REPRESENTATIONS OF QUANTUM CONJUGACY CLASSES OF NON-EXCEPTIONAL QUANTUM GROUPS

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Abstract

Representations of Quantum Conjugacy Classes of Non-Exceptional Quantum Groups

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Let G be a complex non-exceptional simple algebraic group and $\mathfrak{g}$ its Lie algebra. With every point x of the maximal torus $T \subset G$ we associate a highest weight module M_x over the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ and an equivariant quantization of the conjugacy class of x by operators in $\mathrm{End}(M_x)$. These equivariant quantizations are isomorphic for x lying on the same orbit of the Weyl group, and M_x support different exact representations of the same quantum conjugacy class.

This recovers all quantizations of conjugacy classes constructed before, via special x, and also completes the family of conjugacy classes by constructing an equivariant quantization of "borderline" Levi conjugacy classes of the complex orthogonal group $SO(N)$, whose stabilizer contains a Cartesian factor $SO(2) \times SO(P)$, $1 \leqslant P < N$, $P \equiv N \mod 2$.

To achieve this, generators of the Mickelsson algebra $Z_q(\mathfrak{g}, \mathfrak{g}')$, where $\mathfrak{g}' \subset \mathfrak{g}$ is the Lie subalgebra of rank $\mathrm{rk}\,\mathfrak{g}' = \mathrm{rk}\,\mathfrak{g} - 1$ of the same type, were explicitly constructed in terms of Chevalley generators via the R-matrix of $U_q(\mathfrak{g})$.

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Contents

Chapter 1

Introduction

This is a PhD dissertation in equivariant deformation quantization of an important family of homogeneous spaces: semi-simple conjugacy classes of simple complex algebraic groups of infinite series. We treat them in the spirit of classical algebraic geometry, based on the fact that they exhaust the list of closed conjugacy classes, [81]. The main idea is to realize a quantized polynomial ring, by operators in a module over the total quantum group, canonically associated with every point on the Weyl group orbit in a fixed maximal torus.

The concept of quantization goes back to the origin of quantum mechanics in the early 20th century, when the physical world revealed its quantum nature in experiments with electrons, light waves, [61], and black body radiation [75]. It was then realized that the most adequate description of physical processes of micro scales was in terms of vectors in Hilbert spaces and self-adjoint operators acting upon them, [15, 20]. Two formulations of quantum mechanics: the matrix formalism of Heisenberg [14] and wave approach of Schrödinger [76] signified a revolution in traditional mathematical apparatus used for the description of the physical world and takes into account the Heisenberg uncertainty principle [46].

A conventional state space of classical mechanics is a smooth manifold X equipped with a Poisson structure, while the algebra of physical observables like coordinates, momenta, energy, *etc.* are smooth functions, which form a commutative algebra, [3]. The matrix mechanics of Heisenberg treats physical observables as linear operators that close up into a non-commutative algebra. The passage from the classical commutative algebra to the quantum non-commutative one is controlled by a small parameter $\hbar$ called the Planck constant. This is the famous correspondence principle [13] suggesting that, mathematically, such a transition is a deformation of the initial algebraic structure, [42, 43, 44, 45].

The key concept of Schrödinger's approach to quantum mechanics is wave functions, which are descriptions of the quantum state of a system (i.e. the spatial distribution of a quantum particle) [76]. It implies that we get information about surrounding objects, as scattering data upon testing it with probe matter, be it matter, electrons or photons. This also applies to the geometry of space because it is inseparable from physics. It is therefore natural to incorporate geometry into the quantum world once we consistently treat coordinates as operators. The idea to view a geometric object not as a collection of points but as the function algebra on it is not new and underlies classical algebraic geometry, [19, 79, 80]. The next step on that path is to drop the commutativity constraint and regard an arbitrary associative algebra as that of functions on a "quantum space". This idea has lead to the noncommutative geometry of Alain Connes, [17].

From the initial discovery of quantum mechanics, there arose a purely mathematical problem of what should be understood by quantization, [20]. It was initially viewed as some correspondence between the two algebras without a clear prescription of the transition mechanism. One of the requirements was a realization of the new algebra in a Hilbert space. A systematic approach to quantization was developed by F. Bayen, M. Flato, C. Fronsdal, A. Lichneroviz, and D. Sternheimer in [11], who unchained it from a particular representation. The new multiplication of the quantized function algebra $\mathbb{C}[X]$ is a result of the action of a bidifferential operator, $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ (we adopt the Sweedler convention of indexing tensor factors) on a pair of functions, $f, g \in \mathbb{C}[X]$:

$$f * g = (\mathcal{F}_1 f) \cdot (\mathcal{F}_2 g) = \sum_{n=0}^{\infty} \hbar^n (\mathcal{F}_1^n f) \cdot (\mathcal{F}_2^n g)$$

Here $\cdot$ stands for the classical multiplication in $\mathbb{C}[X]$. The correspondence principle is accounted for, by the requirements that $(\mathcal{F}_1^0 f) \cdot (\mathcal{F}_2^0 g) = f \cdot g$ and $(\mathcal{F}_1^1 f) \cdot (\mathcal{F}_2^1 g) - (\mathcal{F}_1^1 g) \cdot (\mathcal{F}_2^1 f)$ is the Poisson structure on X. In this approach, the underlying vector space is *a priori* given and equals $\mathbb{C}[X]$ with the main problem being associativity implying certain constraints on $\mathcal{F}$.

The contemporary understanding of quantization is as follows: given a Poisson algebra $\mathcal{A}$, its quantization is a $\mathbb{C}[\![\hbar]\!]$-algebra $\mathcal{A}_\hbar$ that is flat as a $\mathbb{C}[\![\hbar]\!]$-module and coincides with $\mathcal{A}$ modulo $\hbar\mathcal{A}_\hbar$ as a $\mathbb{C}$-algebra. In this form, it need not be local, and we can return to the language of algebraic geometry for X being a variety and $\mathbb{C}[X]$ its polynomial ring. In this approach, $\mathcal{A}_\hbar$ is presented as a quotient of a free algebra generated by quantum coordinates. Associativity is given for free and the focus of the problem is moved to the size of the quotient. Typically that is resolved via a representation of $\mathcal{A}_\hbar$ by endomorphisms of a $\mathbb{C}[\![\hbar]\!]$-flat module, so we get back to the original representation-theoretical point of view, but at a new level.

Another key concept underlying modern mathematics, is that of symmetry [47]. It plays an important role in physics as well, giving rise to conservation laws, [72, 83]. If a group G acts on the manifold X and preserves its Poisson structure, $x\{f, g\} = \{xf, xg\}$, $\forall x \in G$, then it is natural to seek a quantization $\mathcal{A}_\hbar$ that supports an action of G by algebra automorphisms, which is a deformation of the initial action. This is the conventional approach to equivariant quantization, and it was adopted from the very beginning of quantum mechanics. A typical example is the dual space $\mathfrak{g}^*$ equipped with the Lie-Kirillov-Kostant-Souriau bracket $\{x, y\} = [x, y] \in \mathfrak{g} \in \mathbb{C}[\mathfrak{g}^*]$, for all $x, y \in \mathfrak{g}$, [55]. It restricts to every orbit in $\mathfrak{g}^*$ making it a symplectic homogeneous manifold. While quantization of $\mathbb{C}[\mathfrak{g}^*]$ is delivered by the universal enveloping algebra, $\mathbb{C}_\hbar[\mathfrak{g}^*] \simeq U(\mathfrak{g})$, quantization of coadjoint orbits remains a non-trivial problem.

Conventional equivariant quantization treats G as a "ruler" applied to the manifold X from outside. On the other hand, a smooth group G is an equal participant in the classical geometry. Homogeneous spaces they act upon can be nicely presented as coset spaces in purely group terms. The most distinguished of them, the conjugacy classes, are realized as submanifolds in G: they are orbits under the conjugation action of the group on itself. This observation makes it very natural to include G into the quantum universe. That is done within the theory of quantum groups created by Drinfeld [30] on the base of results obtained by mathematical physicists, [38, 39].

Drinfeld's theory is based on the material accumulated within the quantum inverse scattering approach to integrable spin chains developed by L. Faddeev and his Leningrad school, [58]. The key ingredient of their method is the so called matrix $R \in \mathrm{End}(V) \otimes \mathrm{End}(V)$, satisfying the (quantum) Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Drinfeld's quantum groups appeared to be just right to accommodate this identity. They are deformations of $\mathbb{C}[G]$ or the universal enveloping of its Lie algebra $\mathfrak{g}$ in the class of Hopf algebras. The R-matrix plays a role of an intertwining operator between tensor products of representations in different order.

The quasi-classical limit of a quantum group is a Poisson group G, which is equipped with a Poisson structure preserved under the multiplication map

$$G \times G \to G$$

Here the direct product is equipped with the Poisson structure naturally lifted from the factors. It is a result by Etingof and Kazhdan, [35, 36], that every Poisson group gives rise to a quantum group in a functorial way. They also quantized some coset spaces G/K whose Poisson structure can be projected from G, [37].

In terms of the Lie algebra $\mathfrak{g}$, the Poisson groups correspond to Lie bialgebras, [30]. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ with a cobracket $\mu\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ making $\mathfrak{g}^*$ a Lie algebra by duality. The compatibility condition is that μ is a $\mathfrak{g}\wedge\mathfrak{g}$-valued 1-cocycle. In the case it is coboundary, there is an element $r \in \mathfrak{g}\wedge\mathfrak{g}$ such that $\mu(\xi) = [\xi\otimes 1 + 1\otimes\xi, r]$, $\xi \in \mathfrak{g}$. It solves a modified Yang-Baxter equation, [78], and is regarded as the quasi-classical term of the quantum R-matrix. Then the Poisson bivector field is given by $r^{l,l} - r^{r,r}$, where $(\eta^r f)(x) = \frac{df}{dt} f(e^{t\eta}x)|_{t=0}$, $(\eta^l f)(x) = \frac{df}{dt} f(xe^{t\eta})|_{t=0}$, for $\eta \in \mathfrak{g}$ and $f \in \mathbb{C}[G]$. This Poisson structure is called Drinfeld-Sklyanin bracket.

Poisson groups can act on Poisson spaces in a compatible way. Namely, X is called a (left) Poisson manifold over a Poisson group G if the action map

$$G \times X \to X$$

is Poisson. This definition recovers the usual G-invariance if G is equipped with the zero bracket. In general, the bracket on X is no longer invariant, so the equivariant quantization needs adapting to the new setting. Let $U_\hbar(\mathfrak{g})$ be the quantized universal enveloping algebra (quantum group) of the

Poisson group G. Let X be a Poisson manifold over G and $\mathcal{A}$ its function algebra. A quantization $\mathcal{A}_\hbar$ is called equivariant if it is an $U_\hbar(\mathfrak{g})$-module algebra (the multiplication $*$ is a $U_\hbar(\mathfrak{g})$-morphism) and the action is a deformation of the classical action.

Quantization of a non-degenerate Poisson structure can be constructed by Fedosov's method, [41] (and [21] for its generalization to the locally non-degenerate case). A general quantization of Poisson structure can be locally obtained via Kontsevich's construction, [56]. However, the method of [56] does not respect group action while Fedosov's approach needs an invariant flat connection, whose general existence is questionable. At the same time, the presence of a group creates a very special context making equivariant quantization a part of representation theory.

The most important example for us is $X = G$ equipped with the conjugation action $x\colon a \mapsto xax^{-1}$, $x, a \in G$. Suppose that G is a Poisson group with the Drinfeld-Sklyanin bracket $r^{l,l} - r^{r,r}$. Suppose there is an ad-invariant symmetric tensor $\omega \in \mathfrak{g} \otimes \mathfrak{g}$ such that $r + \omega \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Let $\eta^{\mathrm{ad}} = \xi^l - \xi^r$ denote the adjoint vector field generated by $\eta \in \mathfrak{g}$. Then the bivector field $r^{\mathrm{ad},\mathrm{ad}} + \omega^{r,l} - \omega^{l,r}$ makes $X = G$ a Poisson-Lie manifold over the Drinfeld-Sklyanin Poisson group G. We call it Semenov-Tjan-Shanksky bracket, [77]. A remarkable fact is that it can be restricted to every conjugacy class of G, [2]. In this way, it is a Poisson-Lie analog of the Kirillov-Kostant-Souriau bracket. In this thesis, we address the restriction of this bracket to semi-simple conjugacy classes for the case of standard factorizable matrix r.

Classification of (local) homogeneous Poisson Lie manifolds was done by Drinfeld in [29]. According to his result, they are described by the following data. Every Poisson group corresponds to a pair of Lie algebras, $(\mathfrak{g}, \mathfrak{g}^*)$, such that their sum is also a Lie algebra, $\mathfrak{g} \bowtie \mathfrak{g}^*$, called double of $\mathfrak{g}$. It is uniquely determined by the requirement that the canonical symmetric inner product on $\mathfrak{g} \oplus \mathfrak{g}^*$ is ad-invariant. Suppose that $\mathfrak{l} \subset \mathfrak{g}$ is the isotropy Lie subalgebra. There exists a Lagrangian Lie subalgebra $\mathfrak{m} \subset \mathfrak{g} \bowtie \mathfrak{g}^*$ such that $\mathfrak{m} \cap \mathfrak{g} = \mathfrak{l}$. This data also describes quasi-Lie bialgebras, which are infinitesimal deformations of $U(\mathfrak{g})$ as quasi-Hopf algebras, [31]. Drinfeld called this finding mysterious and posed a problem of its "quantization", i.e. to reveal the role of quasi-Hopf algebras in quantization of Poisson-Lie manifolds.

Although this problem is still open, there is a class of manifolds, for which this relation has been unveiled. That was done in the framework of dynamical Yang-Baxter equations, dynamical twists and, what is essentially the same, Shapovalov form, [1, 25, 32, 33, 52], for a wide class of homogeneous spaces that were not covered by [37]. Namely, those are semi-simple coadjoint orbits of simple Lie groups and conjugacy classes with Levi isotropy subgroups. Up to now, non-Levi conjugacy classes have not been incorporated into that scheme.

The relation between analytic and algebraic approaches to quantization can be illustrated by the following. Suppose that G acts on itself by conjugation and fix an initial point $x \in G$. Let O_x

be the orbit of that action, i.e. $O_x = \{gxg^{-1} | g \in G\}$. Then x gives rise to two G-maps

$$G \twoheadrightarrow O_x \hookrightarrow G, \quad g \mapsto gxg^{-1},$$

where G on the left is equipped with the left multiplication action. The arrow $\twoheadrightarrow$ is a surjection while $\hookrightarrow$ is an embedding. If $K \subset G$ is the centralizer subgroup of x, then the leftmost mapping factors through an isomorphism, $G/K \simeq O_x$, of G-spaces. This way, the point x makes O_x simultaneously a subset and a quotient set of G. In the dual picture, in terms of function algebras, we have

$$\mathbb{C}\left[G\right] \hookleftarrow \mathbb{C}\left[O_x\right] \twoheadleftarrow \mathbb{C}\left[G\right],$$

where the left map is embedding and the right is a projection.

In the dual picture, the point x is a character of the algebra $\mathbb{C}[G]$, i.e. a homomorphism $x\colon \mathbb{C}[G] \to \mathbb{C}$. Then the composition of maps $\mathbb{C}[G] \to \mathbb{C}[G]$ is given by $(x \otimes \mathrm{id}) \circ \delta$, where δ is the (right) coaction that is dual to conjugation action, $\delta\colon \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$. The image of this map is the subalgebra $\mathbb{C}[G]^K$ of K-invariants under the right multiplication, $f(g) \mapsto f(gk)$, $k \in K$, $f \in \mathbb{C}[G]$.

Let $\mathcal{A}^1_\hbar$ and $\mathcal{A}^2_\hbar$ be the quantizations of $\mathbb{C}[G]$ along the Drinfeld-Sklyanin and Semenof-Tjan-Shansky Poisson brackets, respectively. The algebra $\mathcal{A}^1_\hbar$ is determined by the famous RTT=TTR relations, [40], and is in Hopf duality with $U_q(\mathfrak{g})$. The algebra $\mathcal{A}^2_\hbar$ is related to the reflection equation, [57] and is a right comodule over $\mathcal{A}^1_\hbar$. Lifting the dual picture to the quantum setting, one may try to construct the chain of maps

$$\mathcal{A}^1_\hbar \hookleftarrow \mathbb{C}_\hbar\left[G/K\right] \simeq \mathbb{C}_\hbar\left[O_x\right] \twoheadleftarrow \mathcal{A}^2_\hbar,$$

provided there is character of $\mathcal{A}^2_\hbar$, a deformation of the classical point x. Its quasi-classical characterization is that the Poisson bivector vanishes at x. It is known that such points are in short supply, [63], and only a few classes can be treated this way, [27].

The method of quantum points is a (thin) borderline between the two approaches,

$$\mathcal{A}^1_\hbar \hookleftarrow \mathbb{C}_\hbar\left[G/K\right], \qquad \mathbb{C}_\hbar\left[O_x\right] \twoheadleftarrow \mathcal{A}^2_\hbar, \tag{1.1}$$

which represent the two alternative (analytic and algebraic) formulations of quantization. The right arrow yields $\mathbb{C}_\hbar\left[O_x\right]$ as a quotient of $\mathcal{A}^2_\hbar$ by an invariant ideal. This ideal is realized as the annihilator of a certain module, M_x, of the quantum group $U_q(\mathfrak{g})$ via an embedding $\mathcal{A}^2_\hbar \subset U_q(\mathfrak{g})$. For Levi K, the left arrow can be worked out as well, in a more general setting. There is a unique (up to a multiple) invariant bilinear pairing between M_x (highest weight) and an opposite (lowest weight) module, N_x. It is non degenerate over $\mathbb{C}[\![\hbar]\!]$ and has the inverse form. There is a lift $\mathcal{F} \in U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})$, which one can take for a quasi-Hopf algebra twist $U_\hbar(\mathfrak{g})$. Using this twist, one can change the multiplication on $\mathcal{A}^1_\hbar$ via the left co-regular action. Although the new multiplication is not-associative, it becomes so when restricted to the subspace of $U_\hbar(\mathfrak{k})$-invariants, where $\mathfrak{k}$ is the Lie algebra of K. This quantization of $\mathbb{C}_\hbar[G/K]$ is equivariant with respect to the right co-regular action, [1, 25, 32, 33, 52]. This is the partial answer to the Drinfeld problem from [29]. Its extension for non-Levi K is open due to the absence of $U_q(\mathfrak{k})$.

In this dissertation we pursue the algebraic approach to quantization, via the right arrow in (1.1). Here is an overview of the previous work directly related to this project. Classification of Poisson brackets on conjugacy classes of simple groups was done by Donin, Gurevich, and Shnider in [22, 23], see also the work of Karolinsky, [50], which is based on Drinfeld's approach. Quantization of coadjoint orbits as subalgebras of endomorphisms of parabolic Verma modules was realized in [24]. Their annihilators in universal enveloping algebras were explicitly evaluated by Toshio Oshima in [73]. About the same time, similar results were obtained for two parameter quantization of orbits of $GL(n)$ in [26]. That is the only series where the Kirillov-Kostant-Souriau and Semenov-Tjan-Shansky brackets can be united in a two parameter family, [23].

Non exceptional Levi conjugacy classes of infinite series were constructed by Mudrov in [68]. This approach was extended to non-Levi classes of orthogonal and symplectic groups in [65, 69]. All these results were obtained via very special choice of points on the maximal torus. The case of borderline classes of orthogonal groups was missing from this list.

Observe that semisimple conjugacy classes in $SO(N)$ can be categorized by their sets of eigenvalues: whether they include both ±1 or not. The stabilizer subgroup of the second type is Levi, and such a class is isomorphic to an adjoint orbit in $\mathfrak{so}(N)$ as an affine variety. Their quantization has been constructed in [68]. The stabilizer of the first type contains a Cartesian factor $SO(2m) \times SO(P)$, where $2m$ and P are the multiplicities of the eigenvalues -1 and $+1$ respectively and P is of the same parity as N. If $m \geqslant 2$ (one should also assume $P \geqslant 4$ for even N), the stabilizer subgroup is not Levi. Such classes have been quantized in [64]. The remaining classes corresponding to $m = 1$ form a special family, which was not covered before. Due to the isomorphism $GL(1) \simeq SO(2)$, they form a borderline between the Levi and non-Levi families.

In Chapter 3, the quantization method of the borderline Levi classes is similar to that used in [64] and [68]: a realization of its quantized polynomial algebra in a $U_q(\mathfrak{g})$-module of highest weight. In the case of interest, it is a parabolic Verma module of special weight. Due to this constraint, it is not a deformation of a Verma module over $U_q(\mathfrak{g})$. The boundary classes were not covered in [68] because the analysis was based on the properties of the Shapovalov form derived by deformation arguments from its classical counterpart. The specialization of the highest weight in our present approach requires a special study of the module $\mathbb{C}^N \otimes M_\lambda$.

In Chapter 4, generators of Mickelsson algebras for the non-exceptional quantum groups are constructed. In mathematics literature, lowering and raising operators are known as generators of step algebras, which were originally introduced by Mickelsson [60] for reductive pairs of Lie algebras, $\mathfrak{g}' \subset \mathfrak{g}$. These algebras naturally act on $\mathfrak{g}'$-singular vectors in $U(\mathfrak{g})$-modules and are important in representation theory, [62, 84]. The general theory of step algebras for classical universal enveloping algebras was developed in [84, 86] and extended to the special liner and orthogonal quantum groups in [53]. They admit a natural description in terms of extremal projectors, [86], introduced for classical groups in [5, 6] and generalized to the quantum group case in [54, 82]. It is known that the step algebra $Z(\mathfrak{g}, \mathfrak{g}')$ is generated by the image of the orthogonal complement $\mathfrak{g} \ominus \mathfrak{g}'$ under the extremal projector of the $\mathfrak{g}'$. Another description of lowering/raising operators for classical groups was obtained in [62, 71, 74] in an explicit form of polynomials in $\mathfrak{g}$.

A generalization of the results of [71, 74] to quantum $\mathfrak{gl}(n)$ can be found in [4]. In this special case, the lowering operators can be also conveniently expressed through "modified commutators" in the Chevalley generators of $U(\mathfrak{g})$ with coefficients in the field of fractions of $U(\mathfrak{h})$. Extending [62] to orthogonal and symplectic quantum groups is not straightforward, since there are no nilpotent triangular Lie subalgebras $\mathfrak{g}_\pm$ in $U_q(\mathfrak{g})$ but only their deformed associative envelope. The lack of $\mathfrak{g}_\pm$ can be compensated by the entries of the universal R-matrix with one leg projected to the natural representation. Those entries are nicely expressed through modified commutators in the Chevalley generators turning into elements of $\mathfrak{g}_\pm$ in the quasi-classical limit. Their commutation relation with the Chevalley generators modify the classical commutation relations with $\mathfrak{g}_\pm$ in a way, which is easy to control. Thus the results of [62, 71, 74] can be generalized and generators of Mickelsson algebras for the non-exceptional quantum groups can be constructed. Explicit form of these generators is useful because they are related to singular vectors generating certain submodules involved in quantization of conjugacy classes, especially in Chapter 6.

In Chapter 5, let G denote the complex general linear algebraic group $GL(n)$ and let $\mathfrak{g}$ be its Lie algebra $\mathfrak{gl}(n)$. Regard G as a Poisson group relative to the standard classical r-matrix and let $U_\hbar(\mathfrak{g})$ be the corresponding quantum group. Consider a semisimple conjugacy class $O \subset G$, which is an affine subvariety of G. This chapter presents a family of exact representations of $\mathbb{C}_\hbar[O]$ on $U_\hbar(\mathfrak{g})$-modules of highest weight. This family is parameterized by diagonal matrices from O. Equivalently, every diagonal matrix is associated a highest weight module and an equivariant quantization of the conjugacy class of this matrix, through an operator realization on that module. The quantized affine ring depends on O and not on a particular point in it. However, the modules are not isomorphic thus yielding non-equivalent exact representations of the same quantum conjugacy class.

Although the isotropy subgroups of all points in O are isomorphic, not all are strictly compatible with the standard triangular polarization of $\mathfrak{g}$. We call such a stabilizer a Levi subgroup if simple roots of its Lie algebra $\mathfrak{k}$ are simple roots of $\mathfrak{g}$, i.e. $\Pi^+_\mathfrak{k} \subset \Pi^+_\mathfrak{g}$. By this definition, $\mathfrak{k}$ being a Levi subalgebra depends on a polarization of $\mathfrak{g}$ relative to a Cartan subalgebra, which is fixed once and for all. The quantization theory of the corresponding conjugacy class is standard: it can be realized by operators on a parabolic Verma module M_λ relative to $U_q(\mathfrak{k}) \subset U_q(\mathfrak{g})$. General diagonal matrices in O are uniquely parameterized by Weyl group elements σ satisfying $\sigma(R^+_\mathfrak{k}) \subset R^+_\mathfrak{g}$, where R^+ is the set of positive roots. For such σ we construct a highest weight module $M_{\sigma.\lambda}$ and realize the algebra $\mathbb{C}_\hbar[O]$ in $\mathrm{End}(M_{\sigma.\lambda})$. Of course, $M_{\sigma.\lambda}$ is a parabolic Verma module if $\sigma(\Pi^+_\mathfrak{k}) \subset \Pi^+_\mathfrak{g}$.

An interesting feature of the non-parabolic quantization via $M_{\sigma.\lambda}$ is a lack of natural candidate for the quantum isotropy subgroup. In this respect, this quantization may help to understand the properties of quantum conjugacy classes which are essentially non-Levi, that is, their isotropy subgroups are not isomorphic to Levi subgroups, [64, 65, 69]. Such classes are not present in $GL(n)$ but form a large family in symplectic and orthogonal groups.

Chapter 6 can be viewed as a uniform approach to quantization that includes the results of [26, 64, 65, 68] and Chapter 3 as a special case and it is done in the spirit of Chapter 5 devoted to $G = GL(n)$. The earlier constructed quantum conjugacy classes were realized by operators on certain modules of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of the group G. For a large number of examples, this theory is parallel to the $U(\mathfrak{g})$-equivariant quantization of semisimple adjoint orbit in $\mathfrak{g} \simeq \mathfrak{g}^*$, [24, 26, 73]. In both cases, G and $\mathfrak{g}$, the quantized algebra of polynomial functions is represented on parabolic Verma modules, respectively, over $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$. However, adjoint orbits in G are in a greater supply than in $\mathfrak{g}$. Quantization of some of them requires more general modules, which cannot be obtained by induction from a character of the parabolic extension of the stabilizer, [64, 65]. Moreover, the latter itself disappears as a natural subalgebra in $U_q(\mathfrak{g})$. This observation makes us take a more general look at already constructed quantum homogeneous spaces and conclude that they were obtained through a very special choice of the initial point. Such points are distinguished by their isotropy subgroups, whose triangular decomposition perfectly matches the triangular decomposition of G. They are all of Levi type, as for semisimple orbits in $\mathfrak{g}$, and their basis of simple positive roots is a part of the basis of the total group. That is violated for stabilizers of non-Levi type appearing among conjugacy classes in G. At the same time, one can apply a generic Weyl group transformation to the initial point in $\mathfrak{g}$ and break the nice inclusion of root bases even in the Levi case. In this respect, a generic initial point whose stabilizer is isomorphic to a Levi subgroup has much similarity with essentially non-Levi one. It makes sense therefore to extend the original approach to quantization and consider all points on the maximal torus (the Cartan subalgebra) for the initial point. They belong to the same conjugacy class if and only if they lie on the same orbit of the Weyl group. We associate a module of highest weight with every such point and realize the quantization of its conjugacy class by linear operators on that module. Points on the same Weyl group orbit give rise to isomorphic quantizations, which can be regarded as different representations of the same quantum homogeneous space. They can also be thought of as different polarizations of the same algebra.

There are other interesting problems related to quantum homogeneous spaces, such as quantization of associated vector bundles, star product formulation etc. That is well understood for classes with Levi isotropy subgroups, through the mechanism of parabolic induction, [1, 25, 32, 33, 51]. At the same time, the difference between Levi and non-Levi conjugacy classes is qualitative, and alternative representations of Levi classes could be a bridge between the two cases. A uniform approach to quantization may help to understand the non-Levi case too.

In summary, in this project, we

- quantize the borderline Levi conjugacy classes of orthogonal groups,
- associate an exact representation of quantum semi-simple conjugacy classes of simple classical

matrix groups with every point of a fixed maximal torus.

The dissertation is based on four research papers:

1. T. Ashton, A. Mudrov, *On Representations of Quantum Conjugacy Classes of $GL(n)$*. — Lett. Math. Phys. **103** (2013), 1029–1045.
2. T. Ashton, A. Mudrov, *Quantization of Borderline Levi Conjugacy Classes of Orthogonal Groups*. — J. Math. Phys. **55** (2014), 121702.
3. T. Ashton, A. Mudrov, *R-Matrix and Mickelsson Algebras for Orthosymplectic Quantum Groups*. — J.Math. Phys. **56** (2015), 081701.
4. T. Ashton, A. Mudrov, *Representations of Quantum Conjugacy Classes of Orthosymplectic Groups*, J. Math. Sci. **213** (2016), 637– 650

The structure of the thesis is as follows:

In Chapter 2, the preliminary facts about quantum groups, their representations, natural modules and a diagram technique for the analysis of $\mathbb{C}^N \otimes M_\lambda$, where M_λ is a generalized parabolic Verma module of weight λ, are discussed.

In Chapter 3, the quantizations of conjugacy classes is completed by including quantization of borderline Levi orthogonal classes.

In Chapter 4, there is a technical analysis of the R-matrix and Mickelsson algebras for orthogonal and symplectic quantum groups.

In Chapter 5, the representations of quantum conjugacy classes are constructed, under restriction to $GL(n)$.

In Chapter 6, the theory of representations of quantum conjugacy classes will be generalised for orthogonal and symplectic groups.

Chapter 2

Quantum Groups and their Natural Representations

This chapter provides the foundation for the theory in later chapters. Firstly, the construction of quantum groups $U_q(\mathfrak{g})$ from a system of roots via quantum analog of the Chevalley-Serre relations and extension to a Hopf algebra. Then the natural representation $U_q(\mathfrak{g}(N)) \to \mathrm{End}(\mathbb{C}^N)$ is discussed along with the specific action of the Chevalley generators on $\mathbb{C}^N$. At which point, principal monomials are introduced. These principal monomials are an integral part of a diagram technique, formulated as a tool for the analysis of $\mathbb{C}^N \otimes M_\lambda$, where M_λ is a generalized parabolic Verma module of weight λ.

Let G be a complex simple connected algebraic group of classical type, $\mathfrak{g}$ be its Lie algebra and let n designate the rank of $\mathfrak{g}$.

Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the inner product $(.,.)$ on $\mathfrak{h}^*$ normalized to the unit length of the highest weight of the natural representation. By R we denote the root system of $\mathfrak{g}$ with a fixed subsystem of positive roots $\mathrm{R}^+ \subset \mathrm{R}$ and the basis of simple roots $\Pi^+ \subset \mathrm{R}^+$. Using the standard realization of R in a complex Euclidean vector space with the inner product $(.,.)$, we express the simple positive roots in an orthogonal basis $\{\varepsilon_i\}$ by:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad i = 1, \ldots, n-1, \qquad \alpha_n = \begin{cases} \varepsilon_n - \varepsilon_{n+1} & \mathfrak{g} = \mathfrak{sl}(n+1) \\ \varepsilon_n & \mathfrak{g} = \mathfrak{so}(2n+1) \\ 2\varepsilon_n & \mathfrak{g} = \mathfrak{sp}(2n) \\ \varepsilon_{n-1} + \varepsilon_n & \mathfrak{g} = \mathfrak{so}(2n) \end{cases} \tag{2.1}$$

For every $\lambda \in \mathfrak{h}^*$ we denote by h_λ its image under the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$, that is $(\lambda, \beta) = \beta(h_\lambda)$ for all $\beta \in \mathfrak{h}^*$. We put $\rho = \frac{1}{2}\sum_{\alpha \in \mathrm{R}^+} \alpha$ for the Weyl vector (the half-sum of positive roots).

When we need to distinguish the root system of a subgroup, we mark it with the corresponding subscript and thus reserving by default the notation $\mathrm{R} = \mathrm{R}_{\mathfrak{g}}$, $\mathrm{R}^+ = \mathrm{R}^+_{\mathfrak{g}}$ and $\Pi^+ = \Pi^+_{\mathfrak{g}}$.

Consider the polarization $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$, where $\mathfrak{g}_\pm$ are the nilpotent Lie subalgebras of positive and negative root subspaces.

Suppose that $q \in \mathbb{C}$ is not a root of unity. Denote by $U_q(\mathfrak{g}_\pm)$ the $\mathbb{C}$-algebra generated by $e_{\pm\alpha}$, $\alpha \in \Pi^+$, subject to the q-Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array} \right]_{q_{\alpha_i}} e_{\pm\alpha_i}^{1-a_{ij}-k} e_{\pm\alpha_j} e_{\pm\alpha_i}^{k} = 0,$$

where $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, $i,j = 1,\ldots,n = \mathrm{rk}\,\mathfrak{g}$ (the rank of $\mathfrak{g}$), is the Cartan matrix, $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$, and

$$\left[\begin{array}{c} m \\ k \end{array} \right]_q = \frac{[m]_q!}{[k]_q![m-k]_q!}, \quad [m]_q! = [1]_q \cdot [2]_q \ldots [m]_q.$$

Here and further on, $[z]_q = \frac{q^z - q^{-z}}{q-q^{-1}}$ whenever $q^{\pm z}$ make sense.

Denote by $U_q(\mathfrak{h})$ the commutative $\mathbb{C}$-algebra generated by $q^{\pm h_\alpha}$, $\alpha \in \Pi^+$. The quantum group $U_q(\mathfrak{g})$ is a $\mathbb{C}$-algebra generated by $U_q(\mathfrak{g}_\pm)$ and $U_q(\mathfrak{h})$ subject to the relations [30]

$$q^{h_\alpha} e_{\pm\beta} q^{-h_\alpha} = q^{\pm(\alpha,\beta)} e_{\pm\beta}, \quad [e_\alpha, e_{-\beta}] = \delta_{\alpha,\beta} \frac{q^{h_\alpha} - q^{-h_\alpha}}{q_\alpha - q_\alpha^{-1}}.$$

Although $\mathfrak{h}$ is not contained in $U_q(\mathfrak{g})$, it is convenient for us to keep reference to $\mathfrak{h}$.

We use the notation $e_i = e_{\alpha_i}$ and $f_i = e_{-\alpha_i}$ for $\alpha_i \in \Pi^+$, in all cases apart from $i = n$, $\mathfrak{g} = \mathfrak{so}(2n+1)$, where we set $f_n = [\frac{1}{2}]_q e_{-\alpha_n}$. The reason for this is two-fold. Firstly, the natural representation can be defined through the classical assignment on the generators, as given below. Secondly, we get rid of $q_{\alpha_n} = q^{\frac{1}{2}}$ and can work over $\mathbb{C}[q]$, as the relations involved turn into

$$[e_n, f_n] = \frac{q^{h_{\alpha_n}} - q^{-h_{\alpha_n}}}{q - q^{-1}},$$

$$f_n^3 f_{n-1} - (q+1+q^{-1}) f_n^2 f_{n-1} f_n + (q+1+q^{-1}) f_n f_{n-1} f_n^2 - f_{n-1} f_n^3 = 0.$$

It is easy to see that the square root of q disappears from the corresponding factor in the presentation (2.3).

