arXiv:1605.01791v1 [math-ph] 5 May 2016

Functional central limit theorems and $P(\phi)_1$-processes for the classical and relativistic Nelson models

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Abstract

We construct $P(\phi)_1$-processes indexed by the full time-line, separately derived from the functional integral representations of the classical Nelson model and relativistic Nelson model in quantum field theory. Associated with these processes we define a martingale which, under proper scaling, allows to obtain a central limit theorem for additive functionals of the two processes. We show a number of examples by choosing specific functionals related to particle-field operators.

Key-words: classical and relativistic Nelson models, non-local operators, ground states, Feynman-Kac representations, jump processes, functional central limit theorems

2010 MS Classification: 47D07, 47D08, 60J75, 81Q10

1 Introduction

In this paper we are interested in some stochastic properties of the so called Nelson model of an electrically charged spinless quantum particle coupled to a scalar boson field. These properties will be formulated in terms of central limit theorem-type behaviours of functionals of the particle-field operators. While the quantum field models discussed here are defined in terms of self-adjoint operators on a joint particle-field space of functions, for our purposes a Feynman-Kac type approach will be more suitable. Then the related evolution semigroups can be represented in terms of averages over the paths of suitable random processes, which has been much explored lately.

The Nelson model is defined by a self-adjoint operator of the form

$$H_{\mathrm{N}} = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + H_{\mathrm{i}}, \tag{1.1}$$

on a Hilbert space $L^2(\mathbb{R}^d) \otimes \mathscr{F}_{\mathrm{b}}$, where $\mathscr{F}_{\mathrm{b}}$ denotes the boson Fock space over $L^2(\mathbb{R}^d)$, and the components describe the Hamilton operators of the free particle, free field, and particle-field interaction, respectively. The classical and the relativistic Nelson models differ by the choice of the free particle operator H_{p}, which in the classical model is a Schrödinger operator and in the relativistic case a relativistic Schrödinger operator, as given by the expressions (2.1) and (2.2) below. On the functional integral representation level this difference will appear in the fact that a Schrödinger operator generates a diffusion, while a relativistic Schrödinger operator generates a jump process.

A functional CLT for the classical Nelson model has been first established by Betz and Spohn in [2]. They have shown that under the Gibbs measure obtained from taking the marginal over the particle-generated component of the path measure in the functional integral representation of H_{N}, the process scaled by Brownian scaling converges in distribution to Brownian motion having reduced diffusion coefficients. This means that the particle increases its effective mass due to the coupling to the boson field. The main observation in this paper is that one can associate a martingale with functionals of the process, whose long time behaviour can be predicted by using the martingale convergence theorem. The result for more general Markov processes is originally due to Kipnis and Varadhan [14, Theorem 1.8], and similar problems are studied also in [3, 4, 6]. Whenever in H_{p} the external potential V is chosen to be sufficiently regular, the operator semigroup $\{e^{-tH_{\mathrm{p}}} : t \geq 0\}$ can be studied by a Feynman-Kac type representation, i.e., there exists a random process $(Z_t)_{t\geq 0}$ on a suitable probability space such that

$$\left(e^{-tH_{\mathrm{p}}} f\right)(x) = \mathbb{E}^x \left[e^{-\int_0^t V(Z_s)ds} f(Z_t)\right] \tag{1.2}$$

holds for all Borel measurable f on $\mathbb{R}^d$, where the expectation is taken with respect to the path measure of $(Z_t)_{t\geq 0}$. When H_{p} is a classical Schrödinger operator, the process is a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ and the path measure is Wiener measure $\mathcal{W}$ on the space of continuous paths $C((0,\infty),\mathbb{R}^d)$. When, however, H_{p} is a relativistic Schrödinger operator, the process changes to a d-dimensional Lévy process with their corresponding path measures, now on the Skorokhod space $D((0,\infty),\mathbb{R}^d)$ of càdlàg paths (i.e., the paths are discontinuous, but continuous from the left with right limits). Whenever the coupling between the particle and field is turned on, the boson field will contribute by an infinite dimensional Ornstein-Uhlenbeck process so that in the path integral representation of the evolution semigroup $\{e^{-tH_{\mathrm{N}}} : t \geq 0\}$ a two-component random process will appear, as it will be explained below. Due to the linear coupling between particle and field, one can integrate over the OU-component and the marginal distribution for the particle will contain beside the given external potential also an effective pair-interaction potential resulting from the interaction.

Assuming that the external potential is chosen in such a way that the bottom of the spectrum $E_{\mathrm{p}} = \inf \sigma(H_{\mathrm{p}})$ is an isolated eigenvalue, i.e., a ground state φ_{p} (L^2-normalized eigenfunction corresponding to the bottom of the spectrum) of H_{p} exists, by standard methods it can be shown that it is unique and has a strictly positive version, which we will choose throughout. Using the ground state, we can define the unitary operator

$$U : L^2(\mathbb{R}^d, \varphi_{\mathrm{p}}^2 dx) \to L^2(\mathbb{R}^d, dx), \quad f \mapsto \varphi_{\mathrm{p}} f,$$

and consider the self-adjoint operator $U^{-1}(H_{\mathrm{p}} - E_{\mathrm{p}})U$. This operator generates a stationary Markov process, which we denote by $(Y_t)_{t\geq 0}$ and call a $P(\phi)_1$-process associated with H_{p}. We then have with suitable test functions f and g the formula

$$(f\varphi_{\mathrm{p}}, e^{-t(H_{\mathrm{p}}-E_{\mathrm{p}})} g\varphi_{\mathrm{p}})_{L^2(\mathbb{R}^d,dx)} = \int_{\mathbb{R}^d} \mathbb{E}^x[f(Y_0)g(Y_t)]\varphi_{\mathrm{p}}^2(x)dx. \tag{1.3}$$

In what follows, we are interested in constructing a $P(\phi)_1$-process associated with the Nelson Hamiltonian. This will be then obtained by a similar unitary operator, and will give rise to a two-component random process according to the separate contributions of the particle and the fields operators. Our main aim is then to study a FCLT behaviour of this process in the sense of the invariance principle due to [5].

This paper is organized as follows. In Section 2 we discuss the functional integral representations of the classical and relativistic Nelson models. In Section 3, we construct $P(\phi)_1$-process associated with the two models. Section 4 is devoted to proving a functional central limit theorem for additive functionals associated with the Nelson models by using the properties of the $P(\phi)_1$-process. We show some functionals of special interest for both cases and determine explicitly the variance in the related FCLT. Finally, we make some remarks on extensions to related models in Section 5.

2 Functional integral representations of the Nelson model

2.1 Functional integral representation of the free particle Hamiltonians

We will consider the classical and relativistic Nelson models in parallel. In some aspects of the construction the relevant property is the Markov property of the underlying processes, thus the expressions will appear similar with the difference that the appropriate processes are applied, however, in some other aspects the path properties will become crucial and significant differences appear.

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a Borel-measurable function giving the potential. We denote the multiplication operator defined by V by the same label. The energy of the free particle in the classical model is described by the Schrödinger operator acting on $L^2(\mathbb{R}^d)$ and formally written as

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V. \tag{2.1}$$

The relativistic model is described by the relativistic Schrödinger operator acting on $L^2(\mathbb{R}^d)$ and formally written as

$$\tilde{H}_{\mathrm{p}} = \sqrt{-\Delta + m^2} - m + V, \tag{2.2}$$

where the square-root operator is defined by Fourier transform in the standard way, and the parameter $m \geq 0$ is the rest mass of the particle. These Schrödinger operators can be defined in the sense of perturbation theory by choosing suitable conditions on V. However, since we will use methods of functional integration, we are interested to choose V in a way which allows a Feynman-Kac type representation to hold. The natural choice is Kato-class, in each case given in terms of the related random processes.

In order to describe the classical case, consider the space $\mathscr{X} = C(\mathbb{R}^+, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}^+$. Let $(B_t)_{t\geq 0}$ be d-dimensional Brownian motion defined on $(\mathscr{X}, \mathcal{B}(\mathscr{X}))$, where $\mathcal{B}(\mathscr{X})$ is the σ-field generated by the cylinder sets of $\mathscr{X}$, and denote by $\mathcal{W}^x$ the Wiener measure starting from x at $t = 0$. Also, consider $\tilde{\mathscr{X}} = D(\mathbb{R}^+, \mathbb{R}^d)$, the space of càdlàg paths (i.e., continuous from right with left limits) with values in $\mathbb{R}^d$, and $\mathcal{B}(\tilde{\mathscr{X}})$ the σ-field generated by the cylinder sets of $\tilde{\mathscr{X}}$. Let $(b_t)_{t\geq 0}$ denote a d-dimensional rotationally symmetric relativistic Cauchy process generated by (2.2) when $m > 0$, and a rotationally symmetric Cauchy process when $m = 0$. In each case we denote by $\mathscr{P}^x$ the path measure of the process in question starting from x at time $t = 0$. It is well-known that all these processes have the strong Markov property with respect to their natural filtrations. When we do not need to specify the process, we will use the generic notation $(Z_t)_{t\geq 0}$, and it will be understood that expectations are taken with respect to the own path measure of the process.

Definition 2.1 (Kato-class) We say that $V = V_+ - V_-$ is a Kato-class potential with respect to the random process $(Z_t)_{t\geq 0}$ whenever for its positive and negative parts

$$V_- \in \mathcal{K}^Z \quad \text{and} \quad V_+ 1_C \in \mathcal{K}^Z \text{ for every compact set } C \subset \mathbb{R}^d,$$

hold, where $f \in \mathcal{K}^Z$ means that

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x \left[\int_0^t |f(Z_s)| ds \right] = 0. \tag{2.3}$$

When $(Z_t)_{t\geq 0} = (B_t)_{t\geq 0}$, we call this space Kato-class, and when $(Z_t)_{t\geq 0} = (b_t)_{t\geq 0}$, we call it relativistic Kato-class.

By Khasminskii's lemma [15, Lemma 3.37] and its straightforward extension to relativistic Kato-class it follows that the random variables $-\int_0^t V(Z_s)ds$ are exponentially integrable for all $t \geq 0$, and thus we can define the Feynman-Kac semigroup

$$T_t f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(Z_s)ds} f(Z_t) \right], \quad f \in L^2(\mathbb{R}^d),\ t > 0. \tag{2.4}$$

Using the Markov property and stochastic continuity of the process $(Z_t)_{t\geq 0}$, we can show that $\{T_t : t \geq 0\}$ is a strongly continuous one-parameter semigroup of symmetric operators on $L^2(\mathbb{R}^d)$. Then, by the Hille-Yoshida theorem, there exists a self-adjoint operator K bounded from below such that $e^{-tK} = T_t$. Using the generator K, we can give a definition to a classical and a relativistic Schrödinger operator for Kato-class potentials.

Definition 2.2 If $(Z_t)_{t\geq 0} = (B_t)_{t\geq 0}$, then we call the self-adjoint operator K on $L^2(\mathbb{R}^d)$ a Schrödinger operator with Kato-class potential V. If $(Z_t)_{t\geq 0} = (b_t)_{t\geq 0}$, we call K a relativistic Schrödinger operator with Kato-class potential V.

For simplicity, we keep using the notations (2.1) and (2.2). In both the non-relativistic and relativistic cases we have then the following Feynman-Kac formula.

Proposition 2.3 (Functional integral representation) *Let $f, g \in L^2(\mathbb{R}^d)$. If V is of Kato-class, then for any of the operators K defined above we have*

$$\left(f, e^{-tK} g\right)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathbb{E}^x[\bar{f}(Z_0) e^{-\int_0^t V(Z_s)ds} g(Z_t)] dx. \tag{2.5}$$

In particular,

$$(e^{-tK} g)(x) = \mathbb{E}^x[e^{-\int_0^t V(Z_s)ds} g(Z_t)], \quad x \in \mathbb{R}^d. \tag{2.6}$$

Proof: See [15, Sections 3.3, 3.6], [11, Section 4].

Below we will need two-sided processes $(b_t)_{t\in\mathbb{R}}$, i.e., indexed by the time-line $\mathbb{R}$ instead of usually by the semi-axis $\mathbb{R}^+$. These processes can be defined as follows. Consider the measurable space $(\Omega, \mathcal{B}(\Omega))$, with càdlàg space $\Omega = D_{\mathrm{r}}(\mathbb{R}; \mathbb{R}^d)$, as well as $\widehat{\Omega} = D_{\mathrm{r}}(\mathbb{R}^+, \mathbb{R}^d) \times D_{\mathrm{l}}(\mathbb{R}^+, \mathbb{R}^d)$ and $\widehat{\mathscr{P}^x} = \mathscr{P}^x \times \mathscr{P}^x$, where $D_{\mathrm{l}}(\mathbb{R}^+, \mathbb{R}^d)$ denotes càglàd space (i.e., paths continuous from left with right limits). Let $\omega = (\omega_1, \omega_2) \in \widehat{\Omega}$ and define

$$\widehat{b}_t(\omega) = \begin{cases} \omega_1(t), & t \geq 0, \\ \omega_2(-t), & t < 0. \end{cases}$$

Since $\widehat{b}_t(\omega)$ is càdlàg in $t \in \mathbb{R}$ under $\widehat{\mathscr{P}^x}$, we define $b : (\widehat{\Omega}, \mathcal{B}(\widehat{\Omega})) \to (\Omega, \mathcal{B}(\Omega))$ by $b_t(\omega) = \widehat{b}_t(\omega)$. Then we have that $b \in \mathcal{B}(\widehat{\Omega})/\mathcal{B}(\Omega)$ since $b^{-1}(E) \in \mathcal{B}(\widehat{\Omega})$, for all cylinder sets $E \in \mathcal{B}(\Omega)$. Thus b is an Ω-valued random variable on $\widehat{\Omega}$. Denote again the image measure of $\widehat{\mathscr{P}^x}$ on $(\Omega, \mathcal{B}(\Omega))$ with respect to b by $\mathscr{P}^x = \widehat{\mathscr{P}^x} \circ b^{-1}$. The coordinate process denoted by the same symbol $b_t : \omega \in \Omega \mapsto \omega(t) \in \mathbb{R}^d$ is a Cauchy (respectively, relativistic Cauchy) process over $\mathbb{R}$ on $(\Omega, \mathcal{B}(\Omega), \mathscr{P}^x)$. The properties of the so obtained process can be summarized as follows.

Proposition 2.4 *The following hold:*

1. $\mathscr{P}^x(b_0 = x) = 1$
2. *the increments* $(b_{t_i} - b_{t_{i-1}})_{1\leq i\leq n}$ *are independent symmetric Cauchy (respectively, relativistic Cauchy) random variables for any* $0 = t_0 < t_1 < \cdots < t_n$ *with* $b_t - b_s \stackrel{\mathrm{d}}{=} b_{t-s}$ *for* $t > s$
3. *the increments* $(b_{-t_{i-1}} - b_{-t_i})_{1\leq i\leq n}$ *are independent symmetric Cauchy (respectively, relativistic Cauchy) random variables for any* $0 = -t_0 > -t_1 > \cdots > -t_n$ *with* $b_{-t} - b_{-s} \stackrel{\mathrm{d}}{=} b_{s-t}$ *for* $-t > -s$
4. *the function* $\mathbb{R} \ni t \mapsto b_t(\omega) \in \mathbb{R}$ *is càdlàg for every* $\omega \in \Omega$
5. b_t *and* b_s *are independent for* $t > 0$ *and* $s < 0$.

A completely similar construction can be made of two-sided Brownian motion, with simplifications due to path continuity.

