LECTURERS’ TOOLS AND STRATEGIES IN UNIVERSITY MATHEMATICS TEACHING: AN ETHNOGRAPHIC STUDY

by

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For my family
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Abbreviations

DTP  Definition-Theorem-Proof
ESUM  Engineering Students Understanding Mathematics (Jaworski, Robinson, Matthews and Croft, 2012)
MEU  Mathematics Education Unit
SGT(s)  Small Group Tutorial(s)
SPA  Spectrum of Pedagogical Awareness (Nardi, Jaworski & Hegedus, 2005)
TKiP  Teaching Knowledge-in-Practice
VLE  Virtual Learning Environment
ZPD  Zone of Proximal Development (Vygotsky, 1978)
Abstract

The thesis presents the analytical process and the findings of a study on: lecturers’ teaching practice with first year undergraduate mathematics modules; and lecturers’ knowledge for teaching with regard to students’ mathematical meaning making (understanding). Over three academic semesters, I observed and audio-recorded twenty-six lecturers’ teaching to a small group tutorial of two to eight first year students, and I discussed with the lecturers about their underlying considerations for teaching. The analysis of this thesis focuses on a characterisation of each of three (of the twenty-six) lecturers’ teaching, which I observed for more than one semester. I chose the teaching of three experienced lecturers, due to diversity in terms of ways of engaging the students with the mathematics, and due to my consideration of their commitment to teaching for students’ mathematical meaning making.

The distinctive nature of the study is concerned with the conceptualisation of university mathematics teaching practice and knowledge within a Vygotskian perspective. In particular, I used for the characterisation of teaching practice and of teaching knowledge the notions ‘tool-mediation’ and ‘dialectic’ from Vygotskian theory. I also used a coding process grounded to the data and informed by existing research literature in mathematics education. I conceptualised teaching practice into tools for teaching and actions with tools for teaching (namely strategies). I then conceptualised teaching knowledge as the lecturers’ reflection on teaching practice. The thesis contributes to the research literature in mathematics education with an analytical framework of teaching knowledge which is revealed in practice, the ‘Teaching Knowledge-in-Practice’ (TKiP). TKiP analyses specific kinds of lecturer’s knowing for teaching: didactical knowing and pedagogical knowing. The framework includes emerging tools for teaching (e.g. graphical representation, rhetorical question, students’ faces) and emerging strategies for teaching (e.g. creating students’ positive feelings, explaining), which were common or different among the three lecturers’ teaching practice.
Overall, TKiP is produced to offer a dynamic framework for researcher analysis of university mathematics teaching knowledge. Analysis of teaching knowledge is important for gaining insights into why teaching practice happens in certain ways. The findings of the thesis also suggest teaching strategies for the improvement of students’ mathematical meaning making in tutorials.
UNIT I:

Theory and methodology
Chapter 1

INTRODUCTION AND BACKGROUND TO THE STUDY

This thesis is the culmination of four years of research on university mathematics teaching practice for students’ meaning making at a British university. Teaching practice is concerned with “what teachers do and think daily, in class and out, as they perform their teaching work” (Speer, Smith & Horvath, 2010, p. 99). This is an area with reported dearth of research in university mathematics education (e.g. Speer, Smith & Horvath, 2010; Fukawa-Connelly, 2012; Viirman, 2014a; Jaworski, Mali & Petropoulou, 2016). It is an important area as it offers insights into the craft of teaching at university level and the opportunities for students to make mathematical meaning.

I study mathematics teaching practice in the small group tutorial (SGT) setting, which includes 2-8 first year undergraduate students and a tutor. The tutor is not a postgraduate student, but a lecturer in modules offered by the Mathematics Department. I selected the SGT setting as it offers a negotiation of mathematical meaning through dialogue between students and tutor, thus an analysis of teaching practice with regard to students’ mathematical meaning making is possible.

The study focuses on knowledge for teaching through an examination of teaching practice. It builds on the Vygotskian theory of learning and knowing in order to develop an analytical framework of teaching knowledge which is revealed in practice. In particular, the research questions are:

• How is teaching knowledge revealed in teaching practice with first year undergraduate mathematics modules?
• How does teaching knowledge interact with students’ mathematical meaning making?
The research design includes interviews with twenty-six tutors and observations of their teaching, with three extended cases of teaching studied in depth. The study aims to develop a framework for analysis of ‘teaching knowledge in practice’, which will not be in the form of evaluation of teaching.

In this introduction, I provide a detailed description about the university of the study and the SGT setting. Following the description is a discussion on didactics and pedagogy, terms which influenced the conceptualisation of teaching practice and knowledge for teaching in this study. Then, I account for observational studies of tutorial and lecture teaching at university level in order to situate the study in university mathematics education. Finally, I present previous research in knowledge for teaching in order to make explicit connections between this study and earlier studies, and to clarify the contribution.
The structure of the thesis

This first chapter, Chapter I, introduces the study with details about the context, the research focus and the research questions. Through literature reviews, it also situates the study within university mathematics education and discusses its contribution.

Chapter 2 explains the theoretical background of the study, which is the Vygotskian theory. It provides the reader with the connections between different notions and the conceptualisation of learning and knowing.

Chapter 3 discusses my choices within the research design as well as the process of data analysis, with which I addressed the research questions and developed the analytical framework ‘Teaching Knowledge-in-Practice’ (TKiP).

Chapters 4, 5 and 6 are the data analysis chapters; each of them addresses one of the three extended cases of teaching. A comprehensive analysis of findings about teaching practice and knowledge resulted into three long chapters. These chapters explain how the analytical framework emerged from the data.

The last chapter, Chapter 7, reports on findings from the cross-case analysis and draws the conclusions. It synthesises from the three cases of teaching to contribute to the research literature with the analytical framework ‘Teaching Knowledge-in-Practice’. It concludes with a discussion about implications of the study, limitations and future studies.
1.1 The university of the study and the small group tutorial setting

At the university of this study, mathematics students are registered for a single mathematics or a joint mathematics three-year programme. A four-year single mathematics programme is also available and is suitable for students interested in research careers in mathematics. A student’s option in all programmes is to register for a salaried professional placement year before the last year of studies. After graduation, those who decide to become mathematics teachers at schools register for a *Postgraduate Certificate in Education* or choose other routes into teacher training.

Mathematics modules are offered by the mathematics department, which comprises more than 50 academic staff members. Lecturers in mathematics modules are active researchers in mathematics (usually employed by the mathematics department) or active researchers in mathematics education (usually employed in the Mathematics Education Unit). Depending on their experience, the teaching workload usually includes one to three modules per year, a small tutorial group of 2-8 first year students and hours for one to one support to students at one of the two Mathematics Learning Support Centres of the University. This study analyses teaching practice and knowledge in the SGT setting, so the lecturer-participants are referred to as tutors.

At the time of observations, first year mathematics students (usually aged 18-21) were expected to attend lectures in several modules including calculus/analysis\(^1\) and linear algebra, which were core modules. The lectures in each module lasted for three hours per week. The capacity of the lecture theatres was for large cohorts of more than 200 students. Material relating to the modules, such as lecture notes, weekly problem sheets and tests, could be found at the Virtual Learning Environment of the University. The assessment of students’ performance in calculus/analysis and linear algebra included three tests, which were worth 20%-40% depending on the module, and the final exam, which was worth 80%-60%. The final exam adhered to the University’s regulations for equality to the degree of content difficulty each year.

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\(^1\) In the second year of this study, the ‘Calculus’ module was reformed into an ‘Analysis’ module.
Mathematics students were members of a small group tutorial of two to eight students in their first year of undergraduate studies only. Tutorials were 50 minute weekly sessions. Work in tutorials was on the material of the lectures particularly paying attention to students’ difficulties. Students’ attendance was strongly encouraged, although not compulsory. The personal tutor was an active researcher and lecturer in modules offered by the mathematics department. The tutor’s responsibilities included responding to students’ difficulties with mathematics modules, marking the students’ tests and providing the students with pastoral care. Tutors remained in pastoral care as the students progressed through their programme.

The modules that were usually discussed in tutorials were calculus/analysis and linear algebra. Students were expected to work on the material of the lectures beforehand in order to bring their questions to the tutorial. For example, the instructions below were written in the first Problem Sheet of the lectures in analysis:

> Homework problems will be attached to the end of each week’s handout. You should start working on them as soon as you have been to the first lecture(s) in any given week. For those questions you get stuck with, ask your fellow tutees and your tutor for hints and help. For those questions you are pretty confident about, ask your fellow tutees and your tutor to read and critique your answer. You need to get good at writing clear mathematical arguments, and this will help.

I searched for the use of the words “questions” and “problems” in the lecture material, such as the Problem Sheets at the end of each week’s lecture notes. I nevertheless found that the use was quite loose for the needs of this research. So, I decided to refer to tasks rather than “questions” and “problems” of the lecture material. I also refer to questions when a student made an enquiry to the tutor.

Generally, in tutorials, some student groups brought questions or tasks with which they faced difficulties, but largely students did not take the responsibility. In the latter case, the personal tutor made a decision as to which tasks the tutorial group should focus on. These tasks usually came from the lecture material: tests or weekly problem sheets. So, the tutors discussed with the students the tasks and the corresponding mathematical theory to the tasks.
I distinguished the tutorial setting as an opportunity for observation of university mathematics teaching addressed to a small group of students. The small group enabled the tutor and the students to discuss the mathematics, and offered me an opportunity for an analysis of ‘teaching practice’ and ‘knowledge for teaching’ with regard to students’ mathematical meaning making. This opportunity was exceptional for me since I observed university mathematics teaching implemented by different tutors and addressed to student groups with different levels of performance in a “natural” university environment, where I did not intervene to suggest that tutors should work in particular ways with the students. In this way, I investigated ‘knowledge for teaching’ through an examination of each lecturer’s everyday teaching practice in a setting created to foster students’ mathematical meaning making.

1.2 Literature informing the study

The study is triggered by a desire to research into the teaching practice of mathematics tutors and their teaching knowledge, at university level. I investigated teaching knowledge through an examination of teaching practice, discerned in pedagogical practice and didactical practice. In this section, I first illuminate the ways in which the terms ‘pedagogy’ and ‘didactics’ are used in literature, and then I discuss researchers’ perspectives on how pedagogy and didactics are connected with terms relating to teaching knowledge.

Pedagogy derives from adoptions of the ancient Greek [παῖς (child) + ἀγωγή (leading)]. In Understanding pedagogy and its impact on learning (1999), I found a good translation of the ancient Greek for pedagogy, into English: the ‘oversight of a child’ or ‘the leading of a child to school’. The specificity of the term to children encouraged considerations for relative terms adapted from the ancient Greek language, such as ‘andragogy’ (Knowles, 1975) and ‘heutagogy’ (Hase & Kenyon, 2000). Andragogy relates to self-directed adult learning, with or without the assistance of others, outside of formal education and dependent on personality, preference and context (Knowles, 1975; Hiemstra, 1994; Brookfield, 2009).
‘Heutagogy’ is regarded with self-determined learning, revealed by the learner’s capability to use her/his competences in novel as well as familiar circumstances (Stephenson & Weil, 1992; Hase & Kenyon, 2000; Blaschke & Hase, 2015). The principles of ‘heutagogy’, which nurture capability, indicate that the learner reflects on what s/he learned, the way it has been learned and the way that the learning influences her/his values and belief system.

‘Andragogy’ and ‘heutagogy’ are neatly defined but specific to other kinds of education than university mathematics education within the tutorial setting of my study. In contrast to those terms, I faced a difficulty to find a definition for pedagogy in English writing; rather, I found some criticism to pedagogy as a poorly defined idea. Reading research literature in education, I made sense of pedagogy as the craft of teaching:

“any conscious activity by one person designed to enhance learning in another”

(Watkins & Mortimore, 1999, p.3)

This quotation indicates the wide range and complex nature of pedagogy which, in agreement with Watkins and Mortimore, does not imply that the term is ill-defined. The tutor’s pedagogy is connected with learning and the learner; in this study, I take account of as much of the teacher’s perspective on students’ learning as is possible in the tutorial setting. In particular, I recognise that tutorials, with the small group of students, afford possibilities for tutor’s perspectives on their teaching practice with regard to students’ mathematical meaning making. I see meaning making as an enculturation into mathematics (Ben-Zvi & Arcavi, 2001) through dialogue.

Watkins and Mortimore (1999, p.2) view didactics as a “more limited term” than pedagogy because it includes “only the teacher’s role and activity”; thus a study of didactics does not seem to take the learner into account. I found a definition of didactics in the work of Brousseau: “Nous appelons «Didactique des Mathématiques» la science des conditions spécifiques de diffusion (imposée) des savoirs mathématiques utiles aux membres et aux institutions de l’humanité” (Brousseau, 1994, cited in Bosch & Chevallard, 1999, p.77). I translate this definition in English as:
We call «Didactics of Mathematics» the science of the specific conditions of dissemination (via imposition) of mathematical knowledge(s) useful to members and institutions of humanity.

The tutor’s didactics is mathematics-specific, connected to ways of disseminating the principles and content of mathematics. In contrast to pedagogy, which is about ways of teaching to bring about learning, didactics is connected with the subject matter. In his definition, Brousseau referred to mathematics useful to members and institutions of humanity. This indicates that the students learn the mathematics that their teacher designs to teach; thus the mathematics relies on the teacher as well as the institution and its constraints (e.g. content, curriculum).

Herbart’s didactic triangle (Figures 1.1 and 1.2, cited in Kansanen & Meri, 1999) illustrates a possible connection between pedagogy and didactics. It draws on the German Didaktik (Hopmann & Riquarts, 1995); however, Caillot (2007) argued that the term is comparative with the French Didactique. I discuss the didactic triangle to present a possible connection between pedagogy and didactics in literature. Notably I did not use the didactic triangle as a framework in my analysis; however, it enabled me to deepen my sense making of didactics and pedagogy and, in agreement with the aforementioned definitions of pedagogy (Watkins & Mortimore, 1999) and didactics (Brousseau, 1994), to develop meanings of these terms for my study.

Kansanen and Meri (1999) stressed that, in a study with the didactic triangle, it is impossible to consider it as a whole; rather, the triangle can be analysed in pairs starting with the pedagogical relation (Figure 1.1). On the one hand, the pedagogical relation is the relation between the teacher and the student(s); how the teacher makes the content relevant to the students. This is an asymmetric relation because, for instance, the teacher is knowledgeable in mathematics while the student is learning...
the mathematics. On the other hand, the didactic relation (Figure 1.2) is the relation between the teacher and the mathematics; in particular, the goals stated in the curriculum and the teacher’s design. In Figures 1.1 and 1.2 the points of the triangle are drawn with teacher, student and mathematical content. In literature, there are also variations on how the points of the triangle can be interpreted; these are in relation to organisation and society levels (e.g. Paschen, 1979; Künzli, 1998).

Ruthven (2012) analysed the integration of digital technologies into secondary mathematics teaching practice with the didactic triangle. He suggested the addition of a vertex to acknowledge the role of technological resources, including tools which range from arithmetic calculators present in the classroom to “the fundamental machinery of schooling itself” (2012, p.627). His suggestion of a technological vertex, placed at the center of the triangle, indicates the mediating role of technological resources between content, teacher and student. Ruthven presented a possible connection between tools, pedagogy and didactics in literature.

The significance of tools in the tutor’s pedagogy and didactics is evident in my study, as well. In particular, I made sense of the tutor’s didactical practice and pedagogical practice in terms of teaching actions with tools; that is, tool-mediated actions theorised through a Vygotskian perspective. (The term tool-mediated action is discussed in Chapter 2: Section 2.1). The method for data analysis in my study is a grounded theory (Glaser & Strauss, 1967; Charmaz, 2006) analytical approach to observational and interview data of tutorial teaching. So, the ‘teaching actions’ and the associated ‘tools’ first emerged from the data to describe a tutor’s teaching practice; then they were informed by previous studies in literature; and finally they were interpreted in relation to didactical practice and pedagogical practice.

The aim of this study is to examine teaching knowledge which is revealed in the tutors’ teaching practice, namely didactical practice and pedagogical practice. In the next few paragraphs, I offer an exposition of researchers’ perspectives on possible connections between pedagogy, didactics and teaching knowledge.

The German didaktik, on which the didactic triangle draws, shares similarities with Shulman’s ‘pedagogical content knowledge’ (e.g. Bromme, 1995, Westbury, Hopmann & Riquarts, 2000; Kansanen, 2009a). Shulman (1986, 1987) defined
‘pedagogical content knowledge’ as the intersection between ‘content knowledge’ of the subject-matter (here mathematical knowledge) and ‘pedagogical knowledge’ of “classroom management and organization that appear to transcend subject-matter” (1987, p.8).

When I started my study, I considered that the kind of knowledge on which I might focus could be ‘pedagogical content knowledge’. However, in literature, there is inconsistency in using the concepts ‘pedagogical knowledge’ and ‘pedagogical content knowledge’; for instance, Ball, Thames and Phelps (2008) asserted that the latter lacks definition. Kansanen (2009b, p.8) stressed that “even in his own writing, Shulman (1986; 1987) uses these alternatively or without distinguishing between them”.

Kansanen (2009b) attempted to make sense of Shulman’s ‘pedagogical content knowledge’ through the elements content, teacher and student of the didactic triangle and the work of Ball, Thames and Phelps (2008) in teaching knowledge. He used the work of Ball, Thames and Phelps, because it combines “one or more types of knowledge with the content as well as with various point of views related to teaching” (2009b, p.16), offering a larger perspective of content than the one in Shulman’s work. Kansanen (2009b) argued that if the focus of analysis is on the element ‘student’, as it is in a study of pedagogy, then pedagogical knowledge is a broader term than ‘pedagogical content knowledge’. For example, he asserted that ‘curricular knowledge’ (Shulman, 1986) is connected with the teacher and the content, ultimately recognising that ‘pedagogical knowledge’ is an amalgam of all categories of knowledge in Shulman’s work. (Shulman’s categories of knowledge are presented in Appendix A.) However, Kansanen (2009b, p.5) concluded that the “increasing use of pedagogical content knowledge may likely show the way to a more heterogeneous usage of this concept in the future”.

The reported inconsistency in the use of ‘pedagogical content knowledge’ in literature persuaded me that the term was not useful for my study about teaching knowledge. Rather, considering that I was investigating teaching knowledge revealed in the tutors’ didactical practice and pedagogical practice, I was in search of kinds of

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I provide more explanation about Shulman’s contribution to the area of teaching knowledge later in this chapter, where I devote a section on previous studies in teaching knowledge (Section 1.2.2).
knowledge which could be interpreted from my data as ‘didactical’ and ‘pedagogical’, respectively.

In the next sections, I expose previous research literature in observational studies of tutorial and lecture teaching practice at university, as well as different categorisations of teaching knowledge, in order to situate my study with regard to earlier research studies and show its contribution.

1.2.1 Observational studies of tutorial and lecture teaching practice at university level

From a considerable literature on teachers and teaching, I draw on two perspectives to make sense of teaching. The first suggests that that teaching practice is concerned with “what teachers do and think daily, in class and out, as they perform their teaching work” (Speer, Smith & Horvath, 2010, p.99). In other words, teaching practice is concerned with teachers’ thinking, judgments and decision-making in planning, implementing and reflecting on their work. The second perspective is a conceptualisation of teaching practice in terms of teaching actions.

An action might be described as ‘teaching’ if, first, it aims to bring about learning, second, it takes account of where the learner is at, and, third it has regard for the nature of what has to be learnt. (Pring, 2000, p.23)

It seems to me that this conceptualisation is well related to the tutors’ didactical practice and pedagogical practice. It takes learning and the learner into consideration; thus it relates to the tutor’s pedagogical practice. It is also connected with the tutor’s didactical practice, as it is concerned with the nature of the content.

Treffert-Thomas and Jaworski (2015) conducted a literature review in university mathematics teaching; their categorisation is into research, professional, and pedagogical literature, with the last recommending approaches to teaching to practitioners. While recognising that professional literature has a lot to offer in terms of reports and reflections on personal experiences of university mathematics teachers regarding their teaching practice, in this section, the focus is on research literature.
Research literature offers analyses of observed teaching practice at university level, which I present in this section. It also offers established concepts which I used as theoretical codes in my grounded analytical approach to the data. (The term theoretical code is discussed in Chapter 3: Section 3.4.)

Observational studies of tutorial and lecture teaching practice at university level is an area with reported dearth of research in university mathematics education (e.g. Speer, Smith & Horvath, 2010; Fukawa-Connelly, 2012; Viirman, 2014a; Jaworski, Mali & Petropoulou, 2016). After a systematic literature review, Speer, Smith and Horvath (2010) reported very little research focusing on observational studies of teaching practice at university level. In this section, my exposition of the small number of studies that have been conducted, before and after 2010, ranges from observational studies of lecture teaching to observational studies of tutorial teaching, both at university level. In this exposition, I recognise some common themes that underpin the studies; however, I categorise them with regard to didactical or pedagogical practice only when this was expressed by the authors.

1.2.1.1 Observational studies of lecture teaching through sociocultural perspectives

In this section, I present a group of studies which are analysed through sociocultural perspectives, taking account of differing cultures and the social context of the teaching/learning environment. The presentation is in chronological order.

Pampaka, Williams, Hutcheson, Wake, Black, Davis and Hernandez-Martinez (2012) conducted a UK longitudinal study of nine teachers’ pedagogical practice, described in a scale from transmissionist and teacher-centered to connectionist and student-centered. They drew on Williams, Black, Hernandez-Martinez, Davis, Pampaka and Wake (2009) and Williams (2011), who used a range of sociocultural perspectives (discursive psychology, a narrative approach to identity, and cultural-historical activity theory) in the analysis of observational and interview data of teaching, in order to report on the relation between teaching and a range of learning outcomes (e.g. dispositions). Williams et al. (2009) analysed students’ biographical narratives on their cultural models, positions and dispositions towards or against mathematics, as well as their teachers’ lessons and interviews, concluding that students with similar backgrounds positioned themselves differently according to positions offered by the
teachers. Also, based on two teachers’ biographical narratives, Williams (2011) offered insights into the teachers’ connectionist or transmissionist identity with regard to their pedagogical practice.

Artemeva and Fox (2011) also used a range of sociocultural perspectives (rhetorical genre theory, communities of practice, and activity theory) in the analysis of observational and interview data of lecture teaching in seven countries. They investigated the ‘chalk and talk’ pedagogical practice, namely “writing out a mathematical narrative on the board while talking aloud” (p.1), by analysing teaching actions such as verbalising “everything they write on the board (running commentary)”, talking “about what they write on the board (metacommentary)” and using “rhetorical questions to signal transitions, pause the action for reflection, or check student understanding” (p.11). Artemeva and Fox concluded that pedagogical practices, such as ‘chalk and talk’, override local differences, such as use of lecture notes and nature of classroom interaction, across teaching contexts. They also found that experienced teachers, contrary to novices, used considerable time for the arrangement of text in relation to the type of board in the classroom (e.g. number and positioning of panels).

Based on observations of teaching of beginning level calculus for the concepts of limits and continuity in a US university as well as interviews with students, Güçler (2013) used a discursive analytical approach, with the commognitive framework of Sfard (2008), to explore characteristics of the teacher’s and the students’ discourses on limits. She found that the students’ mathematical discourse was not as consistent as the teachers’ discourse, and attributed this to the teacher’s shifts and discrepancies in elements of his discourse. She suggested teachers to pay attention to their alternating ‘metalevel rules’ on limits, such as alternating between graphing and using symbolic representations, or alternating in using metaphors connected to different features of limits in a particular mathematical context (e.g. that of computing limits).

In his doctoral study, Viirman (2014b) used a discursive approach to cognition with the commognitive framework of Sfard, as well. He analysed observational data of seven mathematicians’ lectures in first-semester calculus and algebra at three Swedish universities, focusing on the concept of function. He distinguished between written and oral mathematical discourse (2014a) and pedagogical discourse (2015) in ‘chalk
and talk’ teaching. For instance, he recognised formal language in written mathematical discourse and informal language in oral mathematical discourse. He also identified characteristics of the teachers’ pedagogical discourse such as types of teacher questions: control questions (e.g. Does it make sense?), rhetorical questions, questions asking for facts (e.g. definitions, calculations) and enquiries for reflection on mathematics.

Through an Activity Theory perspective and a grounded analytical approach to observations of first year linear algebra teaching and interviews with the teacher, Treffert-Thomas (2015) conducted a doctoral study which characterised goal-oriented teaching actions and the teacher’s thinking behind them. In her analysis of interviews, she discerned modes of the teacher’s reflection on teaching, into: ‘expository’ (of the mathematical meanings the students should make) and ‘didactic’ (of his goals for teaching and the associated actions) (Jaworski, Treffert-Thomas & Bartsch, 2009). She ultimately found a model of a hierarchy of goals, with teaching actions in relation to each goal, so that students gradually develop meanings in linear algebra.

Petropoulou, Jaworski, Potari and Zachariades (2015) explored first-year calculus teaching in two Greek Universities, and ways in which the teaching could take into account students’ learning needs. This paper is based on the first author’s ongoing doctoral study. The data sources are observations of lectures, reflective discussions on the observations with the teachers who were six mathematicians, and group meetings between some of the teachers and the research team. Through a number of perspectives [activity theory, grounded analytical approach, and the Teaching Triad (Jaworski, 1994)], the first author offers various goal-oriented teaching actions that characterise calculus teaching and their relation to elements of the Teaching Triad (e.g. ‘sensitivity to students’), gaining deep insights into ways of taking students’ learning needs into consideration (e.g. Jaworski, Mali & Petropoulou, 2016). (The ‘Teaching Triad’ is discussed in this chapter: Section 1.2.2.) Petropoulou, Potari and Zachariades (2011) also linked teaching actions in the lecture setting with research, teaching and studying experiences; they found that one of the six teachers drew on experiences from his own research practice in mathematics, his involvement in mathematics education research and his participation in the group meetings of the study in order to design his own teaching.
Despite the small number of studies in this group, some common methods for data analysis can be recognised: activity theory and approaches to analysis of discourse. Studies which analyse teachers’ and students’ discourses, often examine teaching in relation to students’ learning. Studies which analyse teaching through an activity theory approach concentrate on the didactic design: teaching actions in relation to goals for teaching. The researchers of the last studies use approaches to students’ learning in addition to the activity theory approach.

1.2.1.2 Observational studies of lecture teaching through other perspectives

The observational study of a proof-oriented university mathematics course from Weber (2004) is one of the earliest in this group. Findings emerge from a grounded analysis of data in a case study of one teacher’s lecture teaching in a “definition-theorem-proof” (DTP) format. The teacher was a mathematician and Weber unpacked his DTP lectures into three teaching styles. The logico-structural style was regarded with division of the board: a list of assumptions at the top; a desired conclusion at the bottom; and inferences in between, drawn down from the assumptions and up from the conclusions. The teacher’s linear explanation of the final proof on the board was also included. Within the procedural lecture style, the teacher illustrated a proof’s general structure with an incomplete argument on the board and then filled in the gaps while describing the thinking and stressing the techniques and heuristics. Finally, in the semantic style, the teacher intuitively described the meanings of concepts, for example by using diagrams to explain definitions, and then constructed rigorous proofs. Through an analysis of the teacher’s reflection on the three teaching styles, a range from the logico-structural lecture style to the semantic style was suggested so that some students develop a competent performance at proof-writing.

Fukawa-Connelly (2012) investigated one teacher’s presentation of proofs in abstract algebra. The teacher was a mathematician who, as in the previous author’s study, presented modes of thought so that the students develop proof-writing abilities. Analysis revealed a funnelling pattern from “questions that a mathematician should ask while writing proofs, such as, “What does that mean?”, “What comes next?”, and “What do I still need to do?” (p.343) to factual questions, such as asking for definitions, restatements and calculations. Fukawa-Connelly concluded that, due to
the reduced cognitive demand of the factual questions, the students did not engage significantly in proof-writing.

Based on interviews with teachers and observations of teaching, Mesa, Celis and Lande (2014) investigated the relationship between teachers’ descriptions of their practice during interviews and their actual teaching practice in the classrooms. In particular, they analysed the teachers’ interaction with the students and the mathematical content, situated within specific classroom and institutional environments. They distinguished the teaching practice into traditional, meaning-making, or student-support, and classified framing talk and mathematical questioning in the classroom. They ultimately suggested alignment between the declared teaching approaches during interviews and the actual teaching approaches in the classrooms in the case of framing talk, but not in the case of mathematical questioning.

The lecture is still a dominant setting in university mathematics education in spite of doubts about its value regarding the students’ learning (Bligh, 1972; Holton, 2001; Fritze & Nordkvelle, 2003), such as difficulty in paying attention (Bligh, 1972), passive listening to the teacher (Fritze & Nordkvelle, 2003), and policy recommendations to reduce the amount of lecturing and adopt interactive teaching (e.g. Association of American Universities, 2011 and the White House, 2012 cited by Hora, & Ferrare, 2013). The studies in this section (Section 1.2.1.2) analyse lecture teaching, where teachers do not explore innovative approaches to students’ learning. They enable researchers to gain insights into teaching by developing analytical terms such as logico-structural/procedural/semantic DPT teaching, or traditional/meaning-making/student-support teaching, or connectionist/transmissionist teaching, and different characteristics of teaching such as typologies of teacher questions at university level. The next observational studies examine teaching in inquiry-based curriculum reforms and in a teaching experiment (Cobb 2000; Steffe and Thompson 2000), where new approaches to students’ learning are explored and important issues are raised for further investigation.
1.2.1.3 Observational studies of lecture teaching, using an inquiry-based curriculum reform or a teaching experiment to foster students’ learning

This is a collection of a few studies, seeking innovative ways of teaching that fosters students’ learning; they employ different perspectives including sociocultural. In this category, the studies were conducted mainly in the USA.

Larsen and Zandieh (2007) drew on realistic mathematics education to design a teaching experiment in an introductory group theory course. The first author was the teacher for the course, which was videotaped and analysed in relation to processes that resemble those described by Lakatos in his 1976 book *Proofs and Refutations*. The processes included a primitive conjecture, a counterexample to this conjecture, dismissing the counterexample (“monster-barring”), excluding the counterexample from the domain of the theorem (“exception-barring”), noticing where the proof would work (“proof-analysis”) and producing a version of the standard theorem. The authors proposed a guided reinvention of mathematics through these processes, which can provide heuristics in a design of teaching for students’ active engagement in the development of mathematical ideas.

Wagner, Speer and Rossa (2007) and Speer and Wagner (2009) took a cognitive analytical perspective to observational and interview data of teaching with a reformed inquiry-oriented curriculum. The teacher was a mathematician and, for the first time, taught an undergraduate course in differential equations within that curriculum. He faced challenges in relation to inquiry-oriented teaching goals, and in responding to students’ contributions due to unfamiliarity of typical ways that student would think. The authors suggested that, in a study of teaching practice, it is possible to identify essential forms of knowledge for inquiry-oriented teaching, such as “knowledge of typical ways student think (correctly and incorrectly) about the task or content in question” (Speer & Wagner, 2009, p.558); and “the formal mathematical knowledge that mathematicians with advanced degrees have developed through study and/or research” (Speer & Wagner, 2009, p.533)

Johnson and Larsen (2012) drew on theory to teachers’ *listening* to analyse observational and debriefing data of teaching of abstract algebra in an inquiry-based curriculum development project. The three teachers successfully implemented the
curriculum; however, as in the previous authors’ study, they were unable to make sense of students’ expressed difficulties with the mathematics. Further analysis of a case of teaching revealed that the teacher was not able to access the students’ conceptions of the particular mathematics.

In the *Engineering Students Understanding Mathematics (ESUM)* project, Jaworski, Robinson, Matthews and Croft (2012) took a sociocultural analytical perspective to observational data of innovative inquiry-based teaching in a UK university, oral and written reflections of the teacher, surveys of student perceptions, and focus group interviews with students. The teacher was a researcher in mathematics education and a member of the research team. She implemented the innovation into the teaching of mathematics to first year engineering students, and developed her practice through her reflections on it in discussion with the team. An Activity Theory frame indicated tensions between the perspectives of the team in planning and implementing the course and those of students attending the course.

Following her study on calculus teaching for the concepts of limits and continuity, Güçler (2016) designed a teaching experiment to investigate her suggestion to teachers to attend to shifts and discrepancies in elements of their discourse, in order to promote their students’ learning. In the experiment, she studied calculus teaching that explicitly attended to ‘metalevel rules’ (‘metarules’) in the mathematical discourse on functions. For instance, ‘metarules’ included adopting and eliminating assumptions (e.g. assumptions of arbitrariness such as arbitrary correspondence between the domain and the range of functions). Despite some students’ persistent difficulties with various aspects of functions, the findings indicated that such teaching has the potential to foster students’ learning.

The curriculum development projects and teaching experiments are focused on innovations which aim to foster students’ learning. The findings confirm teachers’ difficulty in attending to students’ contributions and expressed difficulties, indicating areas for improvement in teaching. Jaworski, Robinson, Matthews and Croft (2012) suggested development and improvement through the teacher’s reflections on teaching in discussion with the research team, and Güçler (2016) suggested elimination of discrepancies in the teacher’s discourse.
1.2.1.4 Observational studies of tutorial teaching

The tutorial setting is additional to the lecture setting and affords inherent possibilities for teaching that have the potential to foster students’ learning. However, observational studies focusing on tutorial teaching are rather limited compared to studies in the lecture setting.

Based on observations of weekly tutorial teaching and interviews with students, Nardi (1996) conducted her doctoral study to explore students’ cognitive tensions with mathematical abstraction, within the areas of foundational analysis, calculus, linear algebra and group theory. Through cognitive and sociocultural theories of learning applied on episodes from teaching, she found cognitive tensions between Informal/Intuitive/Verbal and Formal/Abstract/Symbolic reasoning, as well as learning difficulties with the nature of rigour in formal mathematics. Subsequently, Nardi (2008) investigated mathematicians’ perceptions of their students’ learning and reflections on their teaching, based on data from their small group tutorials and students’ work.

Jaworski (2003) and Nardi, Jaworski and Hegedus (2005) reported on findings from the University Mathematics Teaching Project, which characterised pedagogy in small group tutorial teaching in a UK university. Data sources were observations of the tutorials and post-tutorial interviews with the teachers, who were six mathematicians. Through a grounded analytical approach and the Teaching Triad (Jaworski, 1994), Jaworski (2003) distinguished teachers’ exposition patterns, such as teacher ‘explanation’, teacher ‘as expert’ and forms of teacher ‘questioning’; in particular, teachers showed and explained mathematics, ensured that they made the student aware of the correct mathematics and gave a degree of independence to the student to leave and make sense of the mathematics. Furthermore, Nardi, Jaworski and Hegedus (2005) studied teachers’ interpretations of episodes from their teaching, discerning them in: teacher’s conceptualisations of students’ difficulties, teacher’s descriptive accounts of pedagogical aims and practices with regard to these difficulties and teacher’s self-reflective accounts with regard to these practices. They concluded by fitting the episodes in the Spectrum of Pedagogical Awareness (SPA) with four dimensions, namely ‘Naive and Dismissive’, ‘Intuitive and Questioning’, ‘Reflective and Analytic’ and ‘Confident and Articulate’.
Jaworski and Didi (2014) conducted a developmental study of tutorial teaching, designed to promote students’ mathematical meaning making. Through a grounded analytical approach and the application of the Teaching Triad (Jaworski, 1994) to observational and interview data, the authors studied the development of the teaching and the teacher’s reflection on it, in order for the teacher to improve access to students’ meanings. Findings include a flexible approach to what had been planned when students did not seem to the teacher to have made meaning, and a questioning style that promoted students’ meaning making in mathematics. The style comprised ‘meaning questions’ seeking students’ expression of meaning, often in response to “why?”; and ‘inviting questions’ asking students to respond, either directly to a particular student or generally to all students.

As part of the ESUM project, Jaworski (2015b) also studied the design of the tutorial teaching to promote first year engineering students’ conceptual understanding of mathematics, using inquiry-based tasks within a GeoGebra environment. She drew on theory of competencies to evaluate the tasks with regard to students’ mathematical meanings. Her findings suggested students who valued a more procedural approach to learning in order to perform well in exams, and some students who did not engage in demanding tasks but attended to more routine ones. While recognising that the nature of the exam that existed before the innovation was not aligned to the inquiry-based teaching, she stressed that institutional constraints prevented other forms of assessment.

Studies in the tutorial setting reveal more opportunity to examine teaching with regard to the students’ meanings and difficulties than in the lecture setting; however, access to students’ actual meanings still remains challenging. This group of studies offers approaches to teaching which take students’ contributions into consideration although the exposition patterns (Jaworski, 2003) resemble the ones in lectures. In this thesis, different approaches to teaching for students’ mathematical meaning making are analysed, providing insights into the practice and the knowledge (other than mathematical knowledge) needed for such approaches. In order to unpack the approaches to teaching, I drew on established concepts from the aforementioned studies that describe teaching practice; for instance, I used types of teacher questions. I also drew on concepts which are established in research in school mathematics.
education due to the reported dearth of research in university mathematics education. (An exhaustive glossary of the concepts I used in the analytical approach can be found in Appendix D.) The next section is an exposition of different categorisations of teaching knowledge, with which I situate my study in relation to earlier research.

1.2.2 Knowledge for teaching

The Encyclopedia of Mathematics Education (Rowland, 2014a) and the MasterClass in Mathematics Education (Rowland, 2014b) report on three studies on knowledge for teaching in mathematics education: the ‘Knowledge Base’ (Shulman, 1987), ‘Mathematical Knowledge for Teaching’ (Ball, Thames & Phelps, 2008) and the ‘Knowledge Quartet’ (Rowland, Huckstep & Thwaites, 2005). In this section, I account for literature in knowledge for teaching, commencing my exposition with these three studies, and expanding it with studies that introduce a range of different types of knowledge for teaching. This account, however, is not intended to be exhaustive.

In the Knowledge Growth in a Profession project, Shulman and his colleagues studied novice teachers learning to teach through a constructivist approach to learning and observations of teachers. They found that teachers with content knowledge of a subject need to transform this knowledge in order to teach the subject (Shulman, 1986) and proposed a Knowledge Base with seven categories of knowledge for teaching including, for example, ‘pedagogical content knowledge’ (Shulman, 1987). (Shulman’s categories are presented in Appendix A.)

The Knowledge Growth in a Profession is a seminal study in knowledge for teaching; in particular, the ‘Knowledge Base’ (Shulman, 1987) has triggered other researchers to investigate teaching knowledge by creating new categories or modifying the existing ones. Influential in this area of study is the Teacher Education and Learning to Teach project from Ball and her colleagues, who distinguished between ‘subject matter content knowledge’ and ‘pedagogical content knowledge’, creating six categories of ‘Mathematical Knowledge for Teaching’ (Ball, Thames & Phelps, 2008), as follows.
Subject matter content knowledge includes:

- Common content knowledge
- Horizon content knowledge
- Specialised content knowledge

Pedagogical content knowledge includes:

- Knowledge of content and students
- Knowledge of content and teaching
- Knowledge of content and curriculum

This categorisation was developed as a means for the evaluation of mathematical knowledge for primary school mathematics teaching. I explain here the first from the six categories, because it has also been used in research in university mathematics education. ‘Common content knowledge’ was initially concerned with school mathematics, applicable to everyday situations and various professions (e.g. calculations and problem solving). In their study of knowledge for university mathematics teaching with a reformed inquiry-oriented curriculum, Speer and Wagner (2009) suggested a relation between mathematicians’ practices of own mathematics teaching, own studying and own mathematical research, using ‘common content knowledge’ as “the formal mathematical knowledge that mathematicians with advanced degrees have developed through study and/or research” (p.533). The “mathematicians with advanced degrees” were “college mathematics teachers, [i.e.] primarily professional mathematicians” with doctorates in mathematics (2009, p.533).

In my study, I also found interplay between mathematicians’ practices of their own research and their own teaching; however, considering the inconsistency in the use of the category ‘pedagogical content knowledge’ in research literature and the constructivist perspective in which the categorisation is embedded, I chose not to use the category ‘common content knowledge’ in my study.

While Ball, Thames and Phelps (2008) unpacked the underdeveloped ‘subject matter content knowledge’ and ‘pedagogical content knowledge’ by distinguishing six subcategories, the Subject Knowledge in Mathematics programme investigated situations in which those two categories of knowledge surface in mathematics
teaching. Through a grounded analytical approach to observational data of primary school mathematics teaching, Rowland, Huckstep and Thwaites (2005) developed the ‘Knowledge Quartet’ with four broad categories of knowledge: ‘foundation’, ‘transformation’, ‘connection’ and ‘contingency’. Each one of these categories includes a set of contributory codes drawn on practice; for instance ‘contingency’, which is the “ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events” (Rowland, 2014b, p.96), includes ‘deviation from agenda’, responding to students’ ideas’, ‘use of opportunities’ and ‘teacher insight during instruction’ (Thwaites, Jared & Rowland, 2011). ‘Contingency’ is of particular importance to my study. My consideration is that, in my study, knowledge which is revealed in practice is contingent in nature, because the tutors usually did not plan the tutorials but responded to the students in the moment. The ‘Knowledge Quartet’ emerged from analysis of primary school mathematics teaching. It was also used to analyse secondary mathematics teaching knowledge (Thwaites, Jared & Rowland, 2011) and university mathematics teaching knowledge (Rowland, 2009).

‘Knowing-to act in the moment’ (Mason & Spence, 1999) is a kind of knowledge which is also concerned with unexpected classroom interactions. This notion is based on the theory of ‘noticing-marking-recording’ (Mason, 1996), according to which if a person notices and marks something, then s/he is in a position to re-mark it to someone else. Mason (2012) recognised that ‘knowing-to act in the moment’ with the mathematics is so automatic for the teacher that s/he might not recognise students’ difficulties. He ultimately suggested that teachers work on the mathematics at own level in order to bring to the surface the awareness that enables action and “keep fresh about what it might be like for students” (2012, p.36).

Looking at the ‘Knowledge Quartet’, I consider that the distinctive nature of the framework is regarded with relating teaching knowledge to an analytical approach to teaching practice. I found that the ‘Teaching Triad’ (Jaworski, 1994) is an analytical framework of ways of teaching, which also relates teaching knowledge to practice. For instance, its element ‘sensitivity to students’ describes the teacher’s knowledge of students’ thinking and the teacher’s attention to their needs. The ‘Teaching Triad’ emerged through a constructivist approach to learning and a grounded analytical approach to observational and interview data of mathematics teaching. In contrast to
the ‘Mathematical Knowledge for Teaching’ and the ‘Knowledge Quartet’, the
‘Teaching Triad’ was developed from analysis of secondary mathematics teaching. Additionally, it has been used to analyse university mathematics teaching in tutorial settings (Jaworski, 2003; Jaworski & Didis, 2014) and in a lecture setting through sociocultural perspectives (Petropoulou, Jaworski, Potari & Zachariades, 2015). My study also investigates teaching knowledge through an examination of teaching practice; I look through a sociocultural lens at learning and knowing in a university mathematics tutorial setting.

I also found in the literature a diverse body of types of teaching knowledge with regard to professional expertise, and particularly a contribution from Eraut (1994) who unified various perspectives under the notion ‘Professional Knowledge’. Eraut discussed a map of ‘Professional Knowledge’ including, for example, propositional knowledge (e.g. specific propositions about particular cases, decisions and actions), personal knowledge and the interpretation of experience, and process knowledge (e.g. deliberative processes such as planning and decision making).

Beyond Eraut’s contribution, I found several other types of teaching knowledge with regard to professional expertise. For example, Polanyi (1967) offered the notion ‘tacit knowledge’ to denote that which we know but we cannot explain or teach, and the notion ‘explicit knowledge’ to define that which we know and we can explain and teach. Elbaz (1983) named ‘practical knowledge’ what teachers know that others do not, and covered knowledge of self, subject matter, curriculum development and instruction, which are represented in practice as rules, practical principles and images. Chapman (2004) used the same term with Elbaz, ‘practical knowledge’, but with this term she labelled the knowledge that guides actual teaching actions of facilitating peer interactions in learning mathematics. Her ‘practical knowledge’ is concerned with teachers’ conceptions that support a social perspective of learning, students’ behaviours and outcomes in peer interactions, learning activities, and teacher’s behaviours that support peer interactions. Schön (1983) highlighted the value of teacher reflection as raising awareness and named ‘knowledge in action’ the knowledge of the reflective practitioner. Ponte (1994) attributed the notion ‘professional knowledge’ to knowing in action, which is grounded on experience, reflection on experience and theoretical knowledge and includes, for example, the
teachers’ way of living the profession. Finally, for Pring (1996), ‘expert knowledge’ is “a body of knowledge which relates to practice” (p.11); and ‘craft knowledge’ is “a different sort of knowledge which teachers claim to have, but which non-teachers on the whole would not claim to have – the practical knowledge or ‘know-how’ which is reflected in the intelligent activities of the classroom.” (p.12).

Although I recognised the importance of all contributions to professional expertise, I could not relate the specific types of knowledge to the findings of this study. This was because my objective was to investigate how knowledge for teaching could be conceptualised within the Vygotskian perspective, and previous studies were conceptualised within other perspectives. I nevertheless included the established types of knowledge in my account in order to highlight aspects (e.g. teacher reflection) which are important in my study, as well. In spite of differences in theoretical perspectives, many conceptualisations of teaching knowledge emerged from analysis of teaching practice. Moreover, although teacher’s reflection on teaching was employed in different ways in relation to the teacher’s experience, in many studies it was a significant aspect for the development of teaching knowledge. Finally, it seems to me that earlier studies expressed an obscure aspect in the nature of knowledge with regard to professional expertise and this aspect relates, for example, to what Mason (2012) referred to as “automatic” in knowing to act in-the-moment and to what Polanyi (1967) offered as “tacit”. I consider that, in my study, the contingent nature of knowledge which is revealed in practice sheds some light into the “automatic” and “tacit”.
1.3 Contribution

As mentioned earlier, Speer and Wagner (2009) highlighted the rather limited number of studies regarding knowledge for teaching in relation to teaching practice in university mathematics education. Considering the dearth of research in teaching practice in university mathematics education more generally (e.g. Speer, Smith & Horvath, 2010; Fukawa-Connelly, 2012; Viirman, 2014a; Jaworski, Mali & Petropoulou, 2016), I think of my study as one of a few studies to date, focusing directly on teaching for students’ mathematical meaning making, offering insights into how and why university mathematics teaching happens in certain ways (Speer, Smith & Horvath, 2010) and illuminating teaching knowledge in relation to teaching practice at this level (Speer & Wagner, 2009).

During the interpretation of teaching practice, I took into account established concepts from research literature in teaching practice and I juxtaposed different notions of teaching knowledge to shed light into the findings. This contributed to an effort of unifying the diverse body of educational research, which has been criticised to be non-cumulative (Wellington, 2000). The study also looks at teaching practice and teaching knowledge with regard to students’ mathematical meaning making from the tutor’s perspective; my consideration is that this is an additional aspect that distinguishes the study in research literature.

In the next chapter, Chapter 2, I discuss the theoretical perspective of the study, explaining various concepts such as ‘tool’ and ‘tool-mediated action’. I also illuminate the ways with which the perspective enabled me to produce an analytical framework of teaching knowledge through an examination of teaching practice.
Chapter 2

THE THEORETICAL APPROACH OF THE STUDY

In this chapter, I present the sociocultural perspectives on which I drew to develop my conceptualisations of teaching practice, teaching knowledge and students’ mathematical meaning making. In particular, I provide an account of the Vygotskian theory of learning and knowing, blending the aforementioned conceptualisations, in order to explain the theoretical approach of the study.

2.1 Vygotsky’s overarching principles

Vygotsky developed his sociocultural theory of learning and knowing, conducting empirical studies in social interactions between individuals, particularly “in small groups or dyads” (Wertsch, 1985, p.26; Albert, 2012, p.12), and drawing on philosophy and psychology from scholars such as Pavlov, Marx and Hegel. I start my account of the sociocultural approach of my study with two overarching principles Vygotsky developed through his work: mediation and the general genetic law of cultural development. These principles form the basis of Vygotsky’s theory of learning and knowing, which I present later in this chapter.
2.1.1 Mediation

Pavlov worked within behaviourism (e.g. the famous Pavlov’s dogs), which was an early influence in Vygotskian theory, particularly in Vygotsky’s notion of mediation. Lerman (2014) accounted for behaviourism that it is a materialist theory, based on the notion that actions (e.g. dogs’ salivation) are responses to material stimulation that is going on around the individual and can be observed (e.g. a sounding buzzer). Behaviourism’s view of learning is represented by a stimulus-response link i.e. ‘stimulus → response’. In the case of Pavlov’s dogs, the dogs’ learning of food time was represented by a sounding buzzer-salivation link.

Vygotsky (1978) used the stimulus-response link to distinguish elementary functions, such as ‘sensation’ and ‘reactive attention’, from higher mental functions, such as ‘deliberate memory’ and ‘logical thinking’. He attributed immediate responses to stimuli, coming from the natural environment, to elementary mental functions. So, observable ‘reactive attention’ to material stimuli such as a buzz of bees, can be interpreted as an elementary mental function. He considered, in contrast, an active individual who uses “artificial stimuli” (Vygotsky, 1978, p.39) in order to develop her/his higher mental functions. For example, an individual binds a handkerchief over a door handle in order to remember to do something within the next day. The handkerchief is the “artificial stimulus”. The individual is active in using the handkerchief, which influences her/his memory and mediates the process of developing her/his higher mental function ‘deliberate memory’.

Vygotsky (1978) severed the stimulus-response link for higher mental functions by inserting mediation i.e. ‘stimulus → mediation → response’, using a triangular representation of mediation (Figure 2.1, cited from Vygotsky, 1978, p.40), where S is for stimulus, R is for response and X is for the action of mediation.

![Figure 2.1: Triangular representation of mediation.](image)
In the triangular representation of mediation, the artificial stimuli become a means which mediates the development of higher mental functions. Vygotsky introduced the notion of *psychological tool* to denote mediational means.

*Psychological tools* are “devices for mastering mental processes” (Daniels, 2001, p.15).

I include this definition of *psychological tools* from Daniels, because it captures the meaning of the handkerchief as a device for mastering ‘deliberate memory’. So, Vygotsky used the notion of mediation to indicate the existence of *psychological tools* for human development beyond instinctive elementary functions. Vygotsky (1981, p.137) offered examples of *psychological tools*: “language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps and mechanical drawings; [and] all sorts of conventional signs”.

Wertsch (1991) linked human action with mediational means, stressing that human action employs mediational means, namely tools, and these mediational means shape the action. So, “the relationship between action and mediational means is so fundamental that it is more appropriate, when referring to the agent involved, to speak of ‘individual(s)-acting with mediational means’ than to speak simply of ‘individual(s)’” (1991, p.12). In my study, the notion of mediation is fundamental. I think of ‘tutors-acting with mediational means’; their actions thus are tool-mediated actions, or simply, actions with tools. ‘Actions with tools’ can be further explained with the first generation activity theory model, below.

Cole (1996) offered a reproduction of Vygotsky’s mediation with human action mediated through culturally available tools. Activity theorists illustrate Cole’s reproduction with the ‘first generation activity theory model’ (Figure 2.2, adapted from Daniels, 2001, p.86).

![Figure 2.2: First generation activity theory model.](image-url)
In my study, I characterise the practice of teaching in terms of teaching actions; in particular, I conceptualise a mathematics teaching action through the first generation activity theory model\(^3\). The subject is the tutor, who acts with tools for teaching towards the object/motive of students’ mathematical meaning making. In other words, students’ mathematical meaning making motivates the tutor’s teaching actions. Being tool-mediated, students’ mathematical meaning making can be seen as a student higher mental function. Also, teaching can be seen as a tutor higher mental function. So, the tutor acts with tools for teaching, which mediate the student development of mathematical meaning making, as well as her/his own development of teaching and mathematical meaning making.

2.1.1.1 Tools and artefacts

Vygotsky (1978) made a distinction between psychological (mental/ideal) tools and physical (material) tools, focusing on the former and naming them ‘signs’. Material tools are means of mastering the nature, transforming “the process of a natural adaption by determining the form of labour operations” (Vygotsky, 1981, p.137); for instance, prehistoric stone tools. As mentioned above, psychological tools are means for mastering mental processes, transforming “the entire flow and structure of mental functions” (Vygotsky, 1981, p.137). Considering that Vygotsky included language in psychological tools, the next quotation suggests that Vygotsky viewed material and psychological tools as artefacts.

Distinctions between tools as means of labour of mastering nature, and language as means of social intercourse become dissolved in the general concept of artefacts or artificial adoptions. (Vygotsky, 1978, p.53-54)

McDonald, Le, Higgins and Podmore (2005) argued that there is lack of agreement on the relation between tools and artefacts, causing their interchangeable uses in published scholarship. Within the Vygotskian perspective, Cole (1996) suggested that the concept of tool is a subcategory of the superordinate notion of artefact since; for example, people may act as mediating artefacts. Kozulin (1998) confirmed that

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\(^3\) There have been three chronological generations of activity theory models, so far. The structure of the last two generations is expanded to consider interrelations between the individual subject(s) and the surrounding community.
Vygotsky included psychological tools and other human beings in mediators. In the tutorial setting, for example, mediating human beings for students’ mathematics learning can be a more informed fellow student, the tutor and even those who are not physically present such as writers of lecture notes and textbooks.

In this study, I follow Cole (1994) who suggested that the separation of tools into material and psychological is impossible. Cole (1994, p.94) argued that a tool carries its material and psychological/ideal nature inherently, exhibiting “a dual nature”. This is in agreement with scholars who relate the ideal to the material (e.g. Bakhurst, 1995; Wertsch, 1998). In a specific time and place, the material includes what can be perceived through senses (e.g. mathematical notation seen on a whiteboard), and the ideal includes what happens in the mind (e.g. making sense of the mathematical notation for the definition of injectivity). So, in Cole’s terms, mathematical notation on the board exhibits a dual material and psychological/ideal nature.

Wertsch (1998) asserted that “materiality is a property of any mediational means” (p.31). He exemplified his assertion about materiality with Vygotsky’s examples of psychological tools, namely ‘maps’ and ‘mechanical drawings’, arguing that “they are physical objects that can be touched and manipulated” (p.32). He also viewed the materiality of tools through their historicity; “they can continue to exist across time and space” (p.31). Wertsch (1998) stressed that materiality is even a property of language. Written language and recorded spoken language have a clear-cut materiality; they can be seen or heard. However, he stressed that materiality is also a property of spoken language which is not recorded; spoken language is historical and its meanings continue to exist across time and space.

In my study, for example, I consider that the tutor’s spoken language has a dual material and psychological nature. The tutor orally explains the mathematics in the specific place and time of the tutorial, and the students sometimes record on their scripts what is being said and other times they recall it, thereby retaining their tutor’s spoken language across space and time (material nature). Also, the tutor’s spoken language can enable the students to make mathematical meaning (psychological nature). Hereafter, I use the notion tool for teaching to include the dual nature of the tutor’s tool and the possibility that the tool can be a human being. The tutors’ specific
tools for teaching emerged through a grounded analytical approach to the data; examples can be found in Chapters 4, 5 and 6.

2.1.2 General genetic law of cultural development

“[Vygotsky] was particularly interested in how social interaction in small groups or dyads leads to higher mental functioning in the individual” (Wertsch, 1985, p.26): “the first problem is to show how the individual response emerges from the forms of collective life” (Vygotsky, 1981, p.165). Vygotsky envisaged his general genetic law of cultural development with the conclusion: “higher mental functions […] are first divided and distributed among people, and then become functions of the individual” (Vygotsky, 1981, p.164). Vygotsky’s conclusions formed an account of the social origin and social nature of higher mental functions which I summarise in the following extracts:

When we speak of a process, “external” means “social.” Any higher mental function was external because it was social at some point before becoming an internal, truly mental function. It was first a social relation between two people. The means of influencing oneself were originally means of influencing others or others’ means of influencing an individual. (Vygotsky, 1981, p.162)

We could formulate the general genetic law of cultural development as follows: Any function in the child’s cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category. (Vygotsky, 1981, p.163)

The word social when applied to our subject has great significance. Above all, in the widest sense of the word, it means that everything that is cultural is social. Culture is the product of social life and human social activity. That is

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4 The translation into English is from Wertsch. In this chapter, there are various translations of Vygotsky’s work from Wertsch. So, parts of my presentation of Vygotsky’s work include my interpretations of Wertsch’s views of Vygotsky.
why just by raising the question of cultural development of behavior we are
directly introducing the social plane of development. […] These higher mental
functions are the basis of the individual’s social structure. Their composition,
genetic structure, and means of action – in a word, their whole nature – is
social. Even when we turn to mental processes, their nature remains quasi-
social. In their own private sphere, human beings retain the functions of social
interaction. (Vygotsky, 1981, p.164, original underline, italics added)

Vygotsky used the social origin and social nature of higher mental functions as a
criterion to distinguish between elementary and higher mental functions (Wertsch,
1985, p.25); mediation was another distinguishing criterion. Commenting on
Vygotsky’s first extract provided above, Wertsch (1981, p.190) highlighted that “by
coming to master the mediational means of social interaction, the child masters the
very means needed for later independent cognitive processing”. In this way, he
stressed the coherence of Vygotsky’s approach, according to which it is through
mediation that the interpsychological functioning [with others, outside] is transformed
into an intrapsychological one [with self, inside].

The general genetic law of cultural development indicates that higher mental
functions, such as students’ mathematical meaning making, are first developed on the
social plane (e.g. in the lecture theatre with the lecturer, in the tutorial with the tutor
and fellow students, at home with those who are not physically present such as writers
of lecture notes and textbooks). Then, the higher mental functions are developed on
the psychological plane (e.g. at university or at home meaning making is developed
by individual solving of mathematical tasks). In particular, the nature of individual
meaning, that is to say the nature of meaning on the psychological plane, is quasi-
social. The word quasi-social is in Vygotsky’s third extract provided above. It means
that individuals make meaning in social interaction with others, and later individual
development retains that social meaning.

My consideration is that, in his three extracts, above, Vygotsky (1981, p.162,163,164)
postulated the difference between other theories (e.g. Piagetian) which have not
ignored social factors and his own sociocultural theory: higher mental functions are
first external, that is to say social and cultural, and then through mediation they
become internal, that is to say mental and individual. Thus Vygotsky's principles,
mediation and general genetic law of cultural development, reveal the sociocultural origins of the ways people understand the world. Thinking of the two principles, Wertsch asserted that:

“different forms of intermental functioning give rise to related differences in the forms of intramental functioning.” (Wertsch, 1991, p.27)

My understanding of this Wertsch assertion is in two levels. In one level, forms of intermental/interpsychological functioning can be the individual’s sociocultural experiences prior to a new social interaction. So each individual brings her/his unique set of prior experiences, from the social formation of preceding cultures she contributed to, into a new social interaction. The set of prior experiences relates to Lerman’s (2000, p.31) term of “each person’s collection of multiple subjectivities” such as “gender roles, ethnic stereotypes, body shape and size, abilities valued by peers”, and leads to differences in the forms of each individual’s “quasi-social” (Vygotsky, 1981) meanings.

In another level, different forms of intermental functioning exist in different social contexts, because of individuals with different subjectivities within those contexts. So each individual occupies a ‘positioning’ (Evans, 1993; Evans & Tsatsaroni, 1994) in the different subjectivities which are exposed in interaction with others. I understand that individuality includes subjectivities (i.e. personal quasi-social meanings of experiences in a particular social context), and positionings (i.e. personal positions in differing subjectivities formed in different social contexts). Harré and Gillet (1994) argued that each person is a unique collection of subjectivities and positionings, each shared with others of common culture (Lerman, 1996).

To conclude, my understanding of the Wertsch assertion is that one level of consideration for the formation of individual “quasi-social” meanings is the person’s collection of sociocultural experiences and another level of consideration is the specific social context(s) where the person functions. Thus “individuals are always situated in time and place” (Lerman, 1996, p.147).
2.2 Vygotsky’s theory of learning

Lerman (2000, p.34) asserted that “Vygotsky provided a mechanism for learning with four key elements: the priority of the intersubjective; internalisation; mediation; and the zone of proximal development (ZPD)”. I presented the first and third above; so, this section is an exposition about internalisation and the ZPD.

Lerman (2014, p.18) comprehended that the general genetic law of cultural development is Vygotsky’s “theory of internalisation”. According to that theory, sociocultural meanings are tool-mediated and internalised by the individual in the form of “quasi-social” meanings (Vygotsky, 1981, p.164), which are situated in the individual’s psychological plane, or put another way in his/her internal plane of consciousness. “Intersubjectivity concerns the meanings human beings make in relation to each other” (Jaworski, 2015, p.171). Lerman (2000, p.34) attributed “the priority of the intersubjective” to internalisation. In other words, intersubjectivity is on the social plane and precedes internalisation, which is on the psychological plane. So, the process of learning begins with intersubjectivity that leads to internalisation through mediational means.

Intersubjectivity sets up the ZPD of the individual (Lerman, 2001, p. 57) through which learning happens (Lerman, 2014, p. 22), thus “learning is from others” (Lerman, 2000, p.35). Vygotsky defined the ZPD of each individual as:

the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or collaboration with more capable peers. (1978, p.86)

Mediational means can include adult(s), more capable peers, language and tools such as lecture notes and textbooks. Vygotsky’s theory of ZPD nevertheless stresses the significance of mediating human beings physically present in a learning process. In other words, it stresses the significance of the social plane, which is formed by the learner’s communication and collaboration with a teacher or a more capable peer.
Thus, within the Vygotskian perspective, “teaching and learning cannot be discussed separately” (Lerman, 1996, p.138) and the understanding of ZPD should not be reduced to individual learning in a social context. So, the process of learning begins with intersubjectivity that sets up the ZPD and leads to internalisation through mediatinal means.

Meira and Lerman (2001) stressed that the ZPD is not a physical space brought by the individual to a learning situation as part of his/her nature; rather, it is a symbolic space created in social interaction and involving individuals, their tool-mediated actions and the context. In the tutorial setting for example, the ZPD is a product of the previous network of experiences and positionings of the students and the tutor (i.e. the individuals). So, an individual’s ZPD includes quasi-social meanings, drawn on other individuals’ tool-mediated actions. These quasi-social meanings describe the world as it is seen through the eyes of others (Lerman, 2000), taking place in a historically and geographically specific location (i.e. the context).

In a classroom, or amongst a group of students, what arises has been described as “multiple, overlapping zones of proximal development” (Brown et al, 1993). […] in a successful learning activity, the child’s response to a question will originate from his/her previous network of experiences but may well pull a second child into her/his ZPD and a second response might well pull both pupils into their ZPDs. […] there is a shared ZPD, created in the learning activity, and the process of internalisation occurs in each child’s ZPD. (Lerman, 2001, p.57-58)

In the above extract, Lerman referred to ‘learning activity’ and described how the intersubjectivity of a group of peer students sets up their ZPDs. I understand the shared ZPD as a mutual orientation towards agreed-upon socially and culturally mediated meanings. However, all participants in a learning activity (e.g. the tutor as well as the more or less capable fellow students than the individual student) can be pulled into their ZPDs and learn. So, for example, despite “imbalances in relationships and knowledge” among the participants (Lerman, 2001, p.61), the students can learn the mathematics in a tutorial, and the tutor can also be the learner on how to work with the mathematics to teach the particular students. Furthermore, “quite often a ZPD is not created” (Lerman, 2001, p.57). This is because the
participants in social interaction may not engage together in the activity (e.g. one may act separately or the other may not act at all). Thus, the ZPD happens when the participants engage in the teaching/learning activity, acting with tools and communicating with each other.

2.3 Vygotsky’s theory of knowing

In this section, I account for two perspectives in Vygotsky’s theory of knowing (i.e. coming to know): Wertsch’s perspective in Vygotsky’s writings about the ‘decontextualisation of mediational means’, and Vygotsky’s ‘dialectical method’ and worldview. In my study, I found the latter perspective useful for the conceptualisation of teaching knowledge. Commencing this section, I distinguish between two important notions: objective meaning on the social plane; and subjective sense which the individual aligns to objective meaning during social interactions. My analysis of teaching practice and teaching knowledge is with regard to students’ mathematical meaning making, taking into consideration of as much of both notions as possible from the tutor’s perspective.

Wertsch (2000, p.23) referred to Vygotsky’s (1987) conception of thought and language, outlining a set of oppositions to discern thought and language:

<table>
<thead>
<tr>
<th>Language</th>
<th>Thought</th>
</tr>
</thead>
<tbody>
<tr>
<td>External social speech</td>
<td>Inner speech</td>
</tr>
<tr>
<td>Written speech</td>
<td>Inner speech</td>
</tr>
<tr>
<td>Meaning</td>
<td>Sense</td>
</tr>
</tbody>
</table>

In this set of oppositions, the elements of the first column are in opposition to the elements of the second column. So, language precedes thought, and individual “quasi-
social” meaning (Vygotsky, 1981, p.164) is named ‘sense’. The latter is an important distinction for my study; I refer to ‘sense making’ for the subjective making of a particular student’s quasi-social meaning, and to ‘meaning making’ for the objective making of the established mathematical meaning within particular contexts (e.g. lectures, tutorials).

Luria (1973) stressed that language is an important mediational means in thought, because it “carries the cultural inheritance of the communities (ethnic, gender, class, etc.) in which the individual grows up” (Lerman, 1996, p.137). In this way, the individual internalises the world negotiated by the communities (ethnic, gender, class, etc.) in which s/he grows up, thereby making sense of the world and aligning it to established meaning.

Wertsch (2000, p.21) asserted that Vygotsky’s connection between subjective mind and objective world in order to make knowledge of the world possible is the notion of ‘abstract thinking’ or the ‘decontextualisation of mediational means’ (Wertsch, 1985, p.33)”. He thought of ‘abstract thinking’ as thinking through and with mediational means, which are de-contextualised, that is to say less and less dependent on the unique spatiotemporal context in which they are used (Wertsch, 1985). Wertsch (2000, p.20) argued that abstract thinking is possible through “referential relationships” between mediational means and the world of objects i.e. the world around us. My consideration is that the relationships are “referential”, because they are possible through social speech and meaning as well as inner speech and ‘sense’ on the one hand; and on the other hand, because they suggest connections between the objects in the world around us. For example, a product of abstract thinking of an object (e.g. a rose in a garden) is a scientific concept (e.g. “flower”). The referential relationships can be among a rose, a violet and a lily, developed in social interaction of a group of individuals. The individuals can be scientists, such as biologists, or any group of individuals who attempts to make meaning of the world. Then, any rose, any violet and any lily can be a mediational means for the meaning of the concept “flower”.

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5 In this study, I use the term individual “quasi-social” meaning to denote that individuals make meaning in social interaction with others, and later individual development retains that social meaning.
The key difference in the psychological nature of these two kinds of concepts [scientific and everyday] is a function of the presence or absence of a system. Concepts stand in a different relationship to the object when they exist outside a system than when they enter one. The relationship of the word “flower” to the object is completely different for the child who does not yet know the words rose, violet, or lily than it is for the child who does. (Vygotsky, 1987, p.234)

Vygotsky (1997) considered that the established knowledge is the space of connections of concepts in which concepts exist. The “system” Vygotsky referred to in the above extract is the space of connections of concepts. Vygotsky (1987) discerned scientific concepts from spontaneous/everyday concepts having the presence or absence of a system as his criterion. For example, my consideration is that the system in which the scientific concept “flower” exists includes the characteristics of “flower”, the connections between the characteristics of “flower”, and the connections between “flower” and other concepts. Thus, making meaning of the system in which “flower” exists requires abstract thinking, in social interaction with others, about the characteristics of a rose, a violet and a lily and about the connections between those characteristics.

My consideration is that, for Vygotsky, the established knowledge of mathematics is the space of connections of mathematical concepts in which these concepts exist. For example, the system in which the concept of injectivity exists includes the properties of injectivity, and the connections between injectivity and other concepts. Considering Wertsch’s view, injectivity can emerge from abstract thinking of examples of functions; that is, thinking through and with examples of functions less and less dependently on the specificities of each example. Mathematician(s) in social interaction (e.g. in dialogue with each other or while studying) can discover connections between the examples of functions. In particular, mathematicians can discover the properties of injectivity through commonalities between the examples. In this way, the examples of functions mediate the meaning of injectivity; in other words, the examples of functions are mediational means, or simply tools, for the meaning of injectivity.
Vygotsky (1986, p.32) stressed that “In our conception, the true direction of the development of thinking is not from the individual to the social, but from the social to the individual”. This direction from the social plane to the psychological/individual plane is allied to Vygotsky’s general genetic law of cultural development. Lerman (2001b, p.89) explained that “Vygotsky’s psychology was an application of Marx’s theories to learning”. “Marx challenged the image of the individual as the source of sense making” (Lerman, 2001b, p.89); in particular, Marx saw consciousness, that is to say the psychological plane in Vygotsky’s terms, as the result of social relations.

It is not the consciousness of men that determines their being but, on the contrary, their social being that determines their consciousness. (Marx, 1859, p.328-329, cited in Lerman, 2001b, p.89)

Pinkard (2000) asserted that Hegel’s philosophy was the forerunner to Marx’s theories of learning. A confirming reference to Pinkard’s assertion is from Derry (2013, p.106), who held that “Hegel’s argument [is] that the mind does not exist a priori but emerges in social activity”. Derry commented that “to grasp the extent to which Vygotsky’s ideas go beyond a limited concept of abstraction and decontextualisation, it is necessary to understand the different philosophical frame and presuppositions in which his thought was developed” (Derry, 2013, p.105).

Derry’s comment, above, triggered my search for an additional perspective of Vygotsky’s theory of knowing to the one that Wertsch presented with the notion of ‘abstract thinking’. This was partly because the notion of ‘abstract thinking’ as ‘decontextualisation’, which was explained by Wertsch, did not enable me to deepen insights into teaching knowledge from the analysis of the kinds of data that I had collected. The additional perspective that I found is Vygotsky’s ‘dialectical method’ and world view.

Daniels (2001, p.36) cited Van der Veer and Valsiner (1991) to assert that “Vygotsky most definitely adopted a dialectical world view. This was the case for his theories as well as his approach to method.” Dialectics is a key theoretical point for my study. In dialectics, human development is seen as driven by “internal contradictions” (Daniels, 2001, p.36), which I explain below drawing on Hegel, Marx and Vygotsky.
Derry (2013) contended that Vygotsky, following Hegel, attributed thinking, which always takes place within a definite historical place, to the development of “quasi-social” meaning within the individual (Vygotsky, 1981, p.164). Derry (2013, p.115) stressed that, for Hegel, in a process of development successive forms of historical consciousness within the individual (i.e. the psychological plane of the individual in Vygotsky’s terms) “arise out of the inadequacies and one-sidedness of those that precede them”. Each form of consciousness is historical since it takes place within a definite historical place, and arises out of the partiality of the previous form of consciousness. The successive forms of consciousness of reality are related with dialectic connections; that is, connections which arise out of contradictions in dialogue with others. So, the internal plane of the individual is successively faced with contradictions (or antinomies) of the relation between the object (e.g. a flower or a mathematical example of a function) and the knowledge of this object i.e. the system of connections.

The whole structure of the successive forms of consciousness of the object and the connections between these forms reveal knowing of this object within the individual. This method of knowing the object (or reality) is the dialectical method, and the connections among the forms of consciousness are called dialectic. The word dialectic first appeared in the Socratic dialogues written by Plato, and then used by philosophers such as Hegel and Marx. The ‘dialectical method’ is a method to establish the truth of the reality through dialogue between individuals whose views contradict. Dialogue is important for an individual’s knowing of reality. The individual’s views of reality are first formed between individuals; that is to say, in agreement with the general genetic law of cultural development. When these views contradict other individuals’ views in dialogue, a successive form of consciousness within the individual arises. In each successive form of consciousness, the individual aligns her/his sense of the world to the established meaning of the communities in which s/he participates.

Marx’s theories also accepted the dialectical method. However, Marx’s opposition to Hegel’s dialectical method was Hegel’s ideal (mental) as “the demiurgos of the real world” (Marx, 1873, p.14). Pinkard (2000, p.ix) explained this opposition by stressing that “unlike Marx who was a materialist, Hegel was an idealist in the sense that he
thought reality was ultimately spiritual”. In order to make sense of what reality might be as “ultimately spiritual”, I looked at the *Cambridge Dictionary of Philosophy* (1999, p.412), which accounts for idealism that “the real objects constituting the ‘external world’ are not independent of cognising minds, but exist only as in some way correlative to mental operations”. So, within idealism reality is ultimately mental. For example, Berkeley, an idealist, articulated the thesis that “to be [real] is to be perceived” (*esse est percipi*); that is to say, nothing real can be unperceived. As a materialist, Marx (1873, p.14) opposed Hegel and idealism, considering that “on the contrary, the ideal is nothing else than the material world reflected by the human mind, and translated into forms of thought”. In this way, Marx advocated that the human mind reflects on reality (the ‘external world’), which existence is independent of thought, in order to come to know.

However, there are philosophers such as Ilyenkov (1977, p.81), who do not polarise the ideal and the material; that is, they do not see the ideal and the material as mutually exclusive opposites. For example, Vygotsky followed both Hegel and Marx to develop his theory of learning and knowing. Following Marx, Vygotsky considered that the humans’ knowledge of the objects in real world is the reflection of reality (Duarte, 2011). In other words, the human mind reflects on the material/external world through dialogue between people whose views contradict, in order to come to know out of the contradictions and the search for common ground.

In the following extract, which I use to exemplify Vygotsky’s perspective on the dialectical method, Vygotsky (1981, p.173-174) spoke about developmental stages rather than Hegel’s forms of consciousness; however, my consideration is that both Vygotsky and Hegel referred to the psychological/internal plane of consciousness. Also for Vygotsky, following Hegel, each successive developmental stage of consciousness arises out of the partiality of the previous one; the previous stage thus exists within the following stage.

The connections among the developmental stages that interest us in child psychology are dialectic. Each successive stage in the development of behaviour is the negation of the preceding stage. It is a negation in the sense that the qualities peculiar to the first stage of development are copied, destroyed or sometimes transferred into a higher stage. […] Thus, any
subsequent stage involves the change or negation of qualities of the preceding stage. On the other hand, however, the preceding stage exists within the following stage. (Vygotsky, 1981, p.173-174)

Hegel and Marx espoused that there is a passage from quantitative changes to qualitative changes, thus quantity provokes changes in quality. They provided the example of phase transition of a thermodynamic system between liquid and gaseous states of matter, where vaporisation was the qualitative change. In the above extract, Vygotsky referred to qualities rather than quantities. In my study, I searched for patterns (quantities) of teaching actions with tools in transcripts of tutorial observations, in order to identify tutor ‘tools’ and ‘strategies’ for teaching (qualities, as conceptualisations of teaching practice). In other words tutor actions with tools, repeatedly used in her/his teaching, provokes the development of specific tutor ‘tools’ and ‘strategies’ for teaching. Thus, in my study the developmental stages within the tutor are qualitative stages, that is to say stages including qualities of the communities in which the individual participates. In particular, the qualities are tools and strategies for teaching, which the tutor develops throughout her/his teaching career and studies in interaction with other individuals such as own teachers, colleagues and students.

Considering the above extract, Vygotsky’s theory of knowing includes dialectic connections between the developmental stages of the individual. In my study, I investigate the tutors’ developmental stages of knowing for teaching that fosters students’ mathematical meaning making. In dialogue with other individuals whose views contradict (e.g. the tutor uses specific tools to enable the students to make mathematical meaning, but the students do not make meaning), the individual (in this study the tutor) negates her/his tools and strategies (e.g. copy, destroy), thereby creating a successive developmental stage of knowing for teaching. In this stage, s/he brings in some additional tools and/or strategies, and s/he abandons (i.e. destroys) or retains (i.e. copies) previously used tools and/or strategies. By copying tools and/or strategies, a previous developmental stage exists within the new stage. Also, the connections between developmental stages are dialectic in terms of arising out of dialogue with students whose views on their meaning making contradict the tutor’s intentions. (A more practical perspective on dialectic connections is presented in Chapters 4, 5 and 6, where the teaching of three tutors is analysed.) This complexity
of the dialectic connections of developmental stages is not present at Wertsch’s writings of abstract thinking. With regard to abstraction, Vygotsky criticised generalisations as direct results of abstraction.

It is completely clear that if the process of generalising is considered as a direct result of abstraction of traits, then we will inevitably come to the conclusion that thinking in concepts is removed from reality […] Others have said that concepts arise in the process of castrating reality. Concrete, diverse phenomena must lose their traits one after the other in order that a concept might be formed. Actually what arises is a dry and empty abstraction in which the diverse, full-blooded reality is impoverished by logical thought. This is the source of the celebrated words of Goethe: ‘Gray is every theory and eternally green is the golden tree of life’. (Vygotsky, 1998, p.53)

Vygotsky asserted that abstraction is impoverished when it is cut off reality and stressed that instead, “a concept includes not only the general, but also the individual and particular” (1998, p.53). This indicates that Vygotsky rejected that reality is ultimately mental. So, for Vygotsky (1998, p.53), there is “an image of the objective” (i.e. the abstract, the general) in reality. Reality nevertheless is complex in terms of connections and relations, and the individuals synthesise this diversity, first through dialogue, in order to make the connection between thought and concept; in order to come to know. Vygotsky did not reject abstraction; however, he asserted that the concept “does not arise […] as a mechanical result of abstraction – it is the result of a long and deep knowledge of the object” (Vygotsky, 1998, p.54). My consideration is that this ‘long and deep knowledge’ of the object reveals the system of connections of concepts. Vygotsky’s following extract is also illuminative to his views on the relation between reality and abstraction.

while the highest scientific abstraction contains an element of reality […]

Even the most immediate, empirical, raw, singular natural scientific fact already contains a first abstraction. (Vygotsky, 1997, p.249)

To conclude, I follow Derry (2013) who asserted that Vygotsky saw a dynamic view of knowledge (i.e. the space of connections of concepts in which concepts exist), and of the process of how people come to know. The first stage of the developmental
process of coming to know for teaching (for students’ meaning making in mathematics) is formed within a definite historical place (e.g. the tutorial classroom) through dialogue, and includes a first abstraction of teaching practice. This abstraction creates the qualities (i.e. tools and strategies for teaching) which exist in that first developmental stage within the tutor. The dialogue is among individuals (e.g. tutor and students) who bring with them unique sets of prior sociocultural experiences (e.g. with mathematics). Through intersubjectivity with individuals and contradictions in each other’s views (e.g. contradictions between tutor’s views on the design of teaching for students’ meaning making and students’ meaning making per se), the tutor constructs successive developmental stages of her/his knowing about teaching practice for students’ meaning making in mathematics. Each successive developmental stage emerges out of contradictions in individuals’ views (e.g. tutor’s and students’ views on students’ meaning making), and results into the negation/change of tools and/or strategies for teaching in the tutor’s preceding developmental stages. So, the successive developmental stage is connected to the tutor’s preceding developmental stages through contradictions in dialogue between individuals (e.g. tutor and students make dialogue about the mathematics and contribute to the dialogue with their meanings). The contradictions in dialogue reveal the dialectic connections between the individual’s developmental stages. In particular, the developmental stages are in the psychological/individual plane of each participant in the dialogue, and even a highest developmental stage of an individual (e.g. a highest developmental stage of the tutor’s knowing for teaching) contains an element of reality (e.g. teaching practice). My consideration is that the whole structure of both the successive developmental stages (each with an abstraction and an element of reality) and the connections between the developmental stages reveal knowing within the individual. Vygotsky’s view of knowledge with the system of connections of concepts and of the process of how people come to know is dynamic, since it emerges as the outcome of a complex developmental process within a definite historical place.
Chapter 3

METHODOLOGY

In this chapter, I set out the methodology of my study, which is a qualitative enquiry about tutors’ teaching practice and knowledge with regard to students’ mathematical meaning making. I begin with an exposition of my research questions, and then, I draw on Chapters 1 and 2 to develop the research design and methods. The research design includes my methodological choices, and implications of the theoretical perspectives for the methodology. It is followed by a detailed report on the research process, which involves a discussion of the methods for data collection and analysis. That is to say, how the methods outlined were carried out in the field; what the data looked like and how the data were analysed. Concluding this chapter, I comment on ethical issues, my role in the research process, and the trustworthiness of the study.

3.1 Overview and research questions

The objectives of my doctoral research were:

1. to connect tutors’ teaching practice with their knowledge for teaching; in particular, to interpret tutors’ knowledge for teaching from analysis of their teaching practice with first year mathematics modules at university level; and
2. to explore the nature of teaching knowledge with regard to students’ mathematical meaning making from the tutor’s perspective.
In order to address my objectives, I developed the following research questions:

1. How is teaching knowledge revealed in teaching practice with first year undergraduate mathematics modules?
2. How does teaching knowledge interact with students’ mathematical meaning making?

I articulated the first research question in a way that would enable me to explore tutors’ knowledge for teaching through their teaching practice at university level. The inspiration came from Rowland, Huckstep and Thwaites’ (2004) meticulous work towards the Knowledge Quartet, which was based on the investigation of ways in which trainee primary school teachers’ mathematics content knowledge “played out” in their teaching. Following my first research question, I developed the second research question with the intention to indicate that I would investigate teaching practice and knowledge with regard to students’ mathematical meaning making.

I chose to study teaching closely, so I started to collect observational data from both lectures and small group tutorials (SGTs) while working closely with my supervisors in joint meetings. (Small Group Tutorials is a setting discussed in Chapter 1: Section 1.1.) From the data I collected, I chose to study SGTs since they provided opportunities for tutor-student dialogue and interaction through which meaning making could be discerned.

I did not refer to the tutorial setting in the research questions since the motivation for my research was to understand teaching at university level in its widest sense; an idea coming from the Undergraduate Mathematics Teaching Project (Jaworski, 2002). Indeed, aspects of tutors’ teaching practice specific to the tutorial setting, and aspects of tutors’ teaching practice not specific to the tutorial setting, emerged in the data analysis.

The difficulty with gathering data to investigate teaching knowledge and meaning making was that these conceptual areas are neither tangible nor visible, thus not observable per se, since they do not exist as substantial phenomena. I used the interpretative paradigm that allowed me to access these areas through my interpretations of tutors’ and students’ actions in the classroom. (The interpretative
paradigm is a research paradigm discussed in this chapter: Section 3.2.) I considered the tutors’ actions to relate to the nature of teaching and the approach, and accord with what the tutor said (I recorded in SGTs and read in transcripts); did (gestures, body language written in field notes); and intentions (I asked about their thinking in interviews or heard in the classroom). I also thought of the students’ actions as what students said and did during the SGTs. Interviewing students was beyond the scope of this study.

The nature of my interpretations was not evaluative of teaching; the tutors were professionals who conducted research in either mathematics or mathematics education, and some of them were very experienced in teaching. Evaluating the teaching was neither appropriate nor in my intentions. Rather, in interviews with the tutors, I teased out the tutors’ thinking that underlaid their teaching; and used it to justify my interpretations about the nature of teaching in observations.

I conducted data analysis during and after the process of collecting observational and interview data. In particular, I took a grounded analytical approach to the data in order to interpret tutors’ and students’ actions in SGTs. (My grounded analytical approach is a method for data analysis discussed in this chapter: Section 3.4.2.) In my analysis, I sought to characterise teaching practice through tutors’ actions, and students’ meaning making of mathematics through students’ actions and interviews with their tutors; thereby exploring tutors’ teaching practice with regard to students’ meaning making of mathematics. Since I did not collect interview data with students, my interpretations of students’ meaning making addressed the tutors’ practice and associated knowledge which fostered students to be participative and make meaning of mathematics. I thus examined students’ meaning making of mathematics from the tutors’ perspective.
3.2 Research design

This was an **empirical research** study, since I collected data through which I generated theory. In Bassey’s terms (1999) in particular, it was an empirical research which was **theoretical**, since it enabled me to create understanding of the social actions and interactions of tutors and students, which I studied in SGTs. Also, this research ultimately enabled me to create understanding of teaching knowledge, teaching practice, and students’ meaning making.

In my research paradigm, which was the **interpretative paradigm**, understanding is interpretation. Schwandt (2000, p.191), who is an interpretivist researcher according to Miles and Huberman (1994, p.277), explained that, within the interpretative paradigm “to understand a particular social action, the inquirer must grasp the meanings of this action” (Schwandt, 2000, p.191). He continued by saying that to grasp the meanings of an action “requires that one interprets in a particular way what the actors are doing”. Thus, according to Schwandt, the focus of research in the interpretative paradigm is the meanings of social actions, which are interpreted by the researcher. The meanings of social actions in my study were the researcher’s (my) meanings of tutors’ and students’ actions in SGTs.

Denzin (1978) stressed that within the interpretative paradigm the researcher’s meaning is social in origin and not intrinsic (origin of meaning); human beings shape it with their actions (nature of meaning):

> The social world of human beings is not made up of objects that have intrinsic meaning. The meaning of objects lies in the actions that human beings take towards them. (Denzin, 1978, p.7, cited in Eisenhart, 1988)

These assumptions for the nature and origin of meaning within the interpretative paradigm are compatible with the sociocultural lens (Simon, 2009) through which I looked in the analysis of my data. As Jaworski (2015, p.176) explained, the researcher’s meanings are “social meanings in so far as they coincide with meanings that are embedded in, or understood within, the communities” of which the researcher
is a part. In my study, I discussed my meanings of tutors’ and students’ actions in SGTs at conferences and supervisory meetings, thus aligning them to social meanings embedded in mathematics education research communities. So in my study, all accounts and analyses I produced of tutors’ and students’ actions contained an interpretive element (Carr & Kemmis, 1986). (In Section 3.4.2 of this chapter, I provide an example of the analysis of Main study 1, with details about the meaning of Zenobia’s and the students’ action to select tasks.)

Taking the argument about social meaning further, it seems to me that the sociocultural lens sometimes indicates the interpretative paradigm as a methodological stance. In her book about Vygotsky’s philosophy, Derry (2013) cited Brockmeier (1996, p.127), who argued about an “interpretative approach that has developed out of” Vygotsky; however, this statement was not unfolded in the book. Furthermore, in his account about learning and knowing mathematics within the sociocultural perspective, and in particular within the Vygotskian approach, Lerman argued that:

Research is always an interpretation and the text that presents the outcome of any study is inevitably affected by the researcher/author, at least from interpretivist and post-positivist methodological positions. (2014, p.23)

Lerman’s quotation “research is always an interpretation” indicates that, in his investigation of learning and knowing mathematics within the Vygotskian approach, a researcher always interprets the meaning of what the actors are doing; and with regard to Denzin (1978) and Jaworski (2015a) this meaning is social. Furthermore, Lerman’s quotation “the text that presents the outcome of any study is inevitably affected by the researcher/author” opens up a discussion on the trustworthiness of a study within the interpretative paradigm.

A traditional technique for the trustworthiness of a study is a replication of the study for achieving the same outcomes. Within the interpretative paradigm, however, there is not only one truth or one perspective on reality, which is independent of interpretations, and thus can be applied to other cases. Willis (2007) stressed that since reality is socially constructed, there are multiple perspectives on reality, so the researcher should not eliminate all but one true reality from study conclusions. In
addition to this, there is not a more correct interpretation or a better or a worse than other, and this results from the fact that knowledge is perspectival and contextual within the paradigm (Schwandt, 2000). Since the context of the study has implications in the interpretations of the researcher; repeating my study implies a different context of the study and thus different outcomes. This nevertheless does not mean that there are not criteria for the trustworthiness in findings and conclusions (e.g. Miles & Huberman, 1994; Guba & Lincoln, 1981; Lincoln, 1990; Lincoln & Guba, 1985). (An account of the criteria of trustworthiness, which I follow in my study, is discussed in this chapter: Section 3.7.) So, the findings of my study have the possibility to be confirmed in other studies.

The interpretive paradigm is also compatible with exploratory case studies and an ethnographic research methodology. My research includes a set of case studies of the tutors’ teaching in SGTs: 26 tutors were observed and interviewed, with three extended cases studied in depth. The case studies are exploratory (Yin, 1993), since the research questions are of an explorative nature, and theory is generated by directly observing SGTs. In particular, tutors’ teaching practice and knowledge was explored through direct observations in SGTs, and interviews with the tutors.

In the Cambridge dictionary of Philosophy, Daniel E. Little defined ethnography, for anthropologists, as an investigation of culture:

Researchers “immerse themselves in the life of a local culture” and “attempt to describe and interpret aspects of the culture”. (1999, p.290)

Little also discerned the research methods of observation and interview for ethnographic studies:

Researchers investigate culture through careful observation and recording of “various features of social life”; and through interview of “beliefs and values of members of the local culture”. (1999, p.290)
Although Little exemplified ethnography for anthropologists, the culture in my study is the teaching culture of the tutor and students in the tutorials where my observations took place. I argue that my study is **ethnographic** since my data collection at the University lasted for three academic semesters with observations of each extended case for more than one semester; thereby “immersing myself” in each tutor’s teaching culture. The “attempt to describe and interpret aspects of the culture” in my study is evident from accounts and analyses I produced of tutors’ and students’ actions for the interpretation of tutors’ teaching practice and knowledge with regard to students’ mathematical meaning making. I also used the commonly adopted methods for ethnographic studies: participant observations and interviews. In observations of teaching in tutorials, I *explored* and *characterised* the tutors’ teaching practice. “Various features of social life” in my study was concerned with the characterisation of each tutor’s teaching with regard to students’ mathematical meaning making, in terms of strategies and tools for teaching. (*Strategies* and *tools* for teaching are seen through the Vygotskian perspective, which is discussed in Chapter 2: Section 2.1.) I also conducted interviews with the tutors about *underlying considerations* of the observed teaching practice in order to *interpret* their knowledge for teaching. So, “beliefs and values of members of the local culture” in my study were tutors’ underlying considerations which included tutors’ thinking, views and values for the teaching culture of their own tutorials.

### 3.3 Methods for data collection

The cases of tutors’ teaching in SGTs were multiple across the stages of the study. The structure of the study included four stages: Pilot study 1 (stage 1); Main study 1 (stage 2); Pilot study 2 (stage 3); and Main study 2 (stage 4). Details can be found in the following Table 3.1 and in Appendix B.
Table 3.1: Details for the structure of the study.

<table>
<thead>
<tr>
<th></th>
<th># of SGT observations</th>
<th># of tutors observed</th>
<th>Details for # of tutors observed</th>
<th>Semester#/Year#</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pilot study 1</strong> (stage 1)</td>
<td>26</td>
<td>7</td>
<td>Multiple observations of 7 tutors’ teaching.</td>
<td>Semester1/Year 1</td>
</tr>
<tr>
<td><strong>Main study 1</strong> (stage 2)</td>
<td>10 (1 of 10 SGTs lasted for 3 hours)</td>
<td>1 (female)</td>
<td>The teaching of 1 of the 7 tutors in Pilot study 1 was observed systematically.</td>
<td>Semester2/Year1</td>
</tr>
<tr>
<td><strong>Pilot study 2</strong> (stage 3)</td>
<td>23</td>
<td>21</td>
<td>In order to select the sample for Main study 2: • 2 of the 21 tutors had also been observed in Pilot study 1; and • 2 of the 21 tutors were observed twice in Pilot study 2.</td>
<td>Semester2/Year1</td>
</tr>
<tr>
<td><strong>Main study 2</strong> (stage 4)</td>
<td>22</td>
<td>2 (male)</td>
<td>The teaching of 2 of the 21 tutors in Pilot study 2 was observed systematically.</td>
<td>Semester1/Year2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>81</td>
<td>26</td>
<td>-</td>
<td>3 semesters</td>
</tr>
</tbody>
</table>

Following each of the vast majority of SGT observations were interviews with the tutors about their underlying considerations. The nature of the interviews was friendly but professional; so discussions with the tutors was a more suitable term than interviews with them.

The common aspect of all sampling methods used for my pilot and main studies was the **theoretically driven sequential sampling** (Miles & Huberman, 1994). In other words, the sampling was not wholly prespecified, but evolved conceptually and sequentially during fieldwork. The levels of sampling were sequential; on the one hand, the initial sampling drove me to observe other cases within the same stage of the study. For example, within Pilot study 1, I started to approach and observe tutors who cooperated with the Mathematics Education Unit (MEU) of the University thus being

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6 Year # is the year # of the study in which the observations took place, and semester # is the academic semester within year #. Lecture observations are not included in this table.
familiar with mathematics education (one level of sampling), and gradually moved to also observe tutors not familiar with mathematics education (another level of sampling). (The MEU is introduced in Chapter 1: Section 1.1.) On the other hand, the sequence was *between the stages of the study*. For example, I selected to study in depth one participant’s teaching from Pilot study 1 (stage 1), and this was the participant for Main study 1 (stage 3). So, there was a sequence between sampling in pilot studies and sampling in main studies.

There was also a *theoretical* process within the sequential sampling; that is, sampling of one stage of the study learnt from sampling of the previous stage of the study. This learning was theoretical, since it was based on issues discerned in analysis of the tutors’ teaching in each stage of the study. For example, when I started to analyse my cases of Main study 2, I had brought the emergent characterisation of teaching into tools and strategies from the case of Main study 1. I thus used the characterisation across the cases of my main studies. This characterisation nevertheless was theory emerged from data, and not prespecified.

Miles and Huberman (1994) stressed that in case theory emerges from data, Glaser’s and Straus’s (1967) term for theoretically driven sampling is *theoretical sampling*. I gradually made the terminology for my sampling more precise by specifying that my theoretically driven sampling is in Glaser’s and Straus’s (1967) term a theoretical sampling.

The sample came from a specific British University; first, because of convenience (distance, networking), and second, because of the profile of the University. The elements of context, which contributed to the profile of the University, were:

- There was an average-sized group of tutors in modules offered by the Mathematics department at the time of data collection (40 tutors).
- Unlike most universities, that university had the structure of SGTs, in which the tutors were not Ph.D. students.
- Tutors came from different educational backgrounds and cultures; the academic staff thus was multicultural.
- There were tutors with doctorates in mathematics as well as tutors with doctorates in mathematics education, and both taught mathematics in SGTs.
Data is qualitative and consists of field notes during observations in SGTs, audio-recordings of SGTs, and audio-recordings of follow up discussions with tutors. Data was collected during the pilot and main studies over three academic semesters.

- **Field notes:** My field notes consisted of the words I captured from tutor and students in a dialogue form; tutor’s and students’ writings on the board; and their body language such as hand movements and facial expressions. I was specifically careful to capture in my notes as much wording as I possibly could from students, since they were usually not speaking loud enough so that I could hear them afterwards in the recordings. The students’ articulations in SGTs usually consisted of some words or phrases some of which were inaudible in recordings. During SGTs, I also wrote comments on the tutors’ actions in the right margin of my field notes. When the students were getting ready to leave the tutorials in the end of the sessions, I revised these comments to organise them for the follow up discussions with the tutors about their underlying considerations for their actions. The discussions in all my studies were unstructured, informal and friendly. I selected this type of interview so that I could go in depth into the conceptual areas I explored through my research questions. (A sample of field notes is in Appendix C.)

- **Audio-recordings:** I audio recorded and transcribed all SGTs from my main studies. My recorder was a small noiseless device of $15 \times 5 \times 1.4$ cm. I also audio recorded those tutorials from my pilot studies for which I had permission from the tutor and the students. I then enhanced all transcripts with information coming from my field notes. (A sample of transcript is in Appendix C.) For the tutorials I did not have permission to record, the most usual reason tutors said was that it was intrusive for students. In all tutorials, however, when I had permission from tutors, I also had permission from students. I had the feeling, but no data to support it, that students were forgetting the existence of the recorder after the first couple of minutes in the tutorial. There were also some discussions with tutors about their SGTs in the pilot studies, where the tutors did not give permission for audio recording. In that case, my endeavour was to capture as much from the tutors’ responses as I possibly could in notes. All tutors, whose teaching I present in this thesis, nevertheless gave me permission to audio record their SGTs and our follow up discussions.
3.4. Methods for data analysis

I refer here again to my research questions in order to report the methods I employed to analyse the data:

1. How is teaching knowledge revealed in teaching practice with first year undergraduate mathematics modules?
2. How does teaching knowledge interact with students’ mathematical meaning making?

In order to address my research questions, I did not test hypotheses based on extant research. This was because research in university mathematics education, in particular observational studies of “what teachers do and think daily, in class and out, as they perform their teaching work”, is rather limited (Speer et al., 2010); and research in additional settings to lectures such as tutorials is far less. Considering the dearth in research, neither did it seem appropriate to me to use a framework with categories for data analysis developed a priori. Rather, I took a grounded approach to analysis of data in order to develop my own grounded theory (Glaser & Strauss, 1967) for the characterisation of mathematics teaching practice and mathematics teaching knowledge at university level. (The research process towards the development of my grounded theory is discussed in some detail in this chapter: Sections 3.4.1-3.4.4.)

I chose the grounded approach to analysis of data, since it offered me a systematic method “for the discovery of concepts and generation of theory” and “not for testing or replicating theory” (Glaser, 1998, p.69). However, I chose not to espouse Glaser’s and Strauss’ view of the researcher’s tabula rasa towards existing research in the field. That is to say, an uncontaminated view according to which a researcher should not review the research literature until late in the analytical process, when theory has already been emerged from data. I based my latter choice on publications from Glaser and Strauss after their first book The Discovery of Grounded Theory in 1967 (e.g. Glaser, 1978, 1992, 1998; Strauss & Corbin, 1990). In these publications, Glaser as well as Strauss recognised factual issues in a research process which might prevent
the researcher’s tabula rasa. I explain how such issues are applicable in the case of my research study.

In my area of research, which is teaching practice and knowledge, there is substantive literature in school level education. I had studied numerous works within this literature as well as some works within university mathematics education literature prior to my doctoral research. This is a possibility for which Strauss and Corbin (1990) recognised that “We all bring to the inquiry a considerable background in professional and disciplinary literature” (p.48). Yet Glaser (1998) identified the possibility of knowledge of literature prior to enquiry; he suggested using it as data for constant comparison with the emerging theory from the data. Charmaz illuminated this process of constant comparison:

Through comparing other scholars' evidence and ideas with your grounded theory, you may show where and how their ideas illuminate your theoretical categories and how your theory extends, transcends, or challenges dominant ideas in your field. (2006, p.165)

In my study, I constantly compared emerging theory from my data with concepts from research in teaching practice at school level education, where the extant literature is extensive. After this process, I sometimes brought in my coding concepts from extant research literature. An example of such concepts is the types of mathematical examples I assigned to my coding when I identified a mathematical example in the data; e.g. generic examples, non-examples and real-world examples. (These types of mathematical examples are presented in Appendix D.) It nevertheless is important that I did not make a literature review in types of mathematical examples in order to align my grounded theory with findings about the use of examples in literature. Rather, I studied thoroughly only the concepts of generic examples, non-examples and real-world examples, which I thought to be of relevance to my data, in order to make meaning of them and enable the process of constant comparison. In this way, I avoided being “derailed from grounded theory by the prism of conceptual grab of the received concepts and problems [other researchers] have written about” (Glaser, 1998, p.70).
In his book *Theoretical Sensitivity*, Glaser (1978) asserted that “It is necessary for the grounded theorist to know many theoretical codes in order to be sensitive to rendering explicitly the subtleties of the relationships in his data” (p.72). An exemplification of theoretical codes is the types of mathematical examples I assigned to my coding. Furthermore, Strauss and Corbin (1990) suggested that “as your theory evolves, you can incorporate seemingly relevant elements of previous theories, but only as they prove themselves to be pertinent to the data gathered in your study” (p.50). My interpretation of Strauss’ and Corbin’s suggestion is that the incorporation of “elements of previous theories” in a grounded theory is the theoretical codes which “prove themselves to be pertinent to the data gathered” through the process of constant comparison. In agreement with Glaser (1978) and Strauss and Corbin (1990), I enriched my knowledge of research literature in school and university mathematics education throughout the stages of my study. In this way, I kept up my theoretical sensitivity for the codes I developed in my coding.

A number of researchers supported the view that it is impossible not to use findings from the literature in research (e.g. Cutcliffe, 2000; Eisenhardt, 2002, cited in Dunne, 2011). Stern (2007, cited in Dunne, 2011) added that a literature review for a grounded theory approach is essential for academic honesty as well as for demonstrating the contribution to extant knowledge within the field. Dunne (2011), however, argued that one of the most problematic issues relates to when theory or findings in existing literature should be used during a grounded theory study. Eventually, he (2011, p.116) recommended an early literature review “before commencing data collection and analysis” and he lists a number of benefits, such as:

1. an early literature review “may promote ‘clarity in thinking about concepts and possible theory development’” (Henwood & Pidgeon, 2006, p.350);
2. an early literature review reveals “how the phenomenon has been studied to date” (Denzin, 2002; McMenamin, 2006); and
3. an early literature review “can ensure the study has not already been done” (Chiovitti & Piran, 2003).

My interpretation is that the first benefit is congruent with Glaser’s *theoretical sensitivity*, since a researcher may identify suitable concepts for theoretical codes in this way. However, Glaser (1998) explicitly disagreed with the second and third
claimed benefits. With regard to the second claimed benefit, he exemplified a researcher’s preconceiving from literature: “I have seen researchers discover ‘awareness’ as a category in their study and then immediately start to use our awareness context theory as if it was the core variable, and it was not. It was just a subcategory. Hence the true core category was never discovered as such” (p.74). (Here Glaser referred to the “awareness context theory”, which is a particular theory within social sciences.) Finally, in relation to the third claimed benefit, he contended that “It is a waylaying deficit. First of all, no one else has done the study. It is impossible with such complex, multivariate work.” (p.74).

In my study, I did not conduct a literature review before data collection and analysis in the sense that I did not use key words in teaching practice and knowledge at university level, and did not study all relevant works that my search would have fed back to me. Rather, prior to the start of my doctoral study, I had studied research publications in the context of my Master’s programme and research dissertation. From those research publications, I used concepts for my theoretical codes in my doctoral study. Also after the start of my doctoral study, I was informed about recent or earlier works in school level and university mathematics education by reading research papers. Those research papers either were circulated in my Department or I searched for them to conduct the constant comparison method. I also attended various conferences throughout the years when I conducted my doctoral research.

I conducted a literature review in teaching practice and knowledge at university level at the write-up phase of the thesis. (My literature review has been discussed in Chapter 1: Section 1.2.) Indeed the developers of grounded theory suggested the position of weaving in literature at this phase of a thesis (Glaser & Strauss, 1967; Glaser, 1978; Strauss & Corbin, 1990; Glaser, 1992; Glaser, 1998). For instance, Glaser (1998) asserted that “the literature review should be performed at the sorting-writing stage of doing grounded theory. Adding to the literature, synthesizing it, transcending it, starting it, not reinvesting it, correcting it and abandoning the reverence of it are important.” (p.79). My reflection on Charmaz’s report on objectives for reading literature within a grounded analytical approach to data is that, in this way, I fulfilled: “to make explicit and compelling connections between [my]
My expertise in the grounded analytical approach and my understanding of the data was evolving over my data analysis. The following sections include my report on the data collection and the associated analytical process throughout the stages of my study.

3.4.1 Pilot study 1

Pilot study 1 commenced soon after the start of my doctoral studies. During this stage, I observed calculus lectures three times a week from Week 3 until Week 11 (i.e. 29 lectures in total) and SGTs every week from Week 4 until Week 11 (i.e. 26 SGTs in total). Various tutorials of each tutor were observed, and discussions with the tutors were fewer in number compared to next stages of the study. (Calendar details about data collection for Pilot study 1 are provided in Appendix B.)

I chose participants based on information which I collected through my supervisors. Goetz and Lecompte (1984, cited in Miles & Huberman, 1994) define sampling which is based on information coming from experts or key informants as reputational case selection. In their terms, my supervisors were my key informants for prospective participants, since they suggested tutors from their networks. In addition to this, the cases have an opportunistic element in their selection because of convenience; hence, the sampling was opportunistic (Kuzel, 1992; Patton, 1990, cited in Miles & Huberman, 1994), as well.

The rationale for participant selection was to be able, preliminarily, to understand teaching through observations of a variation in cases. Through the information I gained from my supervisors, I selected a wide range of tutors in terms of experience in teaching and doctorates; the tutors in Calculus of that semester, who also taught in SGTs; some tutors who had previously cooperated with the MEU; and some tutors who had not. In particular, the information I gained towards participant selection was:
• Lecturer1 and Lecturer3 had written part of the lecture notes in calculus, lectured calculus, cooperated with the MEU and were experienced in teaching. Lecturer1 was involved in the reform of the curriculum for calculus, which was called analysis from the next year onwards. They both did not study in the UK.

• Lecturer2 and Lecturer4 had a doctorate in mathematics education in contrast with the other tutors who had a doctorate in mathematics. My assumption was that “they also have knowledge of mathematics education on which they can potentially draw in their teaching”. Lecturer2 did not study in the UK.

• Lecturer5 was in a responsible position regarding the teaching and learning of mathematics at the University, and was experienced in teaching.

• Lecturer6 and Lecturer7 had not cooperated with the MEU until that time. Lecturer6 was early career and Lecturer7 was experienced in teaching and researching. They both did not study in the UK.

The purposes of this pilot study were to explore the context of the study, and to select a participant for my first main study. In order to address these purposes, I kept field notes during lectures and SGTs. Soon after each observation of a lecture or SGT, I also wrote my reflection of the tutor’s observed teaching based on events and quotes from the tutor’s or the students’ words written in my field notes. In my reflections, I narrated what the tutor did with regard to the mathematics content and the students; and how the students responded to that. My reflections, or else my narratives, were a first level analysis of tutors’ teaching, which enabled me to gain insight into seven tutors’ teaching in lectures or/and SGTs.

From those seven tutors, I selected to invite Lecturer3 to be the participant for my first main study. That would involve a whole semester SGT observation. I selected Lecturer3 to be the participant due to issues which emerged in my narratives of her teaching. The issues were concerned with students’ group work in Lecturer3’s tutorials; Lecturer3’s use of graphs/examples for inductive thinking; and Lecturer3’s mathematical flexibility. (Examples of narratives of Lecturer3’s tutorial teaching and an explanation of the emerging issues are woven in my account of the analysis of Pilot study 1 in Appendix E.) Lecturer3 accepted my invitation, and in response to my question, she chose the pseudonym Zenobia for my study.
While exploring the context of the study in Pilot study 1, I was learning about:

1. mathematical issues (e.g. the level of mathematics taught at the University);
2. teaching/learning issues (e.g. what the tutorial looked like in terms of what the tutors and students did, what the tutors’ reflection on what they did was);
3. methodological issues (e.g. what emerged in my field notes that I could ask in follow up discussions with the tutors); and
4. methodological practicalities (e.g. where the best place for me to sit was in order not to be intrusive and to be able to capture the teaching in notes and audio recordings).

Observing teaching in all its complexity (i.e. mathematical issues, teaching/learning issues, methodological issues and practicalities) in a tutorial classroom or lecture theatre, as well as finding a focus for what might be called “tutors’ teaching knowledge” and “tutors’ teaching practice” was a demanding task. I made some decisions at that point, according to which: the students’ actions were important for feeding back what might be discerned as students’ mathematical meaning making; the tutorial setting was more suitable than the lecture setting for discerning students’ mathematical meaning making; the tutors’ actions were important for what might be called “tutors’ teaching practice”. The following text is an explanation of those decisions in some detail.

I had expressed a focus on teaching with regard to students’ mathematical meaning making in my second research question. However, the lecture setting seemed inappropriate to me for gaining insight into students’ mathematical meaning. There was a large cohort of students (100+ students) in my lecture observations, and students rarely spoke. Only a few students answered the lecturer’s questions but I could not each time identify who the student was in the whole cohort. In addition to this, the lecturer circulated at the time of students’ work on the mathematics, but I could only observe what a few students around me did, and they were not the same students in each observation. Sometimes, the lecturer wrote on the visualiser in a Definition-Theorem-Proof format, and I could only observe students copying on their notes. It thus seemed impossible to me to have evidence for students’ meaning in lectures.
Since data for students was limited in lectures, in my narratives of lectures, it was evident that the focus was on what the lecturer did (sometimes exclusively). I provide two of my narratives written after first observations of Zenobia’s teaching in lectures. I use these narratives to exemplify, first, my focus on what the lecturer did in lecture observations, and second, how narratives were written.

**Narrative1  Lecture observation in October 23rd, 2012**

Zenobia presents the mathematics in slides. Students have access to lecturer’s slides in the Virtual Learning Environment (VLE) of the University, before the implementation of the lecture. Today, Zenobia starts with a revision of the previous lecture. She goes on with new mathematics to students (limits and limit computation), slide after slide. She presents theorems in slides, and simultaneously writes at the overhead projector examples of those theorems. Students can find some of those examples with a solution in VLE. Except for slides, in VLE there are also videos about calculus topics linked to YouTube. During her lecture, Zenobia encourages students to watch those videos at home.

In this narrative, my attention was at the lecture setting: lecture resources such as slides and videos linked to YouTube in VLE, students’ access to lecture resources, and lecturer’s use of lecture resources. I did not refer to the mathematics but rather, to how the lecture looked in terms of what the lecturer did. In particular, the lecturer revised the previous lecture, then presented theorems in slides and examples of those theorems. However, I did not have data to express what students did.

**Narrative2  Lecture observation in October 24th, 2012**

Today’s lecture is on derivatives and continuity. Zenobia asks the students whether they have done the topics at school. She begins the lecture by writing at the overhead projector that:

If \( f \) has a derivative, we say it is differentiable.

Then, she recalls the definition of derivatives, for which

\[
    f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}
\]

The lecture continues with examples and non-examples of derivatives either contributed by her or contributed by students, and always presented at the overhead projector; so, Zenobia draws graphs of tangent lines (also vertical ones); graphs where there is no tangent at a point of a function; and graphs where the curve is not a function. She also finds whether a graph of a function is continuous at a point or discontinuous at a point. Zenobia then
introduces the extreme value theorem and the intermediate value theorem from slides in VLE. She explains the theorems to students by using examples with graphs of functions.

Although both narratives are not as lengthy as my narratives on next observations, they shed light into initial subtleties I picked out of Zenobia’s lectures. In the second narrative, my attention was to the mathematics (e.g. “derivatives and continuity”, “the definition of derivatives”, “the extreme value theorem and the intermediate value theorem” [Narrative2]), as well as to what the tutor did with regard to her demonstration of the mathematics: she attempted to connect derivatives and continuity with students’ experience from school; then presented the definition of derivatives, examples and non-examples of derivatives; and finally presented theorems and examples of those theorems. Her choice to use examples and non-examples of derivatives, as well as examples for theorems, all based on graphical representations, might be distinctive for her lecture teaching. In this narrative, however, I also commented on students’ contributions to the lecture. Those contributions were examples and kinds of examples students offered, which were potential indications of students’ mathematical meaning.

It seemed to me that the tutorial setting was a more suitable context than lectures for the second research question. In contrast to large student cohorts, in SGTs there were 2-8 students, who I could identify. Furthermore, I usually observed dialogue between the tutor and the students about the mathematics; thus obtaining data to express the students’ response to the tutor’s actions. (Narratives of Zenobia’s tutorials are presented in Appendix E due to space limits. In these narratives, I provide the students’ contributions to the tutorial dialogue and to the solution of mathematical tasks. Narratives of tutorials are also presented in Chapters 5 and 6.)
3.4.2 Main study 1

During Main study 1, I observed calculus lectures three times a week from Week 3 until Week 11 (28 lectures in total) and all Zenobia’s SGTs from Week 3 until Week 13 (10 SGTs in total). (Calendar details about data collection for Main study 1 are provided in Appendix B.) In this study, I continued observing lectures in calculus to have access to the lecture materials used in SGTs, and the calculus content taught in lectures. I also discussed with Zenobia after her SGTs about her underlying considerations with regard to the observed teaching.

In Kuzel’s (1992) and Patton’s (1990, cited in Miles & Huberman, 1994) terms, an intensity case is an information-rich case. I considered Zenobia’s teaching as an intensity case. Both in our discussions and during SGTs, Zenobia gave a lot of information on what she did and why. It was the level of the quality and quantity of the commenting of what she thought and did that offered me an extra layer of information which I considered to be rich. (Findings from the case of Zenobia’s teaching in Pilot and Main study 1 are discussed in Chapter 4.)

I produced full transcripts of observations of SGTs for Main study 1. During data collection, I started to use an open coding process (Strauss & Corbin, 1990) for the coding of the transcripts. Strauss and Corbin (1990) explained the process of open coding as:

> During open coding the data are broken down into discrete parts, closely examined, compared for similarities and differences and questions are asked about the phenomena as reflected in the data. (Strauss & Corbin, 1990, p.62)

The level of detail for my open coding was line-by-line. This enabled me to examine my data closely. During the open coding process, I broke down the transcripts of data into sentences or passages (called incidents), to which I assigned a code (called a category when more than two incidents were assigned to the same code). Passages occupied from less than a line to a few lines in transcripts. My open coding involved:
“coding each incident in [my] data into as many categories of analysis as possible, as categories emerge or data emerge that fit an existing category” (Glaser & Strauss, 1967, p.105).

This resulted into generating 81 codes in a qualitative data analysis software (NVivo) for the transcripts of the first two tutorials (a first and a middle SGT, i.e. SGT2, SGT5), which I coded for Zenobia’s teaching. Within the 81 codes, I discerned codes of different nature, which corresponded to the three following types.

1. **Descriptive codes**: an overt description of tutor’s actions (e.g. “selecting tasks”); and students’ actions (e.g. “correcting what is written on the board”).

2. **Interpretative codes**: my interpretation of tutor’s actions (e.g. “encouraging students’”); and students’ actions (e.g. “engaging with the mathematics”).

3. **Theoretical codes** (Glaser, 1978): my use of other researchers’ theory in teaching practice at school level education (i.e. my use of concepts in research literature) to describe or interpret tutors’ actions [e.g. “revoicing” (O’Connor & Michaels, 1993, 1996)]; and students’ actions [e.g. “meaning making” (e.g. Ormell, 1974; Haylock, 1982)].

The following is an incident of Zenobia’s teaching in SGT1. Before that incident, students suggested that in SGT1, work would be on linear algebra; in particular, the first two tasks of an upcoming coursework in linear algebra. Considering that it was an upcoming coursework, Zenobia could not work with the students on its tasks. She could nevertheless select other tasks on the same mathematical topic. After Zenobia’s question whether “the double prime” or “the relation to functions” or “the polynomials” or “the word proof” “scared” the students in those tasks, two students responded “the word proof”. The task with “the word proof” was on vector spaces, and included a question about ‘showing whether sets of polynomials are vector spaces, by showing that they are subspaces’.
Incident 1 (i.e. the first incident included in this thesis) is a passage that occupies 8 lines in the transcript of SGT1. The code of Incident 1 is selecting tasks, and captures the meaning of Zenobia’s and the students’ action to select tasks for SGT1. Within this incident, I also coded other actions by Zenobia and the students than selecting tasks. The codes for the students’ actions (i.e. what they did, body language) were various nods and blank looks [Lines 4-5/Incident 1: “I’m getting a lot of nods from over here, and I’m getting a lot of blank looks.”]. Moreover, the codes for Zenobia’s actions (i.e. what she did and said) were:

- **interpreting students’ response by looking at their faces**
  
  {Lines 3-4/Incident 1: “[Zenobia looked at students’ faces.] Yes? Maybe? I don’t know.”;} Line 6/Incident 1: “[Zenobia looked at students’ faces.] Yeah?”};

- **informing the students that from their faces she tries to interpret their response**
  
  [Lines 4-5/Incident 1: “I’m getting a lot of nods from over here, and I’m getting a lot of blank looks.”];

- **declaring the need for “positive reinforcement”**
  
  [Lines 5-6/Incident 1: “I need positive reinforcement.”]; and

- **devising tasks**
  
  [Lines 7-8/Incident 1: “I am going to define two different sets of polynomials.”].

Since it is essential for Glaser (1992, p.38) that the researcher “starts with no preconceived codes”, my open coding process consisted of codes which emerged from the data. In other words, the codes were neither preconceived from research literature in advance of coding nor critical to the tutors’ teaching due to my values or
beliefs. Rather, they were emerging codes from incidents constantly compared with more incidents or/and concepts in research literature. In other words, while I was coding an incident for a category, I compared it with the previous incidents coded in the same category (Glaser & Strauss, 1967) or/and concepts in research literature.

For instance, I compared the following incident (Incident 2) of Zenobia’s teaching in SGT2 with Incident 1. Before Incident 2 (i.e. the second incident included in this thesis), students suggested work on various tasks from an upcoming coursework in sequences and series. Zenobia summarised the mathematical topic: “max, min, sup and inf” in proofs for sets. She then looked at a problem sheet, in which task 4 included various sequences. The statement of task 4 was: “For each of the following sequences, determine whether the sequence is bounded or unbounded.”

<table>
<thead>
<tr>
<th>Incident 2_SGT2_Selecting tasks</th>
<th>Lines</th>
</tr>
</thead>
</table>
| Zenobia: [...] Being bounded or unbounded as a sequence is the same as being bounded or unbounded as a set. Alright? [...] So, one thing that we could do is to take a look at some of these ones from 4 and think about bounded, unbounded, min and max for some of those [from 4]. [Zenobia looked at students’ faces.] Right? Does that sound reasonable? [Students nodded positively.] OK. [...] | 1  
|                                 | 2  
|                                 | 3  
|                                 | 4  
|                                 | 5  
|                                 | 6  
|                                 | 7 |

Incident 2 occupies 7 lines in the transcript of SGT2. Its code is *selecting tasks*. I coded that incident with the same code with Incident 1, since it captures the same meaning with Incident 1: Zenobia’s and the students’ action to select tasks for an SGT. I coded one more action by Zenobia with a code from Incident 1. That action (i.e. what she did and said) was twofold: Zenobia’s invitation to students for work on task 4, and a look at their faces for their response {Lines 5-6/Incident 2: “[Zenobia looked at students’ faces.] Right? Does that sound reasonable?”}. The corresponding code to Zenobia’s twofold action was *interpreting students’ response by looking at their faces*. The context of Incident 2 contributed to that code; in particular, the students’ re-actions to Zenobia’s invitation. I coded the students’ re-actions (i.e. what they did, body language) as positive *nods*. I also
assigned the code mathematical expertise to Zenobia’s flexibility to suggest work on sequences (rather than sets) for SGT2. Zenobia’s mathematical expertise was evident by her action (i.e. what she said): “Being bounded or unbounded as a sequence is the same as being bounded or unbounded as a set” [Lines 1-2/Incident 2].

Comparing Incident 1 and Incident 2, I discerned a commonality between them. The commonality was Zenobia’s need for students’ “positive reinforcement” in response to her invitations. Zenobia’s term about students’ “positive reinforcement” to her invitations came from lines 5-6 in Incident 1. It was evident in Incident 2 by students’ positive nods. Zenobia provoked the students’ “positive reinforcement” by making the students aware that she needs their “positive reinforcement” in Incident 1 [Lines 5-6/Incident 1: “I need positive reinforcement.”]; and by asking the students whether they accept her invitation in both incidents [Line 6/Incident 1: “Yeah?”; Lines 5-6/Incident 2: “Right? Does that sound reasonable?”]. I created a memo about students’ “positive reinforcement” in the comparison between Incident 1 and Incident 2.

Memoing my reflections on the constant comparison of incidents enabled me to identify commonalities and differences in incidents. For instance, a commonality for the category selecting tasks was students’ “positive reinforcement”. I started making meaning of a category of incidents by identifying commonalities and differences in its incidents. This resulted into retaining codes which captured the meaning I was making of the categories (e.g. selecting tasks); reformulating codes to make the meaning they captured more explicit (e.g. from students’ nods to students’ “positive reinforcement”); and abandoning codes which added bulk to the coded data since their categories did not form patterns in the data. Each code of the new reduced list of codes was the conceptual name of a category of incidents, and captured the meaning of that category. So, the process of my meaning making of categories resulted into “a reduction in the original list of categories for coding” and categories which became “theoretically saturated” (Glaser & Strauss, 1967, p.111).

The categories became theoretically saturated after coding 8 of 10 SGTs for Zenobia’s teaching. (I selected first, middle and final SGTs: SGT1, SGT2, SGT3, SGT5, SGT6, SGT8, SGT9, SGT10.) When I reached the saturation point in the analytical process, the identification of commonalities and differences in incidents of a category
generated underlaid properties of that category (called **dimensions**). For instance, the commonality in incidents of the category *selecting tasks*, was that:

In her action *selecting tasks*, Zenobia acted with the students’ “positive reinforcement” to make the decision for particular tasks.

The commonality in incidents appeared in coding as well, since students’ “positive reinforcement” was a common code for incidents within the category *selecting tasks*. That commonality generated two dimensions:

1. a dimension for *selecting tasks* (called **acting with a tool**); and
2. a dimension for students’ “positive reinforcement” (called **tool**).

In other words, my theorisation of the commonality in incidents of the category *selecting tasks*, was that:

In her action *selecting tasks*, Zenobia acted with **tools**, and one of those **tools** was students’ “positive reinforcement”.

I looked at students’ “positive reinforcement” through the Vygotskian lens. I thus considered it as a Vygotskian **tool**. *(Tool* is a term discussed in Chapter 2: Section 2.1.1.1.) The material nature of students’ “positive reinforcement” was observed students’ **nods**. Its psychological nature was the meaning that was conveyed to Zenobia and I: students agreed with her about the selected tasks.

Glaser and Strauss (1967) referred to *exampleing* as “[a]n opportunistic use of theory” (p.5), where examples of data are selectively chosen for their confirming power of a theory which is speculative and not grounded. In my study, I avoided exampleing by focusing my analysis on the identification and grounded study of **teaching episodes**. During the open coding process, I divided each transcript into parts of Zenobia’s teaching, and those parts were the teaching episodes in her SGTs. A teaching episode captured Zenobia’s and the students’ work (e.g. on a specific mathematical topic or a specific mathematical task) from start until completion; and lasted for a maximum of a couple of minutes of the audio recorded SGT. So, a teaching episode was an example of data with conceptual names of several categories. Considering that each episode started from the next line where the previous episode ended, the teaching
episodes were not selectively chosen; rather, they occupied the whole transcript of a SGT with the intention to remove the bias of exampling.

In the next three chapters (Chapters 4-6), where I discuss findings from the three extended cases of teaching in the main studies, I present teaching episodes to make an exemplification of the data, and to justify the findings. The exemplification of the data with teaching episodes is not exampling, because I chose for each case of teaching three paradigmatic episodes; that is, episodes which included many different conceptual names for the tutors’ teaching strategies and teaching tools. Furthermore, my consideration is that by reading those episodes, the reader develops an image of each tutor’s SGTs. For presentation purposes, however, the episodes I exemplify are not long pieces of dialogue or monologue in transcripts. Rather, they include lines of dialogue coded with various (conceptual names of) categories, as well as text in which I describe the context of that dialogue or monologue.

3.4.3 Pilot study 2

I distinguish between two parts of analysis in my doctoral study. My emerging characterisation of teaching into tools and strategies arose from analysis of the case of Zenobia’s teaching in Pilot and Main study 1 (first part of the study). I then justified its applicability from analysis of two more cases of teaching in Pilot and Main study 2 (second part of the study). The idea for discerning the study into two parts of analysis came from the development of the Teaching Triad (Jaworski, 1994). Teaching Triad was produced from analysis of a case of teaching, and then tested by application to analysis of two further cases of teaching. (The Teaching Triad is discussed in Chapter 1: Section 1.2.2.)

The purposes of Pilot study 2 were 1) to select two participants for Main study 2, and 2) to widen my investigation of university mathematics teaching. Although participant selection was a purpose of Pilot study 1 as well, I considered that the sample of the remaining 6 of 7 tutors’ teaching was limited for the selection of two participants for my second main study. I nevertheless had good reasons to think of Lecturer1 and
Lecturer7 (participants in Pilot study 1) as good candidates for a whole semester observation, thus conducting a single observation of their teaching in Pilot study 2.

I collected data for Pilot study 2 and data for Main study 1 during the same semester. In Pilot study 2, I mainly conducted single observations of 21 tutors’ teaching in SGTs, and follow up interviews with them about underlying considerations. I observed 23 SGTs in total (from Week 7 until Week 11); in particular, 19 tutors’ teaching in SGTs I had never observed before, and Lecturer1’s as well as Lecturer7’s teaching I had also observed in Pilot study 1. (Calendar details about data collection for Pilot study 2 are provided in Appendix B.)

My aim was the sampling of Pilot study 2 to be comprehensive (Goetz & Lecompte, 1984, cited in Miles & Huberman, 1994), that is to say every tutor’s teaching to be examined in the given population (34 tutors’ teaching). So, I contacted 34 tutors and most of them were willing to participate in the study. However, four tutors responded negatively; and four tutors responded positively but I could not observe their teaching due to practical reasons. In the two pilot studies together, I observed 26 tutors’ teaching in SGTs, and conducted follow up interviews with them; 20 tutors held doctorates in mathematics, and 6 tutors held doctorates in mathematics education. Data from 26 tutors’ teaching contributed to a wide insight into university mathematics teaching in SGTs.

Considering that almost all respondents to my invitations were willing to participate in my studies, it could be of value to report the ways I contacted them: through my supervisors, who were in their networks; in a friendly discussion during lunchtime at one of the dining halls of the University; and via email, where I informed them the content and purpose of my study in brief as well as my intention - to observe their SGTs and to discuss with them about teaching afterwards. Some participants told me that they discussed with each other their experience of participating in my study, and/or that in my discussion with them, they thought about interesting things they had never thought before. I thanked all my participants for their kindness and time.
Data analysis in Pilot study 2 involved narratives. This choice was in line with data analysis in Pilot study 1. So, within 2 hours after each observation of a SGT in Pilot study 2, I narrated my reflection on the tutors’ teaching: what the tutor did with regard to the mathematics content and the students; and how the students responded to that. The content of my narratives included an overview of as much data as I could recall in such a short period of time after my experience with the tutors’ teaching. My field notes supported me in this attempt.

The narratives enabled me to select two participants for my second main study: Lecturer7 and Lecturer23. I asked the two tutors to select their pseudonyms; Lecturer7 selected the name Aristophanes, which abbreviation Phanes I use for practical reasons in publications; and Lecturer23 selected the name Alex. Phanes and Alex were non-standard English speakers. My analysis of the case of Zenobia’s teaching in the pilot and main studies indicated that Zenobia was a tutor who had not only a strong mathematical background as a research mathematician, but also a way of engaging students in actively discovering the mathematics and contributing to the tutorial. (The analysis of the case of Zenobia’s teaching is in Chapter 4.) My consideration was that my study of teaching practice and knowledge needed two other contrasting cases of teaching to Zenobia’s case, in the sense that in those two cases the tutor talk should be dominant. Phanes was a research mathematician. My criterion for selecting the case of his teaching was the rich mathematical explanations, which I observed, discussed with Phanes and considered that made the mathematics look simple to his high-achieving student and to me in the pilot study. My consideration about the student came from two facts: she achieved 100% in a coursework in linear algebra; and in tutorials she asked for Phanes’ support by articulating specific questions in mathematics. (The analysis of the case of Phanes’ teaching is in Chapter 5.) Alex’s research was in mathematics education. My criterion for the case of his teaching was the influence of mathematics education research in his teaching for students’ mathematical meaning making, which was evident in the classroom and I discussed with him following up the observation. (The analysis of the case of Alex’s teaching is in Chapter 6.)
When I completed my observations, I discerned commonalities and differences in emerging issues of the 21 tutors’ teaching in Pilot study 2 (e.g. the issue tutors’ ‘views on connections between teaching and own research’). I considered that an in-depth analysis of emerging issues would be promising; I nevertheless left that analysis for future work.

### 3.4.4 Main study 2

During Main study 2, I observed, systematically, Phanes’ SGTs and Alex’s SGTs from Week 2 until Week 12 (22 SGTs in total). (Calendar details about data collection for Main study 2 are provided in Appendix B.) The purpose of Main study 2 was to justify the applicability of the characterisation of teaching into tools and strategies. So, from analysis of Phanes’ and Alex’s teaching, I would:

1. analyse two more cases of teaching; and
2. get insights into whether the characterisation of teaching in Main study 1 made sense beyond Zenobia’s specific case, that is to say to “add confidence to findings” of Main study 1 (Miles & Huberman, 1994, p.29).

My consideration of Phanes’ and Alex’s teaching is that they were critical cases (Kuzel, 1992; Patton, 1990, cited in Miles & Huberman, 1994) in my study. A critical case in Kuzel’s and Patton’s terms is a case which “exemplifies the main findings” (my first purpose) and “serves to increase confidence in conclusions” (my second purpose) (Miles & Huberman, 1994, p.28).

Full transcriptions from audio data of observations were produced for the second main study, as well. During the data collection, I started to code the transcripts using the list of categories for the first main study (pre-determined set of categories) and new categories. I paid attention to constantly compare the incidents and the associated categories of the first main study with the uncoded data of the second main study. I coded 8 of 11 transcripts selecting first, middle and final SGTs from Alex (SGT1, SGT2, SGT3, SGT4, SGT6, SGT7, SGT8, SGT11); and 8 of 11 transcripts selecting
first, middle and final SGTs from Phanes (SGT1, SGT2, SGT3, SGT4, SGT6, SGT7, SGT8, SGT10).

The constant comparison method enabled me to conduct a cross-case analysis with which the categories of the characterisation of teaching into tools and strategies became more lucid. (The cross-case analysis is discussed in Chapter 7.) The cross-case analysis also enabled me to revisit the data of the three cases of teaching, and investigate the ways (characterised into teaching tools and strategies) in which the tutors (re)designed the teaching in order to promote students’ mathematical meaning making. I discerned the tutors’ teaching practice into didactical practice and pedagogical practice in the cross-case analysis, because these two kinds were a commonality among the cases of teaching; all tutors employed some ways to disseminate the principles and content of mathematics (didactical practice), and some ways with which they intended to enable the students to ultimately make meaning of the mathematics (pedagogical practice). Thus, the distinction of teaching practice into didactical practice and pedagogical practice emerged from the cross-case analysis, enabling the analysis of teaching practice across cases in such a way that interpretations of didactical knowing and pedagogical knowing gave rise to the emerging analytical framework ‘Teaching Knowledge-in-Practice’ (TKiP). In particular, the “such a way” or the link between the analysis of practice and the analysis of knowing was the dialectic method. (The dialectic method is discussed in Chapter 2: Section 2.3. Also, a preview of the TKiP is presented in Section 3.4.4.1, below.) The carefully selected sampling of the three tutors’ teaching illuminated different ways of didactical and pedagogical practice, sharpening the theorisation of didactical knowing and pedagogical knowing, respectively.

3.4.4.1 Preview of the emerging analytical framework

In this section, I provide a preview of the analytical framework that emerged from the data through the sociocultural approach that I presented in Chapter 2. In Chapters 4, 5, and 6, I explain the ways with which the framework emerged from the data of three cases of teaching, and the ways with which relevant theoretical concepts to the framework (e.g. tools, strategies, didactical practice, pedagogical knowing) are substantiated in the data.
In this study, the analysis of teaching knowledge is revealed through analysis of teaching practice with regard to students’ mathematical meaning making. I explain the analytical process from the analysis of teaching practice to the analysis of teaching knowledge, below. The outcome of the analytical process is an analytical framework of teaching knowledge which is revealed in practice, the ‘Teaching Knowledge-in-Practice’ (TKiP). TKiP is represented in Figures 3.1, 3.2 and 3.3.

Figure 3.1: Teaching Knowledge-in-practice: analysis of teaching practice for students’ meaning making.

Considering that, within the Vygotskian approach, each successive developmental stage of knowing has an abstraction and an element of reality (see Section 2.3), the model of teaching practice in Figure 3.1 represents the observed elements of teaching practice (i.e. reality) by the researcher. In this figure, the blue stages of the design of teaching and the successive redesigns are the observed developmental stages of the tutor’s teaching practice. Each developmental stage of teaching practice is analysed into the tutor’s tools for teaching (e.g. tutor’s spoken language, graphical representations), and the tutor’s tool-mediated actions for teaching, namely strategies for teaching (i.e. modes of actions with tools). The red arrows represent the dialectic connections between the developmental stages, thus dialogue about mathematical meanings between the tutor and the students. In this dialogue, the students do not make meaning of the mathematics and this contradicts the tutor’s (re)design. So, the tutor successively redesigns until a stage of final redesign, which enables the students to make meaning of the mathematics from the tutor’s perspective.
In the developmental stages of design and redesign(s) towards students’ mathematical meaning making, the tutor could solely act with tools and strategies drawn on the space of mathematics (e.g. graphical representations) in order to mediate mathematical meaning to students and trigger the process of students’ mathematical meaning making. However, in practice, a transmission of mathematical tools (and associated modes of action) to students does not indicate that students are in a position to make mathematical meaning. Students learn through dialogue and interaction with more knowledgeable others (e.g. teacher, fellow students), and the teacher needs to adapt the teaching to the students’ needs in order to enable them to make mathematical meaning. So, the teacher needs to act with tools drawn on an additional space to the space of mathematics, which is the space of teaching/learning. The latter space is situated in the context of the students, including tools such as encouraging statements to students and questions to evaluate students’ sense making.

The two spaces in Figure 3.1, i.e. the space of mathematics and the space of teaching/learning, illustrate the distinction between teaching tools into mathematical tools and teaching/learning tools. The spaces of mathematics and teaching/learning are not straight but interrelated in Figure 3.1. The interrelation of helixes is evident by the researcher, because in mathematics teaching practice some strategies and tools, which are used within and across the developmental stages of design and redesign, are from the space of mathematics, while others are from the space of teaching/learning. (A more practical perspective on the interrelation of the two spaces is presented in Chapters 4, 5 and 6, where the teaching of three tutors is analysed.)

In Chapter 1, I explained that the didactical practice is about ways of disseminating the principles and content of mathematics; thus didactical practice is mathematics-specific. In the model of teaching practice (Figure 3.1), I consider that the didactical practice is about ways of translating the space of mathematics into the context of students. In other words, it is connected with enhancing the tools drawn on the space of mathematics with tools drawn on the space of teaching/learning in each developmental level of teaching.

In Chapter 1, I also explained that the pedagogical practice is about ways of teaching to bring about learning; thus pedagogical practice is student-specific. In the model of teaching practice (Figure 3.1), I consider that the pedagogical practice is about
moving across developmental stages of teaching until a stage which enables the students to make meaning of the mathematics from the tutor’s perspective. ‘Moving across’ does not necessarily mean reducing the mathematical rigour or getting the tutor to do the students’ tasks for them. Rather, it is connected with flexibility in drawing on the students’ responses/silence and repeatedly redesigning the teaching with different tools and strategies until those that meet the students’ sense/meaning making from the tutor’s perspective.

So far, my explanation was for the analysis of the observed teaching practice with regard to students’ mathematical meaning making (Figure 3.1). This section continues with an account of how the analysis of teaching knowledge, distilled into two kinds of tutor’s knowing, is revealed through the analysis of teaching practice.

As mentioned in Section 2.3 Vygotsky, following Marx, considered that knowledge of reality is the reflection of reality (Duarte, 2011). My consideration in this study is that teaching knowledge is the reflection of teaching practice. So, in the model of individual knowing that emerged from the data of this study (Figures 3.2, 3.3), I thought of the tutors’ reflections on the spaces of mathematics and of teaching/learning as the tutors’ epistemology of mathematics and epistemology of teaching/learning, respectively. I also considered that the developmental stages of coming to know for teaching are the tutor’s reflections on the design and redesign(s) for students’ mathematical meaning making, distinguishing the pedagogical knowing (Figure 3.2) and the didactical knowing (Figure 3.3).
In the analysis of my discussion with the participants of this study, I discerned tutors’ views on the space of mathematics and tutors’ views on the space of teaching/learning (i.e. the context of students). From the tutors’ views, I got access to aspects of their personal theories of knowing, namely epistemologies, of mathematics and to aspects of their epistemologies of teaching/learning. An epistemology is a theory of knowing, socio-culturally based, intuitive and holistically inter-connected (Burton, 2004). Socio-culturally based as being based on prior sociocultural experiences of the tutor; intuitive as not always based on established knowledge and as reflecting the tutor’s psychological/individual plane (see Section 2.1 for the term ‘psychological plane’); and holistically inter-connected as forming a connected whole of the tutor’s views. My consideration is that the tutors’ views are formed by the tutors’ reflections on the space of mathematics and the space of teaching/learning through their teaching practice, thus forming their epistemologies of mathematics and teaching/learning. My discussions with the tutors revealed some aspects of the tutors’ views; however, these aspects do not present the entirety of each tutor’s epistemologies.

The analysis of the tutors’ epistemologies is important, because it reveals the tutors’ views as the distillate of their practice (or in other words, their experience) up until the time that they are accessed by the researcher. The data of this study for more than one semester for each case of teaching also enabled me to analyse the ways with
which the tutors come to know their teaching practice, which I discerned into the ways with which they come to know their pedagogical practice (pedagogical knowing) and their didactical practice (didactical knowing).

**Pedagogical knowing** is concerned with knowing ways of moving across developmental stages of teaching until a stage which enables the students to make meaning of the mathematics. The ways are analysed into tools and strategies for teaching. Knowing these ways comes from the tutor’s reflection on the development of past teaching practice to a successful stage for students’ mathematical meaning making (from the tutor’s perspective). In Figure 3.2, the pedagogical knowing is represented with a red arrow from the stage of the initial design to the stage of the final redesign in order to show the developmental stages of coming to know the pedagogical practice.

**Didactical knowing** is concerned with knowing ways of translating the principles and content of mathematics into forms of tutor’s thought in the context of students. The ways are analysed into tools and strategies for teaching. Knowing these ways comes from the tutor’s reflection on the enhancement of tools drawn on the space of mathematics with tools drawn on the space of teaching/learning in each developmental stage of teaching. In Figure 3.3, the didactical knowing is represented with a red arrow from the tutor’s epistemology of mathematics to the tutor’s epistemology of teaching/learning in order to show that the tutor comes to know the didactical practice of disseminating the principles and content of mathematics to the context of the students. Considering that the preceding stage of design exists within the following stage of redesign, the tutor reflects on all developmental stages of design and redesign in order to develop her/his didactical knowing, which s/he puts into future practice for the initial design of the teaching. This is the reason for the upper view in Figure 3.3.

To conclude, the outcome of the analytical process, that is to say the process from the analysis of teaching practice to the analysis of teaching knowledge, is an analytical framework of teaching knowledge which is revealed in practice, the ‘Teaching Knowledge-in-Practice’. I refer to teaching knowledge rather than knowing for teaching, because the framework emerged from data analysis of three carefully selected cases of teaching. In this way, the ‘Teaching Knowledge-in-Practice’ forms a
level of researcher abstract thinking, which reveals a system of connections between concepts, such as pedagogical practice and pedagogical knowing.

3.5 Ethical issues in data collection and data analysis

Tutors’ commitment to my research was to accept me observing their SGTs or/and lectures with the use of an audio recorder, and to have a follow up discussion with me about their teaching. The students’ commitment to my research was to accept me observing and audio-recording the SGTs where they were in.

The first step of my ethical considerations was to adhere to the Ethical Code of the University, and to submit an ethical clearance form regarding observations and audio recordings. I then used consent forms with information about the purpose of the study, the observations and the audio-recordings, the anonymity of the participants, and the use of all data for research purposes. In each first SGT I observed, I introduced myself briefly, and provided the tutor and the students with the consent form in order to read and sign it. In lecture observations, the tutor had signed the consent form in advance, and informed the students about my presence in the lecture theatre and my research. The cohort of students was large in lectures (100+ students), so the advice by the Ethics Committee was that it would be impractical if all students signed forms.

Another ethical issue was the identity of the participants within the university, where I conducted the study; in particular, the amount of information I could publish in my accounts about their professional background or their teaching. The Mathematics Department was of an average size (50 lecturers at that time), and their colleagues might recognise them. In data analysis, I was careful enough to write only information that was not exclusive for them and was needed for my research. Considering the tutors’ and the students’ identity more broadly, both in data analysis and publications, I used data entries with numbers for the tutors of the pilot studies (e.g. Lecturer3) and the students (e.g. St1 for Student1); as well as pseudonyms for the tutors of the main studies (e.g. Zenobia). Moreover, I did not name the university where I conducted the
study.

The presentation of findings in data analysis and publications was an additional ethical consideration. Since analysis was qualitative and paradigmatic teaching episodes were selected for exemplification of findings, the audience in presentations and the readers in publications were provided with a small slice cut from each tutors’ teaching of a whole semester. Although my ethics and position as the researcher was not to be critical to or to evaluate a tutor’s teaching, the audience and readers, at the ease of sitting back and listening or reading, might criticise a tutor’s teaching due to their values and beliefs. I thus was careful to provide as much of the context as I possibly could, and state what research findings the data exemplify at start of presentations.

3.6 Methodological implications of the interpretative paradigm and the grounded analytical approach: My role in the data collection process

While coding transcripts, I listened again to the recordings to note down the voice tone of the tutor’s and students’ utterances. This resulted into making changes to the transcripts, as well; usually altering utterances that I found to have been differently transcribed from what I heard while listening again. Lerman (2001) stressed that the process of formatting transcripts is “never-ending” (p.54), and comprises a layer of interpretation in data; for instance, other researchers may alter more utterances while listening again and so on. I considered my changes in transcripts, and thus my transcripts in general, to be a layer of my interpretation in data. Additional layers of interpretation in my research process were: the place of the recorder which sometimes clearer captured utterances from students close to it; my comments on the tutors’ actions in the right margin of my field notes for later discussion with the tutor; thus my field notes in general; and every account I produced after an episode in observations and associated discussions.
Holland, Lachicotte, Skinner and Cain (1998, cited in Jaworski, 2015) recognised *figured worlds* of interpretation, with each one to be:

> a socially and culturally constructed realm of interpretation in which particular characters and actors are recognised, *significance is assigned* to certain acts, and particular outcomes are *valued* over others. (p.52, italics added)

My aforementioned considerations on layers of interpretation in my study belong in my world (i.e. the researcher’s world). Within that world, I observed SGTs while *assigning significance* to the tutor’s and the students’ actions. My comments on the right margin of field notes reveal that I *valued* some tutors’ actions over others. Yet the place of the recorder indicates that I *valued* those students’ actions which I captured in recordings and/or field notes.

There were also two additional worlds to mine in SGTs: that of the students, and that of the tutor. The students *assigned significance* to the mathematics, and *valued* the tutor’s teaching as well as the solution of tasks in SGTs. The tutor, in her/his effort to teach the mathematics, *assigned significance* to the mathematics, and *valued* some students’ contributions over others.

After her citation regarding *figured worlds*, Jaworski (2015a) stressed that “the rigour of the research lies in justifying interpretations and rooting conjectures” (p.178). My discussions with the tutors after SGTs, added to my analysis the tutors’ layer of interpretation with regard to students’ and their own actions in SGTs. That layer of interpretation, or else the tutor’s world, added rigour to my grounded analytical approach. For instance, I rooted my interpretations of each tutor’s *tools* and *strategies* in observational data. Then, I justified my interpretations with the use of data from my discussions with the tutors.

Lerman (2014) asserted that in research:

> What we produce, in the end, is some form of text that reflects the researcher’s interpretation of how the data function as evidence of knowing and learning mathematics according to the theoretical position adopted. (p.16)
In this chapter, I referred to my choices about the methods I used in analysis; and the data I collected “in order to be able to call it evidence of knowing and learning” (Lerman, 2015, p.16). Furthermore, in the previous chapter (Chapter 2), I explained how I viewed the theoretical position I adopted, which is the Vygotskian approach. My consideration nevertheless is that my theoretical position is not limited to my methodological decisions and the sociocultural perspective. Rather, it also includes what teaching, meaning making and learning meant to me before participant selection, and before my grounded analytical approach to data.

I commenced my doctoral studies when I was an early career tutor with experience ranging from 1-1 teaching to teaching to 90 students. I had experiences of teaching, meaning making, and learning as a pupil and later as a student, as well as a theoretical understanding of those terms from my undergraduate and postgraduate studies. My experiences as a pupil/student indicated to me that I enjoyed teaching in which there was certain mathematical challenge. As a tutor, I felt rewarded when students were optimistic and smiled with their achievements. However, my theoretical underpinnings were those that formed my views for teaching, meaning making, and learning. In particular, having studied various theories of teaching, knowing and learning as a student, I thought of the Vygotskian theory as the most convincing. As a tutor and later as an observer, I valued opportunities provided to students to develop mathematical meaning socially in the classroom or lecture theatre. I also valued the teaching of informed tutors by mathematics education literature, and their effort to enable the students’ meaning by using that information in their everyday practice. I also admired the breadth and depth of the mathematical background of tutors who developed mathematics in their research. I started my observations without being judgmental to tutors’ teaching and without preconceptions of a better teaching than the observed one. Rather, my intention was to seek ways of teaching mathematics and the associated mathematical meaning making within my views of the Vygotskian perspective. I considered the different ways of teaching I observed and analysed as being informing for me as a tutor, as well.

In observations, I dressed neutrally in order not to be intrusive, and to avoid bias from the influence of the researcher. In discussions with the tutors after my observations, I avoided questions with mathematics education terminology, as well as leading
questions that could have implied expected answers. I was also supportive of the tutors, and developed a nice relationship with them.

Guthrie (2010) stressed that the data collection is more complete when the observer is a *non-active* participant; this is because the observer’s attention can then focus only on the observation. So, the observer is less distracted by her own role, as it is restricted to a *non-active* participation in teaching/learning e.g. only taking field notes. My views on my role as an observer were congruent with Guthrie’s *non-active* participant; so, during SGTs, I was taking field notes while sitting in my chair. That was beneficial for my research, since it enabled me to capture as much of the tutorial context as I possibly could in my notes.

Additionally, in Schwandt’s interpretative terminology, there is the *uninvolved observer* (2000, p.194) who intends to be external, i.e. without inducing any change or disturbing during observations and at the same time, without being “literally at a distance or from behind some kind of one-way mirror” (Schwandt’s, 2000, p.207). My experience of observations in Pilot and Main study 1 unveiled that the uninvolved observer was not realistic. Rather, my role was most aligned to a reference from Maxwell:

> As observers and interpreters of the world, we are inextricably part of it; we cannot step outside our own experience to obtain some observer-independent account of what we experience. (1992, p.283)

I could not be an uninvolved observer. I was inevitably part of SGTs. I was cautious not to disturb, and the tutors in the main study ensured me that their teaching did not seem to have any change due to my presence; so, the tutor or the students rarely talked to me during SGTs.
3.7 Trustworthiness of the study

Trustworthiness of a qualitative study regards evidence with which the reader trusts the findings and conclusions of the study. Miles and Huberman (1994, p.277) reviewed “26 tactics for drawing and verifying conclusions”, and produced standards for the quality of findings and conclusions of qualitative research by “pairing traditional terms with those proposed as more viable alternatives for assessing the trustworthiness and authenticity of naturalistic research [Guba & Lincoln, 1981; Lincoln, 1990; Lincoln & Guba, 1985]”. They produced five main overlapping criteria:

1. objectivity/confirmability of qualitative work [i.e. “relative neutrality and reasonable freedom from unacknowledged researcher biases—at the minimum, explicitness about the inevitable biases that exist” (p.278)];
2. reliability/dependability/auditability [i.e. “whether the process of the study is consistent, reasonably stable over time and across researchers and methods” (p.278)];
3. internal validity/credibility/authenticity [i.e. “truth value” of findings (p.278)];
4. external validity/transferability/fittingness [i.e. “whether the conclusions of a study have any larger import” (p.279)]; and
5. utilisation/application/action orientation [i.e. “what the study does for its participants, both researchers and researched” (p.280)].

The following sections form an account of the ways that my study met each of those issues.
3.7.1 Objectivity/confirmability

I established the objectivity/confirmability of my study by describing and explaining in detail my methods for data collection and analysis. In those descriptions and explanations, I presented the actual sequence of how data were collected and analysed towards the emerging analytical framework, ‘Teaching Knowledge-in-Practice’. I was explicit about my interpretations and theorisations, and I discussed in detail about layers of interpretation in my research process. Moreover, I reported on issues I considered to avoid bias, and ethical issues.

3.7.2 Reliability/dependability/auditability

In Chapter 2, I explained my views on the Vygotskian theory, and its connectedness with my research study. In Section 3.6 of this chapter, I described my role in the research process taking into account my methods for data collection, the interpretative paradigm of my research, and my grounded analytical approach. Furthermore, during data collection and data analysis, I worked closely with my supervisors, published articles, and delivered presentations at various national and international conferences. To conclude, the connectedness of theory with my research study, the description of my role in the research process, and the forms of peer and colleague review at conferences, supervisory meetings and publications contributed to the reliability/dependability/auditability of the study.

3.7.3 Internal validity/credibility/authenticity

I collected data of various tutors’ teachings in the tutorial setting over three academic semesters. I established the internal validity/credibility/authenticity of the study by keeping detailed records of my data collection, and by using triangulation. In particular, I applied to the study triangulation of sources (i.e. observational data and interview data with the tutors); and triangulation of methods (i.e. grounded analytical approach to the data in which research literature was embedded).
3.7.4 External validity/transferability/fittingness

Although replication of a research study for achieving the same outcomes is not applicable within the interpretative paradigm, Lincoln and Guba (1985) stressed that a detailed description of the study increases the possibility of transferability of the research findings and outcomes, since other researchers could judge whether a transfer is possible. To this end, I provided a thick description of the tutorial setting, and presented my criteria for participant selection in detail. Those criteria resulted into three cases of teaching with a variation of participants’ research expertise and communication with students, which was crucial for the findings about tutor’s knowing for teaching mathematics. Additionally, a cross-case analysis illuminated the findings about the categories for teaching strategies and tools, and gave rise to the emerging analytical framework.

External validity/transferability/fittingness also relates to the contribution of this research study. In this thesis, I made explicit that the contribution to mathematics education research literature is the analytical framework ‘Teaching Knowledge-in-Practice’. ‘Teaching Knowledge-in-Practice’ analyses university teaching practice and knowledge with regard to students’ mathematical meaning making, through a sociocultural perspective.

3.7.5 Utilisation/application/action orientation

Utilization/application/action orientation of this study is with regard to its participants and potential beneficiaries. I mentioned in Section 3.4.3 that some participants of the second pilot study discussed with enthusiasm with each other about their experience of participating in my study. In addition to this, some participants told me that they would like to read my analyses, and get informed by other tutors’ teaching. In a wider consideration, this thesis is accessible online to potential readers, such as tutors who teach mathematics at university level in various settings. It could also be of interest to researchers who analyse teaching knowledge and teaching practice with regard to students’ mathematical meaning making at university level.
UNIT II:

Data analysis and conclusions
INTRODUCTION TO CHAPTERS 4, 5 AND 6

The next three chapters of the thesis are Chapters 4, 5 and 6. In each of these three chapters, I discuss the findings from one of the three extended cases of teaching. In particular, the structure of each chapter includes:

- a description of the setting of a tutor’s tutorial;
- an analysis of her/his epistemology of teaching/learning and epistemology of mathematics (Part 1);
- an analysis of her/his teaching practice in the main study (Part 2); and
- an analysis of her/his knowing for teaching in the main study (Part 3).

Chapter 4 is the chapter devoted to findings from the case of Zenobia’s teaching.

Chapter 5 is the chapter devoted to findings from the case of Phanes’ teaching.

Chapter 6 is the chapter devoted to findings from the case of Alex’s teaching.

In this introduction, I explain specificities within the structure of each of the three chapters, drawing on the following timeframe of the data collection.

**Timeframe of data collection:**

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**Pilot study 1:**
5 observations of Zenobia’s teaching, & 1 observation of Phanes’ teaching.

**Pilot study 2:**
1 observation of Phanes’ teaching, & 1 observation of Alex’s teaching.
Explanation of specificities within the structure of each chapter:

The initial section of each chapter is the ‘Setting’. It includes a detailed description of who the tutor is, when my observations took place and the layout of the classrooms. This first section is in order for the reader to develop an image of each case of teaching, such as the one that I had when I started the observations.

Following the ‘Setting’ is Part 1. In Part 1, I present my interpretation of aspects of the tutor’s epistemology of teaching/learning and her/his epistemology of mathematics, based on my discussions with the tutors about their views on teaching/learning and on mathematics, respectively. My discussions with the tutors are grounded on the tutors’ teaching practice, which I observed. As my meaning of teaching practice and knowing for teaching was developing throughout the data collection and analysis, I started to analyse data in order to get insights into the tutor’s epistemologies from Semester 2/Year 1 of the study. This resulted into drawing on data of the pilot and/or the main study for the interpretations presented in Part 1.

• In Chapter 4, my interpretation of Zenobia’s epistemologies is from the analysis of data, which is collected in Main Study 1.
• In Chapter 5, my interpretation of Phanes’ epistemologies is from the analysis of data in Pilot Study 1 and 2. Phanes discussed with me his views in our discussion in Pilot study 2; then, he did not have to repeat them to me in the main study.
• In Chapter 6, my interpretation of Alex’s epistemologies is from the analysis of data in Pilot study 2 and Main study 2. Alex shared with me two of his views on teaching/learning in our discussion in Pilot study 2, and more of his views on teaching/learning and on mathematics during Main study 2.

My interpretation of aspects of the tutors’ epistemologies is based on a collection of views in each case of teaching, which are similar or different across cases. For example, the three tutors view connections between teaching and own research, so this is a similar view across cases. Notably, in the case of Zenobia’s and Alex’s teaching each tutor’s view enabled me to access aspects of her/his epistemology of teaching/learning, because in our discussions the tutors focused on their teaching while exposing the connection. In the case of Phanes’ teaching, the tutor focused on
mathematics while exposing the connection, so his exposition enabled me to access aspects of his epistemology of mathematics.

The next part of each chapter is **Part 2**. Part 2 is about the tutors’ teaching practice in the main study. In this part, I include: a table of the conceptual names of categories for the tutor’s teaching practice (i.e. a table with six strategies and associated tools); the conceptual names of categories for the students’ response; and three carefully selected teaching episodes with which I exemplify the categories. Following each episode is a *model of the tutor’s teaching practice* with regard to the students’ mathematical meaning making, and a brief explanatory account of the *model*. (The *model of the tutor’s teaching practice* is introduced in Chapter 3: Section 3.4.4.1, and illustrated in Figure 3.1.) The next six sections illuminate each one of the tutor’s six strategies and associated tools.

The final part of each chapter is **Part 3**. In this part, I synthesise from the analysis of the tutor’s epistemologies and the tutor’s practice, in order to discuss the tutor’s knowing for teaching. In particular, I discuss each tutor’s mathematical knowing, didactical knowing and pedagogical knowing, exemplifying these three kinds through the analysis in Parts 1 and 2. I also illustrate the tutor’s didactical knowing and pedagogical knowing with the *model of the tutor’s knowing for teaching*. (The *model of the tutor’s knowing for teaching* is introduced in Chapter 3: Section 3.4.4.1, and illustrated in Figures 3.2 and 3.3.)

To conclude, the ‘Teaching Knowledge-in-Practice’ contributes to the analysis of the tutor’s teaching practice in Part 2 (of each chapter), and to the analysis of the tutor’s knowing for teaching in Part 3. The ‘Teaching Knowledge-in-Practice’ conceptualises two new types of knowing in research literature: *didactical knowing* and *pedagogical knowing*. It also offers insights into the tutor’s mathematical knowing.
Chapter 4

DISCUSSION OF FINDINGS 1 – The case of Zenobia’s teaching

This is the first of three chapters on the discussion of findings for the main studies. It presents analysis of the case of Zenobia’s teaching. (Case study is a term discussed in Chapter 3: Section 3.2). It commences with a description of the setting of Zenobia’s small group tutorials. Following the description is Part 1, which is a synthesis of my discussions with Zenobia for the analysis of her epistemology of teaching/learning in tutorials and her epistemology of mathematics. Part 2 starts by offering a concise table of the coding for Zenobia’s teaching practice distilled into strategies and tools. (Strategies and tools are terms discussed in Chapter 3: Section 3.4.2.) Then, it exemplifies how the coding emerged with three transcripts of teaching episodes. Part 2 continues with analysis and interpretations made with regard to each strategy for the case of Zenobia’s teaching. Concluding the chapter is Part 3, which presents analysis of Zenobia’s knowing for teaching.

The setting

In this section, I provide the description of the background in which Main study 1 was conducted. This description is about Zenobia, the main study observations, and the layout of the classroom. It enables the reader to locate the study within the setting in which it took place.
Zenobia

Zenobia is a researcher in analysis. She is experienced in both research and teaching. At the time of participating in my study, she had a 20-year research and teaching career. Her teaching responsibilities included large cohorts of students in lectures, and a small group of five first year students in tutorials. She was a lecturer in three mathematics modules. One of the three modules was for first year undergraduate students. Hence, she was lecturer and tutor for the five students in tutorials.

Pilot study and main study observations

The main study of Zenobia’s teaching involved a whole-semester observation. That semester lasted for twelve weeks from February 2013 to May 2013. Before the start of the main study, I had observed five tutorials from Zenobia in the context of the pilot study; thereby being familiar with Zenobia’s students from their first semester at University i.e. from October 2012. The five students were friendly with me during the tutorial time and when I sometimes met them by chance at the University.

The layout of the classrooms

In the pilot study, the classroom was for 20 students, and included two columns of desks with two chairs for each desk. The students sat in front desks facing two whiteboards and Zenobia. Zenobia stood in front of students in order to write the mathematics on the whiteboards. I sat in a chair opposite students in order to observe without being intrusive.

In the main study, there was a large desk in the middle of the classroom and twelve chairs around it. The five students sat around the desk facing two whiteboards and Zenobia. Zenobia sat at the other edge of that desk. She sometimes wrote the mathematics on the whiteboards while sitting in the desk. I placed for me a chair behind and away from the students in order not to be intrusive, and to be able to observe and audio-record.
Analysis of the pilot study

My analysis of the pilot study for the case of Zenobia’s teaching was based on five narratives of SGT observations and a few discussions with Zenobia. The extended observations enabled me to get insights into Zenobia’s ways of working with the students and her mathematical expertise. My interpretation was that Zenobia enabled students’ group work with her invitations to students to go to the board and her questions to students. I also interpreted that Zenobia used heuristics, such as ‘sketch a graph’ and ‘consider special cases’; and mathematical “tricks”, such as the Pascal triangle for the computation of polynomials of degree 5. So, the criteria for inviting Zenobia to be the participant for Main study 1 were Zenobia’s flexibility in engaging students in contributing to the tutorial and her mathematical flexibility in solving tasks. (The analysis of the case of Zenobia’s teaching in the pilot study is discussed in Appendix E.)

During Pilot study 1, I learned to notice the tutors’ and the students’ actions in my observations. The tutors’ actions were important for what I later called ‘tutors’ teaching practice’. Students’ actions were important for my interpretation of students’ mathematical meaning making. Examples of my analysis of Zenobia’s actions in her SGTs in the pilot study are: ‘inviting students to the whiteboard’, and ‘using graphs/examples for inductive thinking’. I also noticed students’ actions such as ‘correctly computing on the board’, and ‘contributing to what would be written on the board’.

The next parts of this chapter, Parts 1, 2 and 3, present the analysis of the main study for the case of Zenobia’s teaching.
Part 1: An interpretation of Zenobia’s epistemologies in analysis of the main study

4.1.1 Zenobia’s epistemology of teaching/learning in tutorials

I interpreted Zenobia’s views for teaching/learning in tutorials from my analysis of interview data, where Zenobia explained to me her underlying considerations and thinking for what she did in observations. I found consistency between what Zenobia did in various observations and what she said in distinct discussions with me. This finding indicates that Zenobia drew on her views for teaching/learning in order to act in tutorials. In other words, these views form her thinking and perception for her teaching actions; and as such, they form her epistemology of teaching/learning in tutorials.

4.1.1.1 Zenobia’s views of small group tutorials and her role as a tutor

I discussed with Zenobia about SGT1 and SGT8 after the end of my last observations of her SGTs. In that discussion, I asked Zenobia why she always starts the tutorials with a discussion with the students about their welfare and their suggestions for group work. Zenobia stressed the pastoral aspect of the SGT setting where the tutor offers week-to-week care to the students about their well-being at the University. Within this setting, she viewed her role as tutor to be about checking that the students’ welfare is OK every week. She also distinguished the tutor from “just another person testing them all the time” by saying that the tutor should not only care pastorally about the students, but should also be “on their side”.

In the same discussion about SGT1 and SGT8, my next question to Zenobia was why during the tutorial she used humour and valued the students, for example, by telling them “good job”. Her response gave me insight into what she meant by a tutor being on the students’ side and, in that case, why she viewed “that the students work harder if they feel like you are on their side and you care about them.” In her response, Zenobia stressed her communication with students which enabled them “to be comfortable enough in that group, to feel safe in that group, not to mind admitting
what they know, what they don’t know, what they want help with or whatever”. So in order to have students who “speak” in the tutorial and “work harder”, she viewed her role as breaking down the barrier of the member of staff and being on the students’ side.

I think it is just if they feel comfortable with me, if they know that I am their friend and I am their advocate, then they are more likely to be willing to admit that they don’t understand something or to ask a question; than, if they are intimidated and see me as the lecturer, the member of staff. So, I am trying to break down that barrier so that we can communicate more effectively.

**Excerpt 1** _Discussion about SGT1 and SGT8_

The section *Creating students’ positive feelings* of this chapter, Section 4.2.2.2, provides evidence of my analysis and interpretations about what Zenobia does in her tutorials to care pastorally about the students and to be “on their side”. In my question why after the pastoral discussion with the students, she talks with the students and together they select a small number of tasks or one task to tackle in the SGT, she said:

Almost all the instruction that they [students] get involves people [lecturers] having decided ahead of time useful examples to show them and going through them for them. And I feel that at least once a week they need a chance for them to direct it and to really check each step that they have really understood what it’s going on.

**Excerpt 2** _Discussion about SGT1 and SGT8_

In Excerpt 2, Zenobia talked about lecture teaching where lecturers choose and use examples which they demonstrate on the board. They usually choose examples before the lecture and demonstrate without being able to check the students’ meaning making. My interpretation from Excerpt 2 is that, contrary to lectures, Zenobia’s views for the small group tutorial is to let students select the task. Her view of her role as tutor is then to check at “each step that they have really understood”. Zenobia’s declaration about students who may “have really understood” is illuminated in the section about her views of what making sense of mathematics is (Section 4.1.1.3).
Furthermore, a first insight into the way she lets students select tasks and thus “direct” the tutorial is in the next section about her views of what teaching is (Section 4.1.1.2). A full analysis is in the section Selecting tasks of this chapter (Section 4.2.2.1).

To conclude, her views of the SGTs include pastoral care for students; and students who “speak” and choose as a group the tasks in the tutorial time. In these SGTs, she views a threefold role as a tutor: to check that the students’ welfare is OK every week; to break down the barrier of the member of staff thus to be on the students’ side; and to check at each step of the task solution that the students have really understood.

4.1.1.2 Zenobia’s views of what teaching is and its connection with mathematical research

In Zenobia’s tutorials, I usually observed that the students discussed with each other and agreed about the selection of a few tasks with which they faced difficulties. Then, they agreed with Zenobia to solve one or two of these tasks in the tutorial. Zenobia discussed with the students each step of the process of the solution of the tasks; in particular, she prompted them to elicit the steps. In discussion with her after SGT2, I asked her why she does not present solutions of tasks on the board in a lecturing format. She responded that she does so only in case her “prodding doesn’t result of anything” and she gives “the next step and the next step”. However, she considered that the students come to the SGT with difficulties in tasks from the lecture material. Her view was that another lecturing hour in which the tutor guides the students through the mathematics might result into the same difficulties for students. In contrast, a tutor who discusses with the students and checks at each step that the students have made sense has the potential to resolve the difficulties. Zenobia also started to explain to me her thinking about what she discusses with the students in her effort to elicit the solutions of tasks.
When I work with students and sort of recognise where they struggle, I have thought, well, when I’ve got a new concept I struggle with how do I tackle it and that has informed the way that I teach now. Like I read something in a paper and I am like ‘Well I have no idea what that means, can I think of a single example that fits that? What’s the simplest possible example I can think of that fits this or what’s the simplest possible example that doesn’t fit to this?

Excerpt 3_Discussion after SGT2

Excerpt 3 is crucial for Zenobia’s design of teaching with regard to the way of working with the mathematics to resolve the students’ difficulties. She declared that the way she works with the mathematics in her research to enable herself to make sense of a new concept has informed the way she works with the mathematics in her teaching to enable the students to make mathematical sense. My analysis of data indicates that various mathematical heuristics are central to this way of working. In the section Decoding the mathematics and encoding the mathematics of this chapter, Section 4.2.2.5, I draw on data to provide insight into the way of working with the mathematics and the nature of the connection between Zenobia’s teaching and mathematical research.

4.1.1.3 Zenobia’s views of what making sense of mathematics is

Observational data from SGT10 sheds light into Zenobia’s views of what making sense of mathematics is; and thus into her declaration in Excerpt 2 about checking whether students “have really understood”. In this tutorial, the following discussion between Zenobia and the students took place:

Zenobia: Do you guys find the way that I use examples to extract your understanding of the definitions and then work back again useful? How do you guys like to understand the definitions? How do you go about understanding a definition?

St: I use ‘The exercise teaches the theory’. The theory doesn’t teach the exercise.
Zenobia: Right, yeah. Exactly. So, doing it in an example is what makes you actually understand what is going on. It’s not that you understand the definition and then the exercise is straightforward. Which is exactly how we design them, right? We do design them to give you context in which to understand.

Excerpt 4_SGT10 Observation

In Excerpt 4, Zenobia referred to her use of examples of a concept that is difficult for students. She said that she uses examples to elicit the students’ sense making of the definition of a concept and then works to solve the task. The student’s perspective was that work with examples of a concept is possible to promote their sense making of the definition of the concept. My interpretation is that at this point of the discussion, Zenobia shared with the students her view of making sense of mathematics is, she said:

“doing it in an example is what makes you actually understand”.

In Excerpt 3, she declared that in her research she devises examples to figure out what works and what does not work for the new concept in order to make sense of it. From Excerpts 3 and 4, I interpret that her view of ‘what making sense of mathematics is’ is informed by her own mathematical research practice. Concluding the above discussion with students, Zenobia stressed a difference between her view of mathematical sense making and views of mathematical sense making in the design of lecture teaching: lecturers (“we” in Excerpt 4) design the presentation of the definitions in a way to give students “context in which to understand” so that “the exercise is straightforward”. However, evidence in Excerpt 4 indicates that within the latter design the student does not consider that the exercise is straightforward.
4.1.2 Zenobia’s epistemology of mathematics

Zenobia continued the discussion with the students, from which Excerpt 4 is a part, by sharing her views on the nature of mathematics. Before starting to talk about the nature of mathematics, she explained to the students the Platonic ideals with the metaphor of “person”.

What is a – what we call – “person”? What is our vision of “person”? So, Plato would have said that there was some sort of Platonic ideal of person and that we recognise anything on Earth as representing a flawed version of that Platonic ideal of “person”. And that’s how we come to recognise that somebody is a person. But at the other end, the more modern version – I don’t remember who said this – is that it’s a cultural consensus to lob a set of objects together and give them a label. But that label is prone to change its meaning as our experience of what those… You know, that we construct a generalisation of “person”, but that generalisation of “person” is not fixed, because if we – for instance – had never met a female person, because we’re mathematicians and we don’t know any women, and then suddenly we meet a female person, it’s not that she’s not a person because she doesn’t fit the ideal of “person” which in our brains is a male person. It’s that – suddenly – the concept of “person” has to change to accommodate a wider set of “person” than we previously had had, right?

Excerpt 5_SGT10 Observation

In this excerpt, Zenobia started to explain her epistemology of mathematics according to which a mathematical object is a Platonic ideal; and people come to make sense of it by familiarising themselves with the cultural consensus of their time about the object. In this discussion with the students, Zenobia’s use of humour is evident in her reference to the cultural group of mathematicians. She said that despite the consensus of “person” as the male person in the cultural group of mathematicians, the female person also belongs in the Platonic ideal of person; resulting into accommodating a wider consensus of “person” in that cultural group. My analysis from Excerpt 5 indicates that Zenobia’s use of humour was towards breaking down the barrier of the member of staff and being on the students’ side. In that discussion, Zenobia then
referred to the history of mathematics to explain to the students her view on the nature of mathematics.

In mathematics, we do rigorously define things. And so, it’s a situation where – from the set of examples that we have – we’ve come up with an ideal idea, and then we can actually rigorously then check that something is in that or not. And what happens in mathematics is that if we see a more general version of things that doesn’t fit that, but that still has some things in common with it, then we create a new definition that’s more general. We come up with new definitions any time we recognise that there are some sets of structures that have some relevance. But it really does emerge out of the examples. And if you look at the history of mathematics, it’s not that people have had the idea of a function. It’s that they’ve had lots of examples of functions and they’ve tried to distil what the critical characteristics of a function are. Does that make sense? So, I think it’s a very natural way to think about the relationship between examples and theories – it’s that we don’t define definitions just off the tops of our heads. We define them because they capture a behaviour we see in examples that have interesting kinds of properties.

Excerpt 6_SGT10 Observation

Excerpt 6, with reference to the history of the development of mathematics, is key for the connection between Zenobia’s reading of the history of mathematics, her views on making mathematical sense, her views on researching mathematics and her views on teaching mathematics. She said that mathematicians come up with a consensus of a mathematical object from sets of relevant structures, or properties, distilled out of examples. That consensus forms a definition of the mathematical object which can be accommodated at a later stage to better describe this ideal object.

Considering my analysis of Excerpts 3-6 into Zenobia’s views, it seems to me that in the case of Zenobia’s teaching a path of informing from her views on the history of mathematics, to her views on the sense-making of mathematics, to her views on conducting her own research in mathematics, to her views on the teaching of mathematics is revealed. In the next sections, I draw on data to unpack Zenobia’s
views with regard to her teaching practice which I distinguish in my analysis into strategies and tools for teaching.
Part 2: Zenobia’s teaching practice in the main study

4.2.1 Data analysis of Zenobia’s teaching practice

In this section, I provide a characterisation of Zenobia’s teaching through a presentation of strategies and tools in her teaching. (Strategies and tools are terms discussed in Chapter 3: Section 3.4.2.) I identified these strategies and tools in my analysis of data from eight tutorials of the main study [SGT1, SGT2, SGT3, SGT5, SGT6, SGT8, SGT9, SGT10]. (The choice of eight tutorials is discussed in Chapter 3: Section 3.4.2.) First, the reader becomes familiar with the conceptual names of categories for Zenobia’s teaching (Table 4.1) and for students’ response to Zenobia’s teaching. Then, the reader gets an insight into the use of conceptual names of categories for Zenobia’s teaching and for students’ responses in transcripts of three teaching episodes. Finally, drawing on Zenobia’s epistemologies, I offer the reader the full analysis and interpretations for the characterisation of Zenobia’s teaching into six strategies and their tools.

The following Table 4.1 presents, in the first column, the conceptual names of categories for strategies of Zenobia’s teaching; and in the second column, the conceptual names of categories for tools. (The conceptual names of categories for the tools numbered 4.1, 5.3, 6.3 and 6.4 are established concepts in research literature, which are presented in a glossary in Appendix D.)

Table 4.1: Analysis of Zenobia’s teaching into strategies and tools for teaching.

<table>
<thead>
<tr>
<th>Conceptual names of strategies for teaching</th>
<th>Conceptual names of tools for teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Creating students’ positive feelings</td>
<td>1.1. Pastoral questions;</td>
</tr>
<tr>
<td></td>
<td>1.2. Humour;</td>
</tr>
<tr>
<td></td>
<td>1.3. Eureka moment;</td>
</tr>
<tr>
<td></td>
<td>1.4. Statements (valuing, encouraging).</td>
</tr>
<tr>
<td>2 Selecting tasks</td>
<td>2.1. Students’ difficulties from teaching experience;</td>
</tr>
<tr>
<td></td>
<td>2.2. Students’ suggestions;</td>
</tr>
<tr>
<td></td>
<td>2.3. Positive reinforcement.</td>
</tr>
<tr>
<td>3 Selecting examples</td>
<td>3.1. Generic set of examples.</td>
</tr>
</tbody>
</table>
| 4 Evaluating students’ sense making of mathematics | 4.1. Questions to evaluate students’ sense making [control questions of students’ sense making (Viirman, 2015, p.1175), inviting questions to students: direct to a student or
Following Table 4.1 are the transcripts of three teaching episodes from observational data of SGT6, SGT8 and SGT10. Attached to each episode is a right margin with conceptual names of categories. The conceptual names of categories for the students’ response to Zenobia’s teaching, through which I characterise the students’ meanings in this study, are:

Correct input; Incorrect input; St difficulty (“St” for student); St question; St sense-making; Positive reinforcement; Reinforcement.

The conceptual names of categories for Zenobia’s teaching correspond to tools from Table 4.1. The inclusion of tools, and not of strategies, in the right margin is for presentation purposes. This is because distinctive teaching tools correspond to one and only one strategy. A brief account after each episode explains the different stages of Zenobia’s “design” and “redesign” for students’ meaning making. After the presentation of the episodes and their brief accounts, I offer explanation, analysis and interpretations I made for each strategy and the associated tools.
<table>
<thead>
<tr>
<th>Episode 1_SGT6_An approach to derivatives</th>
<th>Conceptual names of tools and strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>This episode is situated towards the end of SGT6, and includes the application of the chain rule to multivariable functions. It starts with Zenobia, who writes on the board the following task from a practice test in Calculus:</td>
<td>Procedural task 1:</td>
</tr>
<tr>
<td>$z = f(x, y) = \tan[4\sqrt{(x^2 + y^2)} - 1]$. Find $\frac{\partial f}{\partial x}$ at $(x, y) = (1, 2)$.</td>
<td>$z = f(x, y)$. Find $\frac{\partial z}{\partial x}$</td>
</tr>
<tr>
<td>Zenobia asks the students how they do that task.</td>
<td>Inviting question-gen.</td>
</tr>
<tr>
<td>St5 responds substitution, and St3 suggests the chain rule.</td>
<td>St5, St3 correct inputs</td>
</tr>
<tr>
<td>St5 question</td>
<td>St5 question</td>
</tr>
<tr>
<td>Zenobia writes $\frac{\partial f}{\partial x} = f_x$ on the board.</td>
<td>Symbolic represent.</td>
</tr>
<tr>
<td>Zenobia: That’s right, yeah. [Zenobia writes $\frac{\partial f}{\partial x} = f_x$ on the board.] That’s just two different ways of writing the same thing. That’s OK? [4’’ pause.] Is that OK?</td>
<td>Control questions</td>
</tr>
<tr>
<td>St5: $\frac{df}{dr} \frac{dr}{dx}$ the down thing where you just start writing $f$ with a small</td>
<td>Positive reinforcement</td>
</tr>
<tr>
<td>Zenobia: That’s right, yeah. [Zenobia writes $\frac{df}{dr} \frac{dr}{dx}$ on the board.] That’s just two different ways of writing the same thing. That’s OK?</td>
<td>Pause interval</td>
</tr>
<tr>
<td>St3 difficulty</td>
<td>Positive reinforcement</td>
</tr>
<tr>
<td>Zenobia: You can draw the thing this way, which is that.</td>
<td>Symbolic represent.</td>
</tr>
<tr>
<td>[Zenobia draws Figure 4.1 on the board.] [终于] This $z$ on Figure 4.1 depends on $r$, which depends on $x$ and $y$. [4’’ pause.] you can also think of it as: You’re keeping $y$ constant. So, $df \ dr \frac{\partial f}{\partial r}$ isn’t that bad. Right? [4’’ pause.]</td>
<td>Diagram (Figure 4.1)</td>
</tr>
<tr>
<td>Inviting question-gen.</td>
<td>Inviting question-gen.</td>
</tr>
<tr>
<td>Zenobia writes $\frac{df}{dr} \frac{dr}{dx}$[4’’ pause.] So, what’s $df \ dr \frac{\partial f}{\partial r}$? [6’’ pause.]</td>
<td>Resume interval</td>
</tr>
<tr>
<td>St5 correct input</td>
<td>Symbolic represent.</td>
</tr>
<tr>
<td>Zenobia writes $\frac{df}{dr} = 4\sec^2(4r - 1)$ on the board. St5 suggests the use of implicit differentiation to calculate $\frac{\partial r}{\partial x}$. Zenobia starts by writing on the board $\frac{\partial (r^2)}{\partial x} = \partial \frac{\partial r}{\partial x}(x^2 + y^2)$. St1 suggests that the side $\frac{\partial (r^2)}{\partial x}$ equals $2r \frac{\partial r}{\partial x}$ and Zenobia asks the remaining students if it makes sense. St3 says she is confused with the partial derivative. Zenobia asks her to calculate $\frac{\partial}{\partial x}(x^2 + y^2)$ and St3 responds $2x$. St1 dictates to Zenobia, who</td>
<td>St5 correct input</td>
</tr>
<tr>
<td>St5, St3 correct inputs</td>
<td></td>
</tr>
</tbody>
</table>

120
is writing on the board, that \( \frac{\partial r}{\partial y} = x \).

**Procedural task 2:** Find \( \frac{dz}{dt} \). [Zenobia draws the diagram in Figure 4.2.] In this case [Figure 4.2], we’ve got a situation where we’ve got \( z, x, y, t \). So, \( z \) is the function of both \( x \) and \( y \). And each \( x \) and \( y \) is the function of the \( t \). Are you guys with me? [...] So, in this case [Figure 4.1], if you’re trying to, you can write the dependency of the variables. So, \( z \) depends only \( r – \) directly on \( r \) – and \( r \) depends on \( x \) and \( y \). So, if you want to find the derivative of \( z \) with respect to \( x \), you have to sum the product of partials over each path that goes from \( z \) to \( x \).

In this case [Figure 4.2], there’s only one path that goes from \( z \) to \( x \). It’s that one. But in this setting, here we’ve got \( z \) depending on \( x \) and \( y \), and both \( x \) and \( y \) depend on \( t \). And if I want to find the derivative of \( z \) with respect to \( t \), I again have to sum over each path the product of partials. OK? [2” pause.] Does that make sense? So, each vertex in this diagram is a variable. This edge is going to be \( dz \ dx \ [\frac{\partial z}{\partial x}] \). [Zenobia writes \( \frac{\partial z}{\partial x} \) between the vertices \( z, x \) in Figure 4.2.] And this is going to be \( dx \ dt \ [\frac{\partial x}{\partial t}] \). [Zenobia writes \( \frac{\partial x}{\partial t} \) between the vertices \( x, t \) in Figure 4.2.] What’s this edge going to be [between the vertices \( z, y \) in Figure 4.2]? St1: \( dz, dy \ [\frac{\partial z}{\partial y}] \). dy, dt \( [\frac{\partial y}{\partial t}] \). [Zenobia writes \( \frac{\partial z}{\partial y} \) between the vertices \( y, t \) in Figure 4.2.] Zenobia: So, what’s \( dz, dt \ [\frac{\partial z}{\partial t}] \) going to be? [3” pause.]

St3: \( dz, dx \) timesed by \( dx, dt \) plus \( dz, dy \) timesed by \( dy, dt \ [\frac{\partial y}{\partial t}] \). [Zenobia writes \( \frac{\partial y}{\partial t} \) between the vertices \( y, t \) in Figure 4.2.]

Zenobia: Yeah, perfect. And to be honest, if you’re doing work on your own, it’s not really that critical that you keep track of which ones are partials – which ones are curly \( [\partial] \) and which ones are straight \( [d] \). It’s more important that you keep track of the variables. St2: I think I quite like these diagrams. Where it splits, you know you need a partial, so you know you need a curly. And where it’s dependent on the other thing, then it’s going to be –

Zenobia: Yeah. These are really useful diagrams. Let’s create...
another situation [i.e. another task]. Suppose that we’ve got a function \( z \) that depends on \( x \) and \( y \), and \( x \) and \( y \) both depend on \( t \), and \( y \) also depends on \( s \), and then \( t \) and \( s \) each depend on \( w \) [Figure 4.3]. No problem. So, what’s \( dz, dw \)? [7” pause.] How many paths are there? Well, first, there’s this path. So, that would be \( dz, dx \) times \( dx, dt \) times \( dt, dw \) \[ \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial x} \]. And then, you have to add this path. So, that’s \( dz, dy \); \( dy, dt \); \( dt, dw \) \[ \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial y} \cdot \frac{\partial y}{\partial w} \]. And then, you have to add the last path, which is this one. [Zenobia writes on the board \( dz = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} \cdot \frac{\partial t}{\partial w} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \cdot \frac{\partial t}{\partial w} + \cdots \)] Does that make sense? So, the diagrams are actually really helpful to use to keep track of what all you’ve got to do, so I would really recommend using those. I’m not sure: Has she [the lecturer] talked about those in class?

St3: Yeah. She drew one of the little diagrams at the top of one of the pages.

Zenobia sketched the diagram in Figure 4.4 to create a last situation (i.e. task), where \( z \) depends on \( x \) and \( y \), and \( y \) depends on both \( x \) and \( t \). She said orally what the two paths are for \( dz \). She then told the students that drawing out diagrams of dependencies is a useful way to keep track of variables.

<table>
<thead>
<tr>
<th>Figure 4.1:</th>
<th>Figure 4.2:</th>
<th>Figure 4.3:</th>
<th>Figure 4.4:</th>
</tr>
</thead>
</table>
Brief account of Zenobia’s design and redesign for students’ meaning making: Episode 1

Figure 4.5, below, is a figural representation of Zenobia’s initial design and successive redesigns in Episode 1 in order for the students to make meaning of variable dependencies for the chain rule. In the final redesign, Zenobia’s perspective is that her teaching has reached a stage which enables the students to make meaning of variable dependencies for the chain rule; so, she does not enrich her teaching with a new redesign.

A few words to explain Figure 4.5 are the following. Two helixes represent the space of mathematics and the space of teaching/learning. The two helixes are connected with blue developmental stages. They are developmental of Zenobia’s teaching for students’ meaning making. The first blue stage represents Zenobia’s design to apply substitution and the chain rule to solve Task 1. Zenobia’s tool for her design is symbolic representations of substitution and the chain rule. Zenobia draws on the space of mathematics to act with symbolic representations.

A red arrow represents a dialectic connection between two blue stages; in other words, between design and a successive redesign or between a redesign and a successive redesign. Zenobia designs the teaching in order for the students to make mathematical meaning, so her view is that the students will make meaning with the initial design. However, between Zenobia’s view and what she sees from the students in the tutorial, there is some sort of contradiction. During the implementation of the design, the students do not make mathematical meaning. So, there is a contradiction between Zenobia’s view for students’ meaning making and students’ meaning making per se from Zenobia’s perspective. The dialectic connection arises out of a contradiction in dialogue about mathematical meanings between Zenobia and the students. So, the successive redesign emerges from a contradiction as a change of tools, which are intended to foster students’ mathematical meaning making.

In order to develop her perspective of students’ meaning making, Zenobia acts with questions to evaluate the students’ meanings of the mathematics and pause intervals after her questions. In the dialogue, there are more students’ questions and difficulties than students’ correct inputs. Also, St2 asks for another way of working with the
mathematics. Zenobia’s redesign to work with her Diagram 4.1 emerges from the contradiction between her intention for the initial design and what she sees from the students. Zenobia’s tool is Diagram 4.1 in the redesign stage. She thus transforms the symbolic representations of substitution and the chain rule with a diagram, which represents the variable dependencies for substitution and the chain rule. Zenobia draws on the space of mathematics to act with Diagram 4.1.

In her dialogue with the students in the first redesign stage, she acts with questions to evaluate the students’ sense making of the mathematics and pause intervals; both of which she copied from the preceding stage. She also acts with an encouraging statement. The students offer correct inputs, but St3 expresses a difficulty. Zenobia’s redesign to work with Diagram 4.2 emerges from St3’s difficulty, which is in contradiction with Zenobia’s design for students’ meaning making. Zenobia’s tool is now Diagram 4.2. She also acts with formal language to explain Diagram 4.1 and Diagram 4.2. Zenobia draws on the space of mathematics to act with formal language.

Her dialogue with the students reveals students who have made sense of variable dependencies for the chain rule. They offer correct input and St2 says what sense he made. Zenobia acts with a valuing statement and redesigns to work with Diagram 4.3. She then acts with formal language to explain variable dependencies and with a consolidating statement. In order to act with the consolidating statement, Zenobia steps out of the space of mathematics to consider the context of the students and to consolidate their meaning, thereby drawing on the space of teaching/learning.

Zenobia asks the students whether the lecturer demonstrated the diagrams in the lectures and St3 responds that the lecturer sketched one. Zenobia redesigns again to work with Diagram 4.4 and she acts with a consolidating statement.

Figure 4.5 does not include two straight lines; rather, it includes two interrelated helixes connected with each other by the developmental stages of design and redesign. The helixes are interrelated because in teaching mathematics the space of mathematics and the space of teaching/learning are interrelated. An example of the interrelation between the two helixes is across the stages of design and redesign in Episode 1. In particular, in the stages of design and first redesigns (i.e. first to third blue stages), Zenobia draws on tools of the space of mathematics to enable the
students to make meaning of variable dependencies for the chain rule. So, she acts with symbolic representations, diagrams, and formal mathematical language. However in the next stages of redesign (fourth to fifth blue stages), she steps out of the space of mathematics to consider the context of the students and use language to consolidate their meaning. My interpretation is that Zenobia draws on the space of teaching/learning to act with the consolidating statements, as well as an encouraging statement and a valuing statement for the students. This indicates that in Zenobia’s mathematics teaching the space of mathematics and the space of teaching/learning are interrelated.

**Figure 4.5:** Zenobia’s design and redesign for students’ meaning making in Episode 1.
<table>
<thead>
<tr>
<th><strong>Episode 2_SGT8_Discovering the proof</strong></th>
<th><strong>Conceptual names of tools and strategies</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>This episode is situated in SGT8. SGT8 starts with Zenobia who gives copies of three problem sheets in sequences and series to the students. The students suggest to work on some tasks from those problem sheets. This episode concerns work on the task: If $s_n$ converges to $l$, then every subsequence of $s_n$ also converges to $l$.</td>
<td>Proof task</td>
</tr>
<tr>
<td>St3 is in charge of writing on the board. The remaining students and Zenobia contribute for the writing of the definition of convergence on the board: A sequence $s_n$ converges to $l$ if $\forall \varepsilon &gt; 0 \exists k_0 \in \mathbb{R}$ s.t. $</td>
<td>s_k - l</td>
</tr>
<tr>
<td>Zenobia asks St3 to sketch a convergent sequence on the board. The first two graphs St3 sketches are in Figure 4.6. Zenobia informs the students that both are graphs of functions. St3 says &quot;It would be dots&quot;, and sketches the graph in Figure 4.7.</td>
<td>St3 difficulty</td>
</tr>
<tr>
<td>St3 invites St2 to the board. Zenobia asks St2 to select a subsequence on the graph (Figure 4.7) and define it. The students contribute so that the student on the board writes the definition of subsequence: A subsequence of a sequence $s_n$ is a new sequence $r_m = s_{f(m)}$ where $f$ is an increasing function $f: \mathbb{N} \rightarrow \mathbb{N}$.</td>
<td>Symbolic represent.</td>
</tr>
<tr>
<td>Zenobia asks all students to put values of $r_m$, $\varepsilon$, and $k_0$ on the graph.</td>
<td>Injunction questions about heuristic 'specific case'</td>
</tr>
<tr>
<td>Can you give me a littler epsilon? [5” pause.] Is it still true that $s_k$ is less than $l$ for all $k$ greater than $k_0$?</td>
<td>2 Questions (students to observe)</td>
</tr>
<tr>
<td>St3: $k_0$ would have to be bigger than it.</td>
<td>St3 correct input</td>
</tr>
<tr>
<td>Zenobia: It would have to be bigger, right?</td>
<td>Control question</td>
</tr>
<tr>
<td>St3: Yeah.</td>
<td>Consolidating statement</td>
</tr>
<tr>
<td>Zenobia: OK. So, $k_0$ in particular is going to depend on epsilon.</td>
<td>Injunction questions about heuristic 'specific case'</td>
</tr>
<tr>
<td>Zenobia asks all students to put values of $r_m$ on the graph (Figure 4.7).</td>
<td>Injunction questions about heuristic 'specific case'</td>
</tr>
<tr>
<td>Zenobia: Right. Would you like to give the magic pen to the next victim?</td>
<td>Humour</td>
</tr>
<tr>
<td>Zenobia writes on the board that they know the two aforementioned definitions and they need to prove: $\forall \varepsilon &gt; 0 \exists k_0 &gt; 0 \text{ s.t. } m &gt; k_0 \Rightarrow</td>
<td>m - l</td>
</tr>
<tr>
<td>The students are experimenting with special cases for $k_0$ and $k_0$ on the graph. From special cases, the students suggest that: $f(2) &gt; 3$, and Zenobia writes a summary of students' suggestions on the board “E.g. here, we chose 2 because...” (Figure 4.7).</td>
<td>Symbolic represent.</td>
</tr>
<tr>
<td>Zenobia: How are we going to identify $k_0$?</td>
<td>Question (to observe the relation among $k_0$, $k_0$, $f$ from the graph of Figure 4.7.)</td>
</tr>
<tr>
<td>[Zenobia points to $\exists k_0 &gt; 0$ in (*) above.] We need to show that this thing exists. And the best way to show that it exists is by giving some way to find it.</td>
<td>Heuristic (existence proof)</td>
</tr>
<tr>
<td>A pause, right? To get the property we need. [3” pause.] So, we need to identify. The only thing here is this exists. [Zenobia points to $\exists k_0 &gt; 0$ in (*) above.] We need to show that this thing exists. And the best way to show that it exists is by giving some way to find it.</td>
<td>Pause intervals</td>
</tr>
<tr>
<td>So, we want to know: How do we find $k_0$? [5” pause.] Can you think about it, St4?</td>
<td>Question (to observe the relation among $k_0$, $k_0$, $f$ from the graph of Figure 4.7.)</td>
</tr>
<tr>
<td>Zenobia invites St4 to the board.</td>
<td>Student as tool for teaching</td>
</tr>
<tr>
<td>St4: $f$ of 2 is greater than 3. So, what’s this? [St4 points to $f(2) &gt; 3$ in $f(2) &gt; 3 = k_0$.] This is $f$ of $k_0$.</td>
<td>St4 correct input</td>
</tr>
<tr>
<td>[St4 points to $f$ in $f(2) &gt; 3 = k_0$.] Umm. [4” pause.] It’s $f$ of something in there. [St4 writes $f(k_0) \geq$] Is it $k_0$ hat? [St4 points to $k_0$ in $f(k_0) \geq$] [9” pause.] Umm. [18” pause.] No. I take it that’s not right?</td>
<td></td>
</tr>
<tr>
<td>Zenobia: I’m not even thinking about it. I was just thinking about something else.</td>
<td></td>
</tr>
<tr>
<td>St4: Does anyone have any suggestions about anything? How we can relate –</td>
<td></td>
</tr>
<tr>
<td>Zenobia: Why don’t you just write $f$ of 2 is greater than 3. Greater than or equal to 3. [St4 writes $f(2) \geq 3$ on the board.] Now, in terms of the letters we’re using, what is this? [Zenobia points to 2 in the relation $f(2) \geq 3$ on the board.]</td>
<td>Question (to observe the relation among $k_0$, $k_0$, $f$ from the graph of Figure 4.7.)</td>
</tr>
<tr>
<td>St4: Is that that? [St4 points to 2 and asks if 2 is $k_0$.]</td>
<td></td>
</tr>
<tr>
<td>Zenobia: Yeah.</td>
<td></td>
</tr>
<tr>
<td>St4: And that’s just $k_0$. [St4 points to 3 in $f(2) \geq 3$, and writes $f(k_0) \geq k_0$.]</td>
<td>Symbolic represent.</td>
</tr>
<tr>
<td>Zenobia: OK. That’s a good start. So, now we write the proof. Let epsilon be greater than 0. Start at the left-hand side of the board. Let epsilon be greater than 0...</td>
<td>Encouraging statement</td>
</tr>
</tbody>
</table>
| Zenobia: How are we going to identify $k_0$? | }
You’re going to have a eureka moment in a just a second. You’re going to love it. OK, if epsilon is greater than 0, what do we know is true? [4” pause.]

Inviting question-dir.

Eureka moment

St4 writes the proof (Figure 4.8) on the board. Zenobia intervenes with questions.

<table>
<thead>
<tr>
<th>Figure 4.6: The first two graphs St3 sketched.</th>
<th>Figure 4.7: St3’s third graph on the board at the end of SGT8.</th>
</tr>
</thead>
</table>

Let $\varepsilon > 0$, then $\exists k_0 \in \mathbb{R}$ s.t. $k > k_0 \Rightarrow |s_k - l| < \varepsilon$.

By the definition of a subsequence, $r_m = s_{f(m)}$ for some strictly increasing $f: \mathbb{N} \rightarrow \mathbb{N}$.

Since $f$ strictly increasing, $\exists k_0$ s.t. $f(k_0) > k_0$.

Furthermore, for $m > k_0$, $|r_m - l| = |s_{f(m)} - l| < \varepsilon$ because $f(m) > f(k_0) > k_0$.

| Figure 4.8: St4’s proof on the board. |

Brief account of Zenobia’s design and redesign for students’ meaning making: Episode 2

The following Figure 4.9 is a figural representation of Zenobia’s design and a successive redesign in Episode 2 in order for the students to make meaning of heuristics and the definitions of convergence and subsequence.

In Figure 4.9, the first blue stage represents Zenobia’s design where students experiment with definitions, formulate what is needed to be proved, have insight into the proof for Task 2 and produce the proof. Zenobia’s tools for her design are graphical and symbolic representations, heuristics, consolidating statements, humour and students to the board. My interpretation is that Zenobia draws on the space of mathematics to act with heuristics and mathematical representations (e.g. graphical and symbolic representations). She also steps out of the space of mathematics to consider the context of the students and bring students to the board, consolidate their
meaning and use humour. In this way, she draws on the space of teaching/learning as well as the space of mathematics.

In her dialogue with the students in the stage of design, Zenobia acts with questions about heuristics or questions to observe and pause intervals. The students offer correct inputs; however, neither St1, who is to the board, nor other student offer insight into how to produce the proof. This is in contradiction with Zenobia’s intention for the students’ meaning making of the proof in her design.

Zenobia knows that St4 is a high-achieving student. As she informed me, she brings him to the board because of his mathematical ability and the limited remaining time of the tutorial. Her redesign is St4 to get the insight into the proof and to prove. So, Zenobia acts with St4 to complete the proof, an encouraging statement to St4, questions to St4 and pause intervals. My interpretation is that Zenobia draws on the space of teaching/learning to act with the tools around St4. On the board, St4 acts with heuristics and graphical and symbolic representations. He also succeeds in offering the insight and the proof.

Episode 2 is an episode which indicates that, in analysis of Zenobia’s mathematics teaching, the space of mathematics and the space of teaching/learning are interrelated. This is because in each developmental stage of teaching, Zenobia draws on both spaces to act with tools and to promote students’ meaning making of heuristics and the definitions of convergence and subsequence.
Figure 4.9: Zenobia’s design and redesign for students’ meaning making in Episode 2.

<table>
<thead>
<tr>
<th>Episode 3_SGT10_Selection and use of examples [adapted from Mali, Biza and Jaworski (2014)]</th>
<th>Conceptual names of tools and strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>This is an episode situated in SGT10. SGT10 is about calculus revision towards the approaching semester exams. Zenobia has invited more first year students than her small tutorial group to SGT10. So, St6, St7, St8 and St9 also attend the tutorial. Zenobia and the students select to work on the latest past exam paper. This episode forms part of the students’ and Zenobia’s work for the task:</td>
<td>Proof task</td>
</tr>
<tr>
<td>Let ( f : [0,1] \rightarrow [0,\ln 2] ) be defined by ( f(x) = -\ln \sqrt{1 - \frac{3x^2}{4}} ). Give a short argument explaining how we know that ( f ) is bijective. For this task, Zenobia and the students choose to work first on injectivity.</td>
<td>Conceptual task: Make a conjecture about the relation of injectivity and monotonicity.</td>
</tr>
<tr>
<td>Zenobia: Are there any kinds of functions that you know are going to be injective, for instance? Is there anything about a function that you - [2(^{nd}) pause] OK. So, let’s draw some functions on the board, shall we? So, here’s an example of a function. [The graph of ( f(x) = x^2 ) in Figure 4.10]. And here’s another example of a function. [The graph of ( f(x) = \sin(x) )]</td>
<td>Inviting question-gen.</td>
</tr>
<tr>
<td>Reinforcement Heuristics (sketch a graph, induction)</td>
<td></td>
</tr>
</tbody>
</table>

Design: Students experiment with definitions, formulate what is needed to be proved, have insight into the proof & prove. Redesign: St4 has insight into the proof & proves.
in Figure 4.11.] And here’s an example of a function [the graph of \( f(x) = \ln(x) \) in Figure 4.12], and here’s an example of a function [the graph of \( f(x) = x \) in Figure 4.13]. So, if you wanted to determine some domains on which all of these are injective, how would you do it? How would you do it for this one? [Zenobia points to the graph of \( f(x) = x^2 \)]. How would you find your domain of injectivity? Is it injective on anything?

St5: From 0 to \( \infty \). [Zenobia draws a red line from 0 to \( \infty \) to show the domain on which \( f(x) = x^2 \) is injective.]

Zenobia: Right. This is definitely not injective on the whole thing, right? Because if I go off in opposite directions, I’m going to the same thing, right? OK. But if I go from here on, that’s injective, right? OK. And what about down here? [Zenobia shows the graph of \( f(x) = \sin(x) \).] [5'' pause.] Do you want to have a go at that? [Zenobia looks at St8.] You’re very close. I know you can do it. Just draw a little red line on the domain axis.

St8: I hope I’m right. [15'' pause.] I think. [St8 draws a red line from 0 to \( \infty \).]

Zenobia: You think? OK. So, what does “injective” mean? It means that there shouldn’t be any two points that are at the same height. No, that’s definitely not right.

St8: [17'' pause.] Can you have two parts to the domain? [St8 draws a red line from \( -\pi \) to \( \pi \).]

Zenobia: I guess you could, sure. You just do it. I mean, it’s conventional to choose a connected interval, but you don’t have to.

St8: [10'' pause.] It must be from here to here. [St8 draws a red line from \( -\pi/2 \) to \( \pi/2 \).]

Zenobia: Excellent. Good, good. Right. So, what did you notice? You noticed that you can’t have it go up and down, basically. […] So, what can I say about… OK, what about this function? [Zenobia shows the graph of \( f(x) = \ln(x) \).] Is this injective? Is this an injective function?

St7: Yeah. It is injective.

Zenobia: It is injective. What about this one? [Zenobia shows the graph of \( f(x) = x \).]

St7: Yeah.

Zenobia: OK. So, what can you say about this part of this function, this part of this function, this function and this function? [Zenobia shows the previous functions restricted on the domain of injectivity.] What do they
all have in common? [3" pause.]

St2: They’re monotonically increasing.
Zenobia: They’re monotonically increasing, right. So, a function that’s either monotonically increasing or decreasing could easily have chosen this, instead. [Zenobia plots the graph of \( f(x) = \log_a(x) \) where \( 0 < a < 1 \).] I could have chosen this part instead. [Zenobia shows the graph of \( f(x) = \sin(x) \) restricted on \([\pi/2, 3\pi/2]\).]

So, either monotonically increasing or monotonically decreasing is automatically going to be injective.

**Figure 4.10:** The graph of \( f(x) = x^3 \).

**Figure 4.11:** The graph of \( f(x) = \sin(x) \).

**Figure 4.12:** The graph of \( f(x) = \ln(x) \).

**Figure 4.13:** The graph of \( f(x) = x \).

**Figure 4.14:** The graph of \( f(x) = \sin(x) \).

**Figure 4.15:** The graph of \( f(x) = \ln(x) \).

**Figure 4.16:** The graph of \( f(x) = x \).

**Figure 4.17:** The graph of \( f(x) = x^3 \).

**Figure 4.18:** The graph of \( f(x) = \sin(x) \).

**Figure 4.19:** The graph of \( f(x) = \ln(x) \).

**Figure 4.20:** The graph of \( f(x) = x \).

**Brief account of Zenobia’s design and redesign for students’ meaning making:**

**Episode 3**

Below is Figure 4.14 which is a figural representation of Zenobia’s design and a successive redesign in Episode 3 in order for the students to make meaning of injectivity and the relation between monotonicity and injectivity.

In Figure 4.14, Zenobia’s design is for students to suggest kinds of injective functions and to observe injectivity. She acts with an inviting question to students about kinds of injective functions but no student responds.

Zenobia’s redesign emerges from the contradiction between her expectation for the students’ response and the fact that students offered no response. So, Zenobia sketches by herself graphical representations of injective functions, and then students find domains of injectivity and observe monotonicity. While students work to find domains of injectivity on the board, Zenobia acts with encouraging and valuing statements as well as questions along with their pause intervals. She also acts with heuristics and consolidating statements. Her response to St6’s incorrect inputs and
question includes her ‘explaining’ with informal language and the consolidating statement about injectivity. My interpretation is that Zenobia draws on the body of teaching/learning to act with statements (i.e. consolidating, encouraging and valuing statements), and to the space of mathematics to act with informal mathematical language, heuristics and graphical representations. This indicates that in analysis of her teaching the two spaces are interrelated. Apart from St8’s initial incorrect inputs, the students find correct domains of injectivity and St2 observes monotonicity.

**Figure 4.14:** Zenobia’s design and redesign for students’ meaning making in Episode 3.

### 4.2.2 Zenobia’s strategies for teaching and the associated tools

#### 4.2.2.1 Selecting tasks

This section is an account of analysis for the strategy ‘selecting tasks’ in eight tutorials [SGT1, SGT2, SGT3, SGT5, SGT6, SGT8, SGT9, SGT10]. It contains observational and interview data from those eight tutorials, and focuses on the three teaching episodes from SGT6, SGT8 and SGT10, which I present above.

Zenobia started her tutorials by providing the students with 5 to 15 minutes for pastoral discussion and selection of tasks to tackle. (Her pastoral discussion with the
students is included in the next section which corresponds to the strategy ‘creating students’ positive feelings’. Within the first 5 to 15 minutes of each tutorial, Zenobia asked the students what mathematics they would like to discuss with her. The students first suggested specific lecture material such as tests they had to hand in and/or problem sheets, usually in analysis and linear algebra. Then, they suggested specific tasks from one test or one sheet. Zenobia informs me about the students’ perspective about suggesting tasks in the following email which she sent me after SGT3.

I should mention that I generally don't prepare for tutorials because I have found that the thing students appreciate the most about how I run them is that I let them determine what we should talk about on any given day. So it isn't really possible for me to prepare, as I don't know what they will want to do.

Also, I think it can be useful for them to see how a mathematician approaches an unseen problem, and often if I am going through an unseen problem with them, I can pick out for them what problem solving strategies I have used and discuss them as well as the particular content.

Excerpt 7_Email after SGT3

In this email, she stressed that she did not prepare work for tutorials but let students determine work. This is consistent with her views on tutorials in Excerpt 2 and in my observations. The only tutorial where her students did not have specific suggestions for tasks was SGT5. In this tutorial, Zenobia suggested work on the \( \varepsilon - \delta \) definition of limit by telling them that they had low results in coursework. However, I did not consider in my analysis that students’ low results in coursework tasks was a tool for ‘selecting tasks’, because Zenobia used those results only in SGT5. In contrast, I considered that positive reinforcement was a tool for ‘selecting tasks’, because Zenobia acted with it constantly in her tutorials for the selection of tasks. (Analysis of positive reinforcement as a tool for ‘selecting tasks’ is discussed in Chapter 3: Section 3.4.2.)

In the second paragraph of the email, she referred to “an unseen problem”. By “problem” she meant something that needs a solution, such as a task in the SGT setting. By “unseen” she meant a task the mathematician has not come across so far;
thus developing her/his sense of it and exploring its solution. I found our discussion after SGT3 to be illuminative with regard to what she meant by “problem solving strategies”. She said: “I do have them [basic types of proofs] in my head when I am looking at a problem”. My interpretation is that “basic types of proofs” such as proof by contradiction are examples of “problem solving strategies” for Zenobia, or *heuristics* as I refer to from this part of the thesis on. In particular, I use the term *heuristics* to indicate my interpretation that Zenobia’s “problem solving strategies” draw on Polya’s (1971) heuristics. (Polya’s (1971) heuristics are presented in the glossary in Appendix D. Zenobia’s heuristics are discussed in a next section in this chapter.)

Excerpt 8, below, provides an example for what Zenobia does with the students in order to “let them determine” [Excerpt 7] the work in tutorials. Before Excerpt 8, the students had suggested a few tasks for work. Zenobia listened to them and asked them for the “scariest” task. The “scariest” task was about proving that a particular set of polynomials is a vector space, by showing that it is a subspace.

Zenobia: What is it? What’s scary about it? Oh, it’s double prime. I see. I couldn’t read it before when you handed it to me. I’m going blind. OK. Is it the fact that it relates to functions? Is that what makes it scary – that you’ve got polynomials? Or is it just knowing where to start because it’s a proof, because they say “proof”?

St2: Yes.

Zenobia: Is it the word “proof”?

St3: It’s always the word “proof”.

Excerpt 8_SGT1 Observation

In SGT1, Zenobia selected tasks about proving whether sets of polynomials are vector spaces, by showing that they are subspaces. (For example, two sets of polynomials she selected were $P_n[x] := \{ p \in \mathbb{R}[x] \mid \deg p \leq n \}$, which is a vector space, and $R_n[x] := \{ p \in \mathbb{R}[x] \mid \deg p = n \}$, which is not a vector space.) This indicates that Zenobia’s *tool* for ‘selecting tasks’ is *students’ suggestions*. In our discussion about
SGT1 and SGT8, I asked Zenobia whether she knows from her teaching experience that the word “proof” is a students’ difficulty. Her response included that “Yes. I know that proof always scares people.” This response enabled me to code a second tool for Zenobia’s selection of the “scariest” task in SGT1: *students’ difficulties from teaching experience*.

In my analysis of Excerpts 9 and 10, I interpreted Zenobia’s goals for ‘selecting tasks’ and for teaching, as:

- to enable students to determine their mathematical difficulties in tasks and to resolve them in the tutorial.
- to provide students with heuristics.

In my discussion with Zenobia about SGT1 and SGT8, I asked her about her goals for ‘selecting tasks’ and for teaching, in order to juxtapose them with my interpretation of her goals. She said:

- to enable students “to pass the modules”; and
- “to make students see the culture of mathematics; fundamental topics such as calculus. Not to understand everything in detail, but to understand a few things really really well – what does it really feel to really understand mathematics?”

Excerpt 9_Discussion about SGT1 and SGT8

Zenobia’s goals in Excerpt 9 seem to accord with my interpretation of goals from Excerpt 7 (interview data) and Excerpt 8 (observational data). In Excerpt 9, she informed me about selecting tasks which are key in passing the modules and in making sense of fundamental topics. It seems to me that after the completion of her SGTs, Zenobia’s purpose is to achieve in having students who have resolved their difficulties in tasks that are examined towards a pass for a module; and who have made sense of fundamental topics and mathematical heuristics.

Data analysis of SGT2 and SGT5 offers another example with regard to the interpretation of the tool *students’ difficulties from teaching experience* for Zenobia’s strategy ‘selecting tasks’. In SGT2, the task was “Determine whether the sequence $n^2 (-1)^n$ is bounded or unbounded.” The students had to find the definition of upper
bound and then, rearrange the quantifiers so that they produce the negation of this
definition. In the observation, before a student read the correct definition of upper
bound from lecture notes, a student had offered the definition of maximum and
another student had offered the definition of supremum; both thinking of offering the
definition of upper bound. The students then faced difficulties in producing the
negation of the definition of upper bound. In our discussion, Zenobia’s reflection on
students’ difficulties was the following.

[R]earranging quantifiers is tricky, a lot of students struggle with that and you
know I have a looong experience of teaching calculus and I am seeing students
struggle with this definition [of upper bound] and there is also the definition of
limit.

Excerpt 10_Discussion after SGT2

In SGT5, the tasks were “Show that \( \lim_{(x,y) \to (a,b)} 3y + 1 = 3b + 1 \)” and “Show that
\( \lim_{(x,y) \to (0,0)} \frac{x^2+y^2+3}{4} = \frac{3}{4} \). My interpretation is that, in SGT5, she used students’
difficulties in the \( \varepsilon - \delta \) definition of limit in multivariable functions (known from her
teaching experience and from students’ low results in coursework) as a tool for
‘selecting tasks’. I draw this interpretation on Excerpt 10, where she declared her
teaching experience and on Excerpt 11, below, where she informed the students about
their coursework results.

[W]hat I mostly noticed was difficulty in using definiti
ons. I mean, that’s sort
of one of these things that’s standard for students to have a lot of difficulty
with. It’s very normal to struggle with it. And part of the reason of having a
coursework that’s not worth a whole lot, where you can fail at it – which isn’t
what you should do – is so that when it comes to the exam, you succeed.

Excerpt 11_SGT5 Observation

Below is Table 4.2 with all tasks Zenobia and the students worked on in the 8 SGTs I
analysed for the main study. I distinguish these tasks into: proof tasks (e.g.
“determine”, “prove that”, “show that”), procedural tasks (e.g. “find”, “sketch”), and
conceptual tasks (e.g. “state” the definition of a concept). I distinguish the tasks based
on my interpretation of the nature of their solution. So, the solution of procedural tasks is a taught procedure. For instance, in Episode 1, I consider that finding the partial derivatives with the chain rule and the auxiliary diagrams is a taught procedure. The solution of a conceptual task is to draw on the definition of a concept, to interpret a graph or to make a conjecture for a property. For instance, in Episode 2, Zenobia and the students draw on visual imagery for injectivity to make a conjecture about monotonicity. Finally, I interpret proof tasks to be tasks which solution is a proof, such as the task “Prove that: If $s_n$ converges to $l$, then every subsequence of $s_n$ converges to $l$.” (SGT8).

Table 4.2: The tasks Zenobia used in the eight tutorials.

<table>
<thead>
<tr>
<th>SGT#</th>
<th>Conceptual tasks</th>
<th>Procedural tasks</th>
<th>Proof tasks</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>SGT1</td>
<td>-</td>
<td>-</td>
<td>“Determine whether the sets of polynomials are vector spaces” (3 sets of polynomials, thus 3 tasks)</td>
<td>Linear algebra</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“Prove whether the sets of polynomials are vector spaces, by showing that they are subspaces.” (2 tasks)</td>
<td></td>
</tr>
<tr>
<td>SGT2</td>
<td>-</td>
<td>-</td>
<td>“Determine whether the sequence $n^2 (-1)^n$ is bounded or unbounded.” (1 task)</td>
<td>Calculus</td>
</tr>
<tr>
<td>SGT3</td>
<td>“State the natural domain and the natural image by looking at the contour map” (2 tasks)</td>
<td>“Sketch the contour map” (2 tasks)</td>
<td>-</td>
<td>Calculus</td>
</tr>
<tr>
<td>SGT5</td>
<td>-</td>
<td>-</td>
<td>“Show that $\lim_{(x,y)\to(a,b)} 3y + 1 = 3b + 1$.” (1 task)</td>
<td>Calculus</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + 3}{4} = \frac{3}{4}$.” (1 task)</td>
<td></td>
</tr>
<tr>
<td>SGT6</td>
<td>“Find the limit.” (1 task)</td>
<td>“Is the function continuous?” (1 task)</td>
<td>-</td>
<td>Calculus</td>
</tr>
</tbody>
</table>
In the 8 SGTs I analysed for my main study, Zenobia used 25 tasks in total, from which: \( \frac{11}{25} = 44\% \) were proof tasks (SGT1, SGT2, SGT5, SGT8, SGT9 & SGT10); \( \frac{4}{25} = 16\% \) were conceptual tasks (SGT3 & SGT10); and \( \frac{10}{25} = 40\% \) were procedural tasks (SGT3, SGT6 & SGT10). The topics were in calculus and linear algebra; the ‘Calculus’ module had not been reformed into an ‘Analysis’ module yet. Although the number of proof tasks and procedural tasks is almost the same, the tutorial group worked on proof tasks in 5 of 8 tutorials and on procedural tasks in 3 of 8 tutorials. Notably, proof tasks were usually one to two per tutorial whereas procedural tasks were more per tutorial. This explains why the number of procedural tasks is high compared to proof tasks. My interpretation from the tutorial time that Zenobia and the students devoted for proof tasks is that the focus in tutorials was on proof tasks. It also seems to me that Zenobia’s thinking behind this focus was revealed in my observation of SGT2, when she told the students: “[M]ost of the proofs that you will see this year are of the sort where you’re trying to prove that something satisfies a certain definition.” Indeed, all 11 proof tasks in SGTs were about proving with the use of certain definition(s).
Table 4.2 also demonstrates a focus on analysis (SGT2, SGT3, SGT5, SGT6, SGT8 & SGT10) rather than on linear algebra (SGT1 & SGT9). Considering that my interpretations of the tools for ‘selecting tasks’ in Zenobia’s teaching are positive reinforcement; students’ suggestions; and students’ difficulties from teaching experience, it seems to me that students did not face so many difficulties with linear algebra compared to analysis. Moreover, the following Excerpt 12 from discussion with Zenobia after SGT2 might be illuminative for the tutorial’s focus on analysis, since Zenobia informs me about her teaching experience with students’ difficulties in linear algebra.

[T]he experience I had as a student, I mean, in a lot of situations I really had to learn what it is that students get hung up on because to me especially linear algebra was just obvious. I mean it was simple for me. The first time I saw it I was ‘well of course’, I totally didn’t understand what people were struggling with. And so it took me a while, a sort of analysing, thinking about what students will be getting hung up with in working with students one to one and trying to identify what is it they are getting confused.

Excerpt 12_Discussion after SGT2

In this excerpt, Zenobia explains to me that from her experience as a student she did not come up with difficulties in linear algebra. So later, she had to develop her learning of students’ difficulties; and she did so through analysis of her teaching experience with students who struggled with linear algebra. My interpretation nevertheless is that she did not influence students to focus on analysis in the main study, because the students were the ones who suggested topics and difficulties in the beginning of the tutorials.

4.2.2.2 Creating students’ positive feelings

In the section of this chapter Zenobia’s views on small group tutorials and her role as a tutor (Section 4.1.1.1), my interpretation was that her views of the SGTs include pastoral care for students; and students who “speak” in the tutorial. I also interpreted aspects of her role as a tutor in such a tutorial including the following two: to check that the students’ welfare is OK every week; and to break down the barrier of the
member of staff thus to be on the students’ side. The strategy ‘creating students’ positive feelings’ and its associated tools are consistent with these views from Zenobia on small group tutorials and her role as a tutor. They indeed illuminate what Zenobia does in her tutorials to fulfill her goal that the students should be comfortable enough in that group, feel safe in that group, not mind admitting what they know, what they don’t know, what they want help with [Discussion about SGT1 and SGT8].

As mentioned in the previous section Selecting tasks (Section 4.2.2.1), Zenobia started her tutorials by providing the students with 5 to 15 minutes for pastoral discussion and selection of tasks to tackle. She promoted that few-minute discussion with pastoral questions to students, which is one of the tools for her strategy ‘creating students’ positive feelings’. Examples of pastoral questions in data are:

“How are you guys doing? You seem pretty chipper-ish.” (SGT1);
“Good luck with your interview. When’s your interview?” (SGT2);
“Hey, how did your thing [interview] go?” (SGT3);
“Grammar and Punctuation workshop. Would you like to go?” (SGT6);
“I am worried about St3. Does anybody have her number? […] She won’t be sleeping, will she?” (SGT10).

In all these coded questions, Zenobia asks the students about other aspects of their university lives than those related to studying mathematics; thereby demonstrating her care about them.

Apart from pastoral questions at the beginning of the SGT, Zenobia used the remaining tools for the strategy ‘creating students’ positive feelings’ (Table 4.1) during the whole tutorial time. Humour, or else a form of levity to lift the level of solemnity, was one of her distinctive tools compared to the other two cases of teaching in the main study. For instance, in Excerpt 8 from SGT1, when she asks the students what is “scary” in the task they suggested, she also tells them with a funny facial expression swiping her eyebrows “I couldn’t read it before, I am going blind”. Furthermore, in Episode 2 from SGT8, she tells St2 to give the whiteboard pen to another student in order for the new student to come up to the board, by asking with a funny facial expression: “Would you like to give the magic pen to the next victim?” My interpretation was that Zenobia’s question included a degree of levity/irony, and I
thought of that degree of levity as Zenobia’s humour. In my observations, gradually, the students started to act with levity, as well. So, for instance, in SGT2 the following discussion takes place.

Zenobia: I brought my computer today. Isn’t that good?
St: That’s cool.
Zenobia: Yeah. I’m so prepared.
St: It’s only taken you a couple of weeks of the semester, but now you have done it. You’re all prepared.

Excerpt 13_SGT2 Observation

In SGT2, Zenobia brought her computer to access problem sheets and online tests, and to look at tasks there for the tutorial. In Excerpt 13, she acted with a form of levity to tell the students that she was “so prepared” that she brought her computer. It seemed to me that Zenobia often acted with a form of levity in the tutorial and that enabled the students to answer back with a form of levity, as they did in Excerpt 13; thereby breaking “the barrier of the member of staff” with her humour.

The analysis of data indicated that another tool for the strategy ‘creating students’ positive feelings’ was the eureka moment. This moment can be experienced with euphoria by a student who discovers the solution of a task, or a researcher who discovers a proof or an idea in mathematical research. Episode 2 from SGT8 offers observational data in relation to the eureka moment in Zenobia’s tutorial. In this episode, when St4 is at the board and about to start producing the rigorous proof of Figure 4.8, Zenobia informs him that “You’re going to have a eureka moment in just a second. You’re going to love it.” Here, Zenobia’s statement “You’re going to love it.” informs the student about a positive feeling associated with the eureka moment which he may experience. It seems to me that Zenobia informed St4 about the eureka moment because she knew he was a high-performing student able to discover the proof. Indeed, St4 wrote the proof on the board (Figure 4.8), and achieved more than 90% as an overall grade for the first year of his studies. In the next section of this chapter Decoding the mathematics and encoding the mathematics (Section 4.2.2.5), I analyse the steps Zenobia and the students followed for the discovery of the solution of a proof task.
In observational data, I also identified a number of Zenobia’s statements that valued students and their inputs in the tutorial. Examples of coded *valuing statements* in the data are:

“Yeah, perfect.”  (Zenobia’s response to St3’s correct input in Episode 1)

“Excellent. Good, good. Right.”  (Zenobia’s response to St8’s correct input in Episode 3)

“It’s closed on addition. You even know. You even know how to say it in a fancy way.”  (Zenobia’s response to St4’s correct input in SGT1)

“Forget this whole exceptional business. Alright. Good job. […] Keep up the good work.”  (Zenobia’s response to students’ inputs in SGT2)

“\(v\) equals \(w\). Excellent. Good job. Right.”  (Zenobia’s response to St5’s correct input in SGT9).

The commonality in all these coded *valuing statements* is the context of the dialogue between Zenobia and the students: the students offered input and Zenobia praised them for it. In particular, she used words such as “perfect”, “excellent”, “you even know” and “good job”.

My interpretation is that Zenobia’s statements which encouraged students to offer input in the tutorial is a last tool for the strategy ‘creating students’ positive feelings’. In her *encouraging statements*, Zenobia did not praise the students’ correct inputs but attempted to keep them engaged in the tutorial. An example of an *encouraging statement* is in Excerpt 11 from SGT5, where Zenobia informs the students about their low results in a coursework task regarding the \(\varepsilon-\delta\) definition of limit. In this excerpt, she tells the students that it is normal to struggle with the definition and they can fail in a coursework so that they succeed in exams. My interpretation is that this is an *encouraging statement* which has the potential to keep the students cognitively engaged in spite of their negative feelings about their low results.
Other examples of coded *encouraging statements* are in the three teaching episodes:

“So, df dr \( \frac{df}{dr} \) isn’t that bad. Right? So, what’s df dr?” (Episode 1);

“OK. That’s a good start. So, now we write the proof.” (Episode 2);

“Do you want to have a go at that? You’re very close. I know you can do it. Just draw a little red line on the domain axis.” (Episode 3).

In Episode 1, after St2’s request for an explanation of another way of solving the task “\( f(x, y) = \tan[4 \sqrt{(x^2 + y^2)} - 1] \). Find \( \frac{df}{dx} \) at \( (x, y) = (1, 2) \)”, Zenobia introduces Diagram 1 of Figure 4.1. She then states: “So, df dr \( \frac{df}{dr} \) isn’t that bad.” and asks “So, what’s df dr?” My interpretation from the context of Zenobia’s aforementioned statement and question is that her statement “So, df dr \( \frac{df}{dr} \) isn’t that bad.” encourages students to offer input about \( \frac{df}{dr} \). In Episode 2, before St4 starts to write the formal proof on the board, Zenobia states “OK. That’s a good start.” and then states “So, now we write the proof.” My interpretation is that the first statement encourages students to implement the second statement about writing the proof. I refer here to students and not only St4, because of Zenobia’s use of “we” in the second statement. In Episode 3, Zenobia makes an effort to bring St8 to the board so that he offers input for a domain of injectivity for the graph of \( f(x) = \sin(x) \).

Notably, a difference between the aforementioned *encouraging statements* and *valuing statements* is in the process of coding the data. I coded *valuing statements* in transcripts after students offered correct inputs. These statements were Zenobia’s response to students’ correct inputs in the form of praise. Thus Zenobia was reactive to students’ inputs with her use of *valuing statements*. In contrast, I coded *encouraging statements* before students’ inputs in transcripts. As illustrated in the examples of data analysis above, the next piece of transcript was important for judging whether a statement was *encouraging*, because it included Zenobia’s request for student’s input. So Zenobia was proactive to students’ input in the case of *encouraging statements*. 
4.2.2.3 Selecting examples

In this section, I analyse Zenobia’s strategy ‘selecting examples’. The associated tool to the strategy is a type of example that I identified in $\frac{12}{25} \approx 50\%$ of all tasks Zenobia used in SGTs, and in particular in SGT1, SGT5, SGT6 and SGT10. The type of example that I identified is the *generic set of examples*. I exemplify it here through analysis of Episode 1 and Episode 3.

In Episode 3, Zenobia offers four examples of functions, simultaneously, as a response to students’ difficulty to think of kinds of injective functions. They are the parabola, sin(x), logarithm and linear function (Figures 4.10-4.13). In our discussion for SGT10, Zenobia refers to the four examples as “standard” ones meaning that they “have many applications” and “are very special classes of functions; polynomial, trigonometric and logarithmic functions”. In response to my question about the reasons she chose them, she explains the different features that are illustrated for each class of functions in her four examples.

Everything you see in polynomials is already seen in these two functions [i.e. the parabola, even degree, and the linear function, odd degree] so adding any additional polynomial you don’t get anything new, whereas you never see periodicity or natural domain less than a whole axis in polynomials.

Excerpt 14_Discussion for SGT10

Despite each function in Zenobia’s examples coming from a different class of functions, they nevertheless all have the property that they are injective on certain intervals in their domain. The logarithm function has non-zero curvature and is injective on its domain, which is not the whole $\mathbb{R}$ (Figure 4.12); the linear function is injective on its domain, which is the whole $\mathbb{R}$ (Figure 4.13); the parabola is injective on intervals of its domain (Figure 4.10); and the trigonometric function is periodic as well as injective on intervals of its domain (Figure 4.11). Furthermore, the linear function along with the parabola fit in the class of polynomial functions, and the linear function is odd whereas the parabola is even.
With the range of functions from different classes, Zenobia introduces layers of generality of monotonicity on an interval so that students connect monotonicity on intervals with injectivity. The layers of generality concern the different features that are illustrated for each class of functions in Zenobia’s examples. The layers also concern common properties: all of the functions are injective on intervals, all of the functions are monotonically increasing on intervals, and others are monotonically decreasing on intervals. For instance, the logarithm function of Figure 4.12 and the parabola restricted on \([0, \infty]\) are monotonically increasing functions; in contrast, the logarithm function \(f(x) = \log_a(x)\) where \(0 < a < 1\) and the trigonometric function restricted on \([\pi/2, 3\pi/2]\) are monotonically decreasing functions. Zenobia’s attempt to enable the students to connect monotonicity on intervals with injectivity becomes successful through students’ observation of the common properties in her set of examples. So, different students find domains of injectivity for the set of functions and St2 observes that all four injective functions are strictly monotonic.

The difference between Zenobia’s examples and Mason and Pimm’s (1984) generic examples is that she uses a range of examples instead of a single example that carries the genericity within it. Zenobia’s need for genericity across examples, i.e. the presentation of injectivity and monotonicity across classes of functions, is necessitated by the complexity of the mathematical concepts being taught, which are injectivity and monotonicity, and the complexity of the mathematical context of functions. So, Zenobia’s functions form a generic set of examples (Mali, 2014) rather than distinct generic examples of injectivity and monotonicity on intervals for each function class.

Finally, all Zenobia’s functions are injective and monotonic on an interval; however, they should have a level of generality about them in order not to be prototypical examples. Prototypical examples are particular members of a class of objects, but their features are so specific that cannot characterise the class. (Prototypical example is a term presented in the glossary in Appendix D.) My interpretation is that the linear function is a prototypical example of injectivity and of monotonicity on intervals, because not all injective or strictly monotonic functions are linear. In contrast, the logarithmic function carries more generality of monotonicity on the domain than the linear function, because it has not null curvature and its domain is not the whole \(\mathbb{R}\).
Zenobia’s generic set of the four functions, which are injective and monotonic on intervals, is consistent with her epistemology of mathematics. In Excerpt 6, Zenobia refers to her view of the history of mathematics where “it’s not that people have had the idea of a function. It’s that they’ve had lots of examples of functions and they’ve tried to distil what the critical characteristics of a function are.” In this excerpt, in particular, she makes the connection of historical and contemporary mathematicians (including herself) by saying that “– from the set of examples that we have – we’ve come up with an ideal idea” by recognising “that there are some sets of structures that have some relevance”. Zenobia’s generic set of examples with the four functions indeed is a set of “structures that have some relevance”. The relevance is the “critical characteristics” that all four functions are injective and monotonic on intervals. By observing this relevance, Zenobia and the students distil “what the critical characteristics of” the set of the four functions are and “come up with an ideal idea”. The idea is that “either monotonically increasing or monotonically decreasing is automatically going to be injective” [Episode 3].

Zenobia’s use of the generic set in Episode 3 is connected with mathematician’s research for the discovery of an “ideal idea”. So, its role is to discover the property “a strictly monotonic function is injective”. Polya (1971) attributed the role of discovery to the heuristic induction: “Induction is the process of discovering general laws by the observation and combination of particular instances.” (p.114). In Zenobia’s generic set in Episode 3, the particular instances are the four examples and the general law is the discovered property. Considering my interpretation of Zenobia’s goal to provide students with heuristics [Excerpt 7], Zenobia’s goal in selecting the generic set of monotonicity on intervals seems to be to provide students with the heuristic induction. Also considering my interpretation of her views of what making sense of mathematics is and its connection with the use of examples (see Part 1 of this chapter), another of Zenobia’s goals in selecting the generic set seems to be that ‘the students should make sense of the mathematics’.

In Episode 1, Zenobia offers, not simultaneously, four examples of applying the chain rule in multivariable functions, as a response to students’ difficulty with variable dependencies in the chain rule. The students’ difficulty is expressed by St5’s semi-articulated question to Zenobia “Is $df$ by $dx \left[\frac{\partial f}{\partial x}\right]$ the down thing where you just start
writing $f$ with a small”; St2’s question to Zenobia about another way of finding a partial derivative than substitution; and by St3’s statement twice about her confusion with partial derivatives.

Zenobia’s four examples of applying the chain rule in multivariable functions are the cases illustrated in Figures 4.1-4.4. Figure 4.1 illustrates a case of $r$-substitution and Figures 4.2-4.4 do not illustrate cases of substitution. Also, Figure 4.3 illustrates a more complicated case than Figures 4.2 and 4.4.

- Figure 4.1 illustrates a case where variable $z$ depends on variable $r$; which depends on variables $x$ and $y$.
- Figure 4.2 illustrates a case where variable $z$ depends directly on variables $x$ and $y$; both of which depend on variable $t$.
- Figure 4.4 illustrates a case where variable $z$ again depends directly on variables $x$ and $y$; but here only variable $y$ depends on variable $t$.
- Figure 4.3 illustrates a case where variable $z$ again depends directly on variables $x$ and $y$, both of which depend on variable $t$; but here variable $y$ also depends on variable $s$.

By applying the chain rule in each of the four cases, students have the potential to become aware of a range of variable dependencies for the chain rule. Also, more cases of variable dependencies can be generated as a combination of Zenobia’s four cases. What makes Zenobia’s four cases to be a generic set of examples of variable dependencies for the chain rule is that Zenobia’s cases form the patterns that will be repeated in the aforementioned combinations for more cases of variable dependencies. Considering literature in generic examples, the application of the chain rule is a procedure. My interpretation is that the difference between Zenobia’s four examples and Rowland’s (2002) generic examples is that she uses a range of examples of variable dependencies for the chain rule instead of a single example that carries the genericity within it. It also seems to me that Zenobia’s need for genericity across examples rather than within a distinct example is necessitated by the complexity of the chain rule, and the complexity of the mathematical context of multivariable functions.
In our discussion after SGT6, Zenobia’s comment on Episode 1, which was towards the end of the tutorial, was the following.

That thing at the end with the diagrams, I think that was something that maybe had barely been touched on, but this is really really useful sort of just notational tool for remembering how to do things.

Excerpt 15_Discussion after SGT6

Excerpt 15 includes two important aspects of Zenobia’s thinking about Episode 1. First, she shares with me her reflection on the information she got from St3 and St1 about the lectures, where the lecturer drew only “one of the little diagrams at the top of one of the pages” [Episode 1] despite being “really really useful” [Excerpt 15] in Zenobia’s view. So, in Episode 1, Zenobia offered that “really really useful” to the students. My interpretation is that the “really really useful” for Zenobia is Polya’s (1971) heuristic: ‘Draw a figure.[/diagram].’ This is because, in Episode 1, Zenobia drew diagrams to enable the students to demonstrate the variable dependencies for the chain rule. However, through the diagrams, Zenobia also selected a generic set of examples of variable dependencies for the chain rule. It seems to me that two of Zenobia’s goals for selecting this generic set were ‘to provide students with the heuristic ‘Draw a figure.’” and ‘the students’ sense making of the mathematics’.

A second aspect of Zenobia’s thinking in Excerpt 15 is the following: Zenobia characterises the generic set of examples of variable dependencies for the chain rule as a “notational tool for remembering how to do”. The application of the chain rule indeed is a procedure rather than a concept such as injectivity or monotonicity. So the set of examples of Episode 1 changes in nature from the set of examples of Episode 3. In particular, in Episode 3, the set of examples was a conceptual construct: a set of graphs of functions that belong in different classes of functions but all nevertheless have the property of monotonicity on intervals. In Episode 1, in contrast, it is a procedural construct: a set of diagrams which demonstrate different dependencies of variables; and thus certain procedures which are the applications of the chain rule. My interpretation is that, in Excerpt 15, Zenobia comments on the procedural nature of this generic set of examples by characterising it as a “notational tool for remembering
how to do”. Thus, according to Zenobia, the role of this generic set is “to remember how to do” the chain rule.

4.2.2.4 Evaluating the students’ sense making of mathematics

Analysis of the main study indicates that Zenobia used tools with the goals to interpret and to evaluate students’ sense making of mathematics in tutorials. These are questions to students, pause intervals after questions to students and students’ reinforcement. (Positive reinforcement is a code discussed in Chapter 3: Section 3.4.2. Reinforcement here can be either positive or negative.) In this section, I provide a synthesis of observations and discussions with Zenobia to exemplify these tools.

In observational data, I identified a number of Zenobia’s questions to students to evaluate their mathematical sense making. These questions were in the form of control questions (Viiroman, 2015, p.1175) and inviting questions to students; with the latter being direct to a student or general to all students (Jaworski & Didis, 2014, p.380). Examples of coded control questions and inviting questions: general or direct, are the following from Episode 1. The reader should read the first column with control questions, and then the second one with inviting questions.

| Table 4.3: Coded control questions and inviting questions in Episode 1. |
|------------------------------------------|------------------------------------------|
| **Control questions** | **Inviting questions: general or direct** |
| “That’s just two different ways of writing the same thing. That's OK? [4” pause.] Is that OK?” [St5 responds “Yeah.”] | “Zenobia asks the students how they do that task.” [General question: “St5 responds substitution, and St3 suggests the chain rule.”] |
| “So, you know that the chain rule says that df dx will be df dr times partial r partial x, yeah?” [St2 responds “Yes.”] | “So, what’s df dr?” [6” pause. General question: St1 responds “4sec squared, 4r minus 1.”] |
| “So, df dr \( \frac{dz}{dt} \) isn’t that bad. Right?” [4” pause. Students provide reinforcement.] | “What’s this edge going to be [between the vertices z,y in Figure 4.2]?” [General question: St1 and St3 respond.] |
| “Zenobia asks the remaining students if it makes sense.” [“St3 says she is confused with the partial derivative.”] | “So, what’s dz, dt \( \frac{dz}{dt} \) going to be?” [3” pause. General question: St3 responds.] |
| “Are you guys with me?” “Does that make sense?” [Students provide reinforcement.] | “So, what’s dz, dw \( \frac{dz}{dw} \)?” [7” pause. General question: No one responds.] |
In my data analysis, the context in which Zenobia used control questions corresponds to Viirman’s description of the use of control questions

“when a particularly important or complicated piece of mathematics has been presented” (2015, p.1175).

Viirman nevertheless made this description in the lecture setting. In the tutorial setting of my study, Zenobia’s students came in the class to resolve difficulties, so the piece of mathematics under tutorial discussion was usually complicated for them. Furthermore, as mentioned in section Selecting tasks of this chapter (Section 4.2.2.1), one of Zenobia’s goals for SGTs was to enable students to make sense of fundamental topics [Excerpt 9]. Thus the piece of mathematics under tutorial discussion was important as being a fundamental topic in Zenobia’s view. However, Zenobia did not present the mathematics in SGTs. Rather, she used dialogue with the students to elicit the mathematics and to evaluate the students’ meaning in a process of redesigning the teaching in order to resolve the difficulties.

In the data, Zenobia used control questions to check each step in the process of solving a task, which in Episode 1 resulted into a constant use of control questions. These questions were: “Are you guys with me?” “Does that make sense?” or they ended with “That’s OK?” “yeah?” “Right?” The students usually responded to her control questions by offering reinforcement in the form of a nod, a face expression, a “yes” or their difficulty. When they nodded, expressed or articulated a positive response, meaning that they have made sense, I coded that response as positive reinforcement.

Jaworski and Didis (2014) stressed that the role of inviting questions is to seek students’ articulation of mathematical meaning. Thus, inviting questions reveal students’ difficulties and students’ meaning through students’ expression in their responses. In my analysis of data, the students had to articulate the mathematics in response to Zenobia’s questions. I used the code inviting question when my interpretation of the role of Zenobia’s question was to seek students’ articulation of mathematics and to evaluate their sense making. In Episode 1, Zenobia’s inviting questions ask the students how to start the first task, which corresponds to the case of Figure 4.1, or to make calculations for partial derivatives. Their role is to seek and
evaluate students’ articulation of the application of the chain rule. The task of Episode 1 is coded as procedural task, so the inviting questions are expected to seek students’ sense making of calculations. Also, although all Zenobia’s inviting questions are general, there are students who respond. This indicates that the students are socialised into responding to Zenobia’s questions. In a discussion with Zenobia after the data collection, she said that all students achieved a pass in exams with some of them achieving good marks.

Zenobia’s inviting questions nevertheless require a particular answer from the students. For instance, in Table 4.3 the second question “So, what’s df dr?” requires the particular answer “$4 \text{ sec}^2$, $4r$ minus 1.” In personal communication with Jaworski, I characterised inviting questions which require a particular answer from the students as prompting questions. My interpretation is that all Zenobia inviting questions in Table 4.3 are prompting questions except for the first one. The first question, which is about how the students do a task, can be characterised as an open question, because it does not require a particular answer. So, two students offered two different but correct inputs.

The following Table 4.4 includes the coded control question and inviting question from my analysis of Episode 2. The coded inviting question in Table 4.4 can also be characterised as a prompting question, because it requires a particular answer in relation to the definition of sequence convergence.

<table>
<thead>
<tr>
<th>Control question</th>
<th>Inviting question: general or direct</th>
</tr>
</thead>
<tbody>
<tr>
<td>“It would have to be bigger, right?” [St3 responds “Yeah.”]</td>
<td>“OK, if epsilon is greater than 0, what do we know is true?” [4” pause. Direct: St4 on the board starts writing the proof.]</td>
</tr>
</tbody>
</table>

In Table 4.4, the control question ends with “right?” and its response is coded in the episode as positive reinforcement. The task of Episode 2 is coded as proof task, so the inviting question is expected to ask St4 about the proof. This question is direct to St4, who is to the board. In Episode 2, the coded control and inviting questions are rather few; specifically, they are one of each kind. Considering Zenobia’s questions in this
episode, the majority of them require a particular heuristic as a response. My interpretation of Zenobia’s acting with few questions to evaluate students’ sense making and more questions on heuristics is that her goal in Episode 2 is to enable students to make sense of heuristics [Excerpt 7].

My analysis of Episode 3 resulted into a number of coded control questions and inviting questions, which are included in the following Table 4.5.

<table>
<thead>
<tr>
<th>Control questions</th>
<th>Inviting questions: general or direct</th>
</tr>
</thead>
<tbody>
<tr>
<td>“This is definitely not injective on the whole thing, right?”</td>
<td>“Are there any kinds of functions that you know are going to be injective, for instance? Is there anything about a function that you…” [2” pause. General questions: No one responds.]</td>
</tr>
<tr>
<td>Because if I go off in opposite directions, I’m going to the same thing, right? OK. But if I go from here on, that’s injective, right? OK.” [Zenobia says “OK,” after receiving reinforcement, which is possibly positive from students.]</td>
<td>“So, if you wanted to determine some domains on which all of these are injective, how would you do it? How would you do it for this one? [Zenobia points to the graph of ( f(x) = x^2 ). How would you find your domain of injectivity? Is it injective on anything?” [General questions: St5 responds “From 0 to ( \infty ).”]</td>
</tr>
<tr>
<td>“And what about down here? [Zenobia shows the graph of ( f(x) = \sin(x) ).] [5” pause.] Do you want to have a go at that? [Zenobia looks at St8.]”, [Direct questions: St8 responds.]</td>
<td>“So, what can I say about… OK, what about this function? [Zenobia shows the graph of ( f(x) = \ln(x) ).] Is this injective? Is this an injective function?” “What about this one? [Zenobia shows the graph of ( f(x) = x ).]” [General questions: St3 responds both questions.]</td>
</tr>
</tbody>
</table>

The control questions end with “right?” in this episode, as well. After the last two control questions, Zenobia says “OK”, which indicates students’ response in a form of reinforcement, possibly a positive one. The inviting questions ask the students about injective functions and domains of injectivity. Their role is to seek and evaluate students’ sense making of injectivity from their responses. Most inviting questions are general except for the direct questions to St8.

Considering the pause intervals after Zenobia’s questions in Tables 4.3-4.5, control questions are followed by pauses which last for up to 4 seconds. These 4 seconds usually suffice for the students’ responses which are in the form of reinforcement; a nod, a face expression, a “yes” or a difficulty. In contrast, the pause intervals after
Zenobia’s *inviting questions* usually last for more than 4 seconds and up to 7 seconds. The larger intervals after *inviting questions* are expected because responses to this kind of questions usually require some mathematics. So, the pauses need to be larger than 4 seconds to enable the students to think and respond.

### 4.2.2.5 Decoding the mathematics and encoding the mathematics

In this section, I exemplify the strategy ‘decoding the mathematics and encoding the mathematics’ along with its associated tools. This is a strategy with a number of tools; for this reason, the structure of this section includes subsections. The conceptual name for the strategy comes from my discussions with Zenobia, where she reflected on her teaching practice and she related it to the research mathematicians’ practice of “decoding and encoding”. Drawing on her research practice, she explained:

> The first step [in doing research] is the decoding where you are given a problem and you have to understand what the problem is, what everything means [e.g. by experimenting with images against definition], why it is a problem; the second step is with this picture that you have got from the decoding process, you get some intuition, you play around with things in your head a little bit and then you get this sort of ‘aha I figured it out, I have got this idea now of why that works’ and then you have got the encoding process [i.e. the third step] where you write it down [formally]. […] [In Episode 3,] through examples I tried to extract from the complicated language that core intuition [of step 2]. I tried to teach them to decode the problem to something where they can sort of see ‘oh of course that’s how it works’ and then figure out how to write it in their proof back into a formal language. […] I wanted to explain to them what it is to be a mathematician. […] It is important for students to learn this [encoding and decoding] process because that’s a lot of the process of doing mathematics. And I think that’s a lot of what mathematicians do on a daily basis.

Excerpt 16_Discussion for SGT10
In Excerpt 16, I coded three research steps declared by Zenobia:

- the step of “decoding” with declared research heuristics “understand what the problem is”, “understand what everything means [e.g. by experimenting with images against definition]” and “understand why it is a problem”;
- the step of “intuition” with declared research heuristic “play around with things in your head a little bit”; and
- the step of “encoding”, where you write it down [formally]. (“It” here refers to “intuition”.)

Excerpt 16 demonstrates Zenobia’s use of the same names for her three research steps and her three teaching steps in Episode 3; these are “decoding”, “intuition”, and “encoding”. I used the conceptual name ‘decoding the mathematics and encoding the mathematics’ to denote her strategy to teach the three steps. In observations, Zenobia taught the three steps in tutorial work on proof tasks (e.g. Episode 2 from SGT8, also SGT1, SGT2). Thus in analysis I considered proof tasks to be tools for this strategy.

Excerpt 16 also provides two of Zenobia’s goals in teaching these three steps. My interpretation from Excerpt 16 is that one of Zenobia’s goals is ‘to enculturate students into the process of doing mathematics in the community of mathematicians’. In Excerpt 3 of this chapter, Zenobia also declared that the way she works with the mathematics in her research to enable herself to make sense of a new concept has informed the way she works with the mathematics in her teaching to enable the students to make mathematical sense. It seems to me that the strategy ‘decoding the mathematics and encoding the mathematics’ is a way of working with the mathematics and another of Zenobia’s goals is ‘to enable the students to make mathematical sense’.

In Excerpt 16, Zenobia refers to “intuition” as “this sort of ‘aha I figured it out, I have got this idea now of why that works’”. My interpretation is that intuition for Zenobia is about ‘figuring out the idea’, which seems to happen at the “aha moment”. In our discussion for SGT10, I enquired what intuition and the intuition step are. Zenobia referred to intuition as a “sense of how things work”, or synonymously, a “main core idea” of a proof. She then juxtaposed intuition with the encoding process of a proof by saying that in research “instead of writing down this clear idea, you have to write
down this messy technicality and special formal definitions”. Thus intuition, which is the “main core idea” of a proof, does not include the formal write-up. Rather, the formal write-up is the encoding of the mathematics. Zenobia also exemplified intuition with a reference to research papers where there is “a core basic idea” understood by the authors “but in order to transmit it they have to put it in this complicated language”.

In our discussion for SGT10, Zenobia stressed that “the decoding is where you are reading and encoding is where you are writing”. I asked Zenobia how the intuition step differs from the decoding of the mathematics. She responded that the decoding process is about understanding what the mathematics means in the problem (e.g. with images against definition). She then stressed that the intuition step comes after understanding and is about figuring out the “main core idea” of a proof. In other words, I would say that the intuition step is about getting insight into the proof from the decoding process. In order for me to make sense of her response, she provided me with an example of the three steps (decoding, intuition, encoding) for the mathematical problem of showing that every monotonically increasing function is injective.

There are two different things you have to decode, monotonically increasing and injective. But having decoded both of those you still have to put them together somehow, make that link. So the decoding process would be understanding what’s meant by monotonically increasing and understanding what’s meant by injective. Then maybe just drawing some pictures of monotonically increasing functions maybe [you understand that] that is sort of the image of monotonically increasing and maybe [you understand that] injective is that horizontal line test. Then you put those together and you are like ‘oh yeah of course!’ Then you have to think ‘OK so why is it that?’ Then you have to formally encode that proof.

Excerpt 17_Discussion for SGT10

In Zenobia’s example in Excerpt 17, the decoding process is about understanding the image of monotonically increasing and the horizontal line test of injective. Then, the
intuition step is about figuring out the link between the monotonically increasing and injective. Finally, the encoding process is about formally writing why the link is true.

In Episode 3, the task is the same as the one in Excerpt 17. What Zenobia does is to select the generic set of examples of monotonicity on intervals (the graphs in Figures 4.10-4.13) (strategy 1), and to evaluate three students’ sense making of injectivity by asking them to determine domains of injectivity on the graphs (strategy 2). My interpretation is that the two strategies are in the decoding process and concern students’ sense making of injectivity and monotonicity.

4.2.2.5.1 Tools: Injunction questions about a heuristic, questions to observe and pause intervals

In Episode 3, after coding Zenobia’s strategies ‘selecting examples’ and ‘evaluating students’ sense making of mathematics’, I coded two of Zenobia’s questions to observe monotonicity:

“So, what can you say about this part of this function, this part of this function, this function and this function? [Zenobia shows the previous functions restricted on the domain of injectivity.] What do they all have in common?”

My interpretation is that Zenobia’s questions to observe belong in the intuition step. This is because in these questions, she asks students to “put the graphs together” and to “figure out” the commonality; that is to say, to observe the link between monotonicity and injectivity. St2 offers correct dialogue input as a response to Zenobia’s questions; he says “They’re monotonically increasing.” In this way, he provides evidence of demonstrating his intuition about the link between monotonicity and injectivity.

I considered questions to observe, injunction questions about a heuristic and pause intervals after questions to be tools for Zenobia’s strategy ‘decoding the mathematics and encoding the mathematics’. An exemplification of coded questions is in Episode 2. In this episode, the proof task is “If \( s_n \) converges to \( l \), then every subsequence of \( s_n \) also converges to \( l \)” The definition of convergence is on the board
“A sequence $s_n$ converges to $l$ if $\forall \varepsilon > 0 \exists k_0 \in \mathbb{R}$ s.t. $|s_k - l| < \varepsilon$ $\forall k > k_0$.”

Zenobia asks St3 to sketch a convergent sequence on the board (question 1), and then she asks all students to put values of $s_n$, $\varepsilon$, and $k_0$ on the graph in Figure 4.7 (question 2). I coded both Zenobia’s questions as *injunction questions about heuristics*. In both questions, the *heuristics* are ‘sketch graph’ and ‘consider specific case’. The injunction is for the students to sketch a graph and to put values on it. The graph is in Figure 4.7. It is a specific case of the definition of convergence, because it has a specific selection of values of $s_n$, $\varepsilon$, and $k_0$. My interpretation is that those *injunction questions about heuristics* are in the decoding process and concern students’ sense making of the definition of convergence.

Later in Episode 2, Zenobia makes a series of questions to students on the specific case on the board, in order for them to make sense that in the definition of convergence $k_0$ depends on epsilon. Indeed, she:

asks “So, if I gave you an epsilon, is it true that $s_k$ minus $l$ is less than epsilon for all $k$?” (question 3);
waits for seven seconds (pause interval 1);
expresses that her question was “For all $k$ greater than $k_0$.”;
asks “What if I make epsilon smaller?” (question 4) “Can you give me a littler epsilon?” (question 5);
waits for five seconds (pause interval 2); and
asks “Is it still true that $s_k$ is less than $l$ for all $k$ greater than $k_0$?” (question 6).

I coded Zenobia’s questions 3 and 6 as *questions to observe* that $k_0$ depends on epsilon. After question 3, Zenobia waits for a seven-second *pause interval* which is enough for a student to think and respond to a question; however, the students do not respond with their observation. Question 4 and 5 are *injunction questions about a heuristic*. The *heuristic* is ‘consider special cases’ of values for $\varepsilon$; however, in the data this heuristic is usually about calculating the first few terms (e.g. in a sequence). The injunction is for St3 to put a littler $\varepsilon$ on graph. My interpretation is that questions 3-6 along with the pause intervals are in the decoding process. St3 offers correct dialogue input as a response to Zenobia’s questions; she says “$k_0$ would have to be bigger than it” ($k_0$ would have to be bigger than the value of $k_0$ before decreasing the
size of $\varepsilon$). In this way, St3 provides evidence of making sense of the relation between $\varepsilon$ and $k_0$ in the definition of convergence.

4.2.2.5.2 Tools: Heuristics

In the implementation of the strategy ‘decoding the mathematics and encoding the mathematics’, Zenobia acted with various heuristics which I interpreted as tools for the strategy. In this section, for instance, I exemplified the heuristics ‘consider a specific case’ and ‘consider special cases’. The specific case was a graph for the definition of convergence in Episode 2. The special cases were for $\varepsilon$ on the aforementioned graph. Furthermore in the section Selecting examples (Section 4.2.2.3), I analysed the generic set of examples of monotonicity on intervals in Episode 3. In the case of Zenobia’s teaching, I interpreted this generic set as Polya’s (1971) heuristic ‘induction’. Additionally, in Episode 1, I regarded the generic set of examples of variable dependencies for the chain rule as Polya’s (1971) heuristic ‘draw a figure’. Considering that in the case of Zenobia’s teaching, the figures were graphs or diagrams sketched on the board, I used the conceptual name ‘sketch graph(s)/diagram(s)’ to denote this heuristic from Polya.

Episode 2 provides an example of the heuristic ‘types of proofs’. In this episode, the tutorial group needs to prove that $\forall \varepsilon > 0 \exists k_0 > 0 \text{ s.t. } m > k_0 \Rightarrow |r_m - l| < \varepsilon$. Zenobia points to $\exists k_0 > 0$ and tells students: “We need to show that this thing exists. And the best way to show that it exists is by giving some way to find it.” In this extract, she informs the students about a way of working with existence proofs: finding a value of $k_0$ to prove that $\exists k_0$. I coded this way of working as ‘existence proof’. Another way of working to prove existence is proof by contradiction.

The heuristic ‘know-want’ corresponds to Polya’s (1971) heuristic “What is the unknown? What are the data?” In Episode 2, Zenobia wrote on the board that they know the definition of convergence and the definition of subsequence and they need to prove that “$\forall \varepsilon > 0 \exists k_0 > 0 \text{ s.t. } m > k_0 \Rightarrow |r_m - l| < \varepsilon$”. An example of the heuristic ‘work section-formal write up’ is in the observational data of SGT5. The following excerpt is Zenobia’s explanation of the heuristic to the students.
As I was saying to St1, any time you do a limit proof, there’s sort of the work behind the scenes, but then there’s also the formal write-up. [Zenobia writes “work” for the work section on board:] So, you’re going to do a little work to figure out what numbers you need, what formula you need, and then using that formula you’re going to do the formal write up.

Excerpt 18_SGT5 Observation

In the section Selecting tasks (Section 4.2.2.1), my analysis indicated that the focus in Zenobia’s tutorials was proving with the use of certain definition(s). In tutorials, Zenobia asked students to discern what definitions they need to consider for the solution of a task. I coded these questions as injunction questions about a heuristic. The heuristic was ‘use definition(s)’ and the injunction was for students to offer input in tutorial discussion about those definitions. Sometimes, in tutorials, students could not offer input. Then, Zenobia asked them to find the definitions in their lecture notes. In SGT2, for instance, she asked: “Do you have your notes with you, anybody? Do you want to browse through the notes on here [the laptop]?” I coded such questions as injunction questions about a heuristic. The heuristic here was ‘find definition(s) in lecture notes’ and the injunction was to students. A Polya’s (1971) heuristic is ‘Go back to definitions’. I considered both conceptual names of heuristics to correspond to that heuristic from Polya. For Zenobia, it was important for the students to search in their lecture notes. In our discussion for SGT2, she stressed that the students “need to get the habit of finding the definition” and she “could end up giving a definition that is slightly different from the one of the lecture notes” thereby confusing the students.

In our discussion after SGT2, Zenobia informs me that there are heuristics from her research that she teaches to students in tutorials. Two examples of these heuristics are: ‘consider special cases’ (she said “calculate the first few terms”) and in ‘existence proofs’ that “it doesn’t have to be the best bound, it just has to be some bound”. For the second example, she stressed that “In my research which is Analysis, I am constantly having to find some bound, it’s what Analysis is all about and if I always try to find the best bound I will go nuts”. Indeed, the task in SGT2 was “Determine whether the sequence $n^2(-1)^n$ is bounded or unbounded.” After considering special
cases for the sequence $n^2(-1)^n$, Zenobia and the students worked in order to prove that an upper bound does not exist. During and after this work, Zenobia stated twice that the students do not have to find the best bound thus to struggle for a formula. Rather, they “just have to find a bound”.

4.2.2.5.3 Tools: Students at the board, and a student

Zenobia’s strategy ‘decoding the mathematics and encoding the mathematics’ captures her way of thinking and working with the mathematics in the tutorial. During the implementation of this strategy, the students are usually at the board. In this way, they contribute to the tutorial work by writing Zenobia’s or the fellow students’ contributions for the solution of tasks on the board. My interpretation is that students to board is a tool for the implementation of the strategy ‘decoding the mathematics and encoding the mathematics’. In Episode 2, for example, there are three students to the board in total. St3 is to the board for the definition of convergence and the definition of subsequence (decoding of the mathematics). Then, St1 is to the board for experimenting with the mathematics they need to prove: $\forall \epsilon > 0 \exists k_0 > 0 \text{ s.t. } m > k_0$ $\Rightarrow |r_m - l| < \epsilon$ (intuition step). Finally, St4 is to the board to find the relation between $k_0$ and $\hat{k}_0$ and write the proof (intuition step and encoding of the mathematics). In section Creating students’ positive feelings of this chapter (Section 4.2.2.2), I mentioned that St4 was a particular high-achieving student. After SGT8, I asked Zenobia why she selected him to write down the intuition that $f(\hat{k}_0) \geq k_0$ and the proof. She responded that the remaining tutorial time was limited thus she needed to have the task solved, and St4 was a student capable of offering the proof. My interpretation is that St4 is a tool for the implementation of the strategy ‘decoding the mathematics and encoding the mathematics’. This is because without St4 in the tutorial, the remaining students (e.g. St2) might not be able to complete the solution of the task thus Zenobia might need to offer it by herself.

Finally, in our discussion for SGT1 and SGT8, Zenobia stressed her views on the students’ benefits in a tutorial where the strategy ‘decoding the mathematics and encoding the mathematics’ is implemented: students’ critical evaluation of what they are doing; stronger links with mathematical ideas because of having the moment of discovery; and emphasis that in a large degree they are independent from the tutor.
My interpretation is that Zenobia’s feedback and the fellow students’ inputs in the tutorial dialogue enable the students to evaluate what they are offering. Furthermore, the moment of discovery, which happens in the intuition step, is connected with euphoria and is likely for students to remember the mathematical ideas. It also seems to me that the students indeed are independent from the tutor. This is because they learn various heuristics and the three steps for discovery. However, the strategy ‘decoding the mathematics and encoding the mathematics’ with such an in depth ploughing of mathematical concepts and meaning requires the tutor’s demanding management of the limited small group tutorial time.

4.2.2.6 Explaining

In findings, the tools with which Zenobia acted in her strategy ‘explaining’ are representations and rhetorical questions. In this section, I exemplify analysis and interpretations I made for Zenobia’s ‘explaining’ mainly through Episodes 1, 2 and 3.

Task 1 of Episode 1 is

\[ z = f(x, y) = \tan[4\sqrt{(x^2 + y^2)} - 1]. \text{ Find } \frac{\partial f}{\partial x} \text{ at } (x, y) = (1, 2). \]

Zenobia-students’ discussion for the solution starts with symbolic representations: substitution \([f(x, y) = \tan(4\sqrt{r^2 - 1}), r^2 = x^2 + y^2]\) and the chain rule \([\frac{\partial f}{\partial x} = \frac{df}{dr} \frac{\partial r}{\partial x}].\) They calculate \(\frac{\partial f}{\partial x}\) with a sequence of symbolic representations for the chain rule:

\[ \frac{df}{dr} = 4 \sec^2(4r - 1) \]

\[ \frac{\partial}{\partial x} (r^2) = \frac{\partial}{\partial x} (x^2 + y^2) \]

\[ 2r \frac{\partial r}{\partial x} = 2x \]

\[ \frac{\partial r}{\partial x} = \frac{x}{r} \]

After St2’s question about another solution, Zenobia introduces the diagram of Figure 4.1 (graphical representation) to offer the students a way to remember variable dependencies.
In my analysis, I considered *symbolic representations* to be the mathematics written on the board. I also considered *graphical representations* which were the diagrams and graphs either sketched on the board or represented by gestures. My interpretation was that both types of representations were tools for the strategy ‘explaining’. In data analysis, I nevertheless found Zenobia’s mathematical representations, which were neither symbolic nor graphical. These representations were in the form of *statements, formal or informal language*. I interpreted that their role was to explain the mathematics, but they were not a mere exposition of the mathematics. I coded them as *verbal representations* to indicate their relation to language. My interpretation was that those are also Zenobia’s tools for ‘explaining’.

For example, in Episode 1, after Zenobia introduces the diagram of Figure 4.1, she explains how to find $\frac{dz}{dx}$ by saying:

“‘So, if you want to find the derivative of $z$ with respect to $x$, you have to sum the product of partials over each path that goes from $z$ to $x$.’”

Also, after she introduces the diagram of Figure 4.2, she explains how to find $\frac{dz}{dt}$ by saying:

“‘And if I want to find the derivative of $z$ with respect to $t$, I again have to sum over each path the product of partials.’”

I coded Zenobia’s words in both extracts as two verbal representations (*formal language*), because she did not use exposition for the symbols on the board (e.g. “So, that would be $dz, dx$ times $dx, dt$ times $dt, dw \left[\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} \frac{\partial t}{\partial w}\right]$.”); and did not describe the diagrams (e.g. “In this case [Figure 4.2], we’ve got a situation where we’ve got $z, x, y, t$. So, $z$ is the function of both $x$ and $y$. And each $x$ and $y$ is the function of the $t$.”). Rather, what she did in the two aforementioned coded extracts was to explain in formal language the variable dependencies for the chain rule by talking about ‘the sum of paths’.

My interpretation is that Zenobia’s goal in ‘explaining’ with representations was ‘students should make sense of the mathematics’. After her verbal representation for
the diagram of Figure 4.2, St3 responds by producing the symbolic representation \[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\] indicating that she had made sense of the variable dependencies for the chain rule. Notably, St3 was one of the students who expressed difficulty with partial derivatives in the beginning of Episode 1.

In Episode 2, I coded the definition of convergence “A sequence \( s_n \) converges to \( l \) if \( \forall \varepsilon > 0 \exists k_0 \in \mathbb{R} \text{ s.t. } |s_k - l| < \varepsilon \forall k > k_0 \)” as a symbolic representation. After the students’ contribution to this definition, Zenobia asks them to put values of \( s_n, \varepsilon, \) and \( k_0 \) on the graphical representation in Figure 4.7. Concluding Zenobia and St3’s discussion about those values, Zenobia says “So, \( k_0 \) in particular is going to depend on epsilon.” I coded this extract as a consolidating statement for the relation between \( s_n \) and \( \varepsilon \) in the definition of convergence. The word ‘consolidation’ comes from Zenobia’s discussions with me where she used it to express the role of coded extracts such as the ones I exemplify here. In our discussions, she said that her words in such extracts were towards the consolidation of the students’ meaning in the mathematics of the tasks. Considering the negotiation of meanings in Zenobia’s discussions with the students about the mathematics, it seems to me that consolidating statements were crucial for the students’ mathematical meanings out of the discussions. My interpretation is that Zenobia’s words in the aforementioned extract was towards the consolidation of the students’ meaning of the relation between \( s_n \) and \( \varepsilon \) in the definition of convergence. In my analysis, consolidating statements were some of Zenobia’s tools for ‘explaining’ the mathematics.

Figure 4.8 of Episode 2 demonstrates St4’s proof of “If \( s_n \) converges to \( l \), then every subsequence of \( s_n \) also converges to \( l \).” I coded this proof as a symbolic representation written on the board in a way accepted by the community of mathematicians (e.g. aesthetics) and the institution, which was the university of my study (e.g. notation and particular definitions). Zenobia used the term ‘aesthetics’ when she characterised arguments in her tutorials. An observation excerpt from SGT2, which I coded as a Zenobia aesthetic statement made to students, is the following:

there’s also this idea – aesthetic idea – that you don’t want to bring in a big thing like convergence without… Like, if you don’t need that. It’s like killing a mouse
with an elephant gun. It’s overkill. There is the aesthetic of using minimal information. There’s a reason to that, as well, because it means that your argument is more broadly applicable.

In this excerpt, the aesthetics is about minimal information for proofs. In other tutorials, such as SGT5, the aesthetics was about the formalism in written arguments. The following observation excerpt is about the $\varepsilon$-$\delta$ definition of limit, which Zenobia and the students used for the task “Show that $\lim_{(x,y) \to (0,0)} \frac{x^2+y^2+3}{4} = \frac{3}{4}$.” I coded this excerpt as Zenobia’s aesthetic statement, as well.

You should be able to read this as a sentence. So, let $\delta = 2\sqrt{\varepsilon}$. Then $0 < |(x, y) - (a, b)| < \delta$ implies $0 < |(x, y) - (a, b)| < 2\sqrt{\varepsilon}$. We need the implications, right? Because otherwise, it doesn’t read. It doesn’t scan as a sentence. Now, it looks like a bunch of equations. But in fact, this could be written out as an English sentence – a grammatical English sentence.

Episode 3 includes three last conceptual names of categories, which I interpreted as tools for the strategy ‘explaining’: revoicing statements, rhetorical questions and informal language. In Episode 3, St7 correctly responds to Zenobia about the function in Figure 4.12 that “It is injective”. Zenobia then reutters St7’s response by saying “It is injective”. Afterwards, St2 correctly responds to Zenobia about the commonality of functions in Figures 4.10-4.13 that “They’re monotonically increasing”. Again, Zenobia reutters St2’s response by saying “They’re monotonically increasing”.

I coded Zenobia’s reutterations of students’ contributions as revoicing statements. O’Connor and Michaels (1993, 1996) defined revoicing as the oral or written reuttering of a student’s contribution by another participant in the discussion. Their term revoicing is in the form of reformulation of a student’s contribution. In my study, Zenobia did not reformulate a student’s contribution. Rather, she reuttered it by repeating exactly the same words. I nevertheless used the term revoicing in my study because I interpreted that the roles of coded revoicing statements resembled some of
the roles of revoicing in literature. So, my interpretation was that a role of the two aforementioned Zenobia’s *revoicing statements* is ‘to highlight’ (O’Connor & Michaels, 1996). Also, other roles of the two Zenobia’s statements are ‘to recruit students’ attention to a specific claim’; and to ‘shape students’ follow-up inquiry’ (Park, Kwon, Ju, Park, Rasmussen & Marrongelle, 2007). Specifically, the students’ follow-up inquiry in Episode 3 is about the conjecture that Zenobia consolidated with her statement “So, either monotonically increasing or monotonically decreasing is automatically going to be injective.”

In Episode 3, there are two questions to students that Zenobia makes without expecting a response. These questions are followed by a verbal representation. In the data, after St8’s incorrect input regarding a domain of injectivity for the graph of $f(x) = \sin(x)$, Zenobia explains injectivity in graphs by saying:

“So, what does “injective” mean? It means that there shouldn’t be any two points that are at the same height.”

Then, St8 offers a correct input and she explains injectivity for the graph in Figure 4.11:

“So, what did you notice? You noticed that you can’t have it go up and down, basically.”

In literature, rhetorical questions are questions posed without the requirement of an answer. In the observation extracts, above, I coded the questions as *rhetorical questions*. There is a number of previous research studies on the role of rhetorical questions in university mathematics teaching. For instance, in Fukawa-Connelly’s study (2012) of a lecturer’s demonstration of proofs in teaching abstract algebra, the rhetorical questions were “questions that a mathematician should ask while writing proofs, such as, ‘What does that mean?’, ‘What comes next?’ and ‘What do I still need to do?’” (p.343) Fukawa-Connelly identified that the role of these rhetorical questions was to provide students with modes of thinking about the organisation and structure of proofs. My interpretation is that the role of Zenobia’s *rhetorical question* in the first excerpt accords with Fukawa-Connelly’s identification ‘to provide students with modes of thinking’ that a mathematician should ask while writing proofs. So, St8
should think what injective means to find a domain of injectivity. Then, it seems to me that the role of Zenobia’s rhetorical question in the second excerpt accords with Viirman’s (2015) and Artemeva and Fox’s (2011) identification of the role ‘to direct students’ attention to certain aspects of the mathematics worthy of reflection’. In this excerpt, the mathematics worthy of reflection concern the commonality of the domains of injectivity of \( f(x) = x^2 \) (Figure 4.10) and of \( f(x) = \sin(x) \) (Figure 4.11).

In both excerpts, Zenobia’s verbal representations are in informal language. In our discussion for SGT10, Zenobia provided me with the thinking behind acting with informal language in the decoding of the mathematics.

At this point I am not trying to make them phrase things in a mathematical language. I do that quite a bit when I am trying to get into the intuition first and I really don’t want to burden it with technical vocabulary. I bring up the vocabulary later and by the end I really make them put things in a very strict mathematical formulation.

Excerpt 21_Discussion for SGT10

In Excerpt 21, Zenobia shares with me the role of informal language in the decoding of the mathematics: ‘not to burden the intuition with technical vocabulary’. In the first extract, Zenobia says that injective means “there shouldn’t be any two points that are at the same height.” Her informal language concerns the word height. My interpretation is that in the first extract the intuition is about injectivity. Then, the whole second extract is in informal language; she tells the students that “you can’t have it go up and down”. Zenobia points to the graph of Figure 4.11 when she says “it”. My interpretation is that in the second extract the intuition is about the commonality of the domains of injectivity of \( f(x) = x^2 \) (Figure 4.10) and of \( f(x) = \sin(x) \) (Figure 4.11). Finally, in Excerpt 21, Zenobia also says that “by the end I really make them put things in a very strict mathematical formulation”. It seems to me that her aesthetic statements, such as the ones I exemplified in this section, are towards this “very strict mathematical formulation”.

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Part 3: Zenobia’s knowing for teaching in the main study

4.3.1 Mathematical knowing

Zenobia’s mathematical knowing was evident from the various heuristics she used in her teaching in tutorials, such as ‘sketch a graph’ and ‘consider special cases’. My interpretation is that she drew on Polya’s (1971) heuristics to develop the ones with which she acted in her teaching. Zenobia also declared that she used heuristics for mathematical discovery in her research in analysis.

This study indicated that heuristics were tools for the strategy ‘decoding the mathematics and encoding the mathematics’, which was a teaching and research strategy for Zenobia. Notably work on graphical representations was an integral part of ‘decoding the mathematics’ in tutorials; so, the heuristic ‘sketch a graph’ was dominant over other heuristics.

As mentioned earlier, Zenobia shared with her students one of her meanings of the history of the development of mathematics. This was about the discovery of mathematical concepts through careful observation of sets of examples. Distinctive in her teaching was the generic set of examples, which she used in order for the students to observe commonalities in features of a set of examples and discover mathematical concepts of procedures. (The generic set of examples is discussed in this chapter: Section 4.2.2.3.)

4.3.2 Didactical knowing

In her teaching, Zenobia drew on the space of mathematics to act with tools, such as heuristics, graphical representations and generic sets of examples. Through reflection on her mathematical practice with these tools, Zenobia developed her views on mathematics and thus her epistemology of mathematics. Furthermore, in her teaching Zenobia stepped out of the space of mathematics to select strategies and tools from the context of students (thus from the space of teaching/learning). An example of such
strategies is ‘creating students’ positive feelings’, and an example of such tools is positive reinforcement for selecting tasks. As mentioned earlier in this chapter, ‘creating students’ positive feelings’ and positive reinforcement are compatible with her epistemology of teaching/learning.

In her effort to translate mathematical thinking with heuristics, graphical representations and generic sets of examples into forms of her thought in the context of students, Zenobia enriched the design of her teaching with tools and strategies based on her epistemology of teaching/learning, such as students at the board and rhetorical questions. For instance, she explains the heuristic ‘sketch a graph’ with formal mathematical language, rhetorical questions and students at the board in order for students to make sense of that heuristic (Figure 4.15). So, her design of teaching in order to translate the principles and content of mathematics into forms of her thought in the context of students include a path of informing: from practice drawn on her epistemology of mathematics towards practice drawn on her epistemology of teaching/learning (Figure 4.16).

Zenobia’s didactical knowing is concerned with knowing ways of making the design of teaching in order to translate the principles and content of mathematics into forms of her thought in the context of students. Each time that Zenobia designs her teaching for a particular mathematical topic, the initial design includes those tools and
strategies that have been proved from her teaching experience to be appropriate for such translation to the context of students. In other words, the initial design is the distillate of those tools and strategies that enabled students to make sense of mathematics in Zenobia’s past experiences. However, as a distillate, the initial design includes Zenobia’s reflection on successful past designs and on changes in tools in unsuccessful past designs. So, in Figure 4.16 the initial design includes all past designs and redesigns for a particular mathematical topic.

4.3.3 Pedagogical knowing

Pedagogical knowing is concerned with knowing ways of moving across developmental stages of teaching until a developmental stage which enables the students to make meaning of the mathematics. It is connected with flexibility in drawing on the students’ responses/silence and redesigning the teaching repeatedly with different tools and strategies until those that enable learners to make meaning of the mathematics. The pedagogical knowing depends on the tutor’s strategy ‘evaluating students’ mathematical sense making’ for a judgment as to what stage of the (re)design enables the students to make meaning of the mathematics. Zenobia’s tools for ‘evaluating students’ mathematical sense making’ were: inviting and control questions; and students’ reinforcement.

