal{R}}_{21}) \in \mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{g}_+)$. In this section, we deal only with $\check{R}^-$ and suppress the label "–" for simplicity, $\check{R} = \check{R}^-$.

Denote by N_+ the ring of all upper triangular matrices in $\mathrm{End}(\mathbb{C}^N)$ and by N'_+ its ideal spanned by e_{ij}, $i < j + 1$.

Lemma 2.1. One has

$$\check{R} = 1 \otimes 1 + (q^{1+\delta_{1n}} - q^{-1-\delta_{1n}}) \sum_{i=1}^n \pi(e_i) \otimes f_i \mod N'_+ \otimes U_q(\mathfrak{g}_-),$$

where δ_{1n} is present only for $\mathfrak{g} = \mathfrak{sp}(2n)$.

Proof. For all positive roots α, β, the matrix $\pi(e_\alpha e_\beta)$ belongs to N'_+. Also, $\pi(e_\beta) \in N'_+$ for all $\beta \in \mathrm{R}^+ \backslash \Pi^+$. Therefore, the only terms that contribute to $\mathrm{Span}_{\varepsilon_i - \varepsilon_j \in \Pi^+}\{e_{ij} \otimes U_q(\mathfrak{g}_-)\}$ are those of degree 1 from the series $\exp_{q_\alpha}(1 - q_\alpha^{-2})(e_\alpha \otimes e_{-\alpha})$ with $\alpha \in \Pi^+$. □

Write $\check{R} = \sum_{i,j=1}^N e_{ij} \otimes \check{R}_{ij}$, where $\check{R}_{ij} = 0$ for $i > j$. Due to the $\mathfrak{h}$-invariance of $\check{R}$, the entry $\check{R}_{ij} \in U_q(\mathfrak{g}_-)$ carries weight $\varepsilon_j - \varepsilon_i$.

For all $\mathfrak{g}$, we have $f_{k,k+1} = f_k = f_{k'-1,k'}$ once $k < n$ and $f_{n,n+1} = f_n = f_{n+1,n'}$ for $\mathfrak{g} = \mathfrak{so}(2n+1)$, $f_{n-1,n'} = f_n = f_{n,n'+1}$ for $\mathfrak{g} = \mathfrak{so}(2n)$, and $f_{nn'} = [2]_q f_n$ for $\mathfrak{g} = \mathfrak{sp}(2n)$. We present explicit expressions for the entries f_{ij} in terms of modified commutators in Chevalley generators, $[x, y]_a = xy - ayx$, where a is a scalar; we also put $\bar{q} = q^{-1}$.

Proposition 2.2. Suppose that $\varepsilon_i - \varepsilon_j \in \mathrm{R}^+ \backslash \Pi^+$. Then, the elements f_{ij} are given by the following formulas:

For all $\mathfrak{g}$ and $i + 1 < j \leqslant \frac{N+1}{2}$,

$$f_{ij} = [f_{j-1}, \dots [f_{i+1}, f_i]_{\bar{q}} \dots]_{\bar{q}}, \quad f_{j'i'} = [\dots [f_i, f_{i+1}]_{\bar{q}}, \dots f_{j-1}]_{\bar{q}}. \tag{2.1}$$

Furthermore,

- *For $\mathfrak{g} = \mathfrak{so}(2n+1)$: $f_{nn'} = (q-1) f_n^2$ and*

$$f_{i,n+1} = [f_n, f_{i,n}]_{\bar{q}}, \quad f_{n+1,i'} = [f_{n',i'}, f_n]_{\bar{q}}, \quad i < n,$$
$$f_{ij'} = q^{\delta_{ij}} [f_{n+1,j'}, f_{i,n+1}]_{\bar{q}^{\delta_{ij}}}, \quad i, j < n.$$

- *For* $\mathfrak{g} = \mathfrak{sp}(2n)$*:* $f_{nn'} = [2]_q f_n$ *and*

$$f_{in'} = [f_n, f_{in}]_{\bar{q}^2}, \quad f_{ni'} = [f_{n'i'}, f_n]_{\bar{q}^2}, \quad i < n,$$
$$f_{ij'} = q^{\delta_{ij}}[f_{nj'}, f_{in}]_{\bar{q}^{1+\delta_{ij}}}, \quad i, j < n.$$