2.2 Nelson Hamiltonian in boson Fock space

The Nelson Hamiltonian is defined on a Hilbert space in terms of a self-adjoint operator bounded from below. Consider the boson Fock space $\mathscr{F}_{\mathrm{b}}$ over $L^2(\mathbb{R}^d)$ defined as

$$\mathscr{F}_{\mathrm{b}} = \bigoplus_{n=0}^{\infty} \mathscr{F}_{\mathrm{b}}{}^{(n)},$$

where $\mathscr{F}_{\mathrm{b}}{}^{(n)} = \otimes^n_{\mathrm{sym}} L^2(\mathbb{R}^d)$. The Fock space can be identified with the space of l_2-sequences $(\psi^{(n)})_{n\in\mathbb{N}}$ such that $\psi^{(n)} \in \mathscr{F}_{\mathrm{b}}{}^{(n)}$ and

$$\|\psi\|^2_{\mathscr{F}_{\mathrm{b}}} = \sum_{n=0}^{\infty} \|(\psi^{(n)})\|^2_{\mathscr{F}_{\mathrm{b}}{}^{(n)}} < \infty. \tag{2.7}$$

We denote the "smeared" annihilation and creation operators by $a(f)$ and $a^*(f)$, $f:\mathbb{C}\to\mathbb{R}^d$, $f\in L^2(\mathbb{R}^d)$, respectively, satisfying the canonical commutation relations

$$[a(f),a^*(g)] = (\bar{f},g)\mathbb{1}, \quad [a(f),a(g)] = 0 = [a^*(f),a^*(g)]$$

on a dense domain of $\mathscr{F}_{\rm b}$. Using these operators, the field operator and its conjugate momentum on $\mathscr{F}_{\rm b}$ are defined, respectively, by

$$\Phi(f) = \frac{1}{\sqrt{2}}(a^*(\bar{f})+a(f)) \quad \text{and} \quad \Pi(f) = \frac{i}{\sqrt{2}}(a^*(\bar{f})-a(f)).$$

For real-valued L^2-functions f,g, the commutation relations become

$$[\Phi(f),\Pi(g)] = i(f,g), \quad [\Pi(f),\Pi(g)] = [\Phi(f),\Phi(g)] = 0. \tag{2.8}$$

Denote by $d\Gamma(T):\mathscr{F}_{\rm b}\to\mathscr{F}_{\rm b}$ the second quantization of a self-adjoint operator $T: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$, defined by

$$d\Gamma(T) = 0\oplus\left[\oplus_{n=1}^{\infty}\sum_{j=1}^{n}\underbrace{\mathbb{1}\otimes\cdots\overset{j\text{th}}{T}\cdots\otimes\mathbb{1}}_{n-\text{fold}}\right].$$

The self-adjoint operator

$$H_{\rm f} = d\Gamma(\omega),$$

is the free field Hamiltonian, where

$$\omega(k) = \sqrt{|k|^2+\nu^2} \tag{2.9}$$

is the dispersion relation, and $\nu\geq 0$ denotes the mass of a single boson. Formally, the free field Hamiltonian can be written as

$$H_{\rm f} = \int_{\mathbb{R}^d}\omega(k)a^*(k)a(k)dk. \tag{2.10}$$

Physically, it describes the total energy of the interaction-free boson field since $a^*(k)a(k)$ gives the number of bosons carrying momentum k and $\omega(k)$ is the energy of a single boson. The commutation relations

$$[H_{\rm f},a(f)] = -a(\omega f), \quad [H_{\rm f},a^*(f)] = a^*(\omega f) \tag{2.11}$$

hold for $f\in D(\omega)$ on a dense domain of $\mathscr{F}_{\rm b}$. Hence we deduce that

$$[H_{\rm f},\Phi(f)] = -i\Pi(\omega f). \tag{2.12}$$

Let $\varphi:\mathbb{R}^d\to\mathbb{R}$ be a function describing the charge distribution of the particle, denote by $\widehat{\varphi}$ its Fourier transform, and write $\widetilde{\widehat{\varphi}}(k) = \widehat{\varphi}(-k)$. For every $x\in\mathbb{R}^d$, define

$$H_{\rm i}(x) = \frac{1}{\sqrt{2}}\left(a^*(\widehat{\varphi}e^{-ikx}/\sqrt{\omega}) + a(\widetilde{\widehat{\varphi}}e^{ikx}/\sqrt{\omega})\right). \tag{2.13}$$

Let $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F}_{\rm b}$. We define the interaction Hamiltonian $H_{\rm i} : \mathscr{H} \to \mathscr{H}$ by the constant fiber direct integral $(H_{\rm i}\Psi)(x) = H_{\rm i}(x)\Psi(x)$ for $\Psi \in \mathscr{H}$ such that $\Psi(x) \in D(H_{\rm i}(x))$, for almost every $x \in \mathbb{R}^d$. Here we use the identification

$$\mathscr{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathscr{F}_{\rm b} dx = \Big\{ F : \mathbb{R}^d \to \mathscr{F}_{\rm b} \,\Big|\, \|F\|^2_{\mathscr{H}} = \int_{\mathbb{R}^d} \|F(x)\|^2_{\mathscr{F}_{\rm b}} dx < \infty \Big\}.$$

Formally, this is written as

$$H_{\rm i}(x) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\omega(k)}} (\widehat{\varphi}(k) e^{-ikx} a^*(k) + \widehat{\varphi}(-k) e^{ikx} a(k)) dk. \tag{2.14}$$

The Nelson Hamiltonian describing the interacting particle-field system is then defined by

$$H_{\rm N} = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + H_{\rm i} \tag{2.15}$$

Similarly, the relativistic Nelson Hamiltonian is defined by the operator

$$\tilde{H}_{\rm N} = \tilde{H}_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + H_{\rm i} \tag{2.16}$$

on the space $\mathscr{H}$.

We will use the following standing assumptions throughout this paper.

Assumption 2.5 The following conditions hold:

1. $\overline{\widehat{\varphi}(k)} = \widehat{\varphi}(-k)$ and $\widehat{\varphi}/\sqrt{\omega}, \widehat{\varphi}/\omega \in L^2(\mathbb{R}^d)$.
2. $\widehat{\varphi}/\omega\sqrt{\omega} \in L^2(\mathbb{R}^d)$.
3. The external potential $V = V_+ - V_-$ is of Kato-class in the sense of Definition 2.1.
4. $H_{\rm p}$ has a unique, strictly positive ground state $\varphi_{\rm p} \in D(H_{\rm p})$, with $H_{\rm p}\varphi_{\rm p} = E_{\rm p}\varphi_{\rm p}$, $\|\varphi_{\rm p}\|_{L^2(\mathbb{R}^d)} = 1$, where $E_{\rm p} = \inf \sigma(H_{\rm p})$. Similarly, $\tilde{H}_{\rm p}$ has a unique, strictly positive ground state $\tilde{\varphi}_{\rm p} \in D(\tilde{H}_{\rm p})$, with $\tilde{H}_{\rm p}\tilde{\varphi}_{\rm p} = \tilde{E}_{\rm p}\tilde{\varphi}_{\rm p}$, $\|\tilde{\varphi}_{\rm p}\|_{L^2(\mathbb{R}^d)} = 1$, where $\tilde{E}_{\rm p} = \inf \sigma(\tilde{H}_{\rm p})$.
5. $H_{\rm N}$ has a unique, strictly positive ground state $\varphi_{\rm g} \in D(H_{\rm N})$, with $H_{\rm N}\varphi_{\rm g} = E\varphi_{\rm g}$, $\|\varphi_{\rm g}\|_{\mathscr{H}} = 1$, where $E = \inf \sigma(H_{\rm N})$. Similarly, $\tilde{H}_{\rm N}$ has a unique, strictly positive ground state $\tilde{\varphi}_{\rm g} \in D(\tilde{H}_{\rm N})$ with $\tilde{H}_{\rm N}\tilde{\varphi}_{\rm g} = \tilde{E}\tilde{\varphi}_{\rm g}$, $\|\tilde{\varphi}_{\rm g}\|_{\mathscr{H}} = 1$, where $\tilde{E} = \inf \sigma(\tilde{H}_{\rm N})$.

Denote the "free" operators by

$$H_0 = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} \quad \text{and} \quad \tilde{H}_0 = \tilde{H}_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f}.$$

The spectrum of H_0 can be derived from the spectra of $H_{\rm p}$ and $H_{\rm f}$. We have $\sigma(H_{\rm p}) = [0, \infty)$, $\sigma(H_{\rm f}) = \{0\} \cup [\nu, \infty)$, and in the case $\nu = 0$ the bottom of the spectrum of H_0 is the edge of the continuous spectrum. In general, it is not clear whether the bottom of the spectrum of $H_{\rm N}$ is in the point spectrum or not, however, whenever it is, the eigenfunction associated with this eigenvalue is a ground state. The same considerations hold also for the relativistic operators.

Using Assumption 2.5 it follows that $H_{\rm i}$ is symmetric, and thus $H_{\rm N}$, $\tilde{H}_{\rm N}$ are self-adjoint operators.

Proposition 2.6 *$H_{\rm N}$ is a self-adjoint operator on $D(H_{\rm p}\otimes 1\!\!1)\cap D(1\!\!1\otimes H_{\rm f})$ and essentially self-adjoint on any core of H_0. Similarly, $\tilde{H}_{\rm N}$ is self-adjoint on $D(\tilde{H}_{\rm p}\otimes 1\!\!1)\cap D(1\!\!1\otimes H_{\rm f})$ and essentially self-adjoint on any core of $\tilde{H}_0$.*

Proof: For any $F\in D(H_0)$ we have

$$\|H_{\rm i}F\|\leq 2\|\widehat{\varphi}/2\omega\|\|H_{\rm f}^{1/2}F\|+\|\widehat{\varphi}/\sqrt{2\omega}\|\|F\|.$$

Let $\varepsilon>0$ be arbitrary. We obtain that

$$\|H_{\rm f}^{1/2}F\|^2\leq (F,(H_{\rm p}+H_{\rm f})F)-E_{\rm p}\|F\|^2\leq \varepsilon\|H_0F\|^2+\left(\frac{1}{4\varepsilon}+|E_{\rm p}|\right)\|F\|^2.$$

Thus there exists $b_\varepsilon>0$ such that

$$\|H_{\rm i}F\|\leq \varepsilon\|H_0F\|+b_\varepsilon\|F\|,$$

and the claim follows by the Kato-Rellich theorem, see [15, Theorem 3.11]. The second part of the statement follows similarly. □

2.3 Nelson Hamiltonian in function space

2.3.1 $P(\phi)_1$-process for the free particle operators

Definition 2.7 ($P(\phi)_1$-process) Let $(E,\mathscr{F},P)$ be a probability space and K be a self-adjoint operator in $L^2(E,dP)$, bounded from below. We say that an E-valued stochastic process $(Z_t)_{t\in\mathbb{R}}$ on a probability space $(\mathscr{Y},\mathcal{B},\mathcal{Q}^z)$ is a $P(\phi)_1$-process associated with $((E,\mathscr{F},P),K)$ if conditions 1-4 below are satisfied:

1. $\mathcal{Q}^z(Z_0=z)=1$.
2. **(Reflection symmetry)** $(Z_t)_{t\geq 0}$ and $(Z_t)_{t\leq 0}$ are independent and $Z_t\stackrel{\rm d}{=}Z_{-t}$ for every $t\in\mathbb{R}$.
3. **(Markov property)** $(Z_t)_{t\geq 0}$ and $(Z_t)_{t\leq 0}$ are Markov processes with respect to the fields $\sigma(Z_s,0\leq s\leq t)$ and $\sigma(Z_s,t\leq s\leq 0)$, respectively.
4. **(Shift invariance)** Let $-\infty<t_0\leq t_1<...\leq t_n<\infty$, $f_j\in L^\infty(E,dP)$, $j=1,...,n-1$ and $f_0,f_n\in L^2(E,dP)$. Then for every $s\in\mathbb{R}$,

$$\begin{aligned}\int_E \mathbb{E}_{\mathcal{Q}^z}\left[\prod_{j=0}^n f_j(Z_{t_j})\right]dP(z) &= \int_E \mathbb{E}_{\mathcal{Q}^z}\left[\prod_{j=0}^n f_j(Z_{t_j+s})\right]dP(z)\\ &=(1\!\!1, f_0e^{-(t_1-t_0)K}f_1\cdots f_{n-1}e^{-(t_n-t_{n-1})K}f_n).\end{aligned}\tag{2.17}$$

Denote

$$d{\rm N}(y)=\varphi_{\rm p}^2(y)dy.\tag{2.18}$$

Since by part (4) of Assumption (2.5) the function $\varphi_{\rm p}$ is square integrable and L^2-normalized, $d{\rm N}$ is a probability measure on $\mathbb{R}^d$. Define the unitary operator $U_{\rm p} : L^2(\mathbb{R}^d, d{\rm N}) \to L^2(\mathbb{R}^d, dy)$ by $U_{\rm p} : f \mapsto \varphi_{\rm p} f$. Using that $\varphi_{\rm p}$ is strictly positive, the image

$$L_{\rm p} = U_{\rm p}^{-1}(H_{\rm p} - E_{\rm p})U_{\rm p} = \frac{1}{\varphi_{\rm p}}(H_{\rm p} - E_{\rm p})\varphi_{\rm p}, \tag{2.19}$$

of the Schrödinger operator (2.1) under this map is well-defined and has the domain $D(L_{\rm p}) = \{f \in L^2(\mathbb{R}^d, d{\rm N}) \,|\, f\varphi_{\rm g} \in D(H_{\rm p})\}$. Since $e^{-tL_{\rm p}}\mathbb{1} = \mathbb{1}$ for the identity function $\mathbb{1} \in L^2(\mathbb{R}^d, d{\rm N})$, the operator $L_{\rm p}$ is the generator of a Markov process.

Proposition 2.8 *If V is in Kato class, then*

1. *there exists a probability measure $\mathcal{N}^y$ on $(\mathscr{X}, \mathcal{B}(\mathscr{X}))$ such that the coordinate process $(Y_t)_{t\in\mathbb{R}}$ on $(\mathscr{X}, \mathcal{B}(\mathscr{X}), \mathcal{N}^y)$ is a $P(\phi)_1$-process starting from $y \in \mathbb{R}^d$, associated with the pair $((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), d{\rm N}), L_{\rm p})$*

2. *the function $t \mapsto Y_t$ is almost surely continuous.*

Proof: See [15, Theorem 3.106]. □

We can define a $P(\phi)_1$-process for the relativistic Schrödinger operator (2.2) in a similar way. As above, denote

$$d\tilde{\rm N} = \tilde{\varphi}_{\rm p}^2(x)dx, \tag{2.20}$$

which is a probability measure on $\mathbb{R}^d$ for similar reasons as for the non-relativistic operator. Taking now the unitary map $\tilde{U}_{\rm p} : L^2(\mathbb{R}^d, d\tilde{\rm N}) \to L^2(\mathbb{R}^d, dx)$, $f \mapsto \tilde{\varphi}_{\rm p} f$, we similarly obtain

$$\tilde{L}_{\rm p} = \tilde{U}_{\rm p}^{-1}(\tilde{H}_{\rm p} - \tilde{E}_{\rm p})\tilde{U}_{\rm p} = \frac{1}{\tilde{\varphi}_{\rm p}}(\tilde{H}_{\rm p} - \tilde{E}_{\rm p})\tilde{\varphi}_{\rm p}, \tag{2.21}$$

which is again a Markov generator. Then we have

Theorem 2.9 *If V is in relativistic Kato class, then*

1. *there exists a probability measure $\tilde{\mathcal{N}}^y$ on $(\tilde{\mathscr{X}}, \mathcal{B}(\tilde{\mathscr{X}}))$ such that the coordinate process $(\tilde{Y}_t)_{t\in\mathbb{R}}$ on $(\tilde{\mathscr{X}}, \mathcal{B}(\tilde{\mathscr{X}}), \tilde{\mathcal{N}}^y)$ is a $P(\phi)_1$-process starting from $y \in \mathbb{R}^d$, associated with $((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), d\tilde{\rm N}), \tilde{L}_{\rm p})$*

2. *the function $t \mapsto \tilde{Y}_t$ is almost surely càdlàg.*

Proof: See [12, Theorem 5.1]. □

2.3.2 Infinite dimensional Ornstein-Uhlenbeck process

Let $\mathscr{K}$ be a Hilbert space over $\mathbb{R}$, defined by the completion of $D(1/\sqrt{\omega}) \subset L^2(\mathbb{R}^d)$ with respect to the norm determined by the scalar product

$$(f,g)_{\mathscr{K}} = \int_{\mathbb{R}^d} \overline{\widehat{f}(k)}\widehat{g}(k)\frac{1}{2\omega(k)}dk, \tag{2.22}$$

i.e.,

$$\mathscr{K} = \overline{D(1/\sqrt{\omega})}^{\|\cdot\|_{\mathscr{K}}}.$$

Let $T : \mathscr{K} \to \mathscr{K}$ be a positive self-adjoint operator with Hilbert-Schmidt inverse such that $\sqrt{\omega}T^{-1}$ is bounded. Define the space $C^\infty(T) = \cap_{n=1}^\infty D(T^n)$, and write

$$\mathscr{K}_n = \overline{C^\infty(T)}^{\|T^{n/2}\cdot\|_{\mathscr{K}}}.$$

We construct a triplet $\mathscr{K}_{+2} \subset \mathscr{K} \subset \mathscr{K}_{-2}$, where we identify $\mathscr{K}_{+2}^* = \mathscr{K}_{-2}$. Write $Q = \mathscr{K}_{-2}$, and endow Q with its Borel σ-field $\mathcal{B}(Q)$, defining the measurable space $(Q, \mathcal{B}(Q))$.

Consider the set $\mathscr{Y} = C(\mathbb{R}, Q)$ of continuous functions on $\mathbb{R}$, with values in Q, and denote its Borel σ-field by $\mathcal{B}(\mathscr{Y})$. We define a Q-valued Ornstein-Uhlenbeck process $(\xi_t)_{t\in\mathbb{R}}$,

$$\mathbb{R} \ni t \mapsto \xi_t \in Q$$

on the probability space $(\mathscr{Y}, \mathcal{B}(\mathscr{Y}), \mathcal{G})$ with probability measure $\mathcal{G}$. Let $\xi_t(f) = (\!(\xi_t, f)\!)$ for $f \in \mathcal{M}_{+2}$, where $(\!(.,.)\!)$ denotes the pairing between Q and $\mathcal{M}_{+2}$. Then for every $t \in \mathbb{R}$ and f we have that $\xi_t(f)$ is a Gaussian random variable with mean zero and covariance

$$\mathbb{E}_{\mathcal{G}}[\xi_t(f)\xi_s(g)] = \int_{\mathbb{R}^d} \overline{\widehat{f}(k)}\widehat{g}(k)e^{-|t-s|\omega(k)}\frac{1}{2\omega(k)}dk. \tag{2.23}$$

Note that by (2.23) every $\xi_t(f)$ can be uniquely extended to test functions $f \in \mathcal{M}$, which for simplicity we will denote in the same way.

