THE SPACE BETWEEN THE UNKNOWN AND A VARIABLE

Dave Hewitt

School of Education, University of Birmingham

The meaning given to letters is significant for students' ability to be successful with algebraic tasks. Recent studies have noted that even when students have a sense of generalised number, they often have a natural number bias in the values they think a letter can take. This study analyses interviews from 13 students across two schools to explore the meaning they had for letters. The responses supported the idea that some students have a natural number bias and also that the notion of a letter representing a fraction is problematic. In addition, three other factors emerged which affected the meaning given to a letter: what was mentally stressed; the desire to avoid "messy" calculations; and viewing an equation as an example of a wider class of equations.

BACKGROUND

Several studies have identified difficulties students have with algebra (Herscovics, 1989; Kieran, 1981; Küchemann, 1981). These difficulties relate to a number of factors, including the way in which the equals sign is viewed (Sáenz-Ludlow & Walgamuth, 1998), the need to view an expression both as a process to carry out and as an object in its own right (Sfard, 1991) and the parsing of expressions (Gunnatsson, Hernell, & Sönnerhed, 2012; MacGregor & Stacey, 1997). Another difficulty centres on the meaning given to a letter within an expression. Küchemann's (1981) seminal research identified a hierarchy of six ways in which letters were used by students: *Letter evaluated, Letter not used, Letter as object, Letter as specific unknown, Letter as generalised number*, and *Letter as variable*. A good understanding of the concept of a variable can be core to future success within complex algebraic problems (Trigueros, Ursini, & Escandón, 2012), so students' understanding of letters, or literal symbols, is significant.

The meaning placed given to a literal symbol has changed over time within the history of mathematics (Usiskin, 1988). Ely and Adams (2012) and Christou and Vosniadou (2012) suggest that initially literal symbols only stood for natural numbers and only later was their meaning widened to become the symbolic world of real numbers. Usiskin (1988) suggests that not only has the meaning of a literal symbol changed over the course of history but that it can change according to your conception of what algebra really is.

Recent studies have shown that many students have a natural number bias when considering which numbers a letter might represent (Christou & Vosniadou, 2012; Vamvakoussi, Van Dooren, & Verschaffel, 2012). This suggests a complex journey between having a sense of, in Küchemann's (1981) terms, letter as generalised number and letter as variable.

THE STUDY

This study looked at the meaning students gave to algebraic expressions and equations, including the letters which appeared in them. In all 13 students were interviewed, aged between 12-13 years old, from two non-selective secondary schools in the UK (six from an all-girls school, S1, and seven from a mixed sex school, S2). The questions consisted of presenting students with an expression or an equation and asking them to describe what this meant. With some questions the focus was on the meaning of the letter, or letters, which appeared in that expression or equation. Students were not explicitly asked to solve equations as this might have influenced the meaning they gave for a letter. The questions were presented in two different contexts: the first was simply on a piece of paper, and the second was within a computer environment called *Grid Algebra* which had been used in both schools. Similar questions were presented in each of these environments at different points within the interview. Except for two occasions, there were no differences between the responses students gave to the paper environment compared with the computer environment and as a consequence this is not discussed further in this paper (more detail about the software can be found in Hewitt, 2012). The style of the interviews was semi-structured in that all students were presented with the same questions with additional questions used as appropriate to probe further into the meanings they had. A framework for the interview questions was influenced by Knuth et al. (2005) where an expression or equation was presented and students asked for the meaning they gave to the letter. Follow up questions were guided by the literature on natural number bias (Christou & Vosniadou, 2012) where they had asked students to indicate numbers which could be substituted for a letter. In my case I changed this and offered specific numbers: one larger natural number, one negative number, one decimal and one fraction. In addition, I also presented some expressions and equations and asked what the expression/equation meant. This was to gauge what sense they had of the expression as a whole, whether they interpreted the order of operations correctly and see whether, in the case of equations, they would naturally try to solve the equation without a prompt. Although not the focus of this paper, in general their understanding of order of operations was good.