- *For* $\mathfrak{g} = \mathfrak{so}(2n)$*:* $f_{nn'} = 0$ *and*

$$f_{in'} = [f_n, f_{i,n-1}]_{\bar{q}}, \quad f_{ni'} = [f_{n'+1,i'}, f_n]_{\bar{q}}, \quad i < n-2,$$
$$f_{ji'} = q^{\delta_{ij}}[f_{ni'}, f_{j,n}]_{\bar{q}^{1+\delta_{ij}}}, \quad i, j \leqslant n-1.$$

Proof. The proof is a direct calculation with the use of the identity

$$(f_\alpha \otimes 1)\check{\mathcal{R}} - \check{\mathcal{R}}(f_\alpha \otimes 1) = \check{\mathcal{R}}(q^{-h_\alpha} \otimes f_\alpha) - (q^{h_\alpha} \otimes f_\alpha)\check{\mathcal{R}},$$

which follows from intertwining axiom (1.1) for $x = f_\alpha$. This allows us to construct the elements f_{ij} by induction starting from f_α, $\alpha \in \Pi^+$. □

For each $\alpha \in \Pi^+$, denote by $P(\alpha)$ the set of ordered pairs $l, r = 1, \ldots, N$, with $\varepsilon_l - \varepsilon_r = \alpha$. We call such pairs simple.

Proposition 2.3. The matrix entries $f_{i,j} \in U_q(\mathfrak{g}_-)$ *such that* $\varepsilon_i - \varepsilon_j \notin \Pi^+$ *satisfy the identity*

$$[e_\alpha, f_{ij}] = \sum_{(l,r)\in P(\alpha)} \left(f_{il}\delta_{jr}q^{h_\alpha} - q^{-h_\alpha}\delta_{il}f_{rj}\right),$$

for all simple positive roots α.

Proof. The proof is a straightforward calculation based on intertwining relation (1.1), which is equivalent to

$$(1 \otimes e_\alpha)\check{\mathcal{R}} - \check{\mathcal{R}}(1 \otimes e_\alpha) = \check{\mathcal{R}}(e_\alpha \otimes q^{h_\alpha}) - (e_\alpha \otimes q^{-h_\alpha})\check{\mathcal{R}},$$

for $x = e_\alpha$, $\alpha \in \Pi^+$. Alternatively, one can use the expressions for f_{ij} from Proposition 2.2. □

III. MICKELSSON ALGEBRAS

Consider the Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ corresponding to the root subsystem $\mathrm{R}_{\mathfrak{g}'} \subset \mathrm{R}_{\mathfrak{g}}$ generated by α_i, $i > 1$, and let $\mathfrak{h}' \subset \mathfrak{g}'$ denote its Cartan subalgebra. Let the triangular decomposition $\mathfrak{g}'_- \oplus \mathfrak{h}' \oplus \mathfrak{g}'_+$ be compatible with the triangular decomposition of $\mathfrak{g}$. Recall the definition of step algebra $Z_q(\mathfrak{g}, \mathfrak{g}')$ of the pair $(\mathfrak{g}, \mathfrak{g}')$. Consider the left ideal $J = U_q(\mathfrak{g})\mathfrak{g}'_+$ and its normalizer $\mathcal{N} = \{x \in U_q(\mathfrak{g}) : e_\alpha x \subset J, \forall \alpha \in \Pi^+_{\mathfrak{g}'}\}$. By construction, J is a two-sided ideal in the algebra $\mathcal{N}$. Then, $Z_q(\mathfrak{g}, \mathfrak{g}')$ is the quotient $\mathcal{N}/J$.