DEFINITION 2.1 [59] Hopf algebra H is defined by the following axioms:

1. H is a unital algebra $(H, \cdot, 1)$ over a field k.

2. H is a counital coalgebra (H, Δ, ε) over k. Here the coproduct and counit maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are required to obey $(\Delta \otimes \mathrm{id})\,\Delta = (\mathrm{id} \otimes \Delta)\,\Delta$ and $(\varepsilon \otimes \mathrm{id})\,\Delta = (\mathrm{id} \otimes \varepsilon)\,\Delta = \mathrm{id}$.

3. Δ, ε are algebra homomorphisms.

4. There exists an antipode map $\gamma : H \to H$ obeying $\cdot\,(\mathrm{id} \otimes \gamma)\,\Delta = \cdot\,(\gamma \otimes \mathrm{id})\,\Delta = 1\varepsilon$.

The quantum group $U_q(\mathfrak{g})$ has the following comultiplication Δ, counit ε and antipode γ maps.

$$\begin{array}{lll}\Delta(e_\alpha) = e_\alpha \otimes 1 + q^{h_\alpha} \otimes e_\alpha & \varepsilon(e_\alpha) = 0 & \gamma(e_\alpha) = -q^{-h_\alpha} e_\alpha \\ \Delta(f_\alpha) = f_\alpha \otimes q^{-h_\alpha} + 1 \otimes f_\alpha & \varepsilon(f_\alpha) = 0 & \gamma(f_\alpha) = -f_\alpha q^{h_\alpha} \\ \Delta(h_\alpha) = h_\alpha \otimes 1 + 1 \otimes h_\alpha & \varepsilon(h_\alpha) = 0 & \gamma(h_\alpha) = -h_\alpha\end{array}$$

Thus $U_q(\mathfrak{g})$ becomes a Hopf algebra. We use the Sweedler notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$ for $x \in U_\hbar(\mathfrak{g})$ for the coproduct.

Denote by T the maximal torus of G exponentiating the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Given a point $x \in T$, denote by $K \subset G$ its centralizer subgroup with the Lie algebra $\mathfrak{k}$, which is a reductive subalgebra of maximal rank in $\mathfrak{g}$. The triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{h} \oplus \mathfrak{k}_-$. There are inclusions $\mathrm{R}_\mathfrak{k} \subset \mathrm{R}_\mathfrak{g}$ and $\mathrm{R}^+_\mathfrak{k} \subset \mathrm{R}^+_\mathfrak{g}$, but not $\Pi^+_\mathfrak{k} \subset \Pi^+_\mathfrak{g}$ in general. If the latter holds, K is said to be a regular Levi subgroup of G. If K is not isomorphic to a Levi subgroup, we call it pseudo-Levi. We call it regular if a maximal Levi subgroup among those contained in K is regular. Similar terminology is used for its Lie algebra $\mathfrak{k}$. Collectively we call K and $\mathfrak{k}$ generalized Levi subgroups and subalgebras.

Fix a generalized Levi subalgebra $\mathfrak{k} \subset \mathfrak{g}$. By $\mathfrak{c}^*_\mathfrak{k}$ we denote the set of weights $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \alpha) = 0$ for all $\alpha \in \mathrm{R}_\mathfrak{k}$ and by $\mathfrak{c}^*_{\mathfrak{k},\mathrm{reg}} \subset \mathfrak{c}^*_\mathfrak{k}$ the set of weights such that $(\lambda, \alpha) = 0 \Leftrightarrow \alpha \in \mathrm{R}_\mathfrak{k}$. For each $\lambda \in \mathfrak{c}^*_\mathfrak{k}$ the element $e^{2h_\lambda} \in G$ commutes with K, and $\mathfrak{k}$ is exactly the centralizer Lie algebra of $x = e^{2h_\lambda}$ once $\lambda \in \mathfrak{c}^*_{\mathfrak{k},\mathrm{reg}}$.

The subalgebras $U_q(\mathfrak{b}_\pm) \subset U_q(\mathfrak{g})$ generated by $U_q(\mathfrak{g}_\pm)$ over $U_q(\mathfrak{h})$ are quantized universal enveloping algebras of the Borel subalgebras $\mathfrak{b}_\pm = \mathfrak{h} + \mathfrak{g}_\pm \subset \mathfrak{g}$. We consider a grading in $U_q(\mathfrak{b}_\pm)$ with $\deg e_\alpha = \deg f_\alpha = 1$, $\deg q^{\pm h_\alpha} = 0$, for $\alpha \in \Pi^+$.

The Chevalley generators e_α can be extended to a set of composite root vectors e_β for all $\beta \in \mathrm{R}$. A normally ordered set of root vectors generate a Poincaré-Birkhoff-Witt (PBW) basis of $U_q(\mathfrak{g})$ over $U_q(\mathfrak{h})$, [16]. We will use $\mathfrak{g}_\pm$ to denote the vector space spanned by $\{e_{\pm\beta}\}_{\beta \in \mathrm{R}^+}$.

The universal R-matrix is an element of a certain extension of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. We heavily use the intertwining relation

$$\mathcal{R}\Delta(x) = \Delta^{op}(x)\mathcal{R}, \tag{2.2}$$

between the coproduct and its opposite for all $x \in U_q(\mathfrak{g})$. Let $\{\varepsilon_i\}_{i=1}^n \subset \mathfrak{h}^*$ be the standard orthonormal basis and $\{h_{\varepsilon_i}\}_{i=1}^n$ the corresponding dual basis in $\mathfrak{h}$. The exact expression for $\mathcal{R}$ can be extracted from [16], Theorem 8.3.9, as the ordered product

$$\mathcal{R} = q^{\sum_{i=1}^n h_{\varepsilon_i} \otimes h_{\varepsilon_i}} \prod_\beta \exp_{q_\beta}\Big((1 - q_\beta^{-2})(e_{-\beta} \otimes e_\beta)\Big) \in U_q(\mathfrak{b}_-) \hat{\otimes} U_q(\mathfrak{b}_+), \tag{2.3}$$

where $\exp_q(x) = \sum_{k=0}^\infty q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!}$.

We can also consider $U_q(\mathfrak{g})$ over the ring $\mathbb{C}[q, q^{-1}]$ and its localizations. Further extension over $\mathbb{C}[\![\hbar]\!]$ via $q = e^\hbar$ determines an embedding $U_q(\mathfrak{g}) \subsetneq U_\hbar(\mathfrak{g})$, for which we use the same notation. Then

$U_\hbar(\mathfrak{g})$ is the completion of $U_q(\mathfrak{g})$ in $\hbar$-adic topology. Note that $\mathfrak{h} \subset U_\hbar(\mathfrak{g})$ contrary to $U_q(\mathfrak{g})$. Unless otherwise stated, the quantum group $U_q(\mathfrak{g})$ and its modules are considered over the complex field, upon specialization of q to not a root of unity. We assume that $U_\hbar(\mathfrak{g})$-modules are free over $\mathbb{C}[\![\hbar]\!]$ and their rank will be referred to as dimension. Finite dimensional $U_\hbar(\mathfrak{g})$-modules are deformations of their classical counterparts, and we drop the reference to $\hbar$ to simplify notation.

Let $U_\hbar(\mathfrak{h})$ be the Cartan subalgebra in $U_\hbar(\mathfrak{g})$. We shall deal with $U_\hbar(\mathfrak{h})$-diagonalizable, i.e. weight modules. If V is an $\mathfrak{h}$-invariant subspace, we mean by $[V]_\alpha$ the subspace of weight $\alpha \in \mathfrak{h}^*$. We stick to the additive parametrization of weights facilitated by the embedding $U_q(\mathfrak{h}) \subset U_\hbar(\mathfrak{h})$. Under this convention, weights belong to $\frac{1}{\hbar}\mathfrak{h}^*[\![\hbar]\!]$ and are well defined on $t_{\alpha_i}^{\pm 1} \in q^{\mathfrak{h}}$. It is sufficient for our needs to confine them to the subspace $\frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^* \subset \frac{1}{\hbar}\mathfrak{h}^*[\![\hbar]\!]$.

We denote by $\mathfrak{c}_\mathfrak{k} \subset \mathfrak{h}$ the center of $\mathfrak{k}$ and realize its dual $\mathfrak{c}_\mathfrak{k}^*$ as a subspace in $\mathfrak{h}^*$ thanks to the canonical inner product.

2.1 Natural Representations

Fix $N = n+1$, $N = 2n+1$ or $N = 2n$ when $\mathfrak{g} = \mathfrak{sl}(n+1)$, $\mathfrak{g} = \mathfrak{so}(2n+1)$ or $\mathfrak{g} = \mathfrak{sp}(2n), \mathfrak{so}(2n)$ respectively. The vector space $\mathbb{C}^N$ is regarded as a $U_q(\mathfrak{g})$-module supporting its natural representation.

Let I designate the set of integers $\{1,\ldots,N\}$ and let $\{\varepsilon_i\}_{i\in I}$ be the weights of the natural $U_q(\mathfrak{g})$-module $\mathbb{C}^N$. Then $\{\varepsilon_i\}_{i=1}^{n+\delta_{\mathfrak{g},\mathfrak{sl}}}$, $n = \operatorname{rk}\mathfrak{g}$, form an orthogonal basis in $\mathfrak{h}^*$. The simple positive roots are expressed through $\{\varepsilon_i\}_{i=1}^{n+\delta_{\mathfrak{g},\mathfrak{sl}}}$, as shown in (2.1). Let $i' = N+1-i$ for $i \in I$ and $\varepsilon_{i'} = \varepsilon_{N+1-i} = -\varepsilon_i$ for $\mathfrak{g}$ orthogonal or symplectic.

Denote by $w_i \in \mathbb{C}^N$ the standard basis elements of weight ε_i, $i = 1,\ldots,N$.

By Γ we denote the root lattice $\Gamma = \mathbb{Z}\Pi^+$ with $\Gamma^+ = \mathbb{Z}_+\Pi^+$. For $\beta \in \Gamma^+$ we define $P(\beta)$ to be the set of all ordered pairs $i,j \in I$ such that $\varepsilon_i - \varepsilon_j = \beta$. For each $\alpha \in \Pi^+ \subset \Gamma^+$, the pairs $(i,j) \in P(\alpha)$ are called simple.

Let $e_{ij} \in \operatorname{End}(\mathbb{C}^N)$, $i,j \in I$, denote the standard matrix units. The following assignment $\pi : U_\hbar(\mathfrak{g}) \to \operatorname{End}(\mathbb{C}^N)$ defines a representation of $\mathfrak{g}$, which is equivalent to the natural representation:

$$\pi(e_\alpha) = \sum_{(l,r)\in P(\alpha)} e_{lr}, \quad \pi(f_\alpha) = \sum_{(l,r)\in P(\alpha)} e_{rl}, \quad \pi(h_{\varepsilon_i}) = \begin{cases} e_{ii} & \mathfrak{g} = \mathfrak{sl} \\ e_{ii} - e_{i'i'} & \mathfrak{g} = \mathfrak{so}, \mathfrak{sp} \end{cases}$$

For $\mathfrak{g} \neq \mathfrak{sl}(n+1)$, the assignments expressed explicitly are:

$$\pi(e_i) = e_{i,i+1} + e_{i'-1,i'}, \quad \pi(f_i) = e_{i+1,i} + e_{i',i'-1}, \quad \pi(h_{\alpha_i}) = e_{ii} - e_{i+1,i+1} + e_{i'-1,i'-1} - e_{i'i'},$$

for $i = 1,\ldots,n-1$ and

$$\pi(e_n) = e_{n,n+1} + e_{n'-1,n'}, \quad \pi(f_n) = e_{n+1,n} + e_{n',n'-1}, \quad \pi(h_{\alpha_n}) = e_{nn} - e_{n'n'},$$

$$\pi(e_n) = e_{nn'}, \quad \pi(f_n) = e_{n'n}, \quad \pi(h_{\alpha_n}) = 2e_{nn} - 2e_{n'n'},$$

$$\pi(e_n) = e_{n-1,n'} + e_{n,n'+1}, \quad \pi(f_n) = e_{n',n-1} + e_{n'+1,n}, \quad \pi(h_{\alpha_n}) = e_{n-1,n-1} + e_{nn} - e_{n'n'} - e_{n'+1,n'+1},$$

respectively, for $\mathfrak{g} = \mathfrak{so}(2n+1)$, $\mathfrak{g} = \mathfrak{sp}(2n)$, and $\mathfrak{g} = \mathfrak{so}(2n)$.

Note that $\pi(e_i)$, $\pi(f_i)$ and $\pi(h_{\alpha_i})$ for $i = 1, \ldots, n-1$ defines a direct sum of two representations of $\mathfrak{gl}(n)$. Where $\mathfrak{gl}(n) \subset \mathfrak{g}$ is the Lie subalgebra with the simple roots $\{\alpha_i\}_{i=1}^{n-1}$ and $U_q\bigl(\mathfrak{gl}(n)\bigr)$ the corresponding quantum subgroup in $U_q(\mathfrak{g})$.

A similar assignment for Chevalley generators e_α and f_α changes the summation of the two standard matrix units to their difference, which corresponds to the standard representation that preserves the bilinear form with entries $C_{ij} = \delta_{i'j}$, for $\mathfrak{g} = \mathfrak{so}(N)$, and $C_{ij} = \mathrm{sign}(\frac{N+1}{2} - i)\delta_{i'j}$, for $\mathfrak{g} = \mathfrak{sp}(N)$.

$$\zeta(e_i) = e_{i,i+1} - e_{i'-1,i'}, \quad \zeta(f_i) = e_{i+1,i} - e_{i',i'-1}, \quad \zeta(h_{\alpha_i}) = e_{ii} - e_{i+1,i+1} + e_{i'-1,i'-1} - e_{i'i'},$$

for $i = 1, \ldots, n-1$, and

$$\zeta(e_n) = e_{n,n+1} - e_{n'-1,n'}, \quad \zeta(f_n) = e_{n+1,n} - e_{n',n'-1}, \quad \zeta(h_{\alpha_n}) = e_{nn} - e_{n'n'},$$

$$\zeta(e_n) = e_{nn'}, \quad \zeta(f_n) = e_{n'n}, \quad \zeta(h_{\alpha_n}) = 2e_{nn} - 2e_{n'n'},$$

$$\zeta(e_n) = e_{n-1,n'} - e_{n,n'+1}, \quad \zeta(f_n) = e_{n',n-1} - e_{n'+1,n}, \quad \zeta(h_{\alpha_n}) = e_{n-1,n-1} + e_{nn} - e_{n'n'} - e_{n'+1,n'+1},$$

respectively, for $\mathfrak{g} = \mathfrak{so}(2n+1)$, $\mathfrak{g} = \mathfrak{sp}(2n)$, and $\mathfrak{g} = \mathfrak{so}(2n)$.

Note that Chevalley generators are normalized so that their representation matrices are independent of q.

The algebras $U_q(\mathfrak{g}_\pm)$ are isomorphic via the Chevalley involution $f_\alpha \leftrightarrow e_\alpha$. We call contragredient the representation of $U_q(\mathfrak{g}_\pm)$ on $\mathbb{C}^N$ given by $e_k w_i = \delta_{k,i} w_{i+1}$ and $f_k w_i = -\delta_{k,n-i+1} w_{i-1}$. It factors through the automorphisms $e_i \mapsto e_{n-i}$, $f_i \mapsto f_{n-i}$, (inversion of Dynkin diagram) and the natural representation of $U_q(\mathfrak{b}_\pm)$. Alternatively, it is a composition of the Chevalley involution $f_\alpha \leftrightarrow e_\alpha$ and natural representation.

For any finite dimensional $U_q(\mathfrak{g})$-module W define the (right) dual representation on W^* as $\langle w, x \triangleright u\rangle = \langle \gamma^{-1}(x) \triangleright w, u\rangle$, where $w \in W$, $u \in W^*$, and $x \in U_q(\mathfrak{g})$. Although $U_q(\mathfrak{g}_\pm)$ are not Hopf algebras, their dual representations are still defined through the embedding $U_q(\mathfrak{g}_\pm) \subset U_q(\mathfrak{g})$. Consider another copy of vector space $\mathbb{C}^N$ as dual to initial $\mathbb{C}^N$, with the basis $\{v_i\}$, and the right conatural representation of $U_q(\mathfrak{g}_+)$ on it. Since $\gamma^{-1}(e_\alpha) = -e_\alpha q^{-h_\alpha}$, we have

$$e_\alpha w_i = \sum_{j=1}^{n} \pi(e_\alpha)_{ij} w_j, \quad e_\alpha v_i = -q^{-(\alpha,\varepsilon_i)} \sum_{j=1}^{n} \pi(e_\alpha)_{ji} v_j, \tag{2.4}$$

where $\pi(e_\alpha)$, $\alpha \in \Pi^+$, are the matrices of the natural $\mathfrak{g}_+$-action on $\mathbb{C}^N$.

The representation $\pi : U_q(\mathfrak{g}) \to \mathrm{End}\left(\mathbb{C}^N\right)$ is equivalent to the natural representation. The action of the Chevalley generators, up to scalar multipliers, can be conveniently visualized by the diagrams

$$\mathfrak{g} = \mathfrak{sl}(n+1)$$

$$\begin{tikzcd}
w_{n+1} & w_n \arrow[l, "f_{\alpha_n}"'] & \cdots \arrow[l, "f_{\alpha_{n-1}}"'] & w_2 \arrow[l, "f_{\alpha_2}"'] & w_1 \arrow[l, "f_{\alpha_1}"']
\end{tikzcd}$$

$$\mathfrak{g} = \mathfrak{so}(2n+1)$$

$$\begin{tikzcd}
w_{1'} & \cdots \arrow[l, "f_{\alpha_1}"'] & w_{n'} \arrow[l, "f_{\alpha_{n-1}}"'] & w_{n+1} \arrow[l, "f_{\alpha_n}"'] & w_n \arrow[l, "f_{\alpha_n}"'] & \cdots \arrow[l, "f_{\alpha_{n-1}}"'] & w_1 \arrow[l, "f_{\alpha_1}"']
\end{tikzcd}$$

$$\mathfrak{g} = \mathfrak{sp}(2n)$$

$$\begin{tikzcd}
w_{1'} & \cdots \arrow[l, "f_{\alpha_1}"'] & w_{n'} \arrow[l, "f_{\alpha_{n-1}}"'] & w_n \arrow[l, "f_{\alpha_n}"'] & \cdots \arrow[l, "f_{\alpha_{n-1}}"'] & w_1 \arrow[l, "f_{\alpha_1}"']
\end{tikzcd}$$

$$\mathfrak{g} = \mathfrak{so}(2n)$$

$$\begin{tikzcd}
w_{1'} & \cdots \arrow[l, "f_{\alpha_1}"'] & w_{n'+1} \arrow[l, "f_{\alpha_{n-2}}"'] & w_{n'} \arrow[l, "f_{\alpha_{n-1}}"'] & w_n \arrow[ll, bend right, "f_{\alpha_n}"'] & w_{n-1} \arrow[l, "f_{\alpha_{n-1}}"'] \arrow[ll, bend right, "f_{\alpha_n}"'] & \cdots \arrow[l, "f_{\alpha_{n-2}}"'] & w_1 \arrow[l, "f_{\alpha_1}"']
\end{tikzcd}$$

Reversing the arrows we get the diagrams for positive Chevalley generators of $\mathfrak{g}$.

Similarly, we can consider dual natural representation of $U_q(\mathfrak{g})$ on $\mathbb{C}^N$. In the dual basis $\{v^i\}_{i=1}^N$, the graphs will be similar, with all arrows reversed.

We introduce a partial ordering on the integer interval I by setting $i \preccurlyeq j$ if and only if there is a (monic) Chevalley monomial $\psi \in U_q(\mathfrak{g}_-)$ such that w_j is equal to ψw_i up to an invertible scalar multiplier, $w_j = \psi w_i$. This monomial, if exists, represents a path from w_i to w_j in the representation diagram, which becomes the Hasse diagram of the poset. Such ψ is unique, which is obvious for the series A, B and C and still true for D. Indeed, two different paths from w_{n-1} to w_{n+2} yield the products $f_{\alpha_n} f_{\alpha_{n-1}}$ and $f_{\alpha_{n-1}} f_{\alpha_n}$, which are the same due to Serre relations. We denote this monomial by ψ_{ji}. The relation $\prec$ is consistent with the natural ordering on $\mathbb{Z}$, and coincides with it unless $\mathfrak{g} = \mathfrak{so}(2n)$. In the latter case n and n' are incomparable.

In what follows, we also use the monomial ψ^{ij} obtained from ψ_{ji} by reversing the order of Chevalley generators, so that $v^i = \psi^{ij} v^j$. We also put $\psi^{ii} = 1$ for all i. It is clear that $\psi^{ij} = \psi^{im}\psi^{mj}$ for any m such that $i \preccurlyeq m \preccurlyeq j$.

DEFINITION 2.2 We call ψ^{ij} principal monomial of the pair $i \preccurlyeq j$.

Remark that all Chevalley monomials of weight $\varepsilon_j - \varepsilon_i$ are obtained from ψ^{ij} by permutation of factors.

Consider the left ideal $J \subset U_q(\mathfrak{g}_-)$ generated by $f_{\alpha_1}^2, f_{\alpha_i}$, $i > 1$. Let $\mathcal{N}$ be the quotient module $U_q(\mathfrak{g}_-)/J$. Remark that the automorphism of $U_q(\mathfrak{g}_-)$ defined by $f_{\alpha_i} \mapsto a_i f_{\alpha_i}$ for invertible $a_i \in \mathbb{C}$ leaves J invariant and gives rise to an automorphism of $\mathcal{N}$.

PROPOSITION 2.3 *$\mathcal{N}$ is isomorphic to the natural $U_q(\mathfrak{g}_-)$-module $\mathbb{C}^N$.*

Proof. This is a standard fact about finite-dimensional irreducible quotients of Verma $U_q(\mathfrak{g})$-modules, [48]. The special case of $\mathbb{C}^N$ can be checked directly by constructing the obvious epimorphism $\mathcal{N} \to \mathbb{C}^N$, $1 \mapsto w_1$, and its section $w_1 \mapsto 1 + J$, $w_i \mapsto f_{\alpha_{i-1}} \dots f_{\alpha_1} + J$, $i > 1$. $\square$

COROLLARY 2.4 *The (right or left) conatural and contragredient representations of the algebras $U_q(\mathfrak{g}_\pm)$ on $\mathbb{C}^N$ are isomorphic.*

Proof. Indeed, in the case of $U_q(\mathfrak{g}_+)$, they are cyclic representations generated by the vector w_1 satisfying $e^2_{\alpha_1} w_1 = e_{\alpha_i} w_1 = 0$, $i > 2$. Hence they are quotients of $\mathcal{N}$ by some submodules. Since their dimension is N, those submodules are zero, and the quotients are isomorphic to $\mathcal{N}$. The case of $U_q(\mathfrak{g}_-)$ is checked similarly. $\square$

COROLLARY 2.5 *Let V be a $U_q(\mathfrak{g}_+)$-module and regard $\mathbb{C}^N$ as the conatural $U_q(\mathfrak{g}_+)$-module. Then* $\mathrm{Hom}_{U_q(\mathfrak{g}_+)}(\mathbb{C}^N, V) = \{v \in V | e^2_{\alpha_1} v = e_{\alpha_i} v = 0,\ i > 1\}$.

Proof. Since the module $\mathbb{C}^N$ is cyclic, every homomorphism from $\mathrm{Hom}_{U_q(\mathfrak{g}_+)}(\mathbb{C}^N, V)$ is determined by the assignment $1 \mapsto v$, where the vector v annihilates the ideal J. $\square$

2.2 Diagram Technique

In what follows, we work out a tool for the analysis of $\mathbb{C}^N \otimes M_\lambda$, where M_λ is a generalized parabolic Verma module of weight λ. In this section, we do it for the ordinary Verma module $M_\lambda = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_+)} \mathbb{C}_\lambda$ with $\lambda \in \frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$. An essential part of the technique is a diagram language developed in [10]. We consider the standard filtration $V_\bullet = (V_i)_{i=1}^N$, $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_N = \mathbb{C}^N \otimes M_\lambda$, where V_i is generated by $\{w_j \otimes v_\lambda\}_{j \leqslant i}$. Its graded module $\mathrm{gr} V_\bullet$ is a direct sum of V_j/V_{j-1}, which are isomorphic to the Verma modules $M_{\lambda+\varepsilon_j}$ (the proof of [12], Lemma 5, readily adapts to quantum groups).

Given $\beta \in \Gamma^+$ we define $\Psi_\beta \subset U_q(\mathfrak{g}_-)$ to be the subset of Chevalley monomials of weight β. We assume that a pair $(i,j) \in P(\beta)$ is chosen for this section. Having fixed an order of elementary factors in ψ, we regard it a as path from v_λ to ψv_λ. We associate with ψv_λ a graph H_ψ with nodes $\{v^k\} \in M_\lambda$, $v^j = v_\lambda$, $v^i = \psi v_\lambda$, and arrows being negative Chevalley generators acting on M_λ. For $\psi = \psi^{ij}$, this path is unique in almost all cases (except for type D, where we eliminate the ambiguity by fixing the order as $f_{\alpha_{n-1}} f_{\alpha_n}$). For principal ψ, we are concerned not just with the terminating node ψv_λ, but also in all intermediate nodes. On the contrary, for non-principal ψ, only ψv_λ is important for us, while the specific path is immaterial.

We say that f_α has length 2 if $\alpha = \alpha_n$ and $\mathfrak{g} = \mathfrak{so}(2n)$. All other generators are assigned with length 1. If all factors in ψ have length 1, we write $\psi = \phi_i \dots \phi_{j-1}$ with $\phi_k \in \{f_\alpha\}_{\alpha \in \Pi^+}$, and we set $v^k = \phi_k v^{k+1}$. Then the diagram H_ψ is set to be

$$\begin{tikzcd}
v^i & v^{i+1} \arrow[l, "\phi_i"'] & \cdots \arrow[l, "\phi_{\alpha_{i+1}}"'] & v^{j-1} \arrow[l, "\phi_{j-2}"'] & v^j \arrow[l, "\phi_{j-1}"']
\end{tikzcd}$$

Now suppose that ψ has (exactly one) factor of length 2. Write $\psi = \phi_i \dots \phi_k \phi_{k+2} \dots \phi_{j-1}$, where $\phi_k = f_{\alpha_n}$ (there are $j - i - 1$ factors). Then the graph H_ψ is

$$\begin{tikzcd}
v^i & \cdots \arrow[l, "\phi_i"'] & v^k \arrow[l, "\phi_{k-1}"'] & v^{k+1} & v^{k+2} \arrow[l, dashed] \arrow[ll, bend right, "\phi_k = f_{\alpha_n}"'] & \cdots \arrow[l, "\phi_{k+2}"'] & v^j \arrow[l, "\phi_{j-1}"']
\end{tikzcd}$$

Here we distinguish two cases. If $\psi = \psi^{ij}$, then $\phi_{k-1} = f_{\alpha_{n-1}}$, and the dashed arrow $f_{\alpha_{n-1}}$ is included in H_ψ. The node v^{k+1} is set to $f_{\alpha_{n-1}} v^{k+2}$. For non-principal ψ, the node v^{k+1} is arbitrary (immaterial) and there is no arrow from v^{k+2} to v^{k+1}.

We also consider a graph V_{ij}, which is a part of the natural representation diagram of $U_q(\mathfrak{g}_-)$ that includes all paths from w_i to w_j. We transpose it to make a vertical graph oriented from top w_i to bottom w_j.

We denote by $\mathrm{Arr}(v^m)$ the set of arrows originated at v^m and similarly $\mathrm{Arr}(w_k)$ the set of arrows from w_k. By construction, an arrow from node m to node k has length $k - m$.

Finally, we define tensor product $D_\psi = H_\psi \otimes V_{ij}$ as a graph on a two-dimensional lattice whose nodes are $w_k^m = w_k \otimes v^m \in \mathbb{C}^N \otimes M_\lambda$ and arrows are $\mathrm{Arr}(w_k^m) = \mathrm{Arr}(w_k) \otimes \mathrm{id} \bigcup \mathrm{id} \otimes \mathrm{Arr}(v^m)$, The diagram is oriented so that H_ψ-arrows and V_{ij}-arrows are directed, respectively, leftward and downward; the origin w_i^j is in the right upper corner. We need only the triangular part of the diagram including the nodes w_k^m with $k + m \geqslant i + j$. The set $\{w_k^k\}_{k=i}^j$ is called principal diagonal. With $\psi = \psi^{ij}$, the node w_k^k on the principal diagonal is $w_k \otimes \psi^{kj} v_\lambda$, $k = i, \dots, j$. Here is an example of diagram D_ψ with all arrows of length 1:

$$\begin{tikzcd}
w_i^i & w_i^{i+1} \arrow[l, "\phi_i"'] \arrow[d] & w_i^{i+2} \arrow[l, "\phi_{i+1}"'] \arrow[d] & \cdots \arrow[l, "\phi_{i+2}"'] & w_i^{j-1} \arrow[l, "\phi_{j-2}"'] \arrow[d] & w_i^j \arrow[l, "\phi_{j-1}"'] \arrow[d] \\
 & w_{i+1}^{i+1} & w_{i+1}^{i+2} \arrow[l] \arrow[d] & \cdots \arrow[l] & w_{i+1}^{j-1} \arrow[l] \arrow[d] & w_{i+1}^j \arrow[l] \arrow[d] \\
 & & w_{i+2}^{i+2} & \cdots \arrow[l] & w_{i+2}^{j-1} \arrow[l] \arrow[d] & w_{i+2}^j \arrow[l] \arrow[d] \\
 & & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & & & w_{j-1}^{j-1} & w_{j-1}^j \arrow[l] \arrow[d] \\
 & & & & & w_j^j
\end{tikzcd}$$

The arrows represent the action of the Chevalley generators on the tensor factors $\mathbb{C}^N$ (vertical) and M_λ (horizontal). The following property of this action readily follows from the coproduct of the Chevalley generators: suppose that $\phi \in \mathrm{Arr}(v^m)$ and $\phi \notin \mathrm{Arr}(w_k)$. If $v^r = \phi v^m$, then $\phi(w_k^m) = w_k^r$, i.e., the horizontal arrow yields the action of ϕ on the entire tensor product. In general, $\phi(w_k^m) = w_k^r \mod \mathbb{C} w_s^m$, where $w_s = \phi w_k$.

Suppose that nodes of a column segment BC (with C the bottom node) belong to a $U_q(\mathfrak{g})$-submodule $M \subset \mathbb{C}^N \otimes M_\lambda$. Let ϕ be a Chevalley generator assigned to a horizontal arrow with the origin at this column. Consider the following situations:

1. The length of ϕ is 1.

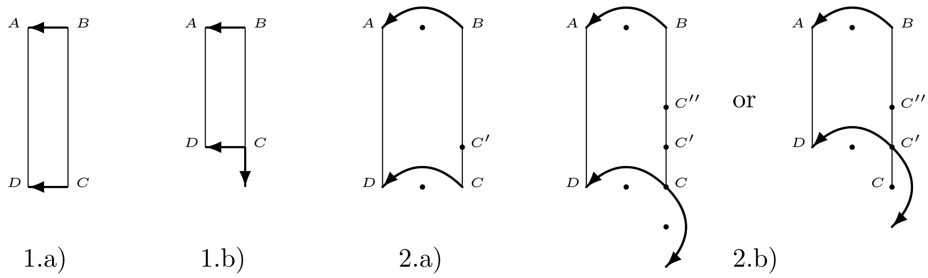
   (a) There is no vertical ϕ-arrow with the origin at C.

   (b) There is a vertical ϕ-arrow with the origin at C.

2. The length of ϕ is 2, and the size of BC is 2 or greater. Let C' and C'' be the nodes 1 and 2 steps up, respectively.

   (a) There is no vertical ϕ-arrow with the origin at C and at C'.

   (b) There is a vertical ϕ-arrow with the origin either at C or at C'.



1.a) 1.b) 2.a) 2.b)

DEFINITION 2.6 We call the transition from column BC to column AD an elementary move or simply a move of length equal to the length of ϕ-arrow. The elementary moves 1.a) and 2.a) are called left. The elementary moves 1.b) and 2.b) are called diagonal.

LEMMA 2.7 (Elementary moves) *Under the conditions above, the column segment AD lies in M.*

Proof. Clear. □

We will use elementary moves to reach a node or collection of nodes in the diagram starting from the rightmost column, which is assumed to be in a submodule M. That way we prove that the target nodes are in M.

Let $D'_\psi \subset D_\psi$ denote the subgraph whose nodes form the triangle lying above the principal diagonal, i.e. $\{w_k^m\}_{k+m>i+j}$.

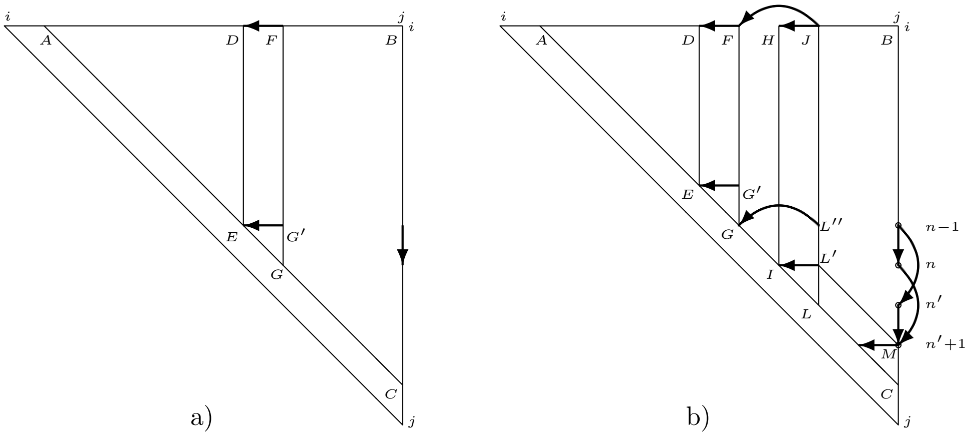
LEMMA 2.8 *Suppose that $\psi = \psi^{ij}$ is a principal monomial. Then the linear span of D'_ψ lies in V_{j-1}.*

Proof. Suppose that all horizontal arrows in D'_ψ have length 1, such as for $\mathfrak{g} = \mathfrak{sl}(2n+1)$, $\mathfrak{g} = \mathfrak{so}(2n+1)$, and $\mathfrak{g} = \mathfrak{sp}(2n)$. Consider the diagram D_ψ on Fig.2.1.a, where D'_ψ is the triangle ABC. The column BC belongs to V_{j-1} by construction. All arrows have length 1. Applying elementary diagonal moves we prove that ABC is in V_{j-1}.