In what follows we will need conditional measures of this Gaussian measure. Since the conditional expectation $\mathbb{E}_{\mathcal{G}}[1_A|\sigma(\xi_0)]$ with respect to $\sigma(\xi_0)$ is trivially $\sigma(\xi_0)$-measurable, there exists a measurable function $h : Q \to \mathbb{R}$ such that $h \circ \xi_0(\omega) = \mathbb{E}_{\mathcal{G}}[1_A|\sigma(\xi_0)](\omega)$. We will use the notation $h(\xi) = \mathcal{G}(A|\xi_0 = \xi)$, however, we remark that $\mathcal{G}(A|\xi_0 = \xi)$ is well defined for $\xi \in Q\setminus N_A$ with a null set N_A only. Nevertheless, since Q is a separable complete metric space, there exists a null set N such that $\mathcal{G}(A|\xi_0 = \xi)$ is well defined for all A and $\xi \in Q \setminus N$. The notation $\mathcal{G}^\xi(\cdot) = \mathcal{G}(\cdot\,|\xi_0 = \xi)$ for the family of conditional probability measures $\mathcal{G}(\cdot|\xi_0 = \xi)$ on $\mathscr{Y}$ with $\xi \in Q\setminus N$ makes then sense, and it is seen that $\mathcal{G}^\xi$ is a regular conditional probability measure. Then we have $\mathbb{E}_{\mathcal{G}}[...] = \int_Q \mathbb{E}_{\mathcal{G}^\xi}[...]d\mathrm{G}(\xi)$, where G is the distribution of the random process $(\xi_t)_{t\in\mathbb{R}}$ on the measurable space $(Q, \mathcal{B}(Q))$, and it is the stationary measure of $\mathcal{G}$. Thus we are led to the probability space $(Q, \mathcal{B}(Q), \mathrm{G})$.

2.3.3 Functional integral representation of the Nelson Hamiltonians

The Wiener-Itô-Segal isomorphism $U_{\rm f}: \mathscr{F}_{\rm b} \longrightarrow L^2(Q, d{\rm G})$, $U_{\rm f}\Phi(f){U_{\rm f}}^{-1} = \xi(f)$, establishes a close connection between $\mathscr{F}_{\rm b}$ and $L^2(Q, d{\rm G})$. Using (2.18) and the stationary measure obtained above, define the product measure

$$\mathrm{P} = \mathrm{N} \otimes \mathrm{G}, \tag{2.24}$$

which is a probability measure on the product space $\mathbb{R}^d \times Q$. The unitary map

$$U_{\rm p} \otimes U_{\rm f} : \mathscr{H} \longrightarrow L^2(\mathbb{R}^d \times Q, d\mathrm{P})$$

establishes a unitary equivalence between $L^2(\mathbb{R}^d \times Q, d\mathrm{P})$ and $\mathscr{H}$, and we make the identification

$$\mathscr{H} \cong L^2(\mathbb{R}^d) \otimes L^2(Q) \cong L^2(\mathbb{R}^d \times Q, d\mathrm{P}). \tag{2.25}$$

For convenience, hereafter we write $L^2(\mathbb{R}^d \times Q, d\mathrm{P})$ simply as $L^2(\mathrm{P})$, moreover $L^2(\mathrm{N})$ and $L^2(\mathrm{G})$ for $L^2(\mathbb{R}^d, d\mathrm{N})$ and $L^2(Q, d\mathrm{G})$, respectively. The images of the free field and interaction Hamiltonians on $L^2(\mathrm{P})$ under this unitary map are given by

$$H_{\rm f}^{U_{\rm f}} = U_{\rm f} H_{\rm f} {U_{\rm f}}^{-1} \tag{2.26}$$

and

$$H_{\rm i}^{U_{\rm f}}(y) = U_{\rm f} H_{\rm i} {U_{\rm f}}^{-1}(y) = \xi(\tilde{\varphi}(\cdot - y)), \quad y \in \mathbb{R}^d. \tag{2.27}$$

Here $\tilde{\varphi}$ is the inverse Fourier transform of $\widehat{\varphi}/\sqrt{\omega}$. To simplify the notations, we write again $H_{\rm f}$ for $H_{\rm f}^{U_{\rm f}}$, and $H_{\rm i}$ for $H_{\rm i}^{U_{\rm f}}$. Then the classical Nelson Hamiltonian $H_{\rm N}$ is unitary equivalent with

$$H = L_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + H_{\rm i} \tag{2.28}$$

acting on $L^2(\mathrm{P})$, where $L_{\rm p}$ is given by (2.19).

Recall that $(Y_t)_{t\in\mathbb{R}}$ is the $P(\phi)_1$-process associated with the pair $((\mathscr{X}, \mathcal{B}(\mathscr{X}), \mathcal{N}^y), L_{\rm p})$, and write

$$d\mathcal{N} = d\mathrm{N}(y) d\mathcal{N}^y.$$

The probability space for the joint system without the particle-field interaction is then the product space $(\mathscr{X} \times \mathscr{Y}, \Sigma, \mathcal{P}_0)$, where $\Sigma = \mathcal{B}(\mathscr{X}) \otimes \mathcal{B}(\mathscr{Y})$ and

$$\mathcal{P}_0 = \mathcal{N} \otimes \mathcal{G}.$$

Define the shift operator $\tau_s : L^2(Q, d\mathrm{G}) \mapsto L^2(Q, d\mathrm{G})$ by $\tau_s \xi(h) = \xi(h(\cdot - s))$. We have then the following functional integral representation for the classical Nelson Hamiltonian H in $L^2(\mathrm{P})$.

Proposition 2.10 (Functional integral representation) *Let $\Phi, \Psi \in L^2(\mathrm{P})$ and suppose that $s \le 0 \le t$. Then*

$$(\Phi, e^{-(t-s)H}\Psi)_{L^2(\mathrm{P})} = \mathbb{E}_{\mathcal{P}_0}[\Phi(Y_s, \xi_s) e^{-\int_s^t \tau_{Y_r} \xi_r(\tilde{\varphi}) dr} \Psi(Y_t, \xi_t)].$$

See [15, Theorem 6.2]. □

Corollary 2.11 *Let* $\Phi, \Psi \in L^2(\mathrm{P})$*,* $t_0 \le ... \le t_n$ *and* $A_0, ..., A_n \in \mathcal{B}(\mathbb{R}^d \times Q)$*. Then,*

$$(\Phi, \mathbb{1}_{A_0} e^{-(t_1-t_0)H} \mathbb{1}_{A_1} e^{-(t_2-t_1)H} \cdots \mathbb{1}_{A_{n-1}} e^{-(t_n-t_{n-1})H} \mathbb{1}_{A_n} \Psi)_{L^2(\mathrm{P})}$$
$$= \mathbb{E}_{\mathcal{P}_0}\left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(Y_{t_j}, \xi_{t_j})\right) \overline{\Phi(Y_{t_0}, \xi_{t_0})} e^{-\int_{t_0}^{t_n} \tau_{Y_s}\xi_s(\tilde{\varphi})ds} \Psi(Y_{t_n}, \xi_{t_n})\right].$$

Proof: We have

$$\begin{aligned}
&(\Phi, \mathbb{1}_{A_0} e^{-sH} \mathbb{1}_{A_1} e^{-tH} \mathbb{1}_{A_2} \Psi) \\
&= \mathbb{E}_{\mathcal{P}_0}[\overline{\Phi(Y_0, \xi_0)} \mathbb{1}_{A_0}(Y_0, \xi_0) e^{-\int_0^s \tau_{Y_r}\xi_r(\tilde{\varphi})dr} \mathbb{1}_{A_1}(Y_s, \xi_s) e^{-\int_s^{s+t} \tau_{Y_r}\xi_r(\tilde{\varphi})dr} \mathbb{1}_{A_2}(Y_{s+t}, \xi_{s+t}) \Psi(Y_{s+t}, \xi_{s+t})] \\
&= \mathbb{E}_{\mathcal{P}_0}[\overline{\Phi(Y_0, \xi_0)} \mathbb{1}_{A_0}(Y_0, \xi_0) \mathbb{1}_{A_1}(Y_s, \xi_s) \mathbb{1}_{A_2}(Y_{s+t}, \xi_{s+t}) e^{-\int_0^{s+t} \tau_{Y_r}\xi_r(\tilde{\varphi})dr} \Psi(Y_{s+t}, \xi_{s+t})].
\end{aligned}$$

By iterating this, we obtain

$$(\Phi, \mathbb{1}_{A_0} e^{-(t_1-t_0)H} \mathbb{1}_{A_1} e^{-(t_2-t_1)H} \cdots \mathbb{1}_{A_{n-1}} e^{-(t_n-t_{n-1})H} \mathbb{1}_{A_n} \Psi)$$
$$= \mathbb{E}_{\mathcal{P}_0}\left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(Y_{t_j-t_0}, \xi_{t_j-t_0})\right) \overline{\Phi(Y_0, \xi_0)} e^{-\int_0^{t_n-t_0} \tau_{Y_s}\xi_s(\tilde{\varphi})ds} \Psi(Y_{t_n-t_0}, \xi_{t_n-t_0})\right].$$

Since both $\mathcal{N}$ and $\mathcal{G}$ are invariant under time shift, we can replace Y_s by Y_{s+t_0} and ξ_s by ξ_{s+t_0}, to find that

$$(\Phi, \mathbb{1}_{A_0} e^{-(t_1-t_0)H} \mathbb{1}_{A_1} e^{-(t_2-t_1)H} \cdots \mathbb{1}_{A_n} e^{-(t_n-t_{n-1})H} \Psi)$$
$$= \mathbb{E}_{\mathcal{P}_0}\left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(Y_{t_j}, \xi_{t_j})\right) \overline{\Phi(Y_{t_0}, \xi_{t_0})} e^{-\int_{t_0}^{t_n} \tau_{Y_s}\xi_s(\tilde{\varphi})ds} \Psi(Y_{t_n}, \xi_{t_n})\right].$$

□

Finally, for later use we quote the following representation formula using Wiener measure instead of the particle $P(\phi)_1$-measure.

Proposition 2.12 *Let* $\Phi, \Psi \in L^2(\mathrm{P})$ *and* $s \le 0 \le t$*. Then*

$$(\Phi, e^{-(t-s)H}\Psi)_{L^2(\mathrm{P})}$$
$$= \int_{\mathbb{R}^d \times Q} \mathbb{E}^{(y,\xi)}_{\mathcal{W}\otimes\mathcal{G}}\left[\Phi(B_s, \xi_s)\varphi_{\mathrm{p}}(B_s) e^{-\int_s^t \tau_{B_r}\xi_r(\tilde{\varphi})dr} \Psi(B_t, \xi_t)\varphi_{\mathrm{p}}(B_t) e^{-\int_s^t V(B_r)dr}\right] dy \otimes d\mathrm{G}.$$

Proof: See [15, Theorem 6.3]. □

For convenience, we write

$$\mathcal{P}_0^{(y,\xi)} = \mathcal{N}^y \otimes \mathcal{G}^\xi, \quad (y, \xi) \in \mathbb{R}^d \times Q,$$

so that $\mathbb{E}_{\mathcal{P}_0}[\cdots] = \int_{\mathbb{R}^d \times Q} \mathbb{E}_{\mathcal{P}_0}^{(y,\xi)}[\cdots] d\mathrm{P}(y,\xi)$. The above transformations and constructions can be repeated for the relativistic model. Then we have

$$\tilde{\mathrm{P}} = \tilde{\mathrm{N}} \otimes \mathrm{G} \tag{2.29}$$

and the relativistic Nelson Hamiltonian $\tilde{H}$ in $L^2(\tilde{\mathrm{P}})$ becomes

$$\tilde{H} = \tilde{L}_\mathrm{p} \otimes \mathbb{1} + \mathbb{1} \otimes H_\mathrm{f} + H_\mathrm{i}, \tag{2.30}$$

using (2.21). We write

$$d\tilde{\mathcal{N}} = d\tilde{\mathrm{N}}(y) d\tilde{\mathcal{N}}^y.$$

We consider the probability space $(\tilde{\mathscr{X}} \otimes \mathscr{Y}, \tilde{\Sigma}, \tilde{\mathcal{P}}_0)$ where $\tilde{\Sigma} = \mathcal{B}(\tilde{\mathscr{X}}) \otimes \mathcal{B}(\mathscr{Y})$ and $\tilde{\mathcal{P}}_0 = \tilde{\mathcal{N}} \otimes \mathcal{G}$. Then we have the following expression for the relativistic Nelson Hamiltonian in $L^2(\tilde{\mathrm{P}})$.

Proposition 2.13 *Let $t_0 \leq \ldots \leq t_n$, $f_0, f_n \in L^2(\tilde{\mathrm{P}})$ and $f_j \in L^\infty(\tilde{\mathrm{P}})$, for $j = 1, \ldots, n-1$. Then*

$$(f_0, e^{-(t_1-t_0)\tilde{H}} f_1 e^{-(t_n-t_{n-1})\tilde{H}} f_n) = \mathbb{E}_{\tilde{\mathcal{P}}_0}\left[\overline{f_0(\tilde{Y}_0, \xi_0)} \left(\prod_{j=1}^n f_j(\tilde{Y}_{t_j}, \xi_{t_j})\right) e^{-\int_{t_0}^{t_n} \tau_{\tilde{Y}_s} \xi_s(\tilde{\varphi}) ds}\right].$$

Proof: The proof is analogous to the proof of Corollary 2.11 (see [15, Theorem 6.2]). □

Now we can give a functional integral representation of $e^{-t\tilde{H}_\mathrm{N}}$ by making use of the Lévy process $(b_t)_{t\in\mathbb{R}}$ and the infinite dimensional OU-process $(\xi_t)_{t\in\mathbb{R}}$.

Proposition 2.14 *Let $\Phi, \Psi \in L^2(\tilde{\mathrm{P}})$ and $s \leq 0 \leq t$. Then*

$$\begin{aligned}
&(\Phi, e^{-(t-s)\tilde{H}} \Psi)_{L^2(\tilde{\mathrm{P}})} \\
&= \int_{\mathbb{R}^d \times Q} \mathbb{E}_{\mathscr{P} \otimes \mathcal{G}}^{(y,\xi)} \left[\Phi(b_s, \xi_s) \tilde{\varphi}_\mathrm{p}(b_s) e^{-\int_s^t \tau_{b_r} \xi_r(\tilde{\varphi}) dr} \Psi(b_t, \xi_t) \tilde{\varphi}_\mathrm{p}(b_t) e^{-\int_s^t V(b_r) dr}\right] dy \otimes d\mathrm{G}.
\end{aligned}$$

Proof: The proof is similar to the proof of Proposition 2.12 (see [15, Theorem 6.3]). □

In what follows we write $\tilde{\mathcal{P}}_0^{(y,\xi)} = \tilde{\mathcal{N}}^y \otimes \mathcal{G}^\xi$ for $(y, \xi) \in \mathbb{R}^d \times Q$.

3 $P(\phi)_1$-processes associated with the Nelson Hamiltonians

3.1 Classical Nelson model

Since the ground state φ_g of H is strictly positive and L^2-normalized, we define the probability measure

$$d\mathrm{M} = \varphi_\mathrm{g}^2 d\mathrm{P}$$

on $\mathbb{R}^d \times Q$. Also, we define the unitary operator $U_\mathrm{g} : L^2(\mathbb{R}^d \times Q, d\mathrm{M}) \to L^2(\mathbb{R}^d \times Q, d\mathrm{P})$ by $U_\mathrm{g} : \Phi \mapsto \varphi_\mathrm{g} \Phi$. We write $\mathscr{K} = L^2(\mathbb{R}^d \times Q, d\mathrm{M})$ and define the self-adjoint operator

$$L_\mathrm{N} = \frac{1}{\varphi_\mathrm{g}}(H - E)\varphi_\mathrm{g}.$$

Let $\mathscr{X}_Q = C(\mathbb{R}; \mathbb{R}^d \times Q)$ be the set of continuous paths with values in $\mathbb{R}^d \times Q$ and indexed by the real line $\mathbb{R}$, and $\mathcal{B}_Q$ the σ-field generated by the cylinder sets. The main result of this section is the following.