For all $\beta_i \in \mathrm{R}^+_{\mathfrak{g}} \backslash \mathrm{R}^+_{\mathfrak{g}'}$ let e_{β_i} be the corresponding PBW generators and let Z be the vector space spanned by $e^{k_l}_{-\beta_l} \ldots e^{k_1}_{-\beta_1} e^{k_0}_0 e^{m_1}_{\beta_1} \ldots e^{m_l}_{\beta_l}$, where $e_0 = q^{h_{\alpha_1}}$, $k_i \in \mathbb{Z}_+$, and $k_0 \in \mathbb{Z}$. The PBW factorization $U_q(\mathfrak{g}) = U_q(\mathfrak{g}'_-) Z U_q(\mathfrak{h}') U_q(\mathfrak{g}'_+)$ gives rise to the decomposition

$$U_q(\mathfrak{g}) = Z U_q(\mathfrak{h}') \oplus (\mathfrak{g}'_- U_q(\mathfrak{g}) + U_q(\mathfrak{g})\mathfrak{g}'_+).$$

Proposition 3.1 (Ref. 5, Theorem 1). The projection $U_q(\mathfrak{g}) \to Z U_q(\mathfrak{h}')$ *implements an embedding of* $Z_q(\mathfrak{g}, \mathfrak{g}')$ *in* $Z U_q(\mathfrak{h}')$.

Proof. The statement is proved in Ref. 5 for the orthogonal and special linear quantum groups but the arguments apply to symplectic groups too. □

The algebra $Z_q(\mathfrak{g}, \mathfrak{g}')$ inherits the adjoint action of the Cartan subalgebra, so one can speak of weights of its elements. It is proved within the theory of extremal projectors that $Z_q(\mathfrak{g}, \mathfrak{g}')$ is generated by elements of weights $\beta \in \mathrm{R}_{\mathfrak{g}} \backslash \mathrm{R}_{\mathfrak{g}'}$ plus $z_0 = q^{h_{\alpha_1}}$. We calculate them in Secs. III A and III B, cf. Propositions 3.5 (negative β) and 3.9 (positive β).

A. Lowering operators

In what follows, we extend $U_q(\mathfrak{g})$ along with its subalgebras containing $U_q(\mathfrak{h})$ over the field of fractions of $U_q(\mathfrak{h})$ and denote such an extension by hat, e.g., $\hat{U}_q(\mathfrak{g})$. In this section, we calculate representatives of the negative generators of $Z_q(\mathfrak{g},\mathfrak{g}')$ in $\hat{U}_q(\mathfrak{b}_-)$.

Set $h_i = h_{\varepsilon_i} \in \mathfrak{h}$ for all $i = 1,\dots,N$ and introduce $\eta_{ij} \in \mathfrak{h} + \mathbb{C}$ for $i,j = 1,\dots,N$, by

$$\eta_{ij} = h_i - h_j + (\varepsilon_i - \varepsilon_j, \rho) - \frac{1}{2}\|\varepsilon_i - \varepsilon_j\|^2. \tag{3.1}$$

Here, $\|\mu\|$ is the Euclidean norm on $\mathfrak{h}^*$.

Lemma 3.2. Suppose that $(l,r) \in P(\alpha)$ for some $\alpha \in \Pi^+$. Then,

(i) if $l < r < j$, then $\eta_{lj} - \eta_{rj} = h_\alpha + (\alpha, \varepsilon_j - \varepsilon_r)$,
(ii) if $i < l < r$, then $\eta_{li} - \eta_{ri} = h_\alpha + (\alpha, \varepsilon_i - \varepsilon_r)$,
(iii) $\eta_{lr} = h_\alpha$.

Proof. We have $(\alpha,\rho) = \frac{1}{2}\|\alpha\|^2$ for all $\alpha \in \Pi^+$. This proves (iii). Further, for $\varepsilon_l - \varepsilon_r = \alpha$,

$$\eta_{lj} - \eta_{rj} = h_\alpha + \frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\varepsilon_j - \varepsilon_r\|^2 - \frac{1}{2}\|\varepsilon_j - \varepsilon_r - \alpha\|^2 = h_\alpha + (\alpha, \varepsilon_j - \varepsilon_r), \quad r < j,$$
$$\eta_{li} - \eta_{ri} = h_\alpha + \frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\varepsilon_i - \varepsilon_r\|^2 - \frac{1}{2}\|\varepsilon_i - \varepsilon_r - \alpha\|^2 = h_\alpha + (\alpha, \varepsilon_i - \varepsilon_r), \quad i < l,$$