Now suppose there is a horizontal arrows of length 2. Assuming $i \leqslant n-1$, $n'+1 \leqslant j$, consider the diagram D_ψ where the triangle D'_ψ is denoted by ABC (cf. Fig.2.1.b). The rightmost column BC

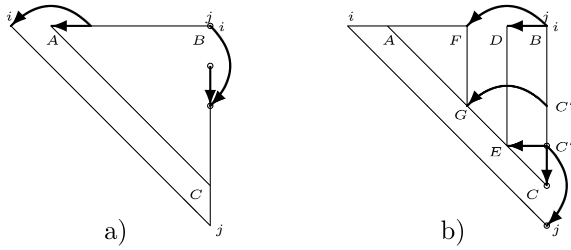
belongs to V_{j-1} by construction. For each node in the trapezoid $JBCL$ there is a horizontal arrow of length 1. Those arrows are distinct from vertical arrows for all nodes in the line $L'M \subset JBCL$. Apply the corresponding left moves to the columns rested on $L'M$. This operation proves that trapezoid $HBCI$ is in V_{j-1}. Then apply the diagonal move of length 2 to the column JL and get $FG \subset V_{j-1}$. All arrows in the triangle AFG have length 1, therefore $AFG \subset V_{j-1}$, via diagonal moves.

Figure 2.1:



The case $i = n$, $n' + 1 \leqslant j$ displayed on Fig.2.2.a is similar to already considered: all horizontal arrows within D'_{ψ} are of length 1. The case $i \leqslant n - 1$, $n' = j$ is displayed on Fig.2.2.b. Apply the diagonal move of length 1 to the column BC' and get $DE \subset V_{j-1}$. Then apply the diagonal move of length 2 to BC' and get $FG \subset V_{j-1}$. Thence the entire triangle AFG is in M. $\square$

Figure 2.2:



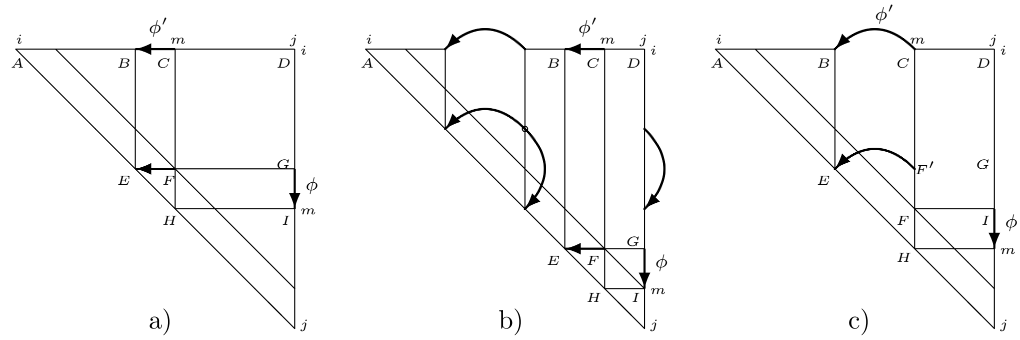
PROPOSITION 2.9 *Suppose $\psi \in \Psi_{\beta}$, $(i, j) \in P(\beta)$, and $\psi \neq \psi^{ij}$. Then $w_i \otimes \psi v_{\lambda} \in V_{j-1}$.*

Proof. Consider a factorization $\psi = \psi' \psi^{mj}$, where m is some integer satisfying $i \prec m \preccurlyeq j$ and $\psi' \in \Psi_{\varepsilon_i - \varepsilon_m}$. Choose m to be the smallest possible. In the factorization $\psi^{ij} = \psi^{im}\psi^{mj}$ let ϕ be the rightmost Chevalley factor in ψ^{im}, while ϕ' the rightmost factor in ψ'. Due to the choice of m, $\phi \neq \phi'$. Further we consider algebras of types A, B, C separately from D.

In diagrams of types A, B and C, all arrows have length 1, Fig.2.3.a. All nodes in the north-east rectangle $CDIH$ are the same as in $D_{\psi^{ij}}$. Therefore $CDGF$ is in V_{j-1}, by Lemma 2.8.

Since $\phi' \neq \phi$, the left move via ϕ' maps CF onto BE, modulo $CF \subset V_{j-1}$, proving $BE \subset V_{j-1}$. Applying diagonal moves to BE we get the triangle $ABE \subset V_{j-1}$ including the node A, which is $w_i^i = w_i \otimes \psi v_\lambda$. Now we look at the type D. We can assume that $i \leqslant n-1, n'+1 \leqslant j$, since otherwise this case reduces to already considered. If the length of ϕ' is 1, the reasoning is the same as above. The only difference is that one may have to use a diagonal move of length 2 in transition from BE to A, see Fig.2.3.b. If the length of ϕ' is 2, then the transition to BE is performed via ϕ' applied to $CF' \subset V_{j-1}$, as shown on Fig.2.3.c. This proves that $BE \subset V_{j-1}$. Further, all horizontal arrows in the triangle ABE are of length 1 (the factor f_{α_n} enters ψ only once). This situation is similar to the types A, B and C considered earlier. Thus, the node $A = w_i^i = w_i \otimes \psi v_\lambda$, belongs to V_{j-1}. □

Figure 2.3:



For $i \preccurlyeq j$ denote by $|i-j|$ the distance (the number of arrows in a path) from i to j on the Hasse diagram of the natural representation of $U_q(\mathfrak{g}_-)$.

Set $h_i = h_{\varepsilon_i} \in \mathfrak{h}$ for all $i \in I$ and introduce $\eta_{ij} \in \mathfrak{h} + \mathbb{C}$ for $i, j \in I$, by

$$\eta_{ij} = h_i - h_j + (\varepsilon_i - \varepsilon_j, \rho) - \frac{1}{2}\|\varepsilon_i - \varepsilon_j\|^2. \tag{2.5}$$

Here $\|\mu\|$ is the Euclidean norm on $\mathfrak{h}^*$.

LEMMA 2.10 *Suppose that $(l,r) \in P(\alpha)$ for some $\alpha \in \Pi^+$. Then*

i) if $l < r < j$, then $\eta_{lj} - \eta_{rj} = h_\alpha + (\alpha, \varepsilon_j - \varepsilon_r)$,

ii) if $i < l < r$, then $\eta_{li} - \eta_{ri} = h_\alpha + (\alpha, \varepsilon_i - \varepsilon_r)$,

iii) $\eta_{lr} = h_\alpha$.

Proof. We have $(\alpha, \rho) = \frac{1}{2}\|\alpha\|^2$ for all $\alpha \in \Pi^+$. This proves iii). Further, for $\varepsilon_l - \varepsilon_r = \alpha$:

$$\begin{aligned}
\eta_{lj} - \eta_{rj} &= h_\alpha + \frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\varepsilon_j - \varepsilon_r\|^2 - \frac{1}{2}\|\varepsilon_j - \varepsilon_r - \alpha\|^2 = h_\alpha + (\alpha, \varepsilon_j - \varepsilon_r), \quad r < j,\\
\eta_{li} - \eta_{ri} &= h_\alpha + \frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\varepsilon_i - \varepsilon_r\|^2 - \frac{1}{2}\|\varepsilon_i - \varepsilon_r - \alpha\|^2 = h_\alpha + (\alpha, \varepsilon_i - \varepsilon_r), \quad i < l,
\end{aligned}$$

which proves i) and ii). □

PROPOSITION 2.11 *Suppose that $i, j \in I$ are such that $i \prec j$. Then*

$$w_i \otimes \psi^{ij} v_\lambda \quad = \quad (-1)^{|i-j|} q^{-(\lambda, \eta_{ij})} w_j \otimes v_\lambda \mod V_{j-1}. \tag{2.6}$$

Proof. Suppose that $\alpha \in \Pi^+$ and $(i,k) \in P(\alpha)$. By Lemma 2.8, the node $w_i \otimes \psi^{kj} v_\lambda \in D'_\psi$ lies in V_{j-1} Applying $\Delta f_\alpha = f_\alpha \otimes q^{-h_\alpha} + 1 \otimes f_\alpha$ to $w_i \otimes \psi^{kj} v_\lambda$ we get

$$w_i \otimes \psi^{ij} v_\lambda = -q^{-(\lambda,\alpha)-(\alpha,\varepsilon_j-\varepsilon_k)} w_k \otimes \psi^{kj} v_\lambda$$

$$= -q^{-(\lambda,\eta_{ij}-\eta_{kj})} w_j \otimes \psi^{kj} v_\lambda \mod V_{j-1}$$

for all $k \preccurlyeq j$. Here we used $f_\alpha w_i = w_k$ and $f_\alpha \psi^{kj} = \psi^{ij}$ for all $k \preccurlyeq j$. Proceeding recursively along the path from i to j with the boundary condition $\eta_{jj} = 0$ we complete the proof. $\square$

Chapter 3

Quantization of Borderline Levi Conjugacy Classes of Orthogonal Groups

This chapter is devoted to quantization of a special family of conjugacy classes in the complex algebraic group $G = SO(N)$. This completes the construction of quantum semisimple conjugacy classes of $SO(N)$ and, generally, of all simple groups of the infinite series. Classes of our present concern have isotropy subgroups with a Cartesian factor $SO(2) \times SO(P)$, where P is of the same parity as N. Due to the isomorphism $GL(1) \simeq SO(2)$, they form a borderline between the Levi and non-Levi families, whose bulk cases have been processed in [65, 68, 69].

Consider the borderline class O_x passing through the diagonal matrix x with entries

$$\underbrace{\mu_1, \dots, \mu_1}_{n_1}, \dots, \underbrace{\mu_\ell, \dots, \mu_\ell}_{n_\ell}, -1, \underbrace{1, \dots, 1}_{P}, -1, \underbrace{\mu_\ell^{-1}, \dots, \mu_\ell^{-1}}_{n_\ell}, \dots, \underbrace{\mu_1^{-1}, \dots, \mu_1^{-1}}_{n_1},$$

where $P = 2p$ if $N = 2n$ and $P = 2p+1$ if $N = 2n+1$. The complex vector $\boldsymbol{\mu} \in \mathbb{C}^{\ell+2}$, where $\mu_{\ell+1} = -1$, $\mu_{\ell+2} = 1$ satisfies the conditions $\mu_i \neq \mu_j^{\pm 1}$ for $i < j \leqslant \ell$ and $\mu_i^2 \neq 1$ for $1 \leqslant i \leqslant \ell$. The centralizer of the point $x \in G$ is the subgroup

$$K = GL(n_1) \times \ldots \times GL(n_\ell) \times SO(2) \times SO(P), \tag{3.1}$$

whose Lie algebra $\mathfrak{k}$ is a Levi subalgebra in $\mathfrak{g}$,

$$\mathfrak{k} = \mathfrak{gl}(n_1) \oplus \cdots \mathfrak{gl}(n_\ell) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(P).$$

The subgroup K is determined by an integer valued vector $\boldsymbol{n} = (n_i)_{i=1}^{\ell+2}$, which is a partition of n. We reserve the integer l for $\sum_{i=1}^{\ell} n_i$, so that $l + 1 + p = n$. Here $n_{\ell+1} = 1$ and $n_{\ell+2} = p$. Let $\mathcal{M}_K$ denote the moduli space of conjugacy classes with the fixed isotropy subgroup (3.1), regarded as Poisson spaces with the bivector field $r^{\mathrm{ad},\mathrm{ad}} + \omega^{r,l} - \omega^{l,r}$ (cf. Introduction). We introduce the subspace $\mathcal{M}'_K$ of classes with $\mu_{\ell+1} = -1$. The sets of all vectors $\boldsymbol{\mu}$ as specified above parameterize $\mathcal{M}_K$ and $\mathcal{M}'_K$ although not uniquely. We denote these sets by $\hat{\mathcal{M}}_K$ and, respectively, $\hat{\mathcal{M}}'_K$.

As a variety, the class O_x associated with $\boldsymbol{\mu}$ and $\boldsymbol{n}$ is determined by the set of equations

$$(X-\mu_1)\dots(X-\mu_\ell)(X+1)(X-1)(X-\mu_\ell^{-1})\dots(X-\mu_1^{-1}) = 0, \tag{3.2}$$

$$\mathrm{Tr}(X^k) = \sum_{i=1}^{\ell} n_i(\mu_i^k + \mu_i^{-k}) + 2(-1)^k + P, \quad k = 1,\dots,N, \tag{3.3}$$

where the matrix multiplication in the first line is understood. This system is polynomial in the matrix entries X_{ij} and defines an ideal of $\mathbb{C}[X_{ij}]$ vanishing on O_x.

THEOREM 3.1 *The system of polynomial relations (3.2) and (3.3) generates the defining ideal of the class O in $\mathbb{C}[SO(N)]$.*

Proof. The proof is similar to [65], Theorem 2.3. □

The goal is a generalization of this statement for the quantized polynomial algebra of O.

3.1 Parabolic Verma Module M_λ

Let $\mathfrak{p}^+ = \mathfrak{k} + \mathfrak{g}_+ \subset \mathfrak{g}$ denote the parabolic subalgebra. An element $\lambda \in \mathfrak{C}^*_{\mathfrak{k}} = \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k}} \oplus \mathfrak{c}^*_{\mathfrak{k}}$ defines a one-dimensional representation of $U_q(\mathfrak{k})$ denoted by $\mathbb{C}_\lambda$. Its restriction to $U_q(\mathfrak{h})$ acts by the assignment $q^{\pm h_\alpha} \mapsto q^{\pm(\alpha,\lambda)}$, $\alpha \in \Pi^+$. Since $q = e^\hbar$, the pole in λ is compensated, and the representation is correctly defined. It extends to $U_q(\mathfrak{p}^+)$ by setting it to zero on $\mathfrak{g}_+ \subset \mathfrak{p}^+$. Denote by M_λ the parabolic Verma module $U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}^+)} \mathbb{C}_\lambda$, [48]. Regarded as a $U_q(\mathfrak{g}_-)$-module by restriction from $U_q(\mathfrak{g})$, M_λ is isomorphic to the quotient $U_q(\mathfrak{g}_-)/U_q(\mathfrak{g}_-)\mathfrak{k}_-$, which we denote by $U^-_{\mathfrak{k}}$.

Of key importance for us is the structure of the tensor product $\mathbb{C}^N \otimes M_\lambda$. The element $\mathcal{R}_{21}\mathcal{R}$ expressed through the universal R-matrix $\mathcal{R} \in U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})$ operates on $\mathbb{C}^N \otimes M_\lambda$ as an invariant matrix $\mathcal{Q} \in \mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{g})$, which commutes with $\Delta(x)$ for all $x \in U_q(\mathfrak{g})$. The normal form of $\mathcal{Q}$ is determined by the submodule structure of $\mathbb{C}^N \otimes M_\lambda$. The eigenvalues of $\mathcal{Q}$ are found in [68]. It is also known that $\mathcal{Q}$ is semisimple for generic $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$. Then we are going to check that $\mathcal{Q}$ remains semisimple for a certain set of λ of our interest.

Note with care that in this section we fix the natural representation ζ with the minus sign, cf. Section 2.1. The natural $U_q(\mathfrak{g})$-module splits into irreducible $U_q(\mathfrak{k})$-modules,

$$\mathbb{C}^N = (\mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_\ell}) \oplus \mathbb{C} \oplus \mathbb{C}^P \oplus \mathbb{C} \oplus (\mathbb{C}^{n_\ell} \oplus \dots \oplus \mathbb{C}^{n_1}), \tag{3.4}$$

which decomposition is compatible with the basis $\{w_i\}_{i=1}^N = \cup_{i=1}^{2\ell+3}\{w_k\}_{k=m_i}^{m_{i+1}-1}$ counting from the left. Here $m_i = n_1 + \dots + n_{i-1} + 1$ for $i = 1,\dots,\ell+2$, and $m_{2\ell+4-i} = N + 1 - \sum_{k=1}^{i} n_k$, $i = 1,\dots,\ell+1$. Note that w_{m_i} is the highest weight vector of the corresponding irreducible $\mathfrak{k}$-submodule in $\mathbb{C}^N$.

For $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$, the operator $\mathcal{Q} \in \mathrm{End}(\mathbb{C}^N \otimes M_\lambda)$ satisfies the equation $\prod_{i=1}^{2\ell+3}(\mathcal{Q} - x_i) = 0$ with the roots

$$\begin{array}{rcl} x_i & = & q^{2(\lambda,\varepsilon_{m_i})-2(m_i-1)},\ i = 1,\dots,\ell+2, \\ x_{2\ell+4-i} & = & q^{-2(\lambda,\varepsilon_{m_i})-2N+2(m_i+n_i)},\ i = 1,\dots,\ell+1, \end{array} \tag{3.5}$$

see [68], Theorem 4.2. The root x_i corresponds to a submodule $M_i \subset \mathbb{C}^N \otimes M_\lambda$, where $\mathcal{Q}$ acts as multiplication by x_i. For generic $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$ and q, the roots x_i are pairwise distinct, and $\mathbb{C}^N \otimes M_\lambda = \oplus_{i=1}^{2\ell+3} M_i$.

In this chapter, we are interested in special λ making $x_{\ell+1} = q^{2(\lambda,\varepsilon_{l+1})-2l}$ equal to $x_{\ell+3} = q^{-2(\lambda,\varepsilon_{l+1})-2l-2P}$. In particular, this condition is satisfied if

$$q^{2(\lambda,\varepsilon_{l+1})} = -q^{-P}. \tag{3.6}$$

Let $\mathfrak{C}^*_{\mathfrak{k},'}$ be the subset of all weights $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$ subject to (3.6). We prove that, for generic $\lambda \in \mathfrak{C}^*_{\mathfrak{k},'}$ and generic q including $q \to 1$, the direct sum decomposition of $\mathbb{C}^N \otimes M_\lambda$ still holds, and the operator $\mathcal{Q}$ is semisimple. To this end, we study the submodules $M_{\ell+1}$ and $M_{\ell+3}$ and show that their sum is direct for all λ satisfying (3.6). The analysis is based on calculation of singular vectors generating $M_{\ell+1}$ and $M_{\ell+3}$.

As in [68], we introduce a subspace of weights that we use for the parametrization of $\mathcal{M}'_K$, the moduli space of borderline conjugacy classes with fixed K. Put $\mu^0_k = e^{2(\lambda,\varepsilon_{m_k})}$, for $k = 1,\dots,\ell+2$. The subset $\mathfrak{c}^*_{\mathfrak{k},'} \subset \mathfrak{c}^*_{\mathfrak{k}}$ is specified by the condition $\mu^0_{\ell+1} = -1$. Let $\mathfrak{c}^*_{\mathfrak{k},reg}$ denote the set of all weights $\lambda \in \mathfrak{c}^*_{\mathfrak{k}}$ such that $\boldsymbol{\mu}^0 \in \hat{\mathcal{M}}_K$ and similarly define $\mathfrak{c}^*_{\mathfrak{k},reg'} \subset \mathfrak{c}^*_{\mathfrak{k},'}$ by the requirement $\boldsymbol{\mu}^0 \in \hat{\mathcal{M}}'_K$. Finally, we introduce $\mathfrak{C}^*_{\mathfrak{k},reg'} = \mathfrak{C}^*_{\mathfrak{k},'} \cap (\frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg'} \oplus \mathfrak{c}^*_{\mathfrak{k}})$. The subset $\mathfrak{C}^*_{\mathfrak{k},reg'}$ is dense in $\mathfrak{C}^*_{\mathfrak{k},'}$.

3.2 Singular Vectors in $\mathbb{C}^N \otimes M_\lambda$

In this section, $\mathfrak{k}$ is the Levi subalgebra $\mathfrak{h} + \mathfrak{so}(P)$, which this assumption can be otherwise put as $\ell = l$. The parabolic Verma module M_λ is relative to this subalgebra. In other words, $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$ if and only if $(\lambda,\varepsilon_i) = 0$, $i = l+2,\dots,n$.

Given weight $\lambda \in \frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$ we denote $\lambda_i = (\lambda,\varepsilon_i)$, for all $i \in I$.

Recall that a non-zero weight vector v in a $U_q(\mathfrak{g})$-module is called singular if it generates the trivial $U_q(\mathfrak{g}_+)$-submodule, i.e. $e_\alpha v = 0$, for all $\alpha \in \Pi^+$. Since the weights of $e_\alpha v$ are pairwise distinct, this is equivalent to the equation $Ev = 0$, where $E = \sum_{m=1}^n e_{\alpha_m}$. We will also work with the operator $E' = \sum_{m=2}^n e_{\alpha_m}$, in view of Corollary 3.3 below.

LEMMA 3.2 *Let W be a finite dimensional $U_q(\mathfrak{g})$-module and W^* its right dual module. Let Y be a $U_q(\mathfrak{g})$-module. Singular vectors in $W \otimes Y$ are parameterized by homomorphisms $W^* \to Y$ of $U_q(\mathfrak{g}_+)$-modules.*

Proof. Choose a weight basis $\{w_i\}_{i=1}^d \subset W$, where $d = \dim W$. Suppose that $u \in W \otimes Y$ is a singular vector, $u = \sum_{i=1}^d w_i \otimes y_i$, for some $y_i \in Y$. Let $\pi\colon U_q(\mathfrak{g}) \to \mathrm{End}(W)$ denote the representation homomorphism, $\pi(u)w_i = \sum_{j=1}^{N} \pi(u)_{ij} w_j$. We have, for $\alpha \in \Pi^+$,

$$e_\alpha u = \sum_{i=1}^{d}\sum_{j=1}^{d} \pi(e_\alpha)_{ij} w_j \otimes y_i + \sum_{i=1}^{d} q^{(\alpha,\varepsilon_i)} w_i \otimes e_\alpha y_i. \qquad (3.7)$$

So $e_\alpha u = 0$ is equivalent to $e_\alpha y_i = -q^{-(\alpha,\varepsilon_i)} \sum_{j=1}^{d} \pi(e_\alpha)_{ji} y_j$. The vector space $\mathrm{Span}\{y_i\}_{i=1}^{d}$ supports the right dual representation of $U_q(\mathfrak{g}_+)$, provided y_i are linear independent. In general, it is a quotient of the right dual representation. □

Formula (3.7) can be more explicitly rewritten as

$$y_j = (-1)^{\epsilon_i+1} q^{(\varepsilon_i-\varepsilon_j,\varepsilon_i)} e_{\varepsilon_i-\varepsilon_j} y_i$$

for all $i, j \in I$ such that $\varepsilon_i - \varepsilon_j \in \Pi^+$. In the following corollary, M_λ is a parabolic Verma module relative to arbitrary $\mathfrak{k}$.

COROLLARY 3.3 *Singular vectors $\{u_i\} \in \mathbb{C}^N \otimes M_\lambda$ are parameterized by weight elements $y \in M_\lambda$ satisfying $e_{\alpha_1}^3 y = 0$ if $N = 3$, $e_{\alpha_1}^2 y = e_{\alpha_2}^2 y = 0$ if $N = 4$ and $e_{\alpha_1}^2 y = E'y = 0$ for $N > 4$.*

Proof. The weight ε_1 is integral dominant. The dual natural representation of $U_q(\mathfrak{g})$ is generated by the vector of lowest weight $-\varepsilon_1$. When restricted to $U_q(\mathfrak{g}_+)$, it is isomorphic to a quotient of the left regular $U_q(\mathfrak{g}_+)$-module. It is the quotient by the left ideal in $U_q(\mathfrak{g}_+)$ generated by $e_{\alpha_1}^3$ if $N = 3$, by $e_{\alpha_1}^2, e_{\alpha_2}^2$ if $N = 4$, and by $e_{\alpha_1}^2, e_{\alpha_i}$, $i = 2, \ldots, n$ if $N > 4$. Therefore, all homomorphisms from the co-natural module to M_λ are generated by the assignment $U_q(\mathfrak{g}_+) \ni 1 \to y \in M_\lambda$, where y satisfies the hypothesis. □

Singular vectors generate $U_q(\mathfrak{g})$-submodules of highest weight. It is known that, for generic λ, singular vectors in $\mathbb{C}^N \otimes M_\lambda$ are parameterized by the highest weights ν of the irreducible $U_q(\mathfrak{g})$-submodules in $\mathbb{C}^N$ and carry the weights $\lambda + \nu$. We denote by u_j the singular vector of weight $\lambda + \varepsilon_j$, $j \in I$, which is defined up to a non-zero scalar factor. We can write

$$u_j = \sum_{i=1}^{N-l} w_i \otimes y_{j,i}, \quad j \in I$$

where $y_{j,i} \in M_\lambda$ is an element of weight $\lambda + \varepsilon_j - \varepsilon_i$, $i \leqslant j$. For each j the linear span $\{y_{j,i}\}_{i=1}^{j}$ supports a quotient of the co-natural representation of $U_q(\mathfrak{g}_+)$, which is cyclically generated by $\{y_{j,1}\}$

Singular vectors u_i, $i = 1, \ldots, n-1$, are related to the subalgebra $\mathfrak{gl}(n) \subset \mathfrak{g}$. They were studied in detail in [7], and can be extracted from Corollary 5.3 below. Singular vector u_{n+1} in the case of $\mathfrak{g} = \mathfrak{so}(2n)$ is related to another copy of $\mathfrak{gl}(n)$ with α_{n-1} replaced by α_n. Singular vector u_{n+1} for $\mathfrak{g} = \mathfrak{so}(2n+1)$ can be constructed as follows. Define "dynamical root vectors" f_{ε_k} by setting $f_{\varepsilon_n} = f_{\alpha_n}$ and

$$f_{\varepsilon_{k-1}} = f_{\alpha_{k-1}} f_{\varepsilon_k} [h_{\varepsilon_k} + n - k + 1]_q - f_{\varepsilon_k} f_{\alpha_{k-1}} [h_{\varepsilon_k} + n - k]_q$$

for all $k = n-1, \ldots, 1$. It is also convenient to put $f_0 = 1$. Let M_λ be the Verma module of highest weight λ and v_λ its canonical generator. The identity

$$e_{\alpha_k} f_{\varepsilon_i} v_\lambda = \delta_{ki}[\lambda_i + n - i] f_{\varepsilon_{i+1}} v_\lambda,$$

can be checked by induction on i. Setting $y_{n+1,1} = f_{\varepsilon_1} v_\lambda$, we obtain $y_{n+1,i} = (-q)^{i-1} \prod_{k=1}^{i-1} [\lambda_i + n - k] f_{\varepsilon_i} v_\lambda$, $i = 1, \ldots n+1$.

We are especially interested in u_{N-l} carrying the weight $\lambda - \varepsilon_{l+1}$. It is expanded over the basis $\{w_i\}_{i=1}^N \subset \mathbb{C}^N$ as $u_{N-l} = \sum_{i=1}^{N-l} w_i \otimes y_i$ with coefficients $y_i = y_{N-l,i}$ of weight $\lambda - \varepsilon_i - \varepsilon_{l+1}$, $i = 1, \ldots, l+1$. They are generated by y_1 via the co-natural action of $U_q(\mathfrak{g}_+)$, so we call y_1 the generating coefficient. The next goal is to evaluate y_1.

Consider the graph corresponding to the co-natural representation of $U_q(\mathfrak{g}_+)$ for $N > 3$.

$$\underset{y_{N-l}}{\circ} \xleftarrow{e_{\alpha_{l+1}}} \quad \cdots \quad \xleftarrow{e_{\alpha_{l+1}}} \underset{y_{l+1}}{\circ} \xleftarrow{e_{\alpha_l}} \underset{y_l}{\circ} \xleftarrow{e_{\alpha_{l-1}}} \quad \cdots \quad \xleftarrow{e_{\alpha_2}} \underset{y_2}{\circ} \xleftarrow{e_{\alpha_1}} \underset{y_1}{\circ}$$

We can readily write down y_i for $l+2 \leqslant i \leqslant N-l$, up to a scalar factor. Indeed, the corresponding weight spaces in M_λ have dimension 1. Suppose that $\psi^{i,N-l} = f_\alpha \psi^{j,N-l}$ for $\alpha = \varepsilon_i - \varepsilon_j \in \Pi^+$ (for odd N, j is always $i-1$, while for even N j may be also $i-2$ for $i = n+1, n+2$). Then $e_\alpha \psi^{i,N-l} v_\lambda \sim \psi^{i+1,N-l} v_\lambda$ and $y_i \sim \psi^{i,N-l} v_\lambda$:

$$y_{N-l} \sim v_\lambda, \quad y_{N-l-1} \sim f_{l+1} v_\lambda, \quad y_{N-l-2} \sim f_{l+2} f_{l+1} v_\lambda, \quad \ldots$$

In particular, $y_{l+1} \sim f_{l+1} \overset{<}{\ldots} f_{n-1} f_n f_n f_{n-1} \overset{>}{\ldots} f_{l+1} v_\lambda$ for odd N and a similar expression with $f_{n-1} f_n$ in place of $f_n f_n$ for even N.

The problem essentially boils down to finding y_i with $i \leqslant l+1$. These coefficients feature the following chain property. Let $\mathfrak{g}'_i \subset \mathfrak{g}$ denote the subalgebra with simple roots $\{\alpha_j\}_{j=i}^n$ and let $M'_{i,\lambda} \subset M_\lambda$ be the $U_q(\mathfrak{g}'_i)$-submodule generated by v_λ. If $y_i \in M'_{i,\lambda}$, then y_i is the generating coefficient for a $U_q(\mathfrak{g}'_i)$-singular vector in $\mathbb{C}^{N-2i+2} \otimes M'_{i,\lambda}$, as follows from the representation graph. This observation enables construction of y_i by descending induction starting from $y_{l+1} \in M'_{l+1,\lambda}$, which is done in the next section.

3.2.1 Symmetric Classes

In this section, we fix $l = 0$ or equivalently $n = 1 + p$. This assumption corresponds to the symmetric conjugacy class of matrices with eigenvalues -1 and $+1$ of multiplicities 2 and P, respectively. The singular vector of interest has weight $\lambda + \varepsilon_{l+1} = \lambda - \varepsilon_1$.

We introduce the following basis in the weight space $[U_{\mathfrak{k}}^-]_{-2\varepsilon_1}$. Observe that $d_P^0 := \dim [U_{\mathfrak{k}}^-]_{-2\varepsilon_1}$

is $p+1$ for odd P and p for even P. Define monomials ϕ_m, $m=1,\ldots,d^0_P$, by

$$
\phi_m \quad = \quad \left\{
\begin{array}{lll}
f_{\alpha_m} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_{m+1}} \overset{>}{\ldots} f_{\alpha_{p+1}} f_{\alpha_{p+1}} \overset{>}{\ldots} f_{\alpha_1}, & 1 \leqslant m \leqslant p+1 & \text{for odd } N, \\
\left.\begin{array}{ll}
f_{\alpha_m} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_{m+1}} \overset{<}{\ldots} f_{\alpha_{p-1}} f_{\alpha_p} f_{\alpha_{p+1}} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1}, & 1 \leqslant m \leqslant p-1 \\
f_{\alpha_p} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_{p+1}} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1}, & m = p \\
f_{\alpha_{p+1}} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_p} f_{\alpha_p-1} \overset{>}{\ldots} f_{\alpha_1}, & m = p+1
\end{array}\right\} & & \text{for even } N.
\end{array}
\right.
$$

All ϕ_m have weight $-2\varepsilon_1$. Using the Serre relations, we can check for even N that $\phi_{p+1} = f_{\alpha_p} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_{p+1}} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1} = \phi_p$, so the number of independent ϕ_m is equal to $d^0_P = \dim[U^-_{\mathfrak{k}}]_{-2\varepsilon_1}$. Still it is convenient to consider both ϕ_p and ϕ_{p+1}.

The leftmost position in all ϕ_m is occupied by f_{α_m}. We define vectors ϕ'_m of weight $-2\varepsilon_1+\alpha_m$ obtained from ϕ_m by deleting this f_{α_m}:

$$
\phi'_m = f_{\alpha_{m-1}} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_{m+1}} \overset{<}{\ldots} f_{\alpha_{p+1}} f_{\alpha_{p+1}} \overset{>}{\ldots} f_{\alpha_1} \quad \text{for odd } N, \text{ and}
$$

$$
\phi'_m = \left\{
\begin{array}{ll}
f_{\alpha_{m-1}} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_{m+1}} \overset{<}{\ldots} f_{\alpha_{p-1}} f_{\alpha_p} f_{\alpha_{p+1}} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1}, & m \leqslant p-1 \\
f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_{p+1}} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1}, & m = p \\
f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1} f_{\alpha_p} f_{\alpha_{p-1}} \overset{>}{\ldots} f_{\alpha_1}, & m = p+1
\end{array}
\right. , \quad \text{for even } N.
$$

Abusing notation, we will also identify ϕ_m and ϕ'_m with their images in the quotient $U^-_{\mathfrak{k}}$.

LEMMA 3.4 *The monomial ϕ'_m spans $[U^-_{\mathfrak{k}}]_{-2\varepsilon_1+\alpha_m}$ for each $m=1,\ldots,p+1$.*

Proof. We can check that $\dim[U^-_{\mathfrak{k}}]_{-2\varepsilon_1+\alpha_m} = 1$, so to prove the statement, we must prove that $\phi'_m \neq 0$. The squared norm $\langle \phi'_m v_\lambda, \phi'_m v_\lambda \rangle$ with respect to the Shapovalov form on M_λ is equal to $[\lambda_1]_q$ for $m=1$ and to $[\lambda_1]_q[\lambda_1-1]_q$ otherwise. It is not zero if $[\lambda_1]_q[\lambda_1-1]_q \neq 0$. Due to the isomorphism $M_\lambda \simeq U^-_{\mathfrak{k}}$, $\phi'_m \neq 0$ as well as its projection in $U^-_{\mathfrak{k}}$ for generic λ. But ϕ'_m is independent of λ, which completes the proof. $\square$

Let Φ^0 denote the linear span of $\{\phi_m v_\lambda\}_{m=1}^{d^0_P} \subset M_\lambda$. Denote by $\hat{E}$ the composition $\mathbb{C}^{d^0_P} \to \Phi^0 \to M_\lambda$ of linear maps, $(A_m) \mapsto \sum_{m=1}^{d^0_P} A_m \phi_m = y \mapsto Ey$. For $N \geqslant 5$, the operator $\hat{E}$ acts on $\mathbb{C}^{d^0_P}$ by $(A_m)_{m=1}^{d^0_P} \mapsto \sum_{m=1}^{p+1} E_m \phi'_m v_\lambda$, where the scalar coefficients E_m are given in the following lemma.