Theorem 3.1 **($P(\phi)_1$-process for the classical Nelson Hamiltonian)** *Let $(y,\xi)\in\mathbb{R}^d\times Q$. Then the following hold.*

1. *There exists a probability measure $\mathcal{P}^{(y,\xi)}$ on $(\mathscr{X}_Q,\mathcal{B}_Q)$ such that the coordinate process $(X_t)_{t\in\mathbb{R}}$ on $(\mathscr{X}_Q,\mathcal{B}_Q,\mathcal{P}^{(y,\xi)})$ is a $P(\phi)_1$-process associated with the pair $\Big((\mathbb{R}^d\times Q,\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(Q),d\mathrm{M}),L_{\mathrm{N}}\Big)$.*
2. *The function $t\mapsto X_t$ is almost surely continuous.*

In order to show this theorem we need a string of lemmas. The idea of proof is taken from [15]. Write $\Sigma'=\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(Q)$ and define the family of set functions $\{\mathcal{M}_\Lambda\,|\,\Lambda\subset[0,\infty),\#\Lambda<\infty\}$ on $\Sigma'^{\#\Lambda}=\underbrace{\Sigma'\times\cdots\times\Sigma'}_{\#\Lambda-\text{times}}$ by

$$\mathcal{M}_\Lambda(A_0\times A_1\times...\times A_n)=\Big(\mathbb{1}_{A_0},e^{-(t_1-t_0)L_{\mathrm{N}}}\mathbb{1}_{A_1}e^{-(t_2-t_1)L_{\mathrm{N}}}\mathbb{1}_{A_1}\cdots\mathbb{1}_{A_{n-1}}e^{-(t_n-t_{n-1})L_{\mathrm{N}}}\mathbb{1}_{A_n}\Big)_{\mathscr{K}}$$

for $\Lambda=\{t_0,\dots,t_n\}$, $n\in\mathbb{N}$. It is straightforward to show that the family of set functions $\mathcal{M}_\Lambda$ satisfies the Kolmogorov consistency relation

$$\mathcal{M}_{\{t_0,t_1,\dots,t_{n+m}\}}((\times_{i=0}^n A_i)\times(\times_{i=n+1}^{n+m}\mathbb{R}^d\times Q))=\mathcal{M}_{\{t_0,t_1,\dots,t_n\}}(\times_{i=0}^n A_i).$$

Define the projection $\pi_\Lambda:(\mathbb{R}^d\times Q)^{[0,\infty)}\longrightarrow(\mathbb{R}^d\times Q)^\Lambda$ by $w\longmapsto(w(t_0),...,w(t_n))$ for $\Lambda=\{t_0,...,t_n\}$, $n\in\mathbb{N}$. Then

$$\mathscr{A}=\{\pi_\Lambda^{-1}(A)\,|\,A\in\Sigma'^{\#\Lambda},\,\#\Lambda<\infty\}$$

is a finitely additive family of sets, and the Kolmogorov extension theorem [13, Theorem 2.2] yields that there exists a unique probability measure $\mathcal{M}$ on $((\mathbb{R}^d\times Q)^{[0,\infty)},\sigma(\mathscr{A}))$ such that

$$\mathcal{M}(\pi_\Lambda^{-1}(A_1\times...\times A_n)=\mathcal{M}_\Lambda(A_1\times...\times A_n),$$

for all $\Lambda\subset[0,\infty)$ with $\#\Lambda<\infty$ and $A_j\in\Sigma'$, and

$$\mathcal{M}_{\{t_0,\dots,t_n\}}(A_0\times\cdots\times A_n)=\mathbb{E}_{\mathcal{M}}\Big[\prod_{j=0}^n\mathbb{1}_{A_{t_j}}(Z_{t_j})\Big]\tag{3.1}$$

holds. Here $(Z_t)_{t\geq0}$ is the coordinate process defined by $Z_t(\omega)=\omega(t)$ for $\omega\in(\mathbb{R}^d\times Q)^{[0,\infty)}$.

Lemma 3.2 *The random process $(Z_t)_{t\geq0}$ on $((\mathbb{R}^d\times Q)^{[0,\infty)},\sigma(\mathscr{A}))$ has a continuous version.*

Proof: We write $Z_t=(x_t,\xi_t)$, where $x_t\in\mathbb{R}^d$ and $\xi_t\in Q$ are the coordinate processes $x_t(\omega)=\omega^1(t)$ and $\xi_t(\omega)=\omega^2(t)$ for all $t\geqslant0$ and $\omega=(\omega^1,\omega^2)\in(\mathbb{R}^d\times Q)^{[0,\infty)}$. Define $\|Z_t\|_{\mathbb{R}^d\times Q}=\sqrt{\|x_t\|^2_{\mathbb{R}^d}+\|\xi_t\|^2_Q}$. Using the Kolmogorov-Čentsov theorem [13, Theorem 2.8], the estimate

$$\mathbb{E}_{\mathcal{M}}[\|Z_t-Z_s\|^4_{\mathbb{R}^d\times Q}]\leq D|t-s|^2\tag{3.2}$$

with some $D > 0$ implies that $(Z_t)_{t\geq 0}$ has a continuous version. Since

$$\|Z_t\|^4_{\mathbb{R}^d\times Q} \leq 2(\|x_t\|^4_{\mathbb{R}^d} + \|\xi_t\|^4_Q),$$

it will suffice to prove the bounds

$$\mathbb{E}_{\mathcal{M}}[\|x_t - x_s\|^4_{\mathbb{R}^d}] \leq D_1|t-s|^2, \tag{3.3}$$

$$\mathbb{E}_{\mathcal{M}}[\|\xi_t - \xi_s\|^4_Q] \leq D_2|t-s|^2. \tag{3.4}$$

To obtain (3.3), recall the moments formula $\mathbb{E}[|B_t - B_s|^{2n}] = K_n|t-s|^n$, with a constant K_n for $n \geq 0$. Let $x_t = (x^1_t, \ldots, x^d_t)$. By using the formula in Proposition 2.12, we have for all $1 \leq i, j \leq d$ that

$$\begin{aligned}
&\mathbb{E}_{\mathcal{M}}[(x^i_t)^n(x^j_s)^m] \\
&= ((x^i)^n\varphi_{\mathrm{g}}, e^{-(t-s)(H-E)}(x^j)^m\varphi_{\mathrm{g}})_{L^2(\mathrm{P})} \\
&= ((x^i)^n\varphi_{\mathrm{p}}\varphi_{\mathrm{g}}, e^{-(t-s)(H_{\mathrm{N}}-E)}(x^j)^m\varphi_{\mathrm{p}}\varphi_{\mathrm{g}})_{L^2(\mathbb{R}^d)\otimes L^2(Q)} \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^x_{\mathcal{W}\times\mathcal{G}}\Big[(B^i_0)^n(B^j_{t-s})^m\varphi_{\mathrm{p}}(B_0)\varphi_{\mathrm{g}}(B_0,\xi_0)\varphi_{\mathrm{p}}(B_{t-s})\varphi_{\mathrm{g}}(B_{t-s},\xi_{t-s}) \\
&\qquad\qquad \times e^{-\int_0^{t-s}\tau_{B_r}\xi_r(\tilde{\varphi}))dr}e^{-\int_0^{t-s}V(B_r)dr}\Big].
\end{aligned}$$

Using the eigenvalue equations and the Feynman-Kac formula for the free particle and the full Nelson Hamiltonians, it follows that

$$\sup_{x\in\mathbb{R}^d}|\varphi_{\mathrm{p}}(x)| = C_1 < \infty \quad \text{and} \quad \sup_{(x,\xi)\in\mathbb{R}^d\times Q}|\varphi_{\mathrm{g}}(x,\xi)| = C_2 < \infty.$$

Thus we have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{M}}[|x_t - x_s|^4] = \int_{\mathbb{R}^d} dx \mathbb{E}^x_{\mathcal{W}\times\mathcal{G}}\left[|B_0 - B_{t-s}|^4\varphi_{\mathrm{p}}(B_0)\varphi_{\mathrm{g}}(B_0,\xi_0)\varphi_{\mathrm{p}}(B_{t-s})\varphi_{\mathrm{g}}(B_{t-s},\xi_{t-s})\right. \\
&\qquad \left.\times e^{-\int_0^{t-s}\tau_{B_r}\xi_r(\tilde{\varphi})}e^{-\int_0^{t-s}V(B_r+x)dr}\right] \\
&\leq C_2^2\int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}\times\mathcal{G}}\varphi_{\mathrm{p}}(x)\Big[\varphi_{\mathrm{p}}(B_{t-s}+x)|B_0 - B_{t-s}|^4 e^{-\int_0^{t-s}\tau_{B_r+x}\xi_r(\tilde{\varphi}))dr}e^{-\int_0^{t-s}V(B_r+x)dr}\Big] \\
&\leq C_2^2\left(\int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}\times\mathcal{G}}\left[|\varphi_{\mathrm{p}}(B_{t-s}+x)|^2\right]\right)^{1/2} \\
&\qquad \times\left(\int_{\mathbb{R}^d} dx|\varphi_{\mathrm{p}}(x)|^2\mathbb{E}_{\mathcal{W}}\left[|B_0 - B_{t-s}|^8 e^{-2\int_0^{t-s}V(B_r+x)dr}\mathbb{E}_{\mathcal{G}}\left[e^{-2\int_0^{t-s}\tau_{B_r+x}\xi_r(\tilde{\varphi}))dr}\right]\right]\right)^{1/2}.
\end{aligned}$$

Since

$$\mathbb{E}_{\mathcal{G}}[e^{-\int_0^{t-s}\tau_{B_r}\xi_r(\tilde{\varphi})dr}] \leq e^{(t-s)\|\widehat{\varphi}/\omega\|^2} = C,$$

we see that

$$\begin{aligned}
\mathbb{E}_{\mathcal{M}}[|x_t - x_s|^4] &\leq C_2^2C\|\varphi_{\mathrm{p}}\|^2\left(\mathbb{E}_{\mathcal{W}}\left[|B_0 - B_{t-s}|^{16}\right]\right)^{1/4}\left(\mathbb{E}_{\mathcal{W}}[e^{-4\int_0^{t-s}V(B_r+x)dr}]\right)^{1/4} \\
&\leq C_2^2CK_8^{1/4}\sup_{x\in\mathbb{R}^d}\left(\mathbb{E}_{\mathcal{W}}[e^{-4\int_0^{t-s}V(B_r+x)dr}]\right)^{1/4}|t-s|^2.
\end{aligned}$$

Thus (3.3) follows. Next we prove (3.4). Let $f \in \mathscr{M}_{+2}$. In the same way as in the proof of (3.3) we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{M}}[\xi_t(f)^n \xi_s(f)^m] &= (\xi(f)^n \varphi_{\rm g}, e^{-(t-s)(H-E)} \xi(f)^m \varphi_{\rm g})_{L^2(d\mathrm{P})} \\
&= (\xi(f)^n \varphi_{\rm p}\varphi_{\rm g}, e^{-(t-s)(H_{\rm N}-E)} \xi(f)^m \varphi_{\rm p}\varphi_{\rm g})_{L^2(\mathbb{R}^d)\otimes L^2(Q)} \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^x_{\mathcal{W}\times\mathcal{G}} \left[\xi_0(f)^n \xi_s(f)^m \varphi_{\rm p}(B_0)\varphi_{\rm g}(B_0,\xi_0)\varphi_{\rm p}(B_{t-s})\varphi_{\rm g}(B_{t-s},\xi_{t-s})\right. \\
&\qquad\qquad \left. \times e^{-\int_0^{t-s}\tau_{B_r}\xi_r(\tilde\varphi))dr} e^{-\int_0^{t-s} V(B_r)dr}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}_{\mathcal{M}}[|\xi_t(f)-\xi_s(f)|^4] \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^x_{\mathcal{W}\times\mathcal{G}} \left[|\xi_0(f)-\xi_{t-s}(f)|^4 \varphi_{\rm p}(B_0)\varphi_{\rm g}(B_0,\xi_0)\varphi_{\rm p}(B_{t-s})\varphi_{\rm g}(B_{t-s},\xi_{t-s})\right. \\
&\qquad\qquad \left. \times e^{-\int_0^{t-s}\tau_{B_r}\xi_r(\tilde\varphi))dr} e^{-\int_0^{t-s} V(B_r+x)dr}\right] \\
&\le C_2^2 \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}\times\mathcal{G}} \left[\varphi_{\rm p}(x)\varphi_{\rm p}(B_{t-s}+x)|\xi_t(f)-\xi_{t-s}(f)|^4\right. \\
&\qquad\qquad \left. \times e^{-\int_0^{t-s}\tau_{B_r+x}\xi_r(\tilde\varphi))dr} e^{-\int_0^{t-s} V(B_r+x)dr}\right] \\
&\le C_2^2 \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}} \left[\varphi_{\rm p}(x)\varphi_{\rm p}(B_{t-s}+x) e^{-\int_0^{t-s} V(B_r+x)dr}\right. \\
&\qquad\qquad \left. \times \mathbb{E}_{\mathcal{G}}\left[|\xi_t(f)-\xi_{t-s}(f)|^4 e^{-\int_0^{t-s}\tau_{B_r+x}\xi_r(\tilde\varphi))dr}\right]\right] \\
&\le C_2^2 \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}} \left[\varphi_{\rm p}(x)\varphi_{\rm p}(B_{t-s}+x) e^{-\int_0^{t-s} V(B_r+x)dr} \left(\mathbb{E}_{\mathcal{G}}\left[|\xi_t(f)-\xi_{t-s}(f)|^8\right]\right)^{1/2}\right. \\
&\qquad\qquad \left. \times \left(\mathbb{E}_{\mathcal{G}}\left[e^{-2\int_0^{t-s}\tau_{B_r+x}\xi_r(\tilde\varphi))dr}\right]\right)^{1/2}\right].
\end{aligned}$$

By Lemma 3.3 below we have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{M}}[|\xi_t(f)-\xi_s(f)|^4] \\
&\le C_2^2 |t-s|^2 \|f\|^2_{\mathscr{M}_{+2}} C \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}}\left[\varphi_{\rm p}(x)\varphi_{\rm p}(B_{t-s}+x) e^{-\int_0^{t-s} V(B_r+x)dr}\right] \\
&\le C_2^2 |t-s|^2 \|f\|^2_{\mathscr{M}_{+2}} C \|\varphi_{\rm p}\|^2 \sup_{x\in\mathbb{R}^d} \left(\mathbb{E}_{\mathcal{W}}[e^{-2\int_0^{t-s} V(B_r+x)dr}]\right)^{1/2}.
\end{aligned}$$

Hence

$$\mathbb{E}_{\mathcal{M}}\left[\frac{|\xi_t(f)-\xi_s(f)|^4}{\|f\|^2_{\mathscr{M}_{+2}}}\right] \le D_2|t-s|^2 \tag{3.5}$$

with a constant D_2. Since $\|\xi_t - \xi_s\|_Q = \sup_{f\neq 0} |\xi_t(f)-\xi_s(f)|/\|f\|_{\mathscr{M}_{+2}}$, for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathscr{M}_{+2}$ such that $\|\xi_t-\xi_s\|_Q \le |\xi_t(f_\varepsilon)-\xi_s(f_\varepsilon)|/\|f_\varepsilon\|_{\mathscr{M}_{+2}} + \varepsilon$. Thus together with (3.5) we have

$$\mathbb{E}_{\mathcal{M}}\left[|\xi_t-\xi_s|^4\right] - \varepsilon \le D_2|t-s|^2, \tag{3.6}$$

and thus (3.4) follows. □

Lemma 3.3 *We have* $\mathbb{E}_{\mathcal{G}}[|\xi_t(f)-\xi_s(f)|^{2n}] \leq D_n|t-s|^n\|f\|^n_{\mathscr{M}_{+2}}$.

