WEIGHT CONJECTURES FOR FUSION SYSTEMS

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Abstract. Many of the conjectures of current interest in the representation theory of finite groups in characteristic p are local-to-global statements, in that they predict consequences for the representations of a finite group G given data about the representations of the p-local subgroups of G. The local structure of a block of a group algebra is encoded in the fusion system of the block together with a compatible family of Külshammer-Puig cohomology classes. Motivated by conjectures in block theory, we state and initiate investigation of a number of seemingly local conjectures for arbitrary triples $(S, \mathcal{F}, \alpha)$ consisting of a saturated fusion system $\mathcal{F}$ on a finite p-group S and a compatible family α.

1. Introduction

Throughout this paper we fix a prime number p and an algebraically closed field k of characteristic p. A block B of a finite group algebra kG gives rise to three fundamental invariants encoding the local structure of B: a defect group S, a saturated fusion system $\mathcal{F}$ on S, and a family $\alpha = (\alpha_Q)_{Q \in \mathcal{F}^c}$ of second cohomology classes $\alpha_Q \in H^2(\mathrm{Out}_{\mathcal{F}}(Q), k^\times)$. The α_Q are called the Külshammer-Puig classes of the block B. They are defined for each $\mathcal{F}$-centric subgroup Q of S, and satisfy a certain compatibility condition (recalled in Section 4). The triple $(S, \mathcal{F}, \alpha)$ is determined by B uniquely up to G-conjugacy. If B is the principal block of kG, then S is a Sylow p-subgroup, $\mathcal{F} = \mathcal{F}_S(G)$, and all the classes α_Q are trivial. In what follows, we freely use standard notation on fusion systems as in [AKO11]. For a finite dimensional k-algebra B, we denote by $\ell(B)$ the number of isomorphism classes of simple B-modules and by $z(B)$ the number of isomorphism classes of simple and projective B-modules. If B is a block of a finite group algebra kG, then we denote by $\mathbf{k}(B)$ the number of ordinary irreducible characters of G associated with B.

The prominent counting conjectures in the block theory of finite groups express numerical invariants of B in terms of $(S, \mathcal{F}, \alpha)$. Alperin's weight conjecture (henceforth abbreviated AWC) predicts the equality

$$\ell(B) = \sum_{Q \in \mathcal{F}^c/\mathcal{F}} z(k_\alpha \, \mathrm{Out}_{\mathcal{F}}(Q)) \ ,$$

where $\mathcal{F}^c/\mathcal{F}$ is a set of representatives of the isomorphism classes in $\mathcal{F}$ of $\mathcal{F}$-centric subgroups of S, and where $k_\alpha \, \mathrm{Out}_{\mathcal{F}}(Q)$ is the group algebra of $\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)/\,\mathrm{Inn}(Q)$ twisted by α_Q. The right side in this version of AWC clearly makes sense for arbitrary saturated fusion systems and arbitrary choices of second cohomology, classes, and this is the starting point of the present paper.

Let $(S, \mathcal{F}, \alpha)$ be a triple consisting of a finite p-group S, a saturated fusion system $\mathcal{F}$ on S, and a family $\alpha = (\alpha_Q)_{Q \in \mathcal{F}^c}$ of classes $\alpha_Q \in H^2(\mathrm{Out}_{\mathcal{F}}(Q); k^\times)$, for any $\mathcal{F}$-centric subgroup Q

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of S, such that the family α is $\mathcal{F}$-compatible in the sense of Definition 4.1 below. If α is the family of Külshammer-Puig classes of a fusion system $\mathcal{F}$ of a block B with defect group S, then α is $\mathcal{F}$-compatible by Theorem [Lin18, 8.14.5]; in that case we will say that the triple $(S, \mathcal{F}, \alpha)$ is *block realizable* and that it is *realized by the block* B.

For any $\mathcal{F}$-centric subgroup Q of S and any subgroup H of $\mathrm{Out}_{\mathcal{F}}(Q)$ or of $\mathrm{Aut}_{\mathcal{F}}(Q)$, by $k_\alpha H$ we will mean the twisted group algebra of H over k with respect to the restriction of α_Q to H. Using the notation in [Lin18, Section 8.15], the *number of weights* of $(S, \mathcal{F}, \alpha)$ is the positive integer $\mathbf{w}(\mathcal{F}, \alpha)$ defined by

$$\mathbf{w}(\mathcal{F}, \alpha) := \sum_{Q \in \mathcal{F}^c/\mathcal{F}} z(k_\alpha \mathrm{Out}_{\mathcal{F}}(Q)) \ ,$$

where the notation $Q \in \mathcal{F}^c/\mathcal{F}$ means that Q runs over a set of representatives of the isomorphism classes in $\mathcal{F}$ of $\mathcal{F}$-centric subgroups of S. Note that $z(k_\alpha \mathrm{Out}_{\mathcal{F}}(Q)) = 0$ unless Q is also $\mathcal{F}$-radical (cf. Lemma 4.11 below), and hence we have $\mathbf{w}(\mathcal{F}, \alpha) = \sum_{Q \in \mathcal{F}^{cr}/\mathcal{F}} z(k_\alpha \mathrm{Out}_{\mathcal{F}}(Q))$. By Proposition 4.5 or the above remarks, if $(S, \mathcal{F}, \alpha)$ is realized by a block B of a finite group algebra, then B satisfies AWC if and only if $\mathbf{w}(\mathcal{F}, \alpha) = \ell(B)$.

If x is an element in S such that $\langle x \rangle$ is fully $\mathcal{F}$-centralized, then $C_{\mathcal{F}}(x)$ is a saturated fusion system on $C_S(x)$, there is a canonical functor $C_{\mathcal{F}}(x)^c \to \mathcal{F}^c$, and restriction along this functor sends the $\mathcal{F}$-compatible family α to a $C_{\mathcal{F}}(x)$-compatible family $\alpha(x)$; see Proposition 4.5 below. Denote by $[S/\mathcal{F}]$ a set of $\mathcal{F}$-conjugacy class representatives of elements of S such that $\langle x \rangle$ fully $\mathcal{F}$-centralized. We set

$$\mathbf{k}(\mathcal{F}, \alpha) := \sum_{x \in [S/\mathcal{F}]} \mathbf{w}(C_{\mathcal{F}}(x), \alpha(x)) \ .$$

By Proposition 4.5, if $(S, \mathcal{F}, \alpha)$ is realized by a block B of a finite group algebra such that B and the B-Brauer pairs satisfy AWC, then $\mathbf{k}(\mathcal{F}, \alpha) = \mathbf{k}(B)$.

For any $\mathcal{F}$-centric subgroup Q of S we define the set $\mathcal{N}_Q$ to be the set of non-empty normal chains σ of p-subgroups of $\mathrm{Out}_{\mathcal{F}}(Q)$ starting at the trivial subgroup; that is, chains of the form

$$\sigma = (1 = X_0 < X_1 < \cdots < X_m)$$

with the property that X_i is normal in X_m for $0 \leqslant i \leqslant m$. We set $|\sigma| = m$, and call m the *length* of σ. We define the following two sets:

$$\mathcal{W}_Q = \mathcal{N}_Q \times \mathrm{Irr}(Q) \ ,$$

$$\mathcal{W}_Q^* = \mathcal{N}_Q \times Q^{\mathrm{cl}} \ ,$$

where $\mathrm{Irr}(Q)$ is the set of ordinary irreducible characters of Q and where Q^{cl} is the set of conjugacy classes of Q. There are obvious actions of the group $\mathrm{Out}_{\mathcal{F}}(Q)$ on the sets $\mathcal{N}_Q$, $\mathrm{Irr}(Q)$, and Q^{cl}, hence on the sets $\mathcal{W}_Q$, $\mathcal{W}_Q^*$. We denote by $I(\sigma, \mu)$ and by $I(\sigma, [x])$ the stabilisers in $\mathrm{Out}_{\mathcal{F}}(Q)$ under these actions, where $(\sigma, \mu) \in \mathcal{W}_Q$ and $(\sigma, [x]) \in \mathcal{W}_Q^*$, with $[x]$ the conjugacy class in Q of an element $x \in Q$. For any $\mathcal{F}$-centric subgroup Q of S we set

$$\mathbf{w}_Q(\mathcal{F}, \alpha) = \sum_{(\sigma,\mu) \in \mathcal{W}_Q/\mathrm{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} z(k_\alpha I(\sigma, \mu))$$

$$\mathbf{w}^*_Q(\mathcal{F},\alpha) = \sum_{(\sigma,[x])\in\mathcal{W}^*_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} z(k_\alpha I(\sigma,[x]))\ ,$$

and we set

$$\mathbf{m}(\mathcal{F},\alpha) = \sum_{Q\in\mathcal{F}^c/\mathcal{F}} \mathbf{w}_Q(\mathcal{F},\alpha)\ ,$$
$$\mathbf{m}^*(\mathcal{F},\alpha) = \sum_{Q\in\mathcal{F}^c/\mathcal{F}} \mathbf{w}^*_Q(\mathcal{F},\alpha)\ .$$

There are refinements of the above numbers which take into account defects of ordinary irreducible characters and which appear in conjectures of Dade and Robinson. These will be considered in Section 2.

Theorem 1.1. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. Then*

$$\boldsymbol{m}^*(\mathcal{F},\alpha) = \boldsymbol{k}(\mathcal{F},\alpha)\,.$$

Theorem 1.1 is a cancellation theorem for arbitrary fusion systems inspired by cancellation theorems of Robinson such as in [Rob96, Theorem 1.2].

Theorem 1.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. If AWC holds, then $\boldsymbol{m}(\mathcal{F},\alpha) = \boldsymbol{m}^*(\mathcal{F},\alpha)$.*

Theorem 1.2 shows that AWC implies an equality (for arbitrary fusion systems) of two numerical invariants dual to each other in the sense that one is obtained by summing over conjugacy classes of p-groups and the other by summing over irreducible characters. Given that the numerical invariants $\mathbf{m}$, $\mathbf{m}^*$, $\mathbf{k}$ are entirely defined at the 'local' level of fusion systems and compatible families, it seems surprising that Alperin's Weight Conjecture is needed to obtain the conclusion of Theorem 1.2.

Corollary 1.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. If AWC holds, then $\boldsymbol{m}(\mathcal{F},\alpha) = \boldsymbol{k}(\mathcal{F},\alpha)$.*

If $(S,\mathcal{F},\alpha)$ is block realizable, then Corollary 1.3 follows from work of Robinson and expresses the fact that a coarse version of the Ordinary Weight Conjecture is implied by AWC (see Theorem 2.4 below).

The paper is organised as follows. Section 2 contains a list of conjectures inspired by their block theoretic counter parts. In Section 3 we collect background material, Section 4 contains relevant properties of $\mathcal{F}$-compatible families, Section 5 contains technicalities needed for the proofs of Theorems 1.1 and 1.2 in Section 6 and Section 7, respectively. In an Appendix, we collect some foundational material from work of Robinson.

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2. Conjectures

We formulate conjectures for fusion systems which are motivated by conjectural or known statements in block theory. For each of these conjectures, the link with a block theoretic conjecture is made either via AWC or via the Ordinary Weight Conjecture, the statement of which will be recalled below. Note that by work of Robinson the Ordinary Weight Conjecture implies the AWC.

These conjectures make precise the idea that the gap between various local-global block theoretic conjectures is purely local. Proving or disproving any of these is a win-win scenario. If one can prove one of these conjectures at the fusion system level, then one would get that AWC (or the ordinary weight conjecture) implies the corresponding block theoretic version. If on the other hand one could disprove any of these, one would either have found a counter example to the corresponding block theoretic conjecture, or one would have found a way to distinguish exotic fusion systems from block realizable fusion systems. Either outcome would be interesting.

We keep the notation of the previous section. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family (see Definition 4.1). Recall from Proposition 4.5 that if $(S, \mathcal{F}, \alpha)$ is realized by a block B which satisfies AWC, then $\mathbf{w}(\mathcal{F}, \alpha) = \ell(B)$, and if all Brauer correspondents of B also satisfy AWC, then $\mathbf{k}(\mathcal{F}, \alpha) = \mathbf{k}(B)$.

Conjecture 2.1. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. Then $\boldsymbol{k}(\mathcal{F}, \alpha) \leqslant |S|$.*

By the above remark, if $(S, \mathcal{F}, \alpha)$ is realizable by a block B such that AWC holds for all B-Brauer pairs, then Conjecture 2.1 holds if and only if B satisfies Brauer's $\mathbf{k}(B)$-conjecture, which predicts the inequality $\mathbf{k}(B) \leqslant |S|$. Also, note that by Theorem 1.1, the inequality of Conjecture 2.1 is equivalent to the inequality $\mathbf{m}(\mathcal{F}, \alpha) \leqslant |S|$. In view of Theorem 1.2 and Corollary 1.3 (see also Conjecture 2.3), one could consider versions of the inequality with $\mathbf{k}(\mathcal{F}, \alpha)$ replaced by $\mathbf{m}(\mathcal{F}, \alpha)$.

Conjecture 2.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. Then $\boldsymbol{w}(\mathcal{F}, \alpha) \leqslant p^s$, where s is the sectional rank of S.*

If $(S, \mathcal{F}, \alpha)$ is realizable by a block B such that AWC holds for B, then the above is equivalent to the statement that B satsifies the conjecture by Malle and Robinson [MR17] predicting the inequality $\ell(B) \leqslant p^s$. Conjecture 2.2 has been shown to hold for the exotic Solomon fusion systems by Lynd and Semeraro [LS17].

Next, we refine the integers $\mathbf{w}(\mathcal{F}, \alpha)$, $\mathbf{m}(\mathcal{F}, \alpha)$, $\mathbf{m}^*(\mathcal{F}, \alpha)$ by taking into account defects of characters. For Q a subgroup of S and d a non-negative integer, we set

$$\mathrm{Irr}_K^d(Q) := \{\mu \in \mathrm{Irr}_K^d(Q) \mid v_p(|Q|/\mu(1)) = d\} \ ;$$

this is the set of ordinary irreducible characters of Q of defect d. Note that this set is $\mathrm{Out}_{\mathcal{F}}(Q)$-stable. As in the previous section, we denote by $\mathcal{N}_Q$ the set of nonempty normal chains of p-subgroups of $\mathrm{Out}_{\mathcal{F}}(Q)$ starting with the trivial subgroup of $\mathrm{Out}_{\mathcal{F}}(Q)$. Given such a chain σ and an irreducible character μ of Q, we denote by $I(\sigma)$ and $I(\sigma,\mu)$ the stabilisers of σ and of the pair (σ,μ) in $\mathrm{Out}_{\mathcal{F}}(Q)$.

Given a saturated fusion system $\mathcal{F}$ on a finite p-group S, an $\mathcal{F}$-compatible family α, and a non-negative integer d, following [AKO11, Part IV, Section 5.7], we set

$$\mathbf{w}_Q(\mathcal{F},\alpha,d) := \sum_{\sigma\in\mathcal{N}_Q/\mathrm{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu\in\mathrm{Irr}^d_K(Q)/I(\sigma)} z(k_\alpha I(\sigma,\mu)).$$

and

$$\mathbf{m}(\mathcal{F},\alpha,d) := \sum_{Q\in\mathcal{F}^c/\mathcal{F}} \mathbf{w}_Q(\mathcal{F},\alpha,d).$$

We clearly have

$$\mathbf{m}(\mathcal{F},\alpha) = \sum_{d\geqslant 0} \mathbf{m}(\mathcal{F},\alpha,d)$$

The Ordinary Weight Conjecture (henceforth abbreviated OWC), first stated in [Rob96] and reformulated in [Rob04], states that if B is a block of the group algebra kG of a finite group G with defect group S, fusion system $\mathcal{F}$ and family of Külshammer–Puig classes α, then for each $d \geqslant 0$, $\mathbf{m}(\mathcal{F},\alpha,d)$ equals the number of ordinary irreducible characters of defect d associated to the block B (cf. [AKO11, IV.5.49]). As noted above, $\mathbf{m}(\mathcal{F},\alpha) = \sum_{d\geqslant 0}\mathbf{m}(\mathcal{F},\alpha,d)$. Thus, OWC implies the following "summed up version" (henceforth abbreviated SOWC): if B is a block of the group algebra kG of a finite group G with defect group S, fusion system $\mathcal{F}$ and family of Külshammer–Puig classes α, then $\mathbf{m}(\mathcal{F},\alpha) = \mathbf{k}(B)$, the number of ordinary irreducible characters of G associated with B. On the other hand, AWC predicts that $\mathbf{k}(\mathcal{F},\alpha)$ equals $\mathbf{k}(B)$. This leads to the following conjecture.

Conjecture 2.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. We have*

$$\boldsymbol{k}(\mathcal{F},\alpha) = \boldsymbol{m}(\mathcal{F},\alpha).$$

Now Corollary 1.3 may be restated as follows.

Theorem 2.4. *Suppose that AWC holds for all blocks. Then Conjecture 2.3 holds for all $(S,\mathcal{F},\alpha)$, S a finite p-group, $\mathcal{F}$ a saturated fusion system on S and α an $\mathcal{F}$-compatible family.*

By [Rob96], [Rob04], AWC is equivalent to SOWC in the sense that a minimal counter-example to AWC is a a minimal counter-example to the other. The difficult implication is that AWC implies SOWC. Theorem 2.4 may be viewed as an extension of Robinson's result to arbitrary fusion systems.

Conjecture 2.5. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. For each positive integer d, we have $\boldsymbol{m}(\mathcal{F},\alpha,d) \geq 0$.*

Remark 2.6. With the above notation, suppose that d is the integer such that $|S| = p^d$. The only chain contributing to the expression for $\mathbf{m}(\mathcal{F},\alpha,d)$ is the chain S of length zero and the contribution of this chain is a strictly positive integer. This is because $\mathrm{Out}_{\mathcal{F}}(S)$ is a p'-group.

We consider next Brauer's height zero conjecture.

Proposition 2.7. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. Suppose that S is abelian of order p^d. Then $\boldsymbol{m}(\mathcal{F}, \alpha, d') = 0$ for all $d' \neq d$.*

Proof. Since S is abelian, S is the only $\mathcal{F}$-centric subgroup of S, and all characters of S are linear, hence of defect d. The result follows. □

Conjecture 2.8. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. Suppose that S is nonabelian of order p^d. Then $\boldsymbol{m}(\mathcal{F}, \alpha, d') \neq 0$ for some $d' \neq d$.*

If S is non-abelian and $(S, \mathcal{F}, \alpha)$ is realized by a block B satisfying OWC, then the above is equivalent to the statement that B satisfies Brauer's height zero conjecture. Note that Navarro and Tiep [NT13] have proved that the height zero conjecture is a consequence of the Dade projective conjecture and of the fact that the Brauer height zero conjecture has been checked for finite quasi-simple groups [KM17]. Eaton has proved in [Eat04] that the Dade projective conjecture is equivalent to the OWC in the sense that a minimal counter-example to one is a minimal counter-example to the other. Thus the above conjecture for block realizable triples is a consequence of OWC.