LEMMA 3.5 *Suppose that $N \geqslant 5$ and $y = \sum_{m=1}^{d^0_P} A_m \phi_m v_\lambda \in \Phi^0$. Then, for all $m=1,\ldots,p+1$, we have $e_m y = E_m \phi'_m v_\lambda$, where the scalar factors E_m are*

$$
\begin{array}{rcl}
E_1 &=& A_1[\lambda_1]_q + A_2[\lambda_1-1]_q, \\
E_i &=& A_{i-1} + [2]_q A_i + A_{i+1}, \quad i = 2,\ldots,d^1_P-1, \\
E_p = E_{p+1} &=& A_{p-1} + [2]_q A_p, \quad \textit{for even } N, \\
E_{p+1} &=& A_p + (1+[2]_q)\, A_{p+1}, \quad \textit{for odd } N.
\end{array}
$$

Proof. A straightforward calculation. $\square$

LEMMA 3.6 *Suppose that $N \geqslant 5$. Then the map $\hat{E}$ is injective for generic λ.*

Proof. Define

$$A_m = (-1)^{m-1}\left[\frac{P}{2} - m + 1\right]_q, \quad m = 1, \dots, d_P^0. \tag{3.8}$$

For $N \geqslant 5$, one can check that (3.8) is a unique solution of the system of equations $E_i = 0$, $i = 2, \dots, p+1$, up to a common scalar factor. This makes $E_1 = A_1[\lambda_1]_q + A_2[\lambda_1 - 1]_q$ into $[\lambda_1 + \frac{P}{2} - 1]_q$, which does not vanish for generic λ. □

COROLLARY 3.7 *a) The system $\{\phi_m v_\lambda\}_{m=1}^{d_P^0}$ forms a basis in $[M_\lambda]_{\lambda-2\varepsilon_1}$. b) The vector $f_{2\varepsilon_1}^{(P)} v_\lambda = \sum_{m=1}^{d_P^0} A_m \phi_m v_\lambda$, where A_m are given by (3.8), is a generating coefficient. c) It is a unique generating coefficient of weight $\lambda - 2\varepsilon_1$, up to a scalar factor.*

Proof. The statement is obvious for $N = 3, 4$ with $p = 0$ and, respectively, $p = 1$. Then $d_P^0 = 1$ and the vectors $f_{2\varepsilon_1}^{(1)} v_\lambda = [\frac{1}{2}]_q f_{\alpha_1}^2 v_\lambda$, $f_{2\varepsilon_1}^{(2)} v_\lambda = f_{\alpha_1} f_{\alpha_2} v_\lambda$ satisfy the conditions $e_{\alpha_1}^3 f_{2\varepsilon_1}^{(1)} v_\lambda = 0$ and $e_{\alpha_1}^2 f_{2\varepsilon_1}^{(2)} v_\lambda = e_{\alpha_2}^2 f_{2\varepsilon_1}^{(2)} v_\lambda = 0$, as required.

Now suppose that $N \geqslant 5$. Since the operator $\hat{E}$ is injective, the map $\mathbb{C}^{d_P^0} \to \Phi^0$ is injective too. It is surjective by construction, hence it is a bijection. For generic λ, the vectors $\{\phi_m v_\lambda\}_{m=1}^{d_P^0}$ form a basis in Φ^0 and hence in $[M_\lambda]_{\lambda-2\varepsilon_1}$, as the latter has dimension d_P^0. The vectors $\{\phi_m\}_{m=1}^{d_P^0}$ form a basis in $[U_{\mathfrak{k}}^-]_{-2\varepsilon_1}$, due to the linear isomorphism $[M_\lambda]_\mu \simeq [U_{\mathfrak{k}}^-]_{\mu-\lambda}$. These vectors are independent of λ, hence they form a basis at all λ, as well as $\{\phi_m v_\lambda\}_{m=1}^{d_P^0}$. This implies that $f_{2\varepsilon_1}^{(P)} v_\lambda \neq 0$, and it is a unique generating coefficient, up to a scalar factor. □

3.2.2 The Case $l = 1$

To keep reference to the symmetric case considered in the previous section, we enumerate the simple roots $\Pi_{\mathfrak{g}} = \{\alpha_i\}_{i=0}^{p+1}$. Then the roots $\{\alpha_i\}_{i=1}^{p+1}$ correspond to the subalgebra $U_q(\mathfrak{g}_1') \subset U_q(\mathfrak{g})$. Under this embedding, we regard ϕ_m and $f_{2\varepsilon_1}^{(P)}$ constructed in the previous section as elements of $U_q(\mathfrak{g})$.

Observe that $d_P^1 := \dim[M_\lambda]_{\lambda-\varepsilon_0-\varepsilon_1}$ is equal to $3p+3$ for odd N and $3p+1$ for even N. The only generator which does not commute with f_{α_0} is f_{α_1}, and it enters ϕ_m twice. There are three possible ways to allocate f_{α_0} relative to these f_{α_1}. We use this observation to construct the basis in $[U_{\mathfrak{k}}^-]_{-\varepsilon_0-\varepsilon_1}$ from the basis in $[U_{\mathfrak{k}}^-]_{-2\varepsilon_1}$. For all $m = 1, \dots, p+1$, define $\phi_m^1 = f_{\alpha_0}\phi_m$ and $\phi_m^3 = \phi_m f_{\alpha_0}$. Define ϕ_m^2 to be the monomial obtained from ϕ_m by replacing the rightmost copy of f_{α_1} with $f_{\alpha_0} f_{\alpha_1}$. For even N, the equality $\phi_{p+1} = \phi_p$ implies $\phi_{p+1}^1 = \phi_p^1$ and $\phi_{p+1}^3 = \phi_p^3$, so we have effectively $3p+1$ monomials for even N and $3p+3$ monomials for odd N.

As in the symmetric case, for all $m \in [1, p+1]$ we define ${\phi'}_m^i \in U_q(\mathfrak{g}_-)$ of weight $-\varepsilon_0 - \varepsilon_1 + \alpha_m$ by deleting the leftmost copy of f_{α_m} from ϕ_m^i. Note that ${\phi'}_1^1 = {\phi'}_1^2$ and, for even N, ${\phi'}_{p+1}^1 = {\phi'}_p^1$, ${\phi'}_{p+1}^3 = {\phi'}_p^3$. Put $r_m = 2$ for $m = 1$ and $r_m = 1$ for $m > 1$.

LEMMA 3.8 *For all $m = 1, \ldots, p+1$, the vectors $\{\phi'^i_m\}_{i=r_m}^3 \subset [U_{\mathfrak{k}}^-]_{-\varepsilon_0-\varepsilon_1+\alpha_m}$ are linearly independent.*

Proof. We can check that the Gram matrix of the system $\{\phi'^i_m v_\lambda\}_{i=r_m}^3$ with respect to the Shapovalov form on M_λ is

$$\begin{pmatrix} [\lambda_1]_q[\lambda_0-\lambda_1+1]_q & [\lambda_1]_q[\lambda_0-\lambda_1]_q \\ [\lambda_1]_q[\lambda_0-\lambda_1]_q & [\lambda_1+1]_q[\lambda_0-\lambda_1]_q \end{pmatrix}, \quad m=1,$$

$$[\lambda_1]_q \begin{pmatrix} [\lambda_1-1]_q[\lambda_0-\lambda_1+2]_q & [\lambda_1-1]_q[\lambda_0-\lambda_1+1]_q & [\lambda_1-1]_q[\lambda_0-\lambda_1]_q \\ [\lambda_1-1]_q[\lambda_0-\lambda_1+1]_q & [\lambda_1]_q[\lambda_0-\lambda_1+1]_q & [\lambda_1]_q[\lambda_0-\lambda_1]_q \\ [\lambda_1-1]_q[\lambda_0-\lambda_1]_q & [\lambda_1]_q[\lambda_0-\lambda_1]_q & [\lambda_1+1]_q[\lambda_0-\lambda_1]_q \end{pmatrix},$$

$m = 2, \ldots, n$, for either parity of N. Its determinant is equal to

$$[\lambda_0-\lambda_1]_q[\lambda_1]_q[\lambda_0+1]_q, \quad m=1,$$
$$[\lambda_0-\lambda_1]_q[\lambda_1]_q^3[\lambda_1-1]_q[\lambda_0+1]_q^2, \quad m=2,\ldots,p+1.$$

It does not vanish for generic λ, hence $\{\phi'^i_m v_\lambda\}_{i=r_m}^3$ are linearly independent. This is also true for all λ, since ϕ'^i_m are independent of λ. □

All $\phi^i_m v_\lambda$ are annihilated by $e^2_{\alpha_0}$, as f_{α_0} enters only once. Therefore their linear combination annihilated by e_{α_i}, $i>1$, is a generating coefficient.

Present $\mathbb{C}^{d^1_P} = \mathbb{C}^{p+1} \oplus \mathbb{C}^{p+1} \oplus \mathbb{C}^{p+1}$ for odd N and $\mathbb{C}^{d^1_P} = \mathbb{C}^{p} \oplus \mathbb{C}^{p+1} \oplus \mathbb{C}^{p}$ for even N. Let the upper index of $(A^i_m) \in \mathbb{C}^{d^1_P}$ label the summand in this decomposition while the lower index mark the coordinate within this summand.

Let Φ^1 denote the linear span of $\{\phi^i_m v_\lambda\}_{m,i} \subset M_\lambda$. Denote by $\hat{E}$ the composition $\mathbb{C}^{d^1_P} \to \Phi^1 \to M_\lambda$ of linear maps acting by $(A^i_m) \mapsto \sum_{m,i} A^i_m \phi^i_m = y \mapsto Ey$. It acts by $\hat{E}\colon (A^i_m) \mapsto \sum_{i=r_m}^3 E^i_m \phi'^i_m v_\lambda$, where the scalar factors E^i_m are given in the following lemma.

LEMMA 3.9 *Suppose that $y = \sum_{m,i} A^i_m \phi^i_m v_\lambda \in \Phi^1$, where $(A^i_m) \in \mathbb{C}^{d^1_P}$. Then, for all $m = 1, \ldots, p+1$, we have $e_{\alpha_m} y = \sum_{i=r_m}^3 E^i_m \phi'^i_m v_\lambda$, where the scalar factors E^i_m are as follows.*
a.1) $P=3$.

$$E^2_1 = A^1_1([\lambda_1]_q + [\lambda_1-1]_q) + A^2_1[\lambda_1]_q, \quad E^3_1 = A^3_1([\lambda_1]_q + [\lambda_1+1]_q) + A^2_1[\lambda_1]_q,$$

a.2) $P = 2p+1 \geqslant 5$

$$\begin{aligned}
E^2_1 &= A^1_1\,[\lambda_1]_q + A^1_2\,[\lambda_1-1]_q + A^2_1\,[\lambda_1+1]_q + A^2_2\,[\lambda_1]_q\,, \\
E^3_1 &= A^3_1\,[\lambda_1+1]_q + A^3_2\,[\lambda_1]_q\,, \\
E^k_m &= A^k_{m-1} + A^k_m\,[2]_q + A^k_{m+1}, && 2 \leqslant m \leqslant p, \\
E^i_{p+1} &= A^i_p + \left(1+[2]_q\right) A^i_{p+1} + A^2_{p+1}, && i = 1,3, \\
E^2_{p+1} &= A^2_p + A^2_{p+1}.
\end{aligned}$$

b.1) $P=4$

$$E_1^2=A_1^1[\lambda_1]_q+A_1^2[\lambda_1+1]_q,\quad E_1^3=A_1^3[\lambda_1+1]_q+A_2^2[\lambda_1]_q,$$
$$E_2^2=A_1^1[\lambda_1]_q+A_1^3[\lambda_1+1]_q,\quad E_2^3=A_1^2[\lambda_1]_q+A_2^2[\lambda_1+1]_q.$$

b.2) $P=2p\geqslant 6$

$$E_i^k \quad=\quad A_{i-1}^k+[2]_qA_i^k+A_{i+1}^k,\quad i=1,\ldots,p-1,$$

whenever the pair (i,k) *is distinct from specified below, in which case* E_i^k *are*

$$\begin{array}{lcll}
E_{p-1}^2 &=& A_{m-3}^2+A_{p-1}^2\,[2]_q+A_p^2+A_{p+1}^2, & \\
E_p^2 &=& A_{p-1}^2+A_p^2\,[2]_q\,, & \\
E_{p+1}^2 &=& A_{p-1}^2+A_{p+1}^2\,[2]_q\,, & \\
E_p^i &=& A_{p-1}^i+A_p^i\,[2]_q+A_{p+1}^2, & i=1,3,\\
E_{p+1}^i &=& A_{p-1}^i+A_p^i\,[2]_q+A_p^2, & i=1,3.
\end{array}$$

Proof. A straightforward brute force calculation. □

Define $f^{(P)}_{\varepsilon_0+\varepsilon_1}=\sum_{m,i}A_m^i\phi_m^i$, where A_m^i are as follows:

$$A_m^i \quad=\quad \begin{cases}(-1)^{m+1}[\lambda_1+P-m]_q[\lambda_1+\frac{P}{2}]_q, & i=1,\\ (-1)^m(q^{m-\frac{P}{2}}+q^{-m+\frac{P}{2}})_q[\lambda_1+\frac{P}{2}-1]_q[\lambda_1+\frac{P}{2}]_q, & i=2,\\ (-1)^{m+1}[\lambda_1+m-1]_q[\lambda_1+\frac{P}{2}-1]_q, & i=3,\end{cases}$$

for $m=1,\ldots,d_P^1$ apart from A_p^2,A_{p+1}^2 for even N, which are set to $(-1)^p[\lambda_1+\frac{P}{2}-1]_q[\lambda_1+\frac{P}{2}]_q$.

LEMMA 3.10 *Up to a scalar factor, the vector* $f^{(P)}_{\varepsilon_0+\varepsilon_1}v_\lambda$ *is a unique solution of the system* $e_{\alpha_i}f^{(P)}_{\varepsilon_0+\varepsilon_1}v_\lambda=0$ *for all* $i=1,\ldots,p+1$*. Furthermore,* $e_{\alpha_0}f^{(P)}_{\varepsilon_0+\varepsilon_1}v_\lambda=[\lambda_0+\lambda_1+P]_qf^{(P)}_{2\varepsilon_1}v_\lambda$.

Proof. The first part of the statement is proved by a lengthy straightforward calculation, which is omitted here. Let us prove the second statement. Observe the identities

$$\sum_{i=1}^3A_m^i[\lambda_0-\lambda_1+3-i]_q=[\lambda_0+\lambda_1+P]_qA_m,$$

which hold for $m=1,\ldots,p+1$, odd N, and for $m=1,\ldots,p-1$, even N. This readily implies the statement for odd N:

$$e_{\alpha_0}f^{(P)}_{\varepsilon_0+\varepsilon_1}=\sum_{m=1}^{p+1}\sum_{i=1}^3A_m^ie_{\alpha_0}\phi_m^iv_\lambda=\sum_{m=1}^{p+1}\left(\sum_{i=1}^3A_m^i[\lambda_0-\lambda_1+3-i]_q\phi_mv_\lambda\right)=[\lambda_0+\lambda_1+P]_qf^{(P)}_{2\varepsilon_1}.$$

If N is even, we have also

$$\sum_{i=1}^3A_p^i[\lambda_0-\lambda_1+3-i]_q+A_{p+1}^2[\lambda_0-\lambda_1+1]=[\lambda_0+\lambda_1+P]_qA_p.$$

Then, for even N,

$$\begin{aligned} e_{\alpha_0} f^{(P)}_{\varepsilon_0+\varepsilon_1} v_\lambda &= \sum_{m=1}^{p}\sum_{i=1}^{3} A^i_m e_{\alpha_0}\phi^i_m v_\lambda + A^2_{p+1} e_{\alpha_0}\phi^2_{p+1} v_\lambda = \sum_{m=1}^{p-1}\left(\sum_{i=1}^{3} A^i_m[\lambda_0-\lambda_1+3-i]_q\phi_m v_\lambda\right) \\ &+ \left(\sum_{i=1}^{3} A^i_p[\lambda_0-\lambda_1+3-i]_q + A^2_{p+1}[\lambda_0-\lambda_1+1]\right)\phi_p v_\lambda = [\lambda_0+\lambda_1+P]_q f^{(P)}_{2\varepsilon_1}, \end{aligned}$$

as required. □

PROPOSITION 3.11 *The vectors ϕ^i_m form a basis in $[U^-_{\mathfrak{k}}]_{-\varepsilon_0-\varepsilon_1}$. Up to a scalar factor, $f^{(P)}_{\varepsilon_0+\varepsilon_1} v_\lambda$ is a unique generating coefficient of the weight $\lambda-\varepsilon_0-\varepsilon_1$.*

Proof. Observe that d^1_P is equal to the dimension of $[U^-_{\mathfrak{k}}]_{-\varepsilon_0-\varepsilon_1}$, so we need to prove only linear independence. Fix a constant c and restrict λ to the hyperplane $\lambda_1 = c$. By Lemma 3.10, the map $\hat{E}\colon \mathbb{C}^{d^1_P} \to \Phi^1 \to M_\lambda$ is injective for all λ such that $[\lambda_0 + c + 2n - 1]_q \neq 0$. Since the map $\mathbb{C}^{d^1_P} \to \Phi^1$ is surjective, the map $E\colon \Phi^1 \to M_\lambda$ is injective too. This implies that $\phi^i_m v_\lambda$ are linearly independent for such λ. Since ϕ^i_m are independent of λ_0, they are linearly independent at all λ subject to $\lambda_1 = c$, and so are $\phi^i_m v_\lambda$. As c is arbitrary, the statement holds true for all λ. □

3.2.3 The Case $l = 2$

In order to relate our calculation to already considered cases $l = 0, 1$, we enumerate the roots as $\alpha_{-1}, \alpha_0, \alpha_1, \dots, \alpha_{p+1}$. We are looking for the generating coefficient of weight $\lambda - \varepsilon_{-1} - \varepsilon_1$. It is an element of M_λ satisfying the equations $e^2_{\alpha_{-1}} y = e_{\alpha_j} y = 0$, $j \geqslant 0$.

Define the element

$$f^{(P)}_{\varepsilon_{-1}+\varepsilon_1} = f_{\alpha_{-1}} f^{(P)}_{\varepsilon_0+\varepsilon_1}[h_{\varepsilon_0}+h_{\varepsilon_1}+P+1]_q - f^{(P)}_{\varepsilon_0+\varepsilon_1} f_{\alpha_{-1}}[h_{\varepsilon_0}+h_{\varepsilon_1}+P]_q \in U_q(\mathfrak{b}_-), \tag{3.9}$$

of weight $-\varepsilon_{-1}-\varepsilon_0-\varepsilon_1$.

PROPOSITION 3.12 *The element $f^{(P)}_{\varepsilon_{-1}+\varepsilon_1} v_\lambda \in M_\lambda$ is a unique generating coefficient of weight $-\varepsilon_{-1}-\varepsilon_0-\varepsilon_1$. Furthermore,*

$$e_{\alpha_{-1}} f^{(P)}_{\varepsilon_{-1}+\varepsilon_1} v_\lambda = [\lambda_{-1}+\lambda_1+P+1]_q f^{(P)}_{\varepsilon_0+\varepsilon_1} v_\lambda.$$

Proof. We are looking for the generating coefficient in the form

$$y = \textstyle\sum_{m,k}(A^{k1}_m f_{\alpha_{-1}}\phi^k_m - A^{k2}_m \phi^k_m f_{\alpha_{-1}}), \tag{3.10}$$

where $(A^{k1}_m), (A^{k2}_m) \in \mathbb{C}^{d^1_P}$. Since $f_{\alpha_{-1}}{\varphi'}^k_m$ and ${\varphi'}^k_m f_{\alpha_{-1}}$ are independent, the conditions $e_{\alpha_m} f^{(P)}_{\varepsilon_{-1}+\varepsilon_1} v_\lambda = 0$ for positive m give $A^{kj}_m = A^k_m C^j$ for some scalars C^j, $j = 1, 2$. That is, $y = C^1 f_{\alpha_{-1}} f^{(P)}_{\varepsilon_0+\varepsilon_1} v_\lambda - C^2 f^{(P)}_{\varepsilon_0+\varepsilon_1} f_{\alpha_{-1}} v_\lambda$.

The coefficients C^1, C^2 are found from the condition $e_{\alpha_0} y = \sum_{m=1}^{n} E_m f_{\alpha_{-1}} \varphi_m = 0$, where E_m are equal to

$$\Big(A^1_m \,[\lambda_0 - \lambda_1 + 2]_q + A^2_m \,[\lambda_0 - \lambda_1 + 1]_q + A^3_m \,[\lambda_0 - \lambda_1]_q\Big)\, C^1$$
$$- \Big(A^1_m \,[\lambda_0 - \lambda_1 + 3]_q + A^2_m \,[\lambda_0 - \lambda_1 + 2]_q + A^3_m \,[\lambda_0 - \lambda_1 + 1]_q\Big)\, C^2$$

This boils down to m equations $E_m = 0$ on C^i. The system can be checked to be consistent and $C^1 = [\lambda_0 + \lambda_1 + P + 1]_q$, $C^2 = [\lambda_0 + \lambda_1 + P]_q$, up to a common scalar factor. Thus, $y = f^{(P)}_{\varepsilon_{-1}+\varepsilon_1} v_\lambda$ is a generating coefficient. □

3.2.4 Generating Coefficients for Arbitrary $l \geqslant 0$

Now we return to the usual enumeration of simple roots, $\alpha_1, \ldots, \alpha_n$. The algebra $\mathfrak{g} = \mathfrak{so}(2l+2+P)$ includes the subalgebra $\mathfrak{so}(6+P)$ via the assignment $\alpha_i \mapsto \alpha_{l+i}$, i.e.

$$\alpha_{-1} \mapsto \alpha_{l-1}, \quad \alpha_0 \mapsto \alpha_l, \quad \ldots, \quad \alpha_{p+1} \mapsto \alpha_{l+p+1} = \alpha_n.$$

Under this embedding, $f^{(P)}_{\varepsilon_{l+1}+\varepsilon_{l+2-i}}(\lambda)$, $i = 1, 2, 3$, become elements of $U_q(\mathfrak{g}_-)$ of weights $-\varepsilon_{l+2-i} - \varepsilon_{l+1}$. The subalgebra $\mathfrak{so}(6+P)$ corresponds to already considered case $l = 2$

Define an element $f^{(P)}_{\varepsilon_{l-1}+\varepsilon_{l+1}} \in U_q(\mathfrak{b}_-)$ by setting

$$f^{(P)}_{\varepsilon_{l-1}+\varepsilon_{l+1}} = f_{\alpha_{l-1}} f^{(P)}_{\varepsilon_l+\varepsilon_{l+1}} [h_{\varepsilon_l} + h_{\varepsilon_{l+1}} + P + 1]_q - f^{(P)}_{\varepsilon_l+\varepsilon_{l+1}} f_{\alpha_{l-1}} [h_{\varepsilon_l} + h_{\varepsilon_{l+1}} + P]_q, \tag{3.11}$$

so that $f^{(P)}_{\varepsilon_{l-1}+\varepsilon_{l+1}}(\lambda)$ is indeed the evaluation of $f^{(P)}_{\varepsilon_{l-1}+\varepsilon_{l+1}}$ at the point $\lambda \in \mathfrak{h}^*$. Observe that

$$e_{\alpha_k} f^{(P)}_{\varepsilon_k+\varepsilon_{l+1}} v_\lambda = [\lambda_k + \lambda_{l+1} + P + l - k]_q f^{(P)}_{\varepsilon_{k+1}+\varepsilon_{l+1}} v_\lambda,$$

once $k = l-1, l$. Suppose we have defined $f_{\varepsilon_{k+1}+\varepsilon_{l+1}}$ for some $k \in [1, l-1]$. Then put

$$\begin{aligned} f^{(P)}_{\varepsilon_k+\varepsilon_{l+1}} &= f_{\alpha_k} f^{(P)}_{\varepsilon_{k+1}+\varepsilon_{l+1}} [h_{\varepsilon_{k+1}} + h_{\varepsilon_{l+1}} + P + l - k]_q \\ &- f^{(P)}_{\varepsilon_{k+1}+\varepsilon_{l+1}} f_{\alpha_k} [h_{\varepsilon_{k+1}} + h_{\varepsilon_{l+1}} + P + l - k - 1]_q. \end{aligned}$$

PROPOSITION 3.13 *The vectors $f^{(P)}_{\varepsilon_k+\varepsilon_{l+1}} v_\lambda \in M_\lambda$ satisfy the equations*

$$\begin{aligned} e_{\alpha_j} f^{(P)}_{\varepsilon_k+\varepsilon_{l+1}} v_\lambda &= \delta_{jk} [\lambda_k + \lambda_{l+1} + P + l - k]_q f^{(P)}_{\varepsilon_{k+1}+\varepsilon_{l+1}} v_\lambda, \quad k = 1, \ldots, l, \\ e_{\alpha_j} f^{(P)}_{2\varepsilon_{l+1}} v_\lambda &= \delta_{j\, l+1} [\lambda_{l+1} + \frac{P}{2} - 1]_q f^{(P)}_{\varepsilon_{l+2}+\varepsilon_{l+1}} v_\lambda, \end{aligned} \tag{3.12}$$

where $f^{(P)}_{\varepsilon_{l+2}+\varepsilon_{l+1}} = \phi'_1$. Then $f^{(P)}_{\varepsilon_1+\varepsilon_{l+1}} v_\lambda$ is a unique generating coefficient of the singular vector in $\mathbb{C}^N \otimes M_\lambda$ of weight $\lambda - \varepsilon_1 - \varepsilon_{l+1}$.

Proof. The case of $k = l-1, l, l+1$ has been worked out in Sections 3.2.1-3.2.3. We suppose that the statement is proved for some $k+1 \leqslant l+1$ and prove it for k. Clearly $e_{\alpha_j} f^{(P)}_{\varepsilon_k+\varepsilon_{l+1}} v_\lambda = 0$ for

$j > k+1$ by the induction assumption and $j < k$ by construction. The element $f^{(P)}_{\varepsilon_{k+1}+\varepsilon_{l+1}}$ of weight $-\varepsilon_{k+1}-\varepsilon_{l+1}$ commutes with e_{α_k} modulo $U_q(\mathfrak{b}^-)e_{\alpha_{k-1}}$, which readily implies the formula for $j = k$. Then the remaining equality $e_{\alpha_{k+1}} f^{(P)}_{\varepsilon_k+\varepsilon_{l+1}} v_\lambda = 0$ easily follows from the induction assumption

$$e_{\alpha_{k+1}} f^{(P)}_{\varepsilon_{k+1}+\varepsilon_{l+1}} = [\lambda_{k+1} + \lambda_{l+1} + P + l - k - 1]_q f^{(P)}_{\varepsilon_{k+2}+\varepsilon_{l+1}} v_\lambda.$$

Finally, we argue that $f^{(P)}_{\varepsilon_1+\varepsilon_{l+1}} v_\lambda$ does not turn zero for all λ. We showed in Sections 3.2.1–3.2.3 that $f^{(P)}_{\varepsilon_k+\varepsilon_{l+1}} v_\lambda \neq 0$ for $k = l, l+1, l+2$. Assuming it is true for all $k \leqslant l$, observe that $f^{(P)}_{\varepsilon_k+\varepsilon_{l+1}}$ is a "modified commutator" of $f^{(P)}_{\varepsilon_{k+1}+\varepsilon_{l+1}}$ with f_{α_k} and that $(\alpha_k, \varepsilon_{k+1} + \varepsilon_{l+1}) \neq 0$. Further arguments are based on [67], Lemma 9.1, and are similar to the proof of Corollary 9.2 therein. $\square$

Next we determine the principal terms of the generating coefficients. This will be of importance for further analysis. Observe that

$$\begin{aligned}
f^{(P)}_{2\varepsilon_{l+1}} v_\lambda &= [\frac{P}{2}]_q \psi^{l+1,N-l} v_\lambda + \ldots, \\
f^{(P)}_{\varepsilon_l+\varepsilon_{l+1}} v_\lambda &= [\lambda_{l+1} + P - 1]_q [\lambda_{l+1} + \frac{P}{2}]_q \psi^{l,N-l} v_\lambda + \ldots, \\
f^{(P)}_{\varepsilon_m+\varepsilon_{l+1}} v_\lambda &= [\lambda_{l+1} + P - 1]_q [\lambda_{l+1} + \frac{P}{2}]_q \prod_{i=m+1}^{l} [\lambda_i + \lambda_{l+1} + P + l - i + 1]_q \psi^{m,N-l} v_\lambda + \ldots,
\end{aligned}$$

where $m < l$. The omitted terms contain only non-principal monomials.

Now we can express the principal terms of the coefficients $y_i = y_{N-l,i}$ of the singular vector u_{N-l}. Introduce scalar coefficients c'_i via the equality $y_i = c'_i \psi^{i,N-l} v_\lambda + \ldots$, where the omitted terms do not contain $\psi^{i,N-l} v_\lambda$. Note that we have exact equality $y_i = c'_i \psi^{i,N-l} v_\lambda$ for $i = l+2, \ldots, N-l$. Formula (3.7) can be rewritten as

$$y_j = (-1)^{\epsilon_i+1} q^{(\varepsilon_i-\varepsilon_j,\varepsilon_i)} e_{\varepsilon_j-\varepsilon_i} y_i = (-1)^{\epsilon_i+1} q^{(\varepsilon_i,\varepsilon_i)} e_{\varepsilon_j-\varepsilon_i} y_i$$

for all $i, j \in [1, N]$ such that $\varepsilon_j - \varepsilon_i \in \Pi^+$. Then

$$c'_m = (-q)^{m-1} [\lambda_{l+1} + P - 1]_q [\lambda_{l+1} + \frac{P}{2}]_q \prod_{i=1}^{m-1} [\lambda_i + \lambda_{l+1} + P + l - i]_q \prod_{i=m+1}^{l} [\lambda_i + \lambda_{l+1} + P + l - i + 1]_q,$$

$$c'_{l+1} = (-q)^l [\frac{P}{2}]_q \prod_{i=1}^{l} [\lambda_i + \lambda_{l+1} + P + l - i]_q,$$

$$c'_{l+2} = (-q)^{l+1} [\lambda_{l+1} + \frac{P}{2} - 1]_q \prod_{i=1}^{l} [\lambda_i + \lambda_{l+1} + P + l - i]_q,$$

where $m = 1, \ldots, l$. Assuming $\mathfrak{g} = \mathfrak{so}(2n+1)$, we continue as

$$\begin{aligned}
c'_{l+2+k} &= (-q)^k c'_{l+2}, & k &= 1, \ldots p, \\
c'_{n+1+k} &= (-q)^{n-l-1} q^{k-1} c'_{l+2}, & k &= 1, \ldots p, \\
c'_{n+2+p} &= (-q)^{n-l-1} q^p c'_{l+2} [\lambda_{l+1}]_q.
\end{aligned} \tag{3.13}$$

For $\mathfrak{g} = \mathfrak{so}(2n)$, we have

$$
\begin{array}{rcll}
c'_{l+2+k} & = & (-q)^k c'_{l+2}, & k = 1, \dots p-1, \\
c'_{n+1+k} & = & (-q)^{n-l-2} q^k c'_{l+2}, & k = 0, \dots p, \\
c'_{n+2+p} & = & (-q)^{n-l-2} q^{p+1} c'_{l+2} [\lambda_{l+1}]_q. &
\end{array}
\tag{3.14}
$$

We use these formulas in the next section.

3.3 Minimal Polynomial for $\mathcal{Q}$.

In this section we deal with two Levi subalgebras, $\mathfrak{k}$ and $\hat{\mathfrak{k}} = \mathfrak{h} + \mathfrak{so}(P) \subset \mathfrak{k}$. All objects related to $\hat{\mathfrak{k}}$ will be marked with a hat. In particular, $\hat{M}_\lambda$ is a parabolic Verma module induced from $U_q(\hat{\mathfrak{k}}+\mathfrak{g}_+)$, while M_λ stands for the one induced from $U_q(\mathfrak{k}+\mathfrak{g}_+)$,

Given a weight $\lambda \in \mathfrak{C}^*_{\hat{\mathfrak{k}}}$ define $\hat{V}_i \subset \mathbb{C}^N \otimes \hat{M}_\lambda$ to be the submodule generated by $\{w_k \otimes v_\lambda\}_{k=1}^i$. The sequence $\{0\} = \hat{V}_0 \subset \hat{V}_1 \subset \ldots \subset \hat{V}_N$ forms a filtration, $\hat{V}_\bullet$, of $\mathbb{C}^N \otimes \hat{M}_\lambda$. Its graded component $\mathrm{gr}\hat{V}_j = \hat{V}_j/\hat{V}_{j-1}$ is generated by (the image of) $w_j \otimes v_\lambda$;

Now assume that $\lambda \in \mathfrak{C}^*_{\mathfrak{k}} \subset \mathfrak{C}^*_{\hat{\mathfrak{k}}}$. Recall that $\{w_{m_i}\}_{i=1}^{2\ell+3}$ are the highest weight vectors of the irreducible $\mathfrak{k}$-blocks in (3.4). Since $\mathrm{Span}\{w_k\}_{k=m_i}^{m_{i+1}-1} = U_q(\mathfrak{k}) w_{m_i}$, the image of $\hat{V}_k$ under projection $\mathbb{C}^n \otimes \hat{M}_\lambda \to \mathbb{C}^n \otimes M_\lambda$ coincides with the image V_{m_i} of $\hat{V}_{m_i}$ for all $k = m_i, \dots, m_{i+1}-1$. The sequence $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_N$ forms a filtration $V_\bullet$ of $\mathbb{C}^N \otimes M_\lambda$ with the graded module

$$
\mathrm{gr}V_\bullet = (\oplus_{i=1}^{\ell} \mathrm{gr}V_i) \oplus \mathrm{gr}V_{l+1} \oplus \mathrm{gr}V_{l+2} \oplus \mathrm{gr}V_{N-l} \oplus (\oplus_{i=1}^{\ell} \mathrm{gr}V_{\ell+3+i}).
\tag{3.15}
$$

The graded components $\mathrm{gr}V_i = V_i/V_{i-1}$ are labelled with irreducible $\mathfrak{k}$-submodules of (3.4), and generated by the images of $w_{m_i} \otimes v_\lambda$ carrying the highest weight $\lambda + \varepsilon_{m_i}$.