which proves (i) and (ii). □

We call a strictly ascending sequence $\vec{m} = (m_1,\dots,m_s)$ of integers a route from m_1 to m_s. We write $m < \vec{m}$ and $\vec{m} < m$ for $m \in \mathbb{Z}$ if, respectively, $m < \min\vec{m}$ and $\max\vec{m} < m$. More generally, we write $\vec{m} < \vec{k}$ if $\max\vec{m} < \min\vec{k}$. In this case, a sequence $(\vec{m},\vec{k})$ is a route from $\min\vec{m}$ to $\max\vec{k}$. We also write $m \le \vec{m}$ if $m = \min\vec{m}$ and $\vec{m} \le m$ if $m = \max\vec{m}$.

Given a route $\vec{m} = (m_1,\dots,m_s)$, define the product $f_{\vec{m}} = f_{m_1,m_2}\cdots f_{m_{s-1},m_s} \in U_q(\mathfrak{g}_-)$. Consider a free right $\hat{U}_q(\mathfrak{h})$-module Φ_{1m} generated by $f_{\vec{m}}$ with $1 \leqslant \vec{m} \leqslant j$ and define an operation $\partial_{lr}\colon \Phi_{1j} \to \hat{U}_q(\mathfrak{b}_-)$ for $(l,r) \in P(\alpha)$ as follows. Assuming $1 \leqslant \vec{\ell} < l < r < \vec{\rho} < j$, set

$$\begin{array}{rcll}
\partial_{lr} f_{(\vec{\ell},l)} f_{(l,r)} f_{(r,\vec{\rho})} &=& f_{(\vec{\ell},l)} f_{(r,\vec{\rho})} [\eta_{lj} - \eta_{rj}]_q, & \\
\partial_{lr} f_{(\vec{\ell},l)} f_{(l,\vec{\rho})} &=& -f_{(\vec{\ell},l)} f_{(r,\vec{\rho})} q^{-\eta_{lj}+\eta_{rj}}, & \\
\partial_{lr} f_{(\vec{\ell},r)} f_{(r,\vec{\rho})} &=& f_{(\vec{\ell},l)} f_{(r,\vec{\rho})} q^{\eta_{lj}-\eta_{rj}}, & \\
\partial_{lr} f_{\vec{m}} &=& 0, & l \notin \vec{m}, \quad r \notin \vec{m}.
\end{array}$$

Extend ∂_{lr} to entire Φ_{1j} by $\hat{U}_q(\mathfrak{h})$-linearity. Let $p\colon \Phi_{1j} \to \hat{U}(\mathfrak{g})$ denote the natural homomorphism of $\hat{U}_q(\mathfrak{h})$-modules.

Lemma 3.3. For all $\alpha \in \Pi^+$ and all $x \in \Phi_{1j}$, $e_\alpha \circ p(x) = \sum_{(l,r)\in P(\alpha)} p \circ \partial_{lr}(x) \bmod \hat{U}_q(\mathfrak{g}) e_\alpha$.

Proof. A straightforward analysis based on Proposition 2.3 and Lemma 3.2. □

To simplify the presentation, we suppress the symbol of projection p in what follows.

Introduce elements $A_r^j \in \hat{U}_q(\mathfrak{h})$ by

$$A_r^j = \frac{q - q^{-1}}{q^{-2\eta_{rj}} - 1}, \tag{3.2}$$

for all $r,j \in [1,N]$ subject to $r < j$. For each simple pair (l,r), we define (l,r)-chains as

$$f_{(\vec{\ell},l)} f_{(l,\vec{\rho})} A_l^j + f_{(\vec{\ell},l)} f_{(l,r)} f_{(r,\vec{\rho})} A_l^j A_r^j + f_{(\vec{\ell},r)} f_{(r,\vec{\rho})} A_r^j, \quad f_{(\vec{\ell},l)} f_{l,j} A_l^j + f_{(\vec{\ell},j)}, \tag{3.3}$$

where $1 \leqslant \vec{\ell} < l$ and $r < \vec{\rho} \leqslant j$. Remark that $f_{(l,r)} = \left[\frac{(\alpha,\alpha)}{2}\right]_q e_{-\alpha}$, where $\alpha = \varepsilon_l - \varepsilon_r$.

Lemma 3.4. The operator ∂_{lr} annihilates (l,r)-chains.

Proof. Applying ∂_{lr} to the 3-chain in (3.3), we get

$$f_{(\vec{\ell},l)}f_{(r,\vec{\rho})}(-q^{-\eta_{lj}+\eta_{rj}}A^j_l + [\eta_{lj}-\eta_{rj}]_q A^j_l A^j_r + q^{\eta_{lj}-\eta_{rj}}A^j_r).$$

The factor in the brackets turns zero on substitution of (3.2).

Now apply ∂_{lj} to the right expression in (3.3) and get

$$f_{(\vec{\ell},l)}([h_\alpha]_q A^j_l + q^{h_\alpha}) = f_{(\vec{\ell},l)}\Big(\frac{q^{h_\alpha}-q^{-h_\alpha}}{q^{-2\eta_{lj}}-1} + q^{h_\alpha}\Big) = f_{(\vec{\ell},l)}\frac{[h_\alpha-\eta_{lj}]_q}{[-\eta_{lj}]_q} = 0,$$

so long as $\eta_{lj} = h_\alpha$ by Lemma 3.2. □

Given a route $\vec{m} = (m_1,\dots,m_s)$, put $A^j_{\vec{m}} = A^j_{m_1}\cdots A^j_{m_s} \in \hat{U}_q(\mathfrak{h})$ (and $A^j_{\vec{m}} = 1$ for the empty route) and define

$$z_{-j+1} = \sum_{1<\vec{m}<j} f_{(1,\vec{m},j)}A^j_{\vec{m}} \in \hat{U}_q(\mathfrak{b}_-), \quad j = 2,\dots,N, \tag{3.4}$$

where the summation is taken over all possible $\vec{m}$ subject to the specified inequalities plus the empty route.

Proposition 3.5. $e_\alpha z_{-j} = 0 \bmod \hat{U}_q(\mathfrak{g})e_\alpha$ for all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and $j = 1,\dots,N-1$.

Proof. Thanks to Lemma 3.3, we can reduce consideration to the action of operators ∂_{lr}, with $(l,r) \in P(\alpha)$. According to the definition of ∂_{lr}, the summands in (3.4) that survive the action of ∂_{lr} can be organized into a linear combination of (l,r)-chains with coefficients in $\hat{U}_q(\mathfrak{h})$. By Lemma 3.4, they are killed by ∂_{lr}. □

The elements z_{-i}, $i = 1,\dots,N-1$, belong to the normalizer $\mathcal{N}$ and form the set of negative generators of $Z_q(\mathfrak{g},\mathfrak{g}')$ for symplectic $\mathfrak{g}$. In the orthogonal case, the negative part of $Z_q(\mathfrak{g},\mathfrak{g}')$ is generated by z_{-i}, $i = 1,\dots,N-2$.

B. Raising operators

In this section, we construct positive generators of $Z_q(\mathfrak{g},\mathfrak{g}')$, which are called raising operators. Consider an algebra automorphism $\omega\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined on the generators by $f_\alpha \leftrightarrow e_\alpha$, $q^{\pm h_\alpha} \mapsto q^{\mp h_\alpha}$. For $i < j$, let g_{ji} be the image of f_{ij} under this isomorphism. The natural representation restricted to $U_q(\mathfrak{g}_\pm)$ intertwines ω and matrix transposition. Since $(\omega\otimes\omega)(\check{\mathcal{R}}) = \check{\mathcal{R}}_{21}$, the matrix $\check{R}^+ = (\pi\otimes \mathrm{id})(\check{\mathcal{R}}_{21})$ is equal to $1\otimes 1 + (q-q^{-1})\sum_{i<j} e_{ji}\otimes g_{ji}$.

Lemma 3.6. For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $i > 1$, $e_\alpha g_{i1} = \sum_{(l,r)\in P(\alpha)} \delta_{il} g_{r1} \bmod \hat{U}_q(\mathfrak{g})e_\alpha$.

Proof. Follows from the intertwining property of the R-matrix. □

Consider the right $\hat{U}_q(\mathfrak{h})$-module Ψ_{i1} freely generated by $f_{(\vec{m},k)}g_{k1}$ with $i \leqslant \vec{m} < k$. We define operators $\partial_{lr}\colon \Psi_{i1} \to \hat{U}_q(\mathfrak{g})$ similarly as we did it for Φ_{1j}. For a simple pair $(l,r) \in P(\alpha)$, put

$$\partial_{l,r} f_{(\vec{m},k)}g_{k1} = \begin{cases} f_{(\vec{m},l)}g_{r1}, & l = k, \\ \big(\partial_{l,r}f_{(\vec{m},k)}\big)g_{k1}, & l \neq k, \end{cases} \quad i \leqslant \vec{m} < r.$$

The Cartan factors appearing in $\partial_{lr}f_{(\vec{m},k)}$ depend on h_α. When pushed to the right-most position, h_α is shifted by $(\alpha,\varepsilon_1-\varepsilon_r)$. We extend ∂_{lr} to an action on Ψ_{i1} by the requirement that ∂_{lr} commutes with the right action of $\hat{U}_q(\mathfrak{h})$. Let p denote the natural homomorphism of $\hat{U}_q(\mathfrak{h})$-modules, $p\colon \Psi_{i1} \to \hat{U}_q(\mathfrak{g})$. One can prove the following analog of Lemma 3.3.

Lemma 3.7. For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $x \in \Psi_{i1}$, $e_\alpha \circ p(x) = \sum_{(l,r)\in P(\alpha)} p\circ\partial_{lr}(x) \bmod \hat{U}_q(\mathfrak{g})e_\alpha$.

Proof. Straightforward. □

We suppress the symbol of projection p to simplify the formulas.

For $i < j$, let $\|i - j\|$ be the number of simple positive roots entering $\varepsilon_i - \varepsilon_j$. For all $i, k = 2, \dots, N$, $i < k$, put

$$A^i_k = \frac{q^{\eta_{k1} - \eta_{i1}}}{[\eta_{i1} - \eta_{k1}]_q}, \quad B^i_k = \frac{(-1)^{\|i-k\|}}{[\eta_{i1} - \eta_{k1}]_q}.$$

For each $(l, r) \in \mathrm{P}(\alpha)$, where $\alpha \in \Pi^+_{\mathfrak{g}'}$, define 3-chains as

$$f_{(i,\vec{m},l)} g_{l1} B^i_l + f_{(i,\vec{m},l)} f_{(l,r)} g_{r1} A^i_l B^i_r + f_{(i,\vec{m},r)} g_{r1} B^i_r, \tag{3.5}$$

with $i < \vec{m} < l < r \leqslant N$ and

$$f_{(i,\vec{\ell},l)} f_{(l,\vec{\rho},k)} g_{k1} A^i_l + f_{(i,\vec{\ell},l)} f_{(l,r)} f_{(r,\vec{\rho},k)} g_{k1} A^i_l A^i_r + f_{(i,\vec{\ell},r)} f_{(r,\vec{\rho},k)} g_{k1} A^i_r \tag{3.6}$$

with $i < \vec{\ell} < l < r < \vec{\rho} < k \leqslant N$. The 2-chains are defined as

$$g_{i1} + f_{(i,r)} g_{r1} B^i_r, \quad f_{(i,\vec{m},k)} g_{k1} + f_{(i,r)} f_{(r,\vec{m},k)} g_{k1} A^i_r, \tag{3.7}$$

where r is such that $\varepsilon_i - \varepsilon_r \in \Pi^+_{\mathfrak{g}'}$ and $i < r < \vec{m} < k \leqslant N$. In all cases, empty $\vec{m}$ are admissible.

Lemma 3.8. For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $(l, r) \in P(\alpha)$, the (l, r)-chains are annihilated by ∂_{lr}.