Conjecture 2.9. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. Suppose that S is nonabelian of order p^d. Let $r > 0$ be the smallest positive integer such that S has a character of degree p^r. Then r is the smallest positive integer such that $\boldsymbol{m}(\mathcal{F}, \alpha, d - r) \neq 0$.*

If $(S, \mathcal{F}, \alpha)$ is realized by a block B satisfying OWC, then the above is equivalent to the statement that B satisfies the conjecture by Eaton and Moreto in [EM14].

Conjecture 2.10. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S of order p^d and let α be an $\mathcal{F}$-compatible family. Then*

(1) $\boldsymbol{k}(\mathcal{F}, \alpha)/\boldsymbol{m}(\mathcal{F}, \alpha, d)$ *is at most the number of conjugacy classes of* $[S, S]$.
(2) $\boldsymbol{k}(\mathcal{F}, \alpha)/\boldsymbol{w}(\mathcal{F}, \alpha)$ *is at most the number of conjugacy classes of* S.

If $(S, \mathcal{F}, \alpha)$ is realized by a block B satisfying OWC, then the above is equivalent to the statement that B satisfies the conjecture of Malle and Navarro in [MN06]. Similar to Conjecture 2.1, one could consider versions of the above inequalities with $\mathbf{k}(\mathcal{F}, \alpha)$ replaced by $\mathbf{m}(\mathcal{F}, \alpha)$ or $\mathbf{m}^*(\mathcal{F}, \alpha)$.

If $\mathcal{F}$ is p-solvable (i.e. if $\mathcal{F}$ is constrained with p-solvable model) then for any $\mathcal{F}$-compatible family α, the triple $(S, \mathcal{F}, \alpha)$ is realizable by a block of a p-solvable group (see Proposition 4.8). The OWC has been shown to hold for blocks of p-solvable groups by Robinson, and AWC for p-solvable groups was proved earlier by Okuyama. The $k(B)$ conjecture for finite p-solvable groups was proved in [GMRS04] and the height zero conjecture for p-solvable groups was shown to hold by Gluck and Wolf [GW84]. Thus Conjectures 2.1, 2.3, 2.5, 2.8 all hold for solvable fusion systems. If moreover $\mathcal{F} = N_{\mathcal{F}}(S)$, then for any $\mathcal{F}$-compatible family α, the triple $(S, \mathcal{F}, \alpha)$ is realizable by a block of a finite group G containing S as a normal (and Sylow) subgroup, hence Conjecture 2.10 holds by [MR17, Theorem 2] and Conjecture 2.9 holds by [EM14].

Let $\mathcal{F}$ be a saturated fusion system on a non-trivial finite p-group S and let $\mathcal{C}$ be the full subcategory of $\mathcal{F}$ of nontrivial subgroups of S. Following the terminology in [Lin09b], briefly reviewed at the end of the next section, we denote by $S_{\triangleleft}(\mathcal{C})$ the subcategory of the subdivision category $S(\mathcal{C})$ of chains

$$\sigma = (Q_0 < Q_1 < \cdots < Q_m)$$

where the Q_i are nontrivial subgroups of S which are normal in the maximal term Q_m. Such a chain σ is called fully *$\mathcal{F}$-normalized* if Q_0 is fully $\mathcal{F}$-normalized, and either $m = 0$ or $\sigma_{\geqslant 1} = (Q_1 < \cdots < Q_m)$ is fully $N_{\mathcal{F}}(Q_0)$-normalized. Denote by $S_{\triangleleft}(\mathcal{C})^f$ the set of all fully $\mathcal{F}$-normalized chains. For $\sigma \in S_{\triangleleft}(\mathcal{C})^f$, we denote by $N_{\mathcal{F}}(\sigma)$ the saturated fusion system on $N_S(\sigma)$ as in [Lin09b, 5.2, 5.3]. By Proposition 4.6 below, an $\mathcal{F}$-compatible family α induces a canonical $N_{\mathcal{F}}(\sigma)$-compatible family $\alpha(\sigma)$, for each fully $\mathcal{F}$-normalised chain σ in $S_{\triangleleft}(\mathcal{C})$. The translation to fusion systems of the Knörr-Robinson reformulation of Alperin's Weight Conjecture in [KR89] reads as follows.

Conjecture 2.11. *Let $\mathcal{F}$ be a saturated fusion system on a finite non-trivial p-group S and let α be an $\mathcal{F}$-compatible family. We have*

$$\boldsymbol{k}(\mathcal{F}, \alpha) = \sum_{\sigma} (-1)^{|\sigma|} \boldsymbol{k}(N_{\mathcal{F}}(\sigma), \alpha(\sigma))$$

where in the sum σ runs over a set of representatives of the isomorphism classes of fully normalised normal chains of non-trivial subgroups of S.

Again, one could consider versions of the above replacing $\mathbf{k}$ with $\mathbf{m}$ or $\mathbf{m}^*$. Taking into account defects of characters, we get the following conjecture, which is an analogue of Dade's ordinary conjecture:

Conjecture 2.12. *Let $\mathcal{F}$ be a saturated fusion system on a finite non-trivial p-group S and let α be an $\mathcal{F}$-compatible family of $\mathcal{F}$. For each $d \geq 0$ we have*

$$\boldsymbol{m}(\mathcal{F}, \alpha, d) = \sum_{\sigma} (-1)^{|\sigma|} \boldsymbol{m}(N_{\mathcal{F}}(\sigma), \alpha(\sigma), d) \ ,$$

where in the sum σ runs over a set of representatives of the isomorphism classes of fully normalised normal chains of non-trivial subgroups of S.

Example 2.13. Let p be an odd prime and let $S \cong p_+^{1+2}$ be an extraspecial group of order p^3 and exponent p. Using the classification of saturated fusion systems on S by Ruiz and Viruel [RV04] (which for $p = 7$ includes three exotic fusion systems), one can show that for any nonconstrained fusion system on S every compatible family α is trivial. Using computations in Magma [BCP97] one can show that for any nonconstrained saturated fusion system $\mathcal{F}$ on S the Conjectures 2.1, 2.2, 2.3, 2.5, 2.8, 2.9 and 2.10 all hold for $\mathcal{F}$. The details for the calculations can be found in Section 8 of [KLLS18].

3. Background material

Lemma 3.1 (Thompson's $A \times B$ Lemma)**.** *Let S be a finite p-group and $A \times B \leq \mathrm{Aut}(S)$ be such that A is a p'-group and B is a p-group. If A centralizes $C_S(B)$, then $A = 1$.*

Proof. See [Gor80, Theorem 5.3.4]. □

We will use standard terminology on saturated fusion systems, as can be found in many sources, including [Cra11], [AKO11]), for instance. We assume familiarity with the notions of centralizers and normalizers in fusion systems.

Lemma 3.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Fix $Q \leq S$ and $K \leq \mathrm{Aut}(Q)$. Assume that Q is fully K-normalized. Then PQ is $\mathcal{F}$-centric for each $N_{\mathcal{F}}^K(Q)$-centric subgroup $P \leq N_S^K(Q)$.*

Proof. The argument given in the proof of [BLO03, Lemma 6.2] generalizes: Let $P \leq N_S^K(Q)$ be an $N_{\mathcal{F}}^K(Q)$-centric subgroup and let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$. Then $\varphi(PQ) \leqslant N_S^{\varphi K \varphi^{-1}}(\varphi(Q))\varphi(Q)$. Since Q is fully K-normalized in $\mathcal{F}$, there is a morphism

$$\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S^{\varphi K \varphi^{-1}}(\varphi(Q))\varphi(Q), S)$$

such that $\psi\varphi(Q) = Q$ and $(\psi\varphi)|_Q \in K$ by [AKO11, Proposition I.5.2(c)]. This means that $\psi\varphi$ is a morphism in $\mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(PQ, S)$. Since $C_S(\varphi(PQ)) \leq N_S^{\varphi K \varphi^{-1}}(\varphi(Q))$,

$$\psi(C_S(\varphi(PQ))) \leq C_S(\psi\varphi(PQ)) \leq C_S(\psi\varphi(P)) \cap N_S^K(Q) \leq \psi\varphi(P),$$

where the middle inequality holds because $\psi\varphi K \varphi^{-1}\psi^{-1} = K$, and where the last inequality holds since P is $N_{\mathcal{F}}^K(Q)$-centric. Hence, $C_S(\varphi(PQ)) \leq \varphi(P) \leqslant \varphi(PQ)$. Since φ was chosen arbitrarily, this shows that PQ is $\mathcal{F}$-centric. □

Lemma 3.3. *Let $x \in S$ be such that $\langle x \rangle$ is fully $\mathcal{F}$-centralized, and fix $Q \leq C_S(x)$. Then Q is $C_{\mathcal{F}}(x)$-centric if and only if Q is $\mathcal{F}$-centric. Moreover, $\mathrm{Out}_{C_{\mathcal{F}}(x)}(Q) = C_{\mathrm{Out}_{\mathcal{F}}(Q)}(x)$ under either of these assumptions.*

Proof. Suppose first that Q is $\mathcal{F}$-centric and let P be $C_{\mathcal{F}}(x)$-conjugate to Q. Then $C_{C_S(x)}(P) \leq C_S(P) \leq P$ and hence Q is $C_{\mathcal{F}}(x)$-centric. Conversely if Q is $C_{\mathcal{F}}(x)$-centric, then $x \in Z(C_S(x)) \leq C_{C_S(x)}(Q) \leq Q$ so $Q = Q\langle x \rangle$ is $\mathcal{F}$-centric by Lemma 3.2 applied in the case $K = 1$. Since $\mathrm{Out}_{\mathcal{F}}(Q)$ acts by conjugation on $Z(Q)$, $C_{\mathrm{Out}_{\mathcal{F}}(Q)}(x)$ is well-defined. Now $\mathrm{Aut}_{C_{\mathcal{F}}(x)}(Q) = C_{\mathrm{Aut}_{\mathcal{F}}(Q)}(x)$ is exactly the set of $\mathcal{F}$-automorphisms of Q which fix x, and this group contains $\mathrm{Inn}(Q)$ by assumption. The lemma follows. □

Given an isomorphism φ in $\mathcal{F}$ from Q to Q', the conjugation map $c_\varphi\colon \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Q')$ given by $\eta \to \varphi\eta\varphi^{-1}$ is an isomorphism which maps $\mathrm{Inn}(Q)$ onto $\mathrm{Inn}(Q')$. Thus, conjugation induces a well-defined isomorphism $\mathrm{Out}_{\mathcal{F}}(Q) \to \mathrm{Out}_{\mathcal{F}}(Q')$, which we denote by $\overline{c}_\varphi$. The following direct application of the extension axiom is needed in Section 5.

Lemma 3.4. *Let Q and Q' be $\mathcal{F}$-centric subgroups of S, and let R be a subgroup of S containing Q as a normal subgroup. Let $\varphi : Q \to Q'$ be an isomorphism in $\mathcal{F}$. Assume that $c_\varphi(\mathrm{Aut}_R(Q)) \leqslant \mathrm{Aut}_S(Q')$, or, equivalently, that $\overline{c}_\varphi(\mathrm{Out}_R(Q)) \leqslant \mathrm{Out}_S(Q')$. Let $R' \leqslant S$ be the inverse image of $c_\varphi(\mathrm{Aut}_R(Q))$ under the canonical homomorphism $N_S(Q') \to \mathrm{Aut}_S(Q')$. Then there exists a morphism $R \to S$ in $\mathcal{F}$ extending φ. Moreover, $\tau(R) = R'$ for any such extension τ.*

Proof. Since $\mathrm{Aut}_S(Q')$ is the full inverse image of $\mathrm{Out}_S(Q')$ under the canonical surjection $\mathrm{Aut}_{\mathcal{F}}(Q') \to \mathrm{Out}_{\mathcal{F}}(Q')$, the two conditions on the image of R are indeed equivalent. Hence, $R \leqslant N_\varphi$ in the notation of [AKO11, Definition 2.2]. Since each $\mathcal{F}$-centric subgroup is fully $\mathcal{F}$-centralised, the extension axiom of saturation yields the first assertion.

If τ and τ' are two $\mathcal{F}$-morphisms extending φ, then one may find $z \in Z(Q)$ such that $\tau' = \tau \circ c_z$ by [BLO03, Lemma A.8]. Since $z \in Q \leqslant R$, this shows the second assertion. $\square$

Let $\mathcal{C}$ be a full subcategory of $\mathcal{F}$ which is closed under isomorphisms and taking supergroups. Following the notation in [Lin18, Section 8.13], we denote by $S(\mathcal{C})$ the subdivision category of $\mathcal{C}$. The objects of $\mathcal{C}$ can be regarded as non-empty chains of non-isomorphisms

$$Q_0 \to Q_1 \to \cdots \to Q_m$$

in $\mathcal{F}$ with Q_i belonging to $\mathcal{C}$. Any homomorphism in $S(\mathcal{C})$ is a composition of a chain preserving isomorphism in $\mathcal{F}$ and an inclusion of a chain as a subchain of another chain. There is a canonical functor $S(\mathcal{C}) \to \mathcal{C}$ mapping a chain to its maximal term.

By [Lin18, Proposition 8.13.3], any chain in $S(\mathcal{C})$ is isomorphic, in $S(\mathcal{C})$, to a chain of proper inclusions

$$Q_0 < Q_1 < \cdots < Q_m$$

of subgroups Q_i of S belonging to $\mathcal{C}$. In other words, the category $S(\mathcal{C})$ is equivalent to its full subcategory, denoted $S_<(\mathcal{C})$ consisting of non-empty chains of proper inclusions of subgroups of S in C. A chain σ above is said to have length m, and we write $|\sigma| = m$. When convenient, we occasionally write Q_σ and Q^σ for the smallest and largest subgroups in σ, respectively.

A morphism between chains $Q_0 < \cdots < Q_m$ and $R_0 < \cdots < R_n$ is a pair consisting of an injective map $\beta\colon \{0, \ldots, m\} \to \{0, \ldots, n\}$ together with a collection of isomorphisms $Q_i \to R_{\beta(i)}$ in $\mathcal{F}$ for each $i \in \{0, \ldots, m\}$ which satisfy the obvious compatibility conditions. Thus, the set of isomorphisms between chains σ, τ in $S_<(\mathcal{C})$ can be identified with the set of chain-preserving isomorphisms $\varphi : Q^\sigma \to Q^\tau$ in $\mathcal{F}$. Whenever $\sigma \in S_<(\mathcal{C})$, let $\mathrm{Aut}_\mathcal{F}(\sigma)$ be the subgroup of $\mathrm{Aut}_\mathcal{F}(Q^\sigma)$ consisting of those automorphisms which preserve each member of the chain. In other words, $\mathrm{Aut}_\mathcal{F}(\sigma)$ is the automorphism group of σ in $S_<(\mathcal{C})$.

We denote by $S_\triangleleft(\mathcal{C})$ the full subcategory of $S_<(\mathcal{C})$ of all chains

$$Q_0 < Q_1 < \cdots < Q_m$$

in $S_<(\mathcal{C})$ satisfying the additional property that the Q_i are normal in the maximal term Q_m, for $0 \leqslant i \leqslant m$.

We denote the set of isomorphism classes of chains in $S(\mathcal{C})$ by $[S(\mathcal{C})]$. Since $\mathcal{C}$, and hence $S(\mathcal{C})$, is an EI-category, the set $[S(\mathcal{C})]$ has a canonical partial order given by $[\sigma] \leqslant [\tau]$, whenever $[\sigma]$, $[\tau]$ are the isomorphism classes of chains σ, τ in $S(\mathcal{C})$ such that $\mathrm{Hom}_{S(\mathcal{C})}(\sigma, \tau)$ is non-empty.

If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G having S as a Sylow p-subgroup, then $[S_<(\mathcal{C})]$ is isomorphic to the poset of G-conjugacy classes of chains of subgroups in $\mathcal{C}$. For a more general statement regarding G-conjugacy classes of chains of Brauer pairs of a block, see [Lin05, Proposition 4.6].

4. Compatible families of second cohomology classes

We describe properties of Külshammer-Puig classes of blocks which are needed to ensure that the conjectures stated for saturated fusion systems do indeed specialize to the block theoretic versions from which they are inspired in case the triple $(S, \mathcal{F}, \alpha)$ under consideration is realized by a block. We briefly review the construction of Külshammer-Puig classes (see e. g. [Lin18, Theorem 5.3.12, Corollary 8.12.9, Section 8.14] for more details and proofs).

Let M be a finite-dimensional simple k-algebra; that is, M is isomorphic to a matrix algebra over k. Let G be a finite group acting on M by algebra automorphisms. By the Skolem-Noether Theorem, every automorphism of M is inner, and hence for any $g \in G$ there is an element $s_g \in M^\times$ such that the action of g is equal to the conjugation action of s_g on M. Since $Z(M) \cong k$, the elements s_g are only unique up to scalars in $k^\times$. Thus for g , $h \in G$ we have $s_g s_h = \alpha(g,h)s_{gh}$ for some $\alpha(g,h) \in k^\times$. The map $\alpha : G \times G \to k^\times$ is then a 2-cocycle whose class in $H^2(G,k^\times)$ is independent of the choices of the s_g. We call this class the *class determined by the action of* G *on* M. If G acts trivially on M, then α is the trivial class.

Suppose now that G has a normal subgroup N such that the action of N on the simple algebra M lifts to a G-stable group homomorphism $\tau : N \to M^\times$. Let $[G/N]$ be a set of representatives of G/N in G. For each $g \in [G/N]$ choose some s_g as above, and for each $h \in N$ set $s_{gh} = s_g\tau(h)$. One checks that the 2-cocycle α determined by this choice has the property that its values $\alpha(g,h)$ depend only on the images of g, h in G/N, for all g, $h \in G$, and hence α induces a 2-cocycle β on G/N whose class in $H^2(G/N,k^\times)$ does not depend on the choices of the s_g (but the class of β does depend on the choice of τ lifting the action of N on M). We call this class *the class determined by the action of* G *on* M *together with the group homomorphism* τ. Even if G acts trivially on M this does not necessarily imply that β is trivial (this depends on whether τ is trivial).