Proof: Since $\xi_t(f)-\xi_s(f)$ is a Gaussian process and $\mathbb{E}_{\mathcal{G}}[e^{iT(\xi_t(f)-\xi_s(f))}] = e^{-\frac{1}{2}CT^2}$, where $C = (\widehat{f}/\sqrt{\omega}, (1-e^{-|t-s|\omega})\widehat{f}/\sqrt{\omega})$, by taking derivatives $2n$ times at $t=0$ on both sides we obtain

$$\begin{aligned}\mathbb{E}_{\mathcal{G}}[|\xi_t(f)-\xi_s(f)|^{2n}] &\leq D_n(\widehat{f}/\sqrt{\omega}, (1-e^{-|t-s|\omega})\widehat{f}/\sqrt{\omega})^n\\ &\leq D_n|t-s|^n\|\widehat{f}\|^n_{L^2}\\ &\leq D_n'|t-s|^n\|f\|^n_{\mathscr{M}_{+2}}.\end{aligned}$$

Here we used that the embedding $i:\mathscr{M}_{+2}\to L^2(\mathbb{R}^d)$ is bounded [15, p288]. $\square$

We denote the continuous version of $(Z_t)_{t\geq 0}$ by $(\bar{Z}_t)_{t\geq 0}$, and the set of $\mathbb{R}^d\times Q$-valued continuous paths by $\mathscr{X}_Q^+ = C([0,\infty),\mathbb{R}^d\times Q)$. Note that $(\bar{Z}_t)_{t\geq 0}$ is a stochastic process on the probability space $((\mathbb{R}^d\times Q)^{[0,\infty)},\sigma(\mathscr{A}),\mathcal{M})$, and the map

$$\bar{Z}_\cdot : ((\mathbb{R}^d\times Q)^{[0,\infty)},\sigma(\mathscr{A}),\mathcal{M}) \to (\mathscr{X}_Q^+,\mathcal{B}^+)$$

is measurable, where $\mathcal{B}^+$ denotes the σ-field generated by cylinder sets. This map induces the image measure $\mathcal{P}_+ = \mathcal{M}\circ\bar{Z}_\cdot^{-1}$ on $(\mathscr{X}_Q^+,\mathcal{B}^+)$. Thus for the coordinate process $(X_t^+)_{t\geq 0}$ on $(\mathscr{X}_Q^+,\mathcal{B}^+,\mathcal{P}_+)$ we have $\bar{Z}_t \stackrel{\mathrm{d}}{=} X_t^+$ for $t\geq 0$. Let $(y,\xi)\in\mathbb{R}^d\times Q$, and define the regular conditional probability measure

$$\mathcal{P}_+^{(y,\xi)}(\cdot) = \mathcal{P}_+(\cdot\,|X_0^+ = (y,\xi))$$

on $(\mathscr{X}_Q^+,\mathcal{B}^+)$. Since the distribution of X_0^+ is $d\mathrm{M}$, we see that

$$\mathcal{P}_+(A) = \int_{\mathbb{R}^d\times Q}\mathbb{E}_{\mathcal{P}_+}^{(y,\xi)}[\mathbb{1}_A]d\mathrm{M}(y,\xi).$$

Using the measure $\mathcal{P}_+$ we have

$$(f,e^{-tL_\mathrm{N}}g)_{\mathscr{K}} = \int_{\mathbb{R}^d\times Q}\mathbb{E}_{\mathcal{P}_+}^{(y,\xi)}[\bar{f}(X_0^+)g(X_t^+)]d\mathrm{M}(y,\xi), \tag{3.7}$$

which implies that

$$\left(e^{-tL_\mathrm{N}}g\right)(y,\xi) = \mathbb{E}_{\mathcal{P}_+}^{(y,\xi)}[g(X_t^+)]. \tag{3.8}$$

Lemma 3.4 *The random process* $(X_t^+)_{t\geqslant 0}$ *on* $(\mathscr{X}_Q^+,\mathcal{B}^+,\mathcal{P}_+^{(y,\xi)})$ *has the Markov property with respect to the natural filtration* $\sigma(X_t^+,\,0\leq s\leq t)$.

Proof: In this proof we set $z=(y,\xi), z_j=(y_j,\xi_j)\in\mathbb{R}^d\times Q$ for notational simplicity. Let

$$p_t(z,A) = (e^{-tL_\mathrm{N}}\mathbb{1}_A)(z), \quad A\in\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(Q). \tag{3.9}$$

Notice that

$$p_t(z,A) = \mathbb{E}_{\mathcal{P}_+}^z[\mathbb{1}_A(X_t^+)] = \mathbb{E}_{\mathcal{P}_+}[\mathbb{1}_A(X_t^+)|X_0^+=z].$$

We show that $p_t(z,A)$ is a probability transition kernel, i.e.,

1. $p_t(z,\cdot)$ is a probability measure on $\mathcal{B}(Q)$
2. the function $z \mapsto p_t(z,A)$ is Borel measurable
3. the Chapman-Kolmogorov identity

$$\int_{\mathbb{R}^d\times Q} p_t(z,A)p_s(z_1,dz) = p_{s+t}(x_1,A) \tag{3.10}$$

is satisfied.

Note first that by (3.7) it is easy to see that $e^{-tL_{\mathrm{N}}}$ is positivity improving. For every function $f \in \mathscr{H}$ such that $0 \le f \le \mathbb{1}$, we have

$$\left(e^{-tL_{\mathrm{N}}}f\right)(z) = \mathbb{E}^z_{\mathcal{P}_+}[f(X^+_t)] \le \mathbb{E}^z_{\mathcal{P}_+}[\mathbb{1}] = 1.$$

Then we deduce that $0 \le e^{tL_{\mathrm{N}}}f \le \mathbb{1}$ and $e^{tL_{\mathrm{N}}}\mathbb{1} = \mathbb{1}$, and (1)-(2) follow. We can also show that the finite dimensional distribution is given by

$$\mathbb{E}_{\mathcal{P}_+}\left[\prod_{j=1}^n \mathbb{1}_{A_j}(X^+_{t_j})\right] = \int_{(\mathbb{R}^d\times Q)^n} \prod_{j=1}^n \mathbb{1}_{A_j}(z_j)\prod_{j=1}^n p_{t_j-t_{j-1}}(z_{j-1},dz_j).$$

Thus $(X^+_t)_{t\ge 0}$ is a Markov process by [15, Proposition 2.17]. □

Now we extend $(X^+_t)_{t\ge 0}$ to a random process indexed by the full real line $\mathbb{R}$. Consider the product probability space $(\widehat{\mathscr{X}^+_Q}, \widehat{\mathcal{B}}^+, \widehat{\mathcal{P}}^{(y,\xi)}_+)$ with $\widehat{\mathscr{X}^+_Q} = \mathscr{X}^+_Q \times \mathscr{X}^+_Q$, $\widehat{\mathcal{B}}^+ = \mathcal{B}^+ \otimes \mathcal{B}^+$ and $\widehat{\mathcal{P}}^{(y,\xi)}_+ = \mathcal{P}^{(y,\xi)}_+ \otimes \mathcal{P}^{(y,\xi)}_+$, and let $(\widehat{X}_t)_{t\in\mathbb{R}}$ be the stochastic process

$$\widehat{X}_t(\omega) = \begin{cases} X^+_t(\omega_1) & t \ge 0, \\ X^+_{-t}(\omega_2) & t \le 0, \end{cases}$$

on the product space, for $\omega = (\omega_1,\omega_2) \in \widehat{\mathscr{X}^+_Q}$.

Lemma 3.5 *It follows that*

(1) $\widehat{X}_0 = (y,\xi)$ *a.s.*

(2) $\widehat{X}_t, t \ge 0$ *and* $\widehat{X}_s, s < 0$ *are independent*

(3) $\widehat{X}_t \stackrel{\mathrm{d}}{=} \widehat{X}_{-t}$ *for all* $t \in \mathbb{R}$,

(4) $(\widehat{X}_t)_{t\ge 0}$ *(resp.* $(\widehat{X}_t)_{t\le 0}$*) is a Markov process with respect to* $\sigma(\widehat{X}_s, 0 \le s \le t)$ *(resp.* $\sigma(\widehat{X}_s, t \le s \le 0)$*)*

(5) *for* $f_0,...,f_n \in \mathscr{K}$ *and* $-t = t_0 \le t_1 \le \ldots \le t_n = t$, *we have*

$$\mathbb{E}_{\widehat{\mathcal{P}}_+}\left[\prod_{j=0}^n f_j(\widehat{X}_{t_j})\right] = \left(f_0, e^{-(t_1+t)L_{\mathrm{N}}}f_1 ... e^{-(t-t_{n-1})L_{\mathrm{N}}}f_n\right)_{\mathscr{K}}. \tag{3.11}$$

Proof: (1)-(3) are straightforward. (4) follows from Lemma 3.4. (5) follows from (3.1) and a simple limiting argument. □

Proof of Theorem 3.1. We show that the stochastic process $(\widehat{X}_t)_{t\in\mathbb{R}}$ defined on $(\widehat{\mathscr{X}_Q^+}, \widehat{\mathcal{B}}, \widehat{\mathcal{P}}_+^{(y,\xi)})$ is a $P(\phi)_1$-process associated with $((\mathbb{R}^d\times Q, \Sigma', \mathrm{M}), L_\mathrm{N})$. The Markov property, reflection symmetry and the shift invariance property follow from Lemma 3.5. Continuity of $t \mapsto \widehat{X}_t$ has been shown in Lemma 3.2. Thus $\widehat{X}_\cdot : (\widehat{\mathscr{X}_Q^+}, \widehat{\mathcal{B}}, \widehat{\mathcal{P}}_+^{(y,\xi)}) \to (\mathscr{X}_Q, \mathcal{B}_Q)$ is measurable and the image measure $\mathcal{P}^{(y,\xi)} = \widehat{\mathcal{P}}_+^{(y,\xi)} \circ \widehat{X}_\cdot^{-1}$ defines a probability measure on $(\mathscr{X}_Q, \mathcal{B}_Q)$. Hence the coordinate process $(X_t)_{t\in\mathbb{R}}$ on $(\mathscr{X}_Q, \mathcal{B}_Q, \mathcal{P}^{(y,\xi)})$ satisfies $X_t \stackrel{\mathrm{d}}{=} \widehat{X}_t$, and then $(X_t)_{t\in\mathbb{R}}$ is a $P(\phi)_1$-process associated with $((\mathbb{R}^d\times Q, \Sigma', d\mathrm{M}_0), L_\mathrm{N})$. □

Lemma 3.6 *The random process $(X_t)_{t\in\mathbb{R}}$ is a reversible Markov process under $\mathcal{P}$, and its stationary measure is* M*, i.e., for every $n \geq 1$ we have that $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ has the same distribution as $(X_{\tau-t_1}, X_{\tau-t_2}, ..., X_{\tau-t_n})$ for all $t_1, ..., t_n, \tau \in \mathbb{R}$.*

Proof: Let $f, g \in \mathscr{K}$. Then $\mathbb{E}_\mathcal{P}[f(X_t)g(X_s)] = \left(f, e^{-|t-s|L_\mathrm{N}} g\right)_\mathscr{K}$. Thus the lemma follows. □

3.2 Relativistic Nelson model

Define a probability measure on $\mathbb{R}^d\times Q$ by

$$d\tilde{\mathrm{M}} = \tilde{\varphi}_\mathrm{g}^2 d\tilde{\mathrm{P}}.$$

The unitary operator $\widetilde{U}_\mathrm{g} : L^2(\mathbb{R}^d\times Q, d\tilde{\mathrm{M}}) \to L^2(\mathbb{R}^d\times Q, d\tilde{\mathrm{P}})$ is defined by $\Phi \mapsto \tilde{\varphi}_\mathrm{g}\Phi$. Let $\tilde{\mathscr{K}} = L^2(\mathbb{R}^d\times Q, d\tilde{\mathrm{M}})$. Define the operator

$$\tilde{L}_\mathrm{N} = \frac{1}{\tilde{\varphi}_\mathrm{g}}(\tilde{H} - \tilde{E})\tilde{\varphi}_\mathrm{g}.$$

Let $\tilde{\mathscr{X}}_Q = D(\mathbb{R}, \mathbb{R}^d\times Q)$ be the space of càdlàg paths with values in $\mathbb{R}^d\times Q$ on the whole real line, and $\tilde{\mathcal{B}}_Q$ the σ-field generated by cylinder sets. Similarly to the classical Nelson model, we can construct a $P(\phi)_1$ process for the relativistic Nelson Hamiltonian.

Theorem 3.7 ($P(\phi)_1$-process for the relativistic Nelson Hamiltonian) *Let $(y,\xi) \in \mathbb{R}^d\times Q$. Then ,*

1. *There exists a probability measure $\tilde{\mathcal{P}}^{(y,\xi)}$ on $(\tilde{\mathscr{X}}_Q, \tilde{\mathcal{B}}_Q)$ such that the coordinate process $(\tilde{X}_t)_{t\in\mathbb{R}}$ on $(\tilde{\mathscr{X}}_Q, \tilde{\mathcal{B}}_Q, \tilde{\mathcal{P}}^{(y,\xi)})$ is $P(\phi)_1$-process associated with the pair $\Big((\mathbb{R}^d\times Q, \mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(Q), d\tilde{\mathrm{M}}), \tilde{L}_\mathrm{N}\Big)$.*

2. *The function $t \mapsto \tilde{X}_t$ is càdlàg a.s.*

The proof of Theorem 3.7 is parallel with that of Theorem 3.1 except for path regularity. We only discuss this part of the proof. Define the family of set functions $\{\tilde{\mathscr{M}}_\Lambda \mid \Lambda \subset \mathbb{R}, \#\Lambda < \infty\}$ on $\Sigma'^{\#\Lambda} = \underbrace{\Sigma' \times \cdots \times \Sigma'}_{\#\Lambda}$ by

$$\tilde{\mathscr{M}}_\Lambda(A_0 \times A_1 \times ... \times A_n) = \Big(1\!\!1_{A_0}, e^{-(t_1-t_0)\tilde{L}_N} 1\!\!1_{A_1} e^{-(t_2-t_1)\tilde{L}_N} ... 1\!\!1_{A_{n-1}} e^{-(t_n-t_{n-1})\tilde{L}_N} 1\!\!1_{A_n}\Big)_{\tilde{\mathscr{K}}}$$

for $\Lambda = \{t_0, \dots, t_n\}$. By the same way as for the classical Nelson Hamiltonian, we define the projection $\pi_\Lambda : (\mathbb{R}^d \times Q)^{[0,\infty)} \longrightarrow (\mathbb{R}^d \times Q)^\Lambda$ by $w \longmapsto (w(t_0), ..., w(t_n))$ for $\Lambda = \{t_0, ..., t_n\}$, and $\mathscr{A} = \{\pi_\Lambda^{-1}(A) \mid A \in \Sigma'^{\#\Lambda}, \#\Lambda < \infty\}$ is a finitely additive family of sets. Using the Kolmogorov extension theorem, there exists a unique probability measure $\tilde{\mathscr{M}}$ on $((\mathbb{R}^d \times Q)^{[0,\infty)}, \sigma(\mathscr{A}))$ such that

$$\begin{aligned} \mathbb{E}_{\tilde{\mathscr{M}}}\Big[\prod_{j=0}^n 1\!\!1_{A_{t_j}}(\tilde{Z}_{t_j})\Big] &= \tilde{\mathscr{M}}_{\{t_0,\cdots,t_n\}}(A_0 \times \cdots \times A_n) \\ &= \Big(1\!\!1_{A_0}, e^{-(t_1-t_0)\tilde{L}_N} 1\!\!1_{A_1} e^{-(t_2-t_1)\tilde{L}_N} 1\!\!1_{A_1} ... 1\!\!1_{A_{n-1}} e^{-(t_n-t_{n-1})\tilde{L}_N} 1\!\!1_{A_n}\Big)_{\tilde{\mathscr{K}}}, \end{aligned} \tag{3.12}$$

where $(\tilde{Z}_t)_{t\geqslant 0}$ is the coordinate process on $((\mathbb{R}^d \times Q)^{[0,\infty)}, \sigma(\mathscr{A}), \tilde{\mathscr{M}})$. The equality (3.12) leads to the following result:

Lemma 3.8 *The stochastic process $(\tilde{Z}_t)_{t\geqslant 0}$ is shift invariant under $\tilde{\mathscr{M}}$, i.e, for $f_0, ..., f_n \in \tilde{\mathscr{K}}$ and $s \geqslant 0$ it follows that*

$$\mathbb{E}_{\tilde{\mathscr{M}}}\Big[\prod_{j=0}^n f_j(\tilde{Z}_{t_j+s})\Big] = \mathbb{E}_{\tilde{\mathscr{M}}}\Big[\prod_{j=0}^n f_j(\tilde{Z}_{t_j})\Big] = \Big(f_0, e^{-(t_1+t)\tilde{L}_N} f_1 \cdots e^{-(t-t_{n-1})\tilde{L}_N} f_n\Big)_{\tilde{\mathscr{K}}}. \tag{3.13}$$

Now we prove that $(\tilde{Z}_t)_{t\geqslant 0}$ has càdlàg version under $\tilde{\mathscr{M}}$. For this purpose, we need the following technical lemmas which make use of the ideas in [18, pp 59-62]. Let $I \subset [0, \infty)$ and $\varepsilon > 0$. We say that $\tilde{Z}.(\omega)$, with ω fixed, has ε-oscillation n times in I, if there exist $t_0, t_1, \dots, t_n \in I$ such that $t_0 < t_1 < t_2 \dots < t_n$ and $\|\tilde{Z}_{t_j} - \tilde{Z}_{t_{j-1}}\|_{\mathbb{R}^d \times Q} > \varepsilon$ for $j = 1, \dots, n$. We say that $\tilde{Z}.(\omega)$ has ε-oscillation infinitely often in I, if, for every n, $\tilde{Z}.(\omega)$ has ε-oscillation n times in I. Let

$$\begin{aligned} \Omega' &= \Big\{\omega \in \Omega \mid \lim_{s\in\mathbb{Q}, s\downarrow t} \tilde{Z}_s(\omega) \text{ and } \lim_{s\in\mathbb{Q}, s\uparrow t} \tilde{Z}_s(\omega) \text{ exist in } \mathbb{R}^d \times Q \text{ for all } t \geq 0\Big\}, \\ A_{N,k} &= \Big\{\omega \in \Omega \mid \tilde{Z}_t(\omega) \text{ does not have } 1/k - \text{oscillation infinitely often in } [0, N] \cap \mathbb{Q}\Big\}, \\ \Omega'' &= \bigcap_{N=1}^\infty \bigcap_{k=1}^\infty A_{N,k}. \end{aligned}$$

Similarly as in [18, Lemma 11.2], we can see that $\Omega'' \subset \Omega'$. Define

$$B(p, \varepsilon, I) = \Big\{\omega \in (\mathbb{R}^d \times Q)^{[0,\infty)} \mid \tilde{Z}_t(\omega) \text{ has } \varepsilon - \text{oscillation } p \text{ times in } I\Big\}.$$

Lemma 3.9 *For every $\varepsilon > 0$ we have* $\lim_{|t-s|\to 0} \tilde{\mathcal{M}}\left(\|\tilde{Z}_t - \tilde{Z}_s\|_{\mathbb{R}^d\times Q} > \varepsilon\right) = 0$ *and* $\tilde{\mathcal{M}}(\Omega'') = 1$.

Proof: In this proof we set

$$\Phi_{\rm pg}(y,\xi) = \tilde{\varphi}_{\rm p}(y)\tilde{\varphi}_{\rm g}(y,\xi)$$

and $\mathbb{E}[\cdots] = \int_{\mathbb{R}^d\times Q} dy \otimes d\mathrm{G}\mathbb{E}^{(y,\xi)}_{\mathscr{P}\times\mathcal{G}}[\cdots]$ for notational simplicity. Consider

$$\begin{aligned} J_\varepsilon &= \sup_{(y,\xi)\in\mathbb{R}^d\times Q} (\mathscr{P}^y \otimes \mathcal{G}^\xi)\left(\tilde{Z}_{t-s} \in B^c((y,\xi),\varepsilon)\right) \\ &= (\mathscr{P}\otimes\mathcal{G})\left(\tilde{Z}_{t-s} \in B^c(0,\varepsilon)\right) = (\mathscr{P}\otimes\mathcal{G})\left(|\tilde{Z}_{t-s}| > \varepsilon\right). \end{aligned}$$

By Proposition 2.14 we have

$$\tilde{\mathcal{M}}\left(\|\tilde{Z}_t - \tilde{Z}_s\|_{\mathbb{R}^d\times Q} > \varepsilon\right) = (\mathbb{1}, e^{-(t-s)\tilde{L}_{\rm N}} \mathbb{1}_{B^c(0,\varepsilon)})_{\mathscr{K}},$$

where $B^c((y',\xi'),\varepsilon) = \{(y,\xi) \in \mathbb{R}^d \times Q \mid \|(y,\xi) - (y',\xi')\|_{\mathbb{R}^d\times Q} \geq \varepsilon\}$. Also, we have

$$\sup_{(y,\xi)\in\mathbb{R}^d\times Q} |\tilde{\varphi}_{\rm g}(y,\xi)| = \sqrt{K}_2 < \infty \quad \text{and} \quad \sup_{y\in\mathbb{R}^d} \mathbb{E}^y_{\mathscr{P}}\left[e^{-4\int_0^{t-s} V(b_r)dr}\right] \leq C_V.$$

We have

$$\begin{aligned} &\tilde{\mathcal{M}}\left(\|\tilde{Z}_t - \tilde{Z}_s\|_{\mathbb{R}^d\times Q} > \varepsilon\right) \\ &= \mathbb{E}\left[\Phi_{\rm pg}(y,\xi) e^{-\int_0^{t-s} \tau_{b_r}\xi_r(\tilde{\varphi})dr} e^{-\int_0^{t-s} V(b_r)dr} \Phi_{\rm pg}(b_{t-s},\xi_{t-s}) \mathbb{1}_{B^c((y,\xi),\varepsilon)}(b_{t-s},\xi_{t-s})\right] e^{(t-s)\tilde{E}}. \end{aligned}$$

The right-hand side can be evaluated as

$$\begin{aligned} &\tilde{\mathcal{M}}\left(\|\tilde{Z}_t - \tilde{Z}_s\|_{\mathbb{R}^d\times Q} > \varepsilon\right) \\ &\leq K_2 \mathbb{E}\left[\tilde{\varphi}_{\rm p}(y) e^{-\int_0^{t-s} \tau_{b_r}\xi_r(\tilde{\varphi})dr} e^{-\int_0^{t-s} V(b_r)dr} \tilde{\varphi}_{\rm p}(b_{t-s}) \mathbb{1}_{B^c((y,\xi),\varepsilon)}(b_{t-s},\xi_{t-s})\right] e^{(t-s)\tilde{E}}. \end{aligned}$$

By using Schwarz inequality twice, we have

$$\begin{aligned} &\tilde{\mathcal{M}}\left(\|\tilde{Z}_t - \tilde{Z}_s\|_{\mathbb{R}^d\times Q} > \varepsilon\right) \\ &\leq K_2 \|\tilde{\varphi}_{\rm p}\| \mathbb{E}\left[|\tilde{\varphi}_{\rm p}(y)|^2 e^{-2\int_0^{t-s} \tau_{b_r}\xi_r(\tilde{\varphi})dr} e^{-2\int_0^{t-s} V(b_r)dr} \mathbb{1}_{B^c((y,\xi),\varepsilon)}(b_{t-s},\xi_{t-s})\right]^{\frac{1}{2}} e^{(t-s)\tilde{E}} \\ &\leq K_2 \|\tilde{\varphi}_{\rm p}\| \mathbb{E}\left[|\tilde{\varphi}_{\rm p}(y)|^2 \mathbb{1}_{B^c((y,\xi),\varepsilon)}(b_{t-s},\xi_{t-s})\right]^{\frac{1}{4}} \\ &\qquad \times \mathbb{E}\left[|\tilde{\varphi}_{\rm p}(y)|^2 e^{-4\int_0^{t-s} \tau_{b_r}\xi_r(\tilde{\varphi})dr} e^{-4\int_0^{t-s} V(b_r)dr}\right]^{\frac{1}{4}} e^{(t-s)\tilde{E}}. \end{aligned}$$

Since

$$\mathbb{E}\left[|\tilde{\varphi}_{\rm p}(y)|^2 \mathbb{1}_{B^c((y,\xi),\varepsilon)}(b_{t-s},\xi_{t-s})\right]^{\frac{1}{4}} \leq J_\varepsilon^{1/4} \|\tilde{\varphi}_{\rm p}\|^{1/2}$$

and

$$\begin{aligned} &\mathbb{E}\left[|\tilde{\varphi}_{\rm p}(y)|^2 e^{-4\int_0^{t-s} \tau_{b_r}\xi_r(\tilde{\varphi})dr} e^{-4\int_0^{t-s} V(b_r)dr}\right] \\ &= \int_{\mathbb{R}^d\times Q} \mathbb{E}^y_{\mathscr{P}}\left[|\tilde{\varphi}_{\rm p}(y)|^2 e^{-4\int_0^{t-s} V(b_r)dr}\right] \mathbb{E}^\xi_{\mathcal{G}}\left[e^{-4\int_0^{t-s} \tau_{b_r}\xi_r(\tilde{\varphi})dr}\right] dy \otimes d\mathrm{G}. \end{aligned}$$

By [15, Section 6.5] we have

$$\mathbb{E}_{\mathcal{G}}^{\xi}\left[e^{-4\int_0^{t-s}\tau_{b_r}\xi_r(\tilde\varphi)dr}\right] \le e^{(t-s)\left(\int_{\mathbb{R}^d}\frac{|\hat\varphi(k)|^2}{\omega(k)^2}dk+\int_{\mathbb{R}^d}\frac{|\hat\varphi(k)|^2}{\omega(k)^3}dk\right)} = C_{t-s}<\infty.$$

We deduce that

$$\begin{aligned}&\mathbb{E}\left[|\tilde\varphi_{\rm p}(y)|^2 e^{-4\int_0^{t-s}\tau_{b_r}\xi_r(\tilde\varphi)dr}e^{-4\int_0^{t-s}V(b_r)dr}\right]\\ &\quad\le C_{t-s}\int_{\mathbb{R}^d\times Q}|\tilde\varphi_{\rm p}(y)|^2\mathbb{E}_{\mathscr{P}}^{y}\left[e^{-4\int_0^{t-s}V(b_r)dr}\right]dy\otimes d\mathrm{G}\le C_{t-s}C_V\|\tilde\varphi_{\rm p}\|^2,\end{aligned}$$

and so we obtain

$$\tilde{\mathscr{M}}\left(\|\tilde Z_t-\tilde Z_s\|_{\mathbb{R}^d\times Q}>\varepsilon\right)\le K_2C_V^{1/4}\|\tilde\varphi_{\rm p}\|^2C_{t-s}J_\varepsilon^{1/4}e^{(t-s)\tilde E}. \tag{3.14}$$

Next we show that $\lim\limits_{|t-s|\to 0}J_\varepsilon=0$. In fact, we have

$$J_\varepsilon=(\mathscr{P}\otimes\mathcal{G})\left(\{\omega=(\omega^1,\omega^2)\in(\mathbb{R}^d\times Q)^{[0,\infty)}|\|b_{t-s}(\omega^1)\|_{\mathbb{R}^d}^2+\|\xi_{t-s}(\omega^2)\|_Q^2>\varepsilon^2\}\right).$$

By the stochastic continuity of the Lévy process $(b_t)_{t\ge0}$ and the OU-process $(\xi_t)_{t\ge0}$ we deduce that $\lim\limits_{|t-s|\to0}J_\varepsilon=0$. Then $\lim\limits_{|t-s|\to0}\tilde{\mathscr{M}}\left(\|\tilde Z_t-\tilde Z_s\|_{\mathbb{R}^d\times Q}>\varepsilon\right)=0$ follows. To see that $\tilde{\mathscr{M}}(\Omega'')$=1, it suffices to show that $\tilde{\mathscr{M}}(A_{N,k}^c)=0$ for any fixed N and k. We have

$$\begin{aligned}&\tilde{\mathscr{M}}(A_{N,k}^c)=\tilde{\mathscr{M}}\left(\left\{\tilde Z_t\text{ has }1/k-\text{oscillation infinitely often in }\big[0,N\big]\cap\mathbb{Q}\right\}\right)\\ &\le\sum_{j=1}^{l}\tilde{\mathscr{M}}\left(\left\{\tilde Z_t\text{ has }1/k-\text{oscillation infinitely often in }\left[\frac{j-1}{l}N,\frac{j}{l}N\right]\cap\mathbb{Q}\right\}\right)\\ &=\sum_{j=1}^{l}\lim_{p\to\infty}\tilde{\mathscr{M}}\left(B\left(p,\frac1k,\left[\frac{j-1}{l}N,\frac{j}{l}N\right]\cap\mathbb{Q}\right)\right).\end{aligned}$$

We enumerate as $\{t_1,\dots,t_n,\dots\}=[\frac{j-1}{l}N,\frac{j}{l}N]\cap\mathbb{Q}$. Thus

$$\tilde{\mathscr{M}}\left(B\left(p,\frac1k,\left[\frac{j-1}{l}N,\frac{j}{l}N\right]\cap\mathbb{Q}\right)\right)=\lim_{n\to\infty}\tilde{\mathscr{M}}\left(B\left(p,\frac1k,\{t_1,\cdots,t_n\}\right)\right).$$

Then by Proposition 2.14 we obtain

$$\begin{aligned}&\tilde{\mathscr{M}}\left(B\left(p,1/k,\{t_1,\dots,t_n\}\right)\right)\\ &=e^{N\tilde E/l}\mathbb{E}\left[\Phi_{\rm pg}(y,\xi)e^{-\int_0^{N/l}\tau_{b_r}\xi_r(\tilde\varphi)dr}\Phi_{\rm pg}(b_{N/l},\xi_{N/l})e^{-\int_0^{N/l}V(b_r)dr}\mathbb{1}_{B(p,1/k,\{t_1,\dots,t_n\})}(b_{N/l},\xi_{N/l})\right].\end{aligned}$$

Hence in the same estimate preceding (3.14) we have

$$\begin{aligned}&\tilde{\mathscr{M}}\left(B\left(p,\frac1k,\{t_1,\dots,t_n\}\right)\right)\\ &\le K_2C_V^{1/4}\|\tilde\varphi_{\rm p}\|^2e^{\frac12C_{N/l}}\left(\sup_{(y,\xi)\in\mathbb{R}^d\times Q}\mathscr{P}^y\otimes\mathcal{G}^\xi B\left(p,\frac1k,\{t_1,\dots,t_n\}\right)\right)^{\frac14}e^{(N/l)\tilde E}.\end{aligned} \tag{3.15}$$

By [18, Lemma 11.4], furthermore we have

$$(\mathscr{P}^y \otimes \mathcal{G}^\xi)\left(B\left(p, \frac{1}{k}, \{t_1, \ldots, t_n\}\right)\right) \leq \left(\sup_{\substack{s,t\in[0,N]\\ t-s\in[0,N/l]}} (\mathscr{P}\otimes\mathcal{G})\left(|(b_s,\xi_s)-(b_t,\xi_t)| \geq \frac{1}{4k}\right)\right)^p. \tag{3.16}$$

Moreover, by stochastic continuity of $((b_t, \xi_t))_{t\geq 0}$, we can prove uniform stochastic continuity, i.e.,

$$\sup_{\substack{s,t\in[0,N]\\ t-s\in[0,N/l]}} (\mathscr{P}\otimes\mathcal{G})\left(|(b_s,\xi_s)-(b_t,\xi_t)| \geq \frac{1}{4k}\right) \to 0 \tag{3.17}$$

as $l \to \infty$ in Lemma 3.10 below. □

Lemma 3.10 *(3.17) holds.*

Proof: For notational simplicity we write $X'_s = (b_s, \xi_s)$. Fix $a > 0$. For any t there exists $\delta_t > 0$ such that $(\mathscr{P}\otimes\mathcal{G})(|X'_t - X'_s| \geq \varepsilon/2) \leq a/2$ for $|t-s| < \delta_t$ by stochastic continuity. Let $I_t = (t - \delta_t/2, t + \delta_t/2)$. Since I_t is compact, there exists a finite covering I_{t_j}, $j = 1, \ldots, n$, such that $\cup_{j=1}^n I_{t_j} \supset [0, N]$. Let $\delta = \min_{j=1,\ldots,n} \delta_{t_j}$. If $|s-t| < \delta$ and $s, t \in [0, N]$, then $t \in I_{t_j}$ for some j, hence $|s - t_j| < \delta_{t_j}$ and

$$(\mathscr{P}\otimes\mathcal{G})(|X'_t - X'_s| \geq \varepsilon) \leq (\mathscr{P}\otimes\mathcal{G})(|X'_t - X'_{t_j}| \geq \varepsilon) + (\mathscr{P}\otimes\mathcal{G})(|X'_{t_j} - X'_s| \geq \varepsilon) < a.$$

Hence the lemma follows. □

Lemma 3.11 *The process $(\tilde{Z}_t)_{t\geq 0}$ has a right continuous version with left limits (càdlàg) with respect to $\tilde{\mathcal{M}}$.*

Proof: Let $(\tilde{Z}')_{t\geqslant 0}$ be a càdlàg process defined by

$$\tilde{Z}'_t(\omega) = \begin{cases} \lim\limits_{s\in\mathbb{Q}, s\downarrow t} \tilde{Z}_s(\omega) & \omega \in \Omega'', \\ 0 & \omega \notin \Omega''. \end{cases} \tag{3.18}$$

By Lemmas 3.9 the process $(\tilde{Z}_t)_{t\geq 0}$ is stochastically continuous, which implies that there exists a sequence s_n such that

$$\lim_{s_n\in\mathbb{Q}, s_n\downarrow t} \tilde{Z}_{s_n}(\omega) = \tilde{Z}_t(\omega) \tag{3.19}$$

for $\omega \in \Omega''' = (\mathbb{R}^d \times Q)^{[0,\infty)} \setminus N_t$ with some null set N_t. We can also see by the definition of the process $(\tilde{Z}')_{t\geqslant 0}$ that

$$\lim_{s_n\in\mathbb{Q}, s_n\downarrow t} \tilde{Z}_{s_n}(\omega) = \tilde{Z}'_t(\omega) \tag{3.20}$$

for $\omega \in \Omega''$, and $\tilde{\mathcal{M}}(\Omega'') = 1$ by Lemma 3.9. For each t by (3.19) and (3.20) we can derive that $\tilde{Z}_t(\omega) = \tilde{Z}'_t(\omega)$ for $\omega \in \Omega'' \cap \Omega'''$, and $\tilde{\mathcal{M}}(\Omega'' \cap \Omega''') = 1$. Then $(\tilde{Z}')_{t\geqslant 0}$ is a càdlàg version of $(\tilde{Z})_{t\geqslant 0}$. □

We denote the càdlàg version of $(\tilde{Z}_t)_{t\geq 0}$ by $(\bar{\tilde{Z}}_t)_{t\geq 0}$, and the set of $\mathbb{R}^d\times Q$-valued càdlàg paths by $\tilde{\mathscr{X}}_Q^+ = D([0,\infty),\mathbb{R}^d\times Q)$. Note that $(\bar{\tilde{Z}}_t)_{t\geq 0}$ is a stochastic process on the probability space $((\mathbb{R}^d\times Q)^{[0,\infty)},\sigma(\mathscr{A}),\tilde{\mathcal{M}})$, and the map

$$\bar{\tilde{Z}}_\cdot : ((\mathbb{R}^d\times Q)^{[0,\infty)},\sigma(\mathscr{A}),\tilde{\mathcal{M}}) \to (\tilde{\mathscr{X}}_Q^+,\tilde{\mathcal{B}}^+)$$

is measurable, where $\tilde{\mathcal{B}}^+$ denotes the σ-field generated by cylinder sets. This map induces the image measure $\tilde{\mathcal{P}}_+ = \tilde{\mathcal{M}}\circ \bar{\tilde{Z}}_\cdot^{-1}$ on $(\tilde{\mathscr{X}}_Q^+,\tilde{\mathcal{B}}^+)$. Then the coordinate process $(\tilde{X}_t^+)_{t\geq 0}$ on $(\tilde{\mathscr{X}}_Q^+,\tilde{\mathcal{B}}^+,\tilde{\mathcal{P}}_+)$ satisfies that $\bar{\tilde{Z}}_t \stackrel{\mathrm{d}}{=} \tilde{X}_t^+$ for $t\geq 0$. Let $(y,\xi)\in\mathbb{R}^d\times Q$ and define the regular conditional probability measure on $(\tilde{\mathscr{X}}_Q^+,\tilde{\mathcal{B}}^+)$ by $\tilde{\mathcal{P}}_+^{(y,\xi)}(\cdot) = \tilde{\mathcal{P}}_+(\cdot\,|\,\tilde{X}_0^+ = (y,\xi))$.

Lemma 3.12 *The process $(\tilde{X}_t^+)_{t\geqslant 0}$ is a Markov process on $(\tilde{\mathscr{X}}_Q^+,\tilde{\mathcal{B}}^+,\tilde{\mathcal{P}}_+^{(y,\xi)})$ with respect to the natural filtration $\sigma(\tilde{X}_t^+,\,0\leq s\leq t)$.*

Proof: The proof is the same as that of Lemma 3.4. □

We extend $(\tilde{X}_t^+)_{t\geq 0}$ to a Markov process to the whole real line $\mathbb{R}$. This can be done in the same way as the extension of $(X_t^+)_{t\geq 0}$ to a process on the whole real line as seen in the case of the classical Nelson Hamiltonian in the previous section. Consider the product probability space $(\hat{\tilde{\mathscr{X}}}_Q^+,\hat{\tilde{\mathcal{B}}}^+,\hat{\tilde{\mathcal{P}}}_+^{(y,\xi)})$ with $\hat{\tilde{\mathscr{X}}}_Q^+ = \tilde{\mathscr{X}}_Q^+\times\tilde{\mathscr{X}}_Q^+$, $\hat{\tilde{\mathcal{B}}}^+ = \tilde{\mathcal{B}}^+\otimes\tilde{\mathcal{B}}^+$, and $\hat{\tilde{\mathcal{P}}}_+^{(y,\xi)} = \tilde{\mathcal{P}}_+^{(y,\xi)}\otimes\tilde{\mathcal{P}}_+^{(y,\xi)}$. Let $(\hat{\tilde{X}}_t)_{t\in\mathbb{R}}$ be a stochastic process on the product space, defined by $\hat{\tilde{X}}_t(\omega) = \tilde{X}_t^+(\omega_1)$ for $t\geq 0$, and $\hat{\tilde{X}}_t(\omega) = \tilde{X}_{-t}^+(\omega_2)$ for $t\leq 0$, with $\omega=(\omega_1,\omega_2)\in\hat{\tilde{\mathscr{X}}}_Q^+$.

Lemma 3.13 *It follows that*

1. $\hat{\tilde{X}}_0 = (y,\xi)$ *a.s.*
2. $\hat{\tilde{X}}_t, t\geq 0$ *and* $\hat{\tilde{X}}_s, s<0$ *are independent*
3. $\hat{\tilde{X}}_t \stackrel{\mathrm{d}}{=} \hat{\tilde{X}}_{-t}$ *for all* $t\in\mathbb{R}$
4. $(\hat{\tilde{X}}_t)_{t\geq 0}$ *(resp. $(\hat{\tilde{X}}_t)_{t\leq 0}$) is a Markov process with respect to $\sigma(\hat{\tilde{X}}_s, 0\leq s\leq t)$ (resp. $\sigma(\hat{\tilde{X}}_s, t\leq s\leq 0)$)*
5. *for $f_0,\ldots,f_n\in\tilde{\mathscr{K}}$ and $-t=t_0\leq t_1\leq\ldots\leq t_n=t$, we have*

$$\mathbb{E}_{\hat{\tilde{\mathcal{P}}}_+}\left[\prod_{j=0}^n f_j(\hat{\tilde{X}}_{t_j})\right] = \Big(f_0, e^{-(t_1+t)\tilde{L}_{\mathrm{N}}}f_1 \ldots e^{-(t-t_{n-1})\tilde{L}_{\mathrm{N}}}f_n\Big)_{\tilde{\mathscr{K}}}. \tag{3.21}$$

Proof: The proof is similar to the proof of Lemma 3.5 □

Proof of Theorem 3.7: We show that the random process $(\hat{\tilde{X}}_t)_{t\in\mathbb{R}}$ defined on $(\hat{\tilde{\mathscr{X}}}_Q^+,\hat{\tilde{\mathcal{B}}}^+,\hat{\tilde{\mathcal{P}}}_+^{(y,\xi)})$ is a $P(\phi)_1$-process associated with $((\mathbb{R}^d\times Q,\Sigma',\tilde{\mathrm{M}}),\tilde{L}_{\mathrm{N}})$. The Markov property, reflection symmetry and the shift invariance follow from Lemma 3.13. The càdlàg property of the path $t \mapsto \widehat{X}_t$, $t \geq 0$, (resp. càglàd property of $t \mapsto \widehat{X}_t$, $t \leq 0$) was shown in Lemma 3.11. Thus the map $\hat{\tilde{X}}_\cdot : (\hat{\tilde{\mathscr{X}}}_Q^+, \hat{\tilde{\mathcal{B}}}^+, \hat{\tilde{\mathcal{P}}}_+^{(y,\xi)}) \to (\tilde{\mathscr{X}}, \tilde{\mathcal{B}}_Q)$ is measurable and the image measure $\tilde{\mathcal{P}}^{(y,\xi)} = \hat{\tilde{\mathcal{P}}}_+^{(y,\xi)} \circ \hat{\tilde{X}}_\cdot^{-1}$ defines a probability measure on $(\tilde{\mathscr{X}}, \tilde{\mathcal{B}}_Q)$. Hence the coordinate process $(\tilde{X}_t)_{t\in\mathbb{R}}$ on $(\tilde{\mathscr{X}}, \tilde{\mathcal{B}}_Q, \tilde{\mathcal{P}}^{(y,\xi)})$ satisfies $\tilde{X}_t \stackrel{\mathrm{d}}{=} \hat{\tilde{X}}_t$, is a $P(\phi)_1$-process associated with the pair $\Big((\mathbb{R}^d \times Q, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(Q), d\tilde{\mathrm{M}}_0), \tilde{L}_{\mathrm{N}}\Big)$. □

4 Functional central limit theorems

4.1 Classical Nelson model

Next we discuss FCLT related to the classical and relativistic Nelson models, starting with the classical case. Let

$$M_t = f(X_t) - f(X_0) + \int_0^t L_{\mathrm{N}} f(X_s) ds, \quad t \geqslant 0, \tag{4.1}$$

where $f \in D(L_{\mathrm{N}}) \subset \mathscr{K}$.

Lemma 4.1 *$(M_t)_{t\geqslant 0}$ is a martingale with stationary increments under $\mathcal{P}$.*

Proof: By Lemma 3.4 $(X_t)_{t\geqslant 0}$ is a Markov process with semigroup $T_t = e^{-tL_{\mathrm{N}}}, t \geqslant 0$. Using the Markov property, we have

$$\mathbb{E}_{\mathcal{P}}[f(X_t)|\mathcal{F}_s] = T_{t-s} f(X_s), \quad 0 \leq s \leq t. \tag{4.2}$$

Since the function $t \to T_t$ is differentiable, we obtain

$$\frac{d}{dt} T_t f = -L_{\mathrm{N}} T_t f = -T_t L_{\mathrm{N}} f \quad \text{and} \quad T_t f - f = -\int_0^t L_{\mathrm{N}} T_s f ds, \quad t \geq 0. \tag{4.3}$$

Hence

$$\mathbb{E}_{\mathcal{P}}[M_t|\mathcal{F}_s] = M_s + \mathbb{E}_{\mathcal{P}}\left[f(X_t) - f(X_s) + \int_s^t L_{\mathrm{N}} f(X_r) dr \,|\mathcal{F}_s\right] \quad \text{a.s.} \tag{4.4}$$

Using (4.2)-(4.3) we show that the second term on the right hand side of (4.4) is zero. Indeed,

$$\begin{aligned}
&\mathbb{E}_{\mathcal{P}}\left[f(X_t) - f(X_s) + \int_s^t L_{\mathrm{N}} f(X_r) dr \,|\mathcal{F}_s\right] \\
&= T_{t-s} f(X_s) - f(X_s) + \int_s^t L_{\mathrm{N}} T_{r-s} f(X_s) dr \\
&= T_{t-s} f(X_s) - f(X_s) + \int_0^{t-s} L_{\mathrm{N}} T_r f(X_s) dr \\
&= T_{t-s} f(X_s) - f(X_s) - T_{t-s} f(X_s) + f(X_s) = 0.
\end{aligned}$$

By the shift invariance of the process $(X_t)_{t\geqslant 0}$, it then follows that $(M_t)_{t\geq 0}$ is a martingale under $\mathcal{P}$, and it has stationary increments. □

We begin by proving a CLT for the process $(M_t)_{t\geqslant 0}$ under $\mathcal{P}$. The fundamental tool will be the following martingale central limit theorem [9, Section 5]:

Proposition 4.2 *Let $(N_t)_{t\in\mathbb{R}}$ be a martingale on a probability space $(\Omega, \mathcal{F}, P)$ with*

$$\alpha^2 = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}_P[N_t^2] < \infty,$$

and assume that $(N_t)_{t\in\mathbb{R}}$ has stationary increments. Then

$$\lim_{s\to\infty} \frac{1}{\sqrt{s}} N_{[st]} \stackrel{\mathrm{d}}{=} \alpha^2 B_t.$$

Lemma 4.3 *If $\mathbb{E}_{\mathcal{P}}[f^2(X_t)] < \infty$ and $\mathbb{E}_{\mathcal{P}}[(L_{\mathrm{N}} f)^2(X_t)] < \infty$ for every $t \geq 0$ and $f \neq 0$, then*

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}_{\mathcal{P}}[M_t^2] = 2\, (f, L_{\mathrm{N}} f)_{\mathscr{K}}\,. \tag{4.5}$$

$$(f, L_{\mathrm{N}} f)_{\mathscr{K}} > 0. \tag{4.6}$$

Proof: We have

$$\begin{aligned} \mathbb{E}_{\mathcal{P}}[M_t^2] &= \mathbb{E}_{\mathcal{P}}[f^2(X_t)] + \mathbb{E}_{\mathcal{P}}[f^2(X_0)] - 2\mathbb{E}_{\mathcal{P}}[f(X_0) f(X_t)]] + 2\mathbb{E}_{\mathcal{P}}\left[f(X_t) \int_0^t dr L_{\mathrm{N}} f(X_r) \right] \\ &\quad - 2\mathbb{E}_{\mathcal{P}}\left[f(X_0) \int_0^t dr L_{\mathrm{N}} f(X_r) \right] + \mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r) \right)^2 \right]. \end{aligned} \tag{4.7}$$

Consider

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r) \right)^2 \right].$$

Writing $T_t = e^{-tL_{\mathrm{N}}}$, and using the shift invariance and Markov properties of $(X_t)_{t\geqslant 0}$, we obtain

$$\begin{aligned} &\mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r) \right)^2 \right] \\ &= \mathbb{E}_{\mathcal{P}}\left[\int_0^t ds \int_0^t dr L_{\mathrm{N}} f(X_r) L_{\mathrm{N}} f(X_s) \right] = \int_0^t ds \int_0^t dr \mathbb{E}_{\mathcal{P}}[L_{\mathrm{N}} f(X_0) L_{\mathrm{N}} f(X_{|r-s|})] \\ &= \int_0^t ds \int_0^t dr \mathbb{E}_{\mathcal{P}}\left[L_{\mathrm{N}} f(X_0) \mathbb{E}_{\mathcal{P}}[L_{\mathrm{N}} f(X_{|r-s|}) \,|\, \mathcal{F}_0] \right] = \int_0^t ds \int_0^t dr \left(T_{|s-r|} L_{\mathrm{N}} f, L_{\mathrm{N}} f \right)_{\mathscr{K}}. \end{aligned} \tag{4.8}$$

Hence

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r) \right)^2 \right] = 2\, (f, L_{\mathrm{N}} f)_{\mathscr{K}}\,. \tag{4.9}$$

By Schwarz inequality, we have

$$\begin{aligned}\left|\mathbb{E}_{\mathcal{P}}\left[f(X_t)\int_0^t dr L_{\mathrm{N}} f(X_r)\right]\right| &\leq \left(\mathbb{E}_{\mathcal{P}}\left[f^2(X_t)\right]\right)^{1/2}\left(\mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r)\right)^2\right]\right)^{1/2} \\ &= \left(\mathbb{E}_{\mathcal{P}}\left[f^2(X_0)\right]\right)^{1/2}\left(\mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r)\right)^2\right]\right)^{1/2}.\end{aligned}$$

Thus we obtain

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{P}}\left[f(X_t)\int_0^t dr L_{\mathrm{N}} f(X_r)\right] = 0. \tag{4.10}$$

Moreover, by the same argument, we have

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{P}}\left[f(X_0)\int_0^t dr L_{\mathrm{N}} f(X_r)\right] = 0. \tag{4.11}$$

Furthermore, by Schwarz inequality again,

$$|\mathbb{E}_{\mathcal{P}}[f(X_0)f(X_t)]| \leq \mathbb{E}_{\mathcal{P}}[f^2(X_t)]^{\frac{1}{2}}\mathbb{E}_{\mathcal{P}}[f^2(X_0)]^{\frac{1}{2}} = \mathbb{E}_{\mathcal{P}}[f^2(X_0)]. \tag{4.12}$$

Thus $\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{P}}[f(X_0)f(X_t)] = 0$. Then by (4.7) we conclude that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{P}}[M_t^2] = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r)\right)^2\right] = 2\left(f, L_{\mathrm{N}} f\right)_{\mathscr{K}}. \tag{4.13}$$

Hence (4.5) follows. Next we prove (4.6). By (4.8) we can write

$$\begin{aligned}&\mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r)\right)^2\right] = \int_0^t dr\int_0^t ds\left(T_{|s-r|}L_{\mathrm{N}} f, L_{\mathrm{N}} f\right)_{\mathscr{K}} \\ &= 2\int_{0\leq r\leq s\leq t} drds\left(T_{s-r}L_{\mathrm{N}} f, L_{\mathrm{N}} f\right)_{\mathscr{K}} = 2\int_{0\leq r\leq s\leq t} drds\left(T_r L_{\mathrm{N}} f, L_{\mathrm{N}} f\right)_{\mathscr{K}} \\ &= 2\int_{0\leq r\leq s\leq t} drds\left(T_{\frac{r}{2}}L_{\mathrm{N}} f, T^*_{\frac{r}{2}}L_{\mathrm{N}} f\right)_{\mathscr{K}} = 2\int_0^t dr(t-r)\left(T_{\frac{r}{2}}L_{\mathrm{N}} f, T^*_{\frac{r}{2}}L_{\mathrm{N}} f\right)_{\mathscr{K}} \\ &= 4\int_0^{\frac{t}{2}} ds(t-2s)\left(T_s L_{\mathrm{N}} f, T^*_s L_{\mathrm{N}} f\right)_{\mathscr{K}}.\end{aligned}$$

We have $L_{\mathrm{N}} = L^*_{\mathrm{N}}$ and thus $T_t = T^*_t$ for all $t \geqslant 0$, hence the invariant probability measure $\mathcal{P}$ is reversible. Using now reversibility of $\mathcal{P}$, we obtain

$$2t\int_0^{\frac{t}{4}} ds\|T_s L_{\mathrm{N}} f\|^2 \leq \mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\mathrm{N}} f(X_r)\right)^2\right] \leq 4t\int_0^{\frac{t}{2}} ds\|T_s L_{\mathrm{N}} f\|^2.$$

This implies (4.6). □

Theorem 4.4 (Functional central limit theorem) *Let $(B_t)_{t\geq 0}$ be standard Brownian motion. Under the assumptions of Lemma 4.3 we have*

$$\lim_{s\to\infty}\frac{1}{\sqrt{s}}M_{[st]}\stackrel{\mathrm{d}}{=}\sigma^2 B_t,$$

where $\sigma^2 = 2\left(f\varphi_{\mathrm{g}}, [H_0, f]\varphi_{\mathrm{g}}\right)_{L^2(\mathrm{P})}$.

Proof: By Lemma 4.1 the process $(M_t)_{t\geqslant 0}$ is a martingale with stationary increments under $\mathcal{P}$. Furthermore, by Lemma 4.3 we have that σ^2 is finite, hence by Proposition 4.2 the theorem follows. To determine σ^2 note that $(H-E)f\varphi_{\mathrm{g}} = [H_0, f]\varphi_{\mathrm{g}}$. Thus $\sigma^2 = 2\left(f, L_{\mathrm{N}}f\right)_{\mathscr{K}} = 2\left(f\varphi_{\mathrm{g}}, [H_0, f]\varphi_{\mathrm{g}}\right)_{L^2(\mathrm{P})}$. □

For suitable f, define

$$L_t = \int_0^t L_{\mathrm{N}}f(X_s)ds, \tag{4.14}$$

which is an additive functional of the reversible Markov process. We can obtain a central limit theorem for such additive functionals by using Theorem 4.4 and the fundamental result below, see [14, Theorem 1.8].

Proposition 4.5 (Kipnis-Varadhan) *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mu)$ be a filtered probability space and (A, μ_0) a measurable space, where μ and μ_0 denote probability measures on Ω and A respectively. Let $(Y_t)_{t\geq 0}$ be an A-valued Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$. Assume that $(Y_t)_{t\geq 0}$ is a reversible and ergodic Markov process with respect to μ. Let $F: A\to\mathbb{R}$ be a μ_0 square integrable function with $\int_A F d\mu_0 = 0$. Suppose in addition that F is in the domain of $L^{-1/2}$, where L is the generator of the process $(Y_t)_{t\geq 0}$. Let*

$$R_t = \int_0^t F(Y_s)ds.$$

Then there exists a square integrable martingale $(N_t)_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$, with stationary increments, such that

$$\lim_{t\to\infty}\frac{1}{\sqrt{t}}\sup_{0\leq s\leq t}|R_s - N_s| = 0 \tag{4.15}$$

in probability with respect to μ, where $R_0 = N_0 = 0$. Moreover,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_\mu[|R_t - N_t|^2] = 0. \tag{4.16}$$

Now we show a central limit theorem for the additive functional L_t.

Theorem 4.6 (Functional central limit theorem) *Under the assumptions of Lemma 4.3 the random process $(L_t)_{t\geq 0}$ satisfies a functional central limit theorem relative to $\mathcal{P}$, and the limit variance is given by $\sigma^2 = 2\left(f\varphi_{\mathrm{g}}, [H_0, f]\varphi_{\mathrm{g}}\right)_{L^2(\mathrm{P})}$.*

Proof: By Lemma 3.6, the process $(X_t)_{t\geq 0}$ is a reversible Markov process under $\mathcal{P}$. On the other hand, we see by Proposition 2.10 that the semigroup $(T_t)_{t\geq 0}$ associated to $(X_t)_{t\geq 0}$ is positive, i.e., the process is ergodic. We have

$$\mathbb{E}_{\mathcal{P}}[L_{\mathrm{N}}f(X_t)] = (\varphi_{\mathrm{g}}, (H-E)f\varphi_{\mathrm{g}}) = ((H-E)\varphi_{\mathrm{g}}, f\varphi_{\mathrm{g}}) = 0.$$

Thus $\mathbb{E}_{\mathcal{P}}[L_t] = 0$, and the assumptions of Proposition 4.5 are satisfied. Then $(L_t)_{t\geq 0}$ is also a martingale up to a correction term that disappears in the scaling limit. In fact, by (4.15) there exists an $(\mathcal{F}_t)$-martingale $(N_t)_{t\geq 0}$ such that

$$\lim_{s\to\infty} \frac{1}{\sqrt{s}} \mathbb{E}_{\mathcal{P}}[\sup_{0\leq t\leq s} |N_t - L_t|] = 0,$$

and by (4.1) we have that $N_t = M_t$, and hence

$$\lim_{s\to\infty} \frac{1}{\sqrt{s}} \mathbb{E}_{\mathcal{P}}[\sup_{0\leq t\leq s} |M_t - L_t|] = 0. \tag{4.17}$$

Moreover, by (4.13) we have

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}_{\mathcal{P}}[|M_t - L_t|^2] = 0. \tag{4.18}$$

Finally, by (4.17) the difference $M_t - L_t$ vanishes in the diffusive limit. By (4.13) and since the martingale $(M_t)_{t\geq 0}$ has stationary increments, we conclude by Theorem 4.4 that

$$\lim_{s\to\infty} \frac{1}{\sqrt{s}} M_{st} \stackrel{\mathcal{P}}{=} \lim_{s\to\infty} \frac{1}{\sqrt{s}} L_{st} \stackrel{\mathcal{P}}{=} \sigma^2 B_t, \quad t \geqslant 0.$$

□

4.2 Examples of the variance σ^2

In this section we give some examples of direct interest of the functions $f \in D(L_{\rm N})$, $f : \mathbb{R}^d \times Q \ni (x,\xi) \mapsto f(x,\xi) \in \mathbb{C}$, in the FCLT, which allows to compute the variance σ^2 explicitly. In what follows we assume that $h \in L^2(\mathbb{R}^d)$ is any test function and $\gamma \in \mathbb{R}^d$ any real vector. Moreover, we will denote the vector in $L^2(Q)$ associated with the conjugate momentum $\Pi(h)$ in $\mathscr{F}_{\rm b}$, with the same symbol $\Pi(h)$, i.e., we have

$$[\xi(h), \Pi(h')] = \frac{1}{2}(h, h'). \tag{4.19}$$

Example 4.7 Let $f(x,\xi) = \gamma \cdot x$ (a related example is given in [2]). We have

$$[H_0, (\gamma\cdot x)] = [-\frac{1}{2}\Delta, (\gamma\cdot x)] = -\gamma\cdot\nabla.$$

Then

$$\sigma^2 = 2\left((\gamma\cdot x)\varphi_{\rm g}, (-\gamma\cdot\nabla\varphi_{\rm g})\right) = -2 \sum_{1\leq j,k\leq d} \gamma_j\gamma_k \left(x_j\varphi_{\rm g}, \nabla_k\varphi_{\rm g}\right)$$

Denote $X_{jk} = (x_j\varphi_{\rm g}, \nabla_k\varphi_{\rm g})$. For $j \neq k$ we have

$$X_{jk} = -\left(\nabla_k x_j\varphi_{\rm g}, \varphi_{\rm g}\right) = -\left(x_j\nabla_k\varphi_{\rm g}, \varphi_{\rm g}\right) = -\left(\nabla_k\varphi_{\rm g}, x_j\varphi_{\rm g}\right) = -\bar{X}_{jk},$$

i.e., $\operatorname{Re} X_{jk} = 0$. For $j = k$ we have $\operatorname{Re} X_{jk} = -\frac{1}{2}$ since

$$X_{jk} = -\left(\nabla_k x_k\varphi_{\rm g}, \varphi_{\rm g}\right) = -\left(\varphi_{\rm g}, \varphi_{\rm g}\right) - \left(x_k\nabla_k\varphi_{\rm g}, \varphi_{\rm g}\right) = -\|\varphi_{\rm g}\|^2 - \left(\nabla_k\varphi_{\rm g}, x_k\varphi_{\rm g}\right) = -1 - \bar{X}_{jk}.$$

Hence finally we get $\sigma^2 = |\gamma|^2$, in particular,

$$|\gamma|^2 - 2(\gamma\cdot\nabla\varphi_{\rm g}, (H-E)^{-1}\gamma\cdot\nabla\varphi_{\rm g}) = 0. \tag{4.20}$$

Example 4.8 Let $f(x,\xi)=\xi(h)$. By (2.12), we have

$$[H_0,f]=[H_{\rm f},\xi(h)]=-i\Pi(\omega h).$$

With $X=2\left(\xi(h)\varphi_{\rm g},-i\Pi(\omega h)\varphi_{\rm g}\right)$, we obtain

$$\begin{aligned}
X&=-2i\left(\varphi_{\rm g},\xi(h)\Pi(\omega h)\varphi_{\rm g}\right)\\
&=-2i\left(\varphi_{\rm g},\Pi(\omega h)\xi(h)\varphi_{\rm g}\right)+2i\left(\varphi_{\rm g},\Pi(\omega h)\xi(h)\varphi_{\rm g}\right)\\
&=-2i\left(\Pi^*(\omega h)\varphi_{\rm g},\xi(h)\varphi_{\rm g}\right)+2i\left(\varphi_{\rm g},-i\|\sqrt{\omega}h\|^2\varphi_{\rm g}\right)\\
&=-2i\left(-\Pi(\omega h)\varphi_{\rm g},\xi(h)\varphi_{\rm g}\right)+2\|\varphi_{\rm g}\|^2_{L^2({\rm P})}\|\sqrt{\omega}h\|^2=2\|\sqrt{\omega}h\|^2-\overline{X},
\end{aligned}$$

hence

$$\sigma^2=\operatorname{Re}X=\|\sqrt{\omega}h\|^2.$$

Example 4.9 Let $f(x,\xi)=(\gamma\cdot x)\xi(h)$. We have

$$\begin{aligned}
[H_0,(\gamma\cdot x)\xi(h)]&=[-\frac{1}{2}\Delta,(\gamma\cdot x)]\xi(h)+(\gamma\cdot x)[H_{\rm f},\xi(h)]\\
&=-\gamma\cdot\nabla\xi(h)-i(\gamma\cdot x)\Pi(\omega h).
\end{aligned}$$

Then

$$\begin{aligned}
\sigma^2&=2\left((\gamma\cdot x)\xi(h)\varphi_{\rm g},\xi(h)(-\gamma\cdot\nabla\varphi_{\rm g})\right)+2\left((\gamma\cdot x)\xi(h)\varphi_{\rm g},-i(\gamma\cdot x)\Pi(\omega h)\varphi_{\rm g}\right) \qquad (4.21)\\
&=-2\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(x_j\xi(h)\varphi_{\rm g},\xi(h)\nabla_k\varphi_{\rm g}\right)+2\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(x_j\xi(h)\varphi_{\rm g},-i\Pi(\omega h)x_k\varphi_{\rm g}\right).
\end{aligned}$$

Denote again $X_{jk}=(x_j\xi(h)\varphi_{\rm g},\xi(h)\nabla_k\varphi_{\rm g})$. For $j\neq k$

$$\begin{aligned}
X_{jk}&=-\left(\xi(h)\nabla_k x_j\varphi_{\rm g},\xi(h)\varphi_{\rm g}\right)=-\left(\xi(h)x_j\nabla_k\varphi_{\rm g},\xi(h)\varphi_{\rm g}\right)\\
&=-\left(\xi(h)\nabla_k\varphi_{\rm g},x_j\xi(h)\varphi_{\rm g}\right)=-\bar{X}_{jk}.
\end{aligned}$$

For the diagonal part we have

$$\begin{aligned}
X_{kk}&=-\left(\xi(h)\nabla_k x_k\varphi_{\rm g},\xi(h)\varphi_{\rm g}\right)\\
&=-\left(\xi(h)\varphi_{\rm g},\xi(h)\varphi_{\rm g}\right)-\left(\xi(h)x_k\nabla_k\varphi_{\rm g},\xi(h)\varphi_{\rm g}\right)\\
&=-\left(\xi(h)\varphi_{\rm g},\xi(h)\varphi_{\rm g}\right)-\left(\xi(h)\nabla_k\varphi_{\rm g},x_k\xi(h)\varphi_{\rm g}\right)=-\|\xi(h)\varphi_{\rm g}\|^2-\bar{X}_{kk},
\end{aligned}$$

i.e., $\operatorname{Re}X_{kk}=-\frac{1}{2}\|\xi(h)\varphi_{\rm g}\|^2$. To determine the second term in (4.21), write now $X_{jk}=(x_j\xi(h)\varphi_{\rm g},-i\Pi(\omega h)x_k\varphi_{\rm g})$. We have

$$\begin{aligned}
X_{jk}&=-i\left(x_j\varphi_{\rm g},x_k\xi(h)\Pi(\omega h)\varphi_{\rm g}\right)\\
&=-i\left(x_j\varphi_{\rm g},x_k\Pi(\omega h)\xi(h)\varphi_{\rm g}\right)+i\left(x_j\varphi_{\rm g},x_k\Pi(\omega h)\xi(h)\times\varphi_{\rm g}\right)\\
&=-i\left(x_k\Pi^*(\omega h)\varphi_{\rm g},x_j\xi(h)\varphi_{\rm g}\right)+\left(x_j\varphi_{\rm g},\|\sqrt{\omega}h\|^2x_k\varphi_{\rm g}\right)=-\bar{X}_{jk}+\left(x_j\varphi_{\rm g},x_k\varphi_{\rm g}\right)\|\sqrt{\omega}h\|^2.
\end{aligned}$$

Hence we finally obtain

$$\sigma^2=|\gamma|^2\|\xi(h)\varphi_{\rm g}\|^2+2\|(\gamma\cdot x)\varphi_{\rm g}\|^2\|\sqrt{\omega}h\|^2.$$

Example 4.10 Let $f(x,\xi)=e^{i\xi(h)}$. We have

$$\begin{aligned}e^{-i\xi(h)}H_{\rm f}e^{i\xi(h)}&=H_{\rm f}+i[H_{\rm f},\xi(h)]+\frac{1}{2}[[H_{\rm f},i\xi(h)],i\xi(h)]\\&=H_{\rm f}+\Pi(\omega h)+\frac{1}{2}\|\sqrt{\omega}h\|^2.\end{aligned}$$

Thus

$$H_{\rm f}e^{i\xi(h)}=e^{i\xi(h)}H_{\rm f}+e^{i\xi(h)}\Pi(\omega h)+\frac{1}{2}\|\sqrt{\omega}h\|^2e^{i\xi(h)}$$

and

$$[H_0,e^{i\xi(h)}]=e^{i\xi(h)}\Pi(\omega h)+\frac{1}{2}\|\sqrt{\omega}h\|^2e^{i\xi(h)}$$

follow. This gives

$$\sigma^2=2\left(e^{i\xi(h)}\varphi_{\rm g},[H_0,e^{i\xi(h)}]\varphi_{\rm g}\right)=2\left(\varphi_{\rm g},\Pi(\omega h)\varphi_{\rm g}\right)+\|\sqrt{\omega}h\|^2.$$

Example 4.11 Let $f(x,\xi)=(\gamma\cdot x)e^{i\xi(h)}$. We have

$$\begin{aligned}[H_0,(\gamma\cdot x)e^{i\xi(h)}]&=[-\frac{1}{2}\Delta+H_{\rm f},(\gamma\cdot x)e^{i\xi(h)}]\\&=[-\frac{1}{2}\Delta,(\gamma\cdot x)]e^{i\xi(h)}+(\gamma\cdot x)[H_{\rm f},e^{i\xi(h)}]\\&=-\gamma\cdot\nabla e^{i\xi(h)}+(\gamma\cdot x)e^{i\xi(h)}\left(\Pi(\omega h)+\frac{1}{2}\|\sqrt{\omega}h\|^2\right).\end{aligned}$$

Then

$$\begin{aligned}\sigma^2&=2\left((\gamma\cdot x)e^{i\xi(h)}\varphi_{\rm g},[H_0,(\gamma\cdot x)e^{i\xi(h)}]\varphi_{\rm g}\right)\\&=2\left((\gamma\cdot x)e^{i\xi(h)}\varphi_{\rm g},-\gamma\cdot\nabla\varphi_ge^{i\xi(h)}+(\gamma\cdot x)e^{i\xi(h)}\left(\Pi(\omega h)+\frac{1}{2}\|\sqrt{\omega}h\|^2\right)\varphi_{\rm g}\right)\\&=2\left((\gamma\cdot x)\varphi_{\rm g},-\gamma\cdot\nabla\varphi_{\rm g}\right)+2\left((\gamma\cdot x)\varphi_{\rm g},(\gamma\cdot x)\Pi(\omega h)\varphi_{\rm g}\right)+\|(\gamma\cdot x)\varphi_{\rm g}\|\|\sqrt{\omega}h\|^2 \qquad (4.22)\\&=\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(x_j\varphi_{\rm g},\nabla_k\varphi_{\rm g}\right)+2\left((\gamma\cdot x)\varphi_{\rm g},(\gamma\cdot x)\Pi(\omega h)\varphi_{\rm g}\right)+\|(\gamma\cdot x)\varphi_{\rm g}\|^2\|\sqrt{\omega}h\|^2.\end{aligned}$$

To get the first term, denote $X_{jk}=-\left(x_j\varphi_{\rm g},\nabla_k\varphi_{\rm g}\right)$. For the off-diagonal part we have

$$X_{jk}=(\nabla_kx_j\varphi_{\rm g},\varphi_{\rm g})=(x_j\nabla_k\varphi_{\rm g},\varphi_{\rm g})=(\nabla_k\varphi_{\rm g},x_j\varphi_{\rm g})=-\bar{X}_{jk},$$

and the diagonal part gives $X_{kk}=(\nabla_kx_k\varphi_{\rm g},\varphi_{\rm g})=1+(\nabla_k\varphi_{\rm g},x_k\varphi_{\rm g})$. Hence in total

$$\sigma^2=|\gamma|^2+\|(\gamma\cdot x)\varphi_{\rm g}\|^2\|\sqrt{\omega}h\|^2+2\left((\gamma\cdot x)\varphi_{\rm g},\Pi(\omega h)(\gamma\cdot x)\varphi_{\rