

Brauer graph algebras

A survey on Brauer graph algebras, associated gentle algebras and their connections to cluster theory

Sibylle Schroll

November 1, 2017

Abstract

These lecture notes on Brauer graph algebras are the result of a series of four lectures given at the CIMPA research school in Mar del Plata, Argentina, in March 2016. After motivating the study of Brauer graph algebras by relating them to special biserial algebras, the definition of Brauer graph algebras is given in great detail with many examples to illustrate the concepts. This is followed by a short section on the interpretation of Brauer graphs as decorated ribbon graphs. A section on gentle algebras and their graphs, trivial extensions of gentle algebras, admissible cuts of Brauer graph algebras and a first connection of Brauer graph algebras with Jacobian algebras associated to triangulations of marked oriented surfaces follows. The interpretation of flips of diagonals in triangulations of marked oriented surfaces as derived equivalences of Brauer graph algebras and the comparison of derived equivalences of Brauer graph algebras with derived equivalences of frozen Jacobian algebras is the topic of the next section. In the last section, after defining Green's walk around the Brauer graph, a complete description of the Auslander-Reiten quiver of a Brauer graph algebra is given.

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Introduction

Brauer graph algebras originate in the modular representation theory of finite groups where they first appear in the form of Brauer tree algebras in the work of Janusz [50] based on work of Dade [23]. Brauer graph algebras in general, are defined by Donovan and Freislich in [28]. In particular, they classify the indecomposable representations of a Brauer graph algebra in terms of canonical modules of the first and second kind (string and band modules respectively). This classification is based on the work of Ringel [67] on indecomposable representations of dihedral 2-groups and the work of Gel'fand and Ponomarev [40] on indecomposable representations of the Lorentz group.

Brauer graph algebras are defined by combinatorial data based on graphs: Underlying every Brauer graph algebra is a finite graph with a cyclic orientation of the edges at every vertex and a multiplicity function. This combinatorial data encodes much of the representation theory of Brauer graph algebras and is part of the reason for the ongoing interest in this class of algebras.

In [61] the point of view of interpreting Brauer graphs as ribbon graphs has been introduced adding a geometric perspective to the representation theory of Brauer graph algebras and relating Brauer graph algebras with surface cluster theory. The idea of relating Brauer graphs to surfaces was already suggested in [28] where a description of Brauer graphs is given as graphs embedded in oriented surfaces. In [1] the geometric approach based on ribbon graphs has been used to classify two-term tilting complexes over Brauer graph algebras.

The class of Brauer graph algebras coincides with the class of symmetric special biserial algebras [74, 77]. This connection has introduced string combinatorics to the subject as well as a large body of literature on biserial and special biserial algebras, for an overview see for example, the webpages maintained by Julian Külshammer at <http://www.iaz.uni-stuttgart.de/LstAGeoAlg/Kuelshammer/topics/biserial.html> and by Jan Schröer at <http://www.math.uni-bonn.de/people/schroer/fd-atlas.html>.

In recent years there has been a renewed interest in Brauer graph algebras. In addition to the results mentioned in these lecture notes, there are new results on 2-term tilting complexes of Brauer graph algebras [1, 86, 87], on Brauer graph algebras associated to partial triangulations [25], on coverings of Brauer graphs [45], on the finite generation of the Yoneda (or Ext) algebra of a Brauer graph algebra [5, 46] and the generalised Koszulity of Brauer graph algebras [46], as well as new results on the non-periodicity of modules of finite complexity over weakly symmetric Brauer graph algebras [35]. Note also that Brauer graph algebras play a central role in the recent survey on the connection between the representation theory of finite groups and the theory of cluster algebras [59].

On the other hand from the point of view of modular representation theory of

finite groups, there has been much recent work to identify specific Brauer trees and Brauer graphs arising in that context, see for example [30, 31, 32, 21].

In these lecture notes, after motivating the study of Brauer graph algebras from the perspective of symmetric special biserial algebras in Section 1, in Section 2 we give a detailed definition of Brauer graph algebras. Section 3 focuses on the connection of Brauer graph algebras with gentle algebras and the connection of Brauer graph algebras with surface cluster theory. In Section 4, mutation of Brauer graph algebras and derived equivalences are discussed. Section 5 describes the Auslander-Reiten theory of Brauer graph algebras.

Acknowledgements. At the origin of these lecture notes is a series of four lectures on Brauer graph algebras and cluster theory that I gave at the CIMPA research school in Mar del Plata in March 2016 and my heartfelt thanks goes to the organisers, Ibrahim Assem, Patrick Le Meur and Sonia Trepode, as well as to the scientific and local organising committees. I also particularly thank Cristian Arturo Chaparro Acosta who contributed all of the tikz-pictures in these lecture notes (mostly at very short notice). Without his extensive tikz skills these lecture notes would not have been possible in this form. I thank Sefi Ladkani for providing me with a summary of the quoted results of his papers [57, 59] and for pointing out the links of this survey with his survey on triangulation quivers [59]. I further thank him, Drew Duffield, Julian Külshammer, Eduardo Marcos, Klaus Lux, Rowena Paget, Rachel Taillefer as well as two anonymous referees for taking the time to read these lecture notes and for the helpful comments and suggestions that they have provided.

1 Motivation and Connections

1.1 Special biserial algebras

Let K be an algebraically closed field. A quiver $Q = (Q_0, Q_1, s, t)$ is given by a set of vertices Q_0 , a set of arrows Q_1 , a function $s : Q_1 \rightarrow Q_0$ denoting the start of an arrow and $t : Q_1 \rightarrow Q_0$ denoting the end of an arrow. A path in Q is a sequence of arrows $p = \alpha_0 \alpha_1 \dots \alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$, for all $0 \leq i \leq n - 1$. We set $s(p) = s(\alpha_0)$ and $t(p) = t(\alpha_n)$. Furthermore, we denote by $\ell(p)$ the length of p , that is $\ell(p) = n$.

The *path algebra* of a quiver KQ is the algebra whose underlying vector space has a basis given by all possible paths in the quiver Q . This includes a trivial path e_v for every vertex $v \in Q_0$. The multiplication is given by linearly extending the multiplication of two paths p and q in Q , which is given by concatenation if possible and zero otherwise. That is,

$$p \cdot q = \begin{cases} pq & \text{if } t(p) = s(q) \\ 0 & \text{otherwise.} \end{cases}$$

The algebra KQ has been extensively studied and we refer the reader to classical representation theory textbooks, see for example [8, 9, 11, 12, 75]. It follows directly from the definition, that if the quiver Q has a loop or an oriented cycle, the algebra KQ is infinite dimensional.

Denote by \mathcal{R} the two-sided ideal of KQ generated by the arrows in Q . We call a two-sided ideal I of KQ *admissible* if there exists a strictly positive integer $n \geq 2$ such that

$$\mathcal{R}^n \subseteq I \subseteq \mathcal{R}^2.$$

If I is an admissible ideal of KQ , then the algebra KQ/I is a finite dimensional algebra and we say that KQ/I is given by quiver and relations. The algebra KQ/I is indecomposable if and only if the quiver Q is connected.

The motivation behind studying the representation theory of algebras by studying algebras given by quiver and relations is the following theorem due to Gabriel.

Theorem 1.1. [39] *Every connected finite dimensional K -algebra is Morita equivalent to an algebra KQ/I for a unique quiver Q and where I is an admissible ideal of KQ .*

The algebras of the form KQ/I , I admissible, are still arbitrarily complicated and therefore subclasses of these algebras are often considered. For example, to restrict the class of finite dimensional algebras considered, further restrictions on Q and on I can be imposed.

One example is the restriction of the number of arrows starting and ending at each vertex in the quiver. If at each vertex $v \in Q_0$ there is at most one arrow starting and at most one arrow ending at v , then KQ/I is monomial and of *finite representation type* that is, up to

isomorphism, there are only finitely many distinct indecomposable KQ/I -modules. These algebras are the so-called *Nakayama algebras* [62, 9, 8]. An algebra KQ/I is *monomial*, if the ideal I is generated by paths. Note that this definition depends on the presentation of the algebra, that is on the choice of the ideal I . In general, there might be many different ideals J , such that $KQ/I \simeq KQ/J$. It is an open question, raised by Auslander, to find a homological characterisation that implies that a given algebra is monomial. Nevertheless, monomial algebras have been extensively studied and many open problems have been answered in the case of monomial algebras.

The next level of complexity is to allow at most two arrows to begin and end at every vertex in Q :

(S0) At every vertex v in Q there are at most two arrows starting at v and there are at most two arrows ending at v .

While this is a very strong restriction on the algebras one can consider, most algebras whose quiver satisfies condition (S0) are of *wild representation type* (see Section 5 for the definition of the representation type of an algebra).

To pose a further restriction on the class of algebras we consider, we set

(S1) For every arrow α in Q there exists at most one arrow β such that $\alpha\beta \notin I$ and there exists at most one arrow γ such that $\gamma\alpha \notin I$.

Almost all algebras satisfying condition (S1) (and not necessarily condition (S0)) are of wild representation type. In fact, algebras satisfying (S1) are called *special multiserial algebras*. They were first defined in [82] and their representation theory has been studied in [82, 41, 42, 43, 44].

Together conditions (S0) and (S1) are very strong and the corresponding class of algebras has many special properties.

Definition 1.2. A finite dimensional K -algebra A is called *special biserial* if there is a quiver Q and an admissible ideal I in KQ such that A is Morita equivalent to KQ/I and such that KQ/I satisfies conditions (S0) and (S1).

Examples of special biserial algebras include the algebras appearing in the work of Gel'fand and Ponomarev [40] on the classification of Harish-Chandra modules over the Lorentz group and they are closely linked to the modular representation theory of finite groups, see for example, Ringel's classification of the indecomposable modules over dihedral 2-groups, see e.g. [67]. Other examples of special biserial algebras are string algebras (monomial special biserial algebras) see for example [18], discrete derived algebras (classified by Vossieck in [83]), Jacobian algebras of triangulations of marked oriented surfaces with boundary where all marked points lie in the boundary [7] and Brauer graph algebras (see Theorem 2.8).

Special biserial algebras have been intensely studied. We now give a list of some of their most important properties and results.

Theorem 1.3. [84] *Special biserial algebras are of tame representation type.*

Wald and Waschbüsch show that special biserial algebras are of tame representation type by classifying their indecomposable representations. These are given by the so-called *string and band modules* (which first appear in [40] as modules of the first and second kind, see also [67, 28] and also [84]). Based on strings and bands a nice combinatorial description of the morphism between string modules is given in [22, 54].

Furthermore, Wald and Waschbüsch [84] and Butler and Ringel [18] give a combinatorial description of the irreducible maps between string modules and between band modules in terms of string combinatorics giving a description of Auslander-Reiten sequences.

Definition 1.4. A finite dimensional K -algebra A is called *biserial* if every indecomposable projective left and right A -module P is such that $\text{rad}(P) = U + V$, where U and V are uniserial modules and $U \cap V$ is either a simple A -module or zero.

Theorem 1.5. [81] *A special biserial algebra is biserial.*

Pogorzaly and Skowroński prove a partial converse in case that the algebra is *standard*, that is if it admits a simply connected Galois covering, see e.g. [80].

Theorem 1.6. [65] *Let A be a representation-infinite self-injective K -algebra. Then A is standard biserial if and only if A is special biserial.*

Recall that a linear form $f : A \rightarrow K$ is *symmetric* if $f(ab) = f(ba)$ for all $a, b \in A$ and that a finite dimensional K -algebra A is *symmetric* if there exists a symmetric linear form $f : A \rightarrow K$ such that the kernel of f contains no non-zero left or right ideal. An equivalent formulation of this is that A considered as an A - A -bimodule is isomorphic to its K -linear dual $D(A) = \text{Hom}_K(A, K)$ as an A - A -bimodule. More details on equivalent definitions can be found, for example, in [73].

Every finite dimensional K -algebra A is a quotient of a symmetric K -algebra, for example, A is a quotient of its trivial extension $T(A)$, where $T(A)$ is given by the semidirect product of A with its minimal injective co-generator $D(A)$ (see Section 3 for more details on trivial extensions). However, the class of special biserial algebras is special in that every special biserial algebra is a quotient of a symmetric algebra, such that the symmetric algebra is again special biserial.

Theorem 1.7. [84] *Every special biserial algebra is a quotient of a symmetric special biserial algebra.*

Symmetric special biserial algebras are well understood and are ubiquitous in the modular representation theory of finite groups where they have appeared under a different name, that of Brauer graph algebras. While we will define Brauer graph algebras in Section 2, we state the connection of special biserial algebras and Brauer graph algebras here.

Theorem 1.8. [74, 77] *Let KQ/I be a finite dimensional K -algebra. Then KQ/I is a symmetric special biserial algebra if and only if KQ/I is a Brauer graph algebra.*

This Theorem was proved in [74] for the case that the algebra KQ/I is such that the quiver Q contains no parallel arrows. In [77] it was proved that the result holds for all quivers. The following directly follows from Theorems 1.7 and 1.8

Corollary 1.9. *Every special biserial algebra is a quotient of a Brauer graph algebra.*

2 Brauer graph algebras

Recall from the introduction that Brauer graph algebras have their origin in the modular representation theory of finite groups. We will start by briefly recalling how they appear in this context.

2.1 Brauer graph algebras and modular representation theory of finite groups

Let G be a finite group. Suppose that the characteristic of K is equal to some prime number p and suppose that p divides the order of G . Then it follows from Maschke's Theorem that the group algebra KG is not semi-simple. Instead it decomposes into a direct sum of indecomposable two-sided ideals B_i such that $KG = B_1 \oplus \dots \oplus B_n$. The identity element e of G decomposes as $e = e_1 + \dots + e_n$ where the e_i are orthogonal central idempotents of KG such that $e_i \in B_i$. Each B_i is called a p -block of G (or of KG) and it is a symmetric finite dimensional K -algebra with identity element e_i . Each p -block has an associated invariant in the form of a p -group D_i called the *defect group of B_i* . Dade showed that if the defect group D of a p -block B of some finite group G is cyclic then B is a Brauer tree algebra [23] and Donovan showed that if the characteristic K is two and if D is a dihedral 2-group then B is a Brauer graph algebra [27]. We note that the blocks with cyclic defect groups are precisely the p -blocks of G which are of finite representation type.

2.2 Definition of Brauer graphs

Definition 2.1. A *Brauer graph* G is a tuple $G = (G_0, G_1, m, \mathfrak{o})$ where

- (G_0, G_1) is a finite (unoriented) connected graph with vertex set G_0 and edge set G_1 ,
- $m : G_0 \rightarrow \mathbb{Z}_{>0}$ is a function, called the *multiplicity* or *multiplicity function* of G ,
- \mathfrak{o} is called the *orientation* of G and is given, for every vertex $v \in G_0$, by a cyclic ordering of the edges incident with v such that if v is a vertex incident to a single edge i then if $m(v) = 1$, the cyclic ordering at v is given by i and if $m(v) > 1$ the cyclic ordering at v is given by $i < i$.

A *Brauer tree* is a Brauer graph $G = (G_0, G_1, m, \mathfrak{o})$ such that (G_0, G_1) is a tree and $m(v) = 1$, for all but at most one $v \in G_0$.

We note that the graph (G_0, G_1) which (by abuse of notation) we will also simply refer to as G may contain loops and multiple edges. Denote by $\text{val}(v)$ the *valency* of the vertex $v \in G_0$. It is defined to be the number of edges in G incident to v , with the convention that a loop is counted twice (see Example 2.2 (2)). Equivalently we can define $\text{val}(v)$ to be the number of half-edges incident with vertex v . We call the edge i with vertex v *truncated at v* if $m(v)\text{val}(v) = 1$. Note that if i is truncated at both vertices v and w , that is if both $m(v)\text{val}(v) = 1$ and $m(w)\text{val}(w) = 1$ then G is the Brauer graph given by a single edge with both vertices v and w of multiplicity 1 and the corresponding Brauer graph algebra is defined to be $k[x]/(x^2)$.

Examples 2.2. In all four examples below, the orientation of the Brauer graph is given by locally embedding each vertex of the Brauer graph into the clockwise oriented plane. If for some $v \in G_0$, we have $m(v) > 1$, then we record the value of $m(v)$ in a square box next to the corresponding vertex on the graph.

(1) This is an example of what is called a *generalised Brauer tree*, that is the underlying graph is a tree with at least two vertices with multiplicity greater than one. For example, let $G = (G_0, G_1, m, \mathfrak{o})$ be given by

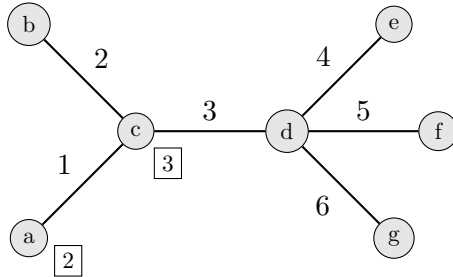


Figure 1: Brauer tree G_1 .

We set $m(i) = 1$ for all $i \in G_0$, $i \neq a, c$ and $m(a) = 2$ and $m(c) = 3$. The cyclic ordering of the edges incident with vertex a is given by $1 < 1$, with vertex b it is given by 2, with vertex c it is given by $1 < 2 < 3 < 1$, with vertex d it is given by $3 < 4 < 5 < 6 < 3$, etc. We have the following valencies for vertices of G : $\text{val}(a) = \text{val}(b) = \text{val}(e) = \text{val}(f) = \text{val}(g) = 1$, $\text{val}(c) = 3$ and $\text{val}(d) = 4$.

(2) Let the Brauer graph $G_2 = (G_0, G_1, m, \mathfrak{o})$ be given by

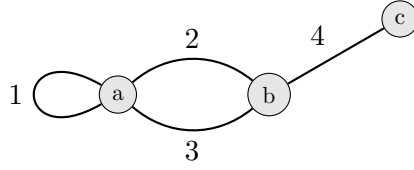


Figure 2: Brauer graph G_2 .

We set $m(i) = 1$ for $i = a, b, c$. The cyclic ordering of the edges incident with vertex a is given by $1 < 1 < 2 < 3 < 1$, with vertex b is given by $2 < 4 < 3 < 2$ and with c is given by 4. We have $\text{val}(a) = 4$, $\text{val}(b) = 3$ and $\text{val}(c) = 1$. Note that the edge 4 is a truncated edge since the vertex c is such that $m(c)\text{val}(c) = 1$.

(3) Let the Brauer graph $G_3 = (G_0, G_1, m, \sigma)$ be given by

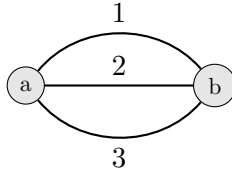


Figure 3: Brauer graph G_2 .

We set $m(a) = m(b) = 1$. The cyclic ordering at vertex a is given by $1 < 2 < 3 < 1$ and at vertex b by $1 < 3 < 2 < 1$ and $\text{val}(a) = \text{val}(b) = 3$.

(4) Let the Brauer graph $G_4 = (G_0, G_1, m, \sigma)$ be given by

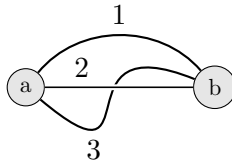


Figure 4: Brauer graph G_4 .

We set $m(a) = m(b) = 1$. The cyclic ordering at vertex a is given by $1 < 2 < 3 < 1$ and at vertex b by $1 < 2 < 3 < 1$ and $\text{val}(a) = \text{val}(b) = 3$.

2.3 Quivers from Brauer graphs

Let a be a vertex of some Brauer graph G , and let i_1, i_2, \dots, i_n be the edges in G incident with a (note that we might have $i_j = i_k$, for some j, k , if the corresponding edge is a

loop). Suppose that the cyclic ordering at a is given by $i_1 < i_2 < \dots < i_n < i_1$. We call i_1, i_2, \dots, i_n the *successor* sequence at a and i_{j+1} is the successor of i_j , for $1 \leq j \leq n-1$ and i_1 is the successor of i_n , and i_{k-1} is the *predecessor* of i_k for $2 \leq k \leq n$ and i_n is the predecessor of i_1 .

Note that if v is a vertex at edge i with $\text{val}(v) = 1$ and if $m(v) > 1$ then $i > i$ and the successor (and predecessor) of i is i , if $m(v) = 1$ then i does not have a successor or predecessor.

In order to specify more precisely the cyclic ordering at a given vertex, especially if there is a loop at that vertex, half-edges are often used. In particular, the language of *ribbon graphs*, see Section 2.7, introduced to Brauer graph algebras in [61], has the advantage of making the notion of half-edge and their cyclic orderings very precise.

Given a Brauer graph $G = (G_0, G_1, m, \sigma)$ we define a quiver $Q_G = (Q_0, Q_1)$ in the following way. The set of vertices Q_0 is given by the set of edges G_1 of G , denoting the vertex in Q_0 corresponding to the edge i in G_1 also by i . The arrows in Q are induced by the orientation σ . More precisely, let i and j be two edges in G_0 incident with a common vertex v and such that j is a direct successor of i in the cyclic ordering of the edges at v . Then there is an arrow $\alpha : i \rightarrow j$ in Q_G .

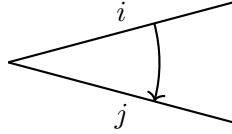


Figure 5: Arrow α defined by successor relation (i, j) .

We say that α is given by the *successor relation* (i, j) . Since every arrow of Q_G starts and ends at an edge of G , there are at most two arrows starting and ending at every vertex of Q_G . Every vertex $v \in G_0$ such that $m(v)\text{val}(v) \geq 2$, gives rise to an oriented cycle C_v in Q_G , which is unique up to cyclic permutation. We call C_v a *special cycle* at v . Let C_v be such a special cycle at v . Then if C_v is a representative in its cyclic permutation class such that $s(C_v) = i = t(C_v)$, $i \in Q_0$, we say that C_v is a *special i -cycle* at v . To simplify notation we will simply write C_v for the special i -cycle at v and specify if necessary that $s(C_v) = i$ or if more detail is needed, we will specify the first or last arrow in C_v .

For $i \in Q_0$ and $v \in G_0$, a special i -cycle at v is not necessarily unique, however, there are at most two special i -cycles at v for any $i \in G_1$ and $v \in G_0$ and this happens exactly when i is a loop. An example of this is given in Example 2.3(2).

Examples 2.3. In this example we give the quivers of the Brauer graphs in Example 2.2 and list some of the special cycles.

(1)

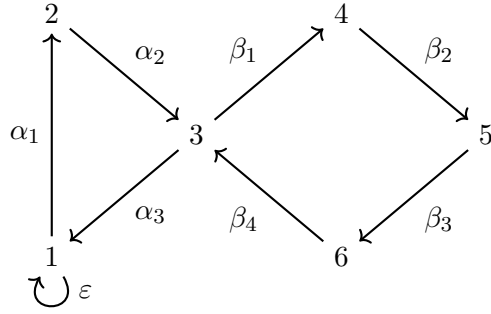


Figure 6: Quiver Q_{G_1} of G_1 .

Special cycles in Q_{G_1} are given by $C_a = \epsilon$, the special 1-cycle at c given by $\alpha_1\alpha_2\alpha_3$, the special 2-cycle at c given by $\alpha_2\alpha_3\alpha_1$, the special 3-cycle at c given by $\alpha_3\alpha_1\alpha_2$, and the special 3-cycle at d given by $\beta_1\beta_2\beta_3\beta_4$, etc.

(2)

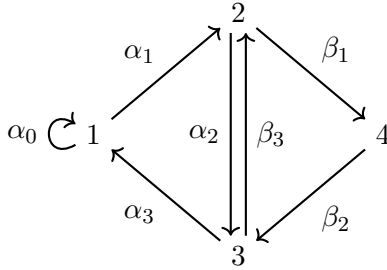


Figure 7: Quiver Q_{G_2} of G_2 .

Special cycles in Q_{G_2} are the special 1-cycles at a corresponding to $\alpha_0\alpha_1\alpha_2\alpha_3$, and $\alpha_1\alpha_2\alpha_3\alpha_0$, the special 2-cycle at a given by $\alpha_2\alpha_3\alpha_0\alpha_1$, the special 2-cycle at b given by $\beta_1\beta_2\beta_3$, etc. Note that there are two distinct 1-cycles at the vertex a , namely $\alpha_0\alpha_1\alpha_2\alpha_3$ and $\alpha_1\alpha_2\alpha_3\alpha_0$.

(3)

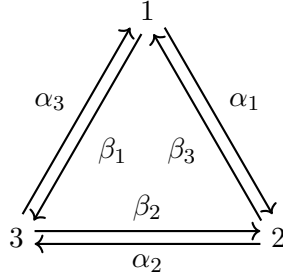


Figure 8: Quiver Q_{G_3} of G_3 .

The special 1-cycle at a is given by $\alpha_1\alpha_2\alpha_3$, the special 1-cycle at b is given by $\beta_1\beta_2\beta_3$, etc.

(4)

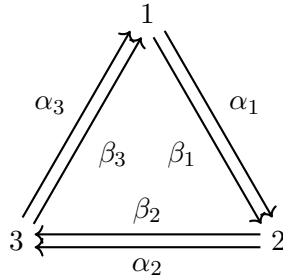


Figure 9: Quiver Q_{G_4} of G_4 .

The special 1-cycle at a is given by $\alpha_1\alpha_2\alpha_3$, the special 1-cycle at b is given by $\beta_1\beta_2\beta_3$, etc.

2.4 Set of relations and definition of Brauer graph algebras

We define an ideal of relations I_G in KQ_G generated by three types of relations. For this recall that we identify the set of edges G_1 of a Brauer graph G with the set of vertices Q_0 of the corresponding quiver Q_G and that we denote the set of vertices of the Brauer graph by G_0 .

Relations of type I:

$$C_v^{m(v)} - C_{v'}^{m(v')},$$

for any $i \in Q_0$ and for any special i -cycles C_v and $C_{v'}$ at v and v' such that both v and v' are not truncated (i.e. $\text{val}(v)m(v) \neq 1$ and $\text{val}(v')m(v') \neq 1$).

Relations of type II:

$$C_v^{m(v)}\alpha_1,$$

for any $i \in Q_0$, any $v \in G_0$ and where $C_v = \alpha_1 \alpha_2 \dots \alpha_n$ is any special i -cycle C_v .

Relations of type III:

$$\alpha\beta,$$

for any $\alpha, \beta \in Q_1$ such that $\alpha\beta$ is not a subpath of any special cycle except if $\alpha = \beta$ is a loop associated to a vertex v of valency one and multiplicity $m(v) > 1$.

The algebra $A_G = KQ_G/I_G$ is called the *Brauer graph algebra* associated to the Brauer graph G .

We note that the relations generating I_G do not constitute a minimal set of relations. Many of the relations, in particular many of the relations of type II, are redundant. In [46] for every Brauer graph algebra a minimal set of relations is determined: The relations of type I and III are always minimal and the only relations of type II appearing in a minimal generating set of relations are those corresponding to an edge i with vertices v and w , such that i is truncated at vertex v and such that the immediate successor of i in the cyclic ordering at w is also truncated at its other endpoint.

Examples 2.4. Sets of relations for the examples in 2.2.

(1) Set of relations of the three types defining the Brauer graph algebra $B_1 = KQ_{G_1}/I_{G_1}$:

$$\text{Type I: } (\alpha_1 \alpha_2 \alpha_3)^3 - \epsilon^2, (\alpha_3 \alpha_1 \alpha_2)^3 - \beta_1 \beta_2 \beta_3 \beta_4$$

$$\text{Type II: } (\alpha_1 \alpha_2 \alpha_3)^3 \alpha_1, (\alpha_2 \alpha_3 \alpha_1)^3 \alpha_2, (\alpha_3 \alpha_1 \alpha_2)^3 \alpha_3, \beta_1 \beta_2 \beta_3 \beta_4 \beta_1, \beta_2 \beta_3 \beta_4 \beta_1 \beta_2, \beta_3 \beta_4 \beta_1 \beta_2 \beta_3, \beta_4 \beta_1 \beta_2 \beta_3 \beta_4, \epsilon^3$$

$$\text{Type III: } \epsilon \alpha_1, \alpha_3 \epsilon, \alpha_2 \beta_1, \beta_4 \alpha_3$$

A minimal set of relations is given by all relations of types I and III and the relations $\beta_2 \beta_3 \beta_4 \beta_1 \beta_2$ and $\beta_3 \beta_4 \beta_1 \beta_2 \beta_3$.

In the following, by abuse of notation, we will abbreviate the relations of type II as follows, for $\beta_i \beta_{i+1} \beta_{i+2} \beta_{i+3} \beta_{i+4}$ we write β^5 , etc.

(2) Set of relations of the three types defining the Brauer graph algebra $B_2 = KQ_{G_2}/I_{G_2}$:

$$\text{Type I: } \alpha_0 \alpha_1 \alpha_2 \alpha_3 - \alpha_1 \alpha_2 \alpha_3 \alpha_0, \alpha_2 \alpha_3 \alpha_0 \alpha_1 - \beta_1 \beta_2 \beta_3, \alpha_3 \alpha_0 \alpha_1 \alpha_2 - \beta_3 \beta_1 \beta_2$$

$$\text{Type II: } \alpha^5, \beta^4$$

$$\text{Type III: } \alpha_0^2, \alpha_1 \beta_1, \alpha_2 \beta_3, \alpha_3 \alpha_1, \beta_2 \alpha_3, \beta_3 \alpha_2$$

A minimal set of relations is given by all relations of types I and III.

(3) Set of relations of the three types defining the Brauer graph algebra $B_3 = KQ_{G_3}/I_{G_3}$:

$$\text{Type I: } \alpha_1 \alpha_2 \alpha_3 - \beta_1 \beta_2 \beta_3, \alpha_2 \alpha_3 \alpha_1 - \beta_3 \beta_1 \beta_2, \alpha_3 \alpha_1 \alpha_2 - \beta_2 \beta_3 \beta_1$$

$$\text{Type II: } \alpha^4, \beta^4$$

$$\text{Type III: } \alpha_1 \beta_3, \beta_3 \alpha_1, \alpha_2 \beta_2, \beta_2 \alpha_2, \alpha_3 \beta_1, \beta_1 \alpha_3$$

A minimal set of relations is given by all relations of types I and III.

(4) Set of relations of the three types defining the Brauer graph algebra $B_4 = KQ_{G_4}/I_{G_4}$:

Type I: $\alpha_1\alpha_2\alpha_3 - \beta_1\beta_2\beta_3, \alpha_2\alpha_3\alpha_1 - \beta_2\beta_3\beta_1, \alpha_3\alpha_1\alpha_2 - \beta_3\beta_1\beta_2$

Type II: α^4, β^4

Type III: $\alpha_i\beta_{i+1}, \beta_i\alpha_{i+1}$ for $i = 1, 2$ and $\alpha_3\beta_1, \beta_3\alpha_1$

A minimal set of relations is given by all relations of types I and III.

Remark 2.5. Any symmetric Nakayama algebra is a Brauer graph algebra. Let Q be the quiver given by a non-zero cycle $\alpha_1\alpha_2 \dots \alpha_n$ with $s(\alpha_1) = t(\alpha_n)$. Let $I = (\alpha^{kn+1})$ for any $k \in \mathbb{Z}_{>0}$. Then $A = KQ/I$ is a Brauer tree algebra with Brauer tree T given by a star with n edges and multiplicity 1 everywhere, except for the central vertex whose multiplicity is equal to k .

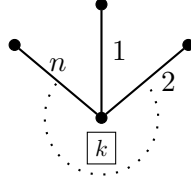


Figure 10: Brauer graph of a Nakayama algebra with n arrows $\alpha_1, \dots, \alpha_n$ and relations of the form $(\alpha_{i_1} \dots \alpha_{i_n})^k \alpha_{i_1}$.

2.5 First properties of Brauer graph algebras

Theorem 2.6. *Given a Brauer graph $G = (G_0, G_1, m, \mathfrak{o})$, the associated Brauer graph algebra $A_G = K_G/I_G$ is a finite dimensional symmetric algebra.*

Proof. We start by showing that I_G is admissible. By the definition of a special cycle C_v , we have that either $\ell(C_v) > 1$ or $m(v) > 1$. Thus $I_G \subset KQ^2$. It follows from the relations of type I, II and III that, for $N = \max_{v \in G_0} \ell(C_v^{m(v)})$, any path p in Q with $\ell(p) \geq N + 1$ is such that $p \in I_G$.

To show that A_G is symmetric, define the following symmetric linear function $f : A_G \rightarrow K$ by setting

$$f(p) = \begin{cases} 1 & \text{if } p = C_v^{m(v)} \text{ for some } v \in G_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then it has no non-zero left-ideal in the kernel and thus A_G is symmetric. \square

Remark 2.7. A Brauer graph algebra is indecomposable if and only if its Brauer graph is connected as a graph.

The following Theorem is well-known. It also follows from recent work on special multi-serial and Brauer configuration algebras in [41, 42].

Theorem 2.8. *Brauer graph algebras are special biserial.*

Corollary 2.9. *Brauer graph algebras are of tame representation type. A Brauer graph algebra is of finite representation type if and only if it is a Brauer tree algebra.*

Remark 2.10. It is possible to introduce scalars in the relations of a Brauer graph algebra. In this case, the resulting Brauer graph algebra is no longer necessarily symmetric, but instead it is weakly symmetric. The case of a weakly symmetric Brauer graph algebra is treated in detail, for example, in [45] and [46]. We note that many interesting examples, such as the existence of non-periodic modules with complexity one appear precisely in the weakly symmetric (non-symmetric) case, see [35]. However, in these notes we will focus on the case of symmetric Brauer graph algebras and refer for a rigorous definition of weakly-symmetric Brauer graph algebras to [45].

2.6 Indecomposable projective modules of Brauer graph algebras

We now list the Loewy series of the projective indecomposable modules of the Brauer graph algebras in the examples above. Since Brauer graph algebras are special biserial, they are biserial and the composition factors of the maximal uniserial submodules of the indecomposable projectives can conveniently be read off the Brauer graph. In the following the projective indecomposable module at vertex i of Q_0 will be denoted by P_i and we denote by i the simple module at vertex i .

Examples 2.11. Let B_1, \dots, B_4 be the Brauer graph algebras with Brauer graphs G_1, \dots, G_4 respectively, given in Example 2.2. Then indecomposable projective modules are given by

(1)

$$\begin{array}{cccccc}
 & \begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{array} & & \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{array} & & \begin{array}{c} 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{array} & & \begin{array}{c} 4 \\ 5 \\ 6 \\ 3 \\ 4 \end{array} & & \begin{array}{c} 5 \\ 6 \\ 3 \\ 4 \\ 5 \end{array} & & \begin{array}{c} 6 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \\
 P_1: & 1 & & P_2: & & P_3: & & P_4: & & P_5: & & P_6:
 \end{array}$$

Figure 11: Indecomposable projective modules over B_1 .

(2)

$$P_1: \begin{array}{c} 1 \\ \frac{1}{2} \quad \frac{2}{3} \\ 3 \quad 1 \\ 1 \end{array} \quad P_2: \begin{array}{c} 2 \\ \frac{3}{1} \quad \frac{4}{3} \\ 1 \quad 2 \end{array} \quad P_3: \begin{array}{c} 3 \\ \frac{1}{2} \quad \frac{2}{4} \\ 2 \quad 3 \end{array} \quad P_4: \begin{array}{c} 4 \\ \frac{3}{2} \\ 2 \quad 4 \end{array}$$

Figure 12: Indecomposable projective modules over B_2 .

(3)

$$P_1: \begin{array}{c} 1 \\ \frac{2}{3} \quad \frac{3}{2} \\ 1 \end{array} \quad P_2: \begin{array}{c} 2 \\ \frac{3}{1} \quad \frac{1}{3} \\ 2 \end{array} \quad P_3: \begin{array}{c} 3 \\ \frac{1}{2} \quad \frac{2}{1} \\ 3 \end{array}$$

Figure 13: Indecomposable projective modules over B_3 .

(4)

$$P_1: \begin{array}{c} 1 \\ \frac{2}{3} \quad \frac{2}{3} \\ 1 \end{array} \quad P_2: \begin{array}{c} 2 \\ \frac{3}{1} \quad \frac{3}{1} \\ 2 \end{array} \quad P_3: \begin{array}{c} 3 \\ \frac{1}{2} \quad \frac{1}{2} \\ 3 \end{array}$$

Figure 14: Indecomposable projective modules over B_1 .

2.7 Brauer graphs as ribbon graphs

In [61] the point of view of interpreting a Brauer graph as a ribbon graph has been introduced. A ribbon graph uniquely defines a compact oriented surface and connects the representation theory of Brauer graph algebras to combinatorial surface geometry. We recall the details of this construction. More details on ribbon graphs and associated surfaces, as well as the proofs of the below statements can be found, for example, in [56].

In order to define ribbon graphs, we start by giving a more formal definition of a graph.

A *graph* is a tuple $\Gamma = (V, E, s, \iota)$ where

1. V is a finite set, called the set of vertices,
2. E is finite set, called the set of half-edges
3. $s : E \rightarrow V$ is a function,
4. $\iota : E \rightarrow E$ is an involution without fixed points.

We think of s as the function sending a half-edge to the vertex it is incident with and for an edge that is not a loop, we think of ι as the function sending a half-edge incident with a vertex to the half-edge incident with the other vertex of the same edge. In the case of a loop, ι sends a half edge incident with the only vertex of the corresponding edge to the

other half-edge of the same edge. That is, a loop in G corresponds to a pair of half-edges $e, e' \in E$ such that $\iota(e) = e'$ and $s(e) = s(\iota(e'))$.

A *ribbon graph* is a graph $\Gamma = (V, E, s, \iota)$ together with a permutation $\sigma : E \rightarrow E$ such that the cycles of σ correspond to the sets $s^{-1}(v)$, for all $v \in V$. In other words a ribbon graph is a graph together with, for every vertex $v \in V$, a cyclic ordering of the half-edges incident with v .

Remark 2.12. 1) It follows from the above definitions that a Brauer graph can be regarded as a (vertex) weighted ribbon graph where the weights are given by the multiplicity function.

2) The language of ribbon graphs in the context of Brauer graph algebras is particularly useful in the treatment of loops in the Brauer graph.

Examples of ribbon graphs are planar graphs and graphs locally embedded in the (oriented) plane such as the Brauer graphs in Example 2.2. Note that as abstract graphs G_3 and G_4 in Example 2.2 are isomorphic, but that as ribbon graphs they are not isomorphic.

Any embedding of a graph into an oriented surface gives a ribbon graph structure on the graph, where the cyclic orderings are induced from the embedding of the edges around each vertex and the orientation of the surface.

On the other hand, we obtain surfaces from ribbon graphs. For this we first define the geometric realisation of a ribbon graph and then we embed it into an open oriented surface.

The *geometric realisation* $|\Gamma|$ of a ribbon graph Γ is the topological space

$$|\Gamma| = (E \times [0, 1]) / \sim$$

where \sim is the equivalence relation defined by $(e, t) \sim (\iota(e), 1 - t)$ for all $t \in [0, 1]$ and $(e, 0) \sim (f, 0)$ if $s(e) = f(e)$ and $(e, 1) \sim (f, 1)$ if $s(\iota(e)) = f(\iota(e))$, for all $e, f \in E$.

Lemma 2.13. [56, 2.2.4] *Every ribbon graph Γ can be embedded in an open oriented surface with boundary in such a way that the cyclic orderings around each of its vertices arise from the orientation of the surface.*

An example of such a surface is the *ribbon surface* S_Γ° of Γ which is constructed in the following way: Firstly, an oriented surface is associated to each vertex and edge of Γ as in Figure 15. Then the structure and orientation of the ribbon graph determines a gluing of the corresponding surfaces giving an oriented surface S_Γ° together with an embedding of Γ into S_Γ° .

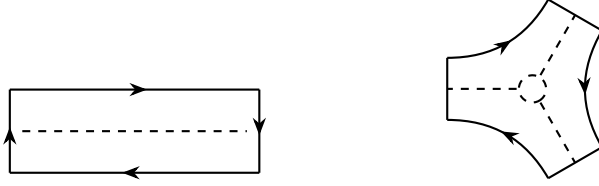


Figure 15: Oriented surfaces corresponding to vertices and edges in Γ .

Note that S_Γ° has a number of boundary components corresponding to the faces of Γ . A *face* of Γ is an equivalence class, up to cyclic permutation, of n -tuples (e_1, e_2, \dots, e_n) of half-edges satisfying

$$e_{p+1} = \begin{cases} \iota(e_p) & \text{if } s(e_p) = s(e_{p-1}) \\ \sigma(e_p) & \text{if } s(e_p) \neq s(e_{p-1}) \end{cases}$$

for all p with $1 \leq p \leq n$ (where subscripts are taken modulo n).

Now let S be a compact oriented surface. An embedding $\Gamma \hookrightarrow S$ is said to be a *filling embedding* if $S \setminus |\Gamma| = \bigsqcup_{f \in F} D_f$, where each D_f is a disc, and F is a finite set. That is, the complement of the embedding is a disjoint union of finitely many discs.

Proposition 2.14. [56, 2.2.7] *Every ribbon graph has a filling embedding into a compact oriented surface such that the connected components of $S \setminus |\Gamma|$ are in bijection with the faces of Γ in the above sense.*

This is proved by gluing discs onto the ribbon surface of the ribbon graph to fill in the boundary components. Such an embedding has the following uniqueness property. We first recall that morphisms and isomorphisms of ribbon graphs are defined in the natural way, that is they are maps of graphs respecting the permutations.

Proposition 2.15. [56, 2.2.10] *Let $\Gamma \rightarrow S$ and $\Gamma' \rightarrow S'$ be filling embeddings of ribbon graphs Γ and Γ' into compact oriented surfaces S and S' and let $f : \Gamma \rightarrow \Gamma'$ be an isomorphism of ribbon graphs. Then f induces an orientation-preserving homeomorphism $f : |\Gamma| \rightarrow |\Gamma'|$ extending to a homeomorphism from S to S' .*

Corollary 2.16. *If Γ is a ribbon graph, then there is a compact oriented surface S_Γ together with a filling embedding $\Gamma \rightarrow S_\Gamma$, unique up to homeomorphism.*

Thus we see that there is a filling embedding of an arbitrary Brauer graph (without, or ignoring, multiplicity) into a compact oriented surface, unique up to homeomorphism, in such a way that the cyclic ordering around each vertex arises from the orientation of the surface. Conversely, we have the following:

Proposition 2.17. [56, 2.2.12] *Every compact oriented surface admits a filling ribbon graph.*

Remark 2.18. (1) Note that in [5] there also is a surface associated to a Brauer graph. This surface arises from a CW-complex associated to the corresponding Brauer graph

algebra. Comparing the definitions, it can be seen that this gives rise to the same surface as the ribbon graph construction in (Corollary 2.16) above. In particular, the G -cycles in [5, §2] correspond to the faces of the Brauer graph as a ribbon graph.

(2) In Sections 3 and 4 (ideal) triangulations of marked oriented surfaces play an important role (we refer the reader to Section 3.3 or to [38] for the definition of an ideal triangulation), these are connected to ribbon graphs in the following way. Let T be a triangulation of a marked oriented surface (S, M) where S might have a boundary and punctures. Then T (including the boundary edges) can be considered as a ribbon graph with boundary where the cyclic ordering of the edges around each vertex is induced by the orientation of S and where the boundary of S induces the set of boundary faces of T . Then there is an embedding $T \hookrightarrow S$ mapping the boundary faces to the boundaries of the boundary components, for more details see for example Section 2.2 in [61].

3 Brauer graph algebras and gentle algebras

In this section we give an explicit construction to show how gentle algebras and Brauer graph algebras are related via trivial extensions. We do this by constructing a ribbon graph for every gentle algebra.

Definition 3.1. A finite dimensional algebra is *gentle* if it is Morita equivalent to a special biserial algebra $A = KQ/I$, that is, (S0) and (S1), hold as well as

(S2) For every arrow α in Q there exists at most one arrow β such that $\alpha\beta \in I$ where $t(\alpha) = s(\beta)$ and there exists at most one arrow γ such that $\gamma\alpha \in I$ where $t(\gamma) = s(\alpha)$.

(S3) The ideal I is generated by paths of length 2.

Example 3.2. Examples of gentle algebras:

(1) Tilted algebras of type A , see for example, [8, Chapter VIII] for a definition of tilted algebras.

(2) Gentle algebras arising as Jacobian algebras associated to marked oriented surfaces where all marked points lie in the boundary (see [7]).

(3) Discrete derived algebras as classified in [83].

The *trivial extension* $T(A) = A \ltimes D(A)$ of an algebra A by its injective cogenerator $D(A) = \text{Hom}_K(A, K)$ is the algebra whose underlying K -vector space is given by $A \oplus D(A)$ and where the multiplication is given as follows:

$$(a, f)(b, g) = (ab, ag + fb) \quad \text{for all } a, b \in A \text{ and } g, f \in D(A).$$

The trivial extension $T(A)$ is symmetric (see for example [75]).

The following result shows the connection between gentle algebras and Brauer graph algebras.

Theorem 3.3. [65] *Let A be a finite dimensional K -algebra. Then A is gentle if and only if $T(A)$ is special biserial.*

Therefore if A is gentle, its trivial extension $T(A)$ is a symmetric special biserial algebra, that is, it is a Brauer graph algebra.

Question 3.4. *Given a gentle algebra, what is the Brauer graph of its trivial extension?*

We will answer this question in Theorem 3.7.

3.1 Graph of a gentle algebra

In [77], to any gentle algebra A is associated a ribbon graph Γ_A whose underlying graph structure is determined by the set of maximal paths in A . The ribbon graph structure in the form of the cyclic orderings of the edges around the vertices are also induced by the maximal paths.

Let $A = KQ/I$ and \mathcal{M} be the set of maximal paths in A . That is \mathcal{M} consists of all images in KQ/I of paths $p \in KQ$ such that $p \notin I$, but $\alpha p \in I$ and $p\alpha \in I$, for all non-trivial $\alpha \in KQ$.

Set $\mathcal{M}_0 = \{e_v, v \in V_0\}$ where V_0 is the set of vertices $v \in Q_0$ such that one of the following holds:

- (1) There exist unique arrows α and β such that $t(\alpha) = v = s(\beta)$ and, furthermore, $\alpha\beta \notin I$.
- (2) v is a source and there is a single arrow starting at v .
- (3) v is a sink and there is a single arrow ending at v .

Set $\overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}_0$.

Example 3.5. Let A be the gentle algebra with quiver

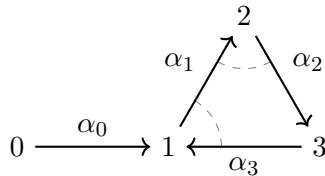


Figure 16: The dashed lines indicate the relations of length 2.

and relations $\alpha_1\alpha_2$ and $\alpha_3\alpha_1$.

Then $V_0 = \{0, 3\}$ and $\overline{\mathcal{M}} = \{e_0, \alpha_0\alpha_1, \alpha_2\alpha_3, e_3\}$.

Construction of the ribbon graph Γ_A of a gentle algebra.

Let $A = KQ/I$ be a gentle algebra. The vertices of Γ_A are in bijection with the elements in $\overline{\mathcal{M}}$ and the edges of Γ_A are in bijection with the vertices of Q_0 . Furthermore, there exists an edge E_v for $v \in Q_0$ between the vertices $m, m' \in \overline{\mathcal{M}}$ if v as a vertex in Q_0 lies in both the paths m and m' . That is if there exists paths p, q, p', q' such that $m = pe_vq$ and $m' = p'e_vq'$ where e_v is the trivial path at v and p, q, p', q' might also possibly be trivial paths at v .

We note that every vertex v in Q_0 lies in exactly two elements of $\overline{\mathcal{M}}$ which are not necessarily distinct. If they are not distinct the corresponding edge E_v in Γ_A is a loop.

Note that if a vertex in Γ_A corresponds to a trivial path in Q then as a vertex in Γ_A it has valency 1, that is, there is only one edge of Γ_A incident with that vertex. In all other case there are at least two edges incident with every vertex of Γ_A .

Let v be a vertex of Γ_A such that there are at least two edges of Γ_A incident at v and let $a_1 \xrightarrow{\alpha_1} a_2 \xrightarrow{\alpha_2} \dots a_{k-1} \xrightarrow{\alpha_{k-1}} a_k$ be the element in $\overline{\mathcal{M}}$ corresponding to v . Let E_1, E_2, \dots, E_k be the edges in Γ_A corresponding to the vertices a_1, a_2, \dots, a_k respectively. Note that they all have the vertex v in common. Then we linearly order E_1, E_2, \dots, E_k as follows: $E_1 < E_2 < \dots < E_k$. We complete this linear order to a cyclic order by adding $E_k < E_1$ and we do this at every vertex of Γ_A . This equips the graph Γ_A with the structure of a ribbon graph.

Example 3.6. For the algebra in example 3.5 above, we get the the following ribbon graph.

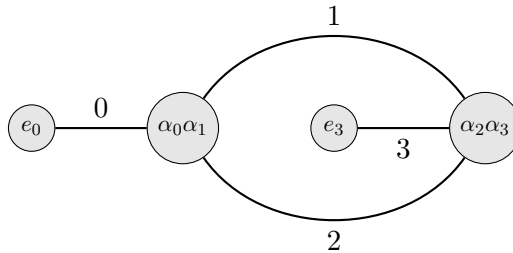


Figure 17: Graph Γ_A of the gentle algebra A in Example 3.5.

The cyclic ordering here corresponds to the ordering given by the embedding into the clockwise oriented plane induced by the maximal paths containing the given vertex. That is at vertex $\alpha_0\alpha_1$ we have cyclic ordering $0 < 1 < 2 < 0$ at vertex $\alpha_2\alpha_3$ we have cyclic ordering $1 < 2 < 3 < 1$, at vertex e_0 it is 0 and at vertex e_3 it is 3.

Theorem 3.7. [77] *Let $A = KQ/I$ be a gentle algebra with ribbon graph Γ_A . Then $T(A)$ is the Brauer graph algebra with Brauer graph Γ_A (and with multiplicity function*

identically equal to one).

There are many situations when there are naturally associated graphs to gentle algebras, given for example by triangulations or partial triangulations of marked unpunctured surfaces. These graphs often coincide with the ribbon graphs defined above. More precisely, using the admissible cuts in the next section we get the following Corollary to Theorem 3.7.

Corollary 3.8. 1) If A is a gentle algebra arising from a triangulation T of an oriented marked surface with marked points in the boundary as defined in [7] then $\Gamma_A = T$ as ribbon graphs.

2) If A is a surface algebra as defined in [24] of a partial triangulation T of an oriented marked surface with marked points in the boundary then $\Gamma_A = T$ as ribbon graphs.

3) If A is a tiling algebra as defined in [20] with partial triangulation T then $\Gamma_A = T$.

We will give three more examples of the graphs of gentle algebras. In particular, the first two examples illustrate each of the statements in Corollary 3.8(1) and (3), respectively. In all three examples the orientation of the graph of the corresponding gentle algebra (induced by the maximal paths) is given by the clockwise orientation of the plane.

Example 3.9. Example illustrating Corollary 3.8 (1). Let $A = KQ/(\alpha_1\beta, \beta\gamma, \gamma\alpha_1)$ be the gentle algebra with quiver Q given by

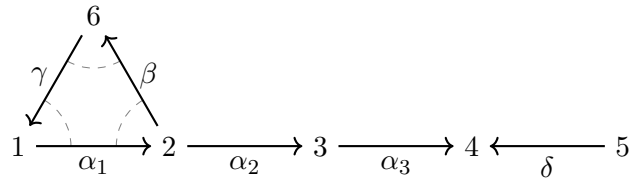


Figure 18: Gentle algebra A associated to the triangulation of the 9-gon in Figure 20.

Then the graph Γ_A of A is given by

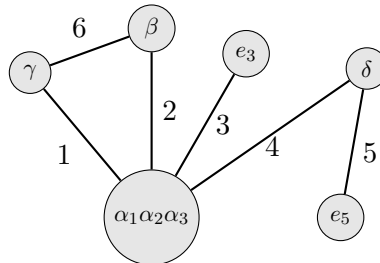


Figure 19: Graph Γ_A of A

The graph Γ_A coincides with the following triangulation of the 9-gon and the gentle algebra A corresponds to the associated Jacobian algebra arising from the associated quiver with potential.

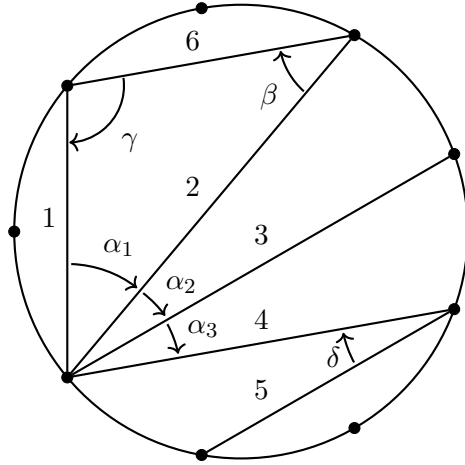


Figure 20: Triangulation of the 9-gon corresponding to Γ_A .

Example 3.10. Example illustrating Corollary 3.8 (3): Example of a tiling algebra as defined in [20] (and using the notation of [20]). In Figure 21 is the algebra A_{10} corresponding to a tiling of $P_{10,1}$ as in Figure 23. The algebra A_{10} is defined to be the algebra KQ_{10}/I where Q_{10} is the quiver in Figure 21 and where $I = (\alpha_1\beta_3, \alpha_2\gamma_1, \gamma_1\delta_2, \delta_2\beta_2, \beta_2\alpha_2, \delta_1\gamma_2, \epsilon\delta_1)$.

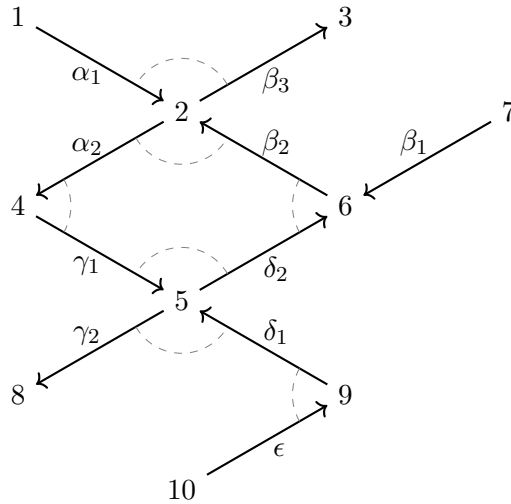


Figure 21: The algebra A_{10} of a tiling of $P_{10,1}$ [20]

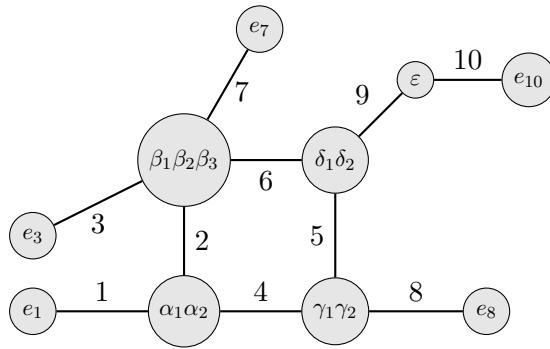


Figure 22: The Graph $\Gamma_{A_{10}}$ of the algebra A_{10} .

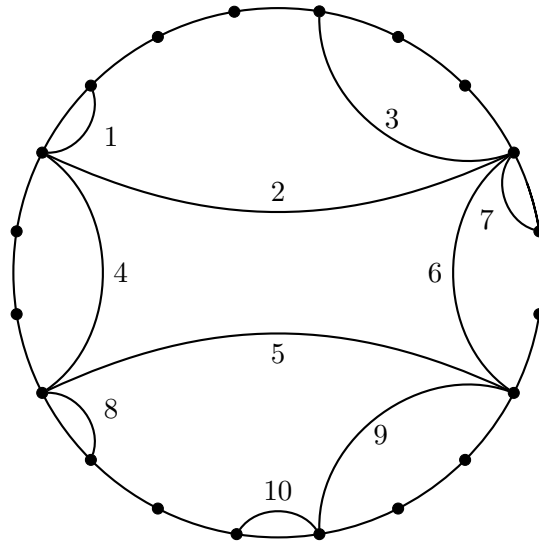


Figure 23: The tiling of $P_{10,1}$ (see [20]) giving rise to the gentle algebra A_{10} . The graph $\Gamma_{A_{10}}$ of A_{10} gives the internal arcs of the tiling of $P_{10,1}$.

Example 3.11. Kalck [51] has shown that the algebras $A_1 = KQ_1/(\alpha_1\alpha_2, \beta_1\beta_2)$ and $A_2 = KQ_2/(\alpha_1\alpha_2, \gamma\alpha_1, \alpha_2\beta)$ are not derived equivalent. However, both algebras have the same Avella-Geiss invariants [10] (these are invariants of gentle algebras such that if two gentle algebras are derived equivalent then they have the same Avella-Geiss invariants) and they also have the same graph as gentle algebras. This graph is the ribbon graph given in Figure 26 (compare also with the Brauer graph in Figure 4. Note that the associated Brauer graph algebra is the trivial extension of both A_1 and A_2).

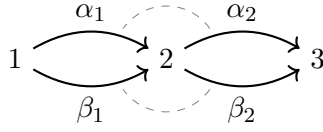


Figure 24: Quiver Q_1

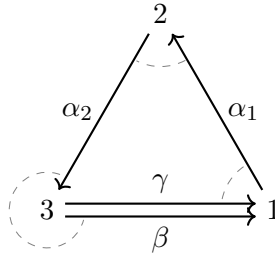


Figure 25: Quiver Q_2

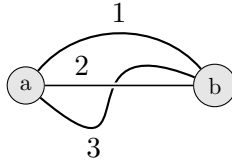


Figure 26: The graph $\Gamma_{A_1} = \Gamma_{A_2}$.

3.2 Gentle algebras from admissible cuts of Brauer graph algebras

Admissible cuts first appear in the PhD thesis of Fernández [37] where they were defined in the context of showing that two schurian algebras Λ and Λ' have isomorphic trivial extensions if and only if Λ' is an admissible cut of $T(\Lambda)$. Recall that Λ is a *schurian algebra* if for each pair of primitive idempotents e and f in Λ , we have $\dim_K(e\Lambda f) \leq 1$.

In [77], *admissible cuts of Brauer graph algebras* are defined as a slight modification of Fernández notion.

Definition 3.12. Let $A = KQ/I$ be a Brauer graph algebra with Brauer graph G and multiplicity function m identically equal to one. An *admissible cut* Δ of Q is a set of arrows containing exactly one arrow from every special cycle C_v (up to rotation), where v is not of valency one as a vertex of G .

We define the *cut algebra with cut* Δ to be the algebra $A_\Delta = KQ/\langle I \cup \Delta \rangle$ where $\langle I \cup \Delta \rangle$ is the ideal of KQ generated by $I \cup \Delta$.

Theorem 3.13. [77] Let $A = KQ/I$ be a Brauer graph algebra with Brauer graph G with multiplicity function identically equal to one. Let Δ be an admissible cut of Q . Then A_Δ is a gentle algebra and $\Gamma_{A_\Delta} = G$.

Corollary 3.14. [77] Every gentle algebra is the cut algebra of a unique Brauer graph algebra, its trivial extension, and conversely, every Brauer graph algebra with multiplicity function identically equal to one is the trivial extension of a (not necessarily unique) gentle algebra.

Example 3.15. Let $\Lambda = KQ/I$ be the Brauer graph algebra with Brauer graph given by

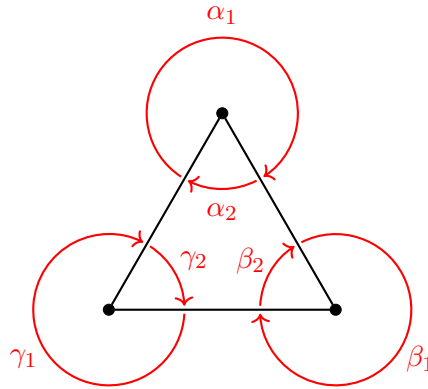


Figure 27: Example 4.14

Let $\Delta_1 = \{\alpha_1, \beta_1, \gamma_1\}$ and $\Delta_2 = \{\alpha_1, \beta_1, \gamma_2\}$ be admissible cuts of Q . Then the cut algebras are isomorphic to the following algebras $A_{\Delta_1} \simeq K(Q \setminus \Delta_1)/(\beta_2\alpha_2, \gamma_2\beta_2, \alpha_2\gamma_2)$ and $A_{\Delta_2} \simeq K(Q \setminus \Delta_2)/(\beta_2\alpha_2)$ and it is easy to see that they are gentle.

Remark 3.16. Let $A = KQ/I$ be a Brauer graph algebra and Δ and Δ' be two distinct admissible cuts of Q . Then A_Δ and $A_{\Delta'}$ have the same number of simple modules but they are not necessarily isomorphic nor derived equivalent. It is possible that A_Δ is of finite global dimension and $A_{\Delta'}$ is of infinite global dimension as is the case in Example 3.15 where A_{Δ_1} is of infinite global dimension whereas A_{Δ_2} is of finite global dimension.

3.3 Gentle surface algebras and quiver mutation

In [57, 58] Ladkani studies when two mutation-equivalent Jacobian algebras arising from quivers with potential associated to marked surfaces with all marked points on the boundary are derived equivalent. Note that in this case the Jacobian algebras are gentle algebras [7]. We will start by recalling Ladkani's result and relate it to the corresponding Brauer graph algebras (given by the trivial extensions of the Jacobian algebras).

For this, recall the Fomin-Zelevinsky quiver mutation.

Definition 3.17. Let Q be a quiver without loops or two cycles and let k be a vertex of Q . The mutation of Q at k is the quiver $\mu_k(Q)$ obtained from Q as follows:

- 1) for each subquiver $i \rightarrow k \rightarrow j$, add a new arrow $i \rightarrow j$;
- 2) all arrows with source or target k are reversed
- 3) remove all newly-created 2-cycles.

Quiver mutation is an involution, that is $\mu_k(\mu_k(Q)) = Q$.

Let S be an oriented surface and let M be a set of marked points in S . An *arc* is a curve without self intersections with endpoints in M , considered up to homotopy, and such that it does not cut out a monogon or bigon and such that it is disjoint from the boundary except for the endpoints. An *ideal triangulation* is a maximal collection of non-crossing arcs. Let T be an ideal triangulation of the *marked surface* (S, M) . We associate to T a quiver $Q(T)$ such that the vertices of $Q(T)$ correspond to the internal (=non-boundary) edges of T and where the arrows are given by the immediate successor relations of two edges incident with a common vertex induced by the orientation of the surface. For more details on triangulations and the associated quivers we refer the reader to [38]. In what follows we will often refer to an ideal triangulation as a triangulation.

Let s be an internal edge in T . Then the *flip of T at s* is defined by removing s and replacing it by s' such that we obtain the triangulation $T' = (T \setminus s) \cup s'$, where $s' \neq s$ is the unique edge completing $T \setminus s$ to a triangulation of (S, M) .

Proposition 3.18. [38] Let T be a triangulation of a marked surface (S, M) and let T' be the triangulation obtained by flipping an internal s edge in T . Then $Q(T') = \mu_s(Q(T))$.

From now on let (S, M) be such that S is a surface with boundary ∂S and M is a set of marked points such that $M \subset \partial S$. Let T be a triangulation of (S, M) and A_T the gentle algebra associated to (S, M, T) [7, 55] with quiver $Q(T)$. Let g be the genus of S and b the number of boundary components in S .

Definition 3.19. 1) An (ideal) triangle in T is a *boundary triangle* if exactly two of its sides are arcs that lie in the boundary of S .

2) Let B_1, \dots, B_b be the boundary components in S . Let n_i be the number of marked points on B_i and let d_i be the number of boundary triangles incident with B_i , that is triangles where exactly two sides lie in the boundary component B_i . Note that the d_i depend on T . We call the sequence

$$g, b, (n_1, d_1), \dots, (n_b, d_b)$$

the *parameters* of T .

For a finite dimensional algebra A denote by $\mathcal{D}^b(A)$ the bounded derived category of finitely generated A -modules. For more details on the definition of bounded derived categories

and their properties we refer the reader to the Appendix in Section 6 and to standard textbooks such as [48, 85].

Theorem 3.20. [58] *Let (S, M) be a marked surface with all marked points in the boundary of (S, M) . Let T and T' be triangulations of (S, M) having the same parameters. Let A_T and $A_{T'}$ be the associated gentle algebras. Then*

$$\mathcal{D}^b(A_T) \simeq \mathcal{D}^b(A_{T'}).$$

Remark 3.21. 1) If (S, M) is a marked disc and T and T' are two triangulations of (S, M) then $\mathcal{D}^b(A_T) \simeq \mathcal{D}^b(A_{T'})$ if and only if T and T' have the same number of boundary triangles. Note that by [7] the algebras A_T and $A_{T'}$ are cluster-tilted algebras of Dynkin type A and their derived equivalence classification was already done in [16]. For the notion of a cluster-tilted algebras, we refer, for example, to [75].

2) If T' is obtained from T by flipping one arc then $\mathcal{D}^b(A_T) \simeq \mathcal{D}^b(A_{T'})$ if and only if T and T' have the same number of boundary triangles.

3) In [58] a boundary triangle is called a *dome*.

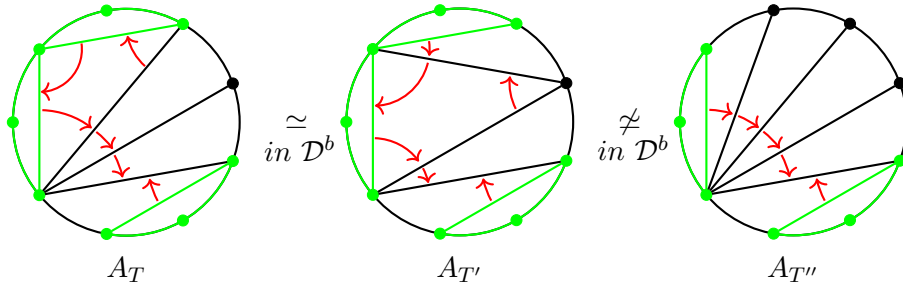
Corollary 3.22. *Let A_T and $A_{T'}$ be gentle algebras associated to triangulations T and T' of (S, M) such that T and T' have the same parameters and let Λ_T and $\Lambda_{T'}$ be the associated Brauer graph algebras, that is $\Lambda_T = A_T \rtimes D(A_T)$ and $\Lambda_{T'} = A_{T'} \rtimes D(A_{T'})$. Then*

$$\mathcal{D}^b(\Lambda_T) \simeq \mathcal{D}^b(\Lambda_{T'}).$$

Proof. This immediately follows from Theorem 3.7 showing that $\Lambda_T = T(A_T)$ and $\Lambda_{T'} = T(A_{T'})$ and the result by Rickard [72] that if two finite dimensional K -algebras A and B are derived equivalent then their trivial extensions $T(A)$ and $T(B)$ are derived equivalent. \square

In particular, in the setting of Corollary 3.22, the quiver $Q_{T'}$ of the Brauer graph algebra $\Lambda_{T'}$ is obtained from the quiver Q_T of the Brauer graph algebra Λ_T by successive Fomin-Zelevinsky quiver mutations.

Example 3.23. Let T, T' and T'' be the following triangulations of the 9-gon giving rise to the associated gentle algebras $A_T, A_{T'}$ and $A_{T''}$.



Then T and T' have the same number of boundary triangles and A_T and $A_{T'}$ are derived equivalent. Furthermore, the corresponding Brauer graph algebras $\Lambda_T = T(A_T)$ and $\Lambda_{T'} = T(A_{T'})$ are derived equivalent. But A_T and $A_{T''}$ are not derived equivalent. Note that here we also have that the corresponding Brauer graph algebras Λ_T and $\Lambda_{T''} = T(A_{T''})$ are not derived equivalent (since T is a graph and T'' is a tree so that Λ_T is of tame representation type whereas $\Lambda_{T''}$ is of finite representation type). However, this does not always hold, see for example in Example 3.15 where the gentle algebras A_{Δ_1} and A_{Δ_2} are not derived equivalent but have isomorphic trivial extensions, i.e. they give rise to the same Brauer graph algebra.

4 Derived equivalences and mutation of Brauer graph algebras

Gentle algebras have a remarkable property. Namely their class is closed under derived equivalence, that is any algebra derived equivalent to a gentle algebra is again a gentle algebra. That is any algebra derived equivalent to a gentle algebra is again a gentle algebra [76].

The same is expected to hold for Brauer graph algebras [6].

4.1 Mutation of Brauer graph algebras

In [52] Kauer defined tilting complexes for Brauer graph algebras and showed that they give rise to derived equivalences in such a way that the endomorphism algebra of the tilting complex is again a Brauer graph algebra where the Brauer graph is obtained from the original one by a simple geometric move on the Brauer graph. We note that the proof at the time of writing was not complete, in that Kauer assumed, but omitted to show, that the tilted algebra has again the structure of a Brauer graph algebra. However, it follows from [70] that this holds for Brauer tree algebras, see also [13]. In the general case of a Brauer graph algebra that is not necessarily a Brauer tree algebra, this has been shown to hold, for example, in [25] and also in [1].

We will describe Kauer's tilting complex construction as well as the corresponding geometric moves on the Brauer graphs below. We note that Kauer's construction corresponds to a more general construction of two-term tilting complexes by Okuyama [63] (based on a manuscript by Rickard [71]). We will state a reformulation of this result due to [60]. For a module M , denote by $P(M)$ the projective cover of M . The definition of a tilting complex is recalled in the Appendix in Section 6.

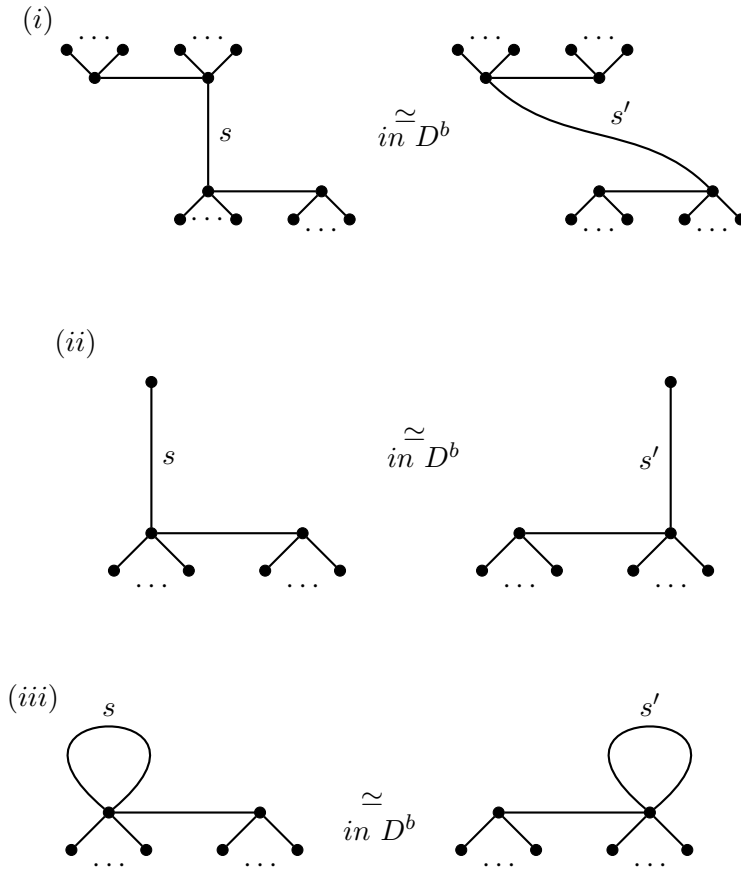
Theorem 4.1. [63, 71] *Let A be a symmetric K -algebra. Let I, I' be disjoint sets of simple A -modules such that $I \cup I'$ is a complete set of representatives of the isomorphism classes of simple A -modules such that for any $S, U \in I$, we have $\text{Ext}_A^1(S, U) = \{0\}$.*

For any $S \in I$, let T_S be the complex $P(\text{rad}(P(S))) \xrightarrow{\pi_S} P(S)$ with non-zero terms in degrees zero and one, and such that $\text{Im } \pi_S = \text{rad}(P(S))$. For any $S' \in I'$ consider $P(S')$ as a complex concentrated in degree zero.

Then the complex $T = (\bigoplus_{S \in I} T_S) \oplus (\bigoplus_{S' \in I'} P(S'))$ is a tilting complex for A .

4.2 Kauer moves on the Brauer graph

An example of an Okuyama-Rickard tilting complex is given by the following complex constructed by Kauer in [52]. Let A be a Brauer graph algebra with Brauer graph G . For $I = \{S_0\}$ where S_0 is a simple A -module such that the corresponding edge in the Brauer graph is not a loop where the corresponding half edges are such that one is a direct successor of the other, set $T = T_{S_0} \oplus \bigoplus_{S \neq S_0, S \text{ simple}} P(S)$. Then in [52] this is shown to be a tilting complex for A by verifying its properties one by one. However, that T is a tilting complex also follows directly from the fact that T is an Okuyama tilting complex for A . Kauer further shows that the Brauer graph algebra $B = \text{End}_{\mathcal{K}^b(P_A)}(T)$ has Brauer graph $G' = (G \setminus s) \cup s'$ where s is the edge in G corresponding to S_0 and where s' is obtained by one of the following local moves on G :



Definition 4.2. We call a local move as in (i)-(iii) above a *Kauer move at s* and we adopt the following notation $\mu_s^+(G) = G' = (G \setminus s) \cup s'$.

Remark 4.3. 1) In [2] Aihara refers to the above moves (i)-(iii) as flips of Brauer graphs and he describes them in terms of quiver combinatorics.

2) In [3] Aihara and Iyama introduce the notion of a (silting) mutation of an algebra in terms of left and right approximations and in [34] the Okuyama-Rickard tilting complexes giving rise to the Kauer moves are expressed in the more general case of weakly symmetric special biserial algebras in terms of silting mutations, that is in terms of left and right approximations.

Example 4.4. Let A be the Brauer graph algebra with Brauer graph as given by the left hand graph in Figure 28. The Kauer move at $s = 0$ gives rise to the Brauer graph on the right hand graph in Figure 28.

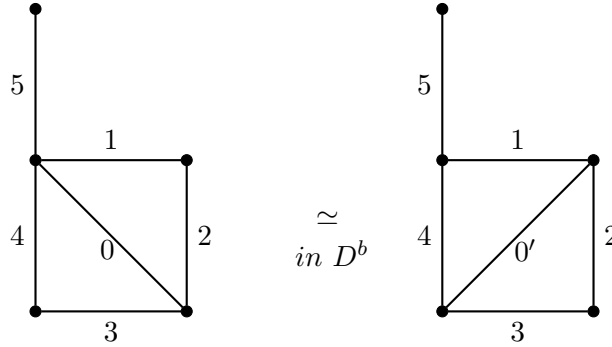


Figure 28: Example of a Kauer move of type (i). The Brauer graph algebras associated to the two graphs are derived equivalent.

The Okuyama-Rickard complex T giving rise to the equivalence of $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$, where $B = \text{End}_{\mathcal{K}^b(A)}(T)$ is given by $T = T_0 \oplus P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5$ where, for $i = 1, 2, 3, 4, 5$, P_i is the stalk complex concentrated in degree zero of the indecomposable projective associated to the edge i and $T_0 = P_4 \oplus P_2 \xrightarrow{\pi} P_0$, that is, T_0 is given by the following complex

$$T_0 = \begin{array}{c} 4 \\ \begin{array}{cc} 5 & 3 \\ 1 & 0 \end{array} \oplus \begin{array}{c} 2 \\ \begin{array}{cc} 3 & 1 \\ 0 & 2 \end{array} \end{array} \xrightarrow{\pi_0} \begin{array}{c} 0 \\ \begin{array}{cc} 4 & 2 \\ 5 & 3 \\ 1 & 0 \end{array} \end{array}$$

concentrated in degrees zero and one and where $\text{Im}(\pi_0) = \text{rad}(P_0) = \begin{array}{c} 4 \\ 5 \\ 1 \\ 0 \end{array} \begin{array}{c} 2 \\ 3 \end{array}$.

4.3 Brauer graph algebras associated to triangulations of marked oriented surfaces

Let S be an oriented surface with boundary and let M be a set of marked points on S . Note here we don't necessarily require that M lies in the boundary of S , however, we do require that on each boundary component of S there is at least one marked point of M . We first recall that any two triangulations of (S, M) are flip-connected, that is, one can be obtained from the other by a series of flips of diagonals, see for example [17, 38].

Theorem 4.5. *Let T and T' be triangulations of a marked oriented surface (S, M) . Then the triangulations T and T' are flip connected.*

Recall that every triangulation of (S, M) is a ribbon graph where the cyclic ordering is induced by the orientation of S . In [61] we consider triangulations T of (S, M) (including the boundary arcs) as Brauer graphs. Note that the Brauer graph algebras we consider in [61] are different from the Brauer graph algebras associated to triangulations in Section 3. Namely, in Section 3, the Brauer graphs consist of the internal arcs of the triangulation whereas in [61] the boundary edges of the triangulation are part of the Brauer graphs, see Example 4.8. It is observed in [61] that when regarding T (including the boundary edges) as a Brauer graph, the flip of T at an internal edge s coincides with applying the Kauer move to T at s . Combining this with Theorem 4.5 we get

Theorem 4.6. *[61] Let T and T' be two triangulations of a marked oriented surface (S, M) . Then the associated Brauer graph algebras Λ_T and $\Lambda_{T'}$ are derived equivalent, that is the bounded derived categories of finitely generated modules $\mathcal{D}^b(\Lambda_T)$ and $\mathcal{D}^b(\Lambda_{T'})$ are equivalent as triangulated categories.*

Furthermore, we have

Proposition 4.7. *[61] Let Q_T be the quiver of the Brauer graph algebra associated to a triangulation T of (S, M) and let $T' = (T \setminus s) \cup s'$. Then $Q_{T'}$ is obtained from Q_T by Fomin-Zelevinsky quiver mutation, that is $Q_{T'} = \mu_s(Q_T)$.*

Note that in Proposition 4.7 we do not have any restrictions on the triangulation T . In particular, it can contain self-folded triangles or punctures with exactly two incident arcs.

Example 4.8. The Brauer graph algebras associated to the following two Brauer graphs given by the triangulations T and T' of a hexagon (together with the boundary arcs) are derived equivalent. Note that here, as opposed to the examples in Section 3.3, T and T' do not have the same number of boundary triangles, yet the corresponding Brauer graph algebras are derived equivalent.

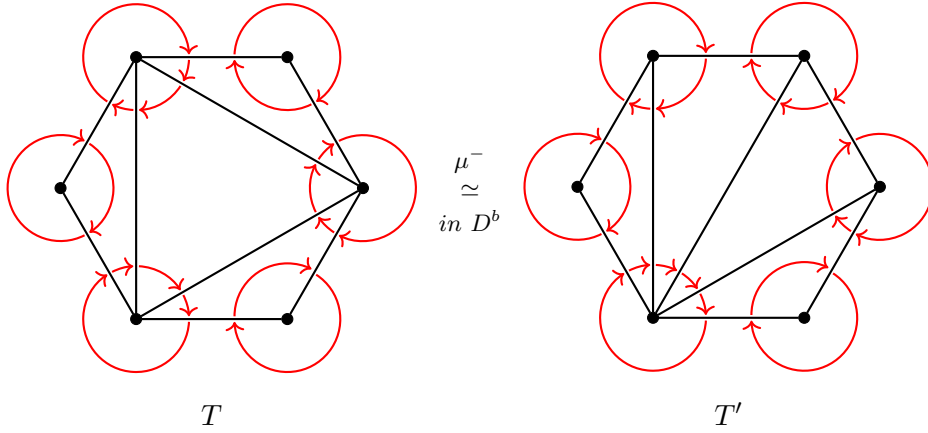


Figure 29: Example of two triangulations where the Brauer graph algebras associated to the Brauer graphs given by the triangulations including the boundary arcs are derived equivalent.

Remark 4.9. By Theorem 4.6 up to derived equivalence there is a unique Brauer graph algebra associated to any marked surface (S, M) . This is similar to the corresponding situation for generalised cluster categories as defined by Amiot, see for example [4]. Namely, if \mathcal{C} and \mathcal{C}' are two cluster categories associated to two quivers with potential (Q, W) and (Q', W') associated to two triangulations T and T' of (S, M) , then \mathcal{C} and \mathcal{C}' are triangle equivalent [53].

4.4 Brauer graph algebras and frozen Jacobian algebras

In this section we show that the frozen Jacobian algebra associated to a triangulation of a marked disc is closely related to the Brauer graph algebra whose Brauer graph corresponds to the triangulation (including the boundary arcs).

Definition 4.10. [15] An *ice quiver with potential* (Q, W, F) is a quiver with potential (Q, W) and a subset F of vertices of Q , called the *frozen vertices*.

The complete path algebra \widehat{KQ} is the completion of the path algebra KQ with respect to the ideal \mathcal{R} generated by the arrows of Q . A potential on Q is an element of the closure $Pot(KQ)$ of the space generated by all non-trivial cyclic paths of Q . We say two potentials are cyclically equivalent if their difference is in the closure of the space generated by all differences $a_1 a_2 \dots a_s - a_2 \dots a_s a_1$, where $a_1 a_2 \dots a_s$ is a cycle.

For a path p in Q , let $\partial_p : Pot(KQ) \rightarrow \widehat{KQ}$ be the unique continuous linear map which for a cycle c is defined by $\partial_p(c) = \sum_{upv=c} vu$ where u and v might be the trivial path at $s(c)$ and $t(c)$, respectively. If $p = a$ for some arrow a in Q then ∂_a is called the *cyclic derivative with respect to a*.

The *frozen Jacobian algebra* of (Q, W, F) is given by $\mathcal{J}(Q, W, F) = \widehat{KQ}/I_F$ where \widehat{KQ} is the complete path algebra, I_F is the closure of the ideal $\langle \partial_a w \mid w \in W, a \in Q_1, s(a) \notin F \text{ or } t(a) \notin F \rangle$.

In the case of a triangulation of a marked disc (S, M, T) , consider the ice quiver (Q, W, F) given by the quiver $Q = (Q_0, Q_1)$ where Q_0 corresponds to the edges of T and where the set F of frozen vertices is given by the boundary arcs and Q_1 is induced by the orientation. So in particular Q_1 includes a boundary arrow for all marked points incident to more than 2 arcs. The potential is given by

$$W = \sum_i C_i - \sum_j D_j$$

where the cycles C_i are given by (all) triangles in T and the cycles D_j are given by the cycles containing a boundary arrow. Then $\mathcal{J}_T = KQ/\langle \partial_a w \mid w \in W, a \in Q_1, s(a) \notin F \text{ or } t(a) \notin F \rangle$ is the frozen Jacobian algebra of (Q, W, F) .

Remark 4.11. All relations in I_F are commutativity relations and the algebra $\mathcal{J}(Q, W, F)$ is infinite dimensional.

Example 4.12. Given the following triangulation of a hexagon, consider the ice quiver with arrows as given in Figure 30 where the frozen vertices correspond to the boundary arcs. We note that the quiver is almost identical to the quiver of the Brauer graph algebra in Example 4.8 associated to the same triangulation of the hexagon, the difference being the absence in the ice quiver of boundary arrows around vertices of valency two.

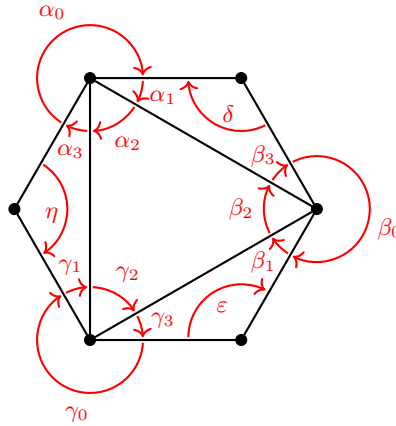


Figure 30: Quiver of the frozen Jacobian algebra associated to this triangulation of the hexagon where the frozen vertices correspond to the boundary arcs of the triangulation.

The arrows $\alpha_0, \beta_0, \gamma_0, \delta, \epsilon, \eta$ are frozen arrows and the boundary arrows are $\alpha_0, \beta_0, \gamma_0$. The

potential is given by

$$W = \alpha_1\beta_3\delta + \gamma_3\varepsilon\beta_1 + \alpha_2\gamma_2\beta_2 + \eta\gamma_1\alpha_3 - \alpha_0\alpha_1\alpha_2\alpha_3 - \beta_0\beta_1\beta_2\beta_3 + \gamma_0\gamma_1\gamma_2\gamma_3.$$

Then

$$I_F = \langle \beta_3\delta - \alpha_2\alpha_3\alpha_0, \gamma_2\beta_2 - \alpha_1\alpha_0\alpha_1, \eta\gamma_1 - \alpha_0\alpha_1\alpha_2, \gamma_3\varepsilon - \beta_2\beta_3\beta_0, \alpha_2\gamma_2 - \beta_3\beta_0\beta_1, \\ \delta\alpha_1 - \beta_0\beta_1\beta_2, \alpha_3\eta - \gamma_2\gamma_3\gamma_0, \beta_2\alpha_2 - \gamma_3\gamma_0\gamma_1, \varepsilon\beta_1 - \gamma_0\gamma_1\gamma_2 \rangle.$$

Let \mathcal{F} be the category of maximal Cohen-Macaulay modules over the Gorenstein tiled $K[x]$ -order defined in [26]. Then Demonet and Luo show the following

Theorem 4.13. [26] *Let P_n be the disc with n marked points in the boundary and let \mathcal{F} be the associated category of maximal Cohen-Macaulay modules over the Gorenstein tiled $K[x]$ -order defined in [26]. Then*

- 1) *The stable category $\underline{\mathcal{F}}$ is triangle equivalent to the cluster category of type A_{n-3} .*
- 2) *There are bijections between the indecomposable objects in \mathcal{F} and the arcs between marked points in P_n , including boundary arcs.*
- 3) *The bijection in 2) induces a bijection between triangulations T in P_n and the cluster-tilting objects M_T in \mathcal{F} .*
- 3) *The frozen Jacobian algebra $\mathcal{J}(Q, W, F)$ associated to a triangulation T of P_n is isomorphic to $\text{End}_{\mathcal{F}}(M_T)^{\text{op}}$.*

We note that $\underline{\mathcal{F}}$ is a Hom-finite 2-Calabi-Yau triangulated category. So the following Theorem by Palu can be applied.

Theorem 4.14. [64] *Let $\underline{\mathcal{C}}$ be a Hom-finite 2-Calabi-Yau triangulated category which is the stable category of a Frobenius category \mathcal{C} . Let $\overline{M}, \overline{M}'$ be cluster-tilting objects in $\underline{\mathcal{C}}$ with pre-images M, M' in \mathcal{C} . Then there is a triangle equivalence*

$$\mathcal{D}(\text{End}_{\mathcal{C}}(M)^{\text{op}}\text{-Mod}) \simeq \mathcal{D}(\text{End}_{\mathcal{C}}(M')^{\text{op}}\text{-Mod})$$

where, for A an algebra, $\mathcal{D}(A\text{-Mod})$ denotes the derived category of all A -modules.

Combining Theorem 4.13 and Theorem 4.14, it was observed in [61] that we obtain the following:

Theorem 4.15. *Let $\mathcal{J}(Q, W, F)$ and $\mathcal{J}(Q', W', F')$ be two frozen Jacobian algebras associated to two triangulations of a polygon. Then there is a triangle equivalence*

$$\mathcal{D}(\mathcal{J}(Q, W, F) - \text{Mod}) \simeq \mathcal{D}(\mathcal{J}(Q', W', F') - \text{Mod}).$$

Summary. We will summarise the comparison between frozen Jacobian algebras and Brauer graph algebras associated to the same triangulation of a polygon. Given a triangulation T of a polygon, let \mathcal{J}_T be the the frozen Jacobian algebra associated to T and

let A_T be the Brauer graph algebra where the Brauer graph is given by T including the boundary arcs. Then the quiver of \mathcal{J}_T is almost identical to the quiver of A_T (compare, for example the quiver in Figure 30 with the quiver in left hand side of Figure 29). The only difference is the absence in the quiver of the frozen Jacobian algebra of boundary arrows around vertices that are incident to only 2 arcs (i.e. around vertices that are not incident with any internal arcs).

Furthermore, given triangulations T, T' of the same polygon, by Theorem 4.15 we have a derived equivalence of the unbounded derived categories of the module category of all modules over the associated frozen Jacobian algebras \mathcal{J}_T and $\mathcal{J}_{T'}$,

$$\mathcal{D}(\mathcal{J}_T - \text{Mod}) \simeq \mathcal{D}(\mathcal{J}_{T'} - \text{Mod})$$

and by Theorem 4.6 we have a derived equivalence of the bounded derived categories of the module categories of finitely generated modules over the associated Brauer graph algebras A_T and $A_{T'}$

$$\mathcal{D}^b(A_T) \simeq \mathcal{D}^b(A_{T'}).$$

This might indicate that there is a structural connection between the frozen Jacobian algebra and the Brauer graph algebra associated to the same triangulation (of a polygon). That this should be the case has also been highlighted by the results in [25] and in [59].

5 Auslander-Reiten components

Auslander-Reiten theory for self-injective special biserial algebras has been well studied. Erdmann and Skowroński classify in [36] the Auslander-Reiten components for self-injective special biserial algebras. As Brauer graph algebras are symmetric special biserial, this gives the general structure of the Auslander-Reiten quiver for Brauer graph algebras. The result for representation-infinite algebras depends on the notions of domestic and polynomial growth. For the convenience of the reader, we recall these notions.

5.1 Finite, tame and wild representation type, domestic algebras and algebras of polynomial growth

Let A be a finite dimensional K -algebra, then we say that A is of *finite representation type*, if up to isomorphism there are only finitely many distinct indecomposable A -modules.

We say A is of *tame representation type*, if for any positive integer n , there exists a finite number of $A - K[x]$ -bimodules M_i , for $1 \leq i \leq d_n$ such that M_i is finitely generated and is free as a left $K[x]$ -module and such that all but a finite number of isomorphism classes of indecomposable n -dimensional A -modules are isomorphic to $M_i \otimes_{K[x]} K[x]/(x - a)$, for $a \in K$ and $1 \leq i \leq d_n$. For each n , let $\mu(n)$ be the least number of such $A - K[x]$ -bimodules. Following [78], we say that A is of *polynomial growth* if there exists a positive

integer m such that $\mu(n) \leq n^m$ for all $n \geq 2$. We say that A is *domestic* if $\mu(n) \leq m$ for all $n \geq 1$ [68] and we say that A is *d-domestic* if d is the least such integer m .

Note that every domestic algebra is of polynomial growth [79].

Let $K\langle x, y \rangle$ be the group algebra of the free group in two generators. We say that A is of *wild representation type* if there is a $K\langle x, y \rangle - A$ bimodule M such that M is free as a $K\langle x, y \rangle$ -module, and if X is an indecomposable $K\langle x, y \rangle$ -module then $M \otimes_K X$ is an indecomposable A -module, and if for some $K\langle x, y \rangle$ -module Y we have $M \otimes_K X \simeq M \otimes_K Y$ then $X \simeq Y$.

By [29] an algebra of infinite representation type that is not of tame representation type is of wild representation type.

5.2 Domestic Brauer graph algebras

In the following let Λ be a representation-infinite Brauer graph algebra with Brauer graph $G = (G_0, G_1, m, \mathfrak{o})$. Bocian and Skowroński give a characterization of the domestic Brauer graph algebras.

Theorem 5.1. [14] *Let Λ be a Brauer graph algebra with Brauer graph G . Then*

(1) Λ is 1-domestic if and only if one of the following holds

- G is a tree with $m(i) = 2$ for exactly two vertices $i_0, i_1 \in G_0$ and $m(i) = 1$ for all $i \in G_0, i \neq i_0, i_1$.
- G is a graph with a unique cycle of odd length and $m \equiv 1$.

(2) Λ is 2-domestic if and only if G is a graph with a unique cycle of even length and $m \equiv 1$.

(3) There are no n -domestic Brauer graph algebras for $n \geq 3$.

Example 5.2. (1) Example of a Brauer graph of a 1-domestic Brauer graph algebra.

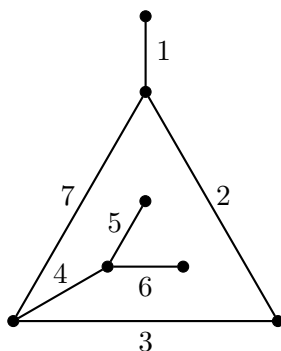


Figure 31: Brauer graph of a 1-domestic Brauer graph algebra

(2) Example of Brauer graph of a 2-domestic Brauer graph algebra.

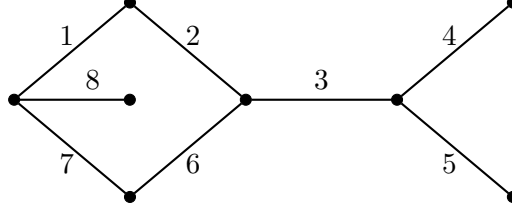


Figure 32: Brauer graph of a 2-domestic Brauer graph algebra

5.3 The stable Auslander-Reiten quiver of a self-injective special biserial algebra

We refer the reader to the textbooks [8, 9, 75] for the definition and general set-up of Auslander-Reiten theory. Let Λ be a self-injective K -algebra (recall that a finite-dimensional K -algebra is self-injective if every projective module is injective). In the following denote by Γ_Λ the Auslander-Reiten quiver of Λ and by ${}_s\Gamma_\Lambda$ the stable Auslander-Reiten quiver of Λ . Denote by $\mathbf{mod} - \Lambda$ the module category of finitely generated Λ -modules and by $\underline{\mathbf{mod}} - \Lambda$ the stable category of Λ . Let $\nu : \Lambda - \mathbf{mod} \rightarrow \Lambda - \mathbf{mod}$ be the Nakayama functor, given by sending $N \in \Lambda - \mathbf{mod}$ to $D(\text{Hom}_\Lambda(N, \Lambda))$. Let Ω be Heller's syzygy functor, that is $\Omega : \underline{\mathbf{mod}} - \Lambda \rightarrow \underline{\mathbf{mod}} - \Lambda$ given by $M \in \underline{\mathbf{mod}}\Lambda$, $\Omega(M) = \ker \pi$ where $\pi : P_M \rightarrow M$ is the projective cover of M . Then the Auslander-Reiten translate τ is given by $\tau M = \nu\Omega^2 M$, for $M \in \mathbf{mod} - \Lambda$. Recall that if Λ is a symmetric K -algebra then $\nu = \text{Id}$ and $\tau = \Omega^2$.

Following Riedtmann [66], any connected component of ${}_s\Gamma_\Lambda$ is of the form $\mathbb{Z}T/G$ where T is an oriented tree and G is an admissible automorphism group. We call T the tree class of ${}_s\Gamma_\Lambda$. The tree class of any component of the stable Auslander-Reiten quiver of an algebra containing a periodic module, that is a module M such that $\tau^n M \simeq M$, for some n , is equal to A_∞ [49]. The shapes of the translations quivers $\mathbb{Z}A_\infty, \mathbb{Z}A_\infty/\langle \tau^n \rangle, \mathbb{Z}A_\infty^\infty, \mathbb{Z}\tilde{A}_{p,q}$ are described in [49]. By $\tilde{A}_{p,q}$ we denote the following orientation of the quiver with underlying extended Dynkin diagram of type \tilde{A}_n

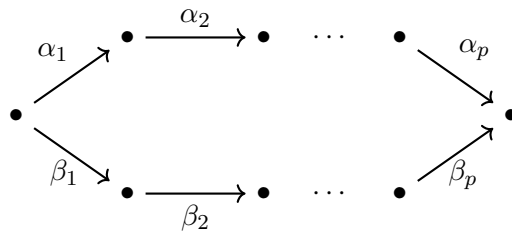


Figure 33: $\tilde{A}_{p,q}$

If a connected component of ${}_s\Gamma_\Lambda$ does not contain any projective or injective module, we call that component *regular*.

The following two theorems give a complete description of the possible components of the stable Auslander-Reiten quiver of a self-injective special biserial algebra of tame representation type.

Theorem 5.3. [36, Theorem 2.1] *Let Λ be a self-injective special biserial algebra. Then the following are equivalent:*

- (1) Λ is representation-infinite domestic.
- (2) Λ is representation-infinite of polynomial growth.
- (3) ${}_s\Gamma_\Lambda$ has a component of the form $\mathbb{Z}\tilde{A}_{p,q}$.
- (4) ${}_s\Gamma_\Lambda$ is infinite but has no component of the form $\mathbb{Z}A_\infty$.
- (5) All but a finite number of components of Γ_Λ are of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$.
- (6) ${}_s\Gamma_\Lambda$ is a disjoint union of m components of the form $\mathbb{Z}\tilde{A}_{p,q}$, m components of the form $\mathbb{Z}A_\infty/\langle\tau^p\rangle$, and m components of the form $\mathbb{Z}A_\infty/\langle\tau^q\rangle$.

Theorem 5.4. [36, Theorem 2.2] *Let Λ be a self-injective special biserial algebra. Then the following are equivalent:*

- (1) Λ is not of polynomial growth.
- (2) ${}_s\Gamma_\Lambda$ has a component of the form $\mathbb{Z}A_\infty$.
- (3) Γ_Λ has infinitely many (regular) components of the form $\mathbb{Z}A_\infty$.
- (4) ${}_s\Gamma_\Lambda$ is a disjoint union of a finite number of components of the form $\mathbb{Z}A_\infty/\langle\tau^n\rangle$, with $n > 1$, infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$ and infinitely many components of the form $\mathbb{Z}A_\infty$.

5.4 Green walks, double stepped Green walks and exceptional tubes

In [47] Sandy Green defined a *walk around the Brauer tree* and showed that for certain modules it encodes their minimal projective resolution.

Let A be a Brauer graph algebra. We call a simple A -module with uniserial projective cover a *uniserial simple A -module*. The uniserial simple A -modules correspond exactly to the truncated edges in the Brauer graph. Recall that a truncated edge is a leaf in the Brauer graph where the leaf vertex has multiplicity one. Green showed that the successive terms in the minimal projective resolution of a uniserial simple A -module are given by 'walking' around the Brauer tree. Roggenkamp showed that the same is true for a Brauer graph algebra [74].

5.4.1 Definition of Green's walk around the Brauer graph

A Green walk on a Brauer graph is given by a graph theoretic path on the graph underlying the Brauer graph. It is defined in terms of successor relations. Let $A = KQ/I$ be a Brauer graph algebra with Brauer graph G .

Definition of Green walks. Suppose first that G consists of a single loop i_0 . Then there are two Green walks of period 1 on G both given by i_0 , see Figure 34.

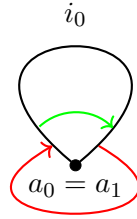


Figure 34: Green walk if the Brauer graph consists of a single loop

Now suppose that G contains more than one edge and let i_0 be an edge in G with vertices a_0 and a_1 . Suppose first that $a_0 \neq a_1$, that is suppose that i_0 is not a loop. Let i_1 be the successor of i_0 at the vertex a_1 . Now let a_2 be the other vertex of i_1 and let i_2 be the successor of i_1 at vertex a_2 . Note that if i_1 is a loop then $a_1 = a_2$ and the successor i_2 of i_1 appears as in Figure 35 (1) and (2).

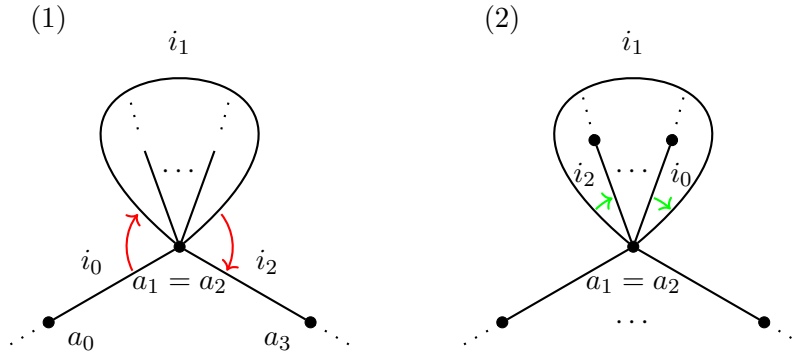


Figure 35:

Let a_3 be the other vertex of i_2 and let i_3 be the successor of i_2 at a_3 (treating the case that i_2 is a loop as above). In this way, we define an infinite sequence of edges i_0, i_1, i_2, \dots , called a *Green walk starting at i_0* . Let k be the minimal number n such that $i_{n+k} = i_k$ for all $k \geq 0$. Then we say that $(i_0, i_1, \dots, i_{k-1})$ is a *Green walk of period k* . Since G is a finite graph, every Green walk is periodic.

Note that if i_0 is a loop, that is if $a_0 = a_1$, then there are two distinct Green walk starting at i_0 depending on the successor i_1 of i_0 chosen. This is illustrated in Figure 36.

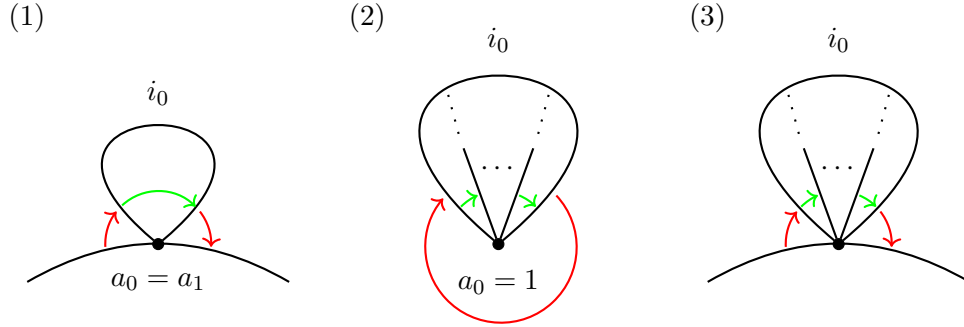


Figure 36: Green walk starting on loops

Every edge appears in exactly two (not necessarily distinct) Green walks, corresponding to the two successors of a given edge. Any edge corresponding to a leaf (truncated or not) appears twice in the same Green walk.

In order to avoid the complication with loops in the definition of a Green walk, many authors consider half-edges instead of edges using, for example, the language of ribbon graphs, see [28, 74, 33, 1].

Remark 5.5. Green walks are independent of the multiplicities in the Brauer graph, they only depend on the graph and the orientation.

Given a Green walk i_0, i_1, i_2, \dots at some edge i_0 of a Brauer graph G , a *double stepped Green walk at i_0* is given by the sequence of edges i_0, i_2, i_4, \dots in G .

Theorem 5.6. [47, 74] Let A be a Brauer graph algebra with Brauer graph G .

(1) Let S be a uniserial simple A -module and let i_0 be the edge in G corresponding to S and let i_0, i_1, \dots be the Green walk starting at i_0 . Then the n th projective in a minimal projective resolution of S is given by P_{i_n} , the indecomposable projective associated to the edge i_n .

(2) Let j_0 be the edge in G corresponding to the simple top of a maximal uniserial submodule M of some projective indecomposable module and let j_0, j_1, \dots be the Green walk starting at j_0 . Then the n th projective in a minimal projective resolution of M is given by P_{j_n} .

Example 5.7. Example of the two Green walks on the 1-domestic Brauer graph given in Example 5.2 and the minimal projective resolution of the simple corresponding to the edge 1 given by the walk starting at 1.

and only if M is uniserial simple or a maximal uniserial submodule of an indecomposable projective A -module.

5.5 Exceptional Tubes

Exceptional tubes are Auslander-Reiten components of the form $\mathbb{Z}A_\infty/\langle\tau^n\rangle$, where the integer $n \geq 1$ denotes the rank of the tube and where for tubes of rank one, we distinguish between the exceptional tubes – that is, the tubes of rank one consisting of string modules, of which there are finitely many – and the homogeneous tubes of rank one, which consist only of band modules.

Theorem 5.10. [33, Theorem 4.3] *Let A be a representation-infinite Brauer graph algebra with Brauer graph G .*

- (1) *The number of exceptional tubes in ${}_s\Gamma_A$ is given by the number of double-stepped Green walks on G .*
- (2) *The length of the double-stepped Green walk gives the rank of the associated exceptional tube.*

Example 5.11. (1) Suppose that A is a Brauer graph algebra with Brauer graph as given in Figure 39. Then there are three distinct Green walks on G of lengths 1, 5, and 6, and thus 4 double-stepped Green walks of lengths 1, 3, 3, and 5 giving rise to four exceptional tubes of respective ranks 1, 3, 3, and 5.

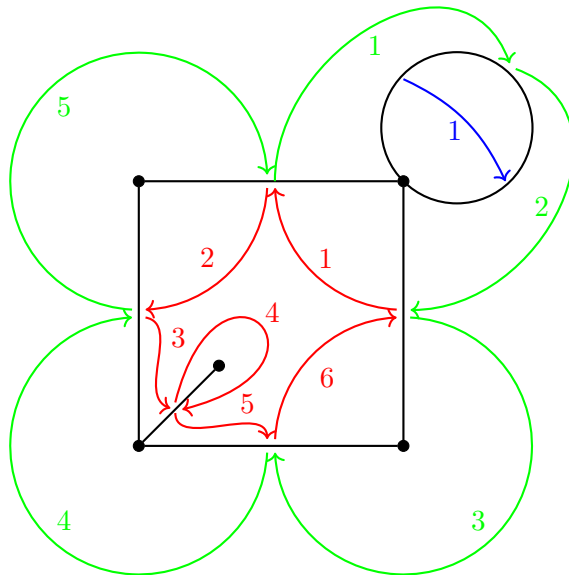


Figure 39: Brauer graph with three distinct Green walks.

More precisely, the stable Auslander-Reiten quiver of A is a disjoint union of infinitely many components of the forms $\mathbb{Z}A_\infty/\langle\tau\rangle$ and $\mathbb{Z}A_\infty^\infty$ and two components of the form $\mathbb{Z}A_\infty/\langle\tau^3\rangle$, one component of the form $\mathbb{Z}A_\infty/\langle\tau^6\rangle$ and one component of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$ (given by a the string corresponding to the arrow inside the unique loop and not a band module as the ones giving rise to the infinitely many other component of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$).

(2) For completeness we include one example of a non-planar Brauer graph G' and the Green walks on G' . Suppose that A' is a Brauer graph algebra with Brauer graph G' .

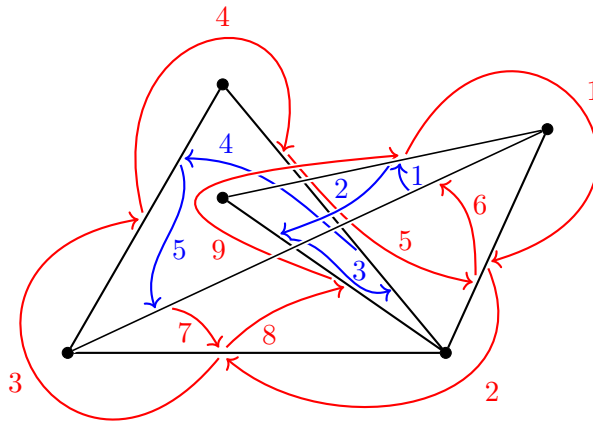


Figure 40: Picture of Brauer graph G' of A' and the two distinct Green walks on G' of lengths 5 and 9, respectively

Then the stable Auslander-Reiten quiver of A' is a disjoint union of infinitely many components of the forms $\mathbb{Z}A_\infty/\langle\tau\rangle$ and $\mathbb{Z}A_\infty^\infty$ and one component of the form $\mathbb{Z}A_\infty/\langle\tau^5\rangle$ and one component of the form $\mathbb{Z}A_\infty/\langle\tau^9\rangle$.

5.6 Auslander-Reiten components of non-domestic Brauer graph algebras

Let A be a representation-infinite non-domestic Brauer graph algebra with Brauer graph G . It follows directly from Theorem 5.4 and Theorem 5.10 that ${}_s\Gamma_\Lambda$ is the disjoint union of

- k components of the form $\mathbb{Z}A_\infty/\langle\tau^{l_i}\rangle$, where k is the number of double stepped Green walks and l_i is the length of the i th double-stepped Green walk, for $1 \leq i \leq k$,

- infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$
- infinitely many components of the form $\mathbb{Z}A_\infty^\infty$.

5.7 Auslander-Reiten components of domestic Brauer graph algebras

Let A be a representation-infinite m -domestic Brauer graph algebra with Brauer graph G . It follows from Theorem 5.3 and Theorem 5.10 that ${}_s\Gamma_\Lambda$ is the disjoint union of

- m components of the form $\mathbb{Z}\tilde{A}_{p,q}$
- m components of the form $\mathbb{Z}A_\infty/\langle\tau^p\rangle$
- m components of the form $\mathbb{Z}A_\infty/\langle\tau^q\rangle$
- infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$.

More precisely,

Theorem 5.12. [33, Theorem 4.4, Corollary 4.5] *Let A be a representation-infinite domestic Brauer graph algebra with Brauer graph G with n edges. If G has a cycle then it is unique and let n_1 be the number of (additional) edges on the inside of the cycle and n_2 the number of (additional) edges on the outside of the cycle. In the notation above:*

- (1) *If A is 1-domestic, then $m = 1$ and $p + q = 2n$. Furthermore,*
 - *if G is a tree, then $p = q = n$,*
 - *if G has a unique cycle (of odd) length l , then $p = l + 2n_1$ and $q = l + 2n_2$.*
- (2) *If A is 2-domestic with unique cycle (of even) length l , then $m = 2$ and $p = \frac{l}{2} + n_1$ and $q = \frac{l}{2} + n_2$ such that $p + q = n$.*

Example 5.13. (1) Example of the stable Auslander-Reiten quiver of a 1-domestic Brauer graph algebra with Brauer graph (including the two distinct Green walks) given by

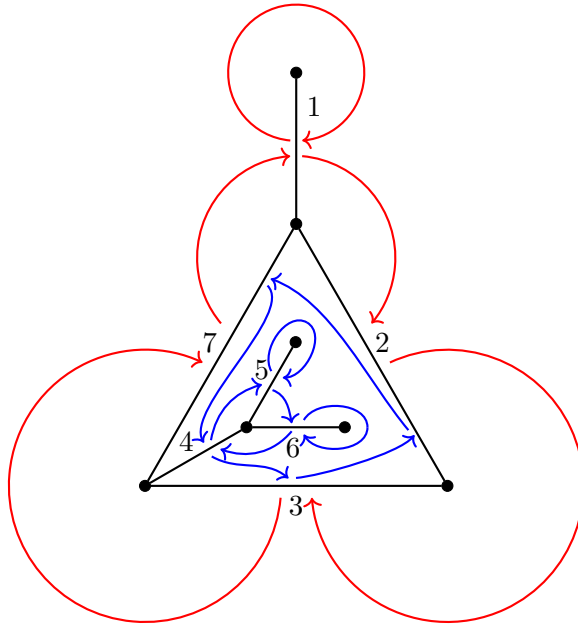


Figure 41: Example 5.5.1

Here $l = 3$, $n_1 = 3$ and $n_2 = 1$ and therefore by Theorem 5.12, $p = 9$ and $q = 5$ and the stable Auslander-Reiten quiver consists of one component of the form $\mathbb{Z}\tilde{A}_{9,5}$, one of the form $\mathbb{Z}A_\infty/\langle\tau^9\rangle$ and one of the form $\mathbb{Z}A_\infty/\langle\tau^5\rangle$ as well as infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$. Note that $p = 9$ is the length of the unique double stepped Green walk on the inside of the 3-cycle and $q = 5$ is the length of the unique double stepped Green walk on the outside of the 3-cycle.

(2) Example of the stable Auslander-Reiten quiver of a 2-domestic Brauer graph algebra with Brauer graph (including the two distinct Green walks) given by

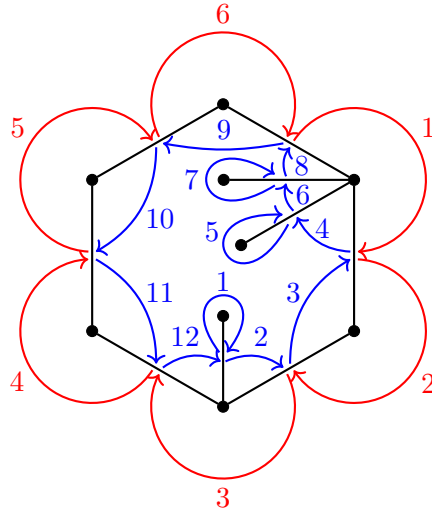


Figure 42: Example 5.8

Here $l = 6$, $n_1 = 3$ and $n_2 = 0$ and therefore by Theorem 5.12, $p = 6$ and $q = 3$ and the stable Auslander-Reiten quiver consists of two components of the form $\mathbb{Z}\tilde{A}_{6,3}$, two of the form $\mathbb{Z}A_\infty/\langle\tau^6\rangle$ and one of the form $\mathbb{Z}A_\infty/\langle\tau^3\rangle$ as well as infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$. Note that $p = 6$ is the length of the two double stepped Green walks on the inside of the 6-cycle and $q = 3$ is the length of the two double stepped Green walks on the outside of the 6-cycle.

5.8 Position of modules in the Auslander-Reiten quiver

In this section we determine, for any representation-infinite Brauer graph algebra (domestic or non-domestic) which simple modules and radicals of indecomposable projective modules (and hence which projective modules) lie in exceptional tubes and which do not. For that we define the notion of exceptional edges and we will see that the simple modules and the radicals of the projective indecomposable modules associated to the exceptional edges lie in the exceptional tubes of ${}_s\Gamma_A$.

Definition 5.14. [33] (1) Let G be a Brauer graph such that G is not a Brauer tree (that is if G is a tree then there are at least two vertices of multiplicity greater than 2). An *exceptional subtree* of G is a subgraph T of G such that

- (i) T is a tree,
- (ii) there exists at unique vertex v in T such that $(G \setminus T) \cup v$ is connected,
- (iii) every vertex of T has multiplicity 1 except perhaps v .

We call v the *connecting vertex* of T .

(2) Let G be a Brauer graph. An edge e in G is called an *exceptional edge* if e lies in an exceptional subtree of G and it is called a *non-exceptional edge* otherwise.

Note that the non-exceptional edges in a Brauer graph G are all connected to each other and that a non-exceptional edge is never truncated.

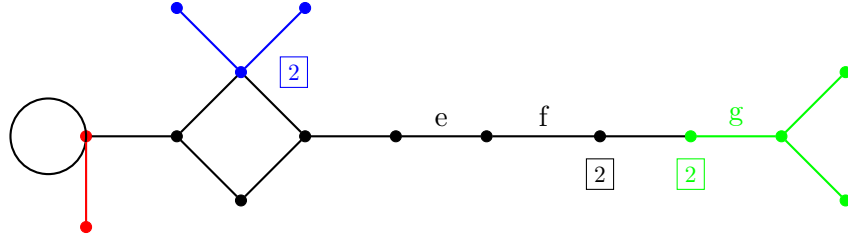


Figure 43: Example of a Brauer graph with three exceptional subtrees

Theorem 5.15. [33, Theorem 4.18] Let A be a representation-infinite Brauer graph algebra with Brauer graph G and let e be an edge in G . Then the simple module S_e associated to e and the radical $\text{rad } P_e$ of the indecomposable projective module P_e associated to e lie in an exceptional tube of ${}_s\Gamma_A$ if and only if e is an exceptional edge.

Furthermore, S_e and $\text{rad } P_e$ lie in the same exceptional tube if and only if e occurs twice within the same double-stepped Green walk.

Theorem 5.16. [33, Theorem 4.21] Let A be a representation-infinite Brauer graph algebra with Brauer graph G and let e and f be two not necessarily distinct non-exceptional edges in $G = (G_0, G_1, m, \mathfrak{o})$. Then S_e and $\text{rad } P_f$ lie in the same component of ${}_s\Gamma_A$ if and only if A is 1-domestic or there exists an even length path

$$\bullet \xrightarrow{e=e_1} v_0 \quad \bullet \xrightarrow{e_2} v_1 \quad \bullet \xrightarrow{\dots} v_2 \quad \dots \quad \bullet \xrightarrow{f=e_{2n}} v_{2n-1} \quad \bullet \xrightarrow{\dots} v_{2n}$$

of non-exceptional edges such that

- (i) none of the edges e_i , for $1 \leq i \leq 2n$, are a loop,
- (ii) e_i and e_{i+1} are the only non-exceptional edges incident with v_i , for $1 \leq i \leq 2n - 1$,
- (iii) $m(v_i) = 1$, for $1 \leq i \leq 2n - 1$, except if $e_i = e_{i+1}$ in which case $m(v_i) = 2$, for $1 \leq i \leq 2n - 1$.

Example 5.17. For the Brauer graph algebra with Brauer graph given by the graph in Figure 43, the simple module S_e associated to the edge e and the $\text{rad } P_f$ where P_f is the indecomposable projective module associated to the edge f , are in the same component of the Auslander-Reiten quiver. Since the edge corresponding to e is not exceptional, this component is not an exceptional tube but of the form $\mathbb{Z}A_\infty$. On the other hand, the simple module S_g as well as $\text{rad } P_g$, the radical of the indecomposable projective associated to

the edge g lie in the same exceptional tube (of rank 27) since the edge g is exceptional and occurs twice in the same double-stepped Green walk.

6 Appendix

6.1 Derived equivalences: Rickard's Theorem

Let A be a finite dimensional K -algebra. We denote by $\mathcal{D}^b(A)$ the bounded derived category of finite dimensional A -modules.

If M is a tilting A -module then the categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$, where $B = \text{End}_A(M)$, are equivalent [48, 19].

Rickard showed that all derived equivalences of finite dimensional algebras are of a similar nature. Namely let $\mathbf{mod}\text{-}A$ be the category of all finitely presented A -modules and P_A the category of all finitely generated projective A -modules. We denote by $\mathcal{K}^b(P_A)$ the homotopy category of bounded complexes in P_A .

Theorem 6.1. [69, 1.1] *Let A and B be two finite-dimensional algebras. The following are equivalent:*

- (a) $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.
- (b) $\mathcal{K}^b(P_A)$ and $\mathcal{K}^b(P_B)$ are equivalent as triangulated categories.
- (c) B is isomorphic to the endomorphism ring of an object T of $\mathcal{K}^b(P_A)$, that is $B \simeq \text{End}_{\mathcal{K}^b(P_A)}(T)$, such that
 - (i) For $n \neq 0$, $\text{Hom}(T, T[n]) = 0$.
 - (ii) $\text{add}(T)$, the full subcategory of $\mathcal{K}^b(P_A)$ consisting of direct summands of direct sums of copies of T , generates $\mathcal{K}^b(P_A)$ as a triangulated category.

Moreover, any equivalence as in (a) restricts to an equivalence between the full subcategories consisting of objects isomorphic to bounded complexes of projectives (which are equivalent to $\mathcal{K}^b(P_A)$ and $\mathcal{K}^b(P_B)$ respectively).

A complex T as in Theorem 6.1 (c) is called a *tilting complex over A* .

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LEICESTER, UNIVERSITY ROAD, LEICESTER LE1 7RH, UNITED KINGDOM
E-mail address: schroll@leicester.ac.uk