This scenario arises if M is a simple algebra quotient of kN by a G-stable maximal ideal in kN. Here the action of G is the conjugation action and the map τ is induced by the canonical algebra surjection $kN \to M$. Any such scenario determines a class β in $H^2(G/N,k^\times)$ whose restriction to G along the canonical surjection $G \to G/N$ is equal to the class α determined by the action of G on M. For technical Clifford theoretic reasons it is usually more convenient to consider the inverse class.

The Külshammer–Puig classes arise in turn as special cases of this construction. Let B be a block of kG with maximal B-Brauer pair (S,e) and associated fusion system $\mathcal{F}$ on S. Let Q be an $\mathcal{F}$-centric subgroup of S. That is, if f is the unique block of $kC_G(Q)$ satisfying $(Q,f) \leqslant (S,e)$, then $Z(Q)$ is a defect group of f (which is clearly central), and hence $kC_G(Q)f$ is a nilpotent block with a unique simple algebra quotient M_Q. The uniqueness ensures that M_Q is $N_G(Q,f)$-stable. By standard facts, M_Q is also the unique simple algebra quotient of $kQC_G(Q)f$. Note that $QC_G(Q)$ is a normal subgroup of $N_G(Q,f)$, and that $N_G(Q,f)/QC_G(Q) \cong \mathrm{Out}_\mathcal{F}(Q)$. Thus the previous scenario with $N_G(Q,f)$ and $QC_G(Q)$ instead of G and N, respectively, yields a canonical class in $H^2(\mathrm{Out}_\mathcal{F}(Q),k^\times)$. The *inverse* of this class is the Külshammer–Puig class α_Q. Using $N_G(Q,f)$ and $C_G(Q)$ would yield the corresponding class, abusively again denoted α_Q, in $H^2(\mathrm{Aut}_\mathcal{F}(Q),k^\times)$.

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. We denote by $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$-centric subgroups of S. For any $Q \in \mathcal{F}^c$, we may (and will) identify without further comment the group $H^2(\mathrm{Out}_\mathcal{F}(Q),k^\times)$ with $H^2(\mathrm{Aut}_\mathcal{F}(Q),k^\times)$ via the isomorphism induced by the canonical surjection $\mathrm{Aut}_\mathcal{F}(Q) \to \mathrm{Out}_\mathcal{F}(Q)$. The assignment $Q \mapsto H^2(\mathrm{Out}_\mathcal{F}(Q),k^\times)$ is not functorial on $\mathcal{F}^c$. In order to interpret certain families of classes in $\prod_{Q\in\mathcal{F}^c} H^2(\mathrm{Out}_\mathcal{F}(Q),k^\times)$ as a limit of a functor, we need to pass to the subdivision category

$S(\mathcal{F}^c)$ of $\mathcal{F}^c$. By [Lin09a, Theorem 1.1], there is a canonical functor $\mathcal{A}^2_{\mathcal{F}}$ from $[S(\mathcal{F}^c)]$ to the category of abelian groups which sends an object τ of $[S(\mathcal{F}^c)]$ to $H^2(\mathrm{Aut}_{S(\mathcal{F}^c)}(\sigma), k^\times)$ for some $\sigma \in S(\mathcal{F}^c)$ such that $\tau = [\sigma]$. The choice of representative σ determines this functor up to unique isomorphism. Let $\alpha = (\alpha_Q)_{Q\in\mathcal{F}^c}$ be a family of classes $\alpha_Q \in H^2(\mathrm{Out}_{\mathcal{F}}(Q), k^\times)$. For each $\tau \in [S(\mathcal{F}^c)]$, define the element $\alpha_\tau \in \mathcal{A}^2_{\mathcal{F}}(\tau)$ to be the restriction of α_{Q_m} to the subgroup $\mathrm{Aut}_{S(\mathcal{F}^c)}(\sigma)$ of $\mathrm{Aut}_{\mathcal{F}}(Q_m)$ where

$$\sigma = (Q_0 \to Q_1 \to \cdots \to Q_m)$$

is the representative of τ in $S[\mathcal{F}^c]$ as above.

Definition 4.1. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. An *$\mathcal{F}$-compatible family* is a family $\alpha = (\alpha_Q)_{Q\in\mathcal{F}^c}$ of classes $\alpha_Q \in H^2(\mathrm{Out}_{\mathcal{F}}(Q), k^\times)$ such that the corresponding family $(\alpha_\tau)_{\tau\in[S(\mathcal{F}^c)]}$ as above belongs to $\lim\limits_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$. In that case, we write $\alpha \in \lim\limits_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$ for short.

The set of $\mathcal{F}$-compatible classes forms a subgroup of the abelian group $\prod_{Q\in\mathcal{F}^c} H^2(\mathrm{Out}_{\mathcal{F}}(Q), k^\times)$.

By [Lin18, Theorem 8.14.5], the family α of Külshammer-Puig classes of a block B of some finite group algebra kG with defect group S and fusion system $\mathcal{F}$ is $\mathcal{F}$-compatible. By [Lin09b, Theorem 4.7] the inclusions of categories $S_\triangleleft(\mathcal{F}^c) \subseteq S_<(\mathcal{C}) \subseteq S(\mathcal{C})$ induce isomorphisms

$$\lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}} \cong \lim_{[S_<(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}} \cong \lim_{[S_\triangleleft(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

Thus to check $\mathcal{F}$-compatibility it suffices to consider normal chains. In fact, it suffices to consider normal chains of length at most 1.

Lemma 4.2 ([Lin18, Theorem 8.14.5] and its proof)**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $\alpha = (\alpha_Q)_{Q\in\mathcal{F}^c}$ with $\alpha_Q \in H^2(\mathrm{Out}_{\mathcal{F}}(Q); k^\times)$ for any $\mathcal{F}$-centric subgroup Q of S. The following are equivalent.*

(1) *The family α is $\mathcal{F}$-compatible.*
(2) *For any proper normal $\mathcal{F}$-centric subgroup Q of an $\mathcal{F}$-centric subgroup R of S, the images of α_Q and α_R in $H^2(\mathrm{Aut}_{S(\mathcal{F}^c)}(Q \triangleleft R), k^\times)$ under the maps induced by the canonical group homomorphisms*

$$\mathrm{Aut}_{S(\mathcal{F}^c)}(Q \triangleleft R) \to \mathrm{Aut}_{\mathcal{F}}(Q)$$
$$\mathrm{Aut}_{S(\mathcal{F}^c)}(Q \triangleleft R) \to \mathrm{Aut}_{\mathcal{F}}(R)$$

are equal.

We need to follow compatible families through passages to centralizers of elements and normalizers of chains of p-subgroups.

Lemma 4.3. [Lin18, Proposition 8.3.7] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let Q be a fully $\mathcal{F}$-centralized subgroup of S. If R is a $C_{\mathcal{F}}(Q)$-centric subgroup of $C_S(Q)$, then QR is an $\mathcal{F}$-centric subgroup of S. The correspondence $R \mapsto QR$ extends to a unique functor*

$$C_{\mathcal{F}}(Q)^c \to \mathcal{F}^c$$

which sends a morphism $\varphi : R \to R'$ in $C_{\mathcal{F}}(Q)^c$ to the unique morphism $\psi : QR \to QR'$ in $\mathcal{F}^c$ which is the identity on Q and coincides with φ on R.

This functor extends obviously to a functor between subdivision categories, and hence this functor sends an $\mathcal{F}$-compatible family α to a $C_{\mathcal{F}}(Q)$-compatible family $\alpha(Q)$. In order to ensure that the conjectures involving this functor specialize to known facts or conjectures, we need to check that if α is realized by a block B of kG, then $\alpha(Q)$ is realized by the corresponding block of $kC_G(Q)$.

Proposition 4.4. *Let G be a finite group, B a block of kG, and (S, e) a maximal B-Brauer pair. Let $\mathcal{F}$ be the fusion system of B on S determined by the choice of e, and let $\alpha = (\alpha_Q)_{Q \in \mathcal{F}^c}$ be the family of Külshammer–Puig classes of B. Denote by e_Q the unique block of $kC_G(Q)$ such that $(Q, e_Q) \leqslant (S, e)$ and by f the unique block of $C_{C_G(Q)}(C_S(Q)) = C_G(QC_S(Q))$ satisfying $(C_S(Q), f) \leqslant (S, e)$. Then $(C_S(Q), f)$ is a maximal $(C_G(Q), e)$-Brauer pair which determines the fusion system $C_{\mathcal{F}}(Q)$ on $C_S(Q)$. The restriction of α to a family $\alpha(Q)$ along the canonical functor $C_{\mathcal{F}}(Q)^c \to \mathcal{F}^c$ is the family of Külshammer–Puig classes of the block $kC_G(Q)e_Q$ with respect to the maximal $(C_G(Q), e_Q)$-Brauer pair $(C_S(Q), f)$.*

Proof. The fact that $(C_S(Q), f)$ is a maximal $(C_G(Q), e)$-Brauer pair which determines the fusion system $C_{\mathcal{F}}(Q)$ on $C_S(Q)$ is well-known, and proved, for instance, in [Lin18, Proposition 8.5.4]. For the statement on Külshammer–Puig classes, we need the contruction of these classes as reviewed at the beginning of this Section. Let R be a $C_{\mathcal{F}}(Q)$-centric subgroup of $C_S(Q)$. By 4.3, QR is $\mathcal{F}$-centric. Note that $C_{C_G(Q)}(R) = C_G(QR)$. Thus if g is the unique block of $kC_G(QR)$ such that $(QR, g) \leqslant (S, e)$, then g is also the unique block of $kC_{C_G(Q)}(R)$ such that $(R, g) \leqslant (C_S(Q), f)$. These blocks have therefore the same unique simple quotient (as they are nilpotent blocks), and clearly $N_{C_G(Q)}(R, f)$ is a subgroup of $N_G(QR, f)$. Since the Külshammer–Puig classes of R and QR for $C_{\mathcal{F}}(Q)$ $\mathcal{F}$ are determined by the respective actions of the groups $N_{C_G(Q)}(R, f)$ and $N_G(QR, f)$ on that simple quotient, it follows that the class of R in $C_{\mathcal{F}}(Q)$ is indeed obtained from restricting the class of QR in $\mathcal{F}$ along the canonical map $\mathrm{Aut}_{C_{\mathcal{F}}(Q)}(R) \to \mathrm{Aut}_{\mathcal{F}}(QR)$. □

We apply this for cyclic Q. Let x be an element in S such that $\langle x \rangle$ is fully $\mathcal{F}$-centralized. For α an $\mathcal{F}$-compatible family, we denote by $\alpha(x)$ the corresponding $C_{\mathcal{F}}(x)$-compatible family, obtained from restricting α along the canonical functor

$$C_{\mathcal{F}}(x)^c \to \mathcal{F}^c$$

from Proposition 4.3 applied with $Q = \langle x \rangle$. By Proposition 4.4, if α is a family of Külshammer–Puig classes of a block, then $\alpha(x)$ is a family of Külshammer–Puig classes of the relevant Brauer correspondent of the block.

Proposition 4.5. *Suppose that $(S, \mathcal{F}, \alpha)$ is realizable by a block B of a finite group algebra kG. Then $\boldsymbol{w}(\mathcal{F}, \alpha)$ is the number of weights associated with B. In particular, AWC holds for B if and only if $\boldsymbol{w}(\mathcal{F}, \alpha) = \ell(B)$. Moreover, if AWC holds for B and all its Brauer pairs, then $\boldsymbol{k}(\mathcal{F}, \alpha) = \boldsymbol{k}(B)$, the number of ordinary irreducible characters associated with B.*

Proof. For the first assertion see for instance [Kes07, Proposition 5.4]. The fusion system $\mathcal{F}$ is determined by a choice of a block e of $kC_G(S)$ such that (S, e) is a maximal B-Brauer pair (see e. g. [Kes07, Definition 3.8]). Let $x \in S$ such that $\langle x \rangle$ is fully $\mathcal{F}$-centralized. Let f be the block of $kC_G(x)$ such that $(\langle x \rangle, f)$ is the unique B-Brauer pair contained in (S, e).

By Proposition 4.4, the triple $(C_S(x), C_{\mathcal{F}}(x), \alpha(x))$ is realized by the block f of $kC_G(x)$, and hence it follows that $\mathbf{w}(C_{\mathcal{F}}(x), \alpha(x)) = \ell(kC_G(x)f)$ thanks to the assumption that B-Brauer pairs satisfy AWC. A theorem of Brauer (cf. [Lin18, Theorem 6.13.12]) now implies the second assertion (see also [AKO11, IV. 5.7]). □

For $\mathcal{F}$ a saturated fusion system on a finite p-group S, denote by $\overline{\mathcal{F}}$ the associated orbit category, obtained from $\mathcal{F}$ by taking as morphisms the orbits $\mathrm{Inn}(R)\backslash \mathrm{Hom}_{\mathcal{F}}(Q, R)$ of morphisms in $\mathcal{F}$ from Q to R modulo inner automorphisms of R, for any two subgroups Q, R of S. In particular, $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$. Recall from [Lin09b, Definition 5.1] that a normal chain

$$\sigma = (Q_0 < Q_1 < \cdots < Q_m) \in S_{\lhd}(\mathcal{F})$$

is called *fully $\mathcal{F}$-normalized* if Q_0 is fully $\mathcal{F}$-normalized and if either $m = 0$ or the chain

$$\sigma_{\geqslant 1} = (Q_1 < \cdots < Q_m)$$

is fully $N_{\mathcal{F}}(Q_0)$-normalized. Every chain in $S_{\lhd}(\mathcal{F})$ is isomorphic to a fully $\mathcal{F}$-normalized chain. Note that since σ is a normal chain, we have $Q_mC_S(Q_m) \leqslant N_S(\sigma)$. We need an analogue of Proposition 4.4 for $N_{\mathcal{F}}(\sigma)$.

Proposition 4.6. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m) \in S_{\lhd}(\mathcal{F})$ be fully $\mathcal{F}$-normalized.*

(1) *For every $P \leq N_S(\sigma)$, if P is $N_{\mathcal{F}}(\sigma)$-centric, then Q_mP is $\mathcal{F}$-centric.*
(2) *Let P, R be $N_{\mathcal{F}}(\sigma)$-centric subgroups of $N_S(\sigma)$, let $\varphi : P \to R$ a morphism in $N_{\mathcal{F}}(\sigma)$, and let $\psi, \psi' : Q_mP \to Q_mR$ be morphisms in $\mathcal{F}$ extending φ and satisfying $\psi(Q_i) = Q_i = \psi'(Q_i)$ for $0 \leqslant i \leqslant m$. Then the classes of ψ and ψ' are conjugate by an element in $Z(P)$. In particular, the correspondence sending φ to any choice of ψ induces a functor*
$$\Psi : N_{\mathcal{F}}(\sigma)^c \to \overline{\mathcal{F}}^c.$$
(3) *For any $N_{\mathcal{F}}(\sigma)$-centric subgroup P of $N_S(\sigma)$, the functor Ψ induces a group homomorphism*
$$\mathrm{Out}_{N_{\mathcal{F}}(\sigma)}(P) \to \mathrm{Out}_{\mathcal{F}}(Q_mP) \ ,$$
and the restriction along these group homomorphisms induces a map from the group of $\mathcal{F}$-compatible families to the group of $N_{\mathcal{F}}(\sigma)$-compatible families.
(4) *If $(S, \mathcal{F}, \alpha)$ is realized by a block B with respect to a maximal B-Brauer pair (S, e), then $(N_S(\sigma), N_{\mathcal{F}}(\sigma), \alpha(\sigma))$ is realized by the block e_m of $kN_G(\sigma, e_m)$ such that $(Q_m, e_m) \leqslant (S, e)$, with respect to the maximal $(N_G(\sigma, e_m), e_m)$-Brauer pair $(N_S(\sigma), f)$, where f is the unique block of $C_{N_G(\sigma)}(N_S(\sigma)) = C_G(N_S(\sigma))$ satisfying $(N_S(\sigma), f) \leqslant (S, e)$.*

Proof. In order to prove the first statement, we argue by induction over the length m of the chain $\sigma = Q_0 < Q_1 < \cdots < Q_m$. Suppose that $m = 0$, so $\sigma = Q_0$, and Q_0 is fully $\mathcal{F}$-normalised. Let P be an $N_{\mathcal{F}}(Q_0)$-centric subgroup of $N_S(Q_0)$. Then Q_0P is $\mathcal{F}$-centric by Lemma 3.2. Suppose now that $m > 0$. Let $P \leqslant N_S(\sigma)$ be $N_{\mathcal{F}}(\sigma)$-centric. Set $\sigma' = Q_0 < Q_1 < \cdots < Q_{m-1}$ and $\mathcal{F}' = N_{\mathcal{F}}(\sigma')$. By [Lin09b, 5.4], σ' is a fully $\mathcal{F}$-normalized chain, and Q_m is fully $\mathcal{F}'$-normalized. By the statement for $m = 0$ applied to $\mathcal{F}'$, it follows that Q_mP is $\mathcal{F}'$-centric. By induction, Q_mP is $\mathcal{F}$-centric.

For the second statement, note that the two extensions ψ, ψ' of φ are both again morphisms in $N_{\mathcal{F}}(\sigma)$, and their restrictions to the $N_{\mathcal{F}}(\sigma)$-centric subgroup P coincide. Thus, by a standard fact (see e. g. [BLO03, Lemma A.8]) they differ by conjugation with an element in $Z(P)$. That means that the image of ψ in the orbit category $\overline{\mathcal{F}}$ is uniquely determined by φ, whence the second statement. The third statement is a formal consequence of the second.