PROPOSITION 3.14 *As a filtration of $U_q(\mathfrak{g}_-)$-modules, $V_\bullet$ is independent of $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$.*

Proof. This statement is true for all parabolic Verma modules relative to a Levi subalgebra $\mathfrak{k}$. Consider the subalgebra $U_q(\mathfrak{g}'_-) \simeq U_q(\mathfrak{g}_-)$ in $U_q(\mathfrak{b}_-)$ generated by $f'_\alpha = q^{h_\alpha} f_\alpha$, $\alpha \in \Pi^+$. The $U_q(\mathfrak{g}'_-)$-module M_λ is isomorphic as a $U_q(\mathfrak{g}_-)$-module to the quotient of $U_q(\mathfrak{g}'_-)$ by the left ideal $\sum_{\alpha \in \sigma \Pi^+_{\mathfrak{k}}} U_q(\mathfrak{g}'_-) f'_\alpha$. This ideal is independent of λ, hence M_λ are isomorphic for all λ. With this isomorphism, the representation of $U_q(\mathfrak{g}'_-)$ on $\mathbb{C}^n \otimes M_\lambda$ is independent of λ since $\Delta(f'_\alpha) = f'_\alpha \otimes 1 + q^{h_\alpha} \otimes f'_\alpha$. On the other hand, $V_i/V_{i-1} \simeq U_q(\mathfrak{b}_-)(w_{m_i} \otimes v_\lambda) = U_q(\mathfrak{g}'_-)(w_{m_i} \otimes v_\lambda)$, $i = 1, \dots, k$ (here we identified $w_{m_i} \otimes v_\lambda$ with its image in V_i/V_{i-1}). $\square$

For generic $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$, the graded component $\mathrm{gr}V_{m_i}$ is a parabolic Verma module induced from $\mathbb{C}_\lambda \otimes \mathbb{C}^{n_i} \subset \mathbb{C}_\lambda \otimes \mathbb{C}^N$, hence that is true for all λ. The operator $\mathcal{Q}$ is scalar on each $\mathrm{gr}V_{m_i}$, which is a cyclic module of highest weight $\lambda + \varepsilon_{m_i}$. Therefore (3.15) determines the spectrum of $\mathcal{Q}$ and a polynomial equation on $\mathcal{Q}$. For generic λ this polynomial is minimal, but may not be so for special values of λ. In particular, that is not the case for $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg'}$.

Suppose that $i \preccurlyeq j$ and fix a path from i to j on the Hasse diagram. We define $\overset{\frown}{\sum}_{m=i}^{j}$ as summation over all nodes m of that path. We shall use it only when it is path-independent.

PROPOSITION 3.15 *Suppose that $i, j \in I$ are such that $i \prec j$. Then*

$$w_i \otimes \psi^{ij} v_\lambda \quad = \quad (-1)^{j-i+\overset{\frown}{\sum}_{k=i}^{j-1}\epsilon_k} q^{(\varepsilon_j-\varepsilon_i,\varepsilon_j)-\overset{\frown}{\sum}_{k=i+1}^{j}(\varepsilon_k,\varepsilon_k)} q^{\lambda_j-\lambda_i} w_j \otimes v_\lambda \mod V_{j-1}. \quad (3.16)$$

If ψ is a Chevalley monomial of weight $\varepsilon_j - \varepsilon_i$ and $\psi \neq \psi^{ij}$, then $w_i \otimes \psi v_\lambda \in V_{j-1}$.

It is also convenient to use an equivalent local version of formula (3.16):

$$w_i \otimes \psi^{ij} v_\lambda = (-1)^{\epsilon_i+1} q^{\lambda_k-\lambda_i+(\varepsilon_k-\varepsilon_i,\varepsilon_j-\varepsilon_k)} w_k \otimes \psi^{kj} v_\lambda \mod V_{j-1} \quad (3.17)$$

where $\varepsilon_i - \varepsilon_k = \alpha \in \Pi^+$ is a positive simple root for some $i, k \in I$, and $j \succcurlyeq k$. Note that (3.16) holds true for $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{k} = \oplus_{i=1}^{\ell+1}\mathfrak{gl}(n_i)$ via the embeddings $U_q(\mathfrak{gl}(n)) \subset U_q(\mathfrak{so}(N))$, $\mathbb{C}^n \subset \mathbb{C}^N$, of algebras and their natural representations.

We consider yet another system of $U_q(\mathfrak{g})$-submodules and compare it with $\{V_i\}_{i=1}^{l+3}$. As we mentioned, for generic λ there are direct sum decompositions

$$\mathbb{C}^N \otimes \hat{M}_\lambda = \oplus_{i=1}^{2l+3}\hat{M}_i, \quad \lambda \in \mathfrak{C}^*_{\hat{\mathfrak{k}}}, \qquad \mathbb{C}^N \otimes M_\lambda = \oplus_{i=1}^{2\ell+3} M_i, \quad \lambda \in \mathfrak{C}^*_{\mathfrak{k}},$$

where $\hat{M}_i$ and M_i are generated by singular vector $\hat{u}_i$ and, respectively, by the projection u_{m_i} of rescaled $\hat{u}_{m_i}$ (which otherwise might turn zero). The submodule M_i (respectively $\hat{M}_i$) is an image of the parabolic Verma module induced from the irreducible $\mathfrak{k}$-submodule of $\mathbb{C}_\lambda \otimes \mathbb{C}^{n_i} \subset \mathbb{C}_\lambda \otimes \mathbb{C}^N$. The left decomposition holds if the Shapovalov forms of $\hat{M}_\lambda$ and all $\hat{M}_i$ are not degenerate; the same is true for the right decomposition. The operator $\mathcal{Q}$ is scalar multiple on $\hat{M}_i$ and M_i with the eigenvalues $\hat{x}_i$ and, respectively, $x_i = \hat{x}_{m_i}$. Denote $W_i = \sum_{k=1}^{i} M_k$.

PROPOSITION 3.16 *There is an inclusion $W_i \subset V_i$. Further, $W_i = V_i$ if and only if $W_i = \oplus_{k=1}^{i} M_k$. Consequently, $W_i = V_i$ if and only if $W_k = V_k$ for all $k \leqslant i$.*

Proof. The inclusion $W_i \subset V_i$ follows from Proposition 3.15. The last statement readily follows from the second. Since M_k and $\mathrm{gr}\, V_k$ are cyclic modules of the same highest weight, either the projection $\pi_k \colon M_k \to \mathrm{gr}\, V_k$ is zero or coincides with $\mathrm{gr}\, V_k$. In the latter case, it is an isomorphism. Indeed, let $\tilde{M}_j$ denote the parabolic Verma module projected onto the submodule M_j. Since $\tilde{M}_j \simeq V_k$, the composition $\tilde{M}_j \to M_j \to V_j$ is an isomorphism once it is surjective. Denote $M'_k = W_{k-1} \cap M_k$. For each k the projection π_k factorizes to the composition

$$M_k \twoheadrightarrow M_k/M'_k \simeq W_k/W_{k-1} \twoheadrightarrow W_k/(W_k \cap V_{k-1}) \hookrightarrow \mathrm{gr} V_k,$$

where the left and middle arrows are surjective and the right one is injective. As argued, π_k is either an isomorphism or $\pi_k = 0$. If $M'_k = \{0\}$ for all $k \leqslant i$, then, by ascending induction on k, all these maps are isomorphisms, and $V_k = W_k$ including $k = i$. Conversely, assuming $V_i = W_i$, we get $M'_i = \{0\}$ and $V_{i-1} = W_{i-1}$. Descending induction on i completes the proof. $\square$

COROLLARY 3.17 *For all $j \in I$, decomposition $W_j = \oplus_{i=1}^{j} M_i$ holds if and only if $\pi_i(u_i) \neq 0$ for all $i = 1, \ldots, j$.*

In particular, if the eigenvalues $\{x_k\}_{k=1}^N$ are pairwise distinct, the sum $W_{2\ell+3} = \oplus_{k=1}^{2\ell+3} M_k$ is direct, and $W_{2\ell+3} = V_{2\ell+3} = \mathbb{C}^N \otimes M_\lambda$. However, we are interested in the situation when $x_{\ell+1} = x_{\ell+3}$. To address this case, we need to calculate the $\pi_{\ell+3}(u_{\ell+3}) \in \mathrm{gr} V_{\ell+3}$.

Let C_i, $i = 1, \ldots, 2\ell+3$, be the scalar coefficient in the presentation $u_i = C_i w_i \otimes v_\lambda \mod V_{i-1}$ and $\hat{C}_i$ be similarly defined for $i = 1, \ldots, 2l+3$. Note that the image of $\hat{u}_i$ may turn zero in $\mathbb{C}^N \otimes M_\lambda$, so u_i is obtained from $\hat{u}_{m_i}$ after an appropriated rescaling. This implies that C_i is proportional to $\hat{C}_{m_i}$ up to a factor turning zero at $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$.

The next goal is to calculate $\hat{C}_i$ for some i of importance. We do it first for $i = n+1$ in the case of odd N. Retaining the principal term, we write

$$y_{n+1,i} = (-q)^{i-1} \prod_{k=1}^{i-1} [\lambda_k + n - k] \prod_{k=i+1}^{n} [\lambda_k + n - k + 1] \psi^{i,n+1} v_\lambda + \cdots$$

PROPOSITION 3.18 $\hat{C}_{n+1} = \prod_{j=1}^n [\lambda_j + 1 + n - j]_q$.

Proof. We can check that

$$\hat{C}_{n+1} = \sum_{i=1}^{n+1} q^{i-1} q^{-\lambda_i + i - n - \delta_{i\, n+1}} \prod_{j=1}^{i-1} [\lambda_j + n - j]_q \prod_{j=i+1}^{n} [\lambda_j + n + 1 - j]_q \tag{3.18}$$

Replacing λ_i with $\lambda_i - \lambda_{n+1}$, we get the expression, which is shown in [7], Lemma 6.1, to be equal to $\prod_{j=1}^n [\lambda_j - \lambda_{n+1} + 1 + n - j]_q$, for any λ_i, $i = 1, \ldots, n+1$. This proves the lemma. $\square$

Next we calculate $\hat{C}_{N-l}$. First we assume $l = 0$. The coefficient $\hat{C}_N$ is $\sum_{i=1}^N c'_i c''_i$, where $c'_1 = [\frac{P}{2}]_q$, $c'_2 = -q[\lambda_1 + \frac{P}{2} - 1]_q$, and c'_i for $i > 2$ are given by formulas (3.13) and (3.14) (one should put $l = 0$ there). The coefficients c''_i are obtained by specialization of (3.16). For $N = 2n+1$ they are $c''_N = 1$ and

$$c''_1 = (-1)^{p-1} q^{-2\lambda_1} q^{-2p+1},\ c''_{1+k} = (-1)^{p-1-k} q^{-\lambda_1} q^{-2p+k},\ c''_{n+m} = q^{-\lambda_1} q^{-p-1+m},\ c''_{N-1} = q^{-\lambda_1},$$

where $k = 1, \ldots, p$, $m = 1, \ldots, m+1$. For $N = 2n$, they are $c''_N = 1$ and

$$c''_1 = (-1)^p q^{-2\lambda_1} q^{-2p+2}, \quad c''_{1+k} = (-1)^{p-k} q^{-\lambda_1} q^{-2p+1+k} \quad c''_{n+k} = q^{-\lambda_1} q^{-p+k}, \quad c''_{N-1} = q^{-\lambda_1},$$

where $k = 1, \ldots, p$.

LEMMA 3.19 *In the symmetric case $l = 0$, the singular vector $\hat{u}_N$ is equal to $\hat{C}_N w_N \otimes v_\lambda$ modulo $\hat{V}_{N-1}$, where $\hat{C}_N = (-1)^{[\frac{P+1}{2}]} [\lambda_1 + \frac{P}{2}]_q [\lambda_1 + P - 1]_q$.*

Proof. The coefficient $(-1)^{[\frac{P+1}{2}]} \hat{C}_N$ is equal to

$$q^{-2\lambda_1 - 2p+1} [\frac{P}{2}]_q + [\lambda_1 + \frac{P}{2} - 1]_q q^{-\lambda_1} \big(-\frac{q^{-2p+1} - q}{q - q^{-1}} + q + \frac{q^{2p+1} - q}{q - q^{-1}} \big) + q^{2p+1} [\lambda_1 + \frac{P}{2} - 1]_q [\lambda_1]_q$$

if $P = 2p + 1$. For $P = 2p$, it is equal to

$$q^{-2\lambda_1-2p+2}[\frac{P}{2}]_q + [\lambda_1 + \frac{P}{2} - 1]_q q^{-\lambda_1} q\big(-\frac{q^{-2p+1} - q}{q - q^{-1}} + \frac{q^{2p-1} - q^{-1}}{q - q^{-1}}\big) + q^{2p}[\lambda_1 + \frac{P}{2} - 1]_q[\lambda_1]_q.$$

Counting the coefficients before $q^{\pm 2\lambda_1}$ and λ-independent terms proves the statement. $\square$

Now consider the general case $l > 0$.

PROPOSITION 3.20 *The singular vector $\hat{u}_{N-l}$ is equal to $\hat{C}_{N-l} w_{N-l} \otimes v_\lambda$ modulo $\hat{V}_{N-1}$, where*

$$\hat{C}_{N-l} = (-1)^{[\frac{P+1}{2}]+l}[\lambda_{l+1} + \frac{P}{2}]_q \prod_{\substack{j=1\\ j\neq l+1}}^{l+2} [\lambda_j + \lambda_{l+1} + P + 1 + l - j]_q.$$

Proof. The second sum in the expansion $\hat{u}_{N-l} = \sum_i^l w_i \otimes y_i + \sum_{i=l+1}^l w_i \otimes y_i$ can be replaced with $(-q)^l \prod_{i=1}^l [\lambda_i + \lambda_{l+1} + P + l - i]_q \hat{C}_{N-l} w_{N-l} \otimes y_{N-l} \mod \hat{V}_{N-l-1}$, where the factor before $\hat{C}_{N-l}$ comes from a different normalization of c'_{l+1} and c'_1 in Lemma 3.19. We have

$$c'_i = (-1)^{[\frac{P+1}{2}]}(-q)^{i-1}\hat{C}_{N-l} \prod_{j=1}^{i-1}[\lambda_j + \lambda_{l+1} + P + l - j]_q \prod_{j=i+1}^{l} [\lambda_j + \lambda_{l+1} + P + l - j + 1]_q$$

and $c''_i = (-1)^{[\frac{P+1}{2}]-l+i-1} q^{-P-l+i+} q^{-\lambda_i-\lambda_{l+1}}$ for $i = 1, \ldots, l$. Note with care that $c''_l = -q^{-2} q^{-\lambda_l+\lambda_{l+1}} c''_{l+1}$. Summing up the products

$$c'_i c''_i = (-1)^l q^{i-1} q^{-l+i} q^{-\lambda_i-\lambda_{l+1}-P} \hat{C}_{N-l} \prod_{j=1}^{i-1}[\lambda_j + \lambda_{l+1} + P + l - j]_q \prod_{j=i+1}^{l} [\lambda_j + \lambda_{l+1} + P + l - j + 1]_q$$

from $i = 1$ to $i = l$ and adding $(-q)^l \prod_{i=1}^l [\lambda_i + \lambda_{l+1} + P + l - i]_q \hat{C}_{N-l}$ we get $\hat{C}_{N-l}(-1)^l$ times the right-hand side of (3.18), where we should replace n with l and λ_i with $\lambda_i + \lambda_{l+1} + P$ for $i = 1, \ldots, l$. Finally, since $\lambda_{l+2} = 0$, the factor $[\lambda_{l+1} + P - 1]_q$ is included in the product as $[\lambda_j + \lambda_{l+1} + P + 1 + l - j]_q$, $j = l + 2$. $\square$

The operator $\mathcal{Q}$ acting on $\mathbb{C}^N \otimes \hat{M}_\lambda$ satisfies the polynomial equation $\prod_{l=1}^{2l+3}(\mathcal{Q} - \hat{x}_i) = 0$. When projected to $\mathrm{End}(\mathbb{C}^N \otimes M_\lambda)$, it satisfies the equation $\prod_{l=1}^{2\ell+3}(\mathcal{Q} - x_i) = 0$, where $x_i = \hat{x}_{m_i}$. Denote by $\bar{C}_{\ell+3}$ the product of $\hat{x}_{l+1} - \hat{x}_k$ over all $k \leqslant l$ such that $k \neq m_i$, $i = 1, \ldots, \ell$. Put $C_{\ell+3} = \frac{\hat{C}_{\ell+3}}{\bar{C}_{\ell+3}}$. Using arguments similar to Lemma 5.13, we can prove that the image of $u_{\ell+3} = \frac{1}{\bar{C}_{\ell+3}} \hat{u}_{\ell+3}$ in $\mathbb{C}^N \otimes M_\lambda$ is regular in q and $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$. Then $u_{\ell+3} = C_{\ell+3} w_{\ell+3} \otimes v_\lambda \mod V_{\ell+2}$ is a singular vector. Similarly we define u_{n+1} for the case $N = 2n + 1$, $P = 1$.

PROPOSITION 3.21 *Suppose that $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$, and $q \in \mathbb{C}$ are such that $\{x_i\}_{i=1}^{2\ell+3} - \{x_{\ell+3}\}$ are pairwise distinct. Then $\mathbb{C}^N \otimes M_\lambda = \oplus_{i=1}^{2l+3} M_i$.*

Proof. All we need to check is that the sum $M_{\ell+1} + M_{\ell+3}$ is direct. We have $W_{\ell+2} = \oplus_{i=1}^{\ell+2} M_i$ hence $W_{\ell+2} = V_{\ell+2}$, by Proposition 3.16. Further, $C_{\ell+3} \neq 0$ implies $W_{\ell+3} = V_{\ell+3}$, hence $M_{\ell+1} \cap M_{\ell+3} \subset W_{\ell+2} \cap M_{\ell+3} = \{0\}$, again by Proposition 3.16. $\square$

COROLLARY 3.22 *For $\lambda \in \mathfrak{C}^*_{\mathfrak{k},'}$, the operator $\mathcal{Q} \in \mathrm{End}(\mathbb{C}^N \otimes M_\lambda)$ satisfies a polynomial equation of degree $2\ell+2$ with roots $\{x_i\}_{i=1}^{2\ell+3} - \{x_{\ell+3}\}$.*

3.4 Quantization of Borderline Levi Classes

Recall that quantization of a commutative $\mathbb{C}$-algebra $\mathcal{A}$ is a $\mathbb{C}[\![\hbar]\!]$-algebra $\mathcal{A}_\hbar$ which is free as a $\mathbb{C}[\![\hbar]\!]$-module and $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \simeq \mathcal{A}$. If $\mathcal{A}$ is a $U(\mathfrak{g})$-algebra, the quantization is called $U_\hbar(\mathfrak{g})$-equivariant if $\mathcal{A}_\hbar$ is a $U_\hbar(\mathfrak{g})$-algebra, and the action of $U_\hbar(\mathfrak{g})$ is a deformation of the $U(\mathfrak{g})$-action. In this section, $\mathcal{A}$ will be the polynomial algebra either on G or on its (borderline Levi) conjugacy class.

Fix $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg'}$ and define $\boldsymbol{\mu} \in \mathbb{C}^{\ell+2}[\![\hbar]\!]$ by

$$\mu_i = x_i, \quad i = 1,\ldots,\ell+2. \tag{3.19}$$

The eigenvalues of $\mathcal{Q}$ on $\mathrm{End}(\mathbb{C}^N \otimes M_\lambda)$ are expressed through $\boldsymbol{\mu}$ by

$$\mu_i, \quad \mu_i^{-1} q^{-2N+2(n_i+1)}, \quad i = 1,\ldots,\ell, \quad \mu_{\ell+1} = -q^{-N+2}, \quad \mu_{\ell+2} = q^{-N+P}, \tag{3.20}$$

cf. (3.5). By construction, $\lim_{\hbar\to 0} \boldsymbol{\mu} \in \hat{\mathcal{M}}'_K$.

Define central elements $\tau_k = \mathrm{Tr}\big(q^{2h_\rho}\mathcal{Q}^k\big) = \mathrm{Tr}_q\big(\mathcal{Q}^k\big) \in U_q(\mathfrak{g})$, [28], where $\rho = \frac{1}{2}\sum_{\alpha\in \mathrm{R}_+}\alpha = \sum_{i=1}^n(\frac{N}{2} - i)\varepsilon_i$ is the half-sum of positive roots. A module of highest weight λ determines a one dimensional representation χ^λ of the center of $U_q(\mathfrak{g})$, which assigns a scalar to each τ_k:

$$\chi^\lambda(\tau_k) = \sum_{i=1}^N q^{2k(\lambda+\rho,\varepsilon_i)-2k(\rho,\varepsilon_1)+k(\varepsilon_i,\varepsilon_i)-k} \prod_{\alpha\in\mathrm{R}_+} \frac{q^{(\lambda+\varepsilon_i+\rho,\alpha)} - q^{-(\lambda+\varepsilon_i+\rho,\alpha)}}{q^{(\lambda+\rho,\alpha)} - q^{-(\lambda+\rho,\alpha)}}, \tag{3.21}$$

cf. [68], formula (24). Restriction of λ to $\mathfrak{C}^*_{\mathfrak{k},reg'}$ makes the right-hand side a function of the vector $\boldsymbol{\mu}$ defined in (3.19). We denote this function by $\vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu})$, where $\boldsymbol{n} = (n_1,\ldots,n_\ell,1,p)$ is the integer valued vector of multiplicities. In the limit $\hbar \to 0$, $\vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu})$ goes over into the right-hand side of (3.3), where $\mu_i = \lim_{h\to 0} q^{2(\lambda,\varepsilon_{m_i})}$, $i = 1,\ldots,\ell$.

In general, $\tau^k \mod \hbar$ do not separate classical conjugacy classes of $SO(2n)$. That is done by an additional invariant which nevertheless turns zero on a class with eigenvalues ± 1. Therefore its quantum counterpart τ^- yielding $\chi^\lambda(\tau^-) = \prod_{i=1}^n (q^{2(\lambda+\rho,\varepsilon_i)} - q^{-2(\lambda+\rho,\varepsilon_i)})$ can be ignored.

Denote by $S \in \mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathbb{C}^N)$ the product PR of the ordinary flip P on $\mathbb{C}^N \otimes \mathbb{C}^N$ and the R-matrix R in the form of [40]. It is $U_\hbar(\mathfrak{g})$-invariant, i.e. commutes with $\Delta(x)$ for all $x \in U_\hbar(\mathfrak{g})$. Let $\kappa \in \mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathbb{C}^N)$ be the one-dimensional projector to the trivial $U_\hbar(\mathfrak{g})$-submodule, [40]. Denote by $\mathbb{C}_\hbar[O(N)]$ the associative $\mathbb{C}[\![\hbar]\!]$-algebra generated by the matrix entries $X = (X_{ij})_{i,j=1}^N \in \mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}_\hbar[O(N)]$ modulo the relations,

$$S_{12}X_2S_{12}X_2 = X_2S_{12}X_2S_{12}, \quad X_2S_{12}X_2\kappa = q^{-N+1}\kappa = \kappa X_2S_{12}X_2. \tag{3.22}$$

These equations are understood in $\mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}_\hbar[O(N)]$, and the indices distinguish the two copies of $\mathrm{End}(\mathbb{C}^N)$, in the usual way.

The algebra $\mathbb{C}_\hbar[O(N)]$ is an equivariant quantization of $\mathbb{C}[O(N)]$, [66]. The algebra $\mathbb{C}_\hbar[G]$, $G = SO(N)$, is a quotient of $\mathbb{C}_\hbar[O(N)]$ setting the quantized determinant (whose existence follows from general deformation arguments, as in [66]) to 1. Its explicit form is immaterial, because it is automatically fixed by the equations of conjugacy class.

The algebra $\mathbb{C}_\hbar[G]$ can be realized as a $U_\hbar(\mathfrak{g})$-invariant subalgebra in $U_q(\mathfrak{g})$ (extension over $\mathbb{C}[\![\hbar]\!]$ understood), with respect to the adjoint action. The embedding is implemented via the assignment

$$\mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}_\hbar[G] \ni X \mapsto \mathcal{Q} \in \mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{g}).$$

Via this inclusion, the $U_q(\mathfrak{g})$-module M_λ extended over $\mathbb{C}[\![\hbar]\!]$ becomes a $\mathbb{C}_\hbar[G]$-module.

THEOREM 3.23 *Suppose that $\lambda = \mathfrak{C}^*_{\mathfrak{k},reg'}$ and let $\boldsymbol{\mu}$ be as in (3.19). The quotient of $\mathbb{C}_\hbar[G]$ by the ideal of relations*

$$\prod_{i=1}^{\ell}(\mathcal{Q} - \mu_i) \times (\mathcal{Q} - \mu_{\ell+1})(\mathcal{Q} - \mu_{\ell+2}) \times \prod_{i=1}^{\ell}(\mathcal{Q} - \mu_i^{-1} q^{-2N+2(n_i+1)}) = 0, \tag{3.23}$$

$$\mathrm{Tr}_q\big(\mathcal{Q}^k\big) = \vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu}), \tag{3.24}$$

is an equivariant quantization of the class $\lim_{\hbar\to 0} \boldsymbol{\mu} \in \hat{\mathcal{M}}'_K$. It is the image of $\mathbb{C}_\hbar[G]$ in the algebra of endomorphisms of the $U_q(\mathfrak{g})$-module M_λ.

Proof. The proof is similar to [64], Theorem 10.1. and [68], Theorem 8.2. Here we indicate the key steps. The algebra $\mathrm{End}(M_\lambda)$ is $\mathbb{C}[\![\hbar]\!]$-free since the module M_λ is $\mathbb{C}[\![\hbar]\!]$-free. The image of $\mathbb{C}_\hbar[G]$ in $\mathrm{End}(M_\lambda)$ factors through a quotient by the ideal generated by $\ker \chi^\lambda$. As a module over $U_\hbar(\mathfrak{g})$, this quotient has $\mathbb{C}[\![\hbar]\!]$-finite isotypic components. Therefore its image in $\mathrm{End}(M_\lambda)$ is $\mathbb{C}[\![\hbar]\!]$-free. The annihilator of M_λ in $\mathbb{C}_\hbar[G]$ contains the $U_\hbar(\mathfrak{g})$-invariant ideal J generated by (3.23) and (3.24). The image of $J/\hbar J$ in $\mathbb{C}[G]$ is exactly the defining ideal of the conjugacy class and therefore a maximal proper $U(\mathfrak{g})$-invariant ideal. This implies that J is exactly the annihilator of M_λ, by the Nakayama lemma, [34]. The quotient of $\mathbb{C}_\hbar[G]$ by J is free over $\mathbb{C}[\![\hbar]\!]$, and its zero fiber $\mod \hbar$ is the polynomial algebra on the conjugacy class. Therefore, $\mathbb{C}_\hbar[G]/J$ is an equivariant quantization of the class. □

Theorem 3.23 describes the ideal of the class in the algebra $\mathbb{C}_\hbar[G]$. To describe its pre-image in $\mathbb{C}_\hbar[O(N)]$, we should replace $\mathcal{Q}$ with X in (3.23) and (3.24) and add the relations (3.22).

Chapter 4

R-Matrix and Mickelsson Algebras for Orthogonal and Symplectic Quantum Groups

In the mathematics literature, lowering and raising operators are known as generators of step algebras, which were originally introduced by Mickelsson [60] for reductive pairs of Lie algebras, $\mathfrak{g}' \subset \mathfrak{g}$. These algebras naturally act on $\mathfrak{g}'$-singular vectors in $U(\mathfrak{g})$-modules and are important in representation theory, [62, 84]. It is known that the step algebra $Z(\mathfrak{g}, \mathfrak{g}')$ is generated by the image of the orthogonal complement $\mathfrak{g} \ominus \mathfrak{g}'$ under the extremal projector of the $\mathfrak{g}'$. Another description of lowering/raising operators for classical groups was obtained in [62, 71, 74] in an explicit form of polynomials in $\mathfrak{g}$.

A generalization of the results of [71, 74] to quantum $\mathfrak{gl}(n)$ can be found in [4]. In this special case, the lowering operators can be also conveniently expressed through "modified commutators" in the Chevalley generators of $U(\mathfrak{g})$ with coefficients in the field of fractions of $U(\mathfrak{h})$. Extending [62] to orthogonal and symplectic quantum groups is not straightforward, since there are no nilpotent triangular Lie subalgebras $\mathfrak{g}_\pm$ in $U_q(\mathfrak{g})$ but only their deformed associative envelope. The lack of $\mathfrak{g}_\pm$ can be compensated by the entries of the universal R-matrix with one leg projected to the natural representation. Those entries are nicely expressed through modified commutators in the Chevalley generators turning into elements of $\mathfrak{g}_\pm$ in the quasi-classical limit. Their commutation relation with the Chevalley generators modify the classical commutation relations with $\mathfrak{g}_\pm$ in a way, which is easy to control. Thus the results of [62, 71, 74] can be generalized and generators of Mickelsson algebras for the non-exceptional quantum groups can be constructed. Explicit form of these generators is useful because they are related to singular vectors generating certain submodules involved in quantization of conjugacy classes, especially in Chapter 6.

4.1 R-Matrix of Non-Exceptional Quantum Groups

In this section, we work with the opposite version of the comultiplication as compared to Chapter 2. The universal R-matrix

$$\mathcal{R} = q^{\sum_{i=1}^n h_{\varepsilon_i}\otimes h_{\varepsilon_i}} \prod_{\beta} \exp_{q_\beta}\Big((1-q_\beta^{-2})(e_\beta\otimes e_{-\beta})\Big) \in U_q(\mathfrak{b}_+)\hat{\otimes} U_q(\mathfrak{b}_-), \tag{4.1}$$

is obtained from (2.3) by the flip of tensor factors.

Define $\check{\mathcal{R}} = q^{-\sum_{i=1}^n h_{\varepsilon_i}\otimes h_{\varepsilon_i}}\mathcal{R}$. Denote by $\check{R}^- = (\pi\otimes \mathrm{id})(\check{\mathcal{R}}) \in \mathrm{End}(\mathbb{C}^N)\otimes U_q(\mathfrak{g}_-)$ and by $\check{R}^+ = (\pi\otimes \mathrm{id})(\check{\mathcal{R}}_{21}) \in \mathrm{End}(\mathbb{C}^N)\otimes U_q(\mathfrak{g}_+)$. In this section, we deal only with $\check{R}^-$ and suppress the label "$-$" for simplicity, $\check{R} = \check{R}^-$.

Denote by N_+ the ring of all upper triangular matrices in $\mathrm{End}(\mathbb{C}^N)$ and by N_+' its ideal spanned by e_{ij}, $i<j+1$.

LEMMA 4.1 *One has*

$$\check{R} = 1\otimes 1 + (q^{1+\delta_{1n}} - q^{-1-\delta_{1n}})\sum_{i=1}^n \pi(e_i)\otimes f_i \mod N_+'\otimes U_q(\mathfrak{g}_-),$$

where δ_{1n} is present only for $\mathfrak{g} = \mathfrak{sp}(2n)$.

Proof. For all positive roots α,β the matrix $\pi(e_\alpha e_\beta)$ belongs to N_+'. Also, $\pi(e_\beta)\in N_+'$ for all $\beta\in \mathrm{R}^+\backslash\Pi^+$. Therefore, the only terms that contribute to $\mathrm{Span}_{\varepsilon_i-\varepsilon_j\in\Pi^+}\{e_{ij}\otimes U_q(\mathfrak{g}_-)\}$ are those of degree 1 from the series $\exp_{q_\alpha}\big((1-q_\alpha^{-2})(e_\alpha\otimes e_{-\alpha})\big)$ with $\alpha\in\Pi^+$. □

Write $\check{R} = \sum_{i,j=1}^N e_{ij}\otimes \check{R}_{ij}$, where $\check{R}_{ij} = 0$ for $i>j$. Due to the $\mathfrak{h}$-invariance of $\check{R}$, the entry $\check{R}_{ij}\in U_q(\mathfrak{g}_-)$ carries weight $\varepsilon_j-\varepsilon_i$.

Introduce a new notation for the negative root vectors $f_{i,j}\in U_q(\mathfrak{g}_-)$, which are indexed by the matrix entry of their natural representation. For all $\mathfrak{g}$, we have $f_{k,k+1} = f_k = f_{k'-1,k'}$ once $k<n$ and $f_{n,n+1} = f_n = f_{n+1,n'}$ for $\mathfrak{g} = \mathfrak{so}(2n+1)$, $f_{n-1,n'} = f_n = f_{n,n'+1}$ for $\mathfrak{g} = \mathfrak{so}(2n)$, and $f_{nn'} = [2]_q f_n$ for $\mathfrak{g} = \mathfrak{sp}(2n)$. We present explicit expressions for the entries f_{ij} in terms of modified commutators in Chevalley generators, $[x,y]_a = xy - ayx$, where a is a scalar; we also put $\bar{q} = q^{-1}$.

PROPOSITION 4.2 *Suppose that $\varepsilon_i-\varepsilon_j\in \mathrm{R}^+\backslash\Pi^+$. Then the elements f_{ij} are given by the following formulas:*
For all $\mathfrak{g}$ and $i+1<j\leqslant \frac{N+1}{2}$:

$$f_{ij} = [f_{j-1},\dots[f_{i+1},f_i]_{\bar{q}}\dots]_{\bar{q}}, \quad f_{j'i'} = [\dots[f_i,f_{i+1}]_{\bar{q}},\dots f_{j-1}]_{\bar{q}}. \tag{4.2}$$

Furthermore,

- *For $\mathfrak{g} = \mathfrak{so}(2n+1)$: $f_{nn'} = (q-1)f_n^2$ and*

$$f_{i,n+1} = [f_n, f_{i,n}]_{\bar{q}}, \quad f_{n+1,i'} = [f_{n',i'}, f_n]_{\bar{q}}, \quad i<n,$$

$$f_{ij'} = q^{\delta_{ij}}[f_{n+1,j'}, f_{i,n+1}]_{\bar{q}^{\delta_{ij}}}, \quad i,j < n.$$

- *For* $\mathfrak{g} = \mathfrak{sp}(2n)$*:* $f_{nn'} = [2]_q f_n$ *and*

$$f_{in'} = [f_n, f_{in}]_{\bar{q}^2}, \quad f_{ni'} = [f_{n'i'}, f_n]_{\bar{q}^2}, \quad i < n,$$

$$f_{ij'} = q^{\delta_{ij}}[f_{nj'}, f_{in}]_{\bar{q}^{1+\delta_{ij}}}, \quad i,j < n.$$

- *For* $\mathfrak{g} = \mathfrak{so}(2n)$*:* $f_{nn'} = 0$ *and*

$$f_{in'} = [f_n, f_{i,n-1}]_{\bar{q}}, \quad f_{ni'} = [f_{n'+1,i'}, f_n]_{\bar{q}}, \quad i < n-2,$$

$$f_{ji'} = q^{\delta_{ij}}[f_{ni'}, f_{j,n}]_{\bar{q}^{1+\delta_{ij}}}, \quad i,j \leqslant n-1.$$

Proof. The proof is a direct calculation with the use of the identity

$$(f_\alpha \otimes 1)\check{\mathcal{R}} - \check{\mathcal{R}}(f_\alpha \otimes 1) = \check{\mathcal{R}}(q^{-h_\alpha} \otimes f_\alpha) - (q^{h_\alpha} \otimes f_\alpha)\check{\mathcal{R}},$$

which follows from the intertwining axiom (2.2) for $x = f_\alpha$. This allows us to construct the elements f_{ij} by induction starting from f_α, $\alpha \in \Pi^+$. □

PROPOSITION 4.3 *The matrix entries* $f_{i,j} \in U_q(\mathfrak{g}_-)$ *such that* $\varepsilon_i - \varepsilon_j \notin \Pi^+$ *satisfy the identity*

$$[e_\alpha, f_{ij}] = \sum_{(l,r)\in P(\alpha)} \big(f_{il}\delta_{jr}q^{h_\alpha} - q^{-h_\alpha}\delta_{il}f_{rj}\big),$$

for all simple positive roots α.

Proof. The proof is a straightforward calculation based on the intertwining relation (2.2), which is equivalent to

$$(1 \otimes e_\alpha)\check{\mathcal{R}} - \check{\mathcal{R}}(1 \otimes e_\alpha) = \check{\mathcal{R}}(e_\alpha \otimes q^{h_\alpha}) - (e_\alpha \otimes q^{-h_\alpha})\check{\mathcal{R}},$$

for $x = e_\alpha$, $\alpha \in \Pi^+$. Alternatively, we can use the expressions for f_{ij} from Proposition 4.2. □

4.2 Mickelsson Algebras

Consider the simple Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ corresponding to the root subsystem $\mathrm{R}_{\mathfrak{g}'} \subset \mathrm{R}_{\mathfrak{g}}$ generated by α_i, $i > 1$, and let $\mathfrak{h}' \subset \mathfrak{g}'$ denote its Cartan subalgebra. Let the triangular decomposition $\mathfrak{g}'_- \oplus \mathfrak{h}' \oplus \mathfrak{g}'_+$ be compatible with the triangular decomposition of $\mathfrak{g}$. Recall the definition of step algebra $Z_q(\mathfrak{g}, \mathfrak{g}')$ of the pair $(\mathfrak{g}, \mathfrak{g}')$. Consider the left ideal $J = U_q(\mathfrak{g})\mathfrak{g}'_+$ and its normalizer $\mathcal{N} = \{x \in U_q(\mathfrak{g}) : e_\alpha x \subset J, \forall \alpha \in \Pi^+_{\mathfrak{g}'}\}$. By construction, J is a two-sided ideal in the algebra $\mathcal{N}$. Then $Z_q(\mathfrak{g}, \mathfrak{g}')$ is the quotient $\mathcal{N}/J$.

For all $\beta_i \in \mathrm{R}^+_{\mathfrak{g}} \backslash \mathrm{R}^+_{\mathfrak{g}'}$ let e_{β_i} be the corresponding PBW generators and let Z be the vector space spanned by $e^{k_l}_{-\beta_l} \dots e^{k_1}_{-\beta_1} e^{k_0}_0 e^{m_1}_{\beta_1} \dots e^{m_l}_{\beta_l}$, were $e_0 = q^{h_{\alpha_1}}$, $k_i, m_i \in \mathbb{Z}_+$, and $k_0 \in \mathbb{Z}$. The PBW factorization $U_q(\mathfrak{g}) = U_q(\mathfrak{g}'_-) Z U_q(\mathfrak{h}') U_q(\mathfrak{g}'_+)$ gives rise to the decomposition

$$U_q(\mathfrak{g}) = Z U_q(\mathfrak{h}') \oplus (\mathfrak{g}'_- U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \mathfrak{g}'_+).$$

PROPOSITION 4.4 ([53], Theorem 1) *The projection $U_q(\mathfrak{g}) \to Z U_q(\mathfrak{h}')$ implements an embedding of $Z_q(\mathfrak{g}, \mathfrak{g}')$ in $Z U_q(\mathfrak{h}')$.*

Proof. The statement is proved in [53] for the orthogonal and special linear quantum groups but the arguments apply to symplectic groups too. □

The algebra $Z_q(\mathfrak{g}, \mathfrak{g}')$ inherits the adjoint action of the Cartan subalgebra, so we can speak of weights of its elements. It is proved within the theory of extremal projectors that $Z_q(\mathfrak{g}, \mathfrak{g}')$ is generated by elements of weights $\beta \in \mathrm{R}_{\mathfrak{g}} \backslash \mathrm{R}_{\mathfrak{g}'}$ plus $z_0 = q^{h_{\alpha_1}}$. We calculate them in the subsequent sections, cf. Propositions 4.7 (negative β) and 4.11 (positive β).

4.2.1 Lowering Operators

In what follows, we extend $U_q(\mathfrak{g})$ along with its subalgebras containing $U_q(\mathfrak{h})$ over the field of fractions of $U_q(\mathfrak{h})$ and denote such an extension by hat, for example $\hat{U}_q(\mathfrak{g})$. In this section we calculate representatives of the negative generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ in $\hat{U}_q(\mathfrak{b}_-)$.

As in Section 2.1, we consider Hasse diagram of the natural representation, with partial ordering $\prec$. We call a strictly ascending sequence $\vec{m} = (m_1, \dots, m_s)$ of integers a route from m_1 to m_s. We write $m \prec \vec{m}$ and $\vec{m} \prec m$ for $m \in \mathbb{Z}$ if, respectively, $m \prec \min \vec{m}$ and $\max \vec{m} \prec m$. More generally, we write $\vec{m} \prec \vec{k}$ if $\max \vec{m} \prec \min \vec{k}$. In this case, a sequence $(\vec{m}, \vec{k})$ is a route from $\min \vec{m}$ to $\max \vec{k}$. We also write $m \preccurlyeq \vec{m}$ if $m = \min \vec{m}$ and $\vec{m} \preccurlyeq m$ if $m = \max \vec{m}$.

Given a route $\vec{m} = (m_1, \dots, m_s)$, define the product $f_{\vec{m}} = f_{m_1, m_2} \cdots f_{m_{s-1}, m_s} \in U_q(\mathfrak{g}_-)$. Consider a free right $\hat{U}_q(\mathfrak{h})$-module, Φ_{1m}, generated by $f_{\vec{m}}$ with $1 \preccurlyeq \vec{m} \preccurlyeq j$ and define an operation $\partial_{lr} \colon \Phi_{1j} \to \hat{U}_q(\mathfrak{b}_-)$ for $(l, r) \in P(\alpha)$ as follows. Assuming $1 \preccurlyeq \vec{\ell} \prec l \prec r \prec \vec{\rho} \prec j$, set

$$\begin{aligned}
\partial_{lr} f_{(\vec{\ell}, l)} f_{(l, r)} f_{(r, \vec{\rho})} &= f_{(\vec{\ell}, l)} f_{(r, \vec{\rho})} [\eta_{lj} - \eta_{rj}]_q, \\
\partial_{lr} f_{(\vec{\ell}, l)} f_{(l, \vec{\rho})} &= -f_{(\vec{\ell}, l)} f_{(r, \vec{\rho})} q^{-\eta_{lj} + \eta_{rj}}, \\
\partial_{lr} f_{(\vec{\ell}, r)} f_{(r, \vec{\rho})} &= f_{(\vec{\ell}, l)} f_{(r, \vec{\rho})} q^{\eta_{lj} - \eta_{rj}}, \\
\partial_{lr} f_{\vec{m}} &= 0, \qquad\qquad l, r \notin \vec{m}
\end{aligned}$$

Extend ∂_{lr} to the entire Φ_{1j} by $\hat{U}_q(\mathfrak{h})$-linearity. Let $p \colon \Phi_{1j} \to \hat{U}(\mathfrak{g})$ denote the natural homomorphism of $\hat{U}_q(\mathfrak{h})$-modules.

LEMMA 4.5 *For all $\alpha \in \Pi^+$ and all $x \in \Phi_{1j}$, $e_\alpha \circ p(x) = \sum_{(l,r) \in P(\alpha)} p \circ \partial_{lr}(x) \mod \hat{U}_q(\mathfrak{g}) e_\alpha$.*

Proof. A straightforward analysis based on Proposition 4.3 and Lemma 2.10. □

To simplify the presentation, we suppress the symbol of projection p in what follows.

Introduce elements $A_r^j \in \hat{U}_q(\mathfrak{h})$ by

$$A_r^j = \frac{q-q^{-1}}{q^{-2\eta_{rj}}-1}, \tag{4.3}$$

for all $r, j \in I$ subject to $r \prec j$. For each simple pair (l, r) we define (l, r)-chains as

$$f_{(\vec{\ell},l)} f_{(l,\vec{\rho})} A_l^j + f_{(\vec{\ell},l)} f_{(l,r)} f_{(r,\vec{\rho})} A_l^j A_r^j + f_{(\vec{\ell},r)} f_{(r,\vec{\rho})} A_r^j, \quad f_{(\vec{\ell},l)} f_{l,j} A_l^j + f_{(\vec{\ell},j)}, \tag{4.4}$$

where $1 \preccurlyeq \vec{\ell} \prec l$ and $r \prec \vec{\rho} \preccurlyeq j$. Remark that $f_{(l,r)} = \left[\frac{(\alpha,\alpha)}{2}\right]_q e_{-\alpha}$, where $\alpha = \varepsilon_l - \varepsilon_r$.

LEMMA 4.6 *The operator ∂_{lr} annihilates (l, r)-chains.*

Proof. Applying ∂_{lr} to the 3-chain in (4.4), we get

$$f_{(\vec{\ell},l)} f_{(r,\vec{\rho})} (-q^{-\eta_{lj}+\eta_{rj}} A_l^j + [\eta_{lj} - \eta_{rj}]_q A_l^j A_r^j + q^{\eta_{lj}-\eta_{rj}} A_r^j).$$

The factor in the brackets turns zero on substitution of 4.3.

Now apply ∂_{lj} to the right expression in (4.4) and get

$$f_{(\vec{\ell},l)} ([h_\alpha]_q A_l^j + q^{h_\alpha}) = f_{(\vec{\ell},l)} (\frac{q^{h_\alpha} - q^{-h_\alpha}}{q^{-2\eta_{lj}} - 1} + q^{h_\alpha}) = f_{(\vec{\ell},l)} \frac{[h_\alpha - \eta_{lj}]_q}{[-\eta_{lj}]_q} = 0,$$

so long as $\eta_{lj} = h_\alpha$ by Lemma 2.10. $\square$

Given a route $\vec{m} = (m_1, \ldots, m_s)$, put $A_{\vec{m}}^j = A_{m_1}^j \cdots A_{m_s}^j \in \hat{U}_q(\mathfrak{h})$ (and $A_{\vec{m}}^j = 1$ for the empty route) and define

$$z_{-j+1} = \sum_{1 \prec \vec{m} \prec j} f_{(1,\vec{m},j)} A_{\vec{m}}^j \in \hat{U}_q(\mathfrak{b}_-), \quad j = 2, \ldots, N, \tag{4.5}$$

where the summation is taken over all possible $\vec{m}$ subject to the specified inequalities plus the empty route.

PROPOSITION 4.7 $e_\alpha z_{-j} = 0 \mod \hat{U}_q(\mathfrak{g}) e_\alpha$ *for all* $\alpha \in \Pi_{\mathfrak{g}'}^+$ *and* $j = 1, \ldots, N-1$.

Proof. Thanks to Lemma 4.5, we can reduce consideration to the action of operators ∂_{lr}, with $(l, r) \in P(\alpha)$. According to the definition of ∂_{lr} the summands in (4.5) that survive the action of ∂_{lr} can be organized into a linear combination of (l, r)-chains with coefficients in $\hat{U}_q(\mathfrak{h})$. By Lemma 4.6 they are killed by ∂_{lr}. $\square$

The elements z_{-i}, $i = 1, \ldots, N-1$, belong to the normalizer $\mathcal{N}$ and form the set of negative generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ for symplectic $\mathfrak{g}$. In the orthogonal case, the negative part of $Z_q(\mathfrak{g}, \mathfrak{g}')$ is generated by z_{-i}, $i = 1, \ldots, N-2$.

4.2.2 Raising Operators

In this section, we regularize Mickelsson generators of positive weights. The assignment $f_\alpha \mapsto e_\alpha$, extends to an anti-algebra isomorphism $\omega \colon U_q(\mathfrak{g}_-) \to U_q(\mathfrak{g}_+)$. Denote $g_{ji} = \omega(f_{ij}) \in U_q(\mathfrak{g}_+)$. The matrix $\check{R}^+ = (\pi \otimes \mathrm{id})(\check{\mathcal{R}}_{21})$ is equal to $1 \otimes 1 + (q - q^{-1}) \sum_{i<j} e_{ji} \otimes g_{ji}$.

LEMMA 4.8 *For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $i > 1$, $e_\alpha g_{i1} = \sum_{(l,r)\in P(\alpha)} \delta_{il} g_{r1} \mod \hat{U}_q(\mathfrak{g})e_\alpha$.*

Proof. Follows from the intertwining property of the R-matrix. $\square$

Consider the right $\hat{U}_q(\mathfrak{h})$-module Ψ_{i1} freely generated by $f_{(\vec{m},k)}g_{k1}$ with $i \preccurlyeq \vec{m} \prec k$. We define operators $\partial_{lr} \colon \Psi_{i1} \to \hat{U}_q(\mathfrak{g})$ similarly as we did it for Φ_{1j}. For a simple pair $(l,r) \in P(\alpha)$, put

$$\partial_{l,r} f_{(\vec{m},k)} g_{k1} = \begin{cases} f_{(\vec{m},l)} g_{r1}, & l = k, \\ \big(\partial_{l,r} f_{(\vec{m},k)}\big) g_{k1}, & l \neq k, \end{cases} \qquad i \preccurlyeq \vec{m} \prec r.$$

The Cartan factors appearing in $\partial_{lr} f_{(\vec{m},k)}$ depend on h_α. When pushed to the right-most position, h_α is shifted by $(\alpha, \varepsilon_1 - \varepsilon_r)$. We extend ∂_{lr} to an action on Ψ_{i1} by the requirement that ∂_{lr} commutes with the right action of $\hat{U}_q(\mathfrak{h})$. Let p denote the natural homomorphism of $\hat{U}_q(\mathfrak{h})$-modules, $p \colon \Psi_{i1} \to \hat{U}_q(\mathfrak{g})$. We can prove the following analog of Lemma 4.5.

LEMMA 4.9 *For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $x \in \Psi_{i1}$, $e_\alpha \circ p(x) = \sum_{(l,r)\in P(\alpha)} p \circ \partial_{lr}(x) \mod \hat{U}_q(\mathfrak{g})e_\alpha$.*

Proof. Straightforward. $\square$

We suppress the symbol of projection p to simplify the formulas.

For $i \prec j$ let $|i-j|$ be the number of simple positive roots entering $\varepsilon_i - \varepsilon_j$. For all $i, k = 2, \ldots, N$, $i \prec k$, put

$$A^i_k = \frac{q^{\eta_{k1} - \eta_{i1}}}{[\eta_{i1} - \eta_{k1}]_q}, \quad B^i_k = \frac{(-1)^{|i-k|}}{[\eta_{i1} - \eta_{k1}]_q},$$

For each (l,r) such that $\varepsilon_l - \varepsilon_r = \alpha$, where $\alpha \in \Pi^+_{\mathfrak{g}'}$, define 3-chains as

$$f_{(i,\vec{m},l)} g_{l1} B^i_l + f_{(i,\vec{m},l)} f_{(l,r)} g_{r1} A^i_l B^i_r + f_{(i,\vec{m},r)} g_{r1} B^i_r, \tag{4.6}$$

with $i \prec \vec{m} \prec l \prec r \preccurlyeq N$ and

$$f_{(i,\vec{\ell},l)} f_{(l,\vec{\rho},k)} g_{k1} A^i_l + f_{(i,\vec{\ell},l)} f_{(l,r)} f_{(r,\vec{\rho},k)} g_{k1} A^i_l A^i_r + f_{(i,\vec{\ell},r)} f_{(r,\vec{\rho},k)} g_{k1} A^i_r \tag{4.7}$$

with $i \prec \vec{\ell} \prec l \prec r \prec \vec{\rho} \prec k \preccurlyeq N$. The 2-chains are defined as

$$g_{i1} + f_{(i,r)} g_{r1} B^i_r, \quad f_{(i,\vec{m},k)} g_{k1} + f_{(i,r)} f_{(r,\vec{m},k)} g_{k1} A^i_r \tag{4.8}$$

where r is such that $\varepsilon_i - \varepsilon_r \in \Pi^+_{\mathfrak{g}'}$ and $i \prec r \prec \vec{m} \prec k \preccurlyeq N$. In all cases, empty $\vec{m}$ are admissible.

LEMMA 4.10 *For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $(l,r) \in P(\alpha)$ the (l,r)-chains are annihilated by ∂_{lr}.*

Proof. Suppose that $i = l$ and apply ∂_{ir} to the left 2-chain in (4.8). The result is

$$g_{r1} + [h_\alpha]_q g_{r1} B^i_r = g_{r1}(1 + [h_\alpha + (\alpha, \varepsilon_1 - \varepsilon_r)]_q B^i_r) = g_{r1}(1 + [\eta_{i1} - \eta_{r1}]_q B^i_r) = 0,$$

by Lemma 2.10. Applying ∂_{ir} to the right 2-chain in (4.8) we get

$$f_{(r,\vec{m},k)}g_{k1}(-q^{-\eta_{i1}+\eta_{r1}}+[\eta_{i1}-\eta_{r1}]_q A^i_r)=0.$$

Now consider 3-chains. The action of ∂_{lr} on the (4.7) produces

$$-f_{(i,\vec{\ell},l)}q^{-h_\alpha}f_{(r,\vec{\rho},k)}g_{k,1}A^i_l+f_{(i,\vec{\ell},l)}[h_\alpha]_q f_{(r,\vec{\rho},k)}g_{k,1}A^i_l A^i_r+f_{(i,\vec{\ell},l)}q^{h_\alpha}f_{(r,\vec{\rho},k)}g_{k,1}A^i_r,$$

which turns zero since $-q^{\eta_{r1}-\eta_{l1}}A^i_l+[\eta_{l1}-\eta_{r1}]_q A^i_l A^i_r+q^{\eta_{l1}-\eta_{r1}}A^i_r=0$. The action of ∂_{lr} on (4.6) yields

$$f_{(i,\vec{m},l)}g_{r1}B^i_l+f_{(i,\vec{m},l)}[h_\alpha]_q g_{r1}A^i_l B^i_r+f_{(i,\vec{m},l)}q^{h_\alpha}g_{r1}B^i_r.$$

This is vanishing since $B^i_l+[\eta_{l1}-\eta_{r1}]_q A^i_l B^i_r+q^{\eta_{l1}-\eta_{r1}}B^i_r=B^i_l+\frac{[\eta_{i1}-\eta_{r1}]_q}{[\eta_{i1}-\eta_{l1}]_q}B^i_r=0$. □

Given a route $\vec{m}=(m_1,\ldots,m_k)$ such that $i\prec\vec{m}$ let $A^i_{\vec{m}}$ denote the product $A^i_{m_1}\ldots A^i_{m_k}$. Introduce elements $z_i\in\hat{U}_q(\mathfrak{g}_-)\mathfrak{g}_+$ of weight $\varepsilon_1-\varepsilon_i$ by

$$z_{i-1}=g_{i1}+\sum_{i\prec\vec{m}\prec k\preccurlyeq N}f_{(i,\vec{m},k)}g_{k1}A^i_{\vec{m}}B^i_k,\quad i=2,\ldots,N.$$

The summation includes empty $\vec{m}$.

PROPOSITION 4.11 $e_\alpha z_i=0 \mod \hat{U}_q(\mathfrak{g})e_\alpha$, *for all* $\alpha\in\Pi^+_{\mathfrak{g}'}$ *and* $i=1,\ldots,N-1$.

Proof. By Lemma 4.8, the vectors $g_{2'1}$ and $z_{N-1}=g_{1'1}$ are normalizing the left ideal $\hat{U}_q(\mathfrak{g})\mathfrak{g}'_+$, so is $z_{N-2}=g_{2'1}+f_1g_{1'1}B^{1'}_{2'}$. Once the cases $i=2',1'$ are proved, we further assume $i\prec 2'$. In view of Lemma 4.9, it is sufficient to show that z_{i-1} is killed, modulo $\hat{U}_q(\mathfrak{g})\mathfrak{g}'_+$, by all ∂_{lr} such that $\varepsilon_l-\varepsilon_r\in\Pi^+_{\mathfrak{g}'}$. Observe that z_{i-1} can be arranged into a linear combination of chains, which are killed by ∂_{lr}, as in Lemma 4.10. □

The elements z_i, $i=1,\ldots,N-1$, belong to the normalizer $\mathcal{N}$. They form the set of positive generators of $Z_q(\mathfrak{g},\mathfrak{g}')$ for symplectic $\mathfrak{g}$. In the orthogonal case, the positive part of $Z_q(\mathfrak{g},\mathfrak{g}')$ is generated by z_i, $i=1,\ldots,N-2$.

Chapter 5

Representations of Quantum Conjugacy Classes of $GL(n)$

Let G denote the complex general linear algebraic group $GL(n)$ and let $\mathfrak{g}$ be its Lie algebra $\mathfrak{gl}(n)$. Regard G as a Poisson group relative to the standard classical r-matrix and let $U_\hbar(\mathfrak{g})$ be the corresponding quantum group. Consider a semisimple conjugacy class $O \subset G$, which is an affine subvariety of G. This chapter presents a family of exact representations of $\mathbb{C}_\hbar[O]$ on $U_\hbar(\mathfrak{g})$-modules of highest weight. This family is parameterized by diagonal matrices from O. Equivalently, every diagonal matrix is associated a highest weight module and an equivariant quantization of the conjugacy class of this matrix, through an operator realization on that module. The quantized affine ring depends on O and not on a particular point in it. However, the modules are not isomorphic thus yielding non-equivalent exact representations of the same quantum conjugacy class.

Although the isotropy subgroups of all points in O are isomorphic, not all are strictly compatible with the standard triangular polarization of $\mathfrak{g}$. We call such a stabilizer a Levi subgroup if simple roots of its Lie algebra $\mathfrak{k}$ are simple roots of $\mathfrak{g}$, i.e. $\Pi^+_\mathfrak{k} \subset \Pi^+_\mathfrak{g}$. By this definition, $\mathfrak{k}$ being a Levi subalgebra depends on a polarization of $\mathfrak{g}$ relative to a Cartan subalgebra, which is fixed once and for all. The quantization theory of the corresponding conjugacy class is standard: it can be realized by operators on a parabolic Verma module M_λ relative to $U_q(\mathfrak{k}) \subset U_q(\mathfrak{g})$. General diagonal matrices in O are uniquely parameterized by Weyl group elements σ satisfying $\sigma(R^+_\mathfrak{k}) \subset R^+_\mathfrak{g}$, where R^+ is the set of positive roots. For such σ we construct a highest weight module $M_{\sigma.\lambda}$ and realize the algebra $\mathbb{C}_\hbar[O]$ in $\mathrm{End}(M_{\sigma.\lambda})$. Of course, $M_{\sigma.\lambda}$ is a parabolic Verma module if $\sigma(\Pi^+_\mathfrak{k}) \subset \Pi^+_\mathfrak{g}$.

This chapter gives a description of singular vectors in the Verma modules and their tensor product with the natural representation of $U_q(\mathfrak{g})$. Then finds the eigenvalues of a "quantum coordinate" matrix acting on the $U_q(\mathfrak{g})$-module $\mathbb{C}^n \otimes M_{\sigma.\lambda}$ and checks that they are independent of σ. This enables us to construct the representation of $\mathbb{C}_\hbar[O]$ in $\mathrm{End}(M_{\sigma.\lambda})$.

5.1 Description of Quantum Conjugacy Classes of $GL(n)$

It is an elementary fact from linear algebra that two semisimple matrices are related by a conjugation if and only if they have the same eigenvalues. So a conjugacy class O_x of an element x is determined by the spectrum of a matrix $X \in O_x$. This spectrum can be described by the complex-valued vector $\boldsymbol{x} = (x_1, \ldots, x_k)$ of pairwise distinct eigenvalues and the integer-valued vector of multiplicities $\boldsymbol{n} = (n_1, \ldots, n_k)$. All x_i are invertible, while $\boldsymbol{n}$ is a partition of n. The integer k is assumed to be from the interval $[2, n]$, as the case $k = 1$ is trivial. The correspondence $(\boldsymbol{x}, \boldsymbol{n}) \mapsto O_x$ goes through the choice of the initial point

$$x = \mathrm{diag}(\underbrace{x_1, \ldots, x_1}_{n_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{n_k}).$$

The centralizer of x in G is the group $K = GL(n_1) \times \ldots \times GL(n_k)$, so O_x is isomorphic to G/K as a G-space. Note that the parametrization $(\boldsymbol{x}, \boldsymbol{n}) \mapsto O_x$ is not one-to-one, as a simultaneous permutation of x_i and n_i gives the same conjugacy class albeit a different initial point.

Restriction of O_x to the maximal torus T of diagonal matrices is an orbit of the Weyl group, which is the symmetric group S_n in the case of study. It acts on diagonal matrices by permutation of entries, $(\sigma X)_{ii} = X_{jj}$, $j = \sigma^{-1}(i)$, where $\sigma \in S_n$. The isotropy subgroup of x in S_n is $S_{\boldsymbol{n}} = S_{n_1} \times \ldots \times S_{n_k}$, thus $O_x \cap T$ is in bijection with $S_n/S_{\boldsymbol{n}}$.

The affine ring $\mathbb{C}[O_x]$ is the quotient of the ring $\mathbb{C}[\mathrm{End}(\mathbb{C}^n)]$ by the ideal of relations

$$(X - x_1) \ldots (X - x_k) = 0, \quad \mathrm{Tr}(X^m) - \sum_{i=1}^{k} n_i x_i^m = 0, \quad m = 1, \ldots, k,$$

where $X = \sum_{i,j=1}^{n} e_{ij} \otimes X_{ij}$ is the matrix of coordinate functions X_{ij}. Here $e_{ij} \in \mathrm{End}(\mathbb{C}^n)$ are the standard matrix units, $e_{ij}e_{lm} = \delta_{jl}e_{im}$. The left equality determines the vector $\boldsymbol{x}$ while the values of $\mathrm{Tr}(X^m)$ fix the vector $\boldsymbol{n}$, up to a simultaneous permutation of their components.

The quantum conjugacy class $\mathbb{C}_\hbar[O_x]$ is described as follows. Let $S \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$ be the Hecke braid matrix associated with $U_q(\mathfrak{g})$, whose explicit form can be extracted from [49]. The quantized polynomial ring $\mathbb{C}_\hbar[\mathrm{End}(\mathbb{C}^n)]$ is generated over $\mathbb{C}[\![\hbar]\!]$ by the matrix entries $(X_{ij})_{i,j=1}^n$ subject to the relations

$$S_{12}X_2S_{12}X_2 = X_2S_{12}X_2S_{12} \tag{5.1}$$

written in the standard form of "reflection equation" in $\mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n) \otimes \mathbb{C}_\hbar[\mathrm{End}(\mathbb{C}^n)]$. The algebra $\mathbb{C}_\hbar[O_x]$ is a quotient of $\mathbb{C}_\hbar[\mathrm{End}(\mathbb{C}^n)]$ by the ideal of relations

$$\prod_{i=1}^{k}(X - x_i) = 0, \quad \mathrm{Tr}_q(X^m) = \sum_{i=1}^{k} x_i^m [n_i]_q \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{q^{n_j}x_i - x_j q^{-n_j}}{x_i - x_j}, \quad m = 1, \ldots, k,$$

with the q-trace of X^n defined as $\mathrm{Tr}_q(X^n) = \sum_{i=1}^{n} q^{n+1-2i} X_{ii}^n$.

Until Proposition 5.15, $U_q(\mathfrak{g})$ is understood as a $\mathbb{C}$-algebra.

The quantum matrix space $\mathbb{C}_\hbar[X_{ij}]$ introduced in (5.1) is a $U_\hbar(\mathfrak{g})$-module algebra. The action is defined on the generators by $(\mathrm{id}\otimes x)(X) = \bigl(\pi\bigl(\gamma(x^{(1)})\bigr)\otimes \mathrm{id}\bigr)(X)\bigl(\pi(x^{(2)})\otimes \mathrm{id}\bigr)$, $x\in U_\hbar(\mathfrak{g})$, and extended to $\mathbb{C}_\hbar[X_{ij}]\ni a,b$ by the "quantum Leibniz rule" $x(ab) = (x^{(1)}a)(x^{(2)}b)$. There exists a homomorphism $\mathbb{C}_\hbar[X_{ij}]\to U_\hbar(\mathfrak{g})$ implemented via the assignment $X_{ij}\mapsto \mathcal{Q}_{ij}$, where $\mathcal{Q}$ is expressed through the universal R-matrix of $U_\hbar(\mathfrak{g})$ by $\mathcal{Q} = (\pi\otimes\mathrm{id})(\mathcal{R}_{21}\mathcal{R})$. The image of this homomorphism is a quantization, $\mathbb{C}_\hbar[G]$, of the coordinate ring of the group G. The algebra $\mathbb{C}_\hbar[O_x]$ is a quotient of $\mathbb{C}_\hbar[G]$.

5.2 Singular Vectors

Denote by $\mathfrak{k} = \mathfrak{gl}(n_1)\oplus\ldots\oplus\mathfrak{gl}(n_k)\subset\mathfrak{gl}(n)$ the stabilizer Lie algebra of the point $x\in O_x$. Put $\mathfrak{p}^\pm = \mathfrak{k}+\mathfrak{g}_\pm$ to be the parabolic subalgebras relative to $\mathfrak{k}$. The universal enveloping algebras $U(\mathfrak{k})$ and $U(\mathfrak{p}^\pm)$ are quantized as Hopf subalgebras in $U_q(\mathfrak{g})$. So $U_q(\mathfrak{k})$ is generated by $\{e_\alpha, f_\alpha\}_{\alpha\in\Pi^+_{\mathfrak{k}}}$ over $U_q(\mathfrak{h})$ and $U_q(\mathfrak{p}^\pm)$ is generated by $U_q(\mathfrak{g}_\pm)$ over $U_q(\mathfrak{k})$.

For every $\lambda\in\frac{1}{\hbar}\mathfrak{h}^*\oplus\mathfrak{h}^*$ define a one-dimensional representation of $U_q(\mathfrak{b}_+)$ by $q^{\pm h_\alpha}\mapsto q^{\pm(\lambda,\alpha)}$, $e_\alpha\mapsto 0$. Denoting it by $\mathbb{C}_\lambda$, consider the Verma module $\hat{M}_\lambda = U_q(\mathfrak{g})\otimes_{U_q(\mathfrak{b}_+)}\mathbb{C}_\lambda$. Let $\mathfrak{c}_{\mathfrak{k}}\subset\mathfrak{h}$ be the center of $\mathfrak{k}$ and $\mathfrak{c}^*_{\mathfrak{k}}\subset\mathfrak{h}^*$ be the subset orthogonal to $\Pi^+_{\mathfrak{k}}$. Suppose that $\lambda\in\frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k}}\oplus\mathfrak{c}^*_{\mathfrak{k}}$, so that $(\lambda,\alpha)=0$ for all $\alpha\in\Pi^+_{\mathfrak{k}}$. For such λ, the $U_q(\mathfrak{b}_+)$-module $\mathbb{C}_\lambda$ extends to a $U_q(\mathfrak{p}^+)$-representation, and $\hat{M}_\lambda$ admits a projection onto the parabolic Verma module $M_\lambda = U_q(\mathfrak{g})\otimes_{U_q(\mathfrak{p}^+)}\mathbb{C}_\lambda$. With $x_i = q^{2(\lambda+\rho,\varepsilon_{m_i})-2(\rho,\varepsilon_1)}$, where $m_i = n_1+\ldots+n_{i-1}+1$, $i=1,\ldots,k$, the quantum conjugacy class $\mathbb{C}_\hbar[O_x]$ is realized by operators on M_λ.

Recall that a non-zero vector v in a $U_q(\mathfrak{g})$-module is called singular if it generates the trivial $U_q(\mathfrak{g}_+)$-submodule, i.e. $e_\alpha v = 0$, for all $\alpha\in\Pi^+$. Also recall Lemma 3.2, in particular, singular vectors in $Y\simeq\mathbb{C}\otimes Y$ generate trivial $U_q(\mathfrak{g}_+)$-modules, which recovers their definition.

Further we describe singular vectors of certain weights in $\hat{M}_\lambda$ and $\mathbb{C}^n\otimes\hat{M}_\lambda$. We need a few technical facts about $\hat{M}_\lambda$. We define "dynamical root vectors" $\check{f}_\alpha\in U_q(\mathfrak{b}_-)$ for all $\alpha\in R^+$. For $\alpha\in\Pi^+$ we put $\check{f}_\alpha = f_\alpha$. If $\alpha = \sum_{k=i}^{j}\alpha_k$ where $\alpha_k\in\Pi^+$ and the ordering i to j coincides with the ordering of simple roots in the Dynkin diagram, then $\alpha = \alpha_i+\beta$, where $\beta = \sum_{k=i+1}^{j}\alpha_k$, we proceed recursively by

$$\check{f}_\alpha = f_{\alpha_i}\check{f}_\beta\frac{q^{h_\beta+(\rho,\beta)}-q^{-h_\beta-(\rho,\beta)}}{q-q^{-1}} - \check{f}_\beta f_{\alpha_i}\frac{q^{h_\beta+(\rho,\beta)-1}-q^{-h_\beta-(\rho,\beta)+1}}{q-q^{-1}} \tag{5.2}$$

Note that q^{h_β} is well defined as an element of $U_q(\mathfrak{h})$ for $\beta\in\mathbb{Z}\Pi^+$. The Cartan coefficients in $\check{f}_\beta$ commute with f_{α_i} and can be gathered on the right. For example, let $j>i+1$ and let $\beta = \alpha_{i+1}+\gamma$, where $\gamma = \sum_{k=i+2}^{j}\alpha_k$ and let $\check{f}_\beta$ be defined similar to $\check{f}_\alpha$ in (5.2). Then the Cartan coefficients $\frac{q^{h_\gamma+(\rho,\gamma)}-q^{-h_\gamma-(\rho,\gamma)}}{q-q^{-1}}$ and $\frac{q^{h_\gamma+(\rho,\gamma)-1}-q^{-h_\gamma-(\rho,\gamma)+1}}{q-q^{-1}}$ commute with f_{α_i} in $\check{f}_\alpha$ because $(\gamma,\alpha_i)=0$. By $\check{f}_\alpha(\lambda)$ we understand an element from $U_q(\mathfrak{g}_-)$ obtained through specialization of the coefficients at weight λ. Clearly $\check{f}_\alpha(\lambda)v_\lambda = \check{f}_\alpha v_\lambda$.

Let $\mathfrak{g}_{ln} \subset \mathfrak{g}$ denote the subalgebra $\mathfrak{gl}(n-l+1)$ with the root system $\{\alpha_l, \ldots, \alpha_{n-1}\}$, $l = 1, \ldots, n-1$. The vectors $\check{f}_\alpha$ are generators of the Mickelsson algebras associated with filtration $\mathfrak{g}_{nn} \subset \ldots \subset \mathfrak{g}_{1n} = \mathfrak{g}$, [85]. Their basic property is the equality

$$e_{\alpha_j} \check{f}_\alpha^m v_\lambda = \delta_{ji} [m]_q [(\lambda + \rho, \alpha) - m]_q \check{f}_\beta \check{f}_\alpha^{m-1} v_\lambda. \tag{5.3}$$

for any Verma module $\hat{M}_\lambda$ and any $m \in \mathbb{N}$, see [67]. It is convenient to extend (5.2) by $\check{f}_\alpha = 1$ for $\alpha = 0$ and $\check{f}_\alpha = 0$ for $\alpha \in -R^+$. Then (5.3) is valid for all $\alpha > 0$.

The following fact about $\check{f}_\alpha v_\lambda$ holds true. Its proof can be found in [67].

PROPOSITION 5.1 *Let $\alpha = \varepsilon_i - \varepsilon_j \in R^+$. The vector $\check{f}_\alpha v_\lambda \in \hat{M}_\lambda$ is not vanishing at all λ. It is singular with respect to $U_q(\mathfrak{g}_{i+1\, n})$. Up to a scalar factor, it is a unique $U_q(\mathfrak{g}_{i+1\, n})$-singular vector of weight $\lambda - \alpha$. It is $U_q(\mathfrak{g})$-singular* iff $q^{2(\lambda+\rho,\alpha)} = q^2$. *Up to a scalar factor, it is a unique singular vector of weight $\lambda - \alpha$.*

Further we apply Lemma 3.2 to $W = \mathbb{C}^n$ and $Y = \hat{M}_\lambda$.

LEMMA 5.2 *For all $l = 1, \ldots, n$, there is a unique $U_q(\mathfrak{b}_+)$-submodule in $\hat{M}_\lambda$ of lowest weight $\lambda + \varepsilon_l - \varepsilon_1$. It is generated by $\check{f}_{\varepsilon_1 - \varepsilon_l} v_\lambda$.*

Proof. Proposition 5.1 implies that $\check{f}_{\varepsilon_1 - \varepsilon_l} v_\lambda$ is a unique, up to a factor, $U_q(\mathfrak{g}_{2\, n})$-singular vector of this weight. It automatically satisfies the equation $e_{\alpha_1}^2 \check{f}_{\varepsilon_1 - \varepsilon_l} v_\lambda = 0$ and generates a unique $U_q(\mathfrak{b}_+)$-submodule, which is a quotient of conatural module, by Corollary 2.5. $\square$

Put $\lambda_i = (\lambda, \varepsilon_i) \in \frac{1}{\hbar}\mathbb{C}$, $i = 1, \ldots, n$, and present the singular vectors in $\mathbb{C}^n \otimes \hat{M}_\lambda$ explicitly.

COROLLARY 5.3 *Up to a scalar factor, the singular vector in $\mathbb{C}^n \otimes \hat{M}_\lambda$ of weight $\lambda + \varepsilon_l$, $l = 1, \ldots, n$, is given by $\hat{u}_l = \sum_{i=1}^{l} (-q)^{i-1} \prod_{j=1}^{i-1} [\lambda_j - \lambda_l + l - j - 1]_q w_i \otimes \check{f}_{\varepsilon_i - \varepsilon_l} v_\lambda$,*

Proof. In accordance with Lemma 5.2, put $y_1 = \check{f}_{\varepsilon_1 - \varepsilon_l} v_\lambda$ in $\hat{u}_l = \sum_{i=1}^n w_i \otimes y_i$ and apply formula (5.3), $m = 1$, to $y_{i+1} = -q e_{\alpha_i} y_i$ for $i > 1$ taking into account $\check{f}_{\varepsilon_i - \varepsilon_l} = 0$, $i > l$. $\square$

5.3 The $U_q(\mathfrak{g})$-module $\mathbb{C}^n \otimes \hat{M}_\lambda$

Define $\hat{V}_j^\lambda \subset \mathbb{C}^n \otimes \hat{M}_\lambda$ as a $U_q(\mathfrak{g}_-)$-submodule generated by $\{w_i \otimes v_\lambda\}_{i=1}^j$. It is also a $U_q(\mathfrak{g})$-submodule, and the sequence $\hat{V}_1^\lambda \subset \ldots \subset \hat{V}_n^\lambda = \mathbb{C}^n \otimes \hat{M}_\lambda$ forms a filtration. Its graded components $\hat{V}_j^\lambda / \hat{V}_{j-1}^\lambda$ are generated by the image of $w_j \otimes v_\lambda$, which is the highest weight vector.

It is known that $\hat{V}_j^\lambda / \hat{V}_{j-1}^\lambda$ are isomorphic to the Verma modules $\hat{M}_{\lambda + \varepsilon_j}$ and determine the spectrum $\{\hat{x}_i\}_{i=1}^n$ of the $U_q(\mathfrak{g})$-invariant operator $\mathcal{Q}$, with $\hat{x}_i = q^{2(\lambda+\rho,\varepsilon_i) - 2(\rho,\varepsilon_1)} = q^{2(\lambda,\varepsilon_i) - 2i + 2}$, [68]. The shifted S_n-action on $\mathfrak{h}^*$ by $\sigma \colon \lambda \mapsto \sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho$ permutes $\hat{x}_i$ to $\hat{x}_{\sigma^{-1}(i)}$.

The initial point $x \in O_x$ determines a partition of the integer interval $[1, n]$ into the disjoint union of k subsets: i, j are in the same subset if and only if $x_i = x_j$. This partition determines a partial ordering on $[1, n]$: we write $i \prec j$ iff i, j are from the same subset and $i < j$. We call a permutation $\sigma \in S_n$ admissible if it respects the ordering, i.e. $\sigma(i) < \sigma(j)$ once $i \prec j$.

LEMMA 5.4 *For every point $a \in O_x \cap T$ there is a unique admissible permutation $\sigma \in S_n$ such that $a = \sigma x$.*

Proof. Indeed, if $i \prec j$ and $\sigma(i) > \sigma(j)$, the sign of inequality can be changed by combining σ with the flip $(i, j) \in S_{\boldsymbol{n}}$. This way, every permutation σ such that $a = \sigma x$ can be adjusted so as to satisfy the required condition. Uniqueness is obvious. $\square$

Lemma 5.4 defines an embedding $S_n/S_{\boldsymbol{n}} \subset S_n$ as a subset of admissible permutations. In terms of root systems, σ is admissible if and only if $\sigma(R^+_{\mathfrak{k}}) \subset R^+_{\mathfrak{g}}$ or, equivalently, $\sigma(\pm R^+_{\mathfrak{k}}) \subset \pm R^+_{\mathfrak{g}}$ or, equivalently, $\sigma(\Pi^+_{\mathfrak{k}}) \subset R^+_{\mathfrak{g}}$. Although the stabilizer of the point $\sigma(x)$ is isomorphic to $\mathfrak{k}$, we call it a Levi subalgebra *only if* $\sigma(\Pi^+_{\mathfrak{k}}) \subset \Pi^+_{\mathfrak{g}}$.

Let $\mathfrak{c}^*_{\mathfrak{k},reg}$ denote the subset in $\mathfrak{c}^*_{\mathfrak{k}}$ such that for $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg}$ the complex numbers $q^{2(\lambda,\varepsilon_{m_i})}$, $i = 1, \ldots, k$, are pairwise distinct.

LEMMA 5.5 *Suppose that $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg}$ and $\sigma \in S_n/S_{\boldsymbol{n}}$. Let $\alpha \in \Pi^+_{\mathfrak{k}}$ and $\sigma(\alpha) = \mu + \nu$ for some $\mu, \nu \in R^+_{\mathfrak{g}}$. Then $\bigl(\sigma(\lambda), \mu\bigr) \neq 0 \neq \bigl(\sigma(\lambda), \nu\bigr)$.*

Proof. For any $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg}$, the equality $\bigl(\sigma(\lambda), \mu\bigr) = 0$ implies $\bigl(\lambda, \sigma^{-1}(\mu)\bigr) = 0 = \bigl(\lambda, \sigma^{-1}(\nu)\bigr)$. Therefore $\sigma^{-1}(\mu)$ and $\sigma^{-1}(\nu)$ belong to $R_{\mathfrak{k}}$ and specifically to $R^+_{\mathfrak{k}}$ since σ is admissible. This contradicts the assumption that α is a simple root. $\square$

PROPOSITION 5.6 *Suppose that $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k}} \oplus \mathfrak{c}^*_{\mathfrak{k}}$ and σ is an admissible permutation. For every $\alpha \in \Pi^+_{\mathfrak{k}}$ the vector $v^\sigma_{\lambda,\alpha} = \check{f}_{\sigma(\alpha)} v_{\sigma\cdot\lambda} \in \hat{M}_{\sigma\cdot\lambda}$ is singular.*

Proof. As follows from (5.3) the vector $v^\sigma_{\lambda,\alpha}$ is singular if and only if

$$0 = [(\sigma \cdot \lambda + \rho, \sigma\alpha) - 1]_q = [\bigl(\sigma(\lambda + \rho), \sigma(\alpha)\bigr) - 1]_q = [(\lambda + \rho, \alpha) - 1]_q.$$

This is the case for all $\alpha \in \Pi^+_{\mathfrak{k}}$ since $\alpha = \varepsilon_i - \varepsilon_{i+1}$ for $i \prec i + 1$ and $(\rho, \alpha) = 1$. $\square$

Introduce the subset $I_{\mathfrak{k}} = \{m_i\}_{i=1}^k \subset [1, n]$ and its complement $\bar{I}_{\mathfrak{k}}$ in $[1, n]$. Elements of $I_{\mathfrak{k}}$ enumerate the highest weights of the irreducible $\mathfrak{k}$-submodules in $\mathbb{C}^n$. For a permutation $\sigma \in S_n/S_{\boldsymbol{n}}$ put $I^\sigma_{\mathfrak{k}} = \sigma(I_{\mathfrak{k}})$ and $\bar{I}^\sigma_{\mathfrak{k}} = \sigma(\bar{I}_{\mathfrak{k}})$. Order the set $I^\sigma_{\mathfrak{k}} = \{m^\sigma_1, \ldots, m^\sigma_k\}$ by $m^\sigma_i < m^\sigma_j$ for $i < j$. Note with care that $m^\sigma_i \neq \sigma(m_i)$.

Suppose that $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k}} \oplus \mathfrak{c}^*_{\mathfrak{k}}$ and σ is an admissible permutation. By Proposition 5.6, the vector $v^\sigma_{\lambda,\alpha}$ is singular in $\hat{M}_{\sigma\cdot\lambda}$ for every $\alpha \in \Pi^+_{\mathfrak{k}}$. Denote by $M_{\sigma\cdot\lambda}$ the $U_q(\mathfrak{g})$-module that is quotient of $\hat{M}_{\sigma\cdot\lambda}$ by the submodule $\sum_{\alpha\in\Pi^+_{\mathfrak{k}}} U_q(\mathfrak{g}) v^\sigma_{\lambda,\alpha}$. Let ϖ be the projector $\hat{M}_{\sigma\cdot\lambda} \to M_{\sigma\cdot\lambda}$. Consider the filtration $(\hat{V}^{\sigma\cdot\lambda}_j)_{j=0}^n$ of $\mathbb{C}^n \otimes \hat{M}_{\sigma\cdot\lambda}$ and put $V^{\sigma\cdot\lambda}_i = (\mathrm{id} \otimes \varpi)(\hat{V}^{\sigma\cdot\lambda}_{m^\sigma_i})$, $i = 1, \ldots, k$.

PROPOSITION 5.7 *For all $m \in \bar{I}^\sigma_{\mathfrak{k}}$, $(\mathrm{id}\otimes\varpi)(\hat{V}^{\sigma\cdot\lambda}_m/\hat{V}^{\sigma\cdot\lambda}_{m-1}) = \{0\}$. The $U_q(\mathfrak{g})$-modules $(V^{\sigma\cdot\lambda}_i)_{i=1}^k$ form a filtration of $\mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$. As a filtration of $U_q(\mathfrak{g}_-)$-modules, it is independent of $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k}} \oplus \mathfrak{c}^*_{\mathfrak{k}}$ once $\sigma(\Pi^+_{\mathfrak{k}}) \subset \Pi^+_{\mathfrak{g}}$.*

Proof. For each $m \in \bar{I}_{\mathfrak{k}}^{\sigma}$ there is a positive integer $l < m$ such that $\alpha = \varepsilon_l - \varepsilon_m \in \sigma\Pi_{\mathfrak{k}}^{+}$. The vector $v_{\lambda,\alpha}^{\sigma}$ is singular in $\hat{M}_{\sigma\cdot\lambda}$ and vanishes in $M_{\sigma\cdot\lambda}$. By Proposition 2.9 and Proposition 2.11,

$$w_l \otimes v_{\lambda,\alpha}^{\sigma} \simeq w_l \otimes \psi^{lm} v_{\sigma\cdot\lambda} \simeq w_m \otimes \hat{v}_{\sigma\cdot\lambda} \mod \hat{V}_{m-1}^{\sigma\cdot\lambda}.$$

Projection to $\mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$ annihilates $w_l \otimes v_{\lambda,\alpha}^{\sigma}$, hence $(\mathrm{id} \otimes \varpi)(w_m \otimes \hat{v}_{\sigma\cdot\lambda}) \in (\mathrm{id} \otimes \varpi)(\hat{V}_{m-1}^{\sigma\cdot\lambda})$. This proves the first and second statements.

The last statement follows from Proposition 3.14. $\square$

Proposition 5.7 gives an upper estimate k for the degree of the minimal polynomial of $\mathcal{Q}$ on $\mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$. To make it exact, we must show that all $V_{i+1}^{\sigma\cdot\lambda}/V_i^{\sigma\cdot\lambda} \neq \{0\}$.

5.4 Realization of $\mathbb{C}_\hbar[O_x]$ in $\mathrm{End}(M_{\sigma\cdot\lambda})$

Until Lemma 5.11, λ is an arbitrary weight from $\frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$. Define $\hat{M}_i^{\lambda} \subset \mathbb{C}^n \otimes \hat{M}_\lambda$ to be the $U_q(\mathfrak{g})$-submodule generated by the singular vector $\hat{u}_i$ of weight $\lambda + \varepsilon_i$, $i = 1, \ldots, n$. The operator $\mathcal{Q}$ restricted to $\hat{M}_i^{\lambda}$ is scalar multiplication by $\hat{x}_i = q^{2(\lambda_i - i + 1)}$. For all $\sigma \in S_n$, the action $\sigma \colon \lambda \mapsto \sigma\cdot\lambda$ gives rise to the permutation $\hat{x}_i \mapsto \hat{x}_{\sigma^{-1}(i)}$.

The contribution of principal monomials to the dynamical root vector gives

$$\check{f}_{\varepsilon_i - \varepsilon_l} v_\lambda = \prod_{j=i+1}^{l-1} [\lambda_j - \lambda_l + l - j]_q \psi_{il} v_\lambda + \ldots, \quad \lambda_i = (\lambda, \varepsilon_i),$$

where non-principal terms are omitted. Applying Proposition 2.9 and Proposition 2.11 we find that $\hat{u}_l$ is equal to $(-1)^{l-1}\hat{C}_l w_l \otimes v_\lambda$ modulo $\hat{V}_{l-1}^{\lambda}$ with some scalar $\hat{C}_l$.

LEMMA 5.8 *For all $l = 1, \ldots, n$, $\hat{C}_l = \prod_{j=1}^{l-1} [\lambda_j - \lambda_l + l - j]_q$.*

Proof. For $l = 2$, we have

$$\hat{u}_2 = -q[\lambda_1 - \lambda_2]_q w_2 \otimes v_\lambda + w_1 \otimes f_1 v_\lambda = -(q[\lambda_1 - \lambda_2]_q + q^{-\lambda_1 + \lambda_2}) w_2 \otimes v_\lambda \mod \hat{V}_1^{\lambda},$$

so $\hat{C}_2 = [\lambda_1 - \lambda_2 + 1]_q$. For $l \geqslant 2$ we find

$$\hat{C}_l = \sum_{i=1}^{l} q^{i-1} q^{\lambda_l - \lambda_i + i - l + 1 - \delta_{il}} \prod_{j=1}^{i-1} [\lambda_j - \lambda_l + l - j - 1]_q \prod_{j=i+1}^{l-1} [\lambda_j - \lambda_l + l - j]_q.$$

Suppose that the lemma is proved for $\hat{C}_{l-1}$. Then we can write

$$
\begin{aligned}
(-1)^{l-1}\hat{C}_l &= \Big(\sum_{i=2}^{l} q^{i-2}q^{\lambda_l-\lambda_i+i-l+1-\delta_{il}}\prod_{j=2}^{i-1}[\lambda_j-\lambda_l+l-j-1]_q\prod_{j=i+1}^{l-1}[\lambda_j-\lambda_l+l-j]_q\Big)\\
&\qquad\times\quad q[\lambda_1-\lambda_l+l-2]_q+q^{\lambda_l-\lambda_1-l+2}\prod_{j=2}^{l-1}[\lambda_j-\lambda_l+l-j]_q\\
&= \big(q[\lambda_1-\lambda_l+l-2]_q+q^{\lambda_l-\lambda_1-l+2}\big)\prod_{j=2}^{l-1}[\lambda_j-\lambda_l+l-j]_q.
\end{aligned}
$$

We applied the induction assumption to the expression in brackets in the top line. The factor in the brackets on the second line is equal to $[\lambda_1-\lambda_l+l-1]_q$, so the statement is proved. $\square$

Up to a non-zero factor, $\hat{C}_l=\prod_{j=1}^{l-1}(\hat{x}_j-\hat{x}_l)$, where $\hat{x}_i$ are the eigenvalues of $\mathcal{Q}$ on $\mathbb{C}^n\otimes\hat{M}_\lambda$.

LEMMA 5.9 *For $i<j$, the submodule $\hat{M}_j^\lambda$ is contained in $\hat{M}_i^\lambda$ if and only if $\hat{x}_i=\hat{x}_j$.*

Proof. "Only if" is obvious. Suppose that $\hat{x}_i=\hat{x}_j$. Then the coefficient $\hat{C}_l$ turns zero and $\hat{u}_l\in\hat{V}_{l-1}^\lambda$. First suppose that all $\hat{x}_l$ are pairwise distinct for $l<j$. Then $\hat{V}_{l-1}^\lambda=\hat{M}_1^\lambda\oplus\ldots\oplus\hat{M}_{l-1}^\lambda$ and $\hat{M}_j^\lambda\subset\hat{M}_i^\lambda$. The vector u_j is singular in $\hat{M}_i^\lambda$, therefore $\hat{u}_j\simeq\check{f}_{\varepsilon_i-\varepsilon_j}\hat{u}_i$. This equality is true for generic λ subject to $\hat{x}_i=\hat{x}_j$, hence for all such λ. $\square$

Define $\hat{W}_i^\lambda=\hat{M}_1^\lambda+\ldots+\hat{M}_i^\lambda$, so that $\hat{W}_i^\lambda\subset\hat{W}_j^\lambda$, $i<j$.

PROPOSITION 5.10 *The submodules $\hat{W}_i^\lambda$ and $\hat{V}_i^\lambda$ coincide if and only if the eigenvalues $\{\hat{x}_l\}_{l=1}^i$ are pairwise distinct.*

Proof. By Proposition 2.9 and Proposition 2.11, $\hat{W}_l^\lambda\subset\hat{V}_l^\lambda$. If the eigenvalues are distinct, $\hat{C}_l\neq0$ for all l and then $\hat{V}_l^\lambda\subset\hat{W}_l^\lambda$. Otherwise $\hat{M}_j^\lambda\subset\hat{M}_i^\lambda$ for some $i<j$, by Proposition 5.9. Then the graded modules $\mathrm{gr}\hat{W}_n^\lambda$ and $\mathrm{gr}\hat{V}_n^\lambda$ are different (recall that the latter is independent of λ as a vector space, by Proposition 5.7 applied to $\mathfrak{k}=\mathfrak{h}$). $\square$

Suppose that the weight λ satisfies the condition $[(\lambda+\rho,\alpha)-1]_q=0$ for $\alpha\in R^+$ and let $\hat{M}_{\lambda-\alpha}\subset\hat{M}_\lambda$ be the submodule generated by the singular vector $\check{f}_\alpha v_\lambda$. Define the quotient module $M_{\lambda,\alpha}=\hat{M}_\lambda/\hat{M}_{\lambda-\alpha}$ and let $\varpi_\alpha\colon\hat{M}_\alpha\to M_{\lambda,\alpha}$ be the projector.

LEMMA 5.11 *If $[(\lambda+\rho,\alpha)-1]_q=0$ for some $\alpha=\varepsilon_i-\varepsilon_j\in R^+$, then $(\mathrm{id}\otimes\varpi_\alpha)(\hat{M}_j^\lambda)=\{0\}$.*

Proof. Let us prove that $\hat{u}_j\in\mathbb{C}^n\otimes\hat{M}_{\lambda-\alpha}\subset\mathbb{C}^n\otimes\hat{M}_\lambda$. This is so if $i=1$ since $\hat{u}_j=w_1\otimes\check{f}_\alpha v_\lambda$. If $i>1$, the definition (5.2) implies $\check{f}_{\varepsilon_{i-1}-\varepsilon_j}v_\lambda=[(\lambda+\rho,\alpha)]_q f_{\alpha_{i-1}}\check{f}_\alpha v_\lambda\in\hat{M}_{\lambda-\alpha}$. Proceeding by descending induction on l we can check that $\check{f}_{\varepsilon_l-\varepsilon_j}v_\lambda\in\hat{M}_{\lambda-\alpha}$ for all $l\leqslant i$. Indeed, all monomials constituting $\check{f}_{\varepsilon_l-\varepsilon_j}$ contain either the factor $f_{\alpha_{i-1}}f_\alpha$ or $f_\alpha f_{\alpha_{i-1}}$, by (5.2). The latter enters with the factor $[h_\alpha+(\rho,\alpha)-1]_q$, which can be pushed to the right and killed by v_λ. The vector $\check{f}_{\varepsilon_l-\varepsilon_j}$ is obtained from $\check{f}_{\varepsilon_{i-1}-\varepsilon_j}$ via generalized commutators with f_{α_m}, $m<i-1$, which commute with $\check{f}_\alpha$. This implies $\hat{u}_j\in\mathbb{C}^n\otimes\hat{M}_{\lambda-\alpha}$ and $(\mathrm{id}\otimes\varpi_\alpha)(\hat{u}_j)=0$, as required. $\square$

COROLLARY 5.12 *Let* $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k}}$ *and* $\sigma \in S_n/S_{\boldsymbol{n}}$. *Then for any* $j \in \bar{I}^\sigma_{\mathfrak{k}}$ *the submodule* $\hat{M}^\lambda_j$ *is annihilated by the projection* $\mathrm{id} \otimes \varpi \colon \mathbb{C}^n \otimes \hat{M}_{\sigma\cdot\lambda} \to \mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$.

Proof. Suppose that $j \in \bar{I}^\sigma_{\mathfrak{k}}$. There exists $i \in [1, n]$ such that $\alpha = \varepsilon_i - \varepsilon_j \in \sigma(\Pi^+_{\mathfrak{k}})$ and $\check{f}_\alpha v_{\sigma\cdot\lambda}$ is singular in $\hat{M}_{\sigma\cdot\lambda}$. As follows from Lemma 5.11, the submodule $\hat{M}^\lambda_j$ is annihilated by the projection $\mathbb{C}^n \otimes \hat{M}_{\sigma\cdot\lambda} \to \mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$. □

Now we turn to the submodules $\hat{M}^\lambda_j$ that survive under $\mathrm{id} \otimes \varpi$; they are of $j \in I^\sigma_{\mathfrak{k}}$.

LEMMA 5.13 *Suppose that* $\alpha = \varepsilon_i - \varepsilon_j$ *and* $\beta = \varepsilon_l - \varepsilon_m$ *are such that* $l \leqslant i < j < m$. *Suppose that* $[(\lambda + \rho, \alpha) - 1]_q = 0$. *Then the vector* $\varpi_\alpha(\check{f}_\beta v_\lambda) \in M_{\lambda,\alpha}$ *has a simple divisor* $\hat{x}_j - \hat{x}_m$.

Proof. We have the only condition on λ, which translates to the $\mathcal{Q}$-eigenvalues as $\hat{x}_i = \hat{x}_j q^2$. For almost all such λ the coefficient $\hat{C}_m \simeq \prod_{r=1}^{m-1}(\hat{x}_m - \hat{x}_r)$ is not zero, therefore $(\mathrm{id} \otimes \varpi_\alpha)(\hat{u}_m) \in \mathbb{C}^n \otimes M_{\lambda,\alpha}$ is not zero. By Lemma 5.9 we have $\hat{M}^\lambda_m \subset \hat{M}^\lambda_j$ at $\hat{x}_m = \hat{x}_j$. Then $(\mathrm{id} \otimes \varpi_\alpha)(\hat{M}^\lambda_m) \subset (\mathrm{id} \otimes \varpi_\alpha)(\hat{M}^\lambda_j) = \{0\}$ and $(\mathrm{id} \otimes \varpi_\alpha)(\hat{u}_m)$ is divisible by $\hat{x}_m - \hat{x}_j$. The degree of this divisor is 1, as it is simple in $\hat{C}_m$. By Corollary 5.3, we can write

$$\hat{u}_m = \sum_{l=1}^{m}(-q)^{l-1}\prod_{s=1}^{l-1}(\hat{x}_s - \hat{x}_m q^2)c_{ms}\, w_l \otimes \check{f}_{\varepsilon_l - \varepsilon_m} v_\lambda, \quad \text{where all } c_{ms} \neq 0.$$

The part of the sum corresponding to $l > i$ is divisible by $\hat{x}_j - \hat{x}_m = \hat{x}_i q^{-2} - \hat{x}_m$. Retaining the terms with $l \leqslant i$ we write $\hat{u}_m = \sum_{l=1}^{i}(-q)^{l-1}\prod_{s=1}^{l-1}(\hat{x}_s - \hat{x}_m q^2)c_{ms}\, w_l \otimes \check{f}_{\varepsilon_l - \varepsilon_m} v_\lambda + \ldots$ Hence the vectors $\varpi_\alpha(\check{f}_{\varepsilon_l - \varepsilon_m} v_\lambda)$ are divisible by $\hat{x}_m - \hat{x}_j$ for all $l = 1, \ldots, i$. Clearly the degree of $\hat{x}_m - \hat{x}_j$ in $\varpi_\alpha(\check{f}_{\varepsilon_1 - \varepsilon_m} v_\lambda)$ is 1 since $\varpi_\alpha(\check{f}_{\varepsilon_1 - \varepsilon_m} v_\lambda)$ generates the other coefficients in $(\mathrm{id} \otimes \varpi_\alpha)(\hat{u}_m)$. By (5.2), $\hat{x}_m - \hat{x}_j$ is a divisor of degree 1 in $\varpi_\alpha(\check{f}_{\varepsilon_l - \varepsilon_m} v_\lambda)$ for all $l = 1, \ldots, i$. □

Assuming $i = 1, \ldots, k$ define the submodules M^λ_i, W^λ_i, and V^λ_i in $\mathbb{C}^n \otimes M_\lambda$ to be the images of $\hat{M}^\lambda_{m_i}$, $\hat{W}^\lambda_{m_i}$, and $\hat{V}^\lambda_{m_i}$ under the projection $\mathbb{C}^n \otimes \hat{M}_\lambda \to \mathbb{C}^n \otimes M_\lambda$. Define $x_i = \hat{x}_{m_i}$ to be the eigenvalues of $\mathcal{Q}$ on $\mathbb{C}^n \otimes M_\lambda$. Put $x^\sigma_i = \hat{x}_{m^\sigma_i}$ and $M^{\sigma\cdot\lambda}_i \subset \mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$ to be the image of $\hat{M}^{\sigma\cdot\lambda}_{m^\sigma_i} \subset \mathbb{C}^n \otimes \hat{M}_{\sigma\cdot\lambda}$.

PROPOSITION 5.14 *The* $U_q(\mathfrak{g})$*-module* $\mathbb{C}^n \otimes M_\lambda$ *splits into the direct sum* $M^{\sigma\cdot\lambda}_1 \oplus \ldots \oplus M^{\sigma\cdot\lambda}_k$ *if and only if the eigenvalues* $\{x_i\}_{i=1}^k$ *are pairwise distinct.*

Proof. Put $C^\sigma_i = \prod_{j=1}^{i-1}(x^\sigma_i - x^\sigma_j)$ and $\bar{C}^\sigma_i = \hat{C}_{m^\sigma_i}/C^\sigma_i \simeq \prod_{j \in [i,j) \cap \bar{I}^\sigma_{\mathfrak{k}}}(\hat{x}_{m^\sigma_i} - \hat{x}_j)$. Define $u^\sigma_i = \frac{1}{\bar{C}^\sigma_i}(\mathrm{id} \otimes \varpi)(\hat{u}_{m^\sigma_i}) \in \mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$. The module $M^{\sigma\cdot\lambda}_i$ is generated by the singular vector u^σ_i, which is a regular function of $\{x^\sigma_j\}$, by Lemma 5.13. We have $u^\sigma_i = C^\sigma_i w_{m^\sigma_i} \otimes v_\lambda \mod V^{\sigma\cdot\lambda}_{i-1}$, Therefore $V^{\sigma\cdot\lambda}_i = W^{\sigma\cdot\lambda}_i = M^{\sigma\cdot\lambda}_1 \oplus \ldots \oplus M^{\sigma\cdot\lambda}_i$ if and only if $C^\sigma_j \neq 0$ for all $j \in [1, i]$. This immediately implies the assertion. □

Choosing λ from $\frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg}$ splits the set of eigenvalues $\{\hat{x}_i = q^{2(\lambda_i - i + 1)}\}_{i=1}^n$ of $\mathcal{Q}$ to k strings

$$(x_1, x_1 q^{-2}, \ldots, x_1 q^{-2(n_1-1)}, \ldots, x_k, x_k q^{-2}, \ldots, x_k q^{-2(n_k-1)}),$$

where $\hat{x}_i, \hat{x}_j$ enter a string if and only if $i \preccurlyeq j$. The lowest term in each string is $x_i = q^{2(\lambda_{m_i} - m_i + 1)}$, $i = 1, \ldots, k$. They are exactly the eigenvalues of $\mathcal{Q}$ that survive in the projection $\mathbb{C}^n \otimes \hat{M}_\lambda \to \mathbb{C}^n \otimes M_\lambda$. The matrix $\mathcal{Q}$ has the same eigenvalues on $\mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$ for all σ. They are exactly the eigenvalues that survive in the projection $\mathbb{C}^n \otimes \hat{M}_{\sigma\cdot\lambda} \to \mathbb{C}^n \otimes M_{\sigma\cdot\lambda}$. The permutation $\sigma \in S_n$ induces a permutation $(x_1, \ldots, x_k) \mapsto (x_1^\sigma, \ldots, x_k^\sigma)$.

At this point we turn to deformations and regard $U_q(\mathfrak{g})$ as a $\mathbb{C}[\![\hbar]\!]$-subalgebra of $U_\hbar(\mathfrak{g})$. Correspondingly, $U_q(\mathfrak{g})$-modules and their quotients are extended over $\mathbb{C}[\![\hbar]\!]$ to become $U_\hbar(\mathfrak{g})$-modules. The standard root vectors $f_\alpha \in U_\hbar(\mathfrak{g}_-)$, $\alpha \in R^+$, generate a PBW-basis in $U_\hbar(\mathfrak{g}_-)$, [16]. This basis establishes a $U_\hbar(\mathfrak{h})$-linear isomorphism $U_\hbar(\mathfrak{g}_-) \simeq U(\mathfrak{g}_-) \otimes \mathbb{C}[\![\hbar]\!]$. The root vectors $f_\alpha \in U_\hbar(\mathfrak{g}_-)$, $\alpha \in \Pi^+$ are deformations of their classical counterparts.

LEMMA 5.15 *Suppose that $\alpha \in \Pi^+_{\mathfrak{k}}$ and $\sigma \in S_n/S_{\boldsymbol{n}}$. For any $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg}$, the specialization $\check{f}_{\sigma(\alpha)}(\sigma\cdot\lambda)$ is a deformation of $f_{\sigma(\alpha)}$, upon a proper rescaling.*

Proof. Let $i < j$ be the pair of integers such that $\sigma(\alpha) = \varepsilon_i - \varepsilon_j$. The statement is trivial if $j - i = 1$, since $\check{f}_{\sigma(\alpha)} = f_{\sigma(\alpha)}$ then. If $j - i > 1$, then, by Lemma 5.5, $q^{2(\sigma(\lambda),\varepsilon_l)} \neq q^{2(\sigma(\lambda),\varepsilon_j)}$ for all l such that $i < l < j$. Hence the modified commutators in $(q - q^{-1})^{j-i-1}\check{f}_{\sigma(\alpha)}(\sigma\cdot\lambda)$ are deformations of ordinary commutators, up to a non-zero multipliers. On the other hand, the standard $f_{\sigma(\alpha)}$ is itself a composition of deformed commutators of Chevalley generators $\square$

PROPOSITION 5.16 *Suppose that $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg}$. Then the $U_q(\mathfrak{g})$-module $M_{\sigma\cdot\lambda}$ is $\mathbb{C}[\![\hbar]\!]$-free.*

Proof. The proof is similar to [65], Proposition 6.2, where it is done for a certain quotient of the parabolic Verma module over $U_q\big(\mathfrak{sp}(n)\big)$. It is based on a construction of a PBW basis in $\hat{M}_\lambda$, see therein, Section 6. Here we indicate only the crucial point: for all $\alpha \in \Pi^+_{\mathfrak{k}}$, $\sigma \in S_n/S_{\boldsymbol{n}}$, and $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg}$ the vectors $\check{f}_{\sigma(\alpha)}v_{\sigma\cdot\lambda}$ can be included in a PBW basis in M_λ when the ring of scalars is $\mathbb{C}[\![\hbar]\!]$. This follows from Lemma 5.15. $\square$

THEOREM 5.17 *For all $\sigma \in S_n/S_{\boldsymbol{n}}$ and $\lambda \in \frac{1}{\hbar}\mathfrak{c}^*_{\mathfrak{k},reg}$ such that $x_i = q^{2\lambda_i - 2m_i + 2}$, $i = 1, \ldots, k$, the homomorphism of $\mathbb{C}_\hbar[\mathrm{End}(\mathbb{C}^n)] \to \mathrm{End}[M_{\sigma\cdot\lambda}]$ factors through an exact representation of $\mathbb{C}_\hbar[O_x]$.*

Proof. The minimal polynomial of $\mathcal{Q}$ and $\mathrm{Tr}_q(\mathcal{Q}^m)$ are independent of σ, hence the homomorphism of $\mathbb{C}_\hbar[G] \to \mathrm{End}[M_{\sigma\cdot\lambda}]$ factors through a homomorphism $\mathbb{C}_\hbar[O_x] \to \mathrm{End}[M_{\sigma\cdot\lambda}]$. In the zero fiber, the kernel of this homomorphism is zero. Indeed, it is a proper invariant ideal in $\mathbb{C}[O_x]$ and it is zero since O_x is a G-orbit. The algebra $\mathbb{C}_\hbar[O_x]$ is $\mathbb{C}[\![\hbar]\!]$-free. It is a direct sum of isotypic $U_\hbar(\mathfrak{g})$-components, which are finite over $\mathbb{C}[\![\hbar]\!]$, [68]. Therefore, the $U_\hbar(\mathfrak{g})$-invariant kernel is free. It is nil since its zero fiber is nil; the representation of $\mathbb{C}_\hbar[O_x]$ in $\mathrm{End}[M_{\sigma\cdot\lambda}]$ is exact. $\square$

5.5 Quantization of the Kirillov Bracket on $\mathfrak{g}^*$

A similar theory can be developed for quantization of the Kirillov bracket on $\mathfrak{g}^*$. The quantum group $U_\hbar(\mathfrak{g})$ is replaced with $U(\mathfrak{g}) \otimes \mathbb{C}[\![t]\!]$, the algebra $\mathbb{C}_\hbar[\mathrm{End}(\mathbb{C}^n)]$ with $U(\mathfrak{g}_t)$ where $\mathfrak{g}_t = \mathfrak{g}[\![t]\!]$

is a $\mathbb{C}[\![t]\!]$-Lie algebra with the commutator $[E_{ij}, E_{lm}]_t = t\delta_{jl}E_{im} - t\delta_{im}E_{lj}$, $E_{ij} \in \mathfrak{g} \subset \mathfrak{g}[\![t]\!]$, $i, j = 1, \ldots, n$. The assignment $E_{ij} \mapsto tE_{ij}$ makes $U(\mathfrak{g}_t)$ a subalgebra in $U(\mathfrak{g}) \otimes \mathbb{C}[\![t]\!]$. The dynamical root vectors are obtained from (5.2) via the limit $q \to 1$. The $U(\mathfrak{g})$-module $M_{\sigma\cdot\lambda}$ is defined similarly. For $\lambda \in \frac{1}{t}\mathfrak{c}^*_{l,reg}$, it is generated over $\mathbb{C}(\!(t)\!)$ by the vector space $U(\mathfrak{g}_-)v_{\sigma\cdot\lambda}$. Its regular part $M^+_{\sigma\cdot\lambda} = U(\mathfrak{g}_-)v_{\sigma\cdot\lambda} \otimes \mathbb{C}[\![t]\!]$ is $U(\mathfrak{g}_t)$-invariant. The algebra $U(\mathfrak{g})$ acts on $\mathrm{End}(M^+_{\sigma\cdot\lambda})$ and on the image of $U(\mathfrak{g}_t)$ in $\mathrm{End}(M^+_{\sigma\cdot\lambda})$. The quantum orbit $\mathbb{C}_t[O_x]$ is described in terms of $E = \sum_{i,j=1}^n e_{ij} \otimes E_{ji} \in \mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{g}_t)$ as the quotient of $U_t[\mathfrak{g}]$ by the ideal of the relations (now x_i may be not invertible but still pairwise distinct)

$$\prod_{i=1}^k (E - x_i) = 0, \quad \mathrm{Tr}(E^m) = \sum_{i=1}^k x_i^m n_i \prod_{\substack{j=1\\ j\neq i}}^k \Bigl(1 + \frac{tn_j}{x_i - x_j}\Bigr), \quad m = 1, \ldots, k.$$

These formulas can also be derived from a two parameter quantization at the limit $\hbar \to 0$. The two parameter quantization can be formally obtained by a shift of the matrix K and its eigenvalues, see [26] for details.

THEOREM 5.18 *For all $\sigma \in S_n/S_{\boldsymbol{n}}$ and $\lambda \in \frac{1}{t}\mathfrak{c}^*_{\mathfrak{k},reg}$ such that $x_i = 2t(\lambda_i - m_i + 1)$, $i = 1, \ldots, k$, the homomorphism of $U(\mathfrak{g}_t) \to \mathrm{End}(M^+_{\sigma\cdot\lambda})$ factors through an exact representation of the algebra $\mathbb{C}_t[O_x]$.*

Chapter 6

Representations of Quantum Conjugacy Classes of Orthosymplectic Groups

Throughout this chapter, $\mathfrak{g}$ is a complex simple Lie algebra of type B, C or D (the A-case has been considered in Chapter 5).

We denote by O_x the conjugacy class of x as before. The coordinate ring $\mathbb{C}[O_x]$ is a quotient of $\mathbb{C}[G]$ by a certain G-invariant ideal. To describe this ideal, observe that x determines a 1-dimensional representation χ_x of the subalgebra of invariants in $\mathbb{C}[G]$ (under the conjugation action). Apart from $SO(2n)$, it is generated by traces of the matrix powers of (X_{ij}), where X_{ij} are the coordinate functions on G. In the special case of $SO(2n)$ one has to add one more invariant that is sensible to the flip of the Dynkin diagram, in order to separate two $SO(2n)$-classes within an $O(2n)$-class whose eigenvalues are all distinct from ± 1. Furthermore, the matrix X, when restricted to O_x, satisfies an equation $p(X) = 0$ with a polynomial p in one variable. The entries of the matrix $p(X)$ are polynomial functions in X_{ij}. The defining ideal of O_x is generated by the entries of $p(X)$ over the kernel of χ_x, provided p is the minimal polynomial for x, [65].

A pseudo-Levi subgroup K contains a Cartesian product of two blocks of the same type as G. They correspond to the eigenvalues ± 1 of the matrix x, which are simultaneously present in its spectrum. For the symplectic group, it is $SP(2m) \times SP(2p)$, where $m, p \geqslant 1$. For the odd orthogonal group, it is $SO(2m) \times SO(2p+1)$, where $m \geqslant 2$, $p \geqslant 0$. For the even orthogonal group, it is $SO(2m) \times SO(2p)$, where $m, p \geqslant 2$. The lower bounds on m, p come from the isomorphism $SO(2) \simeq GL(1)$: if the multiplicities of ± 1 are small, then the isotropy subgroup stays within the Levi type. We distinguished such conjugacy classes as borderline Levi because they share some properties of both types, cf. Chapter 3.

The quantized polynomial algebra $\mathbb{C}_\hbar[O_x]$, $\hbar = \log q$, is described as follows. The algebra $\mathbb{C}[G]$ is replaced with $\mathbb{C}_\hbar[G]$, which is an equivariant quantization of the Poisson bracket $r^{\mathrm{ad},\mathrm{ad}} + \omega^{r,l} - \omega^{l,r}$

on G. This bracket makes G a Poisson–Lie homogeneous space over the Poisson group G equipped with the Drinfeld-Sklyanin bracket $r^{l,l} - r^{r,r}$, with respect to the conjugation action. The algebra $\mathbb{C}_\hbar[G]$ admits an equivariant embedding into the corresponding quantum group $U_\hbar(\mathfrak{g}) \supset U_q(\mathfrak{g})$. As a subalgebra in $U_\hbar(\mathfrak{g})$, it is generated by the entries of the matrix $\mathcal{Q} = (\pi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R})$, where $\mathcal{R}$ is the universal R-matrix of $U_\hbar(\mathfrak{g})$ and π stands for the representation homomorphism $U_\hbar(\mathfrak{g}) \to \mathrm{End}(\mathbb{C}^N)$. The factor $\mathcal{R}_{21}$ is obtained by flip of the tensor legs of $\mathcal{R}$. This embedding makes a $U_\hbar(\mathfrak{g})$-module into a $\mathbb{C}_\hbar[G]$-module and the representation homomorphism of $\mathbb{C}_\hbar[G]$ automatically $U_\hbar(\mathfrak{g})$-equivariant.

The subalgebra of invariants in $\mathbb{C}_\hbar[G]$ coincides with its centre, which is generated by q-traces of the matrix powers of $\mathcal{Q}$ (apart from the special case of $SO(2n)$, as mentioned above). The "quantum initial points" can be described as follows. Regard points of the maximal torus as elements of $U_q(\mathfrak{h})$. With the identification $\mathfrak{h} \simeq \mathfrak{h}^*$ define $\lambda \in \frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$ via the equation $q^{2\lambda} = xq^{2\rho_{\mathfrak{k}} - 2\rho_{\mathfrak{g}}}$, where $\rho_{\mathfrak{k}}$ and $\rho_{\mathfrak{g}}$ are the Weyl vectors (half-sums of positive roots) of the algebras $\mathfrak{k}$ and $\mathfrak{g}$, respectively. In the classical limit $q \to 1$, $q^{2\lambda} \to x$.

Let $\mathfrak{c}_{\mathfrak{k}}^*$ be the orthogonal complement to $\mathbb{C}\Pi_{\mathfrak{k}}^+$. Denote $\mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^* = \frac{1}{\hbar}\mathfrak{c}_{\mathfrak{k},\mathrm{reg}}^* + \mathfrak{c}_{\mathfrak{k}}^* + \rho_{\mathfrak{k}} - \rho$ and $\mathfrak{C}_{\mathfrak{k}}^* = \frac{1}{\hbar}\mathfrak{c}_{\mathfrak{k}}^* + \mathfrak{c}_{\mathfrak{k}}^* + \rho_{\mathfrak{k}} - \rho$. By construction, all $\lambda \in \mathfrak{C}_{\mathfrak{k}}^* \subset \frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$ satisfy $q^{2(\lambda+\rho,\alpha)} = q^{(\alpha,\alpha)}$ for all q if $\alpha \in \Pi_{\mathfrak{k}}^+$ while $\lambda \in \mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^* \subset \mathfrak{C}_{\mathfrak{k}}^*$ satisfies this condition only if $\alpha \in \Pi_{\mathfrak{k}}^+$. Then the Verma module $\hat{M}_\lambda$ has singular vectors $v_{\lambda-\alpha}$ for $\alpha \in \Pi_{\mathfrak{k}}^+$.

With $\lambda \in \mathfrak{C}_{\mathfrak{k},\mathrm{reg}}^*$ we associate a module M_λ of highest weight λ, so that the image of $\mathbb{C}_\hbar[G]$ in $\mathrm{End}(M_\lambda)$ is a quantization of $\mathbb{C}_\hbar[O_x]$. It is a parabolic Verma module if and only if $\mathfrak{k}$ is a regular Levi subalgebra. Irregular Levi subgroups also appear as stabilizers of initial points in $\mathfrak{g}$, so the approach is applicable to the $U(\mathfrak{g})$-equivariant quantization of adjoint orbits in $\mathfrak{g}$ as well.

The highest weight of M_λ defines a central character of $\mathbb{C}_\hbar[G]$, whose kernel is expressed through q-traces of the matrix powers $\mathcal{Q}^k$. The matrix $\mathcal{Q}$ yields an invariant operator on $\mathbb{C}^N \otimes M_\lambda$, and its minimal polynomial is determined by module structure of the tensor product. The annihilator of M_λ is then generated by the entries of the minimal polynomial over the kernel of the central character. The structure of $\mathbb{C}^N \otimes M_\lambda$ is the key point of this approach. This approach makes use of some results on the Mickelsson algebras and Shapovalov inverse from Chapter 4 and [9, 70] and is based on the study of the standard filtration of $\mathbb{C}^N \otimes M_\lambda$ in what follows.

6.1 Reduced Shapovalov Inverse

In this section, we recall a construction of Shapovalov inverse reduced to $\mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{b}_-)$. It is given in Chapter 4 for the natural representation of non-exceptional quantum groups (see also [70] for the general case). Note with care that Chapter 4 and [9, 70] deal with a different comultiplication. To adapt those results to the current setting, we have to twist the coproduct by $q^{\sum_{i=1}^{n} h_{\varepsilon_i} \otimes h_{\varepsilon_i}}$ and replace q with q^{-1}.

Given $\lambda \in \frac{1}{\hbar}\mathfrak{h}^* \oplus \mathfrak{h}^*$ consider a 1-dimensional $U_q(\mathfrak{b}_\pm)$-module $\mathbb{C}_\lambda$ with the representation defined by the assignment $q^{\pm h_\alpha} \mapsto q^{\pm(\lambda,\alpha)}$, $e_\alpha \mapsto 0$ for $\alpha \in \Pi^+$. Denote by M_λ the Verma module

$U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_+)} \mathbb{C}_\lambda$ with the canonical generator v_λ, [48]. Let M^*_λ denote the opposite Verma module $U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_-)} \mathbb{C}_{-\lambda}$ of the lowest weight $-\lambda$. There is a unique (up to a multiplier) invariant pairing $M_\lambda \otimes M^*_\lambda \to \mathbb{C}$, which is equivalent to the contravariant Shapovalov form on M_λ, upon an identification $M^*_\lambda \sim M_\lambda$ through an anti-algebra isomorphism $U_q(\mathfrak{g}_-) \simeq U_q(\mathfrak{g}_+)$, [18]. We also call it Shapovalov form.

Recall that $\mathbb{C}^N$ is regarded as a natural $U_q(\mathfrak{g})$-module and $\{w_j\}_{j\in I} \subset \mathbb{C}^N$ is the standard weight basis. Reduced Shapovalov inverse is a matrix $\hat{F} = \sum_{j=1}^{N} \sum_{i=1}^{j} e_{ij} \otimes \hat{f}_{ij} \in \mathrm{End}(\mathbb{C}^N) \otimes \hat{U}_q(\mathfrak{b}_-)$, where the roof means extension over the field of fractions of $U_q(\mathfrak{h})$. This matrix yields a singular vector $\hat{F}(w_j \otimes v_\lambda)$ in $\mathbb{C}^N \otimes M_\lambda$ for all $j \in I$. For generic λ the matrix $\hat{F}$ is a homomorphic image of the Shapovalov inverse lifted to $\hat{U}_q(\mathfrak{g}_+) \otimes \hat{U}_q(\mathfrak{b}_-)$.

The entries $\hat{f}_{ij}$ can be expressed through the Chevalley generators as follows. Recall from Chapter 4, the elements $f_{ij} \in U_q(\mathfrak{g}_-)$. For all $\mathfrak{g}$, $f_{k,k+1} = f_{k'-1,k'} = f_k$ for $k < n$. The rest of the f_{ij} are expressed as shown in Proposition 4.2 with the change in parameter $\bar{q} \to q$.

We use the notation and terminology of Chapter 4. To every route $\vec{m}$ we assign the products

$$f_{\vec{m}} = f_{m_1,m_2} \dots f_{m_{k-1},m_k}, \quad A^j_{\vec{m}} = A^j_{m_1} \dots A^j_{m_k},$$

where $m_k \prec j$. Put $\rho_i = (\rho, \varepsilon_i)$ and define $\tilde{\rho}_i = \rho_i + \frac{||\varepsilon_i||^2}{2}$ for all $i \in I$. Then $\hat{f}_{ij} = 0$ if $i > j$, $\hat{f}_{ii} = 1$ and $\hat{f}_{ij} = \sum_{i \preccurlyeq \vec{m} \prec j} f_{\vec{m},j} A^j_{\vec{m}} q^{\eta_{ij} - \tilde{\rho}_i + \tilde{\rho}_j}$ for $i < j$, where the summation is done over all routes $(\vec{m}, j)$ from i to j. Note that the factor $q^{\eta_{ij} - \tilde{\rho}_i + \tilde{\rho}_j}$ comes from the opposite multiplication adopted in [7, 70] and Chapter 4.

Recall that there exists a PBW basis in $U_q(\mathfrak{g}_-)$ generated by certain elements labelled by R^+, which can be presented as deformed commutators of the Chevalley generators, [16]. The presence of PBW bases allows to identify $U_q(\mathfrak{g}_-)$ with $U(\mathfrak{g}_-)$ as vector spaces (and $U_q(\mathfrak{h})$-modules). This identification makes $U_q(\mathfrak{g}_-)$ a deformation of $U(\mathfrak{g}_-)$. It follows that f_{ij} are deformations of root vectors from $\mathfrak{g}_-$ (unless $i = j$ in the orthogonal case).

LEMMA 6.1 *Suppose that $\alpha \in \Pi^+_{\mathfrak{k}} \subset \mathrm{R}^+_{\mathfrak{g}}$ and $(i, j) \in P(\alpha)$. For all $\lambda \in \mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$, the specialization $\hat{f}_{ij}[\eta_{ij}]_q$ at weight λ is a deformation of a classical root vector, $-f_\alpha \in \mathfrak{g}_-$.*

Proof. Present λ as $\alpha = \frac{1}{\hbar}\lambda^0 + \lambda^1 \in \mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$, $\lambda^i \in \mathfrak{h}^*$. Observe that a) $e^{2\lambda^0_i} = e^{2\lambda^0_j}$ for all $\alpha = \varepsilon_i - \varepsilon_j \in \Pi^+_{\mathfrak{k}}$ once $\lambda^0 \in \mathfrak{c}^*_{\mathfrak{k}}$ and b) there is no k such that $i \prec k \prec j$ and $e^{2\lambda^0_i} = e^{2\lambda^0_k} = e^{2\lambda^0_j}$ if $\lambda^0 \in \mathfrak{c}^*_{\mathfrak{k},\mathrm{reg}}$. Furthermore, write $\hat{f}_{ij}[\eta_{ij}]_q = -f_{ij} - \sum_{i \prec \vec{m} \preccurlyeq j} f_{i,\vec{m},j} A^j_{\vec{m},j} q^{\tilde{\rho}_j - \tilde{\rho}_i}$, where the sum is taken over non-empty routes $\vec{m}$. For all k subject to $i \prec k \prec j$, the denominator in $A^j_k|_\lambda = -\frac{q - q^{-1}}{q^{2\eta_{kj}|_\lambda} - 1}$ tends to $e^{2\lambda^0_k - 2\lambda^0_j} - 1 \neq 0$ as $q \to 1$. Therefore, the sum vanishes modulo $\hbar$, and f_{ij} tends to a classical root vector. □

Define elements $\check{f}_{ij} = \hat{f}_{ij} \prod_{i \preccurlyeq k \prec j} [\eta_{kj}]_q \in U_q(\mathfrak{b}_-)$ for all $i \prec j$. They satisfy the identity

$$e_\alpha \check{f}_{ij} = -\sum_{(l,r) \in P(\alpha)} \delta_{l,i} q^{-(\alpha,\varepsilon_l)} \check{f}_{r,j} [\eta_{ij}]_q \mod U_q(\mathfrak{g})\mathfrak{g}_+, \quad \forall \alpha \in \Pi^+, \tag{6.1}$$

Fix $(i,j)\in P(\alpha)$ for $\alpha\in \mathrm{R}^+$ and suppose that $\lambda=\frac{1}{\hbar}\lambda^0+\lambda^1$ with $\lambda^i\in\mathfrak{h}^*$ satisfies the condition $[\eta_{ij}|_\lambda]_q=0=[\eta_{j'i'}|_\lambda]_q$. Then there is a singular vector $v_{\lambda-\alpha}$ of weight $\lambda-\alpha$ in the Verma module M_λ. We can take $v_{\lambda-\alpha}=\check{f}_{ij}v_\lambda$ provided it is not zero, since $e_\alpha\check{f}_{ij}v_\lambda=0$ for all $\alpha\in\Pi^+$ by (6.1). If $\check{f}_{ij}v_\lambda=0$ at some λ, we still can obtain $v_{\lambda-\alpha}$ from $\check{f}_{ij}v_\lambda$ (which is polynomial in $e^{\pm2(\lambda^0,\alpha)}$, $\alpha\in\Pi^+$, for fixed λ^1 and q) via renormalization, since singular vectors are defined up to a scalar multiplier. In particular, if $\alpha\in\mathfrak{k}$ for some generalized Levi subalgebra $\mathfrak{k}$ and $\lambda\in\mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$, then $v_{\lambda-\alpha}\simeq f_\alpha v_\lambda$ mod $\hbar$, by Lemma 6.1. Note that $\check{f}_{ij}v_\lambda\simeq\check{f}_{j'i'}v_\lambda$ if $i\neq j'$, as follows from the theory of Mickelsson algebras for quantum groups, [54].

6.2 Standard Filtration on $\mathbb{C}^N\otimes M_\lambda$

Fix a generalized Levi subalgebra $\mathfrak{k}\subset\mathfrak{g}$ and a weight $\lambda\in\mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$. Let $M^{\mathfrak{h}}_\lambda$ denote the Verma module of highest weight λ. For each $\alpha\in\Pi^+_{\mathfrak{k}}$, there is a singular vector $v_{\lambda-\alpha}\in M^{\mathfrak{h}}_\lambda$ generating a submodule $M^{\mathfrak{h}}_{\lambda-\alpha}\subset M^{\mathfrak{h}}_\lambda$, cf. Section 6.1. Set $M^{\mathfrak{k}}_\lambda$ to be the quotient of $M^{\mathfrak{h}}_\lambda$ by the submodule $\sum\limits_{\alpha\in\Pi^+_{\mathfrak{k}}} M^{\mathfrak{h}}_{\lambda-\alpha}$.

We introduce yet another partial ordering on integer interval I that is relative to $\mathfrak{k}$: write $i\lessdot j$ if w_i and $w_j\in U_q(\mathfrak{k}_-)\mathfrak{k}_-w_i$. Clearly $i\lessdot j$ if and only if $i\prec j$ and w_i,w_j belong to an irreducible $\mathfrak{k}$-submodule in $\mathbb{C}^N$. Let $I_{\mathfrak{k}}\subset I$ be the set of all minimal elements with respect to this ordering and $\bar{I}_{\mathfrak{k}}$ be its complement in I. Elements of $I_{\mathfrak{k}}$ label the highest weight vectors of the irreducible $\mathfrak{k}$-submodules in $\mathbb{C}^N$. This notation is compatible with what was used in Chapter 5.

We denote by $V^{\mathfrak{k}}_\bullet=(V^{\mathfrak{k}}_i)_{i=1}^N$ a filtration of $\mathbb{C}^N\otimes M^{\mathfrak{k}}_\lambda$ by the modules $V^{\mathfrak{k}}_i$ generated by $w_k\otimes v_\lambda$, $k=1,\ldots,i$. For $\mathfrak{k}=\mathfrak{h}$ it is the standard filtration considered in the previous chapters. Clearly $V^{\mathfrak{k}}_\bullet$ is obtained from $V^{\mathfrak{h}}_\bullet$ through the projection $\mathbb{C}^N\otimes M^{\mathfrak{h}}_\lambda\to\mathbb{C}^N\otimes M^{\mathfrak{k}}_\lambda$. Further we show that $V^{\mathfrak{k}}_j/V^{\mathfrak{k}}_{j-1}$ vanishes once $j\in\bar{I}_{\mathfrak{k}}$ and q is close to 1.

PROPOSITION 6.2 *For each $\lambda\in\mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$ there exists an open set $\Omega\ni1$ in $\mathbb{C}$ such that the submodule $V^{\mathfrak{k}}_j$ is generated by $w_i\otimes v_\lambda$, $i\leqslant j$, $i\in I_{\mathfrak{k}}$, for all $q\in\Omega$.*

Proof. For all j denote by $V'_j\subset V^{\mathfrak{k}}_j$ the submodule generated by all $w_i\otimes v_\lambda$ with $i\leqslant j$ and $i\in I_{\mathfrak{k}}$. We aim to prove that $V'_j=V^{\mathfrak{k}}_j$.

The statement is trivial for $j=1$. Suppose it is true for all $i<j$. If $j\in I_{\mathfrak{k}}$, then $V^{\mathfrak{k}}_j$ is generated by $w_j\otimes v_\lambda$ and by $V^{\mathfrak{k}}_{j-1}=V'_{j-1}$, hence the proof. Suppose that $j\in\bar{I}_{\mathfrak{k}}$. Choose the greatest i such that $i\lessdot j$. Then $(i,j)\in P(\alpha)$ for some $\alpha\in\Pi^+_{\mathfrak{k}}$. By Lemma 6.1 there exists an open set $\Omega\subset\mathbb{C}$ containing 1 such that the principal term in $\check{f}_{ij}v_\lambda\simeq v_{\lambda-\alpha}$ is not zero for all $q\in\Omega$. Then $w_j\otimes v_\lambda\simeq w_i\otimes\psi^{ij}v_\lambda\simeq w_i\otimes\check{f}_{ij}v_\lambda=0$ modulo $V^{\mathfrak{k}}_{j-1}$, by Propositions 2.11 and 2.9. By the induction assumption, we conclude that $w_j\otimes v_\lambda\in V'_{j-1}$ and $V^{\mathfrak{k}}_j=V'_{j-1}=V'_j$. □

COROLLARY 6.3 *The graded module $\mathrm{gr}V^{\mathfrak{k}}_\bullet$ is isomorphic to the direct sum $\oplus_{j\in I_{\mathfrak{k}}}V^{\mathfrak{k}}_j/V^{\mathfrak{k}}_{j-1}$.*

Recall that the tensor $\mathcal{R}_{21}\mathcal{R}$ commutes with $\Delta(x)$ for all $x\in U_q(\mathfrak{g})$, [30].

PROPOSITION 6.4 *The invariant operator $\mathcal{Q} = (\pi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R})$ preserves the standard filtration. It is scalar on each graded component $V_j^{\mathfrak{k}}/V_{j-1}^{\mathfrak{k}}$, $j \in I_{\mathfrak{k}}$, with the eigenvalue*

$$x_j = q^{2(\lambda+\rho,\varepsilon_j)-2(\rho,\varepsilon_1)+||\varepsilon_j||^2-||\varepsilon_1||^2}, \tag{6.2}$$

unless $V_j^{\mathfrak{k}}/V_{j-1}^{\mathfrak{k}} \neq \{0\}$.

Proof. The operator $\mathcal{Q}$ can be presented as $\Delta(z)(z^{-1} \otimes z^{-1})$, for a certain central element z, [28]. Therefore $\mathcal{Q}$ is a scalar multiple on every submodule and factor module of highest weight of $V_N^{\mathfrak{h}}$. Now we do induction on j. The submodule $V_1^{\mathfrak{h}}$ is of highest weight, thence it is $\mathcal{Q}$-invariant. Suppose that $V_{j-1}^{\mathfrak{h}}$ is $\mathcal{Q}$-invariant for $j > 1$. Since $\mathcal{Q}$ is scalar on $V_j^{\mathfrak{h}}/V_{j-1}^{\mathfrak{h}}$, the submodule $V_j^{\mathfrak{h}}$ is $\mathcal{Q}$-invariant.

The eigenvalue of $\mathcal{Q}$ on $V_j^{\mathfrak{h}}/V_{j-1}^{\mathfrak{h}}$ is determined by its highest weight and equal to (6.2), for all $j \in I$, [68]. So the proposition is proved for $\mathfrak{k} = \mathfrak{h}$. The general case is obtained from this by taking projection to $\mathbb{C}^N \otimes V_\lambda^{\mathfrak{k}}$ and applying Corollary 6.3. □

It follows that $\mathcal{Q}$ satisfies the polynomial equation $\prod_{j\in I_{\mathfrak{k}}} (\mathcal{Q} - x_j) = 0$ on $\mathbb{C}^N \otimes V_\lambda^{\mathfrak{k}}$. We will not address the issue if $V_j^{\mathfrak{k}}/V_{j-1}^{\mathfrak{k}}$ survive for all $j \in I_{\mathfrak{k}}$ as we bypass it in what follows.

6.3 Representations of Quantum Conjugacy Classes

In this section we extend the ground field $\mathbb{C}$ to the local ring $\mathbb{C}[\![\hbar]\!]$ of formal power series in $\hbar$. The quantum group $U_\hbar(\mathfrak{g})$ is a completion of the $\mathbb{C}[q, q^{-1}]$-algebra $U_q(\mathfrak{g})$ in the $\hbar$-adic topology via the extension $q = e^\hbar$. Its Cartan subalgebra $U_\hbar(\mathfrak{h})$ can be generated by $h_\alpha \in \mathfrak{h}$ instead of $q^{\pm h_\alpha}$.

Assuming that $\mathfrak{k}$ is fixed, we suppress the corresponding superscripts and write simply $M_\lambda = V_\lambda^{\mathfrak{k}}$ and $V_\bullet = V_\bullet^{\mathfrak{k}}$.

PROPOSITION 6.5 *Suppose that $\lambda \in \mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$. Then M_λ is $\mathbb{C}[\![\hbar]\!]$-free.*

Proof. The proof is similar to [65], Proposition 6.2, where it is done for a regular pseudo-parabolic Verma module over $U_q\big(\mathfrak{sp}(n)\big)$. The crucial observation is that for all $\alpha \in \Pi_{\mathfrak{k}}^+$ and $\lambda \in \mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$ the vectors $\hat{f}_{ij}(\lambda)$ with $(i,j) \in P(\alpha)$ can be included in a PBW basis in $U_\hbar(\mathfrak{g}_-)$ if the ring of scalars is $\mathbb{C}[\![\hbar]\!]$. This follows from Lemma 6.1. □

Proposition 6.5 implies that the algebra $\mathrm{End}(M_\lambda)$ is also $\mathbb{C}[\![\hbar]\!]$-free. We are going to realize the quantized conjugacy class of a point $x = \lim_{\hbar\to 0} q^{2h_\lambda} \in T$ as a subalgebra in $\mathrm{End}(M_\lambda)$.

Consider the image of the algebra $\mathbb{C}_\hbar[G]$ in $\mathrm{End}(M_\lambda)$ under the composition homomorphism

$$\mathbb{C}_\hbar[G] \to U_q(\mathfrak{g}) \to \mathrm{End}(M_\lambda).$$

Here the algebra $U_q(\mathfrak{g})$ is extended over $\mathbb{C}[\![\hbar]\!]$. This representation induces a character, χ_λ, of the center of $\mathbb{C}_\hbar[G]$. It annihilates the ideal in $\mathbb{C}_\hbar[G]$ generated by the kernel χ_λ and by the entries of the minimal polynomial of $\mathcal{Q} \in \mathrm{End}(\mathbb{C}^N \otimes M_\lambda)$. The center of $\mathbb{C}_\hbar[G]$ is generated by

$$\begin{aligned}
\tau_k &= \mathrm{Tr}_q(\mathcal{Q}^k) := \mathrm{Tr}\big((\pi(q^{2h_\rho}) \otimes 1)\mathcal{Q}^k\big) \in U_\hbar(\mathfrak{g}), \quad k = 1, 2, \ldots, \\
\tau^- &= \mathrm{Tr}_q(\mathcal{Q}_+) - \mathrm{Tr}_q(\mathcal{Q}_-), \quad \text{for} \quad \mathfrak{g} = \mathfrak{so}(2n).
\end{aligned}$$

Here $\mathcal{Q}_\pm$ are the images of $\mathcal{R}_{21}\mathcal{R}$ in $\mathrm{End}(W_\pm) \otimes U_q(\mathfrak{g})$, were $W_\pm \subset \wedge^n(\mathbb{C}^n)$ are finite dimensional irreducible modules of highest weights $\sum_{i=1}^{n-1} \varepsilon_i \pm \varepsilon_n$. In the classical limit, this invariant separates two $SO(2n)$-conjugacy classes whose eigenvalues are all distinct from ± 1. They are flipped by any inversion $x_i \leftrightarrow x_i^{-1}$, $i = 1, \ldots, n$, and amount to an $O(2n)$-conjugacy class. If ± 1 is in the spectrum, the $O(2n)$-conjugacy class is also an $SO(2n)$-class. In this case, τ^- is redundant.

THEOREM 6.6 *Let $\mathfrak{k} \subset \mathfrak{g}$ be a generalized Levi subalgebra, $\lambda \in \mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$, and $M_\lambda = V^{\mathfrak{k}}_\lambda$ the corresponding generalized parabolic Verma module. Then*
i) *the annihilator of M_λ in $\mathbb{C}_\hbar[G]$ is generated by*

$$\begin{aligned}
&\Big(\prod_{i\in I_{\mathfrak{k}}}(\mathcal{Q} - x_i)\Big)_{ij}, && i, j = 1, \ldots, N, \\
&\chi_\lambda(\tau_k) - \sum_{i=1}^{N} x_i^k \prod_{\alpha \in \mathrm{R}_+} \frac{q^{(\lambda+\rho+\varepsilon_i,\alpha)} - q^{-(\lambda+\rho+\varepsilon_i,\alpha)}}{q^{(\lambda+\rho,\alpha)} - q^{-(\lambda+\rho,\alpha)}}, && k = 1, \ldots, N, \\
&\chi_\lambda(\tau^-) - \prod_{i=1}^{n}\big(q^{2(\lambda+\rho,\varepsilon_i)} - q^{-2(\lambda+\rho,\varepsilon_i)}\big), && \mathfrak{g} = \mathfrak{so}(2n),
\end{aligned}$$

where x_i is given by (6.2),

ii) *the image of $\mathbb{C}_\hbar[G]$ in $\mathrm{End}(M_\lambda)$ is an equivariant quantization of $\mathbb{C}_\hbar[O_x]$, $x = \lim_{\hbar\to 0} q^{2h_\lambda}$,*

iii) *this quantization is independent of the choice of initial point and is an exact representation of the unique quantum conjugacy class of x.*

Proof. The statements i) and ii) for all types of classes are proved in [8, 64, 65, 68], for certain regular $\mathfrak{k} = \mathfrak{k}_0$ (cf. Chapter 2 for definition of regular $\mathfrak{k}$). For arbitrary $\mathfrak{k}$ there is an element σ of the Weyl group such that $\mathrm{R}^+_{\mathfrak{k}} = \sigma(\mathrm{R}^+_{\mathfrak{k}_0})$. The shifted action $\lambda_0 \mapsto \sigma(\lambda_0 + \rho) - \rho = \lambda$ takes $\mathfrak{C}^*_{\mathfrak{k}_0,\mathrm{reg}}$ to $\mathfrak{C}^*_{\mathfrak{k},\mathrm{reg}}$. It preserves the central characters and takes the set of eigenvalues of $\mathcal{Q}$ on $\mathbb{C}^N \otimes M^{\mathfrak{h}}_{\lambda_0}$ to eigenvalues on $\mathbb{C}^N \otimes M^{\mathfrak{h}}_\lambda$. Moreover, $\sigma\{x_i\}_{i\in I^{\mathfrak{k}_0}} = \{x_i\}_{i\in I^{\mathfrak{k}}}$ as σ relates the orderings $\lessdot$ relative to $\mathfrak{k}_0$ and $\mathfrak{k}$. This implies that the annihilator of $M^{\mathfrak{k}_0}_\lambda$ in $\mathbb{C}_\hbar[G]$ vanishes on $M^{\mathfrak{k}}_\lambda$, that is, there is an equivariant homomorphism $\mathbb{C}_\hbar[G/K_0] \to \mathrm{End}(M^{\mathfrak{k}}_\lambda)$. In order to complete the proof, we need to show that this homomorphism is an embedding.

Since $\mathbb{C}_\hbar[G/K_0]$ is a direct sum of $\mathbb{C}[\![\hbar]\!]$-finite isotypic $U_\hbar(\mathfrak{g})$-components and $\mathrm{End}(M^{\mathfrak{k}}_\lambda)$ is $\mathbb{C}[\![\hbar]\!]$-free, the image of $\mathbb{C}_\hbar[G/K_0]$ is $\mathbb{C}[\![\hbar]\!]$-free. The algebra $\mathbb{C}[G/K_0]$ has no proper invariant ideals, hence the kernel of the map $\mathbb{C}_\hbar[G/K_0] \to \mathrm{End}(M^{\mathfrak{k}}_\lambda)$ is zero. This completes the proof. $\square$

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