Zenobia’s students were high-achieving students. In teaching episodes, evidence for their mathematical sense making came from their correct mathematical inputs to the tutorial dialogue. Zenobia’s flexibility in drawing on the students’ inputs was concerned with her questions to students, with which she intended to enable them to discover the mathematics in the solutions of tasks (e.g. ‘questions for students to observe’). The students worked as a group to discover the mathematics, but a few times they were not able to make a contribution and remained silent. Zenobia’s flexibility in drawing on the students’ silence was nevertheless evident from bringing St4 at the board; St4 was able to solve tasks when the remaining students were not. Sometimes, Zenobia offered the correct input by herself (e.g. the generic set of examples of monotonicity on intervals in Episode 3). Moreover, due to the openness of her tutorials in terms of the students’ contributions to the solutions, Zenobia’s last
stages of redesign usually included her *consolidating statements* in order to align the students’ meanings to the established ones.

Finally, at times when the students reinforced the teaching by saying that they did not make sense of the mathematics, Zenobia was flexible to change the tools she was using. An example is in Episode 1 where she started the solution of the task with *symbolic representations*. Then the students said that they faced difficulties with the way of working, and Zenobia responded by using a range of *graphical representations* until a stage where the students provided mathematically correct responses (Figure 4.17). This flexibility in approach revealed Zenobia’s pedagogical knowing, i.e. her knowing of ways of creating a developmental stage of teaching which enables the students to make meaning of the mathematics.

![Epistemology of Mathematics Epistemology of Teaching/learning](image)

**Figure 4.17**: Pedagogical knowing.

In the case of Zenobia’s teaching, not only breadth and depth of *mathematical knowing* was evident, but also pedagogical knowing of a variety of strategies and tools was revealed. Furthermore, her didactical knowing was closely connected to her mathematical research practices, which included various heuristics. Zenobia articulated that connection in our discussions, thereby demonstrating her reflection on, and awareness of her teaching. The next chapter, Chapter 5, includes analysis of the case of Phanes’ teaching practice and knowing; Phanes was a research mathematician.
Chapter 5

DISCUSSION OF FINDINGS 2 – The case of Phanes’ teaching

In this chapter, I discuss the findings from analysis of the case of Phanes’ teaching. In the initial section, I provide the setting in which Phanes’ small group tutorials took place. The description of the setting is followed by Part 1. Part 1 is an analysis of Phanes’ epistemologies of teaching/learning and of mathematics based on two narratives of observations and a discussion with Phanes in the pilot study. The next part, Part 2, is a synthesis of my observations regarding strategies and tools of Phanes’ teaching practice in small group tutorials. (Strategies and tools for teaching are terms discussed in Chapter 3: Section 3.4.2.) The synthesis draws upon selected teaching episodes to ground the analysis in data and provide empirical evidence for analysis and interpretations made. The last part, Part 3, presents analysis of knowing for teaching in the case of Phanes’ teaching.

The setting

I start this section with an introduction of who Phanes is and when the pilot and main study observations took place. Then, I provide a description of the layout of the tutorial classrooms in order for the reader to develop an image of Phanes’ tutorials.

Phanes

Phanes is a research ‘Geometer by applications’ and a lecturer in three mathematics modules. One of his modules is for first year undergraduate students. Thus, he is lecturer and tutor for the students in tutorials. His teaching responsibilities include
large cohorts of students in lectures and a small group of first year students in tutorials. Phanes is experienced in both research and teaching. At the time of participating in my study, his research career covered twice as many years as his 15-year teaching career.

**Pilot study and main study observations**

I first observed Phanes’ tutorial during my pilot studies; my first observation was in December 2012 and my second observation was in May 2013. (The pilot studies are discussed in Chapter 3: Sections 3.4.1 & 3.4.3.) The two pilot study observations took place an academic year before I asked Phanes to participate in my main study for a whole-semester observation. That semester lasted for twelve weeks from October 2013 to January 2014. According to agreement with Phanes, I did not observe the first tutorial of the academic year in order for the students to become familiar with the small group tutorial setting.

**The layout of the classrooms**

In the pilot study, the classroom had two columns of desks facing two whiteboards and the lecturer’s desk. The capacity of the classroom was for 20 students. One student sat in one of the two front desks. There were four more students in the group, but I did not meet them in my observations. Phanes stood to write the mathematics on one of the whiteboards. When the student had to solve a task and completed it, Phanes usually shared the desk with the student evaluating the student’s solution. I sat two desks behind the student in order not to be intrusive and to be able to observe and audio-record.

In the main study, the capacity of the classroom was also for 20 students; however, the layout of desks was U shaped. The small group of six students sat around the desks facing a whiteboard and the lecturer’s desk. During the tutorials, Phanes stood to write to the whiteboard and circulated when students had to solve tasks. Occasionally, he sat on a desk in front of all students to talk with them. I sat between the students having the audio-recorder on the desk and taking field notes. The students were friendly with me and smiled, but usually covered their scripts.
Part 1: An interpretation of Phanes’ epistemologies in analysis of the pilot study

My interpretation of aspects of Phanes’ epistemologies of teaching/learning and of mathematics is based on analysis of data in the pilot study, where I discerned his views on teaching/learning and on mathematics. I start my account of the pilot study for Phanes’ teaching by providing narratives of the two observations written during the pilot study, and formatted for the purposes of the presentation of my analysis. The narratives are followed by analysis of Phanes’ discussion with me after the second observation. In transcripts of this discussion, I analysed Phanes’ views of: small group tutorials; his role as a tutor; students’ difficulties with mathematics; what teaching is; what making sense of mathematics is; and connections between teaching and mathematical research. I then used the narratives to provide empirical evidence for Phanes’ teaching with regard to his views. Concluding remarks of this section concern my learning from the pilot study of Phanes’ teaching.

Narrative 1  Observation in December, 2012

Information Phanes shared with me during the tutorial was that initially he had five students in the group; however, two students opted out and only one student always attended the tutorial. In my observation, I met only the latter student. I observed her asking Phanes how to estimate the remainder in Taylor series. The student said Phanes and herself had agreed to work on Taylor series from the previous tutorial. On the whiteboard, Phanes sketched a graphical representation of a random function \( f(x) \) and wrote the symbolic representation with the streams of notation in the Taylor polynomial. He connected the graphical representation and the elements of the first order approximation in the Taylor polynomial by showing what \( f(x), f(a), f'(a), a, x \) are on the graph. He then looked at mathematical tasks and said:

‘I will do this example myself and then you will do something similar.’

In this quote, Phanes used the term ‘example’ to refer to a mathematical task he solved on the board in order to show the student how to calculate a Taylor polynomial. That task was the Taylor polynomial of \( f(x) = \sin(x) \) for \( a = 0 \). He then selected a similar task which was about finding the first three non-zero terms of the Taylor polynomial of \( f(x) = \frac{1}{1+x} \) for \( a = 1 \). After providing the student with time to solve the similar task and after checking her solution, Phanes said:
‘It is correct; you don’t need other examples. So now, let’s go to the remainder!’

Phanes also explained the remainder graphically as well as in its symbolic representation. He, then, solved a first task on the whiteboard by finding the remainder in the Taylor series of \( f(x) = \sin(x) \), \( a = \frac{\pi}{2} \) for the second order approximation on \([0,1]\). At that time however, rather than giving the student one task to solve, he gave her two tasks. He solved the first of the last two tasks on the board while the student was solving it on her script. For the second task, he looked at the student’s script, and informed her that she made some wrong calculations.

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Narrative 2 Observation in May, 2013

In my second observation, the same student was in the tutorial and achieved a 100% mark in a coursework in Linear Algebra. I observed the student showing tasks to Phanes and asking ‘How do I do this?’ The tasks were about finding matrices of linear transformations. On the whiteboard, Phanes showed the student how to do the calculations in one of the tasks regarding linear maps. The linear map was

\[
\varphi: \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ y - z \\ x + z \end{pmatrix}
\]

and the ordered basis was \( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \). He showed her how to find the first column of the matrix of the linear transformation \( \varphi \). He then let the student find the other two columns of the matrix. He checked her calculations and said:

‘Excellent! Very good! Now, I don’t think there is any point in doing a similar question.’

In this quote, my interpretation is that Phanes used the term ‘question’ to refer to a mathematical task. The student nevertheless suggested solving a similar task with polynomials and an ordered basis, which confused her at home. Phanes satisfied the student’s suggestion. On the board, he wrote \( p(x) \to (1 - x^2) \frac{d^2 p}{dx^2} - x \frac{dp}{dx} \), and the standard basis \( \{1, x, x^2, x^3\} \). He then found the first two columns of the matrix of the linear transformation. The student continued solving that task on her script for the third and fourth columns of the matrix. Phanes checked her calculations, said ‘fine’ and wrote the last two columns of the matrix on the whiteboard. Similarly, they solved the task with the same transformation and an ordered basis. After the tutorial finished, Phanes asked the student what she wanted to work on for the next time.
5.1.1 Phanes’ epistemology of teaching/learning in tutorials

In the next sections, I provide an analysis of Phanes’ views of teaching that emerged from the narratives and my discussion with Phanes after the second observation. My interpretation is that the Phanes’ views form his epistemology of teaching/learning in tutorials. I draw this interpretation on analysis I provide in the next sections. This analysis indicates that Phanes’ views are connected with his teaching in observations, thus forming his thinking and perception for his teaching actions.

5.1.1.1 Phanes’ views of small group tutorials and his role as a tutor

Towards the end of Narrative 2, Phanes asked the student what she wanted to work on for the next time. In my discussion with Phanes, I reminded him of that question to the student. In response, Phanes shared with me his views of small group tutorials and his role as a tutor.

I think it is best when students come and have at least questions, rather than me taking random questions and going through them. It is more useful I think for students to ask their specific questions so I am reminding them not to forget to bring questions next time. But this student always comes with questions. I don’t have any problem with her. Some groups never have questions. I don’t want this. I am like a personal tutor only for my group and from the personal tutor they have to take advantage. I am a professional and I can explain to them.

Excerpt 1_Pilot study discussion

The student’s reaction to Phanes’ reminders to “bring questions” was a starting point for an insight into his views, and the actions he was taking with regard to them. Phanes recognised himself as a professional able to explain the mathematics to students, and that was included in his view of his role as a tutor. My interpretation is that the Phanes explanations to students reveal what it meant for him to be a professional. In Narrative 1, for instance, taught procedures concerned calculations about finding a Taylor polynomial or the remainder in a Taylor series. Before Phanes solved the task about finding a Taylor polynomial, he explained both graphically and
symbolically what a Taylor polynomial is. Similarly, before he solved the task about finding the remainder in a Taylor series, he explained both graphically and symbolically what a remainder is.

Phanes’ *view of small group tutorials* included students who bring questions. In the pilot study, I observed these questions were about the mathematics of the lectures with which the student found difficulties. So, my interpretation is that a goal of Phanes’ tutorial teaching was to enable the student to resolve her difficulties, and make sense of the underlying mathematics. One of the actions he was taking toward this goal was “reminding them not to forget to bring questions next time” [Excerpt 1_Pilot study discussion]. Indeed, at the end of the second tutorial, I observed that Phanes asked the student what she wanted to work on for the next time [Narrative 2_Observation in May, 2013]. In another tutorial, the student responded by letting him know the topic, i.e. ‘Taylor series’, and next time she brought her questions [Narrative 1_Observation in December, 2012].

5.1.1.2 Phanes’ *views of students’ difficulties*

In the pilot discussion with Phanes, I commented on my consideration of the outcome of his explanations: the student “understood and the calculations she did were always correct”. I referred to calculations for finding the matrix of linear transformations. Phanes agreed with me. However, he made the distinction between procedures and concepts, and informed me about his teaching experience with students’ difficulties.

But still she struggles with constructions, with the concepts. If you teach a student how to calculate, OK you expand, everything goes to the first column. This, they understand.

Excerpt 2_Pilot study discussion

Phanes’ reply indicated that the student made meaning of his explanation of the procedure of finding the columns of the matrix of linear transformations; however, she still struggled with concepts. From his experience, he generalised that students make sense of procedures (e.g. how to calculate) as opposed to concepts (e.g. span). In the beginning of our pilot study discussion, he indeed raised the issue that students
face difficulties with concepts such as span. He said "students struggle with span: what is span?"

5.1.1.3 Phanes’ views of what teaching is

I noticed a commonality in both observations, which I raised as an issue for my investigation. So in our discussion, I said to Phanes: “you write maybe the first example or you do the first calculation and then you ask the student to continue, to do more examples”. I used the term ‘example’ to be in line with Phanes’ terminology in my observations, where he told the student “I will do this example myself and then you will do something similar.” [Narrative 1_Observation in December, 2012]. His response to the commonality I noticed was:

[I]f the person doesn’t know, you just show, they do mistakes, my example, a simple example and then ask to repeat. This is how you teach anything. It is not about mathematics. You teach how to jump, how to run, how to ski, cycle. It is the same you show by an example and then you ask to repeat. It is the same. I think it is not isolated to mathematics.

Excerpt 3_Pilot study discussion

In Phanes’ terms (in Excerpt 3), his student did not know the remainder in Taylor series, and matrices of linear transformations. My interpretation is that Phanes got to know those difficulties of his student through the questions [Narratives 1 and 2] the student brought in the tutorial.

Phanes’ view of teaching included ‘showing and asking students to repeat’. He described to me this process as “you show by an example and then you ask to repeat” [Excerpt 3_Pilot study discussion]. In ‘showing’, I reported in both observations he selected tasks and then explained the calculations for one of the tasks to the student. In ‘asking students to repeat’, he asked the student to solve one [Narrative 1] or more tasks [Narratives 1 and 2], which he selected under the criterion to be “similar” to the task in ‘showing’ [Excerpt 3_Pilot study discussion]. In Narrative 1, in particular, the tasks Phanes selected were tasks about calculating Taylor polynomials. In ‘showing’, the task was the calculation of the Taylor polynomial of $f(x) = \sin(x)$ for $a = 0$. In
‘asking students to repeat’, the “similar” task was the calculation of the first three non-zero terms of the Taylor polynomial of \( f(x) = \frac{1}{1+x} \) for \( a = 1 \). The two tasks were “similar” in terms of the procedure of carrying out specific calculations to find out the Taylor polynomials.

5.1.1.4 Phanes’ views of what making sense of mathematics is

Phanes’ shared with me his view of sense making of mathematics when, in our pilot study discussion, I suggested that he “could just write the solution on board, from the beginning to the end and then ask: ‘Did you understand it?’ And the students could say yes.” He responded:

[T]here is a big difference between thinking that you understand and actually being able to do yourself. I remember of this lecturer when I was a student. It appeared that I understood everything, everything was crystal clear, he was a really excellent lecturer. And then I tried to do examples and then realised I don’t understand. I had to do examples. So, you understand when you can do examples yourself. I can understand a theorem in general. But it is not enough maybe to then be able to do examples. That’s all. It is very important to do examples with your own hand as many as possible and then you understand. […] And then there is also another possibility. That you may be able to do it yourself and not understand what you are doing. That’s another thing. Because many students can actually solve problems but they do not have any idea why they are doing this and why this is applied and where did it came.

Excerpt 4_Pilot study discussion

In Excerpt 4 Phanes shared with me his view of what making sense of mathematics is, rooted in his experience as an undergraduate student at a foreign university. For Phanes, sense making of mathematics, occurs “when you can do examples […] with your own hand as many as possible”. The specific meaning of doing examples, here, is solving tasks.
Through my sociocultural lenses, I consider the sociocultural context of the experience Phanes had as a student. The meaning of the mathematics was mediated to Phanes through a teaching culture associated with “this lecturer’s” teaching of that time and place. Within that culture, “everything was crystal clear”, and Phanes evaluated “this lecturer” as “really excellent”. However, Phanes had to solve tasks with his own hand, as many as possible, to make sense of the mathematics. Phanes’ learning experiences, over time, formed a view about making sense of mathematics. At the time of our discussion, Phanes’s teaching experience nevertheless indicated “a possibility” which added value to the prior learning experiences: students might have been able to solve the tasks without making sense of proofs and concepts.

‘Solving as many tasks as possible with own hand’ was a learning experience Phanes’ had as a student to make sense of mathematics. Within the time limits of his tutorials and in ‘asking students to repeat’, he offered the students time for ‘solving as many tasks as possible with own hand’ in order for them to make sense of mathematics. My interpretation from this is that Phanes’ goal in ‘showing and asking students to repeat’ is students’ sense making of the mathematics.

A tutor has the expertise to select tasks and to evaluate the students’ sense making of the underlying mathematics. My interpretation is that within the time limits of the tutorial, the ability of the student to solve tasks herself was, for Phanes, an indicator of the student’s sense making of the mathematics. I draw this interpretation from both observations of the pilot study, where Phanes decided to stop providing the student with tasks in order “to repeat” after the student presented a correct solution of a task. He told her “It is correct. You don’t need other examples. So now, let’s go to the remainder!” [Narrative 1] or “Excellent! Very good! Now, I don’t think there is any point in doing a similar question” [Narrative 2].
5.1.2 Phanes’ Epistemology of Mathematics

5.1.2.1 Phanes’ views of connections between teaching and mathematical research

In Excerpt 3, Phanes attributed “simple” to the example in ‘showing’. So, the example, or else the task, he each time selected to show in order for the student “to repeat” satisfied certain criteria of ‘simplicity’. In the pilot study discussion, I gained insight into examples Phanes viewed as “simple but still meaningful”.

One main strategy, one principle I would say, in the research and in teaching is to start with simple examples, which are meaningful, still, not trivial, not completely trivial, they are very simple but they illustrate the concept. […] So [in research] you need to start with a meaningful example and then develop. As many examples in fact as possible. In some cases, examples are more or less enough. You don’t need a general theory because an example is so good that it teaches you. So, to start with examples is the main principle. […] And sometimes [in research], you know, the examples are so convincing that it is of no interest to continue because you understood everything, your general statement and you can skip it, you don’t even get about proving because it is so obvious.

Excerpt 5_Pilot study discussion

In Excerpt 5, Phanes asserted that both in research and teaching he used “simple but still meaningful” examples. In this excerpt, in particular, he informed me about criteria of “simple but still meaningful” examples, which revealed both his views of mathematics and his views of teaching.

To begin with, a criterion for a “simple” example was a mathematically non-trivial example. Furthermore, Phanes attributed three criteria to a “meaningful” example. A “meaningful” example should:

- “illustrate the concept” (first criterion);
- be “so good that it teaches you” (second criterion); and
- sometimes “you don’t even get about proving because [the general statement] is so obvious” (third criterion) [Excerpt 5_Pilot study discussion].
My interpretation is that the epistemological purposes of Phanes’ examples were to explain (first and second criterion) and to convince (third criterion). The third criterion revealed generic proofs, and gave me insight into Phanes’ epistemological views of mathematics. Rowland (2002) stated that in the mathematical community there is a “commonly held view that generic proofs are formally inadequate” (p.179). Phanes, however, did not view mathematics in a conventional way. My interpretation from the third criterion for “meaningful” examples is that generic proofs, or else generic examples, were adequate proofs for him.

Phanes’ “meaningful” examples drew my attention to generic examples (e.g. Mason & Pimm, 1984; Balacheff, 1988; Rowland, 2002). Generic examples nevertheless have different uses in literature. A generic example for Rowland (1999) is “a confirming instance of a proposition, carefully presented so as to provide insight as to why the proposition holds true for that single instance” (p.25). My interpretation is that in case the third criterion is satisfied in Phanes’ “meaningful” examples, then the examples are congruous with Rowland’s consideration of generic examples.

Phanes labeled the examples “meaningful”; so, it seems that either the elementary learner (student) in teaching or the sophisticated learner (researcher) in research would be able to make meaning of the concept (or the procedure) these examples illustrate. In Narrative 2, for instance, Phanes used an example with a linear map and an ordered basis to illustrate the procedure of finding the matrix of a linear transformation (first criterion). Phanes’ explanation for the vector that formed the first column of the matrix could teach the student what the procedure of finding the matrix of a linear transformation is (second criterion). From Phanes’ evaluation of the student’s solution for the next two columns of the matrix (he said “Excellent!” [Narrative 2_Observation in May, 2013]), it seems that the student indeed made sense of the procedure. My interpretation is that Phanes’ purpose with this example was to explain so the third criterion of a “meaningful” example does not seem to have been satisfied by Phanes’ presentation of the example. However, responding in the moment and within limited time to the student’s difficulty (with tasks about finding the matrix of a linear transformation) is a different situation from thinking in advance about what presentation of an example could satisfy all three criteria.
In Excerpt 5, Phanes also talked about the usefulness of having a variety of examples. In Narrative 2, the variety included an example regarding linear maps and examples regarding polynomials. This variety of examples drew my attention to dimensions of variation (Watson & Mason, 2005, adapted from Marton & Booth, 1997). For Marton and Booth (1997):

> To experience a particular situation in terms of general aspects, we have to experience the general aspects. These aspects correspond to dimensions of variation. That which we observe in a specific situation we tacitly experience as values in those dimensions. (p.108)

Watson and Mason (2005) stressed that the variation constitutes a generality, which can be seen through examples lying in these different dimensions. In Narrative 2, the dimensions of variation in Phanes’ examples were two: linear maps and polynomials. In our pilot study discussion, Phanes declared that his most important contribution to research in mathematics is “bridges within different Sciences; between say, Differential Geometry and Differential Equations”. He added that “the most important thing in mathematics is connections”; thereby expressing a connected view of mathematics with connections being within mathematics and between mathematical areas. He also declared a connection between his research and lecture teaching. He told me that, in his module for first year students, he tried “to take examples from all different areas”; and showed “a couple of examples in various levels of complexity” before asking the students “OK can you formulate a general theorem based on the observed?” My interpretation is that, in my second observation, he also used examples “from different areas” (linear maps and polynomials) “in various levels of complexity” (standard basis, ordered basis).
My learning from the pilot study of Phanes’ teaching

From my field experience of observing tutorials, taking notes and audio-recording, I learned that ultimately it was my analysis of data from the pilot study discussion with Phanes that revealed issues in the data; for instance, Phanes’ views of teaching/learning. In contrast with the other two cases of teaching, Phanes shared his views with me in the pilot study. So he did not repeat them to me in the main study. As a result, I drew on my pilot study analysis of Phanes’ views in order to investigate aspects of the thinking behind Phanes’ teaching for the main study. Looking with hindsight at the transcription of the pilot study discussion, I recognised that the questions I asked Phanes had reference to his practices in the two tutorials. So I learned that the questions I would ask Phanes in the main study would evolve gradually during the process of inquiry. Furthermore, sitting behind or far away from the student did not provide me with the opportunity to look at the student’s script, or hear clearly what she said when Phanes sat in her desk. In the main study, I decided to sit in a desk between students.
Part 2: Phanes’ teaching practice in the main study

5.2.1 Data analysis of Phanes’ teaching practice

The small tutorial group of the main study had six students, who were in the BSc Mathematics. Five of the six students attended each of Phanes’ tutorials until the end of the semester. I met the sixth student in the second and third tutorial, but he then opted out.

The following Table 5.1 presents a characterisation of Phanes’ teaching into strategies and tools in his teaching. (Strategies and tools are terms discussed in Chapter 3: Section 3.4.2.) I identified these strategies and tools in my analysis of data from eight tutorials of the main study [SGT1, SGT2, SGT3, SGT4, SGT6, SGT7, SGT8, SGT10]. (The conceptual names of categories for the tools numbered 4.1, 6.3 and 6.4 are established concepts in literature, which are presented in a glossary in Appendix D.)

<table>
<thead>
<tr>
<th>Conceptual names of strategies for teaching</th>
<th>Conceptual names of tools for teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Urging students to bring questions to the tutorial</td>
<td>1.1. Questions to students while commencing the tutorial; 1.2. Statements (encouraging statements during the tutorial, injunction statements towards the end of the tutorial).</td>
</tr>
<tr>
<td>2 Selecting tasks</td>
<td>2.1. Students’ difficulties from teaching experience; 2.2. Students’ suggestions; 2.3. Students’ low results in coursework tasks.</td>
</tr>
<tr>
<td>3 Selecting examples</td>
<td>3.1. Simple but still meaningful examples.</td>
</tr>
<tr>
<td>4 Evaluating students’ sense making of mathematics</td>
<td>4.1. Questions to evaluate students’ sense making [control questions of students’ sense making (Viirman, 2015, p.1175), inviting questions to students: general to all students (Jaworski &amp; Didis, 2014, p.380)] and pause intervals; 4.2. Students’ ability to solve tasks; 4.3. Simple but still meaningful examples.</td>
</tr>
<tr>
<td>5 Showing and asking students to repeat</td>
<td>5.1. Procedural or proof tasks; 5.2. Students’ work on their scripts; 5.3. Tutor.</td>
</tr>
</tbody>
</table>
The conceptual names of categories for the students’ response to Phanes’ teaching, through which I characterise the students’ meanings in this study, are:

Correct input; Relevant input; Incorrect input; St question; Reinforcement.

Empirical evidence for analysis and interpretations I made regarding the strategies and tools of Phanes’ teaching is exemplified through a sample of three teaching episodes. The inclusion of two episodes from SGT1 is because I interpreted various Phanes’ tools in them, and also, the way Phanes teaches the students does not change in nature in the next tutorials. In particular, Episode 1 exemplifies analysis of the strategies ‘showing and asking to repeat’ and ‘explaining’, and Episode 2 exemplifies analysis of the strategy ‘selecting examples’. A brief account after each episode explains the different stages of Phanes’ “design” and “redesign” for students’ meaning making. After the presentation of episodes and their brief accounts, I provide a detailed analysis of each strategy and the associated tools. In this analysis, I mainly draw on the three teaching episodes.

<table>
<thead>
<tr>
<th>Episode 1_SGT1_Rewriting</th>
<th>Conceptual names of tools and strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>[adapted from Mali (2015)]</td>
<td></td>
</tr>
<tr>
<td>This episode is situated in SGT1, after the students asked Phanes to focus on a whole problem sheet in Analysis. It concerns work on a task from the problem sheet: “Rewrite (</td>
<td>x - 1</td>
</tr>
<tr>
<td>Reading the task, Phanes suggested: “we can just sketch the graph of the function”. He wrote (</td>
<td>x - 1</td>
</tr>
<tr>
<td></td>
<td>Symbolic representat. of (</td>
</tr>
<tr>
<td></td>
<td>Verbal representation of (</td>
</tr>
<tr>
<td></td>
<td>Simple but still</td>
</tr>
</tbody>
</table>
reveal the work on modulus signs; he constructed on the board the graph of \(|x|^3\), reflecting the negative part of the graph of \(x^3\) about the x-axis. Then he asked the students to work on their scripts for \(|x| - 1\). He circulated and offered support to the students for 3'20''. For instance, after seeing several students’ scripts and talking separately with some students, he suggested to “express each branch by a formula in terms of \(x\)” and sketched on the board the graph of \(f(x) = |x|\) writing \(y = x\) and \(y = -x\) to each branch of the graph accordingly. Phanes: So, how do I solve this problem? I’ll show you. I saw correct pictures; all of you had correct pictures. So, what am I going to do? I will do it step-by-step. First, I will construct \(|x|\), right? \(|x|\) is this. [Phanes sketched the graph of Figure 5.1] OK? Then, we do \(|x| - 1\), \(|x| - 1\) means that you take \(|x|\) and you shift it down by 1. This means \(-1\), right? So, it gives you this \([g in Figure 5.1]\). These points are 1 and \(-1\). And this point is \(-1\). This is the expression under the modulus sign, and then, you take the modulus of this function and it means that you reflect this negative bit about the x-axis, right? And you get this function. OK? This is the graph of the function. Now, we have to write down the equations for this. You can see that it’s given by different functions on different intervals. For instance, this expression is what \([f in Figure 5.1]\)? This was \(y = x\) \([e in Figure 5.1]\), and then, you shift it by 1, so this is \(x - 1\) \([f in Figure 5.1]\). Is this clear? Please stop me if something is unclear. So, this is \(x - 1\) \([f in Figure 5.1]\). So, what is this \([c in Figure 5.1]\)? What is this – this bit \([c in Figure 5.1]\)? [3' pause.] It has the same slope as \(x - 1\) but it’s shifted it up. St2: It’s \(x + 1\). Phanes: It’s \(x + 1\). So, this bit is \(x + 1\) \([c in Figure 5.1]\). Now, what is this \([a in Figure 5.1]\)? This graph is \(y = -x\) \([b in Figure 5.1]\), and we shift it down, so it’s \(-x - 1\). So, this thing is \(-x - 1\). And what is this \([d in Figure 5.1]\)? [5’ pause.] Phanes: \(-x + 1\). So, what can we now say about this
function \(|x| - 1|\)? It equals. Now, it depends on where \(x\) is, right? So, we know for this function that on this [Phanes points to interval \([1, +\infty)\)], it’s \(x - 1\) if \(x\) is greater than or equal to 1. Agree? It is \(-x + 1\), \(-x + 1\) if \(x\) belongs to \((0,1)\). It is \(x + 1\), \(x + 1\) if \(x\) belongs to \((-1,0)\). And finally, it’s \(-x - 1\) if \(x\) belongs to \((-\infty,-1)\).

![Figure 5.1: Reproduction of graph on the board.](image)

**Brief account of Phanes’ design and redesign for students’ meaning making:**

**Episode 1**

The following Figure 5.2 is a figural representation of Phanes’ stages of design and redesign in Episode 1 in order for the students to make meaning of reasoning by cases with the absolute value function. It is the same figure as the one in the case of Zenobia’s teaching. The stages of design and redesign are developmental of Phanes’ teaching for students’ mathematical meaning making.

In Episode 1, Phanes’ design is first to show to the students how-to sketch the graphical representation of \(|x^3|\) and then, to ask them to sketch the graphical representation of \(||x| - 1|\) and to find the equations. In this developmental stage (the first blue stage in Figure 5.2), he uses formal language to explain reasoning by cases. He also sketches the graphical representation of \(|x^3|\) which is a simple but still meaningful example. (Simple but still meaningful examples are discussed in this chapter: Part 1.)

Figure 5.2 includes two helixes, which are the space of mathematics and the space of teaching/learning. The helixes are interrelated because in teaching mathematics the space of mathematics and the space of teaching/learning are interrelated. An example
of the interrelation between the two helixes is in the stage of design (and similarly in the stage of redesign) in Episode 1. In the stage of design, Phanes draws on the space of mathematics to act with tools which are the formal language, the graphical representation and the simple but still meaningful example. These are tools for the strategy *showing and asking students to repeat*. The strategy nevertheless steps out of the space of mathematics to consider the context of the students and thus the space of teaching/learning. This indicates that in Phanes’ mathematics teaching the space of mathematics and the space of teaching/learning are interrelated. This is why Figure 5.2 does not include two straight lines connected with each other by the developmental stages of design and redesign; rather, it includes two interrelated helixes.

The red arrow represents a dialectic connection between two blue stages; that is to say, between design and redesign. The dialectic connection represents contradiction(s) in dialogue about mathematical meanings between Phanes and the students. Phanes makes dialogue with the students while they are working on $|x| - 1$ on their scripts. In this dialogue, he explains to the students in formal language and shows to them the graphical representation of $|x|$ and the equations for each branch of it. Considering that the students should sketch the graph of $|x| - 1$ on their scripts and find the equations for its branches, my interpretation is that Phanes viewed that students faced difficulties with $|x| - 1$. Thus, he offered them the graphical representation of $|x|$ and the equations for each branch of it. This indicates a contradiction between Phanes’ design for students’ meaning making of reasoning by cases with $|x| - 1$ and the difficulties he views from the students. Phanes finally evaluates that they “had correct pictures” of $|x| - 1$ on their scripts. However, he does not comment on students’ equations for $|x| - 1$ on their scripts.

Based on his dialogue with the students and the contradiction between his intention for students’ meaning making of reasoning by cases and what he views from the students, Phanes decides to redesign and solve the task for $|x| - 1$ on the board. His redesign is first to show to the students the symbolic representations $-x - 1$ and $x - 1$ which are equations for cases of $|x| - 1$, and then to ask them to find the symbolic representations $-x + 1$ and $x + 1$ for $|x| - 1$. Using twice the strategy *showing and asking the students to repeat* for the same task is not what Phanes
usually does in tutorials. In the stage of redesign, he acts with symbolic and graphical representations. He also explains in formal language. He draws on the space of mathematics to act with those tools (symbolic/graphical representations and formal language) and on the space of teaching/learning to use the strategy *showing and asking the students to repeat*.

Phanes makes dialogue with the students (second red arrow) where he invites the students to find \(-x + 1\) and \(x + 1\) for \(|x| - 1\). The students correctly find the two equations. Thus Phanes’ teaching at this developmental stage (the second blue stage) is at an appropriate level for students to make meaning of reasoning by cases with the absolute value function. So, no contradiction between Phanes and students’ views on their meaning making of finding the equations for branches of \(||x| - 1|\) emerges.

![Diagram showing Phanes' design and redesign for students' meaning making in Episode 1.](image)

**Figure 5.2:** Phanes’ design and redesign for students’ meaning making in Episode 1.
This episode is situated in SGT1, after Episode 1. It concerns work on the task in Analysis: “Give an example (written in any representation you wish to use) or state that this is impossible: A sequence that has neither an upper bound nor a lower bound.”

Reading the task, Phanes said: “So, in other words, there’s an element of this sequence which can be as large as you like as long as it’s positive and as small as you like as long as it’s negative. So, it goes to plus infinity or it goes to minus infinity. So, it’s not bounded.”

He wrote $-1, 2, -3, 4, -5, 6, -7, 8$ on the board and suggested: “So, I jump from left to right, from right to left.”

St2: How can we write the formula?

Phanes: I’ll explain to you the procedure. Can you think of a formula to represent what I’ve just said? [1" pause.] So, you’re jumping from left to right, going further and further. It’s something like this. [He moved both his hands in a direction “further and further” away from each other.] Not bounded. How would you write it by a formula? Simple. [3" pause.] As simple as you can. [10" pause.]

St2: It’s like something $-1$ because if you’ve got $a_3$, so $a_1$ as $1$ and then you’ve got $a_2$ as $-1$, then you can have like this sort of...

[...]

Phanes wrote the sequence $a_n = (-1)^n$ on the board and commented “it jumps from $1$ to $-1$, but it doesn’t go to infinity.” A student agreed. Phanes asked: “So, what do you do?” [1" pause.]

St2: You change that $-1$ to anything. So, $-2$ to the power of $n$. [Another student said n simultaneously with St2 confirming it.]

Phanes: $-2$ to the power of $n$. That will do. That will do. I’ll just do it a little bit simpler. But that would do. I would just multiply by $n$. [Phanes wrote $a_n = (-1)^n n$.] So, I take more or less this sequence and I multiply it by $-1$ to the $n$. And this makes it jump from left to right. OK?
Brief account of Phanes’ design and redesign for students’ meaning making: Episode 2

Figure 5.3, below, is a figural representation of Phanes’ initial design and successive redesigns in Episode 2 in order for the students to make meaning of examples of an unbounded sequence.

His design (the first blue stage) is to explain to the students what a “sequence that has neither an upper bound nor a lower bound” is, so that the students suggest a “good” example of such a sequence. The “goodness” of the example relates to belonging in simple but still meaningful examples. So in the stage of design, Phanes draws on the space of mathematics to explain in formal language and to act with the symbolic representation \(-1, 2, -3, 4, -5, 6, -7, 8\).

In dialogue with St2 (the dialectic connection of the first red arrow), St2 asks for the formula of an unbounded sequence. In response, Phanes explains with informal language and with a gesture which indicates the graph of an unbounded sequence. In acting with these tools (informal language and a gesture), Phanes “steps out” of the space of mathematics to bring back tools familiar to the students; thus he draws on the space of teaching/learning. Then, he invites the students to offer the correct formula.

Phanes’ strategy explaining in the stage of the design enables St2 to offer, not correct, but relevant input to an example of an unbounded sequence. This is in contradiction with Phanes’ design, where the students should suggest a “good” example of an unbounded sequence. Phanes redesigns (the second blue stage) to explain again what an unbounded sequence is, so that the students offer a correct example. In explaining, he draws on the space of mathematics to explain with formal language and with the symbolic representation \((-1)^n\).

In dialogue with St2 (the second red arrow), St2 suggests \((-2)^n\), which is a correct example of an unbounded sequence. Thus Phanes’ teaching at this developmental stage (the second blue stage) is at an appropriate level for students to make meaning of an example of an unbounded sequence. However, Phanes redesigns (the third blue stage) to offer to the students a “good” example of an unbounded sequence: this is the simple but still meaningful example \((-1)^n n\). (Simple but still meaningful examples...
are discussed in this chapter: Sections 5.1.2.1 & 5.2.2.3. \((-1)^n\) as a simple but still meaningful example of an unbounded sequence is discussed in Section 5.2.2.3.)

**Figure 5.3:** Phanes’ design and redesign for students’ meaning making in Episode 2.

<table>
<thead>
<tr>
<th><strong>Episode 3_SGT7 and SGT8_ The concept of basis</strong></th>
<th><strong>Conceptual names of tools and strategies</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>This episode includes compartments of SGT7 and SGT8 concerning Phanes’ teaching of span, linear independence and basis.</td>
<td>2 Inviting questions-gen. (basis, span)</td>
</tr>
<tr>
<td>Toward the end of SGT7, Phanes asked the students what basis means in (\mathbb{R}^2) and what span of a system of vectors means. St3 said span is “like how big it is and it can be in there, like it’s a set”. In response, Phanes made an example using his index and middle fingers as vectors and a two dimensional plane formed by his two fingers. The origin of the plane was the link of the two fingers in his palm. Phanes: OK, real space. This is the origin, right, you have two vectors, what is the span of these two vectors? It’s everything that you can obtain by taking the arbitrary linear combinations, right? […] everything which belongs to the plane [formed by his two fingers] […] so in general if you have a vector space, you have certain system of vectors. Then you take all possible linear combinations, what you get is their span, right? And we can that another way br, some other vector, incorrect input. Simple but still meaningful example of span.</td>
<td>Incorrect input</td>
</tr>
</tbody>
</table>
belongs to the span if it can be represented as the linear combination, OK? So basis, you understand what is the basis? [9’’ pause.]

St4 said “I am not sure” and St2 said “3 axes”. Phanes asked for the definition of basis in a vector space. [15’’ pause.] St3 responded “like the floor, everything you add to it”.

Phanes: You are not talking mathematical language, right, it’s not the right language. OK, several vectors are independent, linearly independent, what does it mean? The system of vectors is linearly independent. These are all very important concepts, you need to know these. [10’’ pause.] St2 said he had read this and, from his voice tone (slow pace), it seemed he could not remember it. St4 responded to Phanes’ request of the definition “They can’t be, you can’t make one out of a multiple derivative while they’re together.” Phanes made examples. He used his thumb, his index and middle fingers as vectors and a three dimensional plane formed by his three fingers. He suggested for linear independence that at least two of his three fingers must not be in parallel. He said “you can’t represent one of them as a sum of the other two.” Having exemplified linear independence and span, Phanes asked again the students what basis is. [10’’ pause.] St2 responded “It is when all three of them combine to make” and stopped his phrase at that point. Phanes made again the gesture for the example of using three of his fingers as vectors.

Phanes: [If you force that one [he showed his index finger] to lie on the plane spanned by the other two [he moved his index finger to be in parallel with his middle finger], they [he meant the three finger-vectors] are not the basis. Phanes said what a basis is for two finger-vectors in two dimensional space; for three finger-vectors in three dimensional space; and for four vectors in four dimensional plane.] So a basis consists of linearly independent vectors such that any other vector in the space is their linear combination. Alright? OK, I’ll check how you understand this.

Phanes devised two examples asking the students to show a basis: an example was $V_1 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in Mat_{3 \times 3}(R)$ in the space of skew-symmetric matrices and the other example was $V_2 = \{ax^3 + bx^2 + cx + d\}$ in the space of polynomials. After students’ inability to express the basis for the first
example, Phanes wrote \( (a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1) \) on the board. He said “these three form a basis. Any other vector is their linear combination with coefficients \( a, b, c \).

Is this clear? These are the basis, this is the standard basis of this vector space \([\mathbb{R}^3]\).” St3 then found the basis

\[
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix}
\]

for the first example. For the second example, students indicated some inappropriate suggestions and Phanes wrote the basis \([x^3, x^2, x, 1]\) on the board.

In SGT8, Phanes asked again the students what a basis is in a vector space \([5” pause.\)., and for instance, in a three-dimensional space \([2” pause.\). St2 and St3 made Phanes’ gesture with their thumb, index and middle finger. St2 responded “three vectors” and Phanes asked “Which are?” Many students talked at the same time but they did not offer a response to Phanes. Phanes asked again “in three space, basis consists of three vectors which are?” \([3” pause.\] A student responded “ultra independent”. Phanes corrected the student’s response by stressing “linear independent” and concluded “So in three space it means they don’t lie in one way because what you want from your basis is that any element of your vector space is the sum of these elements from the basis, OK?”

He wrote on the board three vectors \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) and asked if they form a basis in \( R^3 \). \([5” pause.\) St2 suggested Gaussian elimination. Phanes described the procedures for the criterion which is based on Gaussian elimination and said “OK, this is the criterion but what does it mean?” St4 suggested \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) as the result was \( \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \) and suggested “to prove that these three vectors \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) form the basis, what do we need to prove? That any vector, OK, call it \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) in three space can be written as a linear combination of these three

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \]
\]

The students asked how they could
prove if they do not know $a$, $b$, $c$ and Phanes wrote on the board

\[
\begin{bmatrix}
\alpha \\
\beta \\
\gamma 
\end{bmatrix}
\]

Brief account of Phanes’ design and redesign for students’ meaning making: Episode 3

The following Figure 5.4 is a figural representation of Phanes’ stages of design and successive redesigns in Episode 3 in order for the students to make meaning of the concepts of span, linear independence and basis.

In Figure 5.4, Phanes’ design (the first blue stage) is students to articulate what basis, span and linear independence are. In his dialogue with the students (the dialectic connection of the first red arrow), Phanes acts with inviting questions to students about what the three concepts are, but different students offer incorrect inputs. The students’ difficulty with the concepts of span, linear independence and basis is in contradiction with Phanes’ design for students’ meaning making of the concepts. Phanes draws on the space of mathematics to explain in formal language and to use simple but still meaningful examples. He also draws on the space of teaching/learning to act with an encouraging statement to students in order for them to study the three concepts.

In response to the students’ difficulty with the three concepts, Phanes redesigns (second blue stage) to suggest tasks about showing a basis in the space of skew-symmetric matrices and in the space of polynomials. He draws on the space of teaching/learning to show to the students the canonical basis so that they repeat for the space of skew-symmetric matrices. He also draws on the space of mathematics to explain in formal language. In dialogue with Phanes (the second red arrow), St3 reflects on the canonical basis and finds correctly a basis for the space of skew-symmetric matrices. However, the students cannot find a basis for the space of polynomials. This appears to be in contradiction with Phanes’ redesign (second blue stage) for students’ meaning making of the concept of basis. So, Phanes acts with himself as a tutor to find a basis for the space of polynomials.

In response to the students’ difficulty with the concept of basis, Phanes redesigns (third blue stage) so that the students articulate what a basis is in the next tutorial,
which is SGT8. In Phanes’ dialogue with the students (the third red arrow) in SGT8, different students offer incorrect inputs for what basis is. This is again in contradiction with Phanes’ redesign (third blue stage) for students’ meaning making of the concept of basis. In response to students’ incorrect inputs, Phanes redesigns (fourth blue stage) to suggest for students’ work a task about showing a basis.

The dialogue between Phanes and the students about the task is represented in the fourth red arrow. In this dialogue, Phanes invites the students to show a basis and St2 offers correct input about Gaussian elimination. However, the students cannot respond to Phanes’ next question about the criterion for basis, which is based on Gaussian elimination. Phanes draws on the space of mathematics to explain in formal language what it means, thereby offering the response by himself.

**Figure 5.4:** Phanes’ design and redesign for students’ meaning making in Episode 3.
5.2.2 Phanes’ strategies for teaching and the associated tools

5.2.2.1 Urging students to bring questions to the tutorial

‘Urging students to bring questions to the tutorial’ was a strategy evident in the eight selected tutorials for the analysis of Phanes’ teaching [SGT1, SGT2, SGT3, SGT4, SGT6, SGT7, SGT8, SGT10]. In five of the eight tutorials, Phanes asked the students for questions while commencing the tutorial, and/or reminded them to come next time with questions towards the end of the tutorial. Commencing SGT2, for instance, Phanes asked the students “any questions?”; and towards the end of SGT6, his reminder to students was “Come with questions next time”. My interpretation is that for ‘urging students to bring questions to the tutorial’ Phanes acted with the tools: questions to students while commencing the tutorial, as well as injunctions to students which were in the form of reminders towards the end of the tutorial.

‘Urging students to bring questions to the tutorial’ is a strategy in line with Phanes’ views for small group tutorials (e.g. “it is best when students come and have at least questions”) and his role as a tutor (e.g. “I am a professional and I can explain to them”) [Excerpt 1_Pilot study discussion]. In other words, it is a strategy whose use assumes that students work with the mathematics before coming to the tutorial; and view the tutorial as an opportunity to resolve their difficulties with the tutor’s support. The expectation for students’ work is stated to students in lectures, e.g. in analysis (see Chapter 1: Section 1.1). My interpretation from analysis of Excerpt 1 in Part 1 of this chapter is that a Phanes goal for ‘urging students to bring questions to the tutorial’ is ‘students to determine their mathematical difficulties and resolve them in the tutorial’. However, four of the five students in Phanes’ tutorial were so low-performing students who could not satisfy the expectation for work with the mathematics before coming to the tutorial. In this section, I first provide interview and observational data of the students’ performance. Then, I discuss what Phanes did so that students would work with the mathematics and bring difficulties to the tutorial.

5.2.2.1.1 Observational and interview data of the students’ low performance

In all eight tutorials, the students reacted to Phanes’ strategy, ‘urging students to bring questions to the tutorial’, by suggesting whole problem sheets or coursework tasks.
So, the questions they brought into the tutorial were questions about “how to do” a coursework task or a problem sheet. In particular, their reactions to Phanes’ injunctions to bring their questions to the tutorial indicated rather unprepared suggestions or vague difficulties with the mathematics. For instance, in SGT1 a student provided Phanes with a problem sheet in Analysis. Phanes looked at the problem sheet and asked the students for specific questions. A student replied “I was having trouble with quite a bit of it” meaning quite a bit of the problem sheet. (Episode 1 and Episode 2 concern work on tasks from this problem sheet.) Similarly, in SGT3 St4 gave another problem sheet in Analysis to Phanes. She replied to Phanes’ question “So, which one do you want to go over?” by saying “All of them.” Regarding the coursework tasks the students suggested, either they had handed them in or they were going to do so soon.

Phanes was not satisfied with the students’ questions. After SGT10, he informed me: “They don’t even ask me questions, because they can’t formulate a question.” Phanes was not dismissive to the students. It seems to me that, for Phanes, this small tutorial group was one of the groups with students who “never have questions” and do not “take advantage” of the personal tutor [Excerpt 1_Pilot study discussion]; thereby not meeting his expectations for small group tutorials.

Phanes told me that it was the “first time” he had students in such a low performance level: a “not university level” [Discussion after SGT10]. So, our discussions after SGT8-SGT11 were extensive and explorative of the level of the five students’ performance in the first semester. Phanes spoke of a group of four “not strong” students (male and female), and a fifth student (St5) who was “OK”. He said that the four students socialised and did sports but mathematics was not their first priority [Discussions after SGT8, SGT10, SGT11]. Based on the students’ results in pieces of coursework and their inputs in tutorial dialogue, he declared he knew that the students would fail the exams especially in linear algebra. For instance, in our discussion after SGT8 he said “But they may fail, because they are not strong students. I know from their coursework. They may fail first year”. Indeed, at the start of SGT3 and the following SGTs, Phanes usually gave a piece of coursework back to the students and informed them that their results were very low. In SGT8, which was towards the end of the semester, he told the students:
I am not happy with your particular results, your particular group, I am not very happy. Right? Because on the whole I would say our group is below the average. […] but it’s not true for everybody, right? Not all of you.

Excerpt 6_SGT8 Observation

A student, then, asked Phanes whether she “received a pass” and another student said “I don’t want to get it back” meaning getting back the coursework. In SGT9, Phanes commented on their coursework in Analysis as being “not very impressive”. His phrase “not very impressive” was an alternative to very low. Indeed, after the tutorial, he told me that “their coursework was very poor, very weak, apart from St5”. As an observer, I report that St5 solved the tasks on his script, answered correctly to Phanes’ questions and passed the pieces of coursework. However, St5 did not speak as much as the other four students in the tutorials. The other four students wrote on their scripts or looked at the board. They also answered Phanes’ questions about definitions in Linear Algebra using their own words. For instance in SGT7 [Episode 3], St3 said span is “like how big it is and it can be in there, like it’s a set” and basis is “like the floor, everything you add to it”. In SGT10, St2 responded for 1-1 correspondence that “you sort of give a value, you want to include every single. You want to find sort of a notation that sort of includes every single.” In our discussion after SGT10, Phanes’ perspective regarding St2’s response was “it’s still the same picture, he is trying to tell me what 1-1 map is with his own words, and why? He just doesn’t know the definition of 1-1 map.” St2 was one of the students who spoke more than others in the tutorials. I observed he always responded with regard to his perception or even his guess of the mathematics without being based on theory.

The data, so far, begs questions about how the mathematics community fulfills responsibilities to so low-performing students. In particular, how can a tutor help such students and why do the students perform so low in spite of entrance qualifications? Research literature in mathematics education reports on students who traditionally are not well prepared for university mathematics. In the Mathematics Problem, for example, Hawkes and Savage (2000, p.ii) provided evidence of a “decline in students’ mastery of basic mathematical skills”. In the case of Phanes’ teaching, evidence confirmed the Mathematics Problem and indicated that Phanes’ students did not
master the skills that could enable them to articulate definitions and pass the pieces of coursework. My interpretation is that the case of Phanes’ teaching reveals what Treffert-Thomas and Jaworski (2015, p.261) refer to as a challenge for tutors to help ill-prepared students for the transition to university mathematics.

5.2.2.1.2 Phanes’ approach to the students’ low performance

In our discussions after each of SGT8-SGT11, Phanes referred to the students using the phrase “they don’t study”. Phanes’ interpretation was that this was a reason why the students’ performance level was low. It seems to me that students who study was Phanes’ assumption for the students’ learning in the tutorial; in other words, the assumption was that students study and bring their questions to the tutorial, then the tutor “can explain to them” [Excerpt 1_Pilot study discussion]. The following is my analysis of what ‘to study’ meant for Phanes and what he did in order for the students ‘to study’ and bring their questions to the tutorial.

In our discussion after SGT9, Phanes informed me that the five students, who always attended his tutorials, also attended his lectures. It seems to me that, for Phanes, ‘to study’ meant more students’ engagement with mathematics than just lecture and tutorial attendance. After SGT10, he told me that St1 went to his office and said she could not make sense of Linear Algebra. After a discussion with the student, Phanes suggested her ‘to study’:

I said “The solution is to go home, go through the lecture notes, try to understand yourself”. […] she needs to sit and actually study on her own at home. And if she can’t understand still, after three hours or maybe a week, she can come to me.

Excerpt 7_Discussion after SGT10

Considering Excerpt 7, my interpretation is that for Phanes ‘to study’ meant students’ engagement with a topic in mathematics (e.g. going through the lecture notes and trying to make sense of the mathematics) for three hours or maybe a week, on their own at home. The time limit of ‘three hours or maybe a week’ reminded me of Phanes’ discussion with me after SGT7, when he reflected that as a student he studied
at home possibly “the same theory for a week” until he made sense of it. Also, ‘on their own’ was a key phrase in my analysis for:

• Phanes’ views of students’ learning in university education, according to which “they must be able to work on their own with very little guidance” [Discussion after SGT11]; and
• Phanes’ view of what making sense of mathematics is, according to which sense making of mathematics occurs when you can solve tasks “with your own hand as many as possible” [Excerpt 4_Pilot study discussion].

To conclude, my interpretation is that for Phanes, ‘to study’ meant work (theory and tasks) for three hours or maybe a week on the student’s own; then the tutor facilitates student’s sense making of the mathematics by explaining. (The strategy explaining the mathematics is analysed in a next section of this chapter.)

Phanes had advised students on how ‘to study’ since SGT8. In SGT10, which was the tutorial after the discussion with St1 in Phanes’ office, he had a discussion with all five students about studying mathematics. He likened studying mathematics to doing sports. He said at sports, their coach train them by saying to run four laps or push-ups and they do. Phanes encouraged the students that it is the same for mathematics and they have to consider mathematics as a sport; they have to go home and spend time to study algebra. My interpretation is that this was an encouraging statement Phanes used in order for students ‘to study’ and bring questions to the tutorial. In particular, it was a tool for the strategy ‘urging students to bring questions to the tutorial’ in a case of teaching mathematics to low-performing students. Phanes’ encouraging statement could potentially motivate the students ‘to study’ since it included experiences from their everyday life.

However, the students started to query how many modules they could fail and still receive a pass for the first year. St2 responded by declaring his perspective: he could not find the passion and drive to do the mathematics. Phanes attempted to encourage the students more. He asserted that starting to make sense of mathematics comes soon after studying at home and the result is starting to like the mathematics. In my analysis, I considered his assertion to be an encouraging statement, as well.
Phanes used encouraging statements in other occasions, as well. In Episode 1, for instance, he acted with an *encouraging statement* about the students’ work on their scripts. He said “I saw correct pictures; all of you had correct pictures.” In this way, he encouraged the students that they sketched the correct graph for $|x| - 1$. Another instance of an *encouraging statement* is in Episode 3, when he insisted on asking the students about what span, linear independence and basis is. In response to the students’ inability to articulate the definitions, he encouraged them to study by saying that “These are all very important concepts, you need to know these”.

In our discussion after SGT10, Phanes referred to St2’s perspective according to which this student could not find the passion and drive to do the mathematics. He likened his role as a tutor to the role of a doctor and left me with a question.

I can explain everything to everybody if the person is willing to study. If he follows my advice, if he doesn’t I can’t help. It’s like when an ill person doesn’t follow the advice of a doctor. So, how can the doctor help?

Excerpt 8_Discussion after SGT10

It seems to me that Phanes’ question in Excerpt 8 is a natural and important one in a case of teaching mathematics to ill-prepared students, who were interested in mathematics once: How can the tutor help so low-performing students?

In my pilot study observations, I had observed a student who studied, brought questions to the tutorial and achieved 100% in a coursework. That was a contrast to my observations in the main study. Phanes’ approach to teaching the students in the main study included the strategy ‘urging students to bring questions to the tutorial’, and the associated tools: *questions to students* about difficulties while commencing the tutorial; *encouraging statements* during the tutorial; and *injunction statements* for students’ questions in the form of reminders for the next time towards the end of the tutorial. However, this strategy and associated tools did not seem to be successful for this kind of students. After SGT10, I asked Phanes whether he changed his teaching approach with regard to those low-performing students. Phanes’ response was “I was just going to a more basic level trying to explain”. I investigated what a more basic level was in this chapter in section *Explaining* (Section 5.2.2.6). Furthermore, my
analysis of the remaining strategies indicates more tools with which Phanes acted in order to help the students.

5.2.2.2 Selecting tasks

From the eight tutorials I selected to analyse for Phanes’ teaching, the last six included tutorial work on coursework tasks [SGT3, SGT4, SGT6, SGT7, SGT8, SGT10]. Specifically, the percentage of coursework tasks (already handed in by students) over the total number of tasks (solved in all eight tutorials) was $38/79$. That is to say, almost half of all tasks were coursework tasks Phanes retrospectively solved on the board. (Most of the remaining tasks were tasks on regular Problem Sheets.)

It seems to me that Phanes was reactive to solve a total of 38 coursework tasks in order to help the low-performing students in terms of preparing them for the final examinations. So, it seems to me that Phanes’ goal in selecting tasks was ‘students to pass the modules’. In observations, he discussed the coursework tasks with the students by showing a correct solution on the board. At the start of SGT7, for instance, he said to the students that he had bad news concerning their coursework in Linear Algebra. Based on their low results, he suggested them “You do not understand at all what a vector space is” and immediately started to solve a coursework task on vector spaces. In addition to this, towards the end of SGT7, he came back to this students’ difficulty with questions and two tasks about the concepts of vector space, linear independence and basis [Episode 3]. My interpretation from observational data (e.g. from SGT7) is that Phanes’ tool for ‘selecting tasks’ was the students’ low results in coursework tasks. In particular, Phanes suggested to solve 13 coursework tasks in total on the board, whereas the students indicated that they would like to see solutions for 25 more coursework tasks. My analysis also indicates that students’ suggestions of coursework tasks was another Phanes’ tool for ‘selecting tasks’.

As in the case of Zenobia’s teaching, I distinguish all tasks Phanes used into: proof tasks (e.g. “prove that”, “use counterexample for false statement”), procedural tasks (e.g. “compute”, “sketch”), and conceptual tasks (e.g. “give an example for” a concept, “construct definitions” of concepts). In Episode 1, I consider the task “Rewrite $||x| - 1||$ without modulus signs, using several cases where necessary. You
do not need to use lengthy derivations.” as a procedural one. The procedure is the construction of the graph and the identification of functions and intervals; all distilled into steps by Phanes. In Episode 2, I consider the task “Give an example (written in any representation you wish to use) or state that this is impossible: A sequence that has neither an upper bound nor a lower bound.” as a conceptual one. The task is about the concept of sequence and the property regarding unbounded sequences. In Episode 3, I consider that the tasks about showing a basis for the space of skew-symmetric matrices, polynomials and $R^3$ are proof tasks. The presentation of all tasks in Phanes’ eight tutorials is in Table 5.2, below.

<table>
<thead>
<tr>
<th>SGT#</th>
<th>Conceptual tasks</th>
<th>Procedural tasks</th>
<th>Proof tasks</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>SGT1</td>
<td>“Give an example or state that it is impossible.” (10 tasks - not asked to give counterexamples) Inequalities and the triangle inequality (4 tasks)</td>
<td>“Rewrite $</td>
<td>x</td>
<td>- 1$ without modulus signs, using several cases where necessary. You do not need to provide lengthy derivations.” (1 task)</td>
</tr>
<tr>
<td></td>
<td>Finite fields (1 task)</td>
<td>-</td>
<td>-</td>
<td>Linear Algebra</td>
</tr>
<tr>
<td>SGT2</td>
<td>“Construct definitions” for not bounded above sequence, not increasing sequence, sequence which does not tend to infinity (3 tasks)</td>
<td>“Compute the limits” (4 tasks)</td>
<td>“For those that are true, prove it. For those that are false, give a counterexample” (3 tasks)</td>
<td>Analysis</td>
</tr>
<tr>
<td>SGT3</td>
<td>“Construct a definition” for a sequence which tends to minus infinity (1 task)</td>
<td>-</td>
<td>“Prove” (4 tasks)</td>
<td>Analysis</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>“Evaluate the expression $\frac{5x+6}{1-9}$ in $F_{11}$.” (1 task)</td>
<td>-</td>
<td>Linear Algebra</td>
</tr>
<tr>
<td>SGT4</td>
<td>-</td>
<td>“Sketch the graphs of the functions” (3 tasks)</td>
<td>“For those that are true, prove it. For those that are false, give a counterexample” (10 tasks)</td>
<td>Analysis</td>
</tr>
<tr>
<td>SGT6</td>
<td>-</td>
<td>Differential equations (2 tasks)</td>
<td>“For those that are true, prove it. For those that are false, give a counterexample” (7 tasks) “Prove” (2 tasks)</td>
<td>Analysis &amp; Different. equations</td>
</tr>
</tbody>
</table>
Table 5.2, above, presents a reasonable distribution among the different types of tasks. In the 8 SGTs of this table, in particular, Phanes used 79 tasks in total, from which: \( \frac{42}{79} \approx 53\% \) were proof tasks, \( \frac{13}{79} \approx 17\% \) were procedural tasks, and \( \frac{24}{79} \approx 30\% \) were conceptual tasks. So, only 17% of tasks were procedural tasks.

Below is my exposition about Phanes’ reasons for selecting such a balance of tasks for his students.

In Part 1 of this chapter, I provided Phanes’ views of students’ difficulties. I reported Phanes made the distinction between procedures (e.g. how to compute) and concepts, stressing that students struggle with concepts such as span. Phanes commented on students’ difficulties with concepts rather than procedures in my main study, as well. After SGT7, he specified that students face difficulties with concepts in Linear Algebra:
They learn calculations, say compute such and such, but they don’t have conceptual understanding, what they are doing, what it is about. They don’t understand what the basis is, what a vector space is.

Excerpt 9_Discussion after SGT7

He also shared with me that a discussion with the students in SGT10 informed him about the reasons why the particular group of students was more keen on procedures, such as partial differentiation, than concepts, such as linear independence.

As they explain, they are used to this [partial differentiation] from A-levels and with abstract concepts they struggle. They are not used to abstract concepts: what is linear independence? They know how to differentiate a function, they remember the procedure, the rule, but this is an abstract concept, they are not used to it.

Excerpt 10_Discussion after SGT10

Excerpt 10 illuminates another aspect of the Mathematics Problem: ill-prepared students who face difficulties with mathematical concepts. In Part 1 of this chapter and in the section Urging students to bring questions to the tutorial in Part 2 (Section 5.2.2.1), I argued that a goal of Phanes’ tutorial teaching was to enable the students to resolve their difficulties and thus to make sense of the underlying mathematics. A possible interpretation of the high percentage of proof and conceptual tasks, which were solved in Phanes’ tutorials, is that Phanes knew from his teaching experience that students’ struggle with concepts rather than procedures. This interpretation is based on the above Excerpts 9-10 of our discussions in the main study and Excerpt 1 of our discussion in the pilot study. So, a possible interpretation is that Phanes used the students’ difficulties from teaching experience as a tool for ‘selecting tasks’. In other words, based on his teaching experience Phanes was proactive to select these kinds of tasks towards his effort to help the students to resolve difficulties. Earlier in this section, I argued that other Phanes tools for ‘selecting tasks’ were: students’ low results in coursework tasks and students’ suggestions of coursework tasks.
5.2.2.3 Selecting examples

In each of the 8 tutorials in the case of Phanes’ teaching, my analysis indicates that Phanes used at least one example with the conceptual name *simple but still meaningful* example. Some examples were recognised by Phanes as being under this name while the majority were interpreted as such by me. More specifically, \( \frac{32}{79} \approx 40\% \) of the mathematical tasks Phanes and students worked on in SGTs were *simple but still meaningful* examples or included such examples. I start this section by an analysis of Phanes’ view of teaching with regard to students’ sense making of mathematics, where *simple but still meaningful* examples is a key concept. The analysis is followed by an exemplification of this kind of examples through the teaching episodes.

In the previous section, I analysed Excerpts 9 and 10 where Phanes recognised that the group of the main study students struggled with concepts especially in Linear Algebra. After SGT7, when Phanes discussed for the first time the concepts of span, linear independence and basis with the students, I asked him what he does in his teaching to promote students’ meaning making of mathematics when they have difficulties. He informed me that school students are not taught “to think abstractly as mathematicians”; thereby revealing a consideration of the students’ transition between school and university. He then declared that an alternative to teaching in an abstract way is teaching through *simple examples of different levels of complexity from different areas*. For the concepts of span, linear independence and basis, he suggested instances of different mathematical areas: matrices, polynomials, and functions. He stressed:

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[B]y simple examples everybody can understand […] if you consider 10 examples of different levels of complexity from different areas, I think that at the end students will understand.
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Excerpt 11_Discussion after SGT7

My interpretation, from Excerpt 11, is that Phanes indicated *simple examples* as a tutor’s tool for students’ meaning making of mathematics. However, the analysis of the main study indicated that examples *from different areas* did not seem to work for
the low-performing students of his group. In particular, these students were not familiar with different areas of mathematics such as the space of polynomials or the space of skew-symmetric matrices and could not solve the tasks about showing a basis in Episode 3. An important question for the mathematics community, here, is: How do tutors nurture students’ familiarity with different areas in mathematics?

In Excerpt 12 which occurred in the same discussion after Excerpt 11, Phanes explained to me that examples can help students build intuition for abstract concepts thereby promoting mathematical meaning making; at least for himself (a sophisticated learner).

Angeliki: So, is it from your teaching experience that you have seen that examples work well for students to understand the concepts?

Phanes: At least they work for me, simple examples, so they should work for them. It helps me if I consider a very simple example and then I am prepared for an abstract concept after I have seen an example. If I start with an abstract concept, I am not intuited, I don’t know what is in the background, it’s much more difficult.

Angeliki: When you say for me, do you mean in your research or in studying mathematics?

Phanes: In studying, in studying. Of course any research also must start with an example. It’s dangerous to start any research without having a good example. Because it may be after a few ones you see, there are not examples and you are studying an empty set. [Phanes talked about research on an empty set.] So you have to start with an example which is meaningful.

Excerpt 12_Discussion after SGT7

In the Analysis of the Pilot Study of this chapter, Phanes asserted that both in research and teaching he used simple but still meaningful examples. He also added, his lecture teaching was influenced by his research in terms of using examples of different levels of complexity from different areas. In Excerpts 11 and 12, Phanes discussed again
about these assertions from the pilot study. My interpretation, from the context of the main study observations and discussions with Phanes, is that he suggested a kind of examples for mathematical meaning making: *simple but still meaningful* examples. The following is an exemplification of *simple but still meaningful* examples based on interview data and Episodes 2 and 3.

After Excerpt 12, I asked Phanes for an explanation of *meaningful* examples. He said “Not very elementary. Not very difficult, but not elementary at the same time.” It seemed to me that this explanation suited to *simple* examples which criterion was ‘non-trivial’ [Excerpt 15_Pilot study discussion]. I asked Phanes for more explanation of *meaningful* examples. He said:

OK for instance. We do vector spaces. Also, we can study the standard vector space. Three space $[\mathbb{R}^3]$. \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \). It’s \( a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) etc and then find the basis, of course we can do it in this way. But, there are other vector spaces, like polynomials, matrices, some other vector spaces, complex numbers for instance, so you take different examples with vector spaces which look different. They don’t look like vectors in three space. But they are equivalent, all are equivalent. And then non-trivial examples, like skew-symmetric matrices, symmetric matrices or I don’t know, polynomials, or solutions to a certain linear differential equation, also for linear space. This kind of simple examples which accrue in applications.

Excerpt 13_Discussion after SGT7

In Excerpt 13, Phanes provided me with a second criterion for *simple* examples: they “accrue in applications”. So, a *simple* example for Phanes is a non-trivial example which is also useful in various applications. I refer to Episode 2 in order to exemplify Phanes’ *simple* examples and investigate the students’ response with regard to them. In Episode 2, Phanes and the students searched for a formula of a sequence that has neither an upper bound nor a lower bound. For this formula, Phanes requested from the students: “Simple. [3-second pause.] As simple as you can. [10-second pause.]” and waited for the students to respond. When he wrote the sequence \( a_n = (-1)^n \) on the board, three of the six students participated: A student agreed with Phanes that the
sequence \((-1)^n\) “jumps from 1 to \(-1\), but it doesn’t go to infinity”; St2 thought of the logical conclusion \(a_n = (-2)^n\) which was correct for that task; and a third student said \(n\) simultaneously with St2 confirming \((-2)^n\). The students’ increased participation at that point revealed a mutual meaning making of \(a_n = (-2)^n\) as an example of a sequence that has neither an upper bound nor a lower bound; thereby, indicating evidence of intersubjectivity between Phanes and the students. Furthermore, the fact that two students suggested the correct response \((-2)^n\) at the same time indicates that St2’s \(a_n = (-2)^n\) pulled the second student into her ZPD and created a shared ZPD between the students. (Intersubjectivity and ZPD are terms discussed in Chapter 2: Section 2.2.) Phanes’ teaching thus was successful in terms of the students’ meaning making of an example of such a sequence. However, in spite of St2’s thought of \(a_n = (-2)^n\), Phanes suggested the example \(a_n = (-1)^n n\) as a “simpler” one. His use of the word “simpler” challenged me as to what is simple in \(a_n = (-1)^n n\) and not in \(a_n = (-2)^n\). My interpretation is that \(a_n = (-1)^n n\) is simple since it is based on the sequence \((-1)^n\) which could occur in applications. In other words, \((-1)^n\) can provide a basis for a number of alternating sequences with different properties, whereas \((-2)^n\) is just one alternating sequence.

Episode 3 provides another exemplification of simple but still meaningful examples. In this episode, Phanes asked the students for the definitions of span and basis; and students made inappropriate suggestions. For instance, Phanes told me that St3’s response that span is “like how big it is and it can be in there, like it’s a set” did not make sense. In response, Phanes made examples with his fingers and a plane through his fingers to promote students’ intuition for the concepts of span, linear independence and basis. These examples of the concepts of span, linear independence and basis were in \(R^3\) and did not require knowledge of notation. They nevertheless illustrated the assertions in definitions that must hold true for the concepts (first criterion for meaningful examples to “illustrate the concept”). (The criteria for simple but still meaningful examples are presented in this chapter in Part 1.) Also, each example (for span, linear independence and basis) had the potential to be “so good that it teaches you” (second criterion); thereby being adequate to be interpreted as a potential meaningful example for the students. In SGT8, for instance, the students responded to Phanes’ request for the definition of basis by making Phanes’ example-
gesture with three fingers. This indicated that, at least, the students remembered the gesture. So, that example may have been successful for these students.

In Episode 3, Phanes concluded his SGT7 teaching of basis by using examples *from different areas* (i.e. in the space of skew-symmetric matrices and polynomials) so that the students find a basis with their own hand. The examples Phanes used were in *different levels of complexity* from the finger-vectors since they required knowledge of notation. In Excerpt 13, Phanes declared that the spaces of skew-symmetric matrices and polynomials are *simple but still meaningful* examples of vector spaces, where a task could be “find the basis”. He stressed they are non-trivial examples of vector spaces and occur in applications (criteria for *simple* examples). It seems to me that finding a basis for $V_1 = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in \text{Mat}_{3 \times 3}(R) \right\}$ in the space of skew-symmetric matrices and for $V_2 = \{ax^3 + bx^2 + cx + d\}$ in the space of polynomials illustrates how to prove for the concept of basis (first criterion for *meaningful* examples). However, my interpretation is that this was a great conceptual leap for these particular students, who could not use their experience with $V_1$ and $V_2$ in order to find in the next tutorial whether $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ form a basis in $R^3$.

In case of a careful presentation so as to provide insight as to *why* the definition of basis holds true in $V_1$ and $V_2$, the examples could be a generic set of examples of showing a basis. My interpretation is that the difference between Phanes’ examples and Rowland’s (2002) generic examples is that Phanes used a range of examples in *different levels of complexity* rather than a single example that carries the genericity within it. It also seems to me that Phanes’ need for genericity across examples rather than within a distinct example is necessitated by the complexity of the concept of basis, and the complexity of the mathematical context of vector spaces.
5.2.2.4 Evaluating students’ sense making of mathematics

In the Analysis of the Pilot Study of this chapter I argued that, for Phanes, an indicator of the student’s sense making of the mathematics was the ability of the student to solve tasks within the time limits of the tutorial. This section provides observational and interview data as evidence that, in the main study, Phanes based his evaluation of students’ mathematical sense making on their ability to solve tasks. So the students’ ability to solve tasks was Phanes’ tool for ‘evaluating the students’ sense making of mathematics’.

In SGT7, for instance, Phanes said to the students that he would check how they understand basis. He devised two examples of vector spaces (the spaces of skew-symmetric matrices and polynomials) and the task was to show a basis. In my discussion with Phanes after SGT7, I asked him what the ways were with which he checked students’ sense making of the mathematics. He responded “I just give them an example which does not look like three space \([R^3]\)”, such as a set of skew-symmetric matrices or polynomials, and “I want them to tell me what the basis is in this space”. He stressed “If they understand the general concept, they should be able to tell me” (i.e. to solve the task). He explained to me that students are used to \(R^n\) as opposed to the space of skew-symmetric matrices or polynomials. It seems to me that these last two spaces were for getting students used to vector spaces other than \(R^n\).

Furthermore, for Phanes, the spaces of skew-symmetric matrices and polynomials were simple but still meaningful examples of vector spaces [Excerpt 13_Discussion after SGT7]. My interpretation is that he used simple but still meaningful examples as tools not only to promote students’ mathematical sense making but also to evaluate it.

Phanes informed me that in the next tutorial, which was SGT8, he would test again the students’ sense making of basis “by giving them other examples of vector spaces” such as symmetric matrices. I report in Episode 3 that, in SGT8, the task Phanes gave to the students in order to test their sense making of basis was:

\[
\text{Do the vectors } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ form a basis in } R^3? \]
So the example of vector spaces, which he gave to the students, was $\mathbb{R}^3$ even if he recognised students are used to $\mathbb{R}^3$. St2 suggested Gaussian elimination, which was a correct procedure for that task. Phanes raised the question “but what does it mean?” My interpretation is that a student able to answer this question would have made meaning of the concept of basis. However, none of the students could articulate the definition of basis and thus prove with the use of the definition. A student even said “ultra independent” instead of “linear independent”. Phanes’ perspective on the students’ response, which he discussed with me after SGT8, was: “We went through the basis, linear dependence, span, nothing they remember”. Phanes taught basis again in the next tutorial, which was SGT9. As he explained to me, the reason was that “there is no point in doing Linear Algebra, all these problem sheets, coursework, if they don’t understand these basic concepts”. It seems to me that this quotation is another confirming instance that a goal of Phanes’ tutorial teaching was to enable the students to resolve their difficulties and make sense of the underlying mathematics.

While coding Episode 3, I used the conceptual name *inviting question* for Phanes’ question “but what does it mean?” [“It” here refers to the criterion of Gaussian elimination.] This *inviting question* was *general* to all students because Phanes did not ask a particular student in the group. Considering Jaworski and Didis’ (2014) ‘inviting questions’ which role is to seek students’ articulation of mathematical meaning, my interpretation is that Phanes sought students’ articulation of meaning for the concept of basis with his question. Table 5.3, below, demonstrates all Phanes’ *inviting questions* in Episodes 1-3, where he sought students’ articulation of mathematical meaning.

Table 5.3 also presents Phanes’ *control questions* to check that students have made sense of his explanations. It seems to me that the context in which Phanes used *control questions* was when he explained “a particularly important or complicated piece of mathematics” (Viirman, 2015, p.1175). This was also the context in which the lecturers in Viirman’s study used control questions. An example of “a particularly important or complicated piece of mathematics” in the case of Phanes’ teaching is the concept of basis, for which he used time from three consecutive tutorials. Table 5.3, below, illustrates various Phanes’ *inviting* and *control questions* about the concept of basis in Episode 3.

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<table>
<thead>
<tr>
<th>Episode #</th>
<th>Control questions</th>
<th>Inviting questions: general</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>“I will construct [x], right? [1” pause.] [x] is this. [Phanes sketched the graph of Figure 5.1.] OK?” [1” pause.]</td>
<td>“So, what is this [c in Figure 5.1]? What is this – this bit [c in Figure 5.1]?” [3” pause. General question: “St2 responds (x + 1).”]</td>
</tr>
<tr>
<td></td>
<td>“And then, you take the modulus of this function and it means that you reflect this negative bit about the (x) axis, right? [3” pause.] And you get this function. OK?” [4” pause.]</td>
<td>“And what is this [d in Figure 5.1]?” [5” pause. General question: St4 responds (-x + 1).]</td>
</tr>
<tr>
<td></td>
<td>“This was (y = x [c in Figure 5.1]), and then, you shift it by 1, so this is (x - 1 [f in Figure 5.1]). Is this clear?” [3” pause.]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>“Now, it depends on where (x) is, right? So, we know for this function that on this [Phanes points to interval ([1, +\infty)], it’s (x - 1) if (x) is greater than or equal to 1. Agree?” [2” pause.]</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>“So, I take more or less this sequence and I multiply it by (-1) to the (n). And this makes it jump from left to right. OK?” [1” pause.]</td>
<td>“Can you think of a formula to represent what I’ve just said?” [1” pause.] “How would you write it by a formula?” [13” pause. General questions: “St2 offers a relevant input.”]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>“So, what do you do?” [1” pause. General question: St2 responds ((-2)^n).]</td>
</tr>
<tr>
<td>3</td>
<td>“And we say that another vector, some other vector, belongs to the span if it can be represented as the linear combination, OK?” [1” pause.]</td>
<td>“Phanes asked the students what basis means in (R^3) and what span of a system of vectors means.” [General question: “St3 offers an incorrect input.”]</td>
</tr>
<tr>
<td></td>
<td>“So a basis consists of linearly independent vectors such that any other vector in the space is their linear combination. Alright?” [2” pause.]</td>
<td>“So basis, you understand what is the basis?” [9” pause. General question: St4 responds “I am not sure” and St2 responds “3 axes”.]</td>
</tr>
<tr>
<td></td>
<td>“Any other vector is their linear combination with coefficients (a), (b), (c). Is this clear?” [3” pause.]</td>
<td>“Phanes asked for the definition of basis in a vector space.” [15” pause. General question: St3 offers an incorrect input.]</td>
</tr>
<tr>
<td></td>
<td>“So in three space it means they don’t lie in one way because what you want from your basis is that any element of your vector space is the sum of these elements from the basis, OK?” [1” pause.]</td>
<td>“OK, several vectors are independent, linearly independent, what does it mean?” [10” pause. Direct question to St2: St4 offers an incorrect input.]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>“Phanes asked again the students what basis is.” [10” pause. General question: St2 offers an incorrect input.]</td>
</tr>
</tbody>
</table>
|          | | “In SGT8, Phanes asked again the
students what a basis is in a vector space [5’ pause], and for instance, in a three dimensional space. [2’ pause]” [St2 responded “three vectors”]. “Phanes asked ‘Which are?’” [Students did not offer input.] “Phanes asked again ‘in three space, basis consists of three vectors which are?’” [3’ pause. A student responded “ultra independent”].

“OK, this is the criterion but what does it mean?” [St4 offers an incorrect input.]

In Table 5.3, Phanes’ control questions follow after an explanation and end with “OK?”, “Is this clear?”, “Agree?” or “Alright?” For example, in Episode 1 he explains how to find $x - 1$ on the graph by saying “This was $y = x$, and then, you shift it by 1, so this is $x - 1$. Is this clear?” A pause interval after control questions usually lasts for 1” while sometimes it is up to 4”.

Phanes’ inviting questions require a particular answer from the students. For instance, in Table 5.3, the first question “So, what is this [c in Figure 5.1]?” requires the particular answer $x + 1$. In personal communication with Jaworski, I characterised inviting questions which require a particular answer from the students as prompting questions. So, Phanes’ inviting questions are in particular prompting questions. Phanes’ inviting questions usually follow after an explanation, as well. For example, in Episode 2 he tells students “I’ll explain to you the procedure. Can you think of a formula to represent what I’ve just said? [1” pause.] So, you’re jumping from left to right, going further and further. […]” His inviting questions are mostly general to all students; however, there are some instances where Phanes asks direct questions. In Episode 3, for instance, he asks St2 about linear independence. St2 can only say that he had read this and St4 offers an incorrect input.

After Phanes’ inviting questions, the students either respond immediately (e.g. 1” pause interval) or they need plenty of time (e.g. 9”-15” pause intervals). When they need plenty of time, Phanes usually rephrases the question. Similarly with my analysis of Zenobia’s teaching, inviting questions to students require larger pause intervals to enable the students to think and respond. So in contrast to 1”-4” pauses after control
questions, pause intervals after inviting questions can last up to 15”’. My interpretation is that *pause intervals* are also tools for ‘evaluating students’ sense making of the mathematics’ because they enable the students to think for a response.

Earlier in this section, I argued that other Phanes tools for ‘evaluating students’ sense making of mathematics’ were: *simple but still meaningful examples, inviting and control questions* and *students’ ability to solve tasks*. Despite Phanes’ exemplifying, questioning and waiting for students to respond, the students nevertheless usually could not offer correct input to the dialogue about the mathematics. In the next section, I present a key Phanes strategy to enable the students to make mathematical sense and thus offer correct input.

**5.2.2.5 Showing and asking students to repeat**

In Part 1 of this chapter, I analysed Phanes’ *view of teaching*. My interpretation is that the essence of his view is the strategy ‘showing and asking students to repeat’. This strategy is accompanied in the main study by the following order of actions:

- Phanes showed students the solution of a task or a set of tasks on the whiteboard. (This is the component ‘showing’ of the strategy.)
- Then, he provided time for students’ work on “similar” tasks on their scripts. During this time, he circulated giving feedback and support to students. (This is the component ‘asking students to repeat’.)
- Finally, Phanes sometimes used again ‘showing’; he showed the students his thinking and writing up of the solutions of the “similar” tasks on the whiteboard.

The first component of the strategy, ‘showing’, can be expressed with the term ‘parallel modeling’ (e.g. Anghileri, 2006; Grandi & Rowland, 2013). Anghileri (2006) used ‘parallel modeling’ to refer to the situation where the teacher creates and solves a task that shares some of the characteristics of the task for students’ work. Phanes did not usually create a “similar” task to the students’ one; rather, he selected the “similar” task from the lecture material. So it seems to me that observational data of Phanes’ teaching are more congruous with Grandi and Rowland’ use of the term, where ‘parallel modeling’ relates to “modeling the solution to a similar” often simpler task (2013, p.388).
My interpretation is that, in ‘showing and asking students to repeat’, Phanes’ goal was to promote students’ mathematical sense making through parallel modeling and his feedback for the students’ work. In particular, the second component of the strategy, ‘asking students to repeat’, is rooted in Phanes’ view of what sense making of mathematics is (i.e. the ability to solve as many tasks as possible with own hand). As I mentioned in the *Analysis of the Pilot Study*, within the time limits of his tutorials and in ‘asking students to repeat’, Phanes offered the students time for ‘solving as many tasks as possible with own hand’ in order for them to make sense of the mathematics. My observations indicate that Phanes did so in the main study, as well.

Being congruent with his views of teaching and sense making of mathematics, Phanes used the strategy ‘showing and asking students to repeat’ once in each of the first 4 of 8 tutorials. In particular, he provided the students with time for work on 9 tasks over a total of 79 tasks, which is the $\frac{9}{79} \approx 11\%$ of all tasks. My interpretation is that all 9 tasks were procedural or proof tasks. So, a tool associated with ‘showing and asking students to repeat’ was the *procedural or proof tasks* Phanes selected. The mathematical content of the tasks was in analysis: the absolute value (1 task in SGT1); limit computation when $x \to 0$ or $x \to \infty$ (4 tasks in SGT2); the $\varepsilon - \delta$ definition of convergence for sequences that converge to zero (2 tasks in SGT3); and sketching graphs of one variable functions (2 tasks in SGT4). By acting with procedural or proof tasks, Phanes ultimately showed students how-to prove or how-to think in order to solve a task.

In Episode 1, for instance, Phanes created the task about rewriting $|x|^3$ without modulus signs. He showed on the board how-to think in order to construct the graph of $|x|^3$ by reflecting the negative part of the graph of $x^3$ about the $x$-axis. Then he provided the students with time for work on $|x| - 1$ on their scripts. The construction of the graphs of $|x|^3$ and $|x| - 1$ was “similar” in terms of reflecting the negative parts about the $x$-axis; however, the level of complexity for the construction of $|x| - 1$ was higher than the one for the construction of $|x|^3$. It seems to me that ‘showing’ in this episode is congruent with ‘parallel modeling’ (e.g. Anghileri, 2006; Grandi & Rowland, 2013) in terms of creating and solving a simple and similar task to the students’ one.
My interpretation is that Phanes acted with the students’ work on their scripts in order to promote students’ mathematical meaning making. So students’ work on their scripts was a tool for the strategy ‘showing and asking students to repeat’. In Episode 1, Phanes circulated and offered support to the students for the work on their scripts on $|x| - 1$. For instance, he provided a solution step: to “express each branch by a formula in terms of $x$” [Episode 1]. He then offered the example of the graph of $f(x) = |x|$, where $y = x$ and $y = -x$ are the formulas in terms of $x$ for each branch of the graph. The expression of each branch of the graphs of $|x|$ and $|x| - 1$ by a formula is “similar” in terms of the connection between the symbolic and graphical representation of absolute value functions.

Phanes finally wrote up the solution of the task for $|x| - 1$ on the board. He showed on the board how-to think mathematically about rewriting $|x| - 1$ without modulus signs: he adjusted basic graphs (i.e. $|x|$, $x$, $-x$) to construct $|x| - 1$ and from that, he extracted the essential information (i.e. formulas and intervals) for the solution of the specific task. It seems to me that he made this decision because of his view on the importance of sketching graphs in mathematics. Indeed, following Episode 1 in SGT1 was SGT4, where Phanes told the students that “the most important thing in analysis is to graph functions”. Also, in Episode 1, he evaluated that all students had “correct pictures” on their scripts but it could be that they did not have on their scripts the equations or the correct equations.

As mentioned in Chapter 4, analysis of the case of Zenobia’s teaching indicated that Zenobia acted with St4 in order for her to overcome the remaining students’ inability to offer insight into the solution of a challenging task. In other words, St4, who was a particularly high-achieving student, was Zenobia’s tool towards the purpose of not offering the insight by herself. In the case of Phanes’ teaching, the students were low-achieving and Phanes did not have a St4 to offer insight when all other students struggled. My interpretation is that Phanes acted with himself as a tutor in order to offer insight into the solution of tasks. For example, in Episode 1, before he made the general inviting question to the students about what the formula of the branch d is on the graph of $|x| - 1$, he explained by himself how-to find the formula of another branch on the same graph, that of branch b: “This graph is $y = -x$ [b in Figure 5.1], and we shift it down, so it’s $-x - 1$. “ In this way, he paralleled the modelling of the
unknown formula of branch d \([-x + 1]\) with the modelling of the known formula of branch b \([-x - 1]\), and supported the students in offering the correct input \(-x + 1\) for branch d.

To conclude, ‘showing and asking students to repeat’ was a strategy to promote students’ mathematical sense making through parallel modeling and his feedback for the students’ work. His parallel modeling offered the students insight into the solution of tasks in case the students saw the generality in order to solve a new more difficult task. A Phanes tool to offer the students this kind of insight was himself as a tutor; that is, he offered the solutions by himself when the students were not able to do so. Other tools for ‘showing and asking students to repeat’ were procedural or proof tasks and students’ work on their scripts; he circulated and offered support to his students. Finally, ‘explaining’, which I analyse in the next section, was another strategy for offering the students insight into the solution of tasks.

5.2.2.6 Explaining

This section concerns my analysis of the strategy ‘explaining’ in the case of Phanes’ teaching. My interpretation is that the tools, with which Phanes acted, were rhetorical questions (Fukawa-Connelly, 2012; Viirman, 2015) and mathematical representations (e.g. graphical and symbolic representations). In this section, I provide an exemplification of Phanes’ strategy ‘explaining’ through his use of representations and rhetorical questions in Episodes 1, 2 and 3.

In Episode 1, Phanes explains to the students a procedure of processing absolute value expressions based on the piecewise-linear function definition of absolute value (Sierpinska, Bobos & Pruncut, 2011):

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]

The procedure is reasoning by cases and Phanes articulates it with regard to the task “Rewrite \(|x| - 1|\) without modulus signs.” He says: “to get rid of the modulus sign of \(|x|\), you need to know that \(x\) is positive or negative. You have to consider cases. But there is another outer modulus. It’s external. Again, to get rid of it, you need to either consider the case whether the expression inside it is positive or not.” I coded
this extract as a verbal representation in *formal language*, because he explains reasoning by cases with the use of *formal mathematical language* thereby offering to the students more than a mere exposition of the piecewise-linear function definition of absolute value. In other words, Phanes does not repeat that definition to the students; rather, he describes the procedure with which the students can “get rid of the modulus sign”.

Phanes then presents the ways he works through the graphs and symbols dissecting the solution of the task to make its aspects more visible to students. In this way, he explains the construction of graphs for the cases of $|x| - 1$ and encourages the students’ visualization of functions $(x - 1, x + 1, -x - 1, -x + 1)$ and intervals. In his “step-by-step” explanation of the solution, he acts with *graphical representations* (e.g. the graphs of $|x|, |x| - 1, ||x| - 1|$) and *symbolic representations* (e.g. expressing each branch of the graph of $||x| - 1|$ by a formula on the board).

Phanes’ connection between *graphical* and *symbolic* representations in Episode 1 is identified in research literature to promote meaning making of the mathematics. For instance, Haylock (1982) considered meaning making in terms of making connections within mathematics through different representations, such as symbols, diagrams, pictures. The students respond to Phanes’ ‘explaining’ by sketching “correct pictures” on their scripts (according to Phanes) and expressing correctly the formulas: $x + 1$ if $x$ belongs to $(-1,0)$ and $-x + 1$ if $x$ belongs to $(0,1)$.

The students’ visualisation does not nevertheless provide insight into the endpoints of intervals. After the episode, Phanes stresses to students that $||x| - 1|$ is continuous so “it doesn’t matter” if the endpoint is included in the interval. An alternative solution to Phanes’ geometric solution for that specific task is an algebraic application of the piecewise-linear function definition of absolute value, considering cases first for the expression $||x| - 1|$ and then for the expression $|x|$. In our discussion and in response to the question of why he chose a geometric solution when some mathematicians avoid choosing them, Phanes connected his choice with his research area.
It depends on your research area. If you are a Geometer [Phanes is a Geometer], you are happy with geometric solutions; it depends on your background I think. […] You see to me it is easier to see the graph. […] For instance if you are a programmer writing computer programs, then it is more convenient to you to give an algorithm.

Excerpt 14_Discussion after SGT7

An identification of another instance of Phanes’ ‘explaining’ is in Episode 2. In this episode, Phanes explained to the students the property according to which ‘a sequence has neither an upper bound nor a lower bound’. He said: “So, in other words, there’s an element of this sequence which can be as large as you like as long as it’s positive and as small as you like as long as it’s negative. So, it goes to plus infinity or it goes to minus infinity. So, it’s not bounded.” I coded this extract as a verbal representation of the property in formal language. This is because Phanes used formal mathematical language to explain orally the property, but he did not use exposition (e.g. of the negation of the definition of bounded sequence).

Phanes also explained to the students the procedure which gives as an output values of the formula of an unbounded sequence. He said to the students “you’re jumping from left to right, going further and further. It’s something like this.” and moved both his hands in a direction “further and further” away from each other. I coded Phanes’ explanation of the procedure as a verbal representation in informal language, because he used informal language to explain the graph of an unbounded sequence. I also coded his gesture, which represented the graph of an unbounded sequence, as a graphical representation.

Other instances of Phanes’ strategy ‘explaining’ are identified in Episode 3 and concern concepts in linear algebra. After using the examples with his fingers and a plane through his fingers, Phanes generalised from the examples and articulated the definitions without using notation. For instance he said for span “in general if you have a vector space, you have certain system of vectors. Then you take all possible linear combinations, what you get is their span”. Or he said for basis “a basis consists of linearly independent vectors such that any other vector in the space is their linear combination”. I coded both quotations of Phanes’ articulation of the concepts of span
and basis as verbal representations in formal language. This is because he explained orally the concepts in formal mathematical language without using a mere exposition of their definitions.

In the section Urging students to bring questions to the tutorial in Part 2 (Section 5.2.2.1), I reported that Phanes informed me that in order to teach the low-performing students, he was “going to a more basic level trying to explain”. Considering the pilot study and Phanes’ use of graphical and symbolic representations for the explanation of the concept of remainder in Taylor series, it seems to me that the repeating use of verbal representations in the main study was a basic level of explaining. For instance, Phanes’ verbal representations of definitions were different from the ones in the students’ lecture notes in terms of not including notation. Phanes repeatedly acted with those representations to enable the students to make sense of the definitions. He also encouraged the students to study the definitions from the lecture notes. As a result, he could not use that time for tasks in a more advanced level.

The students responded to Phanes’ verbal representations of definitions by saying: “So simple.” (SGT7); “That is it?” (SGT8); and “Oh so it, oh, I think I understand it.” (SGT10). However, when Phanes asked the students to articulate a definition, such as the one for basis in SGT7 and SGT8 [Episode 3], the students were not able to offer correct input. While observing, it seemed to me that although Phanes’ verbal representations of procedures and concepts made sense to the students, soon they forgot them. Indeed, after Phanes’ use of a verbal representation for a definition, one student said to another: “Write it down, before we forget it.”

The last two conceptual names of categories, which I interpreted as tools for the strategy ‘explaining’, are rhetorical questions (Fukawa-Connelly, 2012; Viirman, 2015) and revoicing statements (O’Connor & Michaels, 1993, 1996). Rhetorical questions are questions posed without the requirement of an answer. Similarly with the case of Zenobia’s teaching, analysis of transcripts indicates that Phanes used rhetorical questions before acting with verbal representations. Instances of rhetorical questions are in Episodes 1 and 3. My interpretation is that the role of Phanes’ rhetorical questions “how do I solve this problem?” [Episode 1], “what am I going to do?” [Episode 1] and “what do we need to prove?” [Episode 3] is to provide students with modes of mathematical thinking (Fukawa-Connelly, 2011). In contrast, the role
of Phanes’ *rhetorical questions* in Episode 1 “For instance, this expression is what?”, “Now, what is this?” and “So, what can we now say about this function?” is ‘to direct students’ attention to certain aspects of the mathematics worthy of reflection’ (Viirman, 2015; Artemeva & Fox, 2011). In other words, Phanes uses the last questions to enable the students to reflect on the mathematics he explains to them. Then, the students need to draw on their reflection in order to solve something similar.

I used the term *revoicing statements* to code Phanes’ reutterations of students’ contributions in the form of repeating exactly the same words with the students. For example, in Episode 1, Phanes repeats St2’s correct input “It’s $x + 1$.” and St4’s correct input “$-x + 1$”. Additionally, in Episode 2, he repeats St2’s correct input “$-2$ to the power of $n$.” It seems to me that the role of Phanes’ reutterations of students’ contributions is ‘to highlight’ (O’Connor & Michaels, 1996) and ‘to recruit students’ attention to a specific claim’ (Park, Kwon, Ju, Park, Rasmussen & Marrongelle, 2007). My interpretation thus is that Phanes uses *revoicing statements* to highlight the students’ correct inputs and to recruit the remaining students attention to them.

To conclude, Phanes had a number of tools for the strategy ‘explaining’: *graphical representations* (graphs, diagrams, gestures); *symbolic representations*; *verbal representations* (formal language, informal language and revoicing statements); and *rhetorical questions*. He constantly used the strategy ‘explaining’ and associated tools; in each single explanation, he connected the different kinds of representations, used verbal representations “in a more basic level” for his particular students, and highlighted his telling with rhetorical questions in order for the students to pay attention. My interpretation is that he was a tutor who had well-developed tools for ‘explaining’, although the extreme in terms of performance case of his student group challenged him for a first time attempt to use those tools in a university level teaching to “not university level” students.
Part 3: Phanes’ knowing for teaching in the main study

5.3.1 Mathematical knowing

Vygotsky (1997) connected the tendency to explain from one person to another and/or for oneself with the tendency to generalise and unite knowing (in this case, of the mathematics). For Vygotsky, ‘generalisation of first order’ is explanation of connections within a single area; whereas ‘generalisation of a second higher order’ is explanation of connections beyond the boundaries of a given area. So, generalising tends to unify a single area (first order) or different areas (second order). In the case of Phanes’ teaching, generalising and unifying was with mathematical representations and simple but still meaningful examples through connections and the principle of simplicity. As exemplified in the analysis of Episode 3 in the section Selecting examples (Section 5.2.2.3), Phanes’ simple but still meaningful examples were in different levels of complexity and from different mathematical areas, selected to offer the learner connections (e.g. between mathematical areas, between concepts). Furthermore, in the section Explaining (Section 5.2.2.6), I exemplified his strategy ‘explaining’ with various mathematical representations, which were designed to offer connections among different representations of concepts. My interpretation is that Phanes, with his simple but still meaningful examples and different mathematical representations, intended to stimulate students’ generalisation and unification of a single mathematical area (first order) or different areas (second order). This reveals the breadth and depth of Phanes’ mathematical knowing. Phanes also used simple but still meaningful examples and graphical representations in his own research in mathematics. His mathematical research was about “bridges within different Sciences; between say, Differential Geometry and Differential Equations”. Thus the explanations in his research contributed to ‘generalisation of a second higher order’.
5.3.2 Didactical knowing

In his teaching practice, Phanes draws on the space of mathematics to act with *simple but still meaningful examples* as well as the tools *graphical representations* and *symbolic representations*. Phanes reflects on his mathematical practices with these tools and develops his views on mathematics and thus his epistemology of mathematics. I got access to his epistemology of mathematics through our discussion about his views for connections between his mathematical research and his own teaching. Ultimately his practice, which I observed, with *simple but still meaningful examples* as well as *graphical* and *symbolic representations* is compatible with his epistemology of mathematics; i.e. the connected view and the principle of simplicity.

Phanes also draws on the space of teaching/learning to teach the mathematics to the students; for instance, he uses the strategy ‘showing and asking students to repeat’ and the tools *verbal representations* and *rhetorical questions*. His practice, which I observed, with ‘showing and asking students to repeat’ and the tools *verbal representations* and *rhetorical questions* is compatible with his epistemology of teaching/learning. Phanes uses these tools and strategy to translate mathematical thinking with *simple but still meaningful examples*, *graphical representations* and *symbolic representations* into forms of his thought in the context of students. In particular, he shows *simple but still meaningful examples* and asks students to repeat with more difficult *simple but still meaningful examples* (Figure 5.5). He also explains the *graphical representations* and *symbolic representations* (which are compatible with his epistemology of mathematics) with *verbal representations* and *rhetorical questions* (which are compatible with his epistemology of teaching/learning) (Figure 5.5).
Phanes’ *didactical knowing* is concerned with knowing ways of making the design of teaching in order to translate the principles and content of mathematics into forms of his thought in the context of students. Phanes designs with tools and strategies based on his epistemology of mathematics, such as *graphical representations*, in order to translate mathematical thinking with *graphical representations* into forms of his thought in the context of students. In his effort to make this translation, he enriches the design with tools and strategies based on his epistemology of teaching/learning, such as *verbal representations* and *rhetorical questions*. For instance, he explains the mathematical thinking with *graphical representations* in formal or informal language in order for students to make sense of the *graphical representations*. So, his design of teaching in order to translate the principles and content of mathematics into forms of his thought in the context of students include a path of informing: from practice drawn on his epistemology of mathematics towards practice drawn on his epistemology of teaching/learning (Figure 5.6).

Each time that Phanes designs his teaching for a particular mathematical topic, the initial design includes those tools and strategies that have been proved from preceding designs or redesigns to be appropriate for translating the principles and content of mathematics into forms of his thought in the context of students. In other words, the
initial design is the *distillate* of those tools and strategies that enabled students to make sense of the mathematics in Phanes past experiences. For example, Phanes has developed various tools and strategies for translating the principles and content of mathematics into forms of his thought in the context of students: ‘selecting examples’ with *simple but still meaningful examples* (compatible with his epistemology of mathematics); ‘showing and asking students to repeat’ with different *tasks, students’ work on their scripts* and *himself as tutor* (compatible with his epistemology of teaching/learning); and ‘explaining’ with different *representations* and *rhetorical questions* (compatible with his epistemology of either mathematics or teaching/learning).

Phanes’ *didactical knowing* is evident in his practice from designs of teaching whose repeatedly selected tools and strategies (compatible with his epistemology of mathematics) are enriched with tools and strategies compatible with his epistemology of teaching/learning. An example of such a design may include *simple but still meaningful examples* of a concept, enhanced with the strategy ‘showing and asking to repeat’. This reveals the path of informing: from practice drawn on his epistemology of mathematics towards practice drawn on his epistemology of teaching/learning (Figure 5.6).

### 5.3.3 Pedagogical knowing

*Pedagogical knowing* is concerned with knowing ways of moving across developmental stages of teaching until a developmental stage which enables the students to make meaning of the mathematics. ‘Moving across’ does not necessarily mean reducing the mathematical rigour or getting the tutor to do the students’ tasks for them. Rather, it is connected with flexibility in drawing on the students’ responses/silence and redesigning the teaching repeatedly with different tools and strategies until those that match with the learners’ different cognitive levels as well as thinking and learning styles (e.g. Marton & Säljö, 1976; Felder, 1993; Prosser & Trigwell, 1999).
The *pedagogical knowing* depends on the tutor’s strategy ‘evaluating students’ mathematical sense making’ for a judgment as to what stage of the (re)design enables the students to make meaning of the mathematics. Phanes had various tools for ‘evaluating students’ mathematical sense making’: *inviting* and *control questions*; *students’ ability to solve tasks*; and *simple but still meaningful examples*. In the data, ‘evaluating’ with the use of *simple but still meaningful examples* usually resulted in students who had not made mathematical sense.

Phanes students were low-performing students interested in mathematics once. They fit the category of student lack of basic mathematical skills, described by Hawkes and Savage (2000) in the *Mathematics Problem*. Hawkes and Savage described students who traditionally are not well prepared for university mathematics; so, the phenomenon with such students can be also identified in more cases than the case of Phanes’ teaching. As revealed in our discussions, Phanes was not familiar with working with such low-achieving students. Although he had developed various tools for ‘evaluating students’ mathematical sense making’, he had no previous experience that could inform his designs and redesigns in order to change those tools and strategies that seemed to be unsuccessful for other students. So, his final stage of redesigning sometimes did not seem to match with the students’ sense making. In other occasions the students seemed, from their responses, as if they had made sense in-the-moment but later they were not able to recall.

Phanes developed his teaching to his low-performing students, and in particular his strategy ‘explaining’, by changing his *verbal representations* “in a more basic level” than usual. So, his *pedagogical knowing* was concerned with knowing ways of explaining the mathematics “in a more basic level”, which in some occasions enabled the students to make sense of the mathematics in-the-moment. Figure 5.7 illustrates Phanes’ *pedagogical knowing* with verbal representations “in a more basic level”, which enabled him to ‘step down’ to the cognitive level of his students and work there to enable them to make mathematical sense.
An additional tool, which Phanes implemented in order to ‘step down’ to the context of his students and enable them to make mathematical sense, was *encouraging statements* in order for them to “study”. This was an additional way of developing his teaching for those students, since in my pilot study I had not observed him ‘urging the student to bring her questions to the tutorial’ with *encouraging statements*. The latter student had achieved highly in a piece of coursework, so it seems reasonable to me that she did not need encouragement to “study”.

To summarise, the case of Phanes’ teaching was concerned with breadth and depth of *mathematical knowing* and various tools and strategies revealing his *didactical knowing*. Also, analysis of his way of working with the students and our discussions indicated a developing *pedagogical knowing* in the context of his low-achieving students. That allowed for an analysis of knowing for teaching across the three cases of teaching and a deeper understanding of each type of knowing on the part of the researcher. The next chapter, Chapter 6, includes analysis of the case of Alex’s teaching practice and knowing, who is a researcher in mathematics education.
Chapter 6

DISCUSSION OF FINDINGS 3 – The case of Alex’s teaching

This chapter is devoted to a discussion on findings from analysis of the case of Alex’s teaching (Case study is a term discussed in Chapter 3: Section 3.2). The reader initially becomes familiar with the setting of Alex’s small group tutorials. The description of the setting is followed by Part 1, which includes a narrative of an observation in the pilot study, the follow-up discussion with Alex and an interpretation of aspects of Alex’s epistemology of teaching/learning. Part 1 continues with data from the main study and an analysis of Alex’s epistemologies of teaching/learning and of mathematics. The next part, Part 2, draws upon teaching episodes to exemplify analysis of Alex’s teaching practice into strategies and tools and interpretations. (Strategies and tools are terms discussed in Chapter 3: Section 3.4.2.) The last part is Part 3, which analyses knowing for teaching in the case of Alex’s teaching.

The setting

In this section, I provide the setting of Alex’s small group tutorials by a description about Alex, the pilot and main study observations and the layout of the classrooms.

Alex

Alex is a researcher in mathematics education who looks at the teaching and learning of mathematics through sociocultural lenses. He is a lecturer in two mathematics modules and a mathematics education module. His teaching responsibilities include
large cohorts of students in lectures and a small group of first year students in tutorials. Alex is experienced in both research and teaching. At the time of participating in my study, he had an eight-year research and teaching career.

**Pilot study and main study observations**

My first observation of Alex’s teaching was for the purposes of my pilot study in May 2013. This observation took place an academic year before I asked Alex to become a participant for a whole-semester observation. That semester lasted for twelve weeks from October 2013 to January 2014. Similarly with my observations of Phanes’ teaching, I did not observe Alex’s first tutorial in the main study in order for the students to become familiar with the small group tutorial setting.

**The layout of the classrooms**

In the pilot study, there was a large desk in the middle of the classroom and chairs around it. The capacity of the classroom was for 10 students. One student sat in the desk facing two whiteboards and Alex. Alex sat on the other side of that desk and stood to write the mathematics on the whiteboards. I sat in a chair behind and away from the student in order not to be intrusive, and to be able to observe and audio-record.

In the main study, the capacity of the classroom was for 20 students and the layout of desks was U shaped. In my first observation, I sat in a students’ desk and six students sat around me facing a whiteboard and Alex’s desk. During the tutorials, Alex stood to write to the whiteboard and circulated when students had to solve tasks. The students were friendly and seemed comfortable with me. They did not hesitate to talk to me or to ask me for a pen.
Part 1: An interpretation of Alex’s epistemologies in analysis of the pilot study and the main study

6.1.1 Analysis of the pilot study: Aspects of Alex’s epistemology of teaching/learning

In the pilot study, I started to interpret aspects of Alex’s epistemology of teaching/learning, based on my analysis of his views on teaching/learning. My account of the pilot study for Alex’s teaching starts with a narrative of my pilot study observation. This narrative was written during the pilot study and later, it was enriched for presentation purposes. Following the narrative is analysis of two of Alex’s views that emerged in my discussion with Alex after his tutorial. Concluding this section, I report on my learning from the pilot study.

<table>
<thead>
<tr>
<th>Narrative 1 Observation in May, 2013</th>
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<tbody>
<tr>
<td>During the tutorial, Alex informed me that his group initially included five students. Then, two students opted out and a student stopped attending tutorials. The latter student told Alex that this was because she did not speak to anyone in lectures and her tutorial group was only two students.</td>
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</tbody>
</table>

In my observation, I met only a female student. Her suggestion was to work on implicit differentiation with Alex due to a task she could not solve for an online test. Alex started the tutorial by writing \( x^2 + y^2 = 25, y = f(x) \) on the board and asked the student whether she could recognise the equation. The student said it is a circle. Then, Alex used exposition. He differentiated the equation with respect to \( x \) and after calculations, he concluded: \( \frac{dy}{dx} = -\frac{x}{y} \). He asked “what do we do in the case where we have more than one variable?” and wrote \( x^3 + y^3 + z^3 + 6xyz = 1, z = f(x, y) \) on the board. He differentiated the equation with respect to \( x \) and after calculations, he concluded: \( \frac{dz}{dx} = -\frac{x^2 + 2yz}{z^2 + 2yx} \). He also differentiated the equation with respect to \( y \) and after calculations, he concluded: \( \frac{dz}{dy} = -\frac{y^2 + 2xz}{z^2 + 2yx} \). Alex then introduced the Implicit Differentiation Theorem as a “simpler way”. He wrote: \( F(x, y, f(x, y)) = 0 \), \( \frac{dz}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{x^2 + 2yz}{z^2 + 2yx}, \frac{dz}{dy} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{y^2 + 2xz}{z^2 + 2yx} \). Finally, he stressed to the student that both ways give the same results.
Alex gave a piece of coursework in Linear Algebra back to the student. On the board, he wrote the proof task (of the coursework) the student had not solved. The task was:

Let $V$ be a vector space, and $\varphi: V \to V$ is a linear transformation. Prove that $\text{Im}(\varphi \circ \varphi) \subseteq \text{Im}\varphi$.

Alex stressed that the “important thing” is the subset and sketched the Venn diagram of Figure 6.1. On the board, he wrote $u \in \text{Im}(\varphi \circ \varphi) \rightarrow u \in \text{Im}\varphi$. He told the student that he used time to work out the latter relation thus it was not trivial. In the next couple of minutes, he made four questions to the student for next proof steps and proved the latter relation. Alex finally generalised by saying to the student: “these types of proofs are very common – that we want you to prove that something is a subset of another set. How would you do this? You take an element – any element – in the first set $[\text{Im}(\varphi \circ \varphi)]$ and prove that it’s also an element of the other set $[\text{Im}\varphi]$.”

In the next sections, I provide an analysis of two of Alex’s views of teaching/learning in tutorials. I recognised the two views from the narrative and my discussion with Alex in the pilot study.

6.1.1.1 Alex’s views of students’ difficulties

While I was observing Alex’s teaching of the proof for linear transformations, I took a note for his use of the Venn diagram. In my discussion with Alex after the tutorial, I asked him what the purpose for using the diagram was. Alex responded:

To make the proof more visual. That the relations that I was trying to explain in the proof had a visual reference, so she could see what things actually meant. Because for some students the notation is pointless, it doesn’t have any meaning. What does it mean that the elements are in the image of the linear transformation?
In Excerpt 1, Alex stressed that notation is a difficulty “for some students”. In particular, Alex pointed to a specific difficulty with notation which was relevant to his teaching which I observed: “What does it mean that the elements are in the image of the linear transformation?”

In Narrative 1, notation concerned the linear transformation $\varphi: V \rightarrow V$ and the relation $\text{Im}(\varphi \circ \varphi) \subseteq \text{Im}\varphi$. Alex explained the relation $\text{Im}(\varphi \circ \varphi) \subseteq \text{Im}\varphi$ to the student by using the Venn diagram of Figure 6.1. In that diagram, he illustrated what the notation for $\text{Im}(\varphi \circ \varphi)$, subset $\subseteq$, $\text{Im}\varphi$ and an element $u$ meant. He also illustrated what it means that an element $u$ is in the image of the linear transformation. From the diagram, he then developed an equivalent relation for $\text{Im}(\varphi \circ \varphi) \subseteq \text{Im}\varphi$, which was the relation $u \in \text{Im}(\varphi \circ \varphi) \rightarrow u \in \text{Im}\varphi$. This equivalent relation is a standard method for proving relations with subsets. Alex first said to the student that the subset was “important” and then developed the equivalent relation $u \in \text{Im}(\varphi \circ \varphi) \rightarrow u \in \text{Im}\varphi$.

In Excerpt 1, Alex told me that his purpose of using the diagram was the student’s meaning of notation by providing a visual reference for notation to her. In our pilot study discussion, I asked Alex whether he usually used diagrams or pictures in his teaching. He responded positively by saying that he sketched graphical representations, especially in proofs, in order to “visualise what is happening in the proof”. Observational data for evidence that the student made meaning are limited. Alex made four questions to the student and the student could not respond correctly to two of those questions.

6.1.1.2 Alex’s views of connections between his teaching and his mathematics education research

In the first section of this chapter, I reported that in his research Alex looked at the teaching and learning of mathematics through sociocultural lenses. In our pilot study discussion, I asked Alex whether there is influence of his research in his teaching. Alex’s response was that in his teaching he tried to have “conversation” about the mathematics with the students; so, students who “speak” in the tutorial. His supportive statement was: “usually when you engage in a conversation, you develop your cognitive skills”. This statement indeed is congruent with sociocultural
perspectives. Within a sociocultural perspective, for instance, learning starts in a social context such as a conversation.

In our discussion in the pilot study, Alex shared with me his teaching experience that sometimes it is difficult to engage students in conversation. In our discussion, he referred to “conversation” and his “questions” to students. He recognised that if students “don’t want to answer” his questions, he teaches without insisting in questioning and conversation with students. He explained to me that his aim in tutorials is students’ “learning” and not to “frighten them” because of questioning. It seems to me that Narrative 1 is an instance where Alex interpreted that conversation and questioning might frighten the student. So in the majority of tutorial time, Alex did not insist in questioning but used exposition.

In the past, Alex had been involved in a research project regarding the association of pedagogies and a range of learning outcomes. In our discussion, he brought to the fore research outcomes for pedagogy. He was skeptical with transmissionist pedagogies, and asserted that “connectionist teachers are better than transmissionist teachers”. In transmissionist pedagogies, the lecturer seeks to convey the required mathematical knowledge to student by exposition. So, the students are passive recipients of a fixed package of knowledge. They copy from the board and listen to the lecturer. For instance, the term ‘chalk and talk’ is used for a transmissionist pedagogy; the lecturer writes with chalk on the board and talks. Some pedagogues, such as Alex, consider connectionist pedagogies as more beneficial for students’ learning than transmissionist pedagogies. In connectionist pedagogies, mathematics lecturers try to enable students to make connections; for instance, connections within mathematics.

My interpretation is that in Narrative 1, an aspect of Alex’s pedagogy was transmissionist because his exposition covered the majority of the tutorial time. However, Alex said to me “I am trying not to be a fully transmissionist teacher”. He added to this that he tried, with the students, to make connections. He also provided me with an example in my observation where he made a connection between familiar and unfamiliar mathematics to the student. He claimed:
If you begin with something you know how to do, that you are familiar with [...] and then you move gradually to do something you don’t know, probably there is a connection there.

Excerpt 2_Pilot study discussion

In Excerpt 2, Alex referred to a gradual connection between familiar and unfamiliar mathematics to the student. Thinking of Vygotsky’s definition of ZPD, my interpretation is that Alex intended to enable the student to develop her meaning of differentiation in collaboration with him so that she internalises differentiation of two independent variable equations. (ZPD is a term discussed in Chapter 2: Section 2.2.) Indeed, Alex told me that his design was to ask the student whether she could recognise the equation \( x^2 + y^2 = 25 \), because the equation “comes up quite a lot in this semester” and by that time, the student was able to recognise that it is the equation of a circle. So, he started his teaching in differentiation by the one independent variable equation \( x^2 + y^2 = 25 \) with which the student was familiar. He declared the student made sense of differentiation for \( x^2 + y^2 = 25 \); her difficulty and unfamiliarity was with differentiation for two independent variable equations. So, then Alex demonstrated a more complex differentiation; he differentiated the two independent variable equation \( x^3 + y^3 + z^3 + 6xyz = 1 \).

In our discussion, Alex reflected that the connection between one independent variable differentiation and two independent variable differentiation was successful for the student. He informed me that “she understood how the transition from one variable to two variables was done. So, actually the connection went well.” The evidence he had for the student’s mathematical sense making of the two independent variable differentiation was that the student “didn’t ask anything” and “her face was of understanding”. However, my seat in the tutorial did not enable me to look at the student’s face.
My learning from the pilot study

From my experience of discussing with Alex after my observation, I learnt that asking questions to Alex and receiving his responses was an easier task for me than asking questions and receiving responses from other participants in my study. Alex was a researcher in mathematics education with some common research interests with me, so he analysed his participants’ teaching in his research. My interpretation is that from his research experience, Alex was in a position to analyse his own teaching and to articulate his analysis to me with confidence. Furthermore, considering that both Alex and I belong in the community of researchers in mathematics education, we could communicate with terminology in mathematics education (e.g. transmissionist pedagogy). Moreover, it was evident in our discussions that Alex reflected on research outcomes in mathematics education and tried to put them into practice (e.g. intentions for conversation for students’ cognitive development and a connectionist pedagogy). Finally, my seat behind the student did not provide me with the opportunity to look at the student’s face, which was important for Alex. In the main study, I decided to sit in a desk between the students.

6.1.2 Analysis of the main study: Aspects of Alex’s epistemology of teaching/learning and of mathematics

In the main study, Alex shared with me his views of: small group tutorials; his role as a tutor; what mathematics is; and what mathematical learning is. I synthesised Alex’s quotations from discussions with me after various tutorials to reveal these views. I also used observational data to provide evidence for Alex’s teaching with regard to his views.

Analysis indicates that in both pilot and main studies Alex’s views are connected with his teaching in observations, thus forming his thinking and perception for his teaching. In other words, Alex’s views of students’ difficulties, of connections
between his teaching and his research, of small group tutorials, of his role as a tutor, and of what mathematical learning is, are concerned with his epistemology of teaching/learning in tutorials. Furthermore, I considered that Alex’s view of what mathematics is includes an aspect of his epistemology of mathematics.

6.1.2.1 Alex’s epistemology of teaching/learning in tutorials

6.1.2.1.1 Alex’s views of small group tutorials and his role as tutor

Alex’s views of small group tutorials included students who participate in the tutorial regardless of their potential to offer a mathematical correct contribution. For Alex students’ participation concerned on the one hand, students who prepare work and bring questions about mathematics to the tutorial; and on the other hand, students who “speak” and think in the tutorial. Alex’s views of small group tutorials were revealed across his discussions with me.

In discussion after SGT2, I asked Alex what the thinking was about his questioning when he demonstrated the mathematics on the board. I offered him examples of his questions to students in SGT2, such as: “what do we do here?”, “what do you get there?”, “did you see it in A - levels?”, “do you recognise it?”. His response was:

If [students] sit there and just look at what I do, they won’t learn too much. I don’t believe in transmissionist or at least totally transmissionist pedagogy. […] I like students to at least do something in the tutorial, even if it’s wrong, that’s alright. […] There is the need to be certain dialogue, certain kind of “I ask you something and you have to think about it at least”.

Excerpt 3_Discussion after SGT2

In Excerpt 3, Alex declared that his pedagogy has a positive view of students’ mathematical mistakes. In my observations, students made mathematical mistakes in response to Alex’s invitations to speak. Alex did not usually respond by saying “no” for wrong contributions. Instead, he questioned the students’ contribution (e.g.
“Which square root? Positive or negative?” [SGT1]); or said the correct answer (e.g. “Minus 4, isn’t it?” [SGT7]).

Alex declared he does not believe in a totally transmissionist pedagogy [Excerpt 3_Discussion after SGT2], confirming our discussion in the pilot study and his views on the connection between his research in mathematics education and his own teaching. His view of small group tutorials included students’ thinking about the mathematics, and “dialogue” between the students and the tutor about the mathematics; both for students’ learning of the mathematics [Excerpt 3_Discussion after SGT2]. His questions to students had a dominant role in “dialogue”: “I ask you something and you have to think about it at least” [Excerpt 3_Discussion after SGT2]

Alex’s views of small group tutorials also included students responsible for preparing work and bringing questions about mathematics to the tutorial. Towards the end of SGT6 and SGT7, Alex suggested students to work on specific problem sheets and email him difficulties for the next tutorial. In our discussion after SGT7, I asked Alex what the reason was for these suggestions. He responded that “that gives them a little bit of more responsibility when they come to tutorials.” Students’ responsibility was valued by Alex. In discussion after SGT11, he stressed that students’ responsibility for preparing work and bringing questions to the tutorial was his objective for the tutorial.

[The objective] of the tutorial is to help them in whatever questions they have. The objective of the tutorial is that the question should come from them, they should make the work and if they get stuck, if they have a particular question or they want to understand a concept, then I help them with that. So, it is very good that almost every week, the girls send me an email beforehand saying we stuck in this, can you go through this?

Excerpt 4_Discussion after SGT11

In the above excerpt, Alex explained to me that his role as a tutor was to help students with their work on and questions about mathematics. In our discussions, Alex was explicit about his role as a tutor: “[I]t is their responsibility to prepare for the tutorials and not my responsibility. I am here to help them.” [Discussion after SGT5]; and “My
goal is to help them. That’s it.” [Discussion after SGT10]. Alex also valued “the girls’” responsibility to send him emails with their difficulties [Excerpt 4_Discussion after SGT11]. “The girls” were a group of two students created after SGT5 so that they work on the problem sheets together.

6.1.2.1.2 Alex’s views of what mathematical learning is

In our pilot study discussion, Alex declared that in his teaching he tried to have “conversation” about the mathematics with the students, because “usually when you engage in a conversation, you develop your cognitive skills”. So, Alex’s views included conversation about the mathematics for students’ learning. Excerpt 5, below, is from my discussion with Alex after SGT10 in the main study and concerns a more comprehensive Alex’s view on mathematical learning.

I understand that they need to go work on their own, maybe chat with each other, explain to each other and the ideas evolve. They don’t just come like that ‘oh today I understood what linear independence is’; they evolve. So, I understand that they have to go and think about them, try things. Also my role as a tutor is to help them if they stuck, if they don’t understand something. I move them forward or try to move them forward.

Excerpt 5_Discussion after SGT10

Alex viewed students’ mathematical learning as students’ effort to master the mathematics inside as well as outside the tutorial time. For Alex, mastering the mathematics included conversation among students about the mathematics (“maybe chat with each other, explain to each other and the ideas evolve”); thinking about the mathematics (“they have to go and think about them”); trying out ideas (“try things”) and working on their own (“they need to go work on their own”) [Excerpt 5_Discussion after SGT10]. My interpretation is that, for Alex, mastering the mathematics is a sociocultural practice, since even students’ thinking about the mathematics and working on their own are based on the lecture and tutorial material; thus, on students’ socialisation with the mathematical culture of their institution (or synonymously on students’ enculturation). In Excerpt 5, Alex connected his role as a
tutor with students’ mathematical learning and meaning making, asserting that he helps students to master the mathematics and overcome their difficulties.

6.1.2.2 Alex’s epistemology of mathematics

In Narrative 1 in the pilot study, Alex told the student that he used time to work out the relation \( u \in \text{Im}(\varphi \circ \varphi) \rightarrow u \in \text{Im} \varphi \). In SGT10 in the main study, while Alex was demonstrating the mathematics of a proof for a coursework task on the board, he informed the students that: The proof “took me a while. And I mean I hadn’t realised that for you it would have been really difficult.” In my discussion with Alex after SGT10, I asked him why he shared this information about time with students. The following excerpt is Alex’s response to my question and reveals his view on what mathematics is.

[The students] need to invest some time in understanding and trying things, some things would not work till they find the one that works. And that’s mathematics for me. […] So, that’s what real professional mathematicians do. So, we shouldn’t hide that from students. […] No matter what field you are researching in mathematics. You have to spend a lot of time thinking, ideas don’t come like everyday. They spend months and months, and you see in the history of mathematics to solve certain problems it took centuries.

Excerpt 5_Discussion after SGT10

A meaning Alex gained from the history of mathematics was that mathematics needs time and work to be mastered. Alex explained students’ work on mathematics to me, as “trying things, some things would not work till they find the one that works” [Excerpt 5_Discussion after SGT10]. So, Alex’s view on mathematics included sophisticated (researchers) or elementary (students) learners, who invest time to work on and make sense of the mathematics. From my analysis of Excerpts 4 and 5, it seems to me that Alex expressed a sociocultural view of mathematics connected to the process of mastering it through conversation, thinking, trying out ideas and working on one’s own. So, Alex’s view of ‘what mathematics is’ was revealed to be connected
to his view of ‘what mathematical learning is’, and his role as a tutor-helper for students’ meaning making.
6.2.1 Data analysis of Alex’s teaching practice

The small tutorial group of the main study had seven students, who were in the BSc Programme Mathematics and Accounting and Financial Management. Six of the seven students usually attended Alex’s tutorials for the whole of the semester.

In the main study analysis, I used data from eight tutorials to identify the strategies and tools of Alex’s teaching [SGT1, SGT2, SGT3, SGT4, SGT6, SGT7, SGT8, SGT11]. Table 6.1, below, demonstrates a characterisation of Alex’s teaching into strategies and tools in his teaching. (The conceptual names of categories for the tools numbered 3.1, 3.1, 3.2, 4.1, 5.4, and 6.4 are established concepts in literature, which are presented in a glossary in Appendix D.)

Table 6.1: Analysis of Alex’s teaching into strategies and tools for teaching.

<table>
<thead>
<tr>
<th>Conceptual names of strategies for teaching</th>
<th>Conceptual names of tools for teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introducing students to the tutorial group</td>
<td>1.1. Questions relating to students’ future references; 1.2. Questions relating to students’ suggestions for teaching; 1.3. Injunction statements (about his expectations from students, about emailing him questions for the next tutorial).</td>
</tr>
<tr>
<td>2 Selecting tasks</td>
<td>2.1. Students’ difficulties from teaching experience; 2.2. Students’ suggestions; 2.3. Students’ low results in coursework tasks; 2.4. Upcoming coursework assessment.</td>
</tr>
<tr>
<td>3 Selecting examples</td>
<td>3.1. Real-world examples and real-world non-examples; 3.2. Examples and non-examples.</td>
</tr>
<tr>
<td>4 Evaluating students’ sense making of mathematics</td>
<td>4.1. Questions to evaluate students’ sense making [control questions of students’ sense making (Viirman, 2015, p.1175), inviting questions to students: general to all students (Jaworski &amp; Didis, 2014, p.380)] and pause intervals; 4.2. Students’ mathematical questions; 4.3. Reinforcement [St face(s), St response(s) (St for students)].</td>
</tr>
<tr>
<td>5 Showing and asking students to repeat</td>
<td>5.1. Procedural, proof and conceptual tasks; 5.2. Students’ work on their scripts; 5.3. Tutor; 5.4. Heuristics (find definition(s) in lecture notes, consider special cases).</td>
</tr>
</tbody>
</table>
Explaining

6.1. Graphical representations (graphs, diagrams);  
6.2. Symbolic representations, tabular representations;  
6.3. Verbal representations [formal language, informal language].  
6.4. Rhetorical questions (Fukawa-Connelly, 2012).

The conceptual names of categories for the students’ response to Alex’s teaching, through which I characterise the students’ meanings in this study, are:

Correct input; Relevant input; Incorrect input; No input; St question (St for student);  
St difficulty; Reinforcement [St face(s), St response(s)].

In this section, I report three teaching episodes from the main study to provide empirical evidence for analysis and interpretations I made regarding the strategies and tools of Alex’s teaching. Brief accounts of Alex’s “design” and “redesign” for students’ meaning making follow after the presentation of episodes. Then, I provide my analysis and interpretations for each strategy and its associated tools.

<table>
<thead>
<tr>
<th>Episode 1_SGT2 and SGT3_The concept of injectivity [adapted from Mali (2015)]</th>
<th>Conceptual names of tools and strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>This episode includes compartments of SGT2 and SGT3 concerning Alex’s teaching of the concept of injectivity for one variable functions. Starting SGT2, Alex looked at a problem sheet in analysis which included tasks about finding the inverse of functions. He asked the students “What is an injective function?” In response, St4 made a question “Isn’t injective just (1 - 1)?” Alex said yes and asked the students for the formal definition of injectivity. St2 read first the definition of surjectivity from lecture notes, and then the definition of injectivity: (\forall x, y \in \text{Dom}(f), f(x) = f(y) \Rightarrow x = y). While reading the definition, St2 described how the symbols, which he was not aware of, looked like. Alex wrote the latter definition on the board and sketched the diagram of Figure 6.2. Alex: Let’s say that we have (f(x) = f(y)). Let’s say that (f(x) = f(y)) in the image of (f) in Figure 6.2. Then [‘then’ in Figure 6.2] (x) in here (x) in the domain of (f) in Figure 6.2 has to be?</td>
<td></td>
</tr>
<tr>
<td>Inviting question-gen.</td>
<td>Correct input</td>
</tr>
<tr>
<td>Inviting question-gen.</td>
<td>Correct input</td>
</tr>
<tr>
<td>Correct input</td>
<td>St difficulty</td>
</tr>
<tr>
<td>Symbolic representat.</td>
<td>Diagram-Figure 6.2</td>
</tr>
<tr>
<td>Rhetorical question</td>
<td>---</td>
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</tbody>
</table>
Alex added \( x = y \) on the diagram of Figure 6.2. Then, he sketched the diagram of Figure 6.3.

Alex: What cannot happen is that we have an \( x \) here [\( x \) in Figure 6.3] that takes me to \( f(x) \), and a different \( y \) [\( y \) in Figure 6.3] takes me to \( f(y) \), and this leads to that

\[
f(x) = f(y)
\]

in Figure 6.3]. This is not injective. For example, let’s take \( x^2 \). Would that be injective or not injective? [3” pause.] What do you think, St6?

St6: Not.

Alex: Not injective. Why? [4” pause.]

St6: It should not have two values of \( x \) go to —

Alex wrote on the board: \( x = -1 \mapsto x^2 = 1, x = 1 \mapsto x^2 = 1 \). St2 asked for an example of an injective function and Alex selected the linear function \( f(x) = x \). He said it is injective, since: “If \( f(x) = x \) , each one of these [\( x \in Dom(f) \), Figure 6.2] would reach one target [\( f(x) \in B \), Figure 6.2].” In SGT3, Alex used an example of function he had designed for one of his modules.

Alex: I thought a good example of function is like when you go to the supermarket to buy something and you go to the till and you want to buy let’s say a loaf of bread and you get to the till and the girl or the guy says it is 99p or 1.99.  You say it cannot be possible, it’s one or the other, it’s 99p or 1.99, it cannot have two values, two prices. Well that’s a function. A function is a relationship between a set of inputs, in this case the products in the supermarket, a loaf of bread, and the set of permissible outputs, in this case the prices. So it relates each product to the one, the only one price, it cannot be related to two. So that’s the key idea behind functions.

Alex asked the students to express injectivity in the context of his example. As a response to their inability to do so, he asked them to find the definition of injectivity \( [\forall x, y \in Dom(f) , \] \( f(x) = f(y) \Rightarrow x = y \) \] in their lecture notes. He then wrote the definition of injectivity on the board. In the following extract of the data, Alex implemented the example of the function for the concept of injectivity.

Alex: How would you read that [the definition of injectivity] in the supermarket example? [4” pause.] Which are the \( x \)’s? What’s the domain of the function? [12” pause.] St3, what would the \( x \)’s be in this example in Tesco
<table>
<thead>
<tr>
<th>St3: Products. Products.</th>
<th>Correct input</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alex: The products, exactly. So for all the products in Tesco</td>
<td></td>
</tr>
<tr>
<td>Encouraging statement</td>
<td></td>
</tr>
<tr>
<td>St3: They should be $x$ and $y$.</td>
<td>Relevant input</td>
</tr>
<tr>
<td>Alex: So why would it be $x$ and $y$?</td>
<td>Inviting question-dir.</td>
</tr>
<tr>
<td>St3: Because it’s product and price; $x$ is product, $y$ is price.</td>
<td>Incorrect input</td>
</tr>
<tr>
<td>Alex: $x$ could be bread, $y$ could be milk, mm?</td>
<td>Control question</td>
</tr>
<tr>
<td>St3: Because it’s product and price; $x$ is product, $y$ is price.</td>
<td>Inviting question-gen.</td>
</tr>
<tr>
<td>Alex: What would this mean, this then? [Alex points to ( f(x) = f(y) )] [15” pause.]</td>
<td>Pause interval</td>
</tr>
<tr>
<td>St2: The same price. [St2’s voice is almost inaudible.]</td>
<td>Correct input</td>
</tr>
<tr>
<td>Alex: What would that ( f(x) = f(y) ) mean in the example? [2” pause.] I want you to</td>
<td></td>
</tr>
<tr>
<td>Inviting question-gen.</td>
<td></td>
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<tr>
<td>contextualise a very abstract formal definition so we</td>
<td></td>
</tr>
<tr>
<td>do an everyday job that you can understand; that you</td>
<td></td>
</tr>
<tr>
<td>give some meaning to those things. [8” pause.] Try to</td>
<td></td>
</tr>
<tr>
<td>Pause interval</td>
<td></td>
</tr>
<tr>
<td>think on the example of the supermarket, what would</td>
<td></td>
</tr>
<tr>
<td>Inviting question-gen.</td>
<td></td>
</tr>
<tr>
<td>( f(x) ) equal, what would ( f(y) ) equal, what would $x$ be,</td>
<td></td>
</tr>
<tr>
<td>Control question</td>
<td></td>
</tr>
<tr>
<td>what would $y$ be? [4” pause.]</td>
<td></td>
</tr>
<tr>
<td>St5: ( f(x) ) would be prices.</td>
<td>Correct input</td>
</tr>
<tr>
<td>Alex: Yes, the prices, OK. So it says if the prices are equal,</td>
<td></td>
</tr>
<tr>
<td>Inviting question-gen.</td>
<td></td>
</tr>
<tr>
<td>let’s say 99p, what has to happen to $x$ and $y$? [17” pause.</td>
<td>Diagram-Figure 6.4</td>
</tr>
<tr>
<td>Pause interval</td>
<td></td>
</tr>
<tr>
<td>Alex sketches Figure 6.4 on the board.] Let’s</td>
<td></td>
</tr>
<tr>
<td>Control question</td>
<td></td>
</tr>
<tr>
<td>say $x$ is bread and $y$ is milk, OK? [1” pause.] And I</td>
<td></td>
</tr>
<tr>
<td>Tutor</td>
<td></td>
</tr>
<tr>
<td>notice that the price of the bread and the price of the</td>
<td></td>
</tr>
<tr>
<td>Control question</td>
<td></td>
</tr>
<tr>
<td>milk are the same, they are both 99p. Yes? [2” pause.]</td>
<td></td>
</tr>
<tr>
<td>Tutor</td>
<td></td>
</tr>
<tr>
<td>If this function was injective, then the bread would</td>
<td></td>
</tr>
<tr>
<td>Control question</td>
<td></td>
</tr>
<tr>
<td>have to be milk, well that’s impossible isn’t it? [2”</td>
<td></td>
</tr>
<tr>
<td>Control question</td>
<td></td>
</tr>
<tr>
<td>pause.] Alex deletes milk on Figure 6.4.] In other</td>
<td></td>
</tr>
<tr>
<td>Verbal representation: Informal language</td>
<td></td>
</tr>
<tr>
<td>Control question</td>
<td></td>
</tr>
<tr>
<td>words, I cannot have the price of 99p that belongs to</td>
<td></td>
</tr>
<tr>
<td>Verbal representation: Informal language</td>
<td></td>
</tr>
<tr>
<td>two products, two different products, OK? In the</td>
<td></td>
</tr>
<tr>
<td>Control question</td>
<td></td>
</tr>
<tr>
<td>abstract definition, there is no way that 99p comes</td>
<td></td>
</tr>
<tr>
<td>Reinforcement (St</td>
<td></td>
</tr>
<tr>
<td>from bread and milk. Does that make sense or not? [2”</td>
<td></td>
</tr>
<tr>
<td>faces)</td>
<td></td>
</tr>
<tr>
<td>pause.] Say no if [5” pause] well your faces say no. [3”</td>
<td></td>
</tr>
<tr>
<td>Reinforcement (St</td>
<td></td>
</tr>
<tr>
<td>pause.]</td>
<td></td>
</tr>
<tr>
<td>response)</td>
<td></td>
</tr>
<tr>
<td>St1: No, I’d say no.</td>
<td></td>
</tr>
<tr>
<td>2 Inviting questions</td>
<td></td>
</tr>
<tr>
<td>2 Pause intervals</td>
<td></td>
</tr>
<tr>
<td>Alex: OK. Can you think of another example? […] Do you</td>
<td></td>
</tr>
<tr>
<td>Real-world non-example/ Conceptual</td>
<td></td>
</tr>
<tr>
<td>play a sport? [12” pause.]</td>
<td></td>
</tr>
<tr>
<td>task</td>
<td></td>
</tr>
<tr>
<td>St1 mentioned hockey and Alex devised another example</td>
<td></td>
</tr>
<tr>
<td>Tabular representation</td>
<td></td>
</tr>
<tr>
<td>regarding a function that relates hockey players with their</td>
<td></td>
</tr>
<tr>
<td>scores. This function was expressed by Alex in a tabular</td>
<td></td>
</tr>
</tbody>
</table>
| representation.
Brief account of Alex’s design and redesign for students’ meaning making: Episode 1

The following Figure 6.5 is a figural representation of Alex’s stages of design and successive redesigns in Episode 1 in order for the students to make meaning of injectivity. The stages of design and redesign are developmental of Alex’s teaching for students’ mathematical meaning making. In the final redesign, Alex’s perspective is that his teaching has reached a stage which enables the students to make meaning of injectivity. As a result, he does not enrich his teaching with a new redesign.

In Figure 6.5, the red arrow represents a dialectic connection between two blue stages; that is to say, between design and redesign. The dialectic connection represents contradiction(s) in dialogue about mathematical meanings between Alex and the students. Alex’s design in Episode 1 is that the students articulate the definition of injectivity (first blue stage in Figure 6.5). However, Alex’s dialogue with the students indicates a contradiction between Alex’s design for students’ articulation of the definition of injectivity and St2’s difficulty with notation in the definition of injectivity. Alex redesigns to resolve the difficulty (second blue stage). To support the students in the redesign stage, he writes the symbolic representation of the definition of injectivity on the board; explains in formal language with the accompanying diagrams of Figures 6.2-6.3 to the definition; and uses a non-example and a prototypical example of injectivity. The latter example is in response to St2’s question for an example of injectivity, which indicates a contradiction between Alex’s redesign (second blue stage) for students’ meaning making of injectivity and St2’s difficulty with injectivity. St6 nevertheless offers a correct input.
In response to St2’s question for an example of injectivity, Alex redesigns his teaching to enable the students to make meaning of injectivity in the next tutorial (third blue stage in Figure 6.5). His redesign is, first, to show to students the way of working for the concept of function in a real-word situation in Tesco, and then, in the same real-world situation to ask the students to repeat the work for the concept of injectivity. In the dialectic connection between Alex and the students (third red arrow in Figure 6.5), St2, St3 and St5 offer some correct inputs after long pause intervals from Alex. However, the students are not able to articulate injectivity in the real-world situation and this contradicts Alex’s redesign (third blue stage) for students’ meaning making of injectivity. In response to students’ difficulty, Alex redesigns (fourth blue stage in Figure 6.5) to explain the concept of injectivity and articulate by himself injectivity in the real-world situation.

In the dialectic connection between Alex and the students (fourth red arrow in Figure 6.5), the students offer reinforcement to Alex that they cannot make sense of injectivity in the real-world situation. Again, this contradicts Alex’s redesign (fourth blue stage) for students’ meaning making of injectivity. Alex redesigns (fifth blue stage in Figure 6.5) to enable the students to make meaning of injectivity with another real-world situation relevant to St1’s interests. From his dialogue with the students about the new real-world situation, Alex interprets that the students have made sense of injectivity and he does not redesign.

Figure 6.5 includes two interrelated helixes, not two straight lines connected with each other by the developmental stages of design and redesign. The helixes are interrelated because in teaching mathematics the space of mathematics and the space of teaching/learning are interrelated. An example of the interrelation between the two helixes is across the stages of design and redesign in Episode 1. In particular, in the stages of design and first redesign (first two blue stages), Alex draws on tools of the space of mathematics to enable the students to make meaning of injectivity. So, he acts with the symbolic representation of the definition of injectivity; formal mathematical language; the heuristic ‘find definition in lecture notes’; a non-example and an example of injectivity. However in the next stages of redesign (third to fifth blue stages), he steps out of the space of mathematics to consider the context of the students and select examples there that he can use to parallel injectivity. My
interpretation is that he draws on the space of teaching/learning to step outside into the real world and then bring back parallel examples; to explain injectivity in informal language; and to respond to St3’s correct input with an encouraging statement. This indicates that in Alex’s mathematics teaching the space of mathematics and the space of teaching/learning are interrelated.

Figure 6.5: Alex’s design and redesign for students’ meaning making in Episode 1.
<table>
<thead>
<tr>
<th>Episode 2_SGT8_Proof by definition: Linear map</th>
<th>Conceptual names of tools and strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>This episode is situated in SGT8. It concerns Alex’s teaching for the concept of linear map. The two students (St4 and St5), who formed a group and studied together outside tutorial time, had emailed Alex before SGT8. In this email, they told him about their difficulty with coursework tasks on linear maps and suggested work on this topic for SGT8. In SGT8, Alex suggested the students to get their lecture notes out. He told them that some functions they study in analysis are linear maps. He looked at a student’s problem sheet on linear maps and selected two tasks for his teaching. The first task was to prove that the map ( \phi: \mathbb{C}^3 \to \mathbb{C}^2 ) ((z_1, z_2, z_3) \mapsto (iz_2, z_1)) is linear or explain why it is not. (\mathbb{C}^n) is the (n)-dimensional vector space over the field of complex numbers.</td>
<td>Proof task 1</td>
</tr>
<tr>
<td>(\phi_a: \text{Mat}<em>{2 \times 2}(R) \to R[x], \begin{pmatrix} a &amp; b \ c &amp; d \end{pmatrix} \mapsto (a + d)x^2 - cx + 3b.) ([\text{Mat}</em>{2 \times 2}(R)]) is the four dimensional vector space of (2 \times 2) real matrices. (R[x]) is the vector space of polynomials with real coefficients. On the board, Alex calculated (\phi) for (z_3 = 1, z_2 = i, z_3 = -3) and told the students that (\phi) is a transformation. He then asked the students what the conditions are so that a map is linear. In response to their inability to respond, he discussed privately with two students, and then wrote the conditions on the board.</td>
<td>Proof task 2</td>
</tr>
</tbody>
</table>
| 1) \(\phi(u + v) = \phi(u) + \phi(v)\)  
2) \(\phi(\lambda u) = \lambda \phi(u)\) | Symbolic representat. |
| Alex: So for example, let’s try and prove if this map \([\phi: \mathbb{C}^3 \to \mathbb{C}^2, (z_1, z_2, z_3) \mapsto (iz_2, z_1)]\) is linear. So what will we have to do? [6” pause.] What does that \([\phi(u + v) = \phi(u) + \phi(v)]\) tell you? [6” pause.] It’s just telling you “take any \(u\) and \(v\) in complex numbers, add them, transform [the sum \(u + v\)] and then that \([\phi(u + v)]\) has to be this \([\phi(u) + \phi(v)]\).” Let’s take \(u\) and \(v\) in the domain [Alex writes \(u, v \in \mathbb{C}^3\) in Figure 6.6 below], so what kind of vectors are these \([u, v]\)? \[\text{Alex writes } u = (a, b, c), v = (d, e, f) \text{ in Figure } 6.6.\] Yeah, two vectors \([u, v]\) in general but living in \(\mathbb{C}^3\), then let’s make the sum of them. So how do I add these vectors? [7” pause.] | Verbal representation: Formal language |
| St5: \(a\) plus \(d\), \(b\) plus \(e\), \(c\) plus \(f\). | Rhetorical question |
| Alex exposes the mathematics of the solution and writes on the board the solution of Figure 6.6. For the second task, Alex calculated \(\phi_a\) for \(a = 1, b = 2, c = 3, d = 4\) and told the students that \(\phi_a\) is a transformation. He then provided the students with 7’ 12” to prove on their scripts that \(\phi_a\) is a linear map. Students were writing on their | Heuristic ‘special cases’ |

| | St difficulty |

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scripts or looking at the board. Alex circulated. St5 asked what the second matrix they should take to check the conditions would be. Alex responded by indicating matrices \[
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix}
\] and \[
\begin{pmatrix}
ea_{11} & a_{12} \\
b_{11} & b_{12} \\
\end{pmatrix}
\] or \[
\begin{pmatrix}
eg & f \\
g & h \\
\end{pmatrix}
\] and \[
\begin{pmatrix}
ea_{21} & a_{22} \\
b_{21} & b_{22} \\
\end{pmatrix}
\]. 4' 20" after his response to St5, Alex asked the students whether \(\phi_4\) is a linear map. Several students including St4 and St5 responded that it is.

\[
\begin{align*}
u, v & \in \mathbb{C}^3 \\
u &= (a, b, c) \\
v &= (d, e, f) \\

u + v &= (a + d, b + e, c + f) \\
\phi(u + v) &= (i^3(b + e), (a + d)) \\

\phi(u) &= (i^3b, a) \\
\phi(v) &= (i^3e, d) \\

(i^3(b + e), (a + d)) \\

\phi(\lambda u) &= (i^3\lambda b, \lambda a) = \lambda(i^3b, a) = \lambda\phi(u)
\end{align*}
\]

**Figure 6.6:** Alex’s writing on the board for the solution of the first task.

**Brief account of Alex’s design and redesign for students’ meaning making: Episode 2**

Figure 6.7, below, is a figural representation of Alex’s stages of design and redesign in Episode 2 in order for the students to make meaning of linear maps. His design (the first blue stage) is to select tasks for proving with the definition of linear map. The students’ inability to articulate the conditions of the definition is in contradiction with Alex’s design for students’ meaning making of linear maps (the dialectic connection of the first red arrow). In response to the students’ inability, Alex redesigns the teaching. His redesign (the second blue stage) is to show to the students the way of proving in Task 1 and to ask them to repeat that way of proving in Task 2. While students work on their scripts for Task 2, Alex circulates and makes dialogue with the students (the dialectic connection of the second red arrow). From this dialogue and from the students’ correct response that \(\phi_4\) is a linear map, he decides that the students have made sense of linear maps and he does not redesign.

In the stages of design and redesign, Alex draws on the space of mathematics to act with tools which are symbolic representations, formal language and the heuristic
‘special cases’. These are mathematical tools for the strategy *showing and asking students to repeat*. The strategy nevertheless steps out of the space of mathematics to consider the context of the students and thus the space of teaching/learning. This is an instance that indicates that in Alex’s mathematics teaching the space of mathematics and the space of teaching/learning are interrelated.

**Figure 6.7:** Alex’s design and redesign for students’ meaning making in Episode 2.

### Episode 3_SGT11_Computing the inverse of a linear map

<table>
<thead>
<tr>
<th>Conceptual names of tools and strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>This episode is situated in SGT11. St4 and St5, who had formed a group for studying outside tutorial time, had emailed Alex before SGT11. In this email, they suggested a task with which they found difficulty. The task was from a practice exam sheet in Linear Algebra:</td>
</tr>
<tr>
<td>Is the linear map ( \varphi: \text{GF}_5^3 \to {[t] \mid t \in \text{GF}_5 } ) given by</td>
</tr>
<tr>
<td>Procedural task</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Is the linear map ( \varphi: \text{GF}_5^3 \to {[t] \mid t \in \text{GF}_5 } ) given by</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>In SGT11, Alex started to solve the task on the board, as illustrated in Figure 6.8, while using exposition for the mathematics. When he wrote “( \begin{pmatrix} 1 &amp; 4 &amp; 1 \ 2 &amp; 0 &amp; -3 \ 0 &amp; 2 &amp; 0 \end{pmatrix} ) ( \longrightarrow )” in Figure</td>
</tr>
</tbody>
</table>
6.8. St4 made the following question.

St4: [...] How does that show it was bijective?

Alex: Well what you’re doing here is seeing if this system

\[
\begin{align*}
  a &= x + 4y + z \\
  b &= 2x - 3z \\
  c &= 2y
\end{align*}
\]

has a solution. If I take a general

form \( [v = (a, b, c)] \) and map it through the
transformation \( \varphi[\ldots] \) what I want to do [is to]
establish that there’s a relation that is both injective and
surjective.

Alex stressed that in case the system does not have a solution,
“there are elements in the codomain that are not taken”,
because none of the elements in the domain would go to the
codomain. He added “That would tell me that it’s not surjective
and therefore it’s not bijective.” He then solved the system and
said: “So it has a solution and it is unique. So that tells you that
it is bijective. Let’s say it had a solution but it was not unique,
there’s an infinite number of solutions, then that would tell me
that it is not injective and therefore not bijective.” St2 asked
again what case corresponds to a not injective map, a not
surjective map and a bijective map to make sure he noted them
down correctly.

\[
p \in (F_5)_2[t] \text{ (in codomain)}
\]

\[
v \in F_5^3
\]

such that \( \varphi(v) = p \)

\[
\begin{align*}
  p &= at^2 + bt + c \\
  a &= x + 4y + z \\
  b &= 2x - 3z \\
  c &= 2y
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 4 & 1 \\
  2 & 0 & -3 \\
  0 & 2 & 0
\end{pmatrix} \rightarrow
\]

**Figure 6.8:** Alex’s writing on the board for the solution of the task.

**Brief account of Alex’s design and redesign for students’ meaning making: Episode 3**

The following Figure 6.9 is a figural representation of Alex’s stages of design and
redesign in Episode 3 in order for the students to make meaning of the inverse of a
linear map. His design (the first blue stage) is to select a task for calculating the
inverse of a linear map and to explain with symbolic representations.
St4’s question about the symbolic representation \[
\begin{pmatrix}
1 & 4 & 1 \\
2 & 0 & -3 \\
0 & 2 & 0
\end{pmatrix}
\] is in contradiction with Alex’s design for students’ meaning making of the inverse of a linear map. In response to St4’s question, Alex redesigns the teaching. His redesign (the second blue stage) is to explain in formal language how to find the inverse of the linear map in the task.

In the stages of design and redesign, Alex draws on the space of mathematics to act with tools, which are symbolic representations and formal language. In contrast to analysis of other episodes, he does not draw on tools from the space of teaching/learning in this episode.

Figure 6.9: Alex’s design and redesign for students’ meaning making in Episode 3.
6.2.2 Alex’s strategies for teaching and the associated tools

6.2.2.1 Introducing students to the tutorial group

Analysis in the section Alex’s view of what mathematical learning is in Part 1 of this chapter (Section 6.1.2.1.2) indicates that Alex’s view included conversation about the mathematics for students’ learning; so, students who “speak” in the tutorial. In the main study, Alex’s concern about the students was that they were silent in tutorials. What he did in response was to invite all seven students to attend SGT5 and to design it in order to make students “speak”. I met the seventh student only in that tutorial. In this section, I report Alex’s tools and strategies in order to make students “speak”.

In our discussions after SGT2-SGT5, Alex declared his concerns about his students’ oral participation during tutorial time. In my discussion with Alex after SGT2, he shared with me his experience regarding some tutorial groups being “mostly quiet” in spite of his “questions to them”. SGT4 offered me data to exemplify Alex’s questions to students so that they participate in tutorials. Alex said to me that in response to his students’ low oral participation and declared difficulties with notation in previous SGTs, he designed tasks for SGT4. His declared design was “to connect with what they do outside”; in particular, tasks close to students’ experiences of everyday life. One of the tasks was:

If the proposition is \( P(x) = x \) likes pizza, what does the following mean?

a) \( \forall x \ P(x) = x \)

b) \( \exists x \ P(x) = x \)

c) \( \sim \forall x \ P(x) = x \)

d) \( \sim \exists x \ P(x) = x \)

This task indeed is close to students’ experiences of everyday life: \( x \) could be a student and if \( P(x) = x \) then this student likes pizza.

During the tutorial, Alex made questions to students such as what the quantifiers meant in the context of the task. For instance, he asked: “So St1, in that example of pizza what is \( x \)?” and St1 correctly responded “a person”; or “If I write \( \exists x \ P(x) = x \), what am I saying?” and St5 correctly responded “Someone in the world likes pizza.”
The students’ correct responses revealed they made sense of the quantifiers. However, after SGT4, Alex reflected that although he had designed those tasks “to make [the students] speak”, it was “really hard” for him to actually make this particular group of students speak. He stressed: “They don’t feel comfortable. That’s my perception. [...] It is Week 5 and by this time they should be more open” to participate. In our discussion after the next SGT, Alex generalised his experience with his students saying that the students “speak very little” and “[i]f I do the talking all the time, they don’t even answer my questions”.

My observations gave me insight into Alex’s aforementioned experience with his students and his declared perception regarding students’ uncomfortable situation to “speak”. As an observer, I confirm that in SGT4 the students gave short answers to Alex’s questions, and spoke only when Alex made questions directly to them or generally to all students. I also report that although Alex insisted the students solve the problem sheets at home and suggest tasks or mathematical topics in tutorials, the students did not make suggestions until SGT5.

My interpretation is that a strategy Alex used (so as to make students “speak”) was his design and implementation of tasks. I also interpret that tools for this strategy were: tasks close to students’ experiences and his questions to students in the context of the tasks. However, according to Alex, these tools and strategy did not seem to enable the students to “speak” during the tutorial time. In our discussion after SGT4 he confirmed: “I still don’t know the answer on how to get to them”, meaning how to enable the students to overcome their uncomfortable situation to “speak”.

I reported earlier that Alex invited all seven students to attend SGT5. For this tutorial, he told me that his design was to change the dynamics of the students’ low oral participation: by stressing his expectations to the students; and by inviting each student to introduce herself/himself to the tutorial group. He shared his assumption with me that the students might feel more confident to “speak” in next tutorials if they got to know each other. My interpretation is that Alex’s design was concerned with the strategy introducing students to the tutorial group and was towards the goal to enable the students to “speak”.

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I observed Alex starting SGT5 by stressing his expectations to the students: the students would solve tasks in specific problem sheets agreed from the previous tutorial and email him their questions for the next tutorial. He then said they would do “no maths” in SGT5 and asked the students to “sell themselves”, like they would do in a panel for their dream job, by saying: who they are; what their best attributes are; why they are there; if there is something they find hard/easy/nice; and something important/interesting they would like to share. He also encouraged the students for constructive criticism among them. Alex’s arguments about the necessity of each student introducing herself/himself to the tutorial group was that he could know them and give them good references for jobs in the future. I interpret that tools, with which Alex acted, were questions relating to students’ future references.

The students had five minutes to talk about themselves and Alex was the last one who introduced himself. Almost all students stated they like mathematics and have good marks in A-level Maths. A few students declared their mathematical or study skills such as logical thinking, problem solving and hard working. Four of seven students stressed that sports and sports facilities was a good reason they selected the University. A few students said that the high ranking of the University was a reason they selected it for studies. However, no student made criticism to another student’s sayings and some students were more resistant to talk about themselves. Alex encouraged all students to talk by adding direct questions to them such as: “What do you do in your life?”, “How do you describe yourself as a student?” and “Is there anything you would like to change about university life or teaching?” My interpretation is that these were additional questions relating to students’ future references and questions relating to students’ suggestions for teaching. Two students replied they would like more tasks to work out and appreciate SGTs since they can ask questions there. Closing SGT5 and after Alex talked about himself, he stressed specific problem sheets for students’ homework and the necessity for students to bring their questions in the next tutorial. In this way, he acted with injunction statements about his expectations from students, which I considered as tools for the strategy introducing students to the tutorial group.
From SGT6 to SGT11, Alex gave one or more problem sheets (per tutorial) to students for homework and stressed he did not expect from them to solve all tasks; however, he insisted the students email him their difficulties. My interpretation is that *injunction statements about emailing him questions for the next tutorial* was another tool associated with the strategy *introducing students to the tutorial group*. After SGT7, Alex told me that by asking the students to send him an email with their difficulties, he was “trying to change the dynamics” of the tutorial and give more responsibility to the students. The dynamics of that group was that the students were silent and did not suggest difficulties to work out in next tutorials. Alex explained to me: “I don’t like *me* [stressed voice tone] selecting the exercises [i.e. tasks] because I don’t know if they are really having problems [i.e. difficulties] with those exercises [i.e. tasks].” In my question about how Alex attempted to change the dynamics, he responded:

> In a way that the exercises [i.e. tasks] have to come from them. The problems [i.e. difficulties] have to come from them. They have to tell me I am having trouble with this one, with this exercise [i.e. task], or that one. Instead of *me* [stressed voice tone] saying which ones *I* [stressed voice tone] think they would have problem [i.e. difficulty] with.

Excerpt 6_Discussion after SGT7 [brackets added]

As an observer, I report that the students’ oral participation was raised after SGT5. The majority of students also achieved good marks in the pieces of coursework in a particular module. In SGT6, only two students came to the tutorial. According to Alex, these two students achieved good marks in pieces of coursework and usually did a piece of the homework. Furthermore, in SGT6, a student asked Alex to solve a task of a problem sheet on the board. So, Alex started to succeed in getting students to suggest work for the tutorial. From SGT7 on, five or six students usually attended the tutorials. A group of two students was also created so that they work on the problem sheets together. One of the two students usually sent emails to Alex to inform him about difficulties they would like to discuss in a next tutorial. These difficulties were usually connected to mathematical topics the students were going to be assessed in tests. Finally, the students asked Alex to solve tasks on the board and made mathematical questions to Alex such as “If you are asked to prove that this is a span,
then what would you have to do?” Thus, my observation of the students’ response to the strategy **introducing students to the tutorial group** indicates that it was successful for some students’ participation with regard to “speaking” and doing mathematics during and outside the tutorial time.

### 6.2.2.2 Selecting tasks

From the eight tutorials I analysed for the case of Alex’s teaching in the main study [SGT1, SGT2, SGT3, SGT4, SGT6, SGT7, SGT8, SGT11], the first four included tasks exclusively in analysis, and the last four comprised more tasks in linear algebra than in analysis. For example, 3 of 8 tutorials included tasks on the concepts of injectivity and surjectivity in analysis [SGT2, SGT3, SGT4]. In discussion after SGT2, Alex informed me:

> I know from experience that for them, that injective and surjective doesn’t make any sense. Or some of them say “Why do I need that? If in college I didn’t need it”.

**Excerpt 7_Discussion after SGT2**

In observations, the students’ difficulty was revealed in their responses to Alex. For instance, in Episode 1, the students could not identify some symbols in the definition of injectivity in the supermarket example. In discussion after SGT2, Alex explained to me that the emphasis on analysis tasks was a decision he made in his tutorials.

> In the first semester, I tend to leave linear algebra a little bit relegated. I know that [the lecturer in Linear Algebra] is very good and my previous experience is that students don’t have too much trouble with linear algebra in the first semester.

**Excerpt 8_Discussion after SGT2**

Alex invoked his teaching experience to argue for his decision to focus on analysis (e.g. the concepts of injectivity and surjectivity) in his tutorials. This indicates that an Alex goal in selecting tasks was to enable the students to resolve their difficulties and thus to make sense of the underlying mathematics.
Students suggested a task in analysis in SGT2. They regularly started to suggest work for tutorials after SGT5. They did not have many suggestions; however, they focused on linear algebra. In particular, students suggested a task from a problem sheet in analysis in SGT6; a revision on linear maps for an upcoming coursework in SGT8 and SGT9 [Episode 2]; and a task in linear algebra from a practice exam sheet in SGT11 [Episode 3]. My interpretation is that students’ difficulties from teaching experience, students’ suggestions and upcoming coursework assessment were tools for Alex’s strategy selecting tasks.

I distinguish all tasks Alex used in the main study tutorials into: proof tasks (e.g. “prove that”, “show”), procedural tasks (e.g. “compute”, “sketch”), and conceptual tasks (e.g. non-examples that do not fit the definition of a concept). For instance, in Episode 1, I consider the non-example of injectivity ($f(x) = x^2$), the example of injectivity ($f(x) = x$) and the real-world examples of injectivity (the supermarket example and the hockey example) as four conceptual tasks. They are conceptual since Alex focused on the concept of injectivity. In Episode 2, I consider the tasks ‘Prove that the map $\varphi: \mathbb{C}^3 \rightarrow \mathbb{C}^2, (z_1, z_2, z_3) \mapsto (i^3 z_2, z_1)$ is linear or explain why it is not’ and ‘Prove that the map $\varphi_4: Mat_{2 \times 2}(\mathbb{R}) \rightarrow \mathbb{R}[x], \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a + d)x^2 - cx + 3b$ is linear or explain why it is not’ as proof tasks; they are about proving linear maps. In Episode 3, I consider the task ‘Is the linear map $\varphi$ bijective? If so, compute its inverse’ as a procedural task. The procedure is the solution of a system of linear equations with Gaussian elimination.

In the 8 SGTs I analysed for my main study, Alex used 48 tasks in total, from which: $7/48 \approx 15\%$ were proof tasks, $27/48 \approx 56\%$ were procedural tasks, and $14/48 \approx 29\%$ were conceptual tasks. So, approximately 55% of the tasks were procedural tasks, whereas approximately 45% of tasks were proof and conceptual tasks. Although procedural tasks might be worked out in less time than conceptual and proof tasks, my interpretation is that the high percentage of procedural tasks reveals another Alex’s goal in selecting tasks, which is ‘students to pass the modules’. Table 6.2 demonstrates the 48 tasks Alex used for the eight tutorials in the main study.
<table>
<thead>
<tr>
<th><strong>SGT#</strong></th>
<th><strong>Conceptual tasks</strong></th>
<th><strong>Procedural tasks</strong></th>
<th><strong>Proof tasks</strong></th>
<th><strong>Topic</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>SGT1</td>
<td>-</td>
<td>“Find the natural domain of functions” (6 tasks)</td>
<td>-</td>
<td>Analysis</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Linear algebra</td>
</tr>
<tr>
<td>SGT2</td>
<td>Examples and non-examples of injective and surjective functions, following the respective definitions. (4 tasks)</td>
<td>“Find the inverse of functions” (4 tasks) “Compute the limits” (5 tasks)</td>
<td>-</td>
<td>Analysis</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Linear algebra</td>
</tr>
<tr>
<td>SGT3</td>
<td>Two real-world examples of functions for injectivity and surjectivity (2 tasks)</td>
<td>“Compute the limits” (3 tasks) “Find derivatives of the functions” (3 tasks)</td>
<td>-</td>
<td>Analysis</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Linear algebra</td>
</tr>
<tr>
<td>SGT4</td>
<td>“Using quantifiers and mathematical notation, write the following propositions” “If the proposition is $P(x) = x$ likes pizza, what do the following statements mean?” “If $Q(x) = x$ likes pepperoni, what do the following statements mean?” “How would you read the following statements? [definitions of injectivity and surjectivity]” (4 tasks)</td>
<td>“Sketch the graphs of the functions” (2 tasks)</td>
<td>-</td>
<td>Analysis</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Linear algebra</td>
</tr>
<tr>
<td>SGT6</td>
<td>-</td>
<td>“Compute the integral” (1 task)</td>
<td>-</td>
<td>Analysis</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Linear algebra</td>
</tr>
<tr>
<td>SGT7</td>
<td>“Find a linear combination of $v_1, v_2$ which gives e” (1 task) Non-example of linear independence (1 task)</td>
<td>“Prove the set $S$ is a subspace” (1 task) “Prove the set $S$ is a span” (1 task) “Prove the set $S$ is linear independent” (1 task)</td>
<td>-</td>
<td>Analysis</td>
</tr>
<tr>
<td></td>
<td>“Find derivatives” (1 task) “Find Mclaurin</td>
<td>-</td>
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<td></td>
<td>-</td>
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</tbody>
</table>

Table 6.2: The tasks Alex used in the eight tutorials.
<table>
<thead>
<tr>
<th>Coursework</th>
<th>Tasks</th>
<th>Subjects</th>
</tr>
</thead>
<tbody>
<tr>
<td>SGT8</td>
<td>“Find a basis for the kernel of $\varphi_A$.” (1 task)</td>
<td>Linear algebra</td>
</tr>
<tr>
<td></td>
<td>“Prove that $\varphi, \varphi_4$ are linear maps” (1 task)</td>
<td>Analysis</td>
</tr>
<tr>
<td></td>
<td>“Prove the vectors form a basis” (1 task)</td>
<td>Linear algebra</td>
</tr>
<tr>
<td>SGT11</td>
<td>“Is the linear map $\varphi$ bijective? If so, compute its inverse” (1 task)</td>
<td>Analysis</td>
</tr>
<tr>
<td></td>
<td>“Show the general Riccati equation can be converted into a second order linear homogeneous ODE” (1 task)</td>
<td>Linear algebra</td>
</tr>
</tbody>
</table>

It seems to me that students’ low results in coursework tasks was another tool for Alex’s strategy selecting tasks. This is because 2 of 8 tutorials included coursework tasks [SGT7, SGT11]. As Alex said to the students in the tutorials, his selection of the coursework tasks was due to the students’ low results. In our discussion after SGT7, Alex confirmed that the coursework tasks he solved on the board in SGT7 were the most “problematic for some [students]”, meaning that two students achieved the lowest marks in those tasks. By presenting the solution of those coursework tasks on the board, Alex prepared the students for the final examinations. My interpretation is that the presentation of the solution of the coursework tasks also reveals Alex’s goal in selecting tasks ‘students should pass the modules’.

Last but not least, Alex prepared 5 of 8 tutorials before the start of the tutorials. In particular, SGT3 and SGT4 were on injectivity, surjectivity and notation; SGT6 was on preparation for an upcoming coursework in linear algebra; SGT8 was on students’ suggestion regarding linear maps; and SGT11 was on students’ suggestion regarding a task in linear algebra from a practice exam sheet. My interpretation is that Alex was proactive to work on students’ difficulties with injectivity and surjectivity in SGT2-4, and with notation in SGT4. Alex was also proactive to prepare students with an
upcoming coursework [SGT6]; and reactive to their difficulties with coursework tasks in two tutorials [SGT7, SGT11].

6.2.2.3 Selecting examples

In tutorials, Alex selected real-world examples, real-world non-examples, examples and non-examples. This section is an exemplification of Alex’s examples through teaching episodes.

In SGT2 of Episode 1, Alex wrote the definition of injectivity $\forall x, y \in \text{Dom}(f), f(x) = f(y) \Rightarrow x = y$ on the board. He then used $f(x) = x^2$ as a non-example of an injective function since $x = -1 \mapsto x^2 = 1, x = 1 \mapsto x^2 = 1$; thereby enabling the students to get an insight into what injectivity is not. Although he did not restrict the domain of $f(x) = x^2$ to show a domain of injectivity, he then gave a number of tasks to students about finding the inverse of functions while considering suitable domains.

In SGT2, St6 could articulate why $f(x) = x^2$ is not injective thus indicating some meaning of injectivity. However, it seems that this was not true for St2, who requested an example of an injective function. Alex used the linear function $f(x) = x$ to provide students with intuition of what injectivity is. The linear nature of the graph of $f(x) = x$ indicates that it is a prototypical example (Lakoff, 1987) of injectivity. So, students could potentially consider that all injective functions are linear. In our discussion after SGT2, Alex informed me about his perspective on students’ meaning making of injectivity with regard to his teaching in SGT2.

By the reaction I got when I asked for the definition [of injectivity] the student couldn’t even say what the symbols were there. So, I had to repeat it for him. There was not so much meaning making there. So, that’s why I decided to use examples, use the Venn diagrams for the sets and what exactly it means to be injective and surjective. […] If the students got it, I am not sure about that, because after that they still had the face of ‘what are you talking about?’ So, at that point you say ‘Mmm if I carry on with more examples, eventually they will get it’, because I don’t have any other didactical instrument to make it even clearer for them. Ah in fact when I was preparing my module for another
lecture, I thought of a very good example of the function. When you go to the supermarket and I am going to say to them next time [...] to explain what an injective and a surjective function is. [...] And I think that’s more near the experience of the students, so that they can say “ah yes, I get it now”.

Excerpt 9_Discussion after SGT2

In Excerpt 9, Alex explained to me that he decided to use examples in order to enable students to make meaning of injectivity; in fact, he stressed that examples was his only “didactical instrument” for students’ meaning making. In my terms, Alex indicated examples as his only tool for students’ meaning making of injectivity. It seems to me that Alex’s goal in selecting examples was students’ meaning making of the mathematics.

In the year of data collection of Alex’s teaching, I attended a presentation he delivered about a research project in which he was involved. In this presentation, he informed the audience about how he considered meaning making. He related Skemp’s (1976) relational and instrumental understanding to Hiebert’s (1986) conceptual and procedural knowledge, respectively. Hiebert and Lefevre define conceptual knowledge as “knowledge that is rich in relationships” (1986, p.3), and procedural knowledge as “rules or procedures for solving mathematical problems” (1986, p.7). So, for Alex, relational-conceptual meaning making was Skemp’s “knowing both what to do and why” (1976, p.21) and instrumental-procedural meaning making was Skemp’s knowing “a rule, and ability to use it” (1976, p.21). He also considered meaning making in terms of making connections within mathematics amongst representations, such as symbols, diagrams, pictures (Haylock, 1982); and between mathematics and “other aspects of the world” (Orrell, 1974, p.13), such as real world situations. My interpretation is that Alex attempted to enable students to make relational-conceptual meaning of injectivity by writing the definition of injectivity on the board, and by using \( f(x) = x^2 \) and \( f(x) = x \) as a non-example and an example of injectivity respectively. For \( f(x) = x^2 \), for instance, \( x = -1 \mapsto x^2 = 1, x = 1 \mapsto x^2 = 1 \) could be for students’ ‘knowing what to do’ and the definition of injectivity on the board could be for students’ ‘knowing why’.
In SGT2, Alex interpreted that students might have not made meaning of injectivity [Excerpt 9_Discussion after SGT2]. Since, examples was his only tool for enabling students to make meaning of injectivity, he decided to use two more examples in SGT3. The first was the supermarket example, for which Alex told students: “I want you to contextualise a very abstract formal definition so we do an everyday job that you can understand; that you give some meaning to those things.” [Episode 3]. In this way, he shared with students his goal for meaning making of the definition of injectivity. The “everyday job” he said indicates that Alex potentially characterised the supermarket example as a real-world example. In our discussion after SGT2, I asked Alex whether his research background has an influence in the use of real-world examples.

By making it [the example] nearer to the students’ experience; that comes from mathematics education. […] Because you need to make connections in order to make meaning. To understand something you need to make the appropriate connections from your own experiences.

Excerpt 10_Discussion after SGT2

In Excerpt 10, Alex stressed that “you need to make connections in order to make meaning”. My interpretation is that the supermarket example is congruous with Ormell’s (1974, p.13) conceptualisation of meaning making in terms of making connections between mathematics and “other aspects of the world” such as real world situations. Alex made explicit that in the supermarket example, a real world situation was “students’ experience” [Excerpt 10_Discussion after SGT2]. However, St1 said he could not make sense of injectivity in the supermarket example. Following the supermarket example Alex devised the hockey real-world example, which was close to St1’s interests and own experiences. In our discussion after SGT4, Alex reflected on St1’s meaning making of injectivity with regard to the hockey example.
I thought it went a bit better last time when I asked St1: “What do you do in your life?” I play hockey he said. And it went well I thought; at least they said: “Oh yeah I understand now what you mean.” That’s the design at least to connect with what they do outside.

Excerpt 11_Discussion after SGT4

In Excerpt 11, Alex declared that his design to connect mathematics with students’ own experiences was successful for the students’ meaning of injectivity. However, despite the real world context of the example of function in the supermarket (a product cannot be related simultaneously to two final prices), a function that relates products/players with their prices/scores is not injective in real life since, there, two different products/players can have the same price/score. It seems to me that the supermarket and the hockey examples are real-world non-examples of injectivity.

My interpretation, from the context of the main study observations and discussions with Alex, is that he suggested kinds of examples for mathematical meaning making: non-examples, examples and real-world examples. In all 8 SGTs I analysed for my main study, Alex used 6 tasks which included 6 non-examples of injectivity, surjectivity and linear independence [SGT2, SGT3, SGT6]. In particular, 2 of these 6 tasks included 2 real-world non-examples for the concept of injectivity [SGT3] and 2 real-world examples for surjectivity [SGT3]. Alex used these non-examples and real-world examples after he had written the definition of the concepts on the board. He also used 2 tasks which included 2 real-world examples for notation [SGT4].

6.2.2.4 Evaluating students’ sense making of mathematics

Analysis of the main study indicates that Alex used a number of tools to interpret and evaluate students’ sense making of mathematics in his tutorials. In this section, I provide a synthesis of observations and discussions with Alex to exemplify his tools.

In Episode 1, Alex asked the students to read the definitions of injectivity and surjectivity in order to write them on the board. I coded Alex’s question as an inviting question which is general to all students, because Alex did not ask a particular student in the group. The role of Jaworski and Didis’ (2014) ‘inviting questions’ is to seek
students’ articulation of mathematical meaning. My interpretation is that, with his question, Alex sought students’ articulation of meaning for the definition of injectivity. St2 volunteered to respond Alex’s question; however, he faced difficulties with the symbols in that definition. In our discussion after SGT2, Alex asserted that “[T]he student couldn’t even say what the symbols were there. So, I had to repeat it for him. There was not so much meaning making there.” [Excerpt 9_Discussion after SGT2]. In other words, while commenting on the student’s difficulty with symbols, Alex evaluated the student’s meaning making of injectivity. So, Alex’s assertion is an instance that provides evidence of inviting questions as Alex’s tool for the strategy evaluating students’ sense making of mathematics. The following Table 6.3 demonstrates all Alex’s inviting questions in Episodes 1-2, where he sought students’ articulation of mathematical meaning.

In response to St2’s difficulty with the symbols in Episode 1, Alex used diagrams, examples and non-examples for the definition of injectivity. His evaluation regarding students’ sense making of injectivity in SGT2 was: “If the students got it, I am not sure about that, because after that they still had the face of ‘what are you talking about?’” [Excerpt 9_Discussion after SGT2]. Indeed, I observed that the students’ faces were blank. My interpretation is that in Excerpt 9, Alex referred to a tool for his evaluation of students’ sense making of mathematics. This tool was students’ faces as a means of feeding back or reinforcing his effort for students’ mathematical meaning making. In our discussions after SGT6 and SGT8, Alex also referred to instances where students’ faces were his indicator for their sense making of the mathematics.

In Excerpt 11, Alex offered me another tool for his interpretation of students’ sense making. This was a control question to students about whether his explanation of the mathematics makes sense. For instance, Alex asked the students whether the articulation of injectivity made sense in the context of the hockey example. The students responded that “Oh yeah I understand now what you mean” [Excerpt 11_Discussion after SGT2]; thereby confirming that they made sense of injectivity. Table 6.3, below, illustrates all Alex’s control questions in Episode 1, where he explained “a particularly important or complicated piece of mathematics” (Viirman, 2015, p.1175) and then asked the students whether it made sense. This was also the context in which the lecturers in Viirman’s study used control questions.
In Episode 3, when Alex started to solve a system of linear equations with Gaussian elimination, St4 asked him “How does that show it was bijective?” St4’s question indicates that the student did not make sense of why the solution of the system of linear equations shows that the linear map of the task is bijective. However, St4 was not silent and she was making a broader sense in order to ask such a pertinent question. Furthermore, students’ speaking and asking questions is a difference from earlier tutorials and happens after SGT5 and the implementation of the strategy *introducing students to the tutorial group*. In our discussions, Alex did not comment on St4’s question. However, in his discussion with me after SGT9, he commented on St5’ question in SGT9. In SGT9, Alex was performing elementary row operations to a matrix in order to bring it in Row-Echelon form. St5 asked Alex why he made a zero in the matrix. In our discussion, Alex said “I was a little bit concerned that she couldn’t see that, so actually she hasn’t quite understand what was the whole purpose of putting it to Echelon form.” It thus seems to me that St5’s *question* in SGT9 and St4’s *question* in SGT11 were tools for Alex’s evaluation of students’ sense making of the mathematics he demonstrated on the board.

**Table 6.3:** Coded *control questions* and *inviting questions* in Episodes 1 & 2.

<table>
<thead>
<tr>
<th>Episode #</th>
<th>Control questions</th>
<th>Inviting questions: general</th>
</tr>
</thead>
<tbody>
<tr>
<td>Episode 1</td>
<td>“x could be bread, y could be milk, mm?” [1” pause.]</td>
<td>“What is an injective function?” [General question: “In response, St4 made a question “Isn’t injective just 1 – 1?”] Alex “asked the students for the formal definition of injectivity.”</td>
</tr>
<tr>
<td></td>
<td>“Let’s say x is bread and y is milk, OK?” [1” pause.]</td>
<td>“let’s take x². Would that be injective or not injective? [3” pause.] What do you think, St6?” [Direct question: St6 responds “Not”.] “Not injective. Why? [4” pause.]” [Direct question: St6 responds “It should not have two values of x go to”.]</td>
</tr>
<tr>
<td></td>
<td>“I notice that the price of the bread and the price of the milk are the same, they are both 99p. Yes?” [2” pause.] “If this function was injective, then the bread would have to be milk, well that’s impossible isn’t it?” [2” pause.] “I cannot have the price of 99p that belongs to two products, two different products,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>“How would you read that [the definition of injectivity] in the supermarket example? [4” pause.] Which are the x’s? What’s the domain of the function? [12” pause.]” St3, what would the x’s be in this example in Tesco?” [Direct question: St3 responds “Products. Products..”]</td>
<td></td>
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</table>
Alex’s control questions come after his explanations of the mathematics of tasks in Table 6.3. They end with “OK?”, “Isn’t it?”, “Does that make sense?” or “Mm?” Alex’s “Mm?” is an alternative to “OK?” and it is friendly to students, because it is followed by Alex’s smile. In Episode 1, for instance, Alex explains that “$x$ could be bread, $y$ could be milk”, smiles and asks the students whether his explanation is OK for them; in other words, whether it makes sense for them.

Alex acts with inviting questions to students when he asks them “to respond” (Jaworski & Didis, 2014, p.380). In contrast to control questions, this type of questions does not necessarily follow after an explanation. In Table 6.3, there is a blend of Alex’s general and direct inviting questions to students, with direct questions usually coming after a general question with no students’ response. There are also some inviting questions asking the students “Why?” Jaworski and Didis (2014, p.380) identify ‘meaning questions’ to be lecturers’ questions “overtly seeking students’ expression/articulation of meaning, often in response to the question “why?”” In my study, I identified “why” questions only in the case of Alex’s teaching. So, I decided to use the conceptual name inviting questions for all categories of questions where the tutor sought students’ articulation of mathematical meaning; either overtly or more implicitly.
Alex uses pause intervals of 1”-3” after his control questions, whereas he uses 3”-12” after his inviting questions. In agreement with analysis for the other two cases of teaching in my study, my interpretation is that large pause intervals after inviting questions enable the students to think for a response. In the case of Alex’s teaching, students start to be consistent in responding inviting questions after SGT5.

6.2.2.5 Showing and asking students to repeat

Showing and asking students to repeat is a strategy I first identified in Phanes’ teaching. It concerns the following order of actions: Phanes solved a task or a set of tasks on the whiteboard (showing); gave time to students to solve “similar” task(s) and circulated giving feedback and support to students during the time they worked (asking students to repeat). I identified the strategy showing and asking students to repeat in the case of Alex’s teaching, as well.

In Episode 2, for instance, Alex proved on the board that the map \(\varphi: C^3 \rightarrow C^2, (z_1, z_2, z_3) \mapsto (i^2z_2, z_3)\) is linear (showing); gave 7’12” to students to prove on their scripts that the map \(\varphi_4: Mat_{2\times 2}(R) \rightarrow R[x], \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a + d)x^2 - cx + 3b\) is linear and circulated giving feedback and support to students during the time they worked (asking students to repeat). The task about \(\varphi\) Alex showed and the task for students about \(\varphi_4\) were “similar” in terms of proving that a map (\(\varphi\) or \(\varphi_4\)) is linear by using the definition of linear map. During the 7’12”, Alex circulated and looked at the students’ work on their scripts. My interpretation is that Alex acted with the students’ work on their scripts in order to support the students and to promote their mathematical meaning making. In particular, students’ work on their scripts was a tool in order for Alex to promote the students’ mathematical meaning making. In Episode 2, for instance, Alex looked at St5’s script and answered to St5’s question by indicating the matrices \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) and \(\begin{pmatrix} e & f \\ g & h \end{pmatrix}\) or \(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\) and \(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\) for the conditions of the definition of linear map. In this way, he promoted St5’s meaning of the definition of linear map. Analysis for the case of Phanes’ teaching, in the previous chapter, indicates that students’ work on their scripts was an associated tool to Phanes’ strategy showing and asking students to repeat, as well.
Alex gave time to students to work on 24 tasks over a total of 48 tasks, which is the $\frac{24}{48} = 50\%$ of all tasks. The 24 tasks were 21 procedural tasks, 1 proof task and 2 conceptual tasks; all selected from problem sheets except for the two conceptual tasks which were designed by Alex. So, a tool associated with showing and asking students to repeat was the tasks Alex selected. The mathematical topics were in analysis: natural domain of a function (6 tasks in SGT1); inverse of a function (4 tasks in SGT2); limit computation (7 tasks in SGT2 and SGT3); derivative computation, the concepts of injectivity and surjectivity (5 tasks in SGT3); and sketching graphs of one variable functions (1 tasks in SGT4). There was also a topic in linear algebra: linear map (1 task in SGT8). With regard to limit computation, Alex said to the students during SGT2 that “the whole of calculus is based on the idea of limits”. Alex selected tasks on limit computation in two consecutive tutorials.

In our discussion after SGT2, I said to Alex that for both limit computation and the inverse of a function, he first solved a task on the board and then gave time to students to work on a number of tasks. Alex asserted that:

I first solve one at least and then I let the students work on the others as a way I suppose to show them how it is done and then they can follow that. Because, if I start by letting them solve the first one without me having done any of the [tasks], what should tend to happen is that they don’t know absolutely even how to start […] I will end up to do it myself.

Excerpt 12_Discussion after SGT2

My interpretation is that, for Alex, showing and asking students to repeat was a strategy in order to enable the students to overcome the difficulty on how-to start a task; thereby promoting students’ mathematical meaning making. In Episode 2, for instance, Alex asked the students what the conditions are so that a map is linear. The students did not respond and he wrote the conditions of the definition of linear map for $\varphi$ on the board. He then showed the solution of a first task to students. That solution was based on the conditions written on the board. Alex acted in this way to resolve students’ difficulty on how-to start and to promote their meaning of linear map. In discussion after SGT8, Alex explained to me: “it is my thinking that you need a definition in order to understand the things, but it is also how [the lecturer in linear
algebra] […] always starts ‘Begin your answer by stating what conditions you need to check’.”

In tasks that needed definition(s) for their solution, Alex asked the students to find the definitions in their lecture notes. For instance, he asks the students for the definition of linear map in Episode 2 and for the definition of injectivity in Episode 1. My interpretation is that Alex used the heuristic ‘find definition(s) in lecture notes’ as a tool in his teaching. This is because Alex acted with the heuristic in order to resolve students’ difficulty on how-to start and to promote their meaning making. Zenobia also used that tool in her teaching. Another heuristic with which Alex acted was ‘consider special case(s)’. An exemplification of the latter heuristic is in Episode 2. In this episode, Alex calculates \( \varphi \) for \( z_1 = 1, z_2 = i, z_3 = -3 \) in order to enable the students to make sense that \( \varphi \) is a transformation.

Finally, there were instances in observations where the students could not work out solutions of tasks and could not respond to Alex’s questions for the solutions. An instance in such an observation is in Episode 1, where the students could not work out injectivity in the context of Alex’s supermarket example. So Alex had no choice but to provide the response by himself. He told students: “I notice that the price of the bread and the price of the milk are the same, they are both 99p. […] If this function was injective, then the bread would have to be milk, well that’s impossible”. In such instances, my analysis indicates that Alex acted with a last teaching tool he had: himself as a tutor. This is another similarity with analysis in the case of Phanes’ teaching, where Phanes told the mathematics by himself in response to the students’ inability to offer an input.
6.2.2.6 Explaining

In this section, I discuss Alex’s strategy ‘explaining’ through his use of representations and rhetorical questions. I exemplify this strategy in the case of Alex’s teaching through Episodes 1, 2 and 3.

In Episode 1, while Alex was sketching Figure 6.3 on the board, he said to students:

Alex: What cannot happen is that we have an $x$ here [$x$ in Figure 6.3] that takes me to $f(x)$, and a different $y$ [$y$ in Figure 6.3] takes me to $f(y)$, and this leads to that $[f(x) = f(y)]$ in Figure 6.3. This is not injective.

In this quotation, Alex explained to students the condition under which a function is not injective. My consideration is that Alex’s tool for this explanation is the set diagram illustrated in Figure 6.3. This is because his description of “what cannot happen” in an injective function is based on this graphical representation.

In our pilot study discussion, Alex confirmed his common use of diagrams in his teaching. In the main study observations, evidence is concerned with Alex’s use of set diagrams after he wrote the notation for set theoretic definitions (e.g. injectivity) on the board. In particular, he used nine set diagrams in 5 of 8 tutorials [SGT1, SGT2, SGT3, SGT8, SGT11]. He also connected 12 graphical representations (i.e. graphs of functions, number lines) with symbolic representations (e.g. formulas of functions) in 4 of 8 tutorials [SGT1, SGT2, SGT3, SGT4]. Finally, he used a tabular representation of function for his hockey example in Episode 1. Notably, neither Zenobia nor Phanes used tabular representations of functions in the teaching I observed.

In my discussion with Alex about his use of diagrams, the following dialogue took place:

Angeliki: Is [a diagram] something that helps you when you study mathematics?

Alex: Yes. I think diagrams are very good for explaining a complicated definition, at least they help me and my experience is that some students find
it easier to see what the definition is about if they can see an example in a
diagram.

Excerpt 13_Discussion after SGT2

In Excerpt 13, Alex informed me that he makes diagrams while studying mathematics
and his experience is that some students benefit from the diagrams which represent
the symbolic representations of definitions. My interpretation is that he used set
diagrams as tools for his explanations to students, thereby connecting the symbolic
representations of definitions with graphical representations (i.e. diagrams).

Alex was conscious about connecting mathematical representations for students’
meaning making of mathematics. In Excerpt 9, for instance, he told me that the
student in SGT2 “couldn’t even say what the symbols were” in the definition of
injectivity; “So, that’s why I decided to use examples, use the Venn diagrams for the
sets and what exactly it means to be injective.” It seems to me that Alex’s goal in
using set diagrams was to promote students’ mathematical meaning making. So, in
Episode 1, he started in the abstract mode through symbols; discerned that students
did not make meaning of them; and then brought in a diagram as an alternative way of
representing injectivity.

My interpretation is that Alex’s connections between mathematical representations is
congruous with Haylock’s (1982) meaning making. Haylock conceptualised meaning
making in terms of making connections within mathematics amongst representations,
such as symbols, diagrams, pictures. Considering that Alex was aware of Haylock’s
work, the connections between mathematical representations in his teaching form an
element of an informed teaching by research literature.

In Episodes 2 and 3, Alex was writing on the board and explaining the symbols he
wrote or the notation in definitions. In 4 of 8 tutorials, he wrote and explained the
formalism (i.e. symbols and notation) on the board in a transmissionist mode [SGT6,
SGT7, SGT8, SGT11]; that is, he wrote the statement of the task and exposed the
mathematics of the solution on the board. In Episode 2, for instance, Alex wrote the
definition of linear map on the board and explained to students the first condition
\[ \varphi(u + v) = \varphi(u) + \varphi(v) \].
Alex: What does that \([\varphi(u + v) = \varphi(u) + \varphi(v)]\) tell you? It’s just telling you “take any \(u\) and \(v\) in complex numbers, add them, transform [the sum \(u + v\)] and then that \([\varphi(u + v)]\) has to be this \([\varphi(u) + \varphi(v)]\)”.

I coded this quotation from Alex as a verbal representation and I considered it as Alex’s tool for explaining the first condition in the definition of linear maps. This is because he explained orally the condition in formal mathematical language without using a mere exposition of the definition of linear map.

In Episode 3, St4 asked Alex why the solution of a system of linear equations shows that the associated linear map \(\varphi: F_5^3 \to (F_5)_2[t]\) is bijective. Alex’s explanation was:

Alex: Well what you’re doing here is seeing if this system
\[
\begin{align*}
\alpha &= x + 4y + z \\
\beta &= 2x - 3z \\
\gamma &= 2y
\end{align*}
\]
has a solution. If I take a general form \([v = (\alpha, \beta, \gamma)]\) and map it through the transformation \([\varphi]\) […] what I want to do [is to] establish that there’s a relation that is both injective and surjective.

In this quotation, Alex explained how the solution of the above system of linear equations shows that \(\varphi\) is bijective. My consideration is that his tool for this explanation is a verbal representation in formal mathematical language. This is because he explained orally what would be achieved with the solution of the system of linear equations, without using a mere exposition of symbolic representations of row operations.

Finally, Alex acted with questions for which he did not expect a response. In Episode 1, for instance, he asks while sketching Figure 6.2 for injectivity:

“Let’s say that we have \(f(x) = f(y)\) [in the image of \(f\)]. Then \(x\) in here [\(x\) in the domain of \(f\)] has to be?”

Alex does not expect a response and writes \(x = y\) in the domain of \(f\). Another instance is in Episode 2, where he writes \(u, v \in C^3\) and asks:
“what kind of vectors are these \([u, v]\)?”

Again without expecting a response, he writes \(u = (a, b, c), v = (d, e, f)\).

My interpretation is that the role of Alex’s two *rhetorical questions* in Episodes 1 and 2 is to provide students with modes of mathematical thinking (Fukawa-Connelly, 2011) for injectivity and vectors. This is a role I identified for some rhetorical questions in the cases of Zenobia’s and Phanes’ teaching, as well.
Part 3: Alex’s knowing for teaching in the main study

6.3.1 Mathematical knowing

Alex’s mathematical knowing was revealed in his exposition of the mathematics while solving tasks on the board (e.g. Episodes 2 & 3). Alex solved various tasks while writing on the board and using formal or informal language to explain the mathematics. His use of symbolic representations was dominant (e.g. Figures 6.6 & 6.8). He also acted with additional tools to symbolic representations in order to explain them, such as set-theoretic diagrams and the heuristic ‘consider special cases’.

In contrast to the other two cases of teaching, Alex was a researcher in mathematics education; so, he did not use tools and strategies informed by mathematical research (of his own). He nevertheless acted with tools and strategies informed by research literature in mathematics education, and this contributed to a deeper understanding of didactical and pedagogical knowing in the study.

6.3.2 Didactical knowing

Didactical knowing is concerned with knowing ways of making the design of teaching in order to translate the principles and content of mathematics into forms of tutor’s thought in the context of students. Alex focused the teaching on students’ difficulties, known from his experience and research literature in mathematics education; for instance, notation, proofs and definitions of concepts. He also included in the design various procedures, such as computations. So, the content of mathematics he worked on was notation, proofs, definitions and procedures.

Alex’s epistemology of mathematics included learners, who invest time to work on and make sense of the mathematics in collaboration and in discussion with others. So, he translated the content of mathematics into forms of his thought in the context of students, by telling or discussing it with the students in formal or informal
mathematical language. He enriched that telling or discussing with tools and strategies based on his epistemology of teaching/learning, such as students’ work on their scripts (e.g. Episode 2). For instance, he explained the symbolic representation of the definition of linear map in formal mathematical language, and enriched it with the strategy ‘showing and asking students to repeat’ and its associated tool students’ work on their scripts (Figure 6.10). So, Alex’s design of teaching in order to translate the principles and content of mathematics into forms of his thought in the context of students included a path of informing: from practice drawn on his epistemology of mathematics towards practice drawn on his epistemology of teaching/learning (Figure 6.11).

Alex’s didactical knowing is evident in his practice from designs of teaching whose tools and strategies (compatible with his epistemology of mathematics) are enriched with tools and strategies compatible with his epistemology of teaching/learning. In this way, Alex’s tools drawn on his epistemology of mathematics are interrelated to tools drawn on his epistemology of teaching/learning.
6.3.3 Pedagogical knowing

*Pedagogical knowing* is concerned with knowing ways of redesigning the teaching repeatedly, with different tools and strategies, until a developmental stage of teaching which enables the students to make meaning of the mathematics. A tutor with developed *pedagogical knowing* demonstrates flexibility in drawing on the students’ responses/silence, and redesigning for students’ mathematical meaning making.

The *pedagogical knowing* depends on the tutor’s strategy ‘evaluating students’ mathematical sense making’ for a judgment as to what stage of the (re)design enables the students to make meaning of the mathematics. Alex’s tools for ‘evaluating students’ mathematical sense making’ were: *inviting and control questions; students’ mathematical questions*; and *reinforcement by student faces and/or student responses*.

Alex was flexible to teach notation, a proof, a definition or a procedure by using a variety of tools and strategies in successive developmental stages of his teaching, when students’ responses indicated that they did not make mathematical sense. For example, in Episode 1, he explained the *symbolic representation* of the definition of injectivity with a *set-theoretic diagram, examples*, and two *real-word non-examples* (Figure 6.12). The students did not seem to Alex to make sense of injectivity until the second *real-word non-example*, where they declared they made sense.

![Figure 6.12: Pedagogical knowing.](image-url)
Alex acted with tools and strategies informed by research literature in mathematics education (e.g. real-world examples), which provided him with a variety of tools. My interpretation is that the case of Alex’s teaching was a case with evident pedagogical knowing. This was because of the variety of the different tools, with which he acted among the developmental stages of his teaching, in order for students to make mathematical meaning. The next chapter, Chapter 7, includes the cross-case analysis of teaching, where teaching practice and knowing are examined across cases.
Chapter 7

CONCLUSIONS

The research questions which I set out in the methodology of the study were:

3. How is teaching knowledge revealed in teaching practice with first year undergraduate mathematics modules?
4. How does teaching knowledge interact with students’ mathematical meaning making?

The study, as exposed in the last three chapters (Chapters 4, 5 and 6), focuses on observations of practice, a characterisation of practice in terms of strategies and tools and a conceptualisation of tutor’s knowing for teaching. In this chapter, I summarise in a synopsis the characterisation of teaching practice in the study and then, I synthesise from the analysis of practice across the three cases of teaching. This synthesis leads into a discussion of tutor’s knowing for teaching and, the analytical framework ‘Teaching Knowledge-in-Practice’. Concluding remarks are concerned with implications to teaching and learning of mathematics at university level and methodological implications. The final sections discuss limitations of the research and future studies.
7.1 Mathematics teaching practice at university level

In the methodology of the study, which I presented in Chapter 3, I explained my design of three cases of teaching to address the research questions. They were cases of tutorial teaching observed for more than one semester at a British University. I selected the tutorial setting because it offered me opportunities to talk with the tutors about the students’ mathematical meaning making. The tutors were Zenobia, Phanes and Alex, who usually discussed with the students two first year undergraduate mathematics modules: analysis and linear algebra. I presented a detailed analysis of the teaching practice and knowing of each tutor in Chapters 4, 5 and 6. In particular, I offered a characterisation of each tutor’s teaching practice into specific strategies and tools for teaching, and of each tutor’s knowing for teaching into mathematical, didactical and pedagogical knowing.

This is a study of university mathematics teaching practice and knowing for teaching with regard to students’ mathematical meaning making. It studies three tutors’ teaching practice through the first generation activity theory model, which illustrates Vygotsky’s notion of mediation (Figure 7.1). The theorisation of the study is that the tutor (subject) acts with mediational means (tools) towards the students’ mathematical meaning making (object/motive). In other words, the tutor’s actions with tools, denoted as strategies for teaching, mediate students’ mathematical meaning making.

The tutorial setting, where students’ difficulties were brought into the fore, triggered the tutor’s ‘motive’ of students’ mathematical meaning making. Thus the ‘motive’ is implicit in nature. Students’ mathematical meaning making was nevertheless revealed through my discussions with the tutors as an explicit ‘goal’ for particular strategies, as
well. For instance, in all cases of teaching the tutors informed me that the goal of the strategy ‘selecting examples’ is students’ mathematical meaning making.

Considering the first generation activity theory model (Figure 7.1), I studied students’ mathematical meaning making from the tutors’ perspective. Zooming out for the wider picture, students’ mathematical meaning making was the object of the tutor’s teaching practice. Zooming in on each tutor’s teaching practice, students’ mathematical meaning was a goal for particular strategies. However, students’ mathematical meaning making as the outcome of the tutor’s teaching practice is out of the scope of this study. In other words, I studied the tutor’s strategies and tools for students’ meaning making rather than students’ meaning making per se. This is why the research design did not include data from interviews with the students and/or their marked coursework/exam tasks.

A grounded list of tools with which the three tutors acted is:

- (Types of) mathematical examples;
- (Types of) mathematical representations;
- (Types of) heuristics;
- (Types of) questions to students;
- Pause intervals after questions to students;
- Humour in the form of levity;
- Language (valuing statements, encouraging statements, injunction statements, reminding statements, consolidating statements, revoicing statements);
- Students’ results in coursework;
- Students’ difficulties from teaching experience;
- Students’ mathematical questions;
- Students at the board;
- Students’ inputs in dialogue;
- Students’ work on their scripts (ability to solve);
- A student;
- The tutor;
- Students’ faces.

The rationale for the grounded analytical approach to the data was to identify superordinate categories of tutor’s actions and within them to identify the tools. In the
literature, this way of working resembles the identification of actions as ‘functions’ (e.g. Lobato, Clarke & Ellis, 2005; Baxter & Williams 2010; Grandi & Rowland, 2013). In this study, I identified the tool-mediated strategy as the ‘function’ of a set of teaching tools. In particular, each time I recognised a strategy in the data, I questioned the ‘function’ of the associated tools in order to name that strategy, by asking: *With what tools does the tutor implement the strategy?* For instance, *with what tools does the tutor implement the strategy ‘selecting tasks’?* I addressed the question by recognising a particular set of tools in a case of teaching, such as ‘students’ results in coursework’ and ‘students’ difficulties from teaching experience’. So, the identification of strategies was not the result of a grouping of types of tools for teaching. For instance, I did not recognise a strategy that grouped types of questions or a strategy that grouped types of heuristics.

7.1.1 Students’ meaning making from the tutor’s perspective across cases of teaching

The study links teaching practice and the motive to foster students’ mathematical meaning making. I start this section by pointing out the extent to which I found that the analysed teaching practice achieved its object for students to make mathematical meaning. Then, I juxtapose the different tutors’ strategies for students’ meaning making to discuss issues that reveal commonalities and differences in the three tutors’ teaching practice.

The levels of tutor’s teaching experience and the levels of students’ performance are important in a discussion about teaching practice. This is because students at different levels of performance have different learning needs and tutors with longer teaching careers may use their prior experience with students at different levels to design their practice. The three tutors of this study indeed were in different levels of teaching experience: Zenobia had a 20-year teaching career; Phanes had a 15-year teaching career; and Alex had an 8-year teaching career. They nevertheless all were experienced in teaching. The performance in the groups of their students also varied considerably: Zenobia’s students were high-performing students; Phanes’ students were low-performing students; and Alex’s students were average-performing
students. My discussions with the three tutors indicated that all tutors cared for the students and were aware of their performance level.

In the case of Zenobia’s teaching, the design with advanced mathematical practices, such as the heuristics, enables the constitution of a teaching/learning environment where students and tutor are members of a support community of working in the culture of mathematics. Zenobia’s tutorial was a mathematical learning community for the students. The students and Zenobia worked as a team to solve mathematical tasks on the board and Zenobia supported them with teaching strategies such as ‘creating positive feelings’ and ‘explaining’. Dialogue about the mathematics of the tasks was an integral part of the tutorial with questions usually dominated by Zenobia. Injunction questions about a heuristic and questions to observe the mathematics were distinctive in this case of teaching and offered the students a mediated negotiation of mathematical meaning. Aesthetic statements and consolidating statements were also distinctive. Zenobia acted with aesthetic statements to translate aspects of the culture of mathematics into forms of her thought in the context of students, and with consolidating statements to consolidate the way of working with the mathematics after the mathematical negotiation.

Zenobia’s classroom culture also utilised research practices of mathematicians (e.g. ‘decoding the mathematics and encoding the mathematics’) to develop mathematical knowing that was new to the students and existing for Zenobia. Her tutorials worked on the basis that Burton (2004, p.181) envisaged: students “encouraged and expected to act as researchers in their learning” in order for mathematics to be more accessible and attractive to students. The high level of performance of the students was reflected on their responses to Zenobia’s questions and design. Zenobia also acted with St4 who was a student able to respond correctly despite the remaining students’ silence or difficulty in challenging tasks. In this case of teaching, there is evidence of students’ correct responses in interactions with Zenobia and fellow students, and thus evidence of students’ meaning making from Zenobia’s and my perspective.

In the case of Phanes’ teaching, the design included the tutor showing ways of working with tasks and occasionally asking the students to repeat those ways of working. His practice was consistent during the semester. In contrast to Zenobia’s encoding-decoding thinking, Phanes showed other ways of mathematical thinking.
such as: connections between different mathematical representations; and powerful examples selected under the principle of *simplicity* which enables their use in various applications. His simple examples were designed to be meaningful for the average learner. For the particular group of his low-achieving students, Phanes also explained the mathematics “in a more basic level”. In particular, Phanes described the concepts in a transparent way to the students and me, such as: “you take all possible linear combinations [of vectors], what you get is their span”. I interpret the way as being transparent because, for instance, the students responded to Phanes’ descriptions by saying “So simple.” and “That is it?” The students nevertheless seemed to Phanes and to me as if they made mathematical sense in-the-moment but later they were not able to recall.

In the *Mathematics Problem*, Hawkes and Savage (2000, p.ii) report on evidence of a “decline in students’ mastery of basic mathematical skills”. In the same vein, Treffert-Thomas and Jaworski (2015, p.261) refer to a challenge for tutors to help ill-prepared students for the transition to university mathematics. My interpretation is that despite their entrance qualifications, Phanes’ students did not seem to have the skills and motivation for a smooth transition to mastering university mathematics. Phanes also supported the students with repeated use of the tool *encouraging statement to students to “study”*; however, that tool did not seem to trigger the students’ motivation to “study”. This might be because students were not able to work independently in order to “study” by themselves.

In the case of Alex’s teaching, the design was flexible with different tools and strategies in order for students to make meaning of the mathematics. In our discussions, he clearly stated different ways of teaching with which he intended to enable the students to make meaning as well as his perspective of the students’ actual meaning making after his implementation of these ways. In observations, he started his teaching with an explanation of the definitions, usually by using the tool *diagram*. Then, he applied those definitions in *examples*, which was a next tool with which he usually acted. The students’ feedback, such as facial expressions, indicated to Alex whether he would use additional tools in order to enable the students to make mathematical sense. For instance, he designed *real-world examples* to enable the students to make sense when other tools seemed to fail. Both Zenobia and Phanes
used one or two real-world examples in their teaching; however, in the case of Alex’s teaching, this tool was not only related to a real-word situation, but also to the students’ experiences from everyday life. There were also instances in observations where Alex followed a delivery mode of transmissionist teaching. He was at the board, talking and writing the mathematics, while students were copying to their scripts and occasionally asking questions.

Burton (2004, p.182) asserted that relating mathematics to either the tutor’s or the students’ real life “is certainly one of the ways that more students can be encouraged to enjoy and acquire mathematics and to develop motivation and commitment to pursue the discipline”. She explained that “students say that they do not like learning a subject that is disconnected from their lives or lacks utility in addressing questions that concern them” (2004, p.199). My interpretation is that Alex’s real-world examples had the potential to motivate and commit the students to work with the mathematics. The actual impact of the examples on students is out of the scope of the study; however, my observations also indicate what the students did in Alex’s tutorials. The students initially came to his tutorials embarrassed to speak and to participate. They were average-performing students who nevertheless made an effort to work with the mathematics inside and outside tutorial time, and finally achieved mathematical success. Towards the end of Alex’s tutorials, Alex praised the students because they performed well in pieces of coursework in one of the modules. At that time, they were also more confident to ask Alex questions. My analysis indicated that the change in students’ participation came after Alex’s strategy ‘introducing students to the tutorial group’; so my interpretation is that this strategy indeed enabled the students to participate and to ask Alex questions.

Based on Vygotsky’s general genetic law of cultural development, Wertsch (1985, p.162) offered a theorisation of teaching/learning that enables learners to be successful in learning. This theorisation is useful for a discussion about the three tutors’ teaching with regard to students’ mathematical meaning making. It offers a consideration of different levels of learning at which learners can be, and the roles of the tutors at each of these levels.

In Wertsch’s study, the tutor sets up interpsychological functioning; that is, learning on the social plane. The learners should make the transition from interpsychological
functioning on the social plane to intrapsychological on the individual plane, thus to
learn. Wertsch offered four levels of intersubjectivity between tutor and learners that
enables the transition and thus the success in learning. At the first level of
intersubjectivity, the tutor directs the learner through the strategic steps. However,
communication between tutor and learner is difficult and the learner may not interpret
the tutor’s utterances appropriately. At the second level of intersubjectivity, the
learner seems to share the tutor’s basic sense of objects but fails to embed them within
a whole and to make inferences. This is in contrast to the third level of
intersubjectivity, where the learner can respond appropriately and make inferences.
The third level indicates that intrapsychological functioning on the learner plane is
beginning to account for performance. The tutor does not need to provide all the steps
as the learner functions independently. The tutor nevertheless provides reassurances
of the learner’s correctness. At the fourth and final level of intersubjectivity, the
learner takes over complete responsibility of learning and executing tasks. Thus, in
this theorisation, the levels of intersubjectivity deal with a distribution or a transfer of
responsibility of learning from tutor to learner.

In Wertsch’s study, the tutors were mothers and the learners were preschool children.
The general expectation is that students at the university should be independent
responsible learners; that is, at the third or fourth level of intersubjectivity with the
tutor. However, that expectation sometimes differs from reality, where students can be
at earlier levels of intersubjectivity. Both Alex and Phanes commented in our
discussions that students should be responsible for their own learning and the role of
the tutor is to explain when necessary. My interpretation is that their students were at
early levels of intersubjectivity. Zenobia did not have to make such a comment,
presumably because her students were high performers and responsible for their own
learning. In her tutorials, the students’ learning had shifted from students’ dependency
on her to their own responsibility as members of a support community.

Wertsch’s study, albeit for preschool children, indicated that students, who are
dependent on the tutor for their own learning, need direction from the tutor through all
the steps as long as their dependency holds. It seems to me that examples of such
direction can be found in Alex’s explanation of definitions of concepts; Phanes’
transparent descriptions of concepts; Alex’s and Phanes’ ‘showing and asking
students to repeat’; and Alex’s repertoire of a number of tools for students’ mathematical meaning making. To conclude, in university mathematics education, when students are dependent on the tutor for their own learning, the responsibility of their learning seems to be more on the tutor than on the students. So, the tutor needs to assume more responsibility of the students’ learning, since the students seem unable to take the responsibility for themselves.

7.1.2 Strategies across cases of teaching

Looking at the strategies in the three cases of teaching, four strategies emerged with the same conceptual name across cases. These are: ‘selecting tasks’; ‘selecting examples’; ‘evaluating students’ mathematical sense making’; and ‘explaining’. The remaining strategies differ from tutor to tutor, except for the strategy ‘showing and asking students to repeat’, which is common in the cases of Phanes’ and Alex’s teaching. This section reveals the roots of commonalities and differences in different tutors’ teaching practice.

Table 7.1 illustrates the strategies in the three cases of teaching. The strategies with the same conceptual name across cases are strategies #2, #3, #4 and #6 (Table 7.1), included in a merged cell across cases. Strategy #1 differs from tutor to tutor; so, the cells are not merged. Strategy #5 is common in the cases of Phanes’ teaching and Alex’s teaching, and different in the case of Zenobia’s teaching.

<table>
<thead>
<tr>
<th>#</th>
<th>Zenobia’s strategies</th>
<th>Phanes’ strategies</th>
<th>Alex’s strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Creating students’ positive feelings</td>
<td>Urging students to bring questions to the tutorial</td>
<td>Introducing students to the tutorial group</td>
</tr>
<tr>
<td>2</td>
<td>Selecting tasks</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Selecting examples</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Evaluating students’ mathematical sense making</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Decoding the mathematics and encoding the mathematics</td>
<td>Showing and asking students to repeat</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Explaining</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A first interpretation is that strategies with the same conceptual names were implemented differently from case to case of teaching, because different tutors acted with sets of different tools. For instance, the strategy ‘selecting examples’ (#3 in Table 7.1) with the same conceptual name across cases was implemented with different kinds of examples in each case of teaching. However, different tutors sometimes acted with the same set of tools for a common strategy (i.e. a strategy with the same conceptual name across cases). For instance, all tutors’ tools for the strategy ‘explaining’ (#6 in Table 7.1) were mathematical representations and rhetorical questions. In observations, the strategy ‘explaining’ was nevertheless implemented differently from case to case of teaching, because different tutors repeatedly acted with some tools for ‘explaining’ over others; for instance, Zenobia acted with graphical representations for her heuristics, whereas Alex acted with diagrams to explain set-theoretic definitions.

Another possible interpretation is that some strategies with the same conceptual names were implemented differently from case to case of teaching, because different tutors had different epistemologies of mathematics and different epistemologies of teaching/learning. An epistemology is a theory of knowing, socio-culturally based, intuitive and holistically inter-connected (Burton, 2004): socio-culturally based as being based on prior sociocultural experiences of the tutor; intuitive as reflecting the tutor’s intrapsychological functioning on the individual plane and as not always based on established knowledge; and holistically inter-connected as forming a connected whole of the tutor’s views. My discussions with the tutors revealed some aspects of the tutors’ views; however, these aspects do not present the entirety of each tutor’s epistemologies. Analysis of discussions nevertheless suggests the tutor’s views which made the strategies #1 and #5 be different and the strategies #3 and #6 be implemented differently from case to case of teaching.
Strategies #1 in Table 7.1

In all cases of teaching, the creation of this group of three strategies draws on the tutor’s epistemology of teaching/learning; in particular, on the tutor’s views on students’ mathematical learning and the role of the tutor. All three tutors’ views were that the students should “speak” and participate in the teaching/learning environment of the tutorial. In this section, I summarise the tutors’ different strategies for students who participate in the tutorial. Also, considering the level of achievement at which the students are, I account for the success of the strategies.

Zenobia acted with tools to create students’ positive feelings during the tutorial: pastoral questions, humour, the eureka moment, encouraging and valuing statements. Her view was that if the students feel positive in the tutorial, they feel comfortable and work harder for their learning. She can thus “communicate with them” in the tutorial and they “speak”. My observations confirmed that the students spoke in her tutorials; they suggested tasks, admitted their difficulties, asked questions and offered mathematically correct contributions. My consideration nevertheless is that the feeling of euphoria, which is connected with the eureka moment at the discovery of a solution, attends mostly to high-achieving students. (The eureka moment and the feeling of euphoria are presented in Section 4.2.2.2 about Zenobia’s strategy ‘creating students’ positive feelings’.) This is because students should be able to discover solutions in order to experience a eureka moment.

Phanes acted with tools to urge students to bring questions to the tutorial. His view was that students should “study” and “solve as many tasks as possible with their own hand” in order to learn the mathematics. Then, they should bring their questions and difficulties for work with the tutor. This is a strategy that requires a degree of students’ independency from the tutor and responsibility for their own learning; perhaps students’ willingness and care, as well. Evidence indicates that it was a successful strategy for the student in the pilot study; she suggested work for the tutorial and I observed her working hard. However, the strategy did not seem to work with the low-achieving students of the main study.
Alex implemented different strategies to encourage students to speak in his tutorial. He viewed mathematical learning as being developed through mathematical dialogue between tutor and students. His strategy to design examples in the context of real world situations did not seem to work for mathematical dialogue, until he implemented a strategy concerning the introduction of students to the tutorial group. In the latter strategy, he organised a whole class discussion where every student and Alex talked about her/himself in order to get to know each other, thus feeling comfortable to make mathematical dialogue and to work together. He also set his expectations for students who work and bring questions to the tutorial. The students largely did not attend the next tutorial, but there is no evidence as to whether this was a response to Alex’s strategy. Their introduction to the tutorial group was nevertheless a successful strategy for their hard working, bringing questions to the tutorial, forming a group for work outside the tutorial time and performing well in coursework. Based on evidence that the students were engaged in the tutorial work after Alex used that strategy, my interpretation is that ‘introducing students to the tutorial group’ is a strategy for a tutorial where the tutor meets the students for the first time. This is also because, during the implementation of that strategy, the students get informed about the tutor’s expectations, get to know each other and start to develop an awareness of ‘belonging’ and cultural inclusion, which both offer security and foster responsibility through involvement.

The group of strategies #1 reveals the tutor’s views for students’ learning; how students learn with a degree of independency from the tutor and what opportunities the tutors provide for the students in order for them to work hard and learn. Differences in views explained why the three different tutors’ teaching happened in different ways. For instance, in Zenobia’s and Alex’s tutorials the tutors viewed students’ participation to relate to students who feel comfortable in the group. Also, in Phanes’ and Alex’s tutorials, the tutors’ view was that the students should be responsible to bring questions to the tutorial. In their practice, all tutors used strategies in order for the students to feel comfortable and/or articulate their questions.
In all cases of teaching, the selection of specific kinds of mathematical examples from each tutor draws on the tutor’s own research. The tutors declared in our discussions that they select the examples for students’ mathematical meaning making. In the case of Zenobia’s and Phanes’ teaching, both of whom are mathematicians, the strategy draws on their epistemology of mathematics; whereas in the case of Alex’s teaching, who is a mathematics educator, it draws on his epistemology of teaching/learning. These specific tutors’ epistemologies inform and are informed by their research practice.

Whatever the research area in which the tutor was working, analysis indicates that it influenced the mathematical examples they selected. Zenobia selected *generic sets of examples* and used them in the decoding of the mathematics; a practice she declared she uses in her own research in analysis as well as her teaching. Phanes selected *simple but still meaningful examples* and used them in showing and asking students to repeat. He also declared that he used this kind of examples in his own research on “bridges within different Sciences”. Finally, Alex selected real-word examples and a combination of examples and non-examples; all reported in mathematics education research literature as enhancing the students’ mathematical meaning making (e.g. Ormell, 1975; Zodik & Zaslavsky, 2008).

Zenobia’s epistemology of mathematics, informed by the history of mathematics, included an inductive approach from working with special cases to developing a sociocultural consensus of a mathematical object. The examples served as the special cases towards the generalisation. Phanes’ epistemology of mathematics included connectivities within mathematics and from mathematics to other areas, offered by simple examples that contributed to the generalisation and unification of mathematical areas. Burton (2004, p.191) reported an “almost universal agreement” among the 70 mathematicians of her study of the importance of such connectivities. Alex’s epistemology of teaching/learning included dialogue among learners and tutor and a mediated negotiation of mathematical meaning. He declared in our discussions that an initial tool he designed to encourage dialogue was the real-world examples.
All three tutors’ kinds of examples are related to their research in mathematics or mathematics education. The strategy #3 reveals a path of a double way of informing: from the tutor’s epistemology of her/his discipline to the research area in which the tutor works to her/his teaching practice. Burton (2004) agreed with the first path of informing, that is from the epistemology to the research area. She argued that what mathematicians think mathematics *is* is attached to their research practices. For the second path of informing, that is from the research area to teaching practice, my interpretation is that the two mathematicians select specific kinds of examples to enculturate students into advanced mathematical thinking. These specific kinds of examples are relevant to their epistemologies of mathematics (i.e. Zenobia’s sociocultural consensus of ‘ideal objects’ and Phanes’ connected view within mathematics). In contrast, the mathematics educator selects examples, which are in the context of students and relevant to his epistemology of teaching/learning, in order to bring the mathematical content to students’ culture. So, recognition of kinds of examples repeatedly selected in a tutor’s teaching provide insight into the tutor’s epistemology of her/his discipline.

*Strategies #5 in Table 7.1*

In all cases of teaching, the tutors implemented this group of strategies, drawing on their epistemology of teaching/learning; in particular, on their views on teaching. These strategies occupied most of the tutorial time in each case of teaching.

In the case of Zenobia’s teaching, ‘decoding the mathematics and encoding the mathematics’ relates to her reflection on resolving her own difficulties during her mathematical research. This reflection informed her views on teaching. Burton (2004) explained more generally how the 70 mathematicians in her study discussed about resolving their difficulties.
The mathematicians were clear that when they began a problem in an area that was new for them and they had, as it was, to learn ‘the basics’, they did this by engaging with the problem and searching for ways of understanding and deconstructing it: *To solve a problem, you have to go and find out about some mathematics that you didn’t know.*

(Burton, 2004, p.192)

It seems to me that Burton’s notion of deconstructing is congruous with Zenobia’s notion of decoding. Zenobia’s tools for decoding the mathematics in her teaching and her own research involved various *heuristics*, such as ‘Sketch graph(s)/diagram(s)’ and ‘Consider special cases’.

In the case of Phanes’ and Alex’s teaching, ‘showing and asking students to repeat’ was included in their views of teaching. In our discussion, Phanes stressed that this is how you teach anything and not only mathematics. This strategy reveals an intuitive view of teaching informed by everyday life and, as Phanes stressed, greatly generalisable. It also seems to me that this is a successful teaching strategy independently of the students’ degree of responsibility for their own learning.

The group of strategies #5 is identified after a few repetitions on the part of the tutor, because of the tutorial time it occupies. In discussion with the tutor, it reveals the tutor’s views on teaching. The three tutors were typical of others in the same environment. For instance, Phanes’ teaching was a case with evident breadth of mathematical knowing. Alex’s teaching was a case of a flexible approach to teaching informed by research literature in mathematics education. Although Zenobia’s teaching shares some evidence of Phanes’ breadth of mathematical knowing, analysis indicates a high degree of reflection on her practices, both research and teaching. This provides evidence that Zenobia’s case of teaching was intermediate between the other two cases of teaching in terms of reflecting on mathematical and teaching issues in her research and teaching practice.
Strategy #6 in Table 7.1

In all cases of teaching the selection of the tool *graphical representation*, which is associated with the strategy ‘explaining’, draws on the tutor’s own craft of learning mathematics. Different tutors develop their crafts with regard to the communities in which they participate; such as the research communities. In this study, the three tutors’ crafts of learning differed considerably and differences were observed in their practices with *graphical representations*, as well.

As a researcher of mathematics, Zenobia was a sophisticated learner in mathematics. In particular, she selected *graphical representations* for the implementation of heuristics towards her mathematical discovery in analysis. Heuristic reasoning with *graphical representations* prepared for a rigorous proof in both her research and teaching. Phanes, also as a sophisticated learner and researcher of mathematics, selected *graphical representations* for his convenience in mathematical discovery in geometry.

In the case of Zenobia’s and Phanes’ teaching, the selection of the tool *graphical representation* draws on the tutors’ craft of learning in their own mathematical research. This relates to an intuitive theory of knowing mathematics; an epistemology of mathematics. Thus the repeated selection of the tool *graphical representation* in Zenobia’s and Phanes’ teaching draws on their epistemologies of mathematics.

In contrast to Zenobia and Phanes, Alex was a sophisticated learner as a mathematics tutor, but he was not a researcher in mathematics. He declared that he selected to draw *set diagrams* while studying mathematics. He also repeatedly used set diagrams in his explanations to students, as his teaching experience indicated that some students benefit from the diagrams which represent the symbolic representations of definitions. In the case of Alex’s teaching, the selection of the tool *set diagram* draws on his craft of learning in studying mathematics, thus an epistemology of teaching/learning.

To conclude, the cross case analysis indicated that the tutors develop their epistemologies and practices with regard to the views and values of the communities in which they are members; for instance, with regard to the research communities. Differences in epistemologies explained some differences in strategies and tools that
characterise the teaching of each of the three tutors; in other words, why the three different tutors’ teaching happened in different ways.

**Strategies #2 and #4 in Table 7.1**

The remaining strategies #2 and #4 also indicated important aspects of the three tutors’ teaching; however, my analysis did not indicate their explicit connections with the tutor’s epistemologies. In this section, I account for the aspects that the two strategies revealed.

Strategy #2, which is ‘selecting tasks’, revealed each tutor’s awareness of the students’ difficulties in mathematics. Among the three tutors, there was a consensus that students face difficulties with concepts and perform better in procedures. This is in agreement with research literature on mathematical conceptions and difficulties (e.g. Sfard, 1991; Nardi, Jaworski & Hegedus, 2005; Nardi, 2008).

In the analysis of my discussion with the tutors about ‘selecting tasks’, I was also able to discern the tutors’ goals for teaching in tutorials. My analysis of observational and interview data indicated that the tutors’ practice was oriented to goals, such as: ‘to enable students to pass the exams’; ‘to enable them to resolve difficulties and to make mathematical meaning’. My consideration is that I was able to discern goals connected to the exams and the students’ meaning from analysis of this strategy, because an integral part of the tutorial time was devoted to the solution of tasks. Also, the tutorial setting was established in order for the students to resolve their difficulties; so the tutor, sometimes with the students, selected tasks for discussion, which had the potential to foster students’ mathematical meaning making.

Strategy #4, ‘evaluating students’ sense making’, was a strategy of great importance for the tutor’s decision to redesign the teaching in order to enable the students to make mathematical sense in the tutorial, and to align their meanings to established mathematical meanings. This was the strategy which provided the tutor with the students’ feedback in relation to whether the teaching was successful for goals related to meaning making.
Looking at tools associated with each tutors’ strategy #4, several indicators of students’ mathematical meaning making emerged from the tutor’s perspective. Some indicators in the three cases of teaching were students’ contributions in mathematical dialogue and facial expressions. So, in this strategy the associated tools were the indicators of students’ mathematical meanings.

7.2 Mathematics knowing for teaching at university level

In Chapters 4, 5 and 6, the explanation of strategies and tools for teaching was in each case built on a grounded analysis of eight tutorials and exemplified through a sample of three teaching episodes, which were judged to be paradigmatic of the tutor’s practice. After each episode, I produced a model of the tutor’s teaching practice (Figure 7.2). This model illustrates the stages of design and redesign in a tutor’s teaching. The red arrows represent dialogue about mathematical meanings between the tutor and the students. In the final redesign, the tutor’s perspective is that her/his teaching has reached a stage that enables the students to make meaning of the mathematics.

![Figure 7.2: Model of teaching practice: Tutor’s design and redesign for students’ meaning making.](Image)
In Chapters 4, 5 and 6, I included only the tutor’s tools in the model of Figure 7.2. This was for convenience: distinctive teaching tools correspond to one and only one strategy (e.g. see Table 7.2 below); so, the reader knows the strategy by knowing the associated teaching tool.

The model of teaching practice includes the space of mathematics and the space of teaching/learning in order to illustrate a distinction between teaching tools. My consideration is that the teaching tools which are drawn on the space of mathematics correspond to mathematical practices e.g. in own research or in own teaching. In contrast, the tools drawn on the space of teaching/learning correspond to practices in the context of the students. The spaces of mathematics and teaching/learning are not straight but interrelated in Figure 7.2. This is because in mathematics teaching practice some strategies and tools, which are used in the developmental stages of design and redesign, are from the space of mathematics, while others are from the space of teaching/learning. Table 7.2, below, is an informative table of the distinction between teaching tools drawn on the space of mathematics and teaching tools drawn on the space of teaching/learning.

**Table 7.2:** The distinction in spaces between teaching tools.

<table>
<thead>
<tr>
<th>Strategy #</th>
<th>Tools in the Space of Mathematics</th>
<th>Tools in the Space of Teaching/Learning</th>
<th>Strategy #</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Generic set of examples</td>
<td>Real-world examples</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>Simple but still meaningful</td>
<td>Questions to evaluate students’ sense making</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>examples</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Examples and non-examples</td>
<td>Injunction questions about a heuristic</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>Heuristics</td>
<td>Questions to observe</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>Eureka moment</td>
<td>Rhetorical questions</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>Symbolic representations</td>
<td>Pastoral questions</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>Verbal representations (formal</td>
<td>Questions relating to students’ future references &amp; to students’ suggestions for teaching</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>and informal mathematical</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>language, aesthetic statements)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Graphical representations</td>
<td>Humour in the form of levity</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>Tabular representations</td>
<td>Valuing statements, encouraging</td>
<td>1</td>
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<td></td>
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<td>statements, injunction statements</td>
<td></td>
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<td></td>
<td></td>
<td>Consolidating statements,</td>
<td>6</td>
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</table>
Vygotsky, following Marx, considered knowledge as the reflection of reality (Duarte, 2011). This study enquires into ‘teaching knowledge’ and views it as the reflection of teaching practice. My reference to a person’s knowledge for teaching is with the term *knowing for teaching* in order to stress that *knowing* is on the individual plane. In particular, I consider a tutor’s *knowing for teaching* as a tutor’s reflection on her/his teaching practice. ‘Teaching Knowledge’ is concerned with a generalisation of the analysis of *knowing for teaching* in the three carefully selected cases of teaching and is an outcome of this research.

My assertion in Chapter 2 was that Vygotsky’s philosophy (1981) considered a dynamic view of a person’s knowing with developmental stages connected with each other with dialectic connections. In my study of tutor’s knowing for teaching, a dialectic connection arises out of a contradiction in dialogue about mathematical meanings between the tutor and the students. In particular, the tutor designs the teaching in order for the students to make mathematical meaning, so her/his view is that the students will make meaning with that design. However, during the implementation of the design in the tutorial, there is a contradiction between the tutor’s view for students’ meaning making and students’ meaning making per se from the tutor’s perspective (i.e. what the tutor sees from the students). So, the successive redesign of the teaching emerges from a contradiction as a change of tools, which are intended to foster students’ mathematical meaning making.

<p>| | |</p>
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<tr>
<td><strong>revoicing statements</strong></td>
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<tr>
<td>Students’ results in coursework</td>
<td>2</td>
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<tr>
<td>Students’ difficulties from teaching experience</td>
<td>2</td>
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<tr>
<td>Students’ mathematical questions</td>
<td>4</td>
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<tr>
<td>Students at the board</td>
<td>5</td>
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<tr>
<td>Students’ inputs in dialogue</td>
<td>4</td>
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<tr>
<td>Students’ work on their scripts (ability to solve)</td>
<td>4</td>
</tr>
<tr>
<td>A student</td>
<td>5</td>
</tr>
<tr>
<td>The tutor</td>
<td>5</td>
</tr>
<tr>
<td>Students’ faces</td>
<td>4</td>
</tr>
</tbody>
</table>
In this study, teaching knowledge is the reflection of teaching practice. So, the model of teaching practice (Figure 7.2) offers the foundation for a model of teaching knowledge. I name the latter model ‘Teaching Knowledge-in-practice’ (Figure 7.3), because it illustrates teaching knowledge revealed in practice.

My theorisation of ‘Teaching Knowledge-in-practice’ is the following. I consider the tutor’s theories of knowing the mathematics and knowing the context of the students, thus the tutor’s epistemologies of mathematics and teaching/learning (Figure 7.3), as the tutor’s reflections on the spaces of mathematics and teaching/learning in Figure 7.2. Furthermore, I consider the developmental stages of knowing, or coming to know, for teaching as the tutor’s reflections on the developmental stages of design and redesign in her/his teaching practice. I distinguish between two aspects of tutor’s knowing: the pedagogical knowing (Figure 7.3) and the didactical knowing (Figure 7.4).

*Pedagogical knowing* is concerned with knowing ways of moving across developmental stages of teaching until a developmental stage which enables the students to make meaning of the mathematics from the tutor’s perspective.
Figure 7.3 illustrates *pedagogical knowing* with a red arrow from the stage of design towards the stage of final redesign, where students have made mathematical sense from the tutor’s perspective. This knowing is contingent in nature (e.g. Rowland, Huckstep & Thwaites, 2005; Rowland & Zazkis, 2013), as it draws on dialogue between tutor and students. It thus depends on the tutor’s strategy ‘evaluating students’ mathematical sense making’ for a judgment as to what stage of the (re)design enables the students to make meaning of the mathematics. Considering that learners have different thinking and learning styles (e.g. Marton & Säljö, 1976; Felder, 1993; Prosser & Trigwell, 1999), the *pedagogical knowing* is evident in practice from a flexible design of teaching with a number of tools available for selection in successive redesigns. (An exemplification of such a flexible design of teaching is in the case of Alex’s teaching: Section 6.3.3.)

*Didactical knowing* is concerned with knowing ways of translating the principles and content of mathematics into forms of tutor’s thought in the context of students.

Figure 7.4 illustrates *didactical knowing* with a red arrow from the tutor’s epistemology of mathematics towards the tutor’s epistemology of teaching/learning. So, the red arrow points from the tutor’s reflections on the space of mathematics to the tutor’s reflections on the space of teaching/learning. The *didactical knowing* is concerned with knowing ways of making the design of teaching in order to translate the principles and content of mathematics into the context of students. A tutors’ initial design includes those tools that have been proved from the tutor’s teaching experience to be appropriate for such translating. This is because in any teaching experience, a tutor’s subsequent redesign involves a change in tools of the preceding (re)design and that preceding (re)design exists within the following redesign. In other words, the initial design is the distillate of those tools which enabled students to make sense of the mathematics in a past tutor’s experience. The *didactical knowing* is evident in practice from designs of teaching whose repeatedly selected tools and strategies, which are compatible with the tutor’s epistemology of mathematics, are enriched with tools and strategies compatible with the tutor’s epistemology of teaching/learning. This enrichment is in order for the tutor to translate the principles and content of mathematics into forms of her/his thought in the context of students.
The three cases of teaching offer an exemplification of the pedagogical knowing and the didactical knowing. In the case of Alex’s teaching, his flexible design with different tools and strategies reveals his pedagogical knowing. In the case of Phanes’ and Zenobia’s teaching, his ‘simple but still meaningful examples’ and her ‘encoding and decoding’ are used in both their research practice in mathematics and in their didactical practice. This reveals their deep mathematical knowing which inform their didactical knowing.

7.3 Contribution

The objective of this study was to investigate how teaching knowledge could be conceptualised within the Vygotskian perspective. Central to the conceptualisation of teaching knowledge, with which this study contributes to the research literature, are the concepts of ‘tool’ and ‘tool-mediated action’ (namely ‘strategy’). To date, I am not aware of a study on teaching knowledge that draws on Vygotsky’s theory of learning and knowing in order to view ‘tools’ as the unit of analysis of teaching practice and teaching knowledge. I consider that this is an important contribution of this study.

In the tutorial setting of the study, teaching knowledge was revealed as a reflection of tutors’ practice with a range of strategies and tools for teaching. In particular, the ‘Teaching Knowledge-in-practice’ was produced to offer a framework for researcher analysis of teaching knowledge which is revealed in practice. ‘Teaching Knowledge-in-practice’ presents two aspects of knowing: the pedagogical knowing (Figure 7.3) and the didactical knowing (Figure 7.4); both of which are in interaction with students’ mathematical meaning making. To the best of my awareness, there is no other study in any level of education that investigates within a Vygotskian perspective knowledge for teaching mathematics through an examination of practice with tools for teaching.
The conceptualisation of *didactical knowing* and *pedagogical knowing* was with Vygotsky’s dialectical method, according to which individuals come to know through contradiction(s) in dialogue with other individuals. In particular, although the tutor designed the teaching to enable the students to make mathematical meaning, the students did not always make sense of the mathematics. This was a contradiction between the tutor’s views for her/his design of teaching for students’ mathematical meaning making and the students’ mathematical meaning making with regard to that teaching. My interpretation is that out of successive contradictions of this kind, the tutor comes to know ways of enhancing the design of teaching with tools and strategies drawn on the space of teaching/learning, in order to translate the principles and content of mathematics into forms of her/his thought in the context of students (*didactical knowing*). In this way, the tutor disseminates the mathematics to students. Furthermore, the tutor comes to know ways of being flexible in moving across developmental stages of teaching until a developmental stage which enables the students to make meaning of the mathematics (*pedagogical knowing*). The successive developmental stages of teaching include a variety of tools and strategies drawn on both spaces (of mathematics and of teaching/learning). In this way, the tutor comes to know flexibility in ways of enabling the students to make mathematical meaning.

The in-depth analysis of *didactical knowing* and *pedagogical knowing* in teaching practice also contributed to an understanding of what might be called “automatic” and “tacit” in experienced tutors’ mathematics teaching practice. In particular, the *pedagogical knowing* is revealed in their practice, which is contingent in nature; thus, this kind of knowing is “automatic” and “tacit”. This is because *pedagogical knowing* is concerned with knowing ways of re-acting (with a variety of tools and strategies) to students’ expressed views on the degree to which they have made mathematical meaning with regard to the teaching. In the setting of this study, the *didactical knowing* was usually contingent in nature as well, because the tutors did not usually plan the tutorials but responded to the students’ suggestions for tutorial work in the moment. So, the critical scrutiny through rigorous analysis in this study shed light into specific strategies and tools for teaching that tutors used contingently in order to foster students’ mathematical meaning making. Analysis ultimately revealed that the particularities in the implementation of those strategies and tools, which was
automatic and tacit, was in agreement with the tutors’ epistemologies of mathematics and teaching/learning.

7.4 Implications of The Study

Considerations for mathematics. A sociocultural perspective on mathematics indicates that “mathematics is related to the persons in their sociocultural setting” (Burton, 2004, p.179). In Chapters 4, 5 and 6, I analysed the three tutors’ epistemologies of mathematics which were related to the communities in which the tutors participated; thus to their sociocultural experiences with mathematics within the communities. My analysis confirmed Burton’s (2004) assertion that what mathematicians think mathematics is was attached to their research practices. So, Zenobia’s and Phanes’ epistemologies of mathematics were attached to their research practices. Although Zenobia’s and Alex’s epistemologies of mathematics were informed by the history of mathematics, they were rather different. Zenobia’s epistemology of mathematics, detailed in Chapter 6, included an inductive approach from working with special cases to developing a sociocultural consensus of a mathematical object. In contrast, Alex’s epistemology of mathematics, detailed in Chapter 4, included learners who invest time to work on and master the mathematics. So, the history of mathematics is a source of information for some tutor’s epistemology of mathematics (e.g. Mali, Biza & Jaworski, 2014); however, the meaning that each tutor draws out of history is related to the sociocultural setting within which s/he operates.

Considerations for teaching/learning. In this chapter, the cross case analysis indicated that differences in tutors’ epistemologies explained some differences in strategies and tools that characterise the teaching of each of the three tutors; in other words, why the three different tutors’ teaching happened in different ways. In particular, a path of informing emerged in the analysis of data: from the tutor’s epistemology of her/his discipline (i.e. mathematics or teaching/learning) to the research area in which the tutor works to her/his teaching practice. In the cases of the
two mathematicians’ teaching, practices that they used both in their research and teaching enabled them to translate the principles and content of mathematics into forms of their thought in the context of students, thereby revealing their didactical knowing. So, the didactical knowing is related to these tutors’ research practices in mathematics. This study also indicated that dialogue between tutor and students is integral to students’ mathematical meaning making. Such dialogue involves a mediated negotiation of the mathematics; thus, action with teaching tools on the part of the tutor and feedback on the part of the students. In particular, when the students’ feedback reveals that the students do not make sense of the mathematics, this feedback contradicts the tutor’s design of teaching. The contradiction gives rise to a redesign of the teaching with a change of tools, which are intended to foster students’ mathematical meaning making.

Methodological considerations. Chapters 2 and 3 include a detailed description of the theoretical background of the study and the methods of data analysis. These are a grounded analytical approach to observational and interview data of teaching practice and a sociocultural perspective to teaching practice and teaching knowledge. The difficulty with gathering data to investigate teaching knowledge was that this notion is neither tangible nor visible through the methods for empirical research. Thus, an investigation of knowledge though the examination of practice was a challenge. In order to access teaching knowledge, I needed to determine my interpretative paradigm and learn to work within it; and to unfold teaching practice and teaching knowledge in relation to a sociocultural perspective. In Chapters 4, 5 and 6, analysis using a grounded analytical approach and a Vygotskian perspective led to a conceptualisation of tutor’s teaching practice and knowing for teaching. Also, in this final chapter the cross-case analysis of the three carefully selected cases of teaching led to a more general conceptualisation of teaching knowledge-in-practice. The process of analysing data and reflecting on the methodological approach provided me with grounding in the rigour of working within an interpretative paradigm.
7.5 Limitations

As in any qualitative research, the findings of this study are not generalisable for a large population of tutors; they nevertheless provide in depth insights into the teaching practice and teaching knowledge in three carefully selected cases of teaching. The ‘Teaching Knowledge-in-Practice’ is an analytical framework which will enable researchers to qualitatively analyse teaching practice and teaching knowledge at university level. Analysis of teaching practice is with regard to students’ mathematical meaning making; however, students’ mathematical meaning making is concerned with the object of teaching practice and not the actual outcome in this study. So, the research design did not include interview data with the students and/or their marked coursework/exam tasks. This was because the aim of the study was to characterise teaching practice and teaching knowledge; and not to evaluate it. Finally, albeit the openness of the small group tutorial setting, the experienced tutors in researching and teaching mathematics and the different levels of group performance, the strategies and the tools of this study might need refinement for a characterisation of teaching practice in other settings.

7.6 Future Work

In future, interesting pieces of work will include juxtaposition between cases of tutorial teaching and cases of lecture teaching for a characterisation of university mathematics teaching practice across settings. The ‘Teaching Knowledge-in-Practice’ will not only be an analytical framework of teaching knowledge and practice, but also it may become a developmental framework for tutors’ professional development in their own teaching. Furthermore, evidence from the remaining 23 tutors who participated in the pilot study of this research indicates that didactical knowing is related to the tutors’ research practices. In future, a study on these tutors’ didactical knowing will report on critical aspects of this kind of knowing and on research
practices in mathematics which are revealed in mathematics teaching practice at university level.
References


Declaration of published work

Findings from this thesis are published in research journals.


Furthermore, aspects of the research work for this thesis were presented as working papers at conferences nationally and internationally. The research work was ongoing and the development of ideas is reflected on the papers.

**Peer-reviewed conference papers.**


**Editor-reviewed conference papers.**


**Oral Presentations without publication.**


**Poster presentations.**


The author would like to thank her co-authors for their collaboration.
Appendix A

Shulman’s categories of knowledge

<table>
<thead>
<tr>
<th>Shulman’s Categories of Teacher Content Knowledge (1986, pp. 9-10)</th>
<th>Shulman’s Categories of Knowledge Base (1987, p.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>* subject matter content knowledge;</td>
<td>* content knowledge;</td>
</tr>
<tr>
<td>* pedagogical content knowledge, relating to a) knowledge of “what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them” and b) knowledge of “the ways of representing and formulating the subject that makes it comprehensible to others” e.g. the most useful as well as alternative forms of representations of ideas, analogies, illustrations, examples, explanations as well as strategies for reorganizing the conceptions and preconceptions of learners; and</td>
<td>* general pedagogical knowledge, “with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter”;</td>
</tr>
<tr>
<td>* curricular knowledge, considering knowledge of instructional materials, indications for particular instructional strategies, alternative curriculum materials, curriculum materials in other subjects studied at the same time by students, topics and materials in the same subject taught the preceding and later years.</td>
<td>* curriculum knowledge, “with particular grasp of the materials and programs”;</td>
</tr>
<tr>
<td></td>
<td>* pedagogical content knowledge (PCK), “that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding”;</td>
</tr>
<tr>
<td></td>
<td>* knowledge of learners and their characteristics;</td>
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<tr>
<td></td>
<td>* knowledge of educational contexts, “ranging from the workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures”; and</td>
</tr>
<tr>
<td></td>
<td>* knowledge of educational ends, purposes, and values.</td>
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**Figure A.1:** Shulman’s (1986, p.9-10; 1987, p.8) major categories of teaching knowledge.
## Appendix B

Calendar details of pilot and main studies

**Table B.1:** Calendar details about Pilot Study 1.

<table>
<thead>
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<th>Sample</th>
<th>SGTs</th>
<th>Researcher’s narrative</th>
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Table B.3: Calendar details about Main Study 1.

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Appendix C

Samples of field notes and transcripts

C.1 Sample of field notes

Figure C.1 presents a sample of field notes for approximately six minutes from Zenobia’s SGT5. The tutorial group has just started to prove with the $\varepsilon - \delta$ definition that $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+3}{4} = \frac{3}{4}$. In this sample only, annotations in black letters are added for the reader’s convenience. The ‘Tutor’ is Zenobia, ‘D’ is St2 and ‘J’ is St4.

Figure C.1: Sample of the field notes from Zenobia’s SGT5.
C.2 Sample of transcripts

Figure C.2 presents a sample of the transcript from Zenobia’s SGT5. It is the transcript that corresponds to the sample of field notes in Figure C.1.

| 00:31:15 | Zen.: So, I would suggest that the first thing you do is separate off the last third of the board to be the work space. So, having a look at what we did over here [Zen. points to the solution of the previous task.] – what do you need to write first on that work space?

St2: I think I want to write this thing here from the work space.


| 00:31:15 | St2 is writing on the work section: |

Zen.: You’ve not given yourself a super-big work space.

St2: No, I haven’t.

Zen.: This is a good thing to think about, especially for second-semester Calculus. Don’t skimp on a space. A good way to make sure that you don’t lose stupid points is to give yourself enough room. [Zen. draws a bigger line for the work section] OK, so what’s the next thing?

| 00:31:15 | St2 is writing on the work section: |

St2: Next I am going to write this one here, [that is] to move the 4 out.

| 00:31:15 | St2 is writing on the work section: |

Zen.: Mm hm. OK. So, that’s the first bit of the work. Now, we want to turn it around and do the second bit of the work. So, why don’t you draw a line under that to do the second bit of the work? |
St2: So, I start with 1.

Zen.: We don’t know what delta is yet. Remember, that’s what we put at the very end. Right?

St2: I’m looking for my delta to be

Zen.: Well, you don’t know yet.

St2: Not yet?

Zen.: Not yet. What you need to do is untangle that expression \((x,y) – (a,b)\) now. So, for instance, what are a and b?

St2: What are a and b? Good question. I don’t know. [St2 holds his mouth with his finger.] They’re 0 and 0.

Zen.: They’re 0 and 0. Very good, St2. Excellent.

St2: I have to simplify this.

Zen.: And just to remind you – that comes from the definition – right? – because in the definition, \(|(x,y) – (a,b)|<\delta\), that’s where the limit is \((x,y)\) approaches \((a,b)\). Do you see that? So, whatever is in that limit sign in your problem, that’s what gets plugged in. Now, when we were doing our first example, we were given an arbitrary \((a,b)\) again. We weren’t given a specific set of numbers, so we didn’t have to plug anything in. But often for limits, you are actually going to have to plug in a particular point. So, there are particular numerical values.

St2 is writing under the line he draw in the work section

St2: So, this is to become \((x,y)\), isn’t it? [St2 refers to \(|(x,y) – (a,b)|<\delta\) which has to become \(|(x,y)|<\delta\].] Yeah.

St4: Put the brackets.

St2 corrects his previous inequality

St4: Lovely.

Figure C.2: Sample of the transcript from Zenobia’s SGT5.
Appendix D

Theoretical codes from literature: Glossary of concepts and terminology

In my analysis of the ways the lecturer-participants of this study taught, I use a number of concepts from research literature in school and university mathematics education. These are presented here.

Control questions (Viirman, 2015)

In his discursive analysis with the commognitive framework of Sfard, Viirman studied seven mathematician’s written and oral mathematical discourse (2014) and pedagogical discourse (2015) in “chalk talk” lectures of first-semester calculus and algebra modules at three Swedish universities. Within the pedagogical discourse, he identified a categorisation of questions: control questions, rhetorical questions, questions asking for facts (e.g. terminology, calculations) and enquiries for reflection on mathematics. From these categories of questions, ‘control questions’ is a theoretical code in my data analysis. Viirman (2015) started his account of control questions with examples of this category of questions in his data: ‘Do you follow?’, ‘Do you understand?’ and ‘Are there any questions?’ He then offered the context in his data from which the questions arose: “when a particularly important or complicated piece of mathematics has been presented” or “when the teacher is about to move on from one topic to another” (2015, p. 1175). The context of the data is crucial for the researcher in order to make a judgment while coding as to whether a tutor’s question is a control question.
Petropoulou, Potari and Zachariades (2011) introduced the idea of an example used to illustrate critical characteristics of concepts as one of the lecturer’s ways to foster mathematical meaning in lectures. My interpretation is that the use of an example to illustrate critical characteristics of concepts is what other researchers call a generic example. A generic example is an example that is presented so as to carry the genericity (“the carrier of the general”) inherently (Mason & Pimm, 1984, p. 287). In Mason and Pimm’s terms, a generic example of the concept of even numbers is 6 if it is presented by stressing the key features that make it generic of even numbers (e.g. by rewriting it as 2×3) and by ignoring the irrelevant features regarding 3. In contrast with Mason and Pimm (1984) who considered generic examples of concepts, Rowland (2002) offered generic examples of procedures as the core of generic proofs. Both approaches nevertheless stressed the importance of the presentation of the generic character of such an example.

In Rowland’s approach, the general (argument) is embedded in the generic example “endeavoring to facilitate the identification and transfer of paradigm-yet-arbitrary values and structural invariants within it” (Rowland, 2002, p. 176). An example-of-a-generic-example that Rowland (2002) routinely chooses for the introduction of the notion “generic example” is the calculation of the sum from 1 to 100 with Gauss’ method. Gauss added 1 to 100, 2 to 99 and, so on, and computed fifty 101s. The genericity of his method is that it can be generalised to find the sum of the first 2k positive integers, which is $k(2k + 1)$. The sum from 1 to 100 is a generic example of Gauss’ method. As such, it is “a characteristic representative of the class” (Balacheff, 1988, p.219, cited in Rowland 2002) of the sum of the first 2k positive integers. In his conclusions, Rowland offered three suggestions to teachers, whereby the second was to invite students to connect the generic argument with another particular case so that the students can see the generic character of the generic example. Furthermore, Nardi, Jaworski and Hegedus (2005) reported that the use of generic examples was among the most usual ways that tutors used to enhance their students’ meaning of concepts in tutorials.
Heuristics

Polya said that heuristic was a branch of study with the aim of studying “the methods and rules of discovery and invention” (1971, p. 112). He found traces of such study in the commentators of Euclid such as Pappus; in Descrates; in Leibnitz; and in Bolzano. He viewed his book *How to solve it* (1971) as “an attempt to revive heuristic in a modern and modest form” (p. 113). He referred to heuristics of his book as being:

...mental operations that hint the solution of a mathematical problem (p.130).

My interpretation is that Polya’s term ‘mental operations’ refers to techniques by which the mind can operate to solve a mathematical problem. That is to say, heuristics are techniques for the solution of a problem. In my study of teaching and in this thesis, I use the term task instead of problem. This is because not all tutors viewed the tasks as problems which solution is hinted by heuristics. Consequently, not all tutors said in our discussions that when they approached the solution of a task in the tutorial, they had explicitly in mind heuristics.

Polya produced a “short dictionary of heuristics” (p. 37), where he explained each heuristic he suggested; and four phases in solving a problem. The phases are “understanding the problem”, “devising a plan”, “carrying out the plan” and “looking back” (p. xvi, p. 5). His heuristics are in the form of questions or suggestions. From the various heuristics for the different problem solving phases, I refer in this section to those that the tutors of my study used in their teaching:

“What is the unknown? What are the data?” (Phase 1)

“Draw a figure.” (Phase 1)

“Go back to definitions.” (Phase 2)

“Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.” (Phase 2) (p. xvi)
Polya distinguished heuristic reasoning, which is provisional, from rigorous proof, by saying that a heuristic argument may prepare for a rigorous argument. Heuristic reasoning, or reasoning with the use of heuristics, is connected with discovery. Polya (1971) stressed that “discovery often starts from observation, analogy, and induction” (p. 221). He considered “analogy” and “induction” to be heuristics. For him, the processes of induction and mathematical induction have “very little logical connection” (p. 114). He explained the two heuristics and the heuristic “specialization” as:

“Specialization is passing from the consideration of a given set of objects to that of a smaller set, or of just one object, contained in the given set.” (p. 190)

“Analogy is sort of similarity. Similar objects agree with each other in some respect, analogous objects agree in certain relations of their respective parts.” (p. 37, original italics)

“Induction is the process of discovering general laws by the observation and combination of particular instances.” (p. 114)

My interpretation is that in using specialisation, the problem solver considers special cases (“a smaller set” of objects) or considers a specific case (“just one object, contained in the given set”). Furthermore, induction is “the process of discovering general laws” from particular cases. This process is possible through the observation of similarity; in other words, the observation of “certain relations” in which “analogous objects agree”. So, it seems to me that “specialization”, “analogy” and “induction” are connected with discovery.

Considering the heuristics, Polya (1971) suggested in Phase 2 of problem solving:

“If you cannot solve the problem try to solve first some related problem.” (p. xvi)

He thought that the “related problem” should be more accessible than the original one, thus less complicated. As such, it can be one of the following: “a more general problem”, “a more special problem”, “an analogous problem” or “a part of the problem” (p. xvi). A more general problem reduces the conditions. A more special
problem relates to “specialization”; so the solver needs to consider special cases or a specific case. An analogous problem relates to “analogy”. Polya (1971) provided an example of a “simpler analogous problem” than the original one. He wrote: “In Section 15, our original problem was concerned with the diagonal of a rectangular parallelepiped; the consideration of a simpler analogous problem, concerned with the diagonal of a rectangle, led us to the solution of the original problem.” (p. 38, original italics) So, the “simpler analogous problem” agrees in certain relations of its respective parts with the original problem (e.g. regarding sides of rectangular parallelepiped and sides of rectangle); however, it is less complicated than the original one (e.g. two dimensions rather than three dimensions).

*Inviting questions – general or direct (Jaworski & Didis, 2014)*

Jaworski and Didis (2014) conducted a developmental study of a tutor-researcher’s university teaching of small groups of students. Their study discerned a tutor’s questioning style that fosters students’ meaning making of mathematics. This style includes two types of questions which role is to seek students’ articulation of mathematical meaning. One of the two types is ‘inviting questions – general or direct’ and it is a theoretical code in my study. Jaworski and Didis determined ‘inviting questions’ to be questions with which the tutor asks “students to respond” (p. 380). For them, an ‘inviting question’ can be ‘direct’ or ‘general’. ‘Inviting questions’, which are ‘direct’, require the response from a specific student. Examples of ‘direct questions’ in the data of Jaworski and Didis are:

“Alun. What is, what do you mean, if you write $f_x$ and $f_y$ ?”;

“OK, how about you Erik?”; and

“Alun, do you think the function is the middle one or would you say one of the others?” (2014, p. 380-1)

In these three questions, the researcher can identify a student’s name. So, the first and the third question required the response from Alun, whereas the second question required the response from Erik. In contrast, ‘inviting questions’, which are ‘general’, are posed to the group as a whole. Jaworski and Didis’ example of such a question is:
“So in the question then, we have three graphs; one of them is a function \( f \) and the other two are the partial derivatives \( df/dx \) and \( df/dy \). Now, which is which?” (2014, p. 381)

This is a question which is not posed directly to a student thus the researcher cannot identify a student’s name in it. Jaworski and Didis stressed that ‘inviting questions’ reveal students’ difficulties and students’ meaning through students’ expression in their responses. In their study, they reported that “[s]tudents are unused to such expressing” and “[s]tudents respond only tentatively to the tutor’s questions; responses are not articulate” (p. 383).

**Prototypical examples (Lakoff, 1987)**

Mason and Pimm (1984) introduced the notion of generic example as the specialisation/example that refers to a class of objects. A few years after Mason and Pimm’s contribution to generic examples, Lakoff (1987) focused on the production and the internal structure of that class of objects. He considered that rules or a general principle (which apply in a particular member of the class) take this particular member as input and yield the entire class as output; thereby producing and characterising the class. This particular member is the prototypical example of the class. For instance, considering that a square is a particular member that characterises the class quadrilateral:

- the rules that apply in the class quadrilateral can be four sides of equal length as well as four equal angles; or alternatively,
- the general principle of the class quadrilateral is that of similarity with a square.

However, the aforementioned rules or general principle of square do not indicate that a rhombus is a member of the class quadrilateral, because all angles of a rhombus are not necessarily equal. A common pitfall regarding the use of prototypical examples is that if another example (member of the same class as the prototypical example) does not comply with the prototypical example, then the new example cannot be recognised as member of the same class. A kite is another example of the class quadrilateral, which does not comply with a square, because all sides of a kite do not have the same length and all angles are not equal. Lakoff (1987) stressed that the
prototypical examples are ‘superficial’; put another way, they do not carry layers of generality within them. My interpretation is that the difference between prototypical examples and Mason and Pimm’s (1984) generic examples is the ‘superficiality’ of prototypical examples.

Revoicing (O’Connor & Michaels, 1993, 1996)

O’Connor and Michaels (1993, 1996) defined revoicing as the oral or written reuttering of a student’s contribution by another participant in the discussion. One of their examples of revoicing is:

Steven: I think that I don’t agree with Janette’s idea that we don’t need to use Paulina’s concentrate value, because I think it would be unfair to just not use Paulina’s concentrate . . . if you make a concentrate over with different amounts of lemon juice and sugar, then it’ll just be a totally different concentrate, like Ted said. (O’Connor & Michaels, 1996, p. 63)

In this transcript, Steven is the teacher and Janette, Paulina and Ted are students. Steven orally reutters the three students’ inputs in their discussion about a concentrate. His reutteration is in the form of reformulation. So, he reformulates what the students have already offered in the discussion. Steven’s reutteration is also in the form of creating students’ alignments and oppositions. For instance, he stresses that Janette’s idea is in opposition with Paulina’s concentrate. Reformulations as well as creating students’ alignments and oppositions within an argument are functions of revoicing that O’Connor and Michaels (1996) identified.

O’Connor and Michaels (1996) also recognised the roles of revoicing, some of which are: to highlight; to rebroadcast to reach a wider audience; and to socialise students into discussion and the process of making, analysing and evaluating claims, conjectures and arguments. In another study, which is about university teaching of differential equations, Park, Kwon, Ju, Park, Rasmussen and Marrongelle (2007) found additional revoicing roles. Two of these are: to recruit students’ attention to a specific claim; and to shape students’ follow-up inquiry.
Rhetorical questions (Mason, 2000; Artemeva & Fox, 2011; Fukawa-Connelly, 2012; Viirman, 2015)

Rhetorical questions are questions posed without the requirement of an answer. There is a number of previous research on the role of rhetorical questions in university mathematics teaching. For instance, Viirman (2015) reported the roles of rhetorical questions as recognised in the discourse of the lecturers of his study. He related two of these roles to Mason’s (2000) consideration of asking as focusing. In this way, he identified the roles:

“to direct students’ attention to specific steps in the reasoning”; and

“to direct students’ attention to certain aspects of the mathematics worthy of reflection” (2015, p. 1176).

Additionally, in Artemeva and Fox’s study (2011) of teaching university mathematics by writing a mathematical narrative on the board and talking aloud, the roles of rhetorical questions were:

“to signal a transition in the disciplinary narrative or to pause the action for reflection” (p. 362).

It seems to me that Artemeva and Fox’s roles of rhetorical questions resemble the roles that I cited for Viirman’s study. However, in Viirman’s study, Mason’s consideration of asking as focusing is stressed, thereby revealing a study of teaching with regard to students.

Finally in Fukawa-Connelly’s study (2012) of a lecturer’s demonstration of proofs in teaching abstract algebra, the rhetorical questions were “questions that a mathematician should ask while writing proofs, such as, ‘What does that mean?’, ‘What comes next?’ and ‘What do I still need to do?’” (p. 343) Fukawa-Connelly identified that the role of these rhetorical questions was to provide students with modes of thinking about the organisation and structure of proofs.
Appendix E

The analysis of the pilot study for the case of
Zenobia’s teaching

This section starts with an example of a narrative of Zenobia’s SGT, which reveals
dialogue and interaction between tutor and students. I wrote this narrative soon after
the observation, and reformulated it for presentation purposes of analysis in this
thesis.

I offer the narrative, firstly, to exemplify narratives and analysis of SGT observations
in Pilot study 1, and secondly, to explain issues that emerged in narratives of
Zenobia’s observations. These emerging issues were my criteria for inviting Zenobia
to be the participant of Main study 1.

<table>
<thead>
<tr>
<th>Narrative 3  Tutorial observation in October 29th, 2012</th>
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</thead>
<tbody>
<tr>
<td>This is my first observation of Zenobia’s tutorial. There are four students inside the classroom. Zenobia asks the four students whether they have questions. A student has printed a past coursework test in calculus, and asks how to solve the tasks:</td>
</tr>
<tr>
<td>(a) Compute the following limit: ( \lim_{x \to 0} \frac{(1+x)^5-1}{x} ).</td>
</tr>
<tr>
<td>(b) Suppose that ( f: [c, +\infty] \to \mathbb{R} ), where ( c \in \mathbb{R} ). State the definition of ( \lim_{x \to \infty} f(x) = \infty ).</td>
</tr>
<tr>
<td>Zenobia asks all students:</td>
</tr>
<tr>
<td>‘When you see a limit, what do you do?’</td>
</tr>
<tr>
<td>The students make some suggestions; Zenobia responds to the students’ suggestions with a ‘yes’ or a ‘no’. She chooses the student that asked about the tasks to write the limit of task (a) on the whiteboard. Zenobia suggests to develop the identity ( (1 + x)^5 ) to compute the limit, and shares with the students what she calls ‘a trick’: Pascal’s triangle. Another student comes up to the whiteboard, writes Pascal’s triangle by heart and computes the limit correctly. Zenobia informs students about a students’ common mistake from her experience: When they compute a limit, they...</td>
</tr>
</tbody>
</table>
forget to write $\lim_{x \to 0} \frac{(1+x)^5-1}{x}$. One student suggests ‘a point in a graph’ and then ‘a derivative’. Zenobia responds by saying ‘suppose we want to find $f'$ at $a$’. Two students dictate $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ to the student who writes it on the board. Zenobia asks the students to compare $\lim_{x \to 0} \frac{(1+x)^5-1}{x}$ and $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$. The students do not respond. Zenobia looks at a silent student, and says:

‘You are the most insecure, go to the whiteboard.’

The student writes on the board what Zenobia suggests: $x = h, a = 1$, and he transforms $\lim_{x \to 0} \frac{(1+x)^5-1}{x}$ to $\lim_{x \to 0} \frac{f(1+x)-f(1)}{x}$. After some questions from Zenobia, the student writes on the board: $f(x) = x^5, f'(x) = 5x^4$ when $x = 1, f'(1) = 5$. After computing the limit with both ways, Zenobia makes a summary of the techniques students can use when they have to compute a limit, such as ‘plugging in’, ‘expanding’ or ‘the definition of the derivative’.

For task (b), Zenobia asks from a student to draw on the board a graph of a function for which $\lim_{x \to \infty} f(x) = \infty$ is true. The student draws a function which looks like $f(x) = \frac{1}{x}, x > 0$, and copies on the board the definition of $\lim_{x \to \infty} f(x) = b$ from his lecture notes:

$$\forall \varepsilon > 0 \exists N > 0 \ |f(x) - b| < \varepsilon \text{ for all } x \in A: x > N.$$ 

He then writes the definition of $\lim_{x \to \infty} f(x) = 0$ on the board:

$$\forall \varepsilon > 0 \exists N > 0 \ |f(x)| < \varepsilon \text{ for all } x \in A: x > N.$$ 

Zenobia puts the values of $\varepsilon, N$ on the student’s graph, and explains what the definition says for that graph. The student then draws on the board the graph of a second function which looks like $f(x) = e^x$. Based on students’ contributions to how the definition should change for the second graph, Zenobia writes on the board the definition of $\lim_{x \to \infty} f(x) = \infty$ with regard to the second graph:

$$\forall M > 0 \exists N > 0 \text{ such that } x > N \Rightarrow f(x) > M.$$ 

A student corrects that it is not $\exists N > 0$, but $\exists N > c$.

In the next sections, I chose to present issues that emerged from analysis of narratives of observations of Zenobia’s SGTs during Pilot study 1. In this presentation, I use excerpts from Narrative 3, and include evidence from data of the other four observations of her teaching. Considering that I write the following account of my
pilot study analysis with hindsight, inevitably there are aspects of my current understanding in this presentation.

**Issue 1: Group work in Zenobia’s tutorials**

In Zenobia’s tutorials, I observed a group working collaboratively on mathematics. There were one to three tasks suggested by the students, and one task after the other was under whole group conversation. Zenobia’s conversation with all students, and the easiness with which students stood up to write on the board, were distinctive compared to other tutors’ SGTs. It seemed to me that Zenobia taught in a way which benefited from the fact that the group of students was small (5 students).

In Narrative 3, there are two coursework tasks. For the first task, three students write on the board. In particular, Zenobia “chooses the student that asked about the tasks to write the limit of task (a) on the whiteboard” [Narrative 3]. “Another student comes up to the whiteboard, writes Pascal’s triangle by heart and computes the limit correctly” [Narrative 3]. The latter student writes $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ for a second way of computing $\lim_{x \to 0} \frac{(1+x)^{5-1}}{x}$, based on Zenobia’s and two students’ contributions. Zenobia invites a third student to compute the limit with the second way by saying “You are the most insecure, go to the whiteboard”. A fourth student draws graphs for the second task.

In our discussion after the tutorial, I asked Zenobia why she chose “the most insecure” student to go to the whiteboard. Her response was that that she always chooses for the whiteboard the student she interprets as being the most insecure, since by being there that student is engaged with mathematics, tries to think, and can finally benefits from the work s/he does. It seemed to me that Zenobia’s response indicated that in her SGT, the whiteboard was a means or a tool for engagement with and thinking about the mathematics. Thinking of a tool for Zenobia’s teaching, the Vygotskian theory offered me a theoretical perspective for tools. At later stages of the study when I interpreted tools for Zenobia’s teaching, I looked at how my interpretation of tools could be theorised with regard to Vygotskian tools.
In the tutorial of Narrative 3, all students write on the board once, as well as in the third tutorial I observed. In the other tutorials, there is a student (not always the same for a whole tutorial) or Zenobia to the board. The remaining students contribute with ideas that are written on the board. It seemed to me that Zenobia had developed a group of tutor and students, rather than a tutor and individual students, for thinking and working on mathematics. The whiteboard was fundamental in the process of thinking and working on mathematics, since what was written on the board was a result of group work. In the fourth tutorial I observed, for instance, while a student is going to the board for a task, Zenobia says to the students: ‘Think as a group now. Think.’

In Narrative 3, Zenobia suggests the idea to develop the identity \((1 + x)^5\) using Pascal’s triangle to compute \(\lim_{x \to 0} \frac{(1+x)^5-1}{x}\) for the first task, and the student who goes to the board correctly computes \(\lim_{x \to 0} \frac{(1+x)^5-1}{x}\). Then, a student interprets \(\lim_{x \to 0} \frac{(1+x)^5-1}{x}\) as a derivative, and two students dictate \(f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}\) to the student to the board for a second way of computing that limit. My interpretation is that the student who was to the board was not under assessment. Rather, s/he had to be active and implement her/his idea for the work on the mathematics of the task, or the idea of another member of the group including Zenobia. Also, the four students were able to respond to the fellow students’ ideas as well as to Zenobia’s questions or contributions of ideas. For instance, the student to the board correctly expanded \((1 + x)^5\), and knew the Pascal’s triangle by heart.

Zenobia “asks” the students whether someone can interpret \(\lim_{x \to 0} \frac{(1+x)^5-1}{x}\) for task (a), and “asks” a student to draw on the board a graph of a function for which \(\lim_{x \to \infty} f(x) = \infty\) is true” for task (b) [Narrative 3]. It seemed to me that both of her questions were towards the group’s awareness of mathematical ways of work. Her first question was towards the connection between limit computation and derivative computation; and her second question was towards the heuristic of sketching a graph. With her questions, Zenobia orchestrated the whole group conversation of the tutorial into directions of mathematical enquiry she decided. So, it seemed to me that
Zenobia’s questioning was critical for the whole group conversation about the mathematics of the task, and the students’ mathematical meaning making.

**Issue 2: Graphs/examples for inductive thinking**

My explanation of an instance of students and Zenobia’s *inductive thinking* is towards the development of the definition of \( \lim_{x \to \infty} f(x) = \infty \).

Task (b) is about the definition of:

\[
\lim_{x \to \infty} f(x) = \infty, f: [c, +\infty] \to R.
\]

Zenobia suggests the *heuristic of sketching a graph*. For the definition of

\[
\lim_{x \to \infty} f(x) = 0
\]

\[
[\forall \epsilon > 0 \exists N > 0 \, |f(x)| < \epsilon \text{ for all } x \in A: x > N]
\]

Zenobia puts the values \( \epsilon, N \) on the student’s graph on the board. Then, she interprets the latter definition and its notation with regard to that student’s graph. The students look at the board which has

1. the stream of notation \( \forall \epsilon > 0 \exists N > 0 \, |f(x)| < \epsilon \text{ for all } x \in A: x > N \), which is the definition of \( \lim_{x \to \infty} f(x) = 0 \);
2. a first graph where \( \lim_{x \to \infty} f(x) = 0 \) is true; and
3. a second graph where \( \lim_{x \to \infty} f(x) = \infty \) is true.

The group of students has to *think inductively* on the graphs to produce the definition of

\[
\lim_{x \to \infty} f(x) = \infty.
\]

The students make contributions to how the definition of

\[
\lim_{x \to \infty} f(x) = 0
\]

\[
[\forall \epsilon > 0 \exists N > 0 \, |f(x)| < \epsilon \text{ for all } x \in A: x > N]\]
should change with regard to the second graph. The final *inductive product* of the whole group conversation is the definition of $\lim_{x \to \infty} f(x) = \infty$, $f : [c, +\infty] \to R$, which Zenobia writes on the board:

$$\forall M > 0 \exists N > c \text{ such that } x > N \Rightarrow f(x) > M.$$ 

I observed another instance of students and Zenobia’s *inductive thinking* in the fifth SGT. The following is a part of the narrative from that tutorial.

<table>
<thead>
<tr>
<th>Part of Narrative. Tutorial observation in December 10th, 2012</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zenobia writes on the board:</td>
</tr>
<tr>
<td>span</td>
</tr>
<tr>
<td>linear independence</td>
</tr>
<tr>
<td>basis</td>
</tr>
<tr>
<td>1) What is a linear map?</td>
</tr>
<tr>
<td>2) What do we say “define a linear map”?</td>
</tr>
<tr>
<td>She asks the students:</td>
</tr>
<tr>
<td>‘Do you have any thoughts about how do those things get together?’</td>
</tr>
<tr>
<td>She pauses, and says:</td>
</tr>
<tr>
<td>‘Start by considering an example.’</td>
</tr>
<tr>
<td>Zenobia and students’ contributions towards the production of the example are written on the board by Zenobia: $L: R^2 \to R^2$, matrix $M$, $L(\vec{v}) = M\vec{v}$, $\vec{v}_1 = (1,0)$, $\vec{v}_2 = (0,1)$, $L(\vec{v}_1) = \left(\begin{array}{c} 2 \ 3 \end{array} \right)$, $L(\vec{v}_2) = \left(\begin{array}{c} 1 \ 4 \end{array} \right)$, $M = \left(\begin{array}{cc} a &amp; b \ c &amp; d \end{array} \right)$ A student goes to the board, and correctly computes $M$.</td>
</tr>
<tr>
<td>Zenobia concludes that ‘as long as we find a basis, we can define a linear map’, and stresses that ‘it is an example, useful but not a proof’. She writes on the board $L: R^n \to R^m$, and asks:</td>
</tr>
<tr>
<td>‘How many input vectors do we need? Because in $L: R^2 \to R^2$ it is not clear for which two we are talking about.’</td>
</tr>
<tr>
<td>A student responds $n$, and Zenobia produces a second example on which they work:</td>
</tr>
</tbody>
</table>
I included, above, a part of the narrative of Zenobia’s teaching, which corresponds to approximately 10 minutes from the fifth SGT. In this part of narrative, Zenobia’s suggestion is to

‘start by considering an example’ of a linear map

in order for students to answer her question:

‘Do you have any thoughts about how do those things get together?’

The example is \( L: R^2 \rightarrow R^2 \), \( L(\vec{v}) = M\vec{v} \), \( \vec{v}_1 = (1,0) \), \( \vec{v}_2 = (0,1) \), \( L(\vec{v}_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), \( L(\vec{v}_2) = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \), \( M = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \). This example contributes to Zenobia’s inductive thinking towards the conclusion that ‘as long as we find a basis, we can define a linear map’. This conclusion is an answer for her question above; however, there is no evidence whether the students think inductively as well as Zenobia. Following the example with \( L: R^2 \rightarrow R^2 \), there nevertheless is a consecutive example with a slightly more complex linear map than \( L: R^2 \rightarrow R^2 \). The latter example potentially enables the students to think inductively.

Concluding my analysis, I viewed inductive thinking as a commonality between the heuristic ‘sketch a graph’ in Narrative 3, and the heuristic ‘start by considering an example’ in Part of narrative of the fifth tutorial. In our discussion after the fifth SGT, Zenobia informed me that she starts her own mathematical research by considering examples, and from examples, she generalises and discovers theory. My analysis of observations of the first and fifth tutorial exemplified Zenobia’s use of special cases (i.e. examples of linear maps and graphs of functions) for inductive thinking in her teaching. I thought that it would be interesting to have data with regard to the connection between her research and her teaching for a whole semester observation.
**Issue 3: Zenobia’s mathematical flexibility**

Zenobia did not design her teaching in SGTs. Rather, she asked the students about their questions in the beginning of SGTs. Students questions were on how to solve specific tasks from lecture materials. In the five tutorials I observed, the mathematical topics were in calculus, linear algebra, and differential equations. Zenobia knew the mathematics, and offered the students ways of working with the tasks they suggested. Some of those ways of working were specialised to the particular mathematical topics.

In Narrative 3, for instance, a student asked about two coursework tasks, and one was on limit computation. Zenobia suggested the students work on the limit by expanding \((1 + x)^5\), and for this expansion she shared what she called ‘a trick’: Pascal’s triangle. She then suggested a second way of working with the limit: the definition of derivative. Pascal’s triangle and the definition of derivative are two specialised ways for technical work on limit computation, or else two techniques for limit computation.

Instances of specialised ways for technical work on mathematical topics are also in the fourth tutorial I observed. In that tutorial, the students suggested three tasks, for which Zenobia said that ‘each of these has a little trick’. In the following Part of narrative, I refer to the third of those tasks, which was:

\[
\text{Find } y = \frac{d}{dx} \left[ \int_{x^2}^{9x} \sin (t^2 + t) \, dt \right].
\]

<table>
<thead>
<tr>
<th>Part of Narrative</th>
<th>Tutorial observation in December 3rd, 2012</th>
</tr>
</thead>
<tbody>
<tr>
<td>A student goes to the board for the third task, and another student suggests a way of working with the mathematics by saying:</td>
<td></td>
</tr>
<tr>
<td>‘Split into integrals – substitution in general.’</td>
<td></td>
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<tr>
<td>In response, Zenobia stresses that ‘there is a phrase I am searching here’, and the student who made the suggestion adds ‘Oh! FTC (Fundamental Theorem of Calculus).’ Zenobia shares with the students that:</td>
<td></td>
</tr>
<tr>
<td>‘Everytime you see a [task] with both a derivative and an integral, you think of the FTC.’</td>
<td></td>
</tr>
</tbody>
</table>
In this Part of narrative, my attention is not to what Zenobia might be calling a trick. Rather, my attention is to her suggestion per se that ‘Everytime you see a [task] with both a derivative and an integral, you think of the FTC.’ In this suggestion, Zenobia shared with the students a specialised way for work on derivative computation.

In both the first and fourth tutorial I observed, although Zenobia did not design the tasks, she shared with the students specialised ways for work on the mathematical topics of the tasks. In this way, she demonstrated her mathematical expertise and her mathematical flexibility in teaching, which was not designed. In other words, Zenobia demonstrated what I might evidence as ‘strong mathematical knowledge’. Also in Issue 1, above, I exemplified Zenobia’s questions to students. Through her questions, Zenobia orchestrated a whole group mathematical enquiry towards limit computation. The five observations of Zenobia’s teaching in Pilot study 1 enabled me to think of Zenobia’s teaching as a case where ‘a tutor with strong mathematical knowledge’ used whole group conversation in her tutorials to draw directions of mathematical enquiry. I was interested to observe such a case of teaching for a whole semester.