For the proof of the fourth statement, note first that this makes sense: we have $C_G(Q_m) \leqslant N_G(\sigma, e_m) \leqslant N_G(Q_m, e_m)$, and $\mathrm{Aut}_{S(\mathcal{F})}(\sigma) \cong N_G(\sigma, e_m)/C_G(Q_m)$. In particular, by standard block theory, e_m remains a block of $kN_G(\sigma, e_m)$. An interated application of [Lin18, Proposition 8.5.4] shows that $N_{\mathcal{F}}(\sigma)$ is the fusion system of this block with respect to the maximal Brauer pair as stated. The same argument as at the end of the proof of Proposition 4.4 shows that restricting α yields the family of Külshammer–Puig classes of e_m as a block of $kN_G(\sigma, e_m)$. □

Recall that a saturated fusion system $\mathcal{F}$ on a finite p-group S is *constrained* if $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal $\mathcal{F}$-centric subgroup Q of S. In that case, by [BCG+05, Proposition C] (see [AKO11, Theorem 4.9]), $\mathcal{F}$ is the fusion system of a finite group L with S as Sylow p-subgroup, such that Q is normal in L satisfying $C_L(Q) = Z(Q)$; that is, L is p-constrained. In particular, we have canonical isomorphisms $L/Q \cong \mathrm{Out}_{\mathcal{F}}(Q)$ and $L/Z(Q) \cong \mathrm{Aut}_{\mathcal{F}}(Q)$. The group L is called a model for $\mathcal{F}$.

Proposition 4.7 ([Lin09a, Section 6])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal $\mathcal{F}$-centric subgroup Q of S. Let L be a finite group such that S is a Sylow p-subgroup of L, such that Q is normal in L satisfying $C_L(Q) = Z(Q)$, and such that $\mathcal{F} = \mathcal{F}_S(L)$. The restriction from $\mathcal{F}^c$ to $\mathrm{Aut}_{\mathcal{F}}(Q)$ and the canonical map $L \to \mathrm{Aut}_{\mathcal{F}}(Q)$ induce isomorphisms*

$$H^2(\mathcal{F}^c, k^\times) \cong H^2(\mathrm{Aut}_{\mathcal{F}}(Q), k^\times) \cong H^2(L, k^\times).$$

In particular, any $\mathcal{F}$-compatible family α is uniquely determined by the component α_Q.

Proposition 4.8 (cf. [AKO11, Proposition IV.5.34], [Lin04, 5.3])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal $\mathcal{F}$-centric subgroup Q of S. Let α be an $\mathcal{F}$-compatible family. Let L be a finite group such that S is a Sylow p-subgroup of L, such that Q is normal in L satisfying $C_L(Q) = Z(Q)$, and such that $\mathcal{F} = \mathcal{F}_S(L)$. Choose a finite cyclic subgroup Y of $k^\times$ containing all values of a 2-cocycle representing the class α_Q. Then $(S, \mathcal{F}, \alpha)$ is realized by a block of the central extension $\widehat{L}$ of L by Y determined by α_Q, regarded as a class in $H^2(L, Y)$.*

In particular $\alpha = 0$ if and only if b is the principal block of $k\widehat{L}$ (which is isomorphic to the principal block of kL). More generally, the blocks arising in the previous Proposition are twisted group algebras of L; we lay out the connection between p'-central extensions and twisted group algebras in the next result

Proposition 4.9. *Let G be a finite group, and $\alpha \in H^2(G, k^\times)$.*

(1) *There exists a central extension*

$$1 \to Z \to \widetilde{G} \to G \to 1$$

where Z is a cyclic group of order prime to p and a primitive idempotent e of kZ such for any subgroup L of G, we have an isomorphism $k_\alpha L \cong k\widetilde{L}e$, where $\widetilde{L}$ is the inverse image of L in $\widetilde{G}$. In particular $\ell(k_\alpha L) = \ell(k\widetilde{L}e)$ and $z(k_\alpha L) = z(k\widetilde{L}e)$.

(2) *Suppose that there exists a normal p-subgroup Q of G such that $C_G(Q) = Z(Q)$. Identify α with the corresponding element of $H^2(G/Q, k^\times)$. Let L be a subgroup of G containing Q, S a Sylow p-subgroup of L, and $\widehat{S}$ the Sylow p-subgroup of the inverse image of S in $\widetilde{L}$. Denote also by α the $\mathcal{F}_S(L)$-compatible family determined by the restriction of α to L as in Proposition 4.7. Then, $k\widetilde{L}e$ is a block of $k\widetilde{L}$ realizing $(S, \mathcal{F}_S(L), \alpha)$ through the canonical isomorphism $\widehat{S} \cong S$. Moreover, AWC holds for $k\widetilde{L}e$ if and only if*

$$\ell(k_\alpha L) = \sum_R z(k_\alpha N_{L/Q}(R)/R)$$

where R runs over a set of representatives of the L/Q-classes of p-subgroups of L/Q.

Proof. Since k is algebraically closed it is well-known that $H^2(G, k^\times)$ is finite, and hence α can be represented by a 2-cocycle, abusively still denoted by α, with values in a finite subgroup Z of $k^\times$. Then Z is cyclic of order prime to p, since k is a field of characteristic p. Represent α by a central extension

$$1 \to Z \to \widetilde{G} \to G \to 1$$

and denote, for any $x \in G$, by $\tilde{x}$ an inverse image of x in $\widetilde{G}$ satisfying $\tilde{x}\tilde{y} = \alpha(x, y)\widetilde{xy}$ for all x, $y \in G$. We regard the elements of Z as elements in the centre of $\widetilde{G}$ and not as scalars; if we do want to consider the elements of Z as scalars, we denote this via the inclusion map $\iota : Z \to k^\times$. Set $e = \frac{1}{|Z|}\sum_{z\in Z} \iota(z^{-1})z$. This is a primitive idempotent in kZ, and kZe is 1-dimenional. An easy verification shows that the map sending $\tilde{x}e \in k\widetilde{G}e$ to x induces an algebra isomorphism $k\widetilde{G}e \cong k_\alpha G$. This isomorphism restricts to an isomorphism $k\widetilde{L}e \cong k_\alpha L$, for any subgroup L of G. Statement (1) follows.

Let $\widehat{Q}$ be the Sylow p-subgroup of the inverse image $\widetilde{Q}$ of Q in $\widetilde{L}$. Then $\widetilde{Q} = Z \times \widehat{Q}$, and hence $\widehat{Q}$ is normal in $\widetilde{L}$. Thus all block idempotents of $k\widetilde{L}$ lie in $kC_{\widetilde{L}}(\widehat{Q}) = k(Z(\widehat{Q}) \times Z)$. In other words, the block idempotents of $k\widetilde{L}$ are precisely the primitive idempotents of kZ. In particular, $k\widetilde{L}e$ is a block of $k\widetilde{L}$. One easily checks that this block has defect group $\widehat{S}$, which is isomorphic to S, and through this isomorphism, $\mathcal{F} = \mathcal{F}_S(L)$ is the (in this situation unique) fusion system on S of the block e of $\widetilde{L}$. We need to show that α is the family of Külshammer–Puig classes of this block. By Proposition 4.7, it suffices to show this for the class $\alpha_{\widehat{Q}}$. We write again α instead of α_Q, and consider α as a class of $H^2(L, k^\times)$ whenever appropriate. Note that e remains the unique block of $C_{\widetilde{L}}(\widehat{Q}) = Z(Q) \times Z$ such that $(\widehat{Q}, e)$ is a $(\widetilde{L}, e)$-Brauer pair. So the construction of the Külshammer–Puig class at $\widehat{Q}$ is obtained as the special case of the construction described at the beginning of this section with $\widetilde{L}$ and $Z \times \widehat{Q}$ instead of G and N, respectively, and with the 1-dimensional quotient $M \cong k$ of $k(Z \times \widehat{Q})$ given by the map $\iota : Z \to k^\times$ extended trivially to $\widehat{Q}$, still denoted by ι. Since any group action on a 1-dimensional algebra is trivial, we may choose $s_x = 1$ for x running over a set of representatives of $\widetilde{L}/(\widehat{Q} \times Z) \cong L/Q$. Then also $s_x = 1$ for x running over a set of representatives of $\widetilde{L}/Z \cong L$, because ι is extended trivially to $\widehat{Q}$. Thus, for a general element of the form $\tilde{x}z$, with $x \in L$ and $z \in Z$, we may choose $s_{\tilde{x}z} = \iota(z)$; in particular, $s_{\tilde{x}} = 1$ for $x \in L$. We need to show that this determines α^{-1}. Note that α is determined by its restriction to L via the map $L \to L/Q$. Let x, $y \in L$. By construction, we have $s_{\tilde{x}} = s_{\tilde{y}} = s_{\widetilde{xy}} = 1$.

Since $\tilde{x}\tilde{y} = \widetilde{xy}\alpha(x,y)$, it follows that

$$s_{\tilde{x}\tilde{y}} = s_{\widetilde{xy}}\iota(\alpha(x,y)) = \iota(\alpha(x,y))$$

and hence (writing α instead of $\iota \circ \alpha$) we have

$$1 = s_{\tilde{x}} s_{\tilde{y}} = \alpha(x,y)^{-1} s_{\tilde{x}\tilde{y}}$$

This shows that α is the Külshammer–Puig class of this block at $\widehat{Q}$. Note that by the first statement we have $k\widetilde{L}e \cong k_\alpha L$. The last statement on AWC follows from the fact that if $P \leqslant S$ is $\mathcal{F}_S(L)$-centric radical, then P contains Q and if $Q \leqslant P \leqslant S$, then $N_{L/Q}(P/Q)/(P/Q)) \cong N_L(P)/P = \mathrm{Out}_{\mathcal{F}_S(L)}(P)$. □

Lemma 4.10. *Let G be a finite group with normal subgroup N. Fix a cohomology class $\alpha \in H^2(G, k^\times)$ and write also α for the restriction to N. If $z(k_\alpha G) \neq 0$, then $z(k_\alpha N) \neq 0$.*

Proof. Using Proposition 4.9, we fix a p'-central extension $1 \to Z \to \widehat{G} \to G \to 1$ corresponding to α and a central idempotent $e \in kZ$ such that $k_\alpha G \cong k\widehat{G}e$. Then the restriction α is the class corresponding to the induced central extension $\widehat{N}$ of N, and $k_\alpha N \cong k\widehat{N}e$. Assume now that $k_\alpha G$ has a projective simple module. Then $k\widehat{G}e$, and hence $k\widehat{G}$, has a projective simple module, say M. The restriction of M to $\widehat{N}$ is both projective and semisimple. Hence, any simple summand of $\mathrm{Res}^{\widehat{G}}_{\widehat{N}} M$ is projective. Since e still acts as the identity on the restriction of M, we see that $k\widehat{N}e$ has a projective simple module, and hence so does $k_\alpha N$. □

Lemma 4.11. *Let G be a finite group and $\alpha \in H^2(G, k^\times)$. If $O_p(G) \neq 1$, then $z(k_\alpha G) = 0$.*

Proof. As in the proof of Lemma 4.10, let $1 \to Z \to \widehat{G} \to G \to 1$ be a p'-central extension of G determined by α, and let $e \in kZ$ be a central idempotent in $k\widehat{G}$ such that $k_\alpha G \cong k\widehat{G}e$. Let $P = O_p(G)$ and $\widehat{P}$ be the preimage under the quotient map. Since Z is a p'-group, the restriction of α to P is trivial, and so $\widehat{P} = Z \times P_0$ with P_0 mapping isomorphically to P. Then $O_p(\widehat{P}) = P_0 \neq 1$ is a normal p-subgroup of $\widehat{G}$. Thus, as kP has no projective simple module, neither does $k\widehat{G}$ by Lemma 4.10. Hence neither does $k\widehat{G}e \cong k_\alpha G$. □

Fix a finite group G and an abelian group A. Let $\mathcal{P}$ be the set of all chains of proper inclusions

$$Q_0 = 1 < Q_1 \cdots < Q_m$$

of p-subgroups of G. This is a G-set with respect to the conjugation action of G on chains, and we denote by $N_G(\sigma)$ the stabilizer of σ in G. Let $\mathcal{N}$ be the subset of all such chains satisfying in addition $Q_i \trianglelefteq Q_m$ for each $0 \leq i \leq m$. Let $\mathcal{E}$ be the set of chains in $\mathcal{N}$ consisting of elementary abelian subgroups. Both $\mathcal{N}$ and $\mathcal{E}$ are G-subsets of $\mathcal{P}$. For the purpose of calculating alternating sums indexed by chains, we can pass between $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{E}$:

Lemma 4.12 ([KR89, Proposition 3.3])**.** *Let G, A, $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{E}$ be as above. Let $\boldsymbol{f}$ be a function from the set of subgroups of G to A such that $\boldsymbol{f}$ is constant on conjugacy classes of subgroups of G. Then*

$$\sum_{\sigma \in \mathcal{P}/G} (-1)^{|\sigma|} \boldsymbol{f}(N_G(\sigma)) = \sum_{\sigma \in \mathcal{N}/G} (-1)^{|\sigma|} \boldsymbol{f}(N_G(\sigma)) = \sum_{\sigma \in \mathcal{E}/G} (-1)^{|\sigma|} \boldsymbol{f}(N_G(\sigma)).$$

We shall need the following well-known Lemma in Section 5.

Lemma 4.13 ([Thé92, Lemma 2.1], [KR89, Proposition 3.3])**.** *Let G, A, and $\mathcal{N}$ be as above and let $\boldsymbol{f}$ be a function from the set of subgroups of G to A such that $\boldsymbol{f}$ is constant on conjugacy classes of subgroups of G. If $O_p(G) \neq 1$, then*

$$\sum_{\sigma\in\mathcal{N}/G} (-1)^{|\sigma|}\boldsymbol{f}(N_G(\sigma)) = 0.$$

Proof. We sketch the proof for the convenience of the reader. Set $R := O_p(G)$ and assume that $R > 1$. We show that there exists a G-invariant involution $\eta\colon \mathcal{N} \to \mathcal{N}$ where $N_G(\sigma) = N_G(\eta(\sigma))$ and $|\eta(\sigma)| = |\sigma| \pm 1$. Given $\sigma = (Q_0 < Q_1 < \cdots < Q_m) \in \mathcal{N}$, choose i maximal with the property that $R \nleq Q_i$. Since $R \nleq 1 = Q_0$, we see that there is such an i. By choice of i, we have $Q_i < Q_iR$, and we have $Q_iR \leqslant Q_{i+1}$ if $i < m$. Define

$$\eta(\sigma) = \begin{cases} Q_0 < \cdots < Q_m < Q_mR & \text{if } i = m, \\ Q_0 < \cdots < Q_i < Q_{i+2} < \cdots < Q_m & \text{if } Q_iR = Q_{i+1}, \text{ and} \\ Q_0 < \cdots Q_i < Q_iR < Q_{i+1} < \cdots < Q_m & \text{if } Q_iR < Q_{i+1}. \end{cases}$$

Then $\eta(\sigma) \in \mathcal{N}$ and $N_G(\sigma) = N_G(\eta(\sigma))$ for each $\sigma \in \mathcal{N}$, since R is a normal p-subgroup of G. Also, $|\eta(\sigma)| = |\sigma| \pm 1$. It is a momentary exercise to verify that η is an involution on $\mathcal{N}$. Hence, the alternating sum vanishes as claimed. □

Remark 4.14. We finish this section with a mention of a recurrent elementary tool for reordering sums indexed by two or more sets acted upon by a finite group G, which we will use without much further comment. Let X, Y be finite G-sets and denote by $\pi_X : X\times Y \to X$, $\pi_Y : Y \times X$ the projection maps. Let A be a G-invariant subset of $X \times Y$ under the diagonal action of G on $X \times Y$. Suppose that for any $(x, y) \in X \times Y$ we have an element $\alpha(x, y)$ in some abelian group depending only on the G-orbit of (x, y). Then

$$\sum_{(x,y)\in A/G} \alpha(x, y)$$

is equal to any of the following double sums

$$\sum_{x\in X/G} \quad \sum_{y\in\pi_Y(\pi_X^{-1}(x)\cap A)/G_x} \alpha(x, y)$$

$$\sum_{y\in Y/G} \quad \sum_{x\in\pi_X(\pi_Y^{-1}(y)\cap A)/G_y} \alpha(x, y).$$

Note that the two double sums make sense as by the G-invariance of A, for each $x \in X$, $\pi_Y(\pi_X^{-1}(x) \cap A)$ is G_x-invariant and for each $y \in Y$, $\pi_X(\pi_Y^{-1}(y) \cap A)$ is G_y-invariant. Let $\mathcal{X}$ be a set of representatives of the G-orbits of X and for each $x \in \mathcal{X}$, let $\mathcal{Y}_x$ be a set of representatives of the G_x-orbits of $\mathcal{X}$ and set

$$U := \{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}_x\}.$$

Then, $U \subseteq A$. We will show that U is a set of representatives of the G-orbits of A, and this will yield the equality of $\sum_{(x,y)\in A/G} \alpha(x, y)$ with the first double sum. Suppose that $x, x' \in \mathcal{X}$, $y, y' \in \mathcal{Y}_x$ are such that (x, y) and (x', y') are in the same G-orbit and let $g \in G$ be such that $(x', y') = {}^g(x, y)$. By comparing the first components, it follows that x' and x are in the same G-orbit of X, hence $x' = x$ and $g \in G_x$. Now comparing the second components implies $y' = y$. Conversely, let $(x_0, y_0) \in A$. We will show that (x_0, y_0) is G-conjugate to an element of U. By definition of $\mathcal{X}$, there exists $g \in G$ and $x \in \mathcal{X}$ such that $x_0 = {}^g x$, hence by replacing (x_0, y_0) by ${}^g(x_0, y_0)$ we may assume that $x_0 \in \mathcal{X}$. Since $(x_0, y_0) \in A$, $y_0 \in \pi_Y(\pi_X^{-1}(x_0) \cap A)$. Hence by the definition of $\mathcal{Y}_{x_0}$, y_0 is G_{x_0}-conjugate to some element of $\mathcal{Y}_{x_0}$, say $z_0 = {}^h y_0$ with $h \in G_{x_0}$, $z_0 \in \mathcal{Y}_{x_0}$. Then

$${}^h(x_0, y_0) = ({}^h x_0, {}^h y_0) = (x_0, z_0) \in U$$

as required. The proof of the equality with the second sum is entirely analogous.

5. Towards Theorem 1.1

Throughout this section let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let α be an $\mathcal{F}$-compatible family.