The interviews lasted between 20-30 minutes and were audio recorded. They were all transcribed and initial analysis was carried out on whether there were significant differences between the responses to similar questions within the two environments. This was to see whether learning had remained context dependant or whether students were able to transfer their learning from the computer environment to the traditional paper environment. As reported above, students invariably responded in a similar way to both environments. Additional analysis was then carried out with a Grounded Theory approach (Strauss & Corbin, 1990) focused on the meanings students gave to the letters involved in expressions and equations. More emphasis was given in the analysis to this than whether their arithmetic, for example, was accurate or not. Thus if they showed an awareness that they needed to carry out inverse operations to solve an equation, and that this meant the letter represented one particular value, then this was considered to be of more interest than whether they did the inverse operations in the correct order or whether they made an arithmetic error. Indeed, at times during the interview I offered to be a human calculator for the odd student who was struggling with arithmetic calculations. Throughout this analysis, a number of themes developed and sections of the transcripts were coded accordingly.

RESULTS

Some students showed a clear difference between their meaning of a letter within an expression, such as $4x+2$, and an equation such as $2(x+3)=14$. For example, Joanna (S1, pseudonyms used for all students) said in relation to the expression that "it [*x*] can mean any number in the world. It's kind of the substitution for a number and you can put any number and replace *x*". In relation to the equation above, she said, after solving the equation, that "*x* has to be four because if you put any other number in then it wouldn't equal 14." Sharon (S1) also talked about this same equation and when asked whether *x* could be any number she said "Yeah, as long as it makes 14" and thought that there could be two or three ways of making 14. She showed awareness that the equation is essentially a statement which says that these calculations have to equal 14. It is another awareness altogether that with such a linear equation there would only be one such value which would achieve this. She still had a sense that *x* stood for a determined value or values, and that there was not free choice as to the value *x* could take.

Sylvia (S2) said the following when talking about the expression $4x+2$: "It means any number, for an equation, for anything. Like if you don't know what you've got, how much a specific number is, you put a letter for it". In her case the language shifts from "*any number*" to a "*specific number*". Her use of the word "*equation*" also raised questions about how she was viewing this expression. With another student, Myra (S1), she was quite unsure whether *x* could take any other value than three in the equation $4x+2=14$, saying "I'm not sure. I don't think so. I'm not sure, I don't think so. Well it could be something like...no, I'm not sure. I don't know."

The interviews revealed some interesting thinking with regard to how a letter was viewed and three themes emerged: *What does 'any number' mean?*; *Seeing a class of possible equations*; and *Temporal viewpoints*.

What does 'any number' mean?

Chris (S2) felt that *x* could be any number with the expression $2(x+3)$. However, I continued by asking him whether it could be 562 and he replied "no". Upon further questioning it appeared that he felt this was too big a number.

Matt (S2) talked about the meaning of *f* in $2\left(\frac{f-6}{2}+2\right)$ and said "*f* is like any number. So, it could be like 1, 2, 3 or 4, 5." This list was a list of natural numbers and it could have been just a convenient list to offer as examples or it could have been more about him feeling that *f* had to be a natural number. This issue appeared in other expressions put in front of him where he would start off saying that the letter could be any number but end up restricting the possibilities to natural numbers. For example, with $4x+2$ he said initially that x could be any number. However, when asked whether it could be 532 he said "Probably yeah. But it'll be an odd question." He was uncertain whether it could be negative five and felt it couldn't be 1.8. He ended up feeling that it had to be a "whole number". This was the case when he considered $\frac{3(n+2)}{6}-1$ as well, n had to be a whole number.

Myra (S1) started off saying x could be "any number... it's just any sort of random number" when talking about $4x+2$. However, she then continued to say that it had to be an even number "because it doesn't really work as well with odd numbers. It's got to be even." Her reason for this was because it was easier to divide and times by even numbers than by odd numbers. This sense of something becoming more difficult or 'messy' influenced some of the thinking as to what value a letter could take. So, although eventually agreeing that 2.8 could work she talked about it not working "as well as" other numbers because it was a decimal.

Sharon (S1) felt that although the letters in $k\left(\frac{4}{t}+6\right)-p$ can take different values "it has to like make sense, if it doesn't it's just going to be wrong." So this, in his view, restricted the values to not allowing "weird" [his word] numbers.

With expressions, where the letter represented a variable, the letter taking on the value of *a half* seemed to be particularly problematic for seven of the 13 students interviewed. For most of these seven, they were quite happy that a letter could be 562 or -5 or even 1.8. However, a half was another matter. Four students said "no" immediately and the other three students hesitated before replying or indicated that they were uncertain. For example, Sarah (S2) was quite happy that x in $4x+2$ could be 562, -5 or 1.8 but when asked about a half she said "probably". When asked whether that meant probably yes or probably no, she said "no".

Abigail (S2) also felt that x could not be a half in $2(x+3)$ "because that's put as a like one dash two instead, but if it was half of like in numbers" then it would be fine. The notational form of a half as $\frac{1}{2}$ seemed to be problematic as opposed to the decimal form of 0.5. Even one the students, Romana (S1), who responded positively quite quickly to the possibility of the letter being a half, still re-phrased my wording of "a half" when agreeing: "point five, yeah".

Seeing a class of possible equations

With some of the interview questions, I presented an expression, such as $2(x+3)$, asking them about what the letter x means, followed by the same expression but with it equal to a numerical value, such as $2(x+3)=14$. Some of the students' responses to the equation indicated that they were taking this equation as just an example of equations in general, rather than treating this as a particular case. Matt (S2) felt that the situation regarding what x meant had not changed going from the expression to the equation "because x plus 3 would be x plus 3 and then you do x times 2. So, yeah it'd be the same but the answer would just be different." Although his statement about the operations was not quite correct, the relevant point here was that he was seeing that x could still be any number, it was just that the 14 at the end would have to be a different number. So, he saw the 14 as an example of 'an answer' rather than it being a particular requirement that $2(x+3)$ *must* be 14. Sarah (S2) also seemed to be thinking the same when she responded to being asked whether x could be 562. She said "it depends" and on further questioning it became clear that it depended upon what number was placed after the equals sign.

Myra (S1) seemed to consider keeping the 'answer' of 14 the same with $4x+2=14$ but explored changing the operations carried out on x in order that x could take on different values whilst still ending up with 14. She started off saying that x had to be three but then decided it could be a different number completely if you could work out the operations to make it equal 14 in the end. She felt that "if you worked it out hard enough then I suppose you could do it [make x have a different value]".

Temporal viewpoints

One student, Rebecca (S1), considered which values x could take with the equation $2(x+3)=14$. After much discussion about what x could stand for, Rebecca gave a clear articulation which summed up her thoughts: "When you look at it, it's like, it could be any number. You don't know, you can guess. And then when you work it out it would be one certain number." Here she gave me a sense that her answer to my question would change according to her state of mind at that particular moment in time. Initially, it could be any number as she had not started working it out yet. However, once it had been worked out, it was one particular number.

DISCUSSION

There were a few students who felt that the letter within an expression could stand for "any number" and so appeared to have a sense of generalised number (Küchemann, 1981) but then revealed that by "any number" they meant natural numbers. This fits in with the natural number bias identified by Christou and Vosniadou (2012) and Vamvakoussi et al. (2012). However, students also talked about not including certain numbers due it becoming "messy" or "not working so well". This seemed to indicate that it was not only a matter of the letter itself being a natural number but that, whichever value the letter took, the ensuing calculations should involve only natural numbers. This led to more restrictive domains for the possible values of the letter. For example, Myra wanted the letter not to be an odd number as division was involved and she felt division was "easier" with even numbers.

Many of the students felt that the letter could not represent $\frac{1}{2}$ but some felt that it could represent 0.5. This raised the issue of how they viewed fractions. Stafylidou and Vosniadou (2004) point out that the development of students' concept of fraction is different to that for natural numbers due to its particular notation. Students have difficulty relating the two numbers involved with the numerator and denominator and as such they "think of fractions as pairs of whole numbers and not as single numbers" (Christou & Vosniadou, 2012, p. 515). This might account for why the students rejected "a half" as a possible value for x since they might have viewed the fractional form of a half as not a single number.

Three of the students saw a particular equation as a representation of a class of equations, where either "the answer" (the number to the right of the equals sign in these cases) could be changed, or the operations carried out on the letter could be changed. By considering a class of equations they felt that x could take on different values.

Lastly, Rebecca's response to $2(x+3)=14$ gave a sense of her state of mind at particular moments in time. On first seeing the equation, perhaps before taking on board the particular numbers and operations involved, she had an initial feeling that she did not know what the letter was. So, at that moment in time, x could be anything. there was a sense of the *potential* held within the letter x. However, at a later point in time, when she had been able to note the particular operations involved, she was able to establish the particular value of the letter. As a consequence the potential ("any number") shifts into the actual ("a particular number"). Bardini et al. (2005, p. 129, their emphasis) commented that for some students "it [a variable] is merely a *temporally indeterminate number* whose fate is to become determinate at a certain point." My understanding of this comment is that more information may be provided in the future which will determine the value of a letter. However, in Rebecca's case it was not a matter of more information arriving but that she shifted her attention onto parts of the information which was already currently available (i.e. the particular operations). Thus the shift from *temporally indeterminate number* to *determinate* was one which reflected her thoughts at particular moments in time and was determined by what she chose to stress at that moment.

The responses here not only support some earlier studies regarding natural number bias and the reluctance to consider fractions, but also offer three other ways in which students' thinking can affect the way letters are viewed in the space between unknown and variable. These are: firstly, how the meaning for a letter can be a temporal matter reflecting a state of mind at a particular point in time; secondly, how the wish to avoid "messy" calculations can restrict the domain even further than that of natural numbers; and thirdly, the meaning for a letter can be affected by seeing an equation as an example of a wider class of equations where the role of a particular letter is considered across the class rather than purely within the particular equation in view.

References

Bardini, C., Radford, L., & Sabena, C. (2005). Struggling with variables, parameters, and indeterminate objects or how to go insane in mathematics. In H. L. Chick & J. L. Vincent (Eds.), *Proc. 29th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, 129-136). Melbourne, Australia: PME.

Christou, K. P., & Vosniadou, S. (2012). What kinds of numbers do students assign to literal symbols? Aspects of the transition from arithmetic to algebra. *Mathematical Thinking and Learning*, *14,* 1-27.

Ely, R., & Adams, A. E. (2012). Unknown, placeholder, or variable: What is x? *Mathematics Education Research Journal*, *24*(1), 19-38.

Gunnatsson, R., Hernell, B., & Sönnerhed, W. W. (2012). Useless brackets in arithmetic expressions with mixed operations. In T.-Y. Tso (Ed.), *Proc. 36th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, 275-282). Taipei, Taiwan: PME.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 60-86). Reston, Virginia: National Council of Teachers of Mathematics.

Hewitt, D. (2012). Young students learning formal algebraic notation and solving linear equations: are commonly experienced difficulties avoidable? *Educational Studies in Mathematics*, *81*(2), 139-159. doi: 10.1007/s10649-012-9394-x

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317-326.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: equivalence & variable. *ZDM*, *37*(1), 68-76.

Küchemann, D. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London: John Murray.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics*, *33*, 1-19.

Sáenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretations or equality and the equal symbol. *Educational Studies in Mathematics*, *35*, 153-187.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1-36.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*, 503-518.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded Theory procedures and techniques*. London: SAGE publications Ltd.

Trigueros, M., Ursini, S., & Escandón, C. (2012). Aspects that play an important role in the solution of complex algebraic problems. In T.-Y. Tso (Ed.), *Proc. 36th Conf. of the Int.*

Group for the Psychology of Mathematics Education (Vol. 4, 147-153). Taipei, Taiwan: PME.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 8-19). Reston, Virginia: NCTM.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *Journal of Mathematical Behavior*, *31*, 344-355.