Proof. Suppose that $i = l$ and apply ∂_{ir} to the left 2-chain in (3.7). The result is

$$g_{r1} + [h_\alpha]_q g_{r1} B^i_r = g_{r1}(1 + [h_\alpha + (\alpha, \varepsilon_1 - \varepsilon_r)]_q B^i_r) = g_{r1}(1 + [\eta_{i1} - \eta_{r1}]_q B^i_r) = 0$$

by Lemma 3.2. Applying ∂_{ir} to the right 2-chain in (3.7) we get

$$f_{(r,\vec{m},k)} g_{k1}(-q^{-\eta_{i1} + \eta_{r1}} + [\eta_{i1} - \eta_{r1}]_q A^i_r) = 0.$$

Now consider 3-chains. The action of ∂_{lr} on (3.6) produces

$$-f_{(i,\vec{\ell},l)} q^{-h_\alpha} f_{(r,\vec{\rho},k)} g_{k,1} A^i_l + f_{(i,\vec{\ell},l)} [h_\alpha]_q f_{(r,\vec{\rho},k)} g_{k,1} A^i_l A^i_r + f_{(i,\vec{\ell},l)} q^{h_\alpha} f_{(r,\vec{\rho},k)} g_{k,1} A^i_r,$$

which turns zero since $-q^{\eta_{r1} - \eta_{l1}} A^i_l + [\eta_{l1} - \eta_{r1}]_q A^i_l A^i_r + q^{\eta_{l1} - \eta_{r1}} A^i_r = 0$. The action of ∂_{lr} on (3.5) yields

$$f_{(i,\vec{m},l)} g_{r1} B^i_l + f_{(i,\vec{m},l)} [h_\alpha] g_{r1} A^i_l B^i_r + f_{(i,\vec{m},l)} q^{h_\alpha} g_{r1} B^i_r.$$

This is vanishing since $B^i_l + [\eta_{l1} - \eta_{r1}] A^i_l B^i_r + q^{\eta_{l1} - \eta_{r1}} B^i_r = B^i_l + \frac{[\eta_{i1} - \eta_{r1}]_q}{[\eta_{i1} - \eta_{l1}]_q} B^i_r = 0.$ □

Given a route $\vec{m} = (m_1, \dots, m_k)$ such that $i < \vec{m}$ let $A^i_{\vec{m}}$ denote the product $A^i_{m_1} \dots A^i_{m_k}$. Introduce elements $z_i \in \hat{U}_q(\mathfrak{g}_-)\mathfrak{g}_+$ of weight $\varepsilon_1 - \varepsilon_i$ by

$$z_{i-1} = g_{i1} + \sum_{i < \vec{m} < k \leqslant N} f_{(i,\vec{m},k)} g_{k1} A^i_{\vec{m}} B^i_k, \quad i = 2, \dots, N.$$

The summation includes empty $\vec{m}$.

Proposition 3.9. $e_\alpha z_i = 0 \bmod \hat{U}_q(\mathfrak{g}) e_\alpha$, for all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and $i = 1, \dots, N-1$.

Proof. By Lemma 3.6, the vectors $g_{2'1}$ and $z_{N-1} = g_{1'1}$ are normalizing the left ideal $\hat{U}_q(\mathfrak{g})\mathfrak{g}'_+$, so is $z_{N-2} = g_{2'1} + f_1 g_{1'1} B^{1'}_{2'}$. Once the cases $i = 2', 1'$ are proved, we further assume $i < 2'$. In view of Lemma 3.7, it is sufficient to show that z_{i-1} is killed, modulo $\hat{U}_q(\mathfrak{g})\mathfrak{g}'_+$, by all ∂_{lr} such that $\varepsilon_l - \varepsilon_r \in \Pi^+_{\mathfrak{g}'}$. Observe that z_{i-1} can be arranged into a linear combination of chains, which are killed by ∂_{lr}, as in Lemma 3.8. □

The elements z_i, $i = 1, \dots, N-1$, belong to the normalizer $\mathcal{N}$. They form the set of positive generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ for symplectic $\mathfrak{g}$. In the orthogonal case, the positive part of $Z_q(\mathfrak{g}, \mathfrak{g}')$ is generated by z_i, $i = 1, \dots, N-2$.

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