Our first goal will be to reformulate $\mathbf{m}^*(\mathcal{F}, \alpha)$ by reindexing the sum over objects in the full subcategory $S_\triangleleft(\mathcal{F}^c)$ of the subdivision category of the category of $\mathcal{F}$-centric subgroups. Recall from Section 3 that $S_\triangleleft(\mathcal{F}^c)$ has as objects chains of proper inclusions

$$Q_0 < Q_1 < \cdots < Q_m$$

of $\mathcal{F}$-centric subgroups with the property that the Q_i are normal in the maximal term Q_m, for each $0 \leq i \leq m$. Consider the following sets

$$\begin{aligned} \mathcal{M} &:= \{(Q, \sigma, [x]) \mid Q \in \mathcal{F}^c, \sigma \in \mathcal{N}_Q, [x] \in Q^{\mathrm{cl}}\}, \\ \widetilde{\mathcal{M}} &:= \{(\sigma, x) \mid \sigma \in S_\triangleleft(\mathcal{F}^c), x \in Q_\sigma\}. \end{aligned}$$

The set $\mathcal{M}$ is equipped with the equivalence relation

$$(Q, \sigma, [x]) \sim_{\mathcal{M}} (R, \tau, [y])$$

whenever there exists an isomorphism $\varphi : Q \to R$ in $\mathcal{F}$ such that $\overline{c}_\varphi(\sigma) = \tau$ and such that $\varphi([x]) = [y]$. Here $\overline{c}_\varphi$ is as defined before Lemma 3.4 and we use $\overline{c}_\varphi(\sigma)$ to denote the image of σ under the natural extension of $\overline{c}_\varphi$ to a map from the set of chains of subgroups of $\mathrm{Out}_\mathcal{F}(Q)$ to the set of chains of subgroups of $\mathrm{Out}_\mathcal{F}(R)$. The set $\widetilde{\mathcal{M}}$ is equipped with the equivalence relation

$$(\sigma, x) \sim_{\widetilde{\mathcal{M}}} (\tau, y)$$

whenever there exists an isomorphism $\varphi : \sigma \to \tau$ in $S_\triangleleft(\mathcal{F}^c)$ such that $\varphi(x) = y$.

Proposition 5.1. *We have*

$$\boldsymbol{m}^*(\mathcal{F}, \alpha) = \sum_{(\sigma,x) \in \widetilde{\mathcal{M}}/\sim} (-1)^{|\sigma|} z(k_\alpha C_{\mathrm{Aut}_\mathcal{F}(\sigma)}(x) \operatorname{Aut}_{Q_\sigma}(Q^\sigma) / \operatorname{Aut}_{Q_\sigma}(Q^\sigma)).$$

Proof. This follows from Lemmas 5.4 and 5.5 below. □

We rewrite $\mathbf{m}^*(\mathcal{F}, \alpha)$ in terms of $(\mathcal{M}, \sim)$.

Lemma 5.2.

$$\boldsymbol{m}^*(\mathcal{F}, \alpha) = \sum_{(Q,\sigma,[x]) \in \mathcal{M}/\sim} z(k_\alpha C_{I(\sigma)}([x])).$$

Proof. Let $\mathcal{X}$ be a set of representatives of $\mathcal{F}$-classes in $\mathcal{F}^c$ and for each $Q \in \mathcal{X}$, let $\mathcal{Y}_Q$ be a set of $\mathrm{Out}_{\mathcal{F}}(Q)$ representatives of $\mathcal{W}_Q^*$. Then $\{(Q, \sigma, [x]) : Q \in \mathcal{X}, (\sigma, [x]) \in \mathcal{Y}_Q\}$ is a set of representatives of the $\sim$-equivalence classes of $\mathcal{M}$ and the result follows. □

A normal chain $\sigma = (Q_0 < \cdots < Q_m)$ in $S_{\triangleleft}(\mathcal{F}^c)$ induces a normal chain $\mathrm{Aut}_\sigma(Q_0) := (\mathrm{Aut}_{Q_0}(Q_0) < \cdots < \mathrm{Aut}_{Q_m}(Q_0))$ of p-subgroups in $\mathrm{Aut}_{\mathcal{F}}(Q)$, and a corresponding normal chain $\mathrm{Out}_\sigma(Q_0) \in \mathcal{N}_{Q_0}$ upon factoring by $\mathrm{Inn}(Q_0)$. In this context, bars will denote quotients by $\mathrm{Inn}(Q_0)$. That is, we set $\overline{Q}_i := \mathrm{Out}_{Q_i}(Q_0)$ for each $0 \leq i \leq m$ and we set

$$\overline{\sigma} := \mathrm{Out}_\sigma(Q_0) = (\overline{Q}_0 < \overline{Q}_1 < \cdots < \overline{Q}_m)$$

for short. Note that $\overline{Q}_0$ is trivial.

Lemma 5.3. *The map $\widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ which sends (σ, x) to $(Q_\sigma, \overline{\sigma}, [x])$ induces a bijection between $\widetilde{\mathcal{M}}/\!\sim$ and $\mathcal{M}/\!\sim$.*

Proof. We first show that the map is well-defined. Let $(\sigma, x) \sim (\tau, y)$ in $\widetilde{\mathcal{M}}$. Fix an isomorphism $\varphi\colon \sigma \to \tau$ in $S_{\triangleleft}(\mathcal{F}^c)$ such that $\varphi(x) = y$. Then $(Q_\sigma, \overline{\sigma}, [x]) \sim (Q_\tau, \overline{\tau}, [y])$ via the restriction of φ to Q_σ.

Next, suppose $(Q_\sigma, \overline{\sigma}, [x]), (Q_\tau, \overline{\tau}, [y]) \in \mathcal{M}$ are $\mathcal{M}$-equivalent. Let $\psi\colon Q_\sigma \to Q_\tau$ be an $\mathcal{F}$-isomorphism such that $\overline{c}_\psi(\overline{\sigma}) = \overline{\tau}$ and $\psi([x]) = [y]$. By Lemma 3.4, ψ extends to a chain isomorphism $\widehat{\psi}\colon \sigma \to \tau$. Since $\psi([x]) = [y]$, we have $\psi(x) = uyu^{-1}$ for some $u \in Q_\tau$. Let $\delta\colon Q^\sigma \to Q^\tau$ be the composition of $\widehat{\psi}$ with conjugation by u. Then (σ, x) and (τ, y) are $\widetilde{\mathcal{M}}$-equivalent via δ. This proves injectivity.

It remains to show that whenever $(R, \rho, [z]) \in \mathcal{M}$, there exists $(\sigma, x) \in \widetilde{\mathcal{M}}$ such that $(Q_\sigma, \overline{\sigma}, [x])$ is $\mathcal{M}$-equivalent to $(R, \rho, [z])$. Let $\rho = (1 < X_1 < \cdots < X_m) \in \mathcal{N}_R$. Let $\alpha\colon R \to R'$ be an $\mathcal{F}$-isomorphism with R' fully $\mathcal{F}$-normalised, and consider the chain

$$\overline{c}_\alpha(\rho) = (1 < \overline{c}_\alpha(X_1) < \cdots < \overline{c}_\alpha(X_m)).$$

Since R' is fully $\mathcal{F}$-normalised and $\mathcal{F}$ is saturated, $\mathrm{Out}_S(R')$ is a Sylow p-subgroup of $\mathrm{Out}_{\mathcal{F}}(R')$, so by Sylow's theorem we may fix $\beta \in \mathrm{Out}_{\mathcal{F}}(R')$ such that $\beta\overline{c}_\alpha(X_m)\beta^{-1} \leqslant \mathrm{Out}_S(R')$. Denote by R_i' the inverse image of $\beta\overline{c}_\alpha(X_i)\beta^{-1}$ in $N_S(R')$, and set

$$\sigma := (R' < R_1' < \cdots < R_m') \quad \text{and} \quad x := \widehat{\beta}\alpha(z),$$

where $\widehat{\beta} \in \mathrm{Aut}_{\mathcal{F}}(R')$ is any lift of β. Then $(\sigma, x) \in \widetilde{\mathcal{M}}$, and $(Q_\sigma, \overline{\sigma}, [x])$ is $\mathcal{M}$-equivalent to $(R, \rho, [z])$ via $\widehat{\beta}\alpha$. □

The following lemma is now immediate from Lemmas 5.2 and 5.3.

Lemma 5.4. *We have*

$$\boldsymbol{m}^*(\mathcal{F}, \alpha) = \sum_{(\sigma, x) \in \widetilde{\mathcal{M}}/\sim} (-1)^{|\sigma|} z(k_\alpha C_{I(\overline{\sigma})}([x])).$$

To complete the proof of Proposition 5.1, we give an interpretation of $z(k_\alpha C_{I(\overline{\sigma})}([x])$ in terms of the automisers of chains in $S_{\triangleleft}(\mathcal{F}^c)$.

Lemma 5.5. *Fix $\sigma = (Q_0 < \cdots < Q_m) \in S_{\triangleleft}(\mathcal{F}^c)$, and let π be the composite*

$$\mathrm{Aut}_{\mathcal{F}}(\sigma) \xrightarrow{\text{res}} N_{\mathrm{Aut}_{\mathcal{F}}(Q_0)}(\mathrm{Aut}_\sigma(Q_0)) \longrightarrow I(\overline{\sigma}) \overset{\text{def}}{=} N_{\mathrm{Out}_{\mathcal{F}}(Q_0)}(\mathrm{Out}_\sigma(Q_0)),$$

which restricts to Q_0 *and then factors by* $\mathrm{Aut}_{Q_0}(Q_0)$. *Then*

(1) π *is surjective,*
(2) $\ker(\pi) = \mathrm{Aut}_{Q_0}(Q_m)$, *and*
(3) *for each* $x \in Q_0$, *the group* $C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x)\,\mathrm{Aut}_{Q_0}(Q_m)$ *is the inverse image of* $C_{I(\overline{\sigma})}(x)$ *under* π.

Proof. To prove (1), it suffices to show that the restriction map res: $\mathrm{Aut}_{\mathcal{F}}(\sigma) \to N_{\mathrm{Aut}_{\mathcal{F}}(Q_0)}(\mathrm{Aut}_\sigma(Q_0))$ is surjective. Let $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(Q_0)}(\mathrm{Aut}_\sigma(Q_0))$. Then $c_\alpha(\mathrm{Aut}_{Q_i}(Q_0)) \leqslant \mathrm{Aut}_{Q_i}(Q_0)$ for all $0 \leqslant i \leqslant m$. The first conclusion of Lemma 3.4 then yields an extension $\widetilde{\alpha}$ of α to Q_m.

Fix i with $0 \leqslant i \leqslant m$, and fix $u \in Q_i$. Then since $\widetilde{\alpha}$ is defined on u, we have

$$c_{\widetilde{\alpha}(u)}|_{Q_0} = \alpha(c_u|_{Q_0})\alpha^{-1} \in \mathrm{Aut}_{Q_i}(Q_0)$$

by assumption. Hence, $\widetilde{\alpha}(u)$ lies in the full inverse image of $\mathrm{Aut}_{Q_i}(Q_0)$ under $N_S(Q_0) \to \mathrm{Aut}_S(Q_0)$, which is Q_i because Q_0 is centric. This shows that $\widetilde{\alpha}(Q_i) = Q_i$ for each i, and thus the surjectivity of the restriction map.

That $\mathrm{Aut}_{Q_0}(Q_m) \leqslant \ker(\pi)$ is clear. To see the other inclusion in (2), fix $\varphi \in \ker(\pi)$. Then $\varphi|_{Q_0} = c_u$ for some $u \in Q_0$, so we may fix $z \in Z(Q_0)$ such that $\varphi = c_u c_z = c_{uz}$ by [BLO03, Lemma A.8]. Thus, $\varphi \in \mathrm{Aut}_{Q_0}(Q_m)$, as desired.

Finally, (3) holds because $\ker(\pi) = \mathrm{Aut}_{Q_0}(Q_m)$ acts transitively on the Q_0-class $[x]$. □

Define the following subsets of $\widetilde{\mathcal{M}}$:

(1) $\widetilde{\mathcal{M}}^e$ is the subset of $\widetilde{\mathcal{M}}$ consisting of those (σ, x) for which Q^σ/Q_σ is elementary abelian.
(2) $\widetilde{\mathcal{M}}^\circ$ is the subset of $\widetilde{\mathcal{M}}$ consisting of those (σ, x) for which $C_{Q^\sigma}(x) \leqslant Q_\sigma$.
(3) $\widetilde{\mathcal{M}}^{e,\circ}$ is the intersection of $\widetilde{\mathcal{M}}^e$ and $\widetilde{\mathcal{M}}^\circ$.
(4) $\widetilde{\mathcal{M}}^{e,\circ,c}$ is the subset of $\widetilde{\mathcal{M}}^{e,\circ}$ consisting of those (σ, x) for which $C_{Q^\sigma}(x)\Phi(Q^\sigma)$ is $\mathcal{F}$-centric.

Observe that all these subsets are unions of $\widetilde{\mathcal{M}}$-equivalence classes. Let

$$\mathbf{m}^e(\mathcal{F}, \alpha) := \sum_{(\sigma,x)\in\widetilde{\mathcal{M}}^e/\sim} (-1)^{|\sigma|} z(k_\alpha C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x)\,\mathrm{Aut}_{Q_\sigma}(Q^\sigma)/\,\mathrm{Aut}_{Q_\sigma}(Q^\sigma))$$

and define $\mathbf{m}^\circ(\mathcal{F}, \alpha)$, $\mathbf{m}^{e,\circ}(\mathcal{F}, \alpha)$, and $\mathbf{m}^{e,\circ,c}(\mathcal{F}, \alpha)$ analogously.

Proposition 5.6. *The following hold.*

(1) $\boldsymbol{m}^*(\mathcal{F}, \alpha) = \boldsymbol{m}^e(\mathcal{F}, \alpha)$.
(2) $\boldsymbol{m}^*(\mathcal{F}, \alpha) = \boldsymbol{m}^\circ(\mathcal{F}, \alpha)$.
(3) $\boldsymbol{m}^*(\mathcal{F}, \alpha) = \boldsymbol{m}^{e,\circ}(\mathcal{F}, \alpha)$.

Proof. By Lemma 5.5, Remark 4.14, the obvious analogue of Lemma 5.2 for elementary abelian chains, and by restricting the inverse of the bijection of Lemma 5.3 to classes of elements of $\widetilde{\mathcal{M}}^e$, we have

$$\mathbf{m}^e(\mathcal{F}, \alpha) = \sum_{Q\in\mathcal{F}^c} \sum_{\sigma\in\mathcal{E}_Q/\,\mathrm{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{[x]\in Q^{\mathrm{cl}}/I(\sigma)} z(k_\alpha C_{I(\sigma)}([x])),$$

where $\mathcal{E}_Q \subseteq \mathcal{N}_Q$ is the set of all elementary abelian chains. Thus (1) follows on applying Lemma 4.12 with $G = \mathrm{Out}_{\mathcal{F}}(Q)$ for each $Q \in \mathcal{F}^c$. We next prove (2). Note that if $(\sigma, x) \in \widetilde{\mathcal{M}}$ and $C_{Q^\sigma}(x)$ is not contained in Q_σ, then $C_{\mathrm{Inn}(Q^\sigma)}(x)\,\mathrm{Aut}_{Q_\sigma}(Q^\sigma)/\,\mathrm{Aut}_{Q_\sigma}(Q^\sigma) \cong C_{Q^\sigma}(x)/C_{Q_\sigma}(x)$ is a non-trivial normal subgroup of $C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x)\,\mathrm{Aut}_{Q_\sigma}(Q^\sigma)/\,\mathrm{Aut}_{Q_\sigma}(Q^\sigma)$ and the result follows from Proposition 5.1(3) and Lemma 4.11. The same argument holds with $(\sigma, x) \in \widetilde{\mathcal{M}}^e$, so (3) follows from (1). □

Recall that

$$\mathbf{k}(\mathcal{F}, \alpha) = \sum_{x \in [S/\mathcal{F}]} \sum_{Q \in C_{\mathcal{F}}(x)^c / C_{\mathcal{F}}(x)} z(k_\alpha \,\mathrm{Out}_{C_{\mathcal{F}}(x)}(Q)), \tag{5.1}$$

where $[S/\mathcal{F}] \subseteq S$ is a fixed set of fully $\mathcal{F}$-centralized $\mathcal{F}$-conjugacy class representatives of the elements of S. Define

$$\begin{aligned} \mathcal{C} &:= \{(Q, x) \mid x \in [S/\mathcal{F}], Q \in C_{\mathcal{F}}(x)^c\}, \text{ and} \\ \mathcal{D} &:= \{(Q, x) \mid x \in Z(Q), Q \in \mathcal{F}^c\} \end{aligned}$$

and equivalence relations

$$\begin{aligned} (Q, x) \sim_{\mathcal{C}} (R, y) &\iff x = y \text{ and } \mathrm{Iso}_{C_{\mathcal{F}}(x)}(Q, R) \neq \varnothing, \text{ and} \\ (Q, x) \sim_{\mathcal{D}} (R, y) &\iff \text{there exists } \varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, R) \text{ such that } \varphi(x) = y. \end{aligned}$$

Thus, $\mathcal{C}/\!\sim$ may be viewed as an indexing set for $\mathbf{k}(\mathcal{F}, \alpha)$. Also, $x \in Z(Q)$ whenever $Q \in C_{\mathcal{F}}(x)^c$, so that $\mathcal{C}$ is a subset of $\mathcal{D}$.

Lemma 5.7. *The inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ induces a bijection between $\mathcal{C}/\!\sim$ and $\mathcal{D}/\!\sim$; in particular,*

$$\boldsymbol{k}(\mathcal{F}, \alpha) = \sum_{Q \in \mathcal{F}^c/\mathcal{F}} \sum_{x \in Z(Q)/\,\mathrm{Out}_{\mathcal{F}}(Q)} z(k_\alpha C_{\mathrm{Out}_{\mathcal{F}}(Q)}(x)). \tag{5.2}$$

Proof. If $(Q, x) \sim_{\mathcal{C}} (R, y)$, then $x = y$ and there is an $\mathcal{F}$-isomorphism from Q to R which centralizes x, so that $(Q, x) \sim_{\mathcal{D}} (R, y)$. There is indeed a well-defined map on equivalence classes induced by the inclusion.

Conversely, assume that $(Q, x), (R, y) \in \mathcal{C}$ are $\mathcal{D}$-equivalent. Fix an $\mathcal{F}$-isomorphism φ from Q to R with $\varphi(x) = y$. As $x, y \in [S/\mathcal{F}]$ are $\mathcal{F}$-conjugate, we have $x = y$, and so Q and R are $C_{\mathcal{F}}(x)$-conjugate. This shows that $(Q, x) \sim_{\mathcal{C}} (R, y)$, so the induced map is injective.

To complete the proof of the first assertion, it remains to show that each element of $\mathcal{D}$ is $\mathcal{D}$-equivalent to a member of $\mathcal{C}$. Fix $(R, y) \in \mathcal{D}$. Let $x \in [S/\mathcal{F}]$ be the unique element which is $\mathcal{F}$-conjugate to y. Since $\langle x\rangle$ is fully $\mathcal{F}$-centralized, we may choose a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(C_S(\langle y\rangle), C_S(\langle x\rangle))$ such that $\alpha(y) = x$ by [AKO11, I.2.6(c)]. Set $Q = \alpha(R)$. Then $(R, y) \sim_{\mathcal{D}} (Q, x)$ via α. Since R is $\mathcal{F}$-centric, also Q is $\mathcal{F}$-centric, so that Q is $C_{\mathcal{F}}(x)$-centric by Lemma 3.3. This yields $(Q, x) \in \mathcal{C}$ and completes the proof of the first assertion.

Now $\mathrm{Out}_{C_{\mathcal{F}}(x)}(Q) = C_{\mathrm{Out}_{\mathcal{F}}(Q)}(x)$ for each $x \in Z(Q)$ by Lemma 3.3. Hence, as $\mathcal{C}/\!\sim$ is an indexing set for a single sum computing $\mathbf{k}(\mathcal{F}, \alpha)$ as in (5.1), and as $\mathcal{D}/\!\sim$ is an indexing set for a single sum computing the right hand side of (5.2), we have that (5.2) follows from (5.1). □

Proposition 5.8. *We have, $\boldsymbol{k}(\mathcal{F}, \alpha) = \boldsymbol{m}^{e,\circ,c}(\mathcal{F}, \alpha)$.*

Proof. Let $\mathcal{D}'$ be the subset of $\widetilde{\mathcal{M}}^{e,\circ,c}$ consisting of the pairs (σ, x) such that $|\sigma| = 0$ and $x \in Z(Q_\sigma)$. Then $\mathcal{D}'$ is a union of $\widetilde{\mathcal{M}}$-equivalence classes. Regarding an $\mathcal{F}$-centric subgroup Q as a chain of length zero yields a canonical bijection $\mathcal{D}/{\sim_{\mathcal{D}}} \to \mathcal{D}'/{\sim_{\widetilde{\mathcal{M}}}}$, and so we may regard $\mathbf{k}(\mathcal{F}, \alpha)$ as indexed over $\mathcal{D}'/{\sim_{\widetilde{M}}}$. We use chain pairing to remove the terms from $\mathbf{m}^{e,\circ,c}(\mathcal{F}, \alpha)$ not in $\mathcal{D}'$. This will yield

$$\mathbf{m}^{e,\circ,c}(\mathcal{F}, \alpha) = \sum_{(\sigma,x)\in\mathcal{D}'/\sim} (-1)^{|\sigma|} z(k_\alpha C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x) \mathrm{Aut}_{Q_\sigma}(Q^\sigma)/ \mathrm{Aut}_{Q_\sigma}(Q^\sigma)).$$

The Proposition then follows from the expression for $\mathbf{k}(\mathcal{F}, \alpha)$ in Lemma 5.7, along with Lemma 5.5(3).

For each $\sigma = (Q_0 < \cdots < Q_m) \in S_\triangleleft(\mathcal{F}^c)$, we let $Q_{-1} := C_{Q_m}(x)\Phi(Q_m)$. Define a map

$$\eta \colon \widetilde{\mathcal{M}}^{e,\circ,c} \backslash \mathcal{D}' \longrightarrow \widetilde{\mathcal{M}}^{e,\circ,c} \backslash \mathcal{D}'$$

via

$$(\sigma, x) \longmapsto (\eta(\sigma), x),$$

where

$$\eta(\sigma) = \begin{cases} Q_{-1} < Q_0 < \cdots < Q_m & \text{if } Q_{-1} < Q_0, \text{ and} \\ Q_1 < \cdots < Q_m & \text{if } Q_{-1} = Q_0. \end{cases}$$

It is straightforward to see that η is an involution that preserves $\widetilde{\mathcal{M}}$-equivalence classes if well-defined.

To prove that η is well-defined, we assert three points for a given pair $(\sigma, x) \notin \mathcal{D}'$ with σ as above. First, observe that $\eta(\sigma)$ is a first component of some member of $\widetilde{\mathcal{M}}^{e,\circ,c}$ by definition of Q_{-1} and the fact that $Q_{-1} \in \mathcal{F}^c$ by assumption. In particular, $\eta(\sigma)$ is never the empty chain: if σ has length zero, then $C_{Q_0}(x) = C_{Q_m}(x) < Q_0$ as $(\sigma, x) \notin \mathcal{D}'$, so also $Q_{-1} = C_{Q_0}(x)\Phi(Q_0) < Q_0$, and hence $\eta(\sigma)$ has length 1. Second, note that $x \in C_{Q_m}(x) \leqslant Q_{-1}$ in case Q_{-1} is contained properly in Q_0, so that indeed $(\eta(\sigma), x) \in \widetilde{\mathcal{M}}^{e,\circ,c}$. Lastly, continue to consider a pair (σ, x) not in $\mathcal{D}'$. We claim that $(\eta(\sigma), x)$ is not in $\mathcal{D}'$, and the only case where this is not immediate has $|\sigma| = 1$ and $|\eta(\sigma)| = 0$. In this case either x is not in $Z(Q_0)$, in which case x is likewise not in $Z(Q_1) \leqslant Z(Q_0)$, or $x \in Z(Q_0)$, in which case $C_{Q_1}(x) = C_{Q_0}(x) = Q_0 < Q_1$ so that again x is not in $Z(Q_1)$. This shows that $(\eta(\sigma), x) \notin \mathcal{D}'$ and completes the proof of the last point.

Having shown that η is a well-defined involution, it remains to prove that it preserves the value of each summand appearing in Proposition 5.1. To establish this, it suffices to show that

$$C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x) = C_{\mathrm{Aut}_{\mathcal{F}}(\eta(\sigma))}(x) \quad \text{and} \quad \mathrm{Aut}_{C_{Q_0}(x)}(Q_m) = \mathrm{Aut}_{C_{Q_{-1}}(x)}(Q_m).$$

As Q_{-1} is invariant under $\mathrm{Aut}_{\mathcal{F}}(\sigma)$, one has $\mathrm{Aut}_{\mathcal{F}}(\sigma) \leqslant \mathrm{Aut}_{\mathcal{F}}(\eta(\sigma))$ if $\eta(\sigma)$ has length one more than σ. Also, one has the same containment if $\eta(\sigma)$ has length one less, since $\eta(\sigma)$ is a subchain of σ in that case. Equality therefore holds in both cases, because η is an involution. This completes the proof of the first displayed equality. Finally, the second equality holds since $C_{Q_0}(x) = C_{Q_{-1}}(x)$ for each $(\sigma, x) \in \widetilde{\mathcal{M}}^{e,\circ,c}$. □

6. Proof of Theorem 1.1

In light of Propositions 5.8 and 5.6(3), to complete the proof of Theorem 1.1 it suffices to establish an equality between $\mathbf{m}^{e,\circ}(\mathcal{F},\alpha)$ and $\mathbf{m}^{e,\circ,c}(\mathcal{F},\alpha)$. We will achieve that in this section.

If G is a finite group and σ is a chain of p-subgroups in G such that the first subgroup is a normal subgroup of the last subgroup, then we denote by $G_\sigma \leqslant N_G(Q^\sigma)$ the stabiliser in G of the chain and by $\mathrm{Aut}_G(\sigma)$ the image of G_σ in $N_G(Q^\sigma)/C_G(Q^\sigma)$.

Lemma 6.1. *Let $\sigma = (Q_0 < \cdots < Q_m)$ be a chain of proper inclusions of subgroups of S such that Q_i is normal in Q_m for each $0 \leqslant i \leqslant m$, and let $x \in Q_0$ be such that $C_{Q_m}(x) \leqslant Q_0$. Suppose that Q_m is $\mathcal{F}$-centric, Q_0 is fully $\mathcal{F}$-normalised, and Q_m is fully $N_{\mathcal{F}}(Q_0)$-normalised. Let L be a model of $N_{N_{\mathcal{F}}(Q_0)}(Q_m)$. The following hold:*

1. *$C_{L_\sigma}(x)Q_0/Q_0 \cong C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x)\,\mathrm{Aut}_{Q_0}(Q_m)/\mathrm{Aut}_{Q_0}(Q_m)$.*
2. *If Q_0 is not $\mathcal{F}$-centric, then $z(k_\alpha(C_{L_\sigma}(x)Q_0/Q_0)) = 0$.*

Proof. Set $Q_\sigma = Q_0$ and $Q^\sigma = Q_m$. We have

$$L_\sigma/Z(Q_m) \cong \mathrm{Aut}_L(\sigma) = \mathrm{Aut}_{N_{N_{\mathcal{F}}(Q_0)}(Q_m)}(\sigma) = \mathrm{Aut}_{N_{\mathcal{F}}(Q_0)}(\sigma) = \mathrm{Aut}_{\mathcal{F}}(\sigma).$$

The quotient map $\pi\colon L_\sigma \to \mathrm{Aut}_{\mathcal{F}}(\sigma)$ sends $C_{L_\sigma}(x)$ to $C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x)$. It also sends Q_m to $\mathrm{Aut}_{Q_0}(Q_m)$, since $Z(Q_m) \le C_{Q_m}(x) \le Q_0$ by assumption. Part (1) follows from this.

We now turn to (2), where we first claim that $C_L(Q_0)$ is a p-group under the given assumptions. Let y be an element of $C_L(Q_0)$ of order prime to p, and let c_y be the image of y in $\mathrm{Aut}(Q_m)$. Since $C_{Q_m}(\mathrm{Aut}_{Q_0}(Q_m)) = C_{Q_m}(Q_0) \leqslant C_{Q_m}(x) \leqslant Q_0$, we have

$$[c_y, C_{Q_m}(\mathrm{Aut}_{Q_0}(Q_m))] \le [c_y, Q_0] = 1.$$

Now Lemma 3.1 implies that $c_y = \mathrm{Id}_{Q_m}$, so that $y \in C_L(Q_m) \leqslant Q_m$ is of order a power of p, since Q_m is self-centralising in L. Hence, $y = 1$.

Assume that $z(k_\alpha(C_{L_\sigma}(x)Q_0/Q_0)) \neq 0$. As Q_0 is normal in L_σ, we know that $C_{L_\sigma}(Q_0)Q_0$ is likewise normal in L_σ. But $C_{L_\sigma}(Q_0) \leqslant C_{L_\sigma}(x)$, so $C_{L_\sigma}(Q_0)Q_0$ is normal in $C_{L_\sigma}(x)Q_0$. Hence, $z(k_\alpha(C_{L_\sigma}(Q_0)Q_0/Q_0)) \neq 0$ by Lemma 4.10. It was just shown that $C_{L_\sigma}(Q_0)$ is a p-group, so we have $C_{L_\sigma}(Q_0) \leqslant Q_0$ by Lemma 4.11. In other words, Q_0 is $N_{\mathcal{F}}^K(Q_0)$-centric, where $K \leqslant \mathrm{Aut}_{\mathcal{F}}(Q_0)$ is the subgroup consisting of those automorphisms which extend to automorphisms of σ. Hence, Q_0 is $\mathcal{F}$-centric by Lemma 3.2. □

Lemma 6.2. *Let $(\sigma, x) \in \widetilde{M}^{e,\circ}$, with $\sigma = (Q_0 < \cdots < Q_m)$ as before. If $C_{Q_m}(x)\Phi(Q_m)$ is not $\mathcal{F}$-centric, then $z(k_\alpha C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x)\,\mathrm{Aut}_{Q_0}(Q_m)/\mathrm{Aut}_{Q_0}(Q_m)) = 0$.*

Proof. Write $Q_{-1} = C_{Q_m}(x)\Phi(Q_m)$, and recall that $Q_{-1} \leqslant Q_0$ by definition of $\widetilde{\mathcal{M}}^{e,\circ}$. Using [AKO11, I.2.6(c)], we choose a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q_m, S)$ with $\varphi(R)$ fully $\mathcal{F}$-normalized, and then a morphism $\psi \in \mathrm{Hom}_{N_{\mathcal{F}}(\varphi(R))}(Q_m, N_S(\varphi(R)))$ with $\psi\varphi(Q_m)$ fully $N_{\mathcal{F}}(\varphi(R))$-normalized. Set $\tau = \psi\varphi(\sigma)$ and $y = \psi\varphi(x)$. Conjugation by $\psi\varphi$ yields an isomorphism

$$C_{\mathrm{Aut}_{\mathcal{F}}(\sigma)}(x)\,\mathrm{Aut}_{Q_\sigma}(Q^\sigma)/\mathrm{Aut}_{Q_\sigma}(Q^\sigma) \cong C_{\mathrm{Aut}_{\mathcal{F}}(\tau)}(y)\,\mathrm{Aut}_{Q_\tau}(Q^\tau)/\mathrm{Aut}_{Q_\tau}(Q^\tau).$$

Upon replacing (σ, x) by (τ, y), we may therefore assume Q_{-1} to be fully $\mathcal{F}$-normalized and Q_m to be fully $N_{\mathcal{F}}(Q_{-1})$-normalized.

Assume on the contrary that $Q_{-1} \stackrel{\text{def}}{=} C_{Q_m}(x)\Phi(Q_m)$ is not $\mathcal{F}$-centric, but that $z(k_\alpha C_{\mathrm{Aut}_{\mathcal{F}(\sigma)}}(x)\,\mathrm{Aut}_{Q_0}(Q_m)/\,\mathrm{Aut}_{Q_0}(Q_m)) \neq 0$. As Q_0 is $\mathcal{F}$-centric, Q_{-1} is a proper subgroup of Q_0. Consider the chain

$$\sigma' = (Q_{-1} < Q_0 < \cdots Q_m).$$

It was shown in the last part of the proof of Lemma 5.8 that

$$C_{\mathrm{Aut}_{\mathcal{F}(\sigma)}}(x)\,\mathrm{Aut}_{Q_0}(Q_m)/\,\mathrm{Aut}_{Q_0}(Q_m) = C_{\mathrm{Aut}_{\mathcal{F}(\sigma')}}(x)\,\mathrm{Aut}_{Q_{-1}}(Q_m)/\,\mathrm{Aut}_{Q_{-1}}(Q_m),$$

and that argument did not require $(\sigma, x) \in \widetilde{\mathcal{M}}^c$. But then from Lemma 6.1 applied to σ', we conclude that Q_{-1} is $\mathcal{F}$-centric after all, a contradiction. □

Proof of Theorem 1.1. By Proposition 5.1, Proposition 5.6(3), and Lemma 6.2, we have $\mathbf{m}^*(\mathcal{F},\alpha) = \mathbf{m}^{e,\circ,c}(\mathcal{F},\alpha)$. The result now follows from Proposition 5.8. □

7. Proof of Theorem 1.2

Lemma 7.1 (Robinson)**.** *Suppose that G is a finite group, $Q \trianglelefteq G$ is a p-subgroup and $\alpha \in H^2(G/Q, k^\times)$. We have*

$$\sum_{[x]\in Q^{\mathrm{cl}}/G} \ell(k_\alpha C_G([x])) = \sum_{\mu\in \mathrm{Irr}(Q)/G} \ell(k_\alpha C_G(\mu)) \ .$$

This Lemma is due to Robinson, and it is obtained as a combination of [Rob87], [RS90] (see discussion before Theorem 1.2 of [Rob96]). As a convenience to the reader, the main ideas of the proof are presented in the Appendix.

For a finite group H denote by $\mathcal{S}(H)$ the poset of p-subgroups of H (including the trivial subgroup - so notation is not standard). If Q is a normal p-subgroup of a finite group G, then for any $[x] \in Q^{\mathrm{cl}}$ (respectively $\mu \in \mathrm{Irr}(Q)$), we denote by $I([x])$ (respectively $I(\mu)$) the stabiliser in G/Q of $[x]$ (respectively μ) under the action of G/Q and for any subgroup R of G/Q, we denote by $I([x], R)$ the intersection of $I([x])$ with $N_{G/Q}(R)$ etc.

Lemma 7.2. *Suppose that G is a finite group, $Q \trianglelefteq G$ is a p-subgroup and $\alpha \in H^2(G/Q, k^\times)$. Suppose that $C_G(Q) \leqslant Q$. If AWC holds, then*

$$\sum_{[x]\in Q^{\mathrm{cl}}/G} \ \sum_{R\in \mathcal{S}(I([x]))/I([x])} z(k_\alpha(I([x],R)/R)) = \sum_{\mu\in \mathrm{Irr}(Q)/G} \ \sum_{R\in \mathcal{S}(I(\mu))/I(\mu)} z(k_\alpha(I(\mu,R)/R)).$$

Proof. Let $\mu \in \mathrm{Irr}(Q)$. The full inverse image of $I(\mu) \leqslant G/Q$ in G is $C_G(\mu)$ and for any p-subgroup R of $L/Q = I(\mu)$, $I(\mu, R) = N_{L/Q}(R)$. Hence, by AWC and Proposition 4.9 applied with $L = C_G(\mu)$, we have that

$$\ell(k_\alpha(kC_G(\mu)) = \sum_{R\in \mathcal{S}(I(\mu))/I(\mu)} z(k_\alpha(I(R,\mu)/R)).$$

Similarly, let $x \in Q^{\mathrm{cl}}$. The full inverse image of $I([x]) \leqslant G/Q$ in G is $C_G([x])$. Thus, by AWC and Proposition 4.9 applied with $L = C_G([x])$, we have that

$$\ell(k_\alpha(kC_G([x]) = \sum_{R\in \mathcal{S}(I([x]))/I([x])} z(k_\alpha(I(R,[x])/R)).$$

The result follows by Lemma 7.1. □

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. We recall some earlier notation. For any $\mathcal{F}$-centric $Q \leqslant S$, by Remark 4.14, we have

$$\mathbf{w}_Q(\mathcal{F},\alpha) = \sum_{\sigma\in\mathcal{N}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu\in\operatorname{Irr}_K(Q)/I(\sigma)} z(k_{\alpha_Q} C_{I(\sigma)}(\mu)) \tag{7.1}$$

$$\mathbf{w}^*_Q(\mathcal{F},\alpha) = \sum_{\sigma\in\mathcal{N}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{[x]\in Q^{\mathrm{cl}}/I(\sigma)} z(k_{\alpha} C_{I(\sigma)}([x])) \tag{7.2}$$

Also, since $\operatorname{Out}_{\mathcal{F}}(Q) = \operatorname{Out}_{N_{\mathcal{F}}(Q)}(Q)$ we have

$$\mathbf{w}_Q(\mathcal{F},\alpha) = \mathbf{w}_Q(N_{\mathcal{F}}(Q),\alpha) \quad\text{and}\quad \mathbf{w}^*_Q(\mathcal{F},\alpha) = \mathbf{w}^*_Q(N_{\mathcal{F}}(Q),\alpha). \tag{7.3}$$

Lemma 7.3. *Suppose that G is a finite group and $Q \trianglelefteq G$ is a p-subgroup with $C_G(Q) \leq Q$. Let S be a Sylow p-subgroup of G, $\mathcal{F} = \mathcal{F}_S(G)$, $\overline{G} = G/Q$ and let $\mathcal{P}_Q$ denote the set of all strictly increasing chains of p-subgroups in $\operatorname{Out}_{\mathcal{F}}(Q)$ starting at 1. Then,*

$$\boldsymbol{w}_Q(\mathcal{F},\alpha) = \sum_{\sigma\in\mathcal{P}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu\in\operatorname{Irr}(Q)/I(\sigma)} \sum_{R\in\mathcal{S}(I(\sigma,\mu))/I(\sigma,\mu)} z(k_\alpha(I(R,\sigma,\mu)/R))$$

and

$$\boldsymbol{w}^*_Q(\mathcal{F},\alpha) = \sum_{\sigma\in\mathcal{P}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{[x]\in Q^{\mathrm{cl}}/I(\sigma)} \sum_{R\in\mathcal{S}(I(\sigma,[x]))/I(\sigma,[x])} z(k_\alpha(I(R,\sigma,[x])/R)).$$

Proof. By definition

$$\mathbf{w}_Q(\mathcal{F},\alpha) = \sum_{\sigma\in\mathcal{N}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu\in\operatorname{Irr}(Q)/I(\sigma)} z(k_\alpha(I(\sigma,\mu))).$$

We claim that

$$\mathbf{w}_Q(\mathcal{F},\alpha) = \sum_{\sigma\in\mathcal{P}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu\in\operatorname{Irr}(Q)/I(\sigma)} z(k_\alpha(I(\sigma,\mu))).$$

Indeed, this follows immediately from Lemma 4.12 (or [KR89, Proposition 3.3]). Next, interchanging the order of summation on the right hand side of the above equation we obtain

$$\mathbf{w}_Q(\mathcal{F},\alpha) = \sum_{\mu\in\operatorname{Irr}(Q)/\operatorname{Out}_{\mathcal{F}}(Q)} \sum_{\sigma\in\mathcal{P}_Q/I(\mu)} (-1)^{|\sigma|} z(k_\alpha(I(\sigma,\mu))).$$

Now we claim that

$$\mathbf{w}_Q(\mathcal{F},\alpha) = \sum_{\mu\in\operatorname{Irr}(Q)/\operatorname{Out}_{\mathcal{F}}(Q)} \sum_{\sigma\in\mathcal{P}_Q/I(\mu)} \sum_{R\in\mathcal{S}(I(\sigma,\mu))/I(\sigma,\mu)} (-1)^{|\sigma|} z(k_\alpha(I(R,\sigma,\mu)/R)). \tag{7.4}$$

To prove the claim, let $\mu \in \operatorname{Irr}(Q)$ and for R a p-subgroup of $I(\mu)$, let $\mathcal{P}^R_Q$ be the subset of $\mathcal{P}_Q$ consisting of those chains which are normalised by R, i.e. those chains σ such that $R \leqslant I(\sigma)$. Then

$$\sum_{\sigma\in\mathcal{P}_Q/I(\mu)} \sum_{R\in\mathcal{S}(I(\sigma,\mu))/I(\sigma,\mu)} (-1)^{|\sigma|} z(k_\alpha(I(R,\sigma,\mu)/R))$$

is equal to

$$\sum_{R\in\mathcal{S}(I(\mu))/I(\mu)} \sum_{\sigma\in\mathcal{P}^R_Q/I(R,\mu)} (-1)^{|\sigma|} z(k_\alpha(I(R,\sigma,\mu)/R)) \ ,$$

where we use Remark 4.14 with $G = I(\mu)$, $X = \mathcal{P}_Q$, $Y = \mathcal{S}(I(\mu))$ and A equal to the subset of $X \times Y$ consisting of pairs (σ, R) such that $R \leqslant I(\mu, \sigma)$.

Suppose that $R \neq 1$ and let $\sigma = Q_0 := 1 < Q_1 < \cdots < Q_n$ be an element of $\mathcal{P}_Q^R$. If R is not contained in Q_n, let σ' be the chain obtained from σ by appending Q_nR. Otherwise, let j be the smallest integer such that R is contained in Q_j. Note that $j \neq 0$ since $R > 1$. If $Q_{j-1}R = Q_j$, then let σ' be the chain obtained from σ by deleting Q_j. Otherwise, let σ' be obtained from σ by inserting $Q_{j-1}R$ in between Q_{j-1} and Q_j. Then the pairing $\sigma \to \sigma'$ kills

$$\sum_{R \in \mathcal{S}(I(\mu))/I(\mu)} \sum_{\sigma \in \mathcal{P}_Q^R/I(R,\mu)} (-1)^{|\sigma|} z(k_\alpha(I(R,\sigma,\mu)/R)) \ .$$

Hence, only the terms with $R = 1$ survive, and the claim follows. Interchanging the order of summation in the outer two terms of Equation 7.4 gives the desired expression for $\mathbf{w}_Q(\mathcal{F}, \alpha)$. The proof for $\mathbf{w}_Q^*(\mathcal{F}, \alpha)$ is entirely similar. □

Proposition 7.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let α be an $\mathcal{F}$-compatible family. Suppose that AWC holds. Then $\boldsymbol{w}_Q(\mathcal{F}, \alpha) = \boldsymbol{w}_Q^*(\mathcal{F}, \alpha)$ for all $\mathcal{F}$-centric subgroups Q of S.*

Proof. Let $Q \leqslant S$ be $\mathcal{F}$-centric. By Equation 7.3 we may assume that $\mathcal{F} = N_{\mathcal{F}}(Q)$ and hence by [BCG+05, Proposition C] that $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with S as a Sylow p-subgroup and containing Q as a normal subgroup with $C_G(Q) = Z(Q)$. By Lemma 7.3, we have

$$\mathbf{w}_Q(\mathcal{F}, \alpha) = \sum_{\sigma \in \mathcal{P}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu \in \operatorname{Irr}(Q)/I(\sigma)} \sum_{R \in \mathcal{S}(I(\sigma,\mu))/I(\sigma,\mu)} z(k_\alpha(I(R,\sigma,\mu)/R))$$

and

$$\mathbf{w}_Q^*(\mathcal{F}, \alpha) = \sum_{\sigma \in \mathcal{P}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{[x] \in Q^{\mathrm{cl}}/I(\sigma)} \sum_{R \in \mathcal{S}(I(\sigma,[x]))/I(\sigma,[x])} z(k_\alpha(I(R,\sigma,[x])/\ R)).$$

Let $\sigma \in \mathcal{P}_Q$. By applying Lemma 7.2 to the inverse image $N_G(\sigma)$ of $I(\sigma)$ in G, we obtain

$$\sum_{\mu \in \operatorname{Irr}(Q)/I(\sigma)} \sum_{R \in \mathcal{S}(I(\sigma,\mu))/I(\sigma,\mu)} z(k_\alpha(I(R,\sigma,\mu)/R)) =$$

$$\sum_{[x] \in Q^{\mathrm{cl}}/I(\sigma)} \sum_{R \in \mathcal{S}(I(\sigma,[x]))/I(\sigma,[x])} z(k_\alpha(I(R,\sigma,[x])/\ R)).$$

The result follows. □

Proof of Theorem 1.2. This is immediate from Proposition 7.4. □

We present an alternate proof of Theorem 1.2 which is shorter but makes use of the fact, due to Robinson [Rob96], that AWC implies SOWC. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let α be an $\mathcal{F}$-compatible family. As a consequence of Lemma 4.13, the quantities $\mathbf{m}(\mathcal{F}, \alpha)$, $\mathbf{m}^*(\mathcal{F}, \alpha)$, and $\mathbf{m}(\mathcal{F}, \alpha, d)$ remain unchanged under restricting the sums over isomorphism classes of $\mathcal{F}$-centric subgroups of S to $\mathcal{F}$-centric radical subgroups. We spell this out.

Lemma 7.5. *Let Q be an $\mathcal{F}$-centric subgroup of S and let d be a non-negative integer. Suppose that Q is not $\mathcal{F}$-radical. Then*

$$\boldsymbol{w}_Q(\mathcal{F},\alpha) = \boldsymbol{w}^*_Q(\mathcal{F},\alpha) = \boldsymbol{w}_Q(\mathcal{F},\alpha,d) = 0\ .$$

Proof. Using Remark 4.14, we have

$$\mathbf{w}_Q(\mathcal{F},\alpha) = \sum_{\sigma\in\mathcal{N}_Q/\operatorname{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu\in\operatorname{Irr}(Q)/I(\sigma)} z(k_\alpha I(\sigma,\mu))$$

The quantity in the second sum depends only on $I(\sigma)$. Since Q is not radical, we have $O_p(\operatorname{Out}_{\mathcal{F}}(Q)) \neq 1$. Thus Lemma 4.13, applied to the group $G = \operatorname{Out}_{\mathcal{F}}(Q)$ and the function $\mathbf{f}$ on subgroups of G defined by

$$\mathbf{f}(H) := \begin{cases} \displaystyle\sum_{\mu\in\operatorname{Irr}(Q)/I(\sigma)} z(k_\alpha I(\sigma,\mu)) & \text{if } H = I(\sigma) \text{ for some } \sigma\in\mathcal{N}_Q \\ 0 & \text{otherwise} \end{cases}$$

implies that $\mathbf{w}_Q(\mathcal{F},\alpha) = 0$. Similar arguments show that $\mathbf{w}^*_Q(\mathcal{F},\alpha) = \mathbf{w}_Q(\mathcal{F},\alpha,d) = 0$. □

Note that by Lemma 7.5, we have

$$\mathbf{m}(\mathcal{F},\alpha) = \sum_{Q\in\mathcal{F}^{cr}/\mathcal{F}} \mathbf{w}_Q(\mathcal{F},\alpha) \quad\text{and}\quad \mathbf{m}^*(\mathcal{F},\alpha) = \sum_{Q\in\mathcal{F}^{cr}/\mathcal{F}} \mathbf{w}^*_Q(\mathcal{F},\alpha). \tag{7.5}$$

Lemma 7.6. *Suppose that $\boldsymbol{m}^*(\mathcal{G},\beta) = \boldsymbol{m}(\mathcal{G},\beta)$ for all pairs $(\mathcal{G},\beta)$, where $\mathcal{G}$ is a saturated constrained fusion system and β is a $\mathcal{G}$-compatible family. Then $\boldsymbol{m}(\mathcal{F},\alpha) = \boldsymbol{m}^*(\mathcal{F},\alpha)$.*

Proof. We prove that $\mathbf{w}_Q(\mathcal{F},\alpha) = \mathbf{w}^*_Q(\mathcal{F},\alpha)$ for each fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup $Q \leq S$. Since $\mathcal{F}$ is saturated, there is a fully $\mathcal{F}$-normalized subgroup in each $\mathcal{F}$-conjugacy class, and so the result will then follow from (7.5).

Suppose the above assertion is false, so that $\mathbf{w}_Q(\mathcal{F},\alpha) \neq \mathbf{w}^*_Q(\mathcal{F},\alpha)$ for some Q. Among all such counterexamples $\mathcal{F}$ and Q, choose one such that $|\mathcal{F}| + |S : Q|$ is minimal, where $|\mathcal{F}|$ denotes the number of morphisms in $\mathcal{F}$. Note that $\operatorname{Out}_{N_{\mathcal{F}}(Q)}(Q) = \operatorname{Out}_{\mathcal{F}}(Q)$, and Q is also fully $N_{\mathcal{F}}(Q)$-normalized, $N_{\mathcal{F}}(Q)$-radical, and $N_{\mathcal{F}}(Q)$-centric. Since the sums $\mathbf{w}_Q(\mathcal{F},\alpha)$ and $\mathbf{w}^*_Q(\mathcal{F},\alpha)$ depend only on Q and $\operatorname{Out}_{\mathcal{F}}(Q)$ and not on $\mathcal{F}$, it follows by minimality that $\mathcal{F} = N_{\mathcal{F}}(Q)$.

We have shown that $\mathcal{F}$ is constrained with normal centric subgroup Q. In particular, $\mathbf{m}(\mathcal{F},\alpha) = \mathbf{m}^*(\mathcal{F},\alpha)$ by assumption, and Q is contained in every $\mathcal{F}$-centric radical subgroup (see e. g. [LS17, Lemma 2.4]). From (7.5), $\mathbf{m}(\mathcal{F},\alpha)$ is the sum of $\mathbf{w}_Q(\mathcal{F},\alpha)$ and $\mathbf{w}_R(\mathcal{F},\alpha)$ as R ranges over the fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric radical subgroups with $R > Q$. The same holds for $\mathbf{w}^*_Q(\mathcal{F},\alpha)$. By induction $\mathbf{w}_R(\mathcal{F},\alpha) = \mathbf{w}^*_R(\mathcal{F},\alpha)$ for each such $R > Q$ (since $N_{\mathcal{F}}(R) \subsetneq \mathcal{F}$). It follows that $\mathbf{w}_Q(\mathcal{F},\alpha) = \mathbf{w}^*_Q(\mathcal{F},\alpha)$ after all, a contradiction. □

It thus suffices by Lemma 7.6 to prove $\mathbf{m}(\mathcal{F},\alpha) = \mathbf{m}^*(\mathcal{F},\alpha)$ in the case where $\mathcal{F}$ is constrained.

Proposition 7.7. *Suppose AWC holds for all blocks of all finite groups. If $\mathcal{F}$ is constrained, then $\boldsymbol{k}(\mathcal{F},\alpha) = \boldsymbol{m}(\mathcal{F},\alpha)$.*

Proof. Assume that $\mathcal{F}$ is constrained. By Proposition 4.8, we may fix a model G for $\mathcal{F}$, a p'-central extension $\widehat{G}$ of G, and a block b of $k\widehat{G}$ such that $(\mathcal{F}, \alpha)$ is realized by $k\widehat{G}b$. By Proposition 4.5, since AWC holds for all blocks, we have

$$\mathbf{k}(\mathcal{F}, \alpha) = \mathbf{k}(B).$$

On the other hand, again since AWC holds for all blocks, the results of [Rob96], [Rob04] show that $\mathbf{m}(\mathcal{F}, \alpha) = \mathbf{k}(B)$. □

Proof of Theorem 1.2. Assume AWC holds for all blocks of finite groups. By Theorem 1.1, we have $\mathbf{k}(\mathcal{F}, \alpha) = \mathbf{m}^*(\mathcal{F}, \alpha)$. Hence, $\mathbf{m}(\mathcal{F}, \alpha) = \mathbf{m}^*(\mathcal{F}, \alpha)$ whenever $\mathcal{F}$ is constrained by Proposition 7.7 and assumption. Therefore, $\mathbf{m}(\mathcal{F}, \alpha) = \mathbf{m}^*(\mathcal{F}, \alpha)$ by Lemma 7.6. □

Appendix A. On Lemma 7.1

By Proposition 4.9, Lemma 7.1 is equivalent to the following.

Lemma A.1. *Let G be a finite group, Q a normal p-subgroup of G, Z a central p'-subgroup of G and e a central idempotent of kZ. Then*

$$\sum_{[x]\in Q^{\mathrm{cl}}/G} \ell(kC_G([x])e) = \sum_{\mu\in \mathrm{Irr}(Q)/G} \ell(kC_G(\mu)e). \tag{A.1}$$

The rest of the section is devoted to a proof of Lemma A.1. The basic idea is that, when $e = 1_{kZ}$, then both sides count the number of p-sections in G of elements of Q, or the dimension of the space of ordinary class functions of G vanishing outside p-sections of elements of Q.

Notation. Let $(K, \mathcal{O}, k)$ be a p-modular system which we assume is big enough for the finite groups considered in this section. Denote by $\mathcal{C}(G)$ the K-vector space of all K-valued class functions on G and by $\mathrm{Irr}(G) \subset \mathcal{C}(G)$ the set of ordinary irreducible characters of G viewed as K-valued functions.

For $X \subset G$, denote by $d^X : \mathcal{C}(G) \to \mathcal{C}(G)$, the K-linear map defined by $\varphi \to d^X(\varphi), \varphi \in \mathcal{C}(G)$ where $d^X(\varphi)(g) = 0$ if g_p is not conjugate to an element of X and $d^X(\varphi)(g) = \varphi(g)$ otherwise. Thus, $d^X(\mathcal{C}(G))$ is the subspace of all class functions which vanish outside the p-sections of elements of X, that is those class functions φ such that $\varphi(x) = 0$ unless x_p is conjugate to an element of X.

If $X = \{x\}$ we write d^x for d^X. For general X and $x \in X$, $d^x(\mathcal{C}(G))$ is a subspace of $d^X(\mathcal{C}(G))$ and $d^X(\mathcal{C}(G)) = \oplus_x\, d^x(\mathcal{C}(G))$, where x runs over a set of conjugacy class representatives of p-elements in X. Note that if X is a normal p-subgroup of G, then $d^X\mathcal{C}(G)$ consists of precisely those functions which take the value zero on elements g such that $g_p \notin Q$.

For a central idempotent f of KG denote by $\mathrm{Irr}(G, f)$ the subset of ordinary irreducible characters of G corresponding to simple KGf modules and by $\mathcal{C}(G, f)$ the subspace of $\mathcal{C}(G)$ consisting of those class functions which are in the K-span of $\mathrm{Irr}(G, f)$. Recall that the canonical surjection $\mathcal{O}G \to kG$ induces a bijection between the set of central idempotents of $\mathcal{O}G$ and of kG. By abuse of notation, if e is a central idempotent of kG corresponding to the central idempotent $\widehat{e}$ of $\mathcal{O}G$ we write $\mathrm{Irr}(G, e)$ for $\mathrm{Irr}(G, \widehat{e})$ and $\mathcal{C}(G, e)$ for $\mathcal{C}(G, \widehat{e})$. Thus, if e is a block of kG, then $\mathrm{Irr}(G, e)$ is the subset of ordinary irreducible characters of G belonging to $\widehat{e}$. For N a normal subgroup of G and $\mu \in \mathrm{Irr}(N)$, let $\mathcal{C}(G, \mu)$ denote the subspace of

$\mathcal{C}(G)$ consisting of those class functions which are in the K-span of irreducible characters of G which cover μ and for f a central idempotent of KG (or kG) denote by $\mathcal{C}(G,\mu,f)$ the intersection of $\mathcal{C}(G,\mu)$ and $\mathcal{C}(G,f)$.

The following gives the desired interpretation of the left hand side of Lemma A.1. When $e = 1_{kZ}$, the statement is elementary. Passage to arbitrary e requires an application of Brauer's second main theorem which we now recall. Denote by $\mathrm{IBr}(G)$ the set of Brauer characters of simple kG-modules viewed as K-valued class functions on $G_{p'}$, the set of p-regular elements of G. For $x \in G$ a p-element, $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(C_G(x))$ denote by $d^x_{\chi,\varphi}$ the corresponding generalised decomposition number. By Brauer's second main theorem, if b is the block of kG containing χ, then $d^x_{\chi,\varphi}$ is zero unless φ is the Brauer character of a simple $kC_G(x)$ module lying in a block c of $kC_G(x)$ which is in Brauer correspondence with b. In other words, for all $y \in C_G(x)_{p'}$ we have that

$$\chi(xy) = \sum_{\varphi} d^x_{\chi,\varphi}\varphi(y) \ ,$$

where φ runs over the set of irreducible Brauer characters of $C_G(x)$ lying in Brauer correspondents of b.

Lemma A.2. *Let x be a p-element of G. Let $Z \leqslant G$ be a central p'-subgroup of G and e a central idempotent of kZ. Then,*

$$\dim_K d^x(\mathcal{C}(G,e)) = \ell(kC_G(x)e). \tag{A.2}$$

If Q is a normal p-subgroup of G, then

$$\dim_K d^Q(\mathcal{C}(G,e)) = \sum_{x \in Q^{cl}/G} \ell(kC_G([x])e). \tag{A.3}$$

Proof. The space $d^x(\mathcal{C}(G))$ consists of the class functions on G which vanish outside the p-section of x, hence $\dim_K d^x(\mathcal{C}(G))$ equals the number of p'-conjugacy classes of $C_G(x)$ and this number is in turn equal to the number of isomorphism classes of simple $kC_G(x)$-modules. This proves that the first equation holds when $e = 1_{kZ} = 1_{kG}$. For the general case, first note that since Z is central in G, e is a central idempotent of kG and of $kC_G(x)$ and $\mathrm{Br}_{\langle x\rangle}(e) = e$, where $\mathrm{Br}_{\langle x\rangle} : (kG)^{\langle x\rangle} \to kC_G(x)$ denotes the Brauer homomorphism. We claim that if b is a block of kG such that $be = b$ and c is a block of $kC_G(x)$ in Brauer correspondence with b, then $ce = c$. Indeed, by the uniqueness of central idempotent decompositions and the primitivity of b, we have $be = b$. By definition of Brauer correspondence, $\mathrm{Br}_{\langle x\rangle}(b)c = c$. Hence

$$c = \mathrm{Br}_{\langle x\rangle}(b)c = \mathrm{Br}_{\langle x\rangle}(be)c = \mathrm{Br}_{\langle x\rangle}(b)ec = cec = ce \ ,$$

proving the claim. It follows from the claim that all simple $kC_G(x)c$-modules are $kC_G(x)e$-modules. Thus by Brauer's second main theorem (and the linearity of d^x), if $\tau \in (\mathcal{C}(G,e))$, then for all $y \in C_G(x)_{p'}$ we have

$$\tau(xy) = \sum_{\varphi} d^x_{\chi,\varphi}\varphi,$$

where φ runs over the set of Brauer characters of simple $kC_G(x)e$=modules. Since $d^x\tau$ is determined by its restriction to the subset of $C_G(x)$ consisting of elements whose p-part is x, it follows that $\dim_K d^x(\mathcal{C}(G,e)) \leqslant \ell(kC_G(x)e)$. By the same considerations,

$\dim_K d^x(\mathcal{C}(G,1-e)) \leqslant \ell(kC_G(x)(1-e))$. Since $\mathcal{C}(G) = \mathcal{C}(G,e) \oplus \mathcal{C}(G,1-e)$, $\dim_K d^x(\mathcal{C}(G)) \leqslant \dim_K d^x(\mathcal{C}(G,e)) + \dim_K d^x(\mathcal{C}(G,1-e))$. The first equation now follows from the case $e = 1_{kZ}$.

Let $\overline{e}$ be the image of e under the canonical surjection of $kG \to k(G/Q)$. Recall that restriction along $kG \to kG/Q$ induces a bijection between the set of isomorphism classes of simple kG/Q-modules and kG-modules sending simple $k(G/Q)\overline{e}$-modules to kGe-modules. Also, for any $x \in Q$, e is a central idempotent of $kC_G(x)$ and identifying $C_G(x)/C_G(x) \cap Q$ with $C_G(x)Q/Q$ via the isomorphism induced by inclusion of $C_G(x)$ in $C_G(x)Q$, the image of e in $k(C_G(x)/C_G(x) \cap Q)$ is $\overline{e}$. Hence

$$\ell(kC_G([x])e) = \ell(kC_G(x)Qe) = \ell(k(C_G(x)Q/Q)\overline{e}) = \ell(k(C_G(x)/C_G(x) \cap Q)\overline{e}) = \ell(kC_G(x)e).$$

Now the second equation follows from the first since

$$d^Q(\mathcal{C}(G,e)) = \bigoplus_{[x] \in Q^{cl}/G} d^x(\mathcal{C}(G,e)).$$

□

Lemma A.3. *Let Z be a central p'-subgroup of G and e a central idempotent of kZ. Let Q be a normal p-subgroup of G. Then*

$$d^Q(\mathcal{C}(G,e)) = \bigoplus_{\mu \in \mathrm{Irr}(Q)/G} d^Q(\mathcal{C}(G,\mu,e)). \tag{A.4}$$

Proof. Since

$$\mathcal{C}(G) = \bigoplus_{\mu \in \mathrm{Irr}(Q)/G} \mathcal{C}(G,\mu),$$

we have

$$d^Q(\mathcal{C}(G)) = \sum_{\mu \in \mathrm{Irr}(Q)/G} d^Q(\mathcal{C}(G,\mu)).$$

We show that the sum on the right of the second equation is direct. First note that if φ is an element of $\mathcal{C}(G,Q)$, then $d^Q(\varphi) = 0$ if and only if the restriction of φ to all subgroups H containing Q as a Sylow p-subgroup equals zero. Now suppose that $\varphi_\mu \in \mathcal{C}(G,\mu)$, $\mu \in \mathrm{Irr}(Q)/G$ are such that $\sum_{\mu \in \mathrm{Irr}(Q)/G} d^Q(\varphi_\mu) = 0$ and let H be a subgroup of G containing Q as a Sylow p-subgroup. Then the restriction of $\sum_{\mu \in \mathrm{Irr}(Q)/G} \varphi_\mu = 0$. But it is easy to see that the restriction of φ_μ to H is in the K-span of irreducible characters of H which cover G-conjugates of μ. In particular the restriction of φ_μ and $\varphi_{\mu'}$ for $\mu' \neq \mu$ are orthogonal class functions on H. Hence the restriction of φ_μ to H equals zero for all H and all μ. It follows that $d^Q(\varphi_\mu) = 0$ for all μ. Thus

$$d^Q(\mathcal{C}(G)) = \bigoplus_{\mu \in \mathrm{Irr}(Q)/G} d^Q(\mathcal{C}(G,\mu)). \tag{A.5}$$

The assertion of the lemma now follows as $\mathcal{C}(G,e)$ is the direct sum $\bigoplus_{\mu \in \mathrm{Irr}(Q)/G} \mathcal{C}(G,\mu,e)$. □

Given the above Lemma, it remains to analyse $d^Q(\mathcal{C}(G,\mu,e))$ for each irreducible character μ of Q. This is done via standard Clifford theoretic reductions.

Lemma A.4. *Let Z be a central p'-subgroup of G and e a central idempotent of kZ. Let Q be a normal p-subgroup of G and let $\mu \in \mathrm{Irr}(Q)$. Then $\dim_K d^Q(\mathcal{C}(G,\mu,e)) = \dim_K d^Q(\mathcal{C}(C_G(\mu),\mu,e))$.*

Proof. Induction from $C_G(\mu)$ to G induces a bijection between $\mathrm{Irr}(\mathcal{C}_G(\mu))$ and $\mathrm{Irr}(G)$. Since $Z \leqslant C_G(\mu)$, if $\chi \in \mathrm{Irr}(G,\mu,e)$, then $\mathrm{Ind}_{C_G(\mu)}^G(\chi) \in \mathrm{Irr}(G,\mu,e)$. Hence induction induces an isometric isomorphism between $\mathcal{C}(C_G(\mu),\mu,e)$ and $\mathcal{C}(G,\mu,e)$. Further, it is easy to check from the induction formula that $d^Q(\mathrm{Ind}_{C_G(\mu)}^G(\tau)) = \mathrm{Ind}_{C_G(\mu)}^G(d^Q(\tau))$ for all τ in $\mathcal{C}(C_G(\mu))$. The result follows. □

Lemma A.5. *Let Q be a normal p-subgroup of G and let μ be a G-stable irreducible character of Q. There exist a central extension*

$$1 \to Y \to \widetilde{G} \xrightarrow{\pi} G \to 1,$$

an irreducible character $\widetilde{\mu}$ of $\widetilde{G}$ and a one dimensional character η of Y such that the following holds.

1. *Y is a finite p-group, the inverse image of Q in $\widetilde{G}$ is a direct product of Y with a normal subgroup Q' of $\widetilde{G}$ such that π maps Q' isomorphically onto Q.*
2. *Identifying Q' with Q through π, there exists a bijection*
$$\mathrm{Irr}(G,\mu) \to \mathrm{Irr}(\widetilde{G},\eta^{-1}1_Q), \quad \chi \to \chi_0$$
such that for any $g \in G$ and $\widetilde{g} \in \widetilde{G}$ lifting g
$$\chi(g) = \widetilde{\mu}(\widetilde{g})\chi_0(\widetilde{g}).$$
3. *Suppose that Z is a central p'-subgroup of G, and e is a central idempotent of kZ. Let $\widetilde{Z}$ be the inverse image of Z in $\widetilde{G}$. Then $\widetilde{Z} = Y \times Z'$, where Z' is a central p'-subgroup of $\widetilde{G}$ mapping isomorphically onto Z by π. Identifying Z' with Z the bijection $\chi \to \chi_0$ restricts to a bijection between $\mathrm{Irr}(G,\mu,e)$ and $\mathrm{Irr}(\widetilde{G},\eta^{-1}1_Q,e)$.*

Proof. The proof combines elements of standard Clifford theory. We briefly sketch the basic constructions. Let m be the dimension of μ and let e_μ be the central idempotent of KQ corresponding to μ Then $S = KQe_\mu$ is a matrix algebra of dimension m^2. Since μ is G-stable, the conjugation action of G on KG induces an action of G on S. The group $\widetilde{G}$ is constructed as a subgroup of $G \times S^\times$. Let $\pi : G \times S^\times \to G$ and $\pi' : G \times S^\times$ be the projections onto the first and second components respectively and identify K with the scalar matrices in S. Let $\widehat{G}$ be the the subgroup of $G \times S^\times$ consisting of all elements of the form (x,s), $x \in G$ and $s \in S^\times$ such that $s_x a s_x^{-1} = xax^{-1}$ for all $a \in S$. Since the action of each element of G on S is by an inner automorphism and $K = Z(S)$, the restriction of π to $\widehat{G}$ is a surjective homomorphism with kernel $1 \times K^\times$.

Choose a transversal I for Q in G containing $O_{p'}(G)$. In particular, I contains every central p'-element of G. For each $x \in I$, choose $s_x \in S^\times$ such that $(x,s_x) \in \widehat{G}$ and such that the determinant $\det(s_x)$ of s_x equals 1. This can be achieved by replacing K by a suitable extension containing the m-th roots of $\det(s_x)$, $x \in G$. Further, if $z \in I$ is a central p'-element of G, we choose s_z to be the identity. Extend the map $x \to s_x$ to $s : G \to S^\times$ by setting $s_g = us_x$ if $g = ux$, $u \in Q$, $x \in I$. For all $g,h \in G$, we have $s_g s_h s_{gh}^{-1} \in K^\times$ is a scalar matrix. Note that since $u^{|Q|} = 1$ for all $y \in Q$, we have that $\det(s_g)^{|Q|} = 1$ for all $g \in G$ and consequently by taking determinants we see that $(s_g s_h s_{gh}^{-1})^{m^2|Q|} = 1$ for all $g,h \in G$.

Let $\widetilde{G}$ be the subgroup of $\widehat{G}$ generated by (s_g, g), $g \in G$. The restriction $\pi : \widetilde{G} \to G$ of pi to $\widetilde{g}$ is surjective. Let $Y \leqslant 1 \times K^\times$ be the kernel of π. For $g, h \in \widetilde{G}$,

$$(g, s_g)(h, s_h) = (gh, s_g s_h) = (1, s_g s_h s_{gh}^{-1})(gh, s_{gh}),$$

$$(g, s_g)^{-1} = (1, s_g s_{g^{-1}})(g^{-1}, s_{g^{-1}}) = (1, s_g s_{g^{-1}} s_{gg^{-1}})(g^{-1}, s_{g^{-1}}).$$

It follows that $Y = \langle (1, s_g s_h s_{gh}^{-1}), g, h \in G \rangle$. As noted above, Y has exponent dividing $m^2|Q|$. Since Y is isomorphic to a subgroup of the multiplicative group of a field, Y is cyclic of order dividing $m^2|Q|$. In particular, Y is a finite p-group. Let $Q' = \{(u, s_u) : u \in Q\}$. Since $s_u s_v = uv$ for all $u, v \in Q$, Q' is a subgroup of $\widetilde{G}$ with the required properties. This proves (1).

Let $\eta : Y \to K^\times$ be the irreducible character of Y which sends $(1, \lambda \cdot \mathrm{id}_S)$ to λ. The map $\pi' : \widetilde{G} \to S^\times$ defines a representation of $\widetilde{G}$ whose restriction to YQ equals $\eta\mu$. Let $\widetilde{\mu}$ be the corresponding character. Then $\widetilde{\mu}$ is irreducible and covers $\eta\mu$. Let $\tau = \frac{1}{|Y||Q|}\sum_{y \in Y, u \in Q} \eta^{-1}(y)(uy)^{-1}$ be the central idempotent of KYQ corresponding to $\eta^{-1}1_Q$. There is a K-algebra isomorphism

$$\varphi : KGe_\mu \to S \otimes_K K\widetilde{G}\tau \tag{A.6}$$

satisfying

$$\varphi(ge_\mu) = s_g \otimes (g, s_g)\tau, \quad g \in G.$$

Let $g \in G$ and let $\widetilde{g} \in \widetilde{G}$ be a lift of g. Then $\widetilde{g} = y(g, s_g)$ for some $y \in Y$. Since $ys_g = \eta(y)s_g$ and $y(g, s_g)\tau = \eta^{-1}(g, s_g)\tau$ it follows that $s_g \otimes (g, s_g)\tau = \pi_2(\widetilde{g}) \otimes \widetilde{g}\tau$. Now (2) follows since $\mathrm{Irr}(\widetilde{G}, \eta^{-1}1_Q)$ coincides with the set of irreducible $K\widetilde{G}\tau$ characters.

Let Z be a central p'-subgroup of G. By our choices above, s_z is the identity matrix for all $z \in Z$. Hence $Z' := \{(z, 1) : z \in Z\}$ is a central subgroup of $\widetilde{G}$ and the inverse image $\widetilde{Z}$ of Z in $\widetilde{G}$ is a direct product $\widetilde{Z} = Y \times Z'$. Identifying Z' with Z, the image of the idempotent ee_μ under the isomorphism A.6 is $\mathrm{id_S} \otimes e\tau$, proving (3). □

Lemma A.6. *Let Z be a central p'-subgroup of G and e a central idempotent of kZ. Let Q be a normal p-subgroup of G and let μ be a G-stable irreducible character of Q. Then*

$$\dim_K d^Q(\mathcal{C}(G, \mu, e)) = \ell(kGe). \tag{A.7}$$

Proof. Let $\widetilde{G}$, Y, η and $\widetilde{\mu}$ be as in Lemma A.5. The bijection $\chi \to \chi_0$ extends by linearity to a K-linear isomorphism $i : \mathcal{C}(G, \mu, e) \to \mathcal{C}(\widetilde{G}, \eta^{-1}1_Q, e)$ defined by

$$\varphi(g) = \widetilde{\mu}(\widetilde{g})i(\varphi)(\widetilde{g}), \quad i^{-1}(\psi)(g) = \widetilde{\mu}(\widetilde{g})\psi(\widetilde{g})$$

for all $\varphi \in \mathcal{C}(G, \mu, e)$, $\psi \in \mathcal{C}(\widetilde{G}, \eta^{-1}1_Q, e)$. $g \in G$ and $\widetilde{g} \in \widetilde{G}$ lifting $\widetilde{g}$. Now $g_p \in Q$ if and only if $(\widetilde{g})_p \in YQ$. It follows that

$$i^{-1} \circ d^{YQ} \circ i = d^Q,$$

hence

$$d^{YQ} \circ i = i \circ d^Q,$$

where by d^{YQ} we mean the relevant map on class functions on $\widetilde{G}$. In particular, $\dim_K d^Q(\mathcal{C}(G, \mu, e)) = \dim_K d^Q(\mathcal{C}(\widetilde{G}, \eta^{-1}1_Q, e)$.

Let $\psi \in \mathcal{C}(\widetilde{G}, \eta^{-1}1_Q, e)$. For any $u \in Q$, $y \in Y$, $\widetilde{g} \in \widetilde{G}$, we have $\psi(yu\widetilde{g}) = \eta(y)\psi(\widetilde{g})$ from which it follows that

$$\dim_K d^{YQ}\mathcal{C}(\widetilde{G}, \eta^{-1}1_Q, e) = \dim_K d^1\mathcal{C}(\widetilde{G}, e) = \ell(k\widetilde{G}e) = \ell(kGe)$$

where the second equality holds by Lemma A.2 and the last equality holds since every simple $k\widetilde{G}e$-module has Y in its kernel. □

Proof of Lemma A.1. This follows from Lemmas A.2, A.3, A.4 and A.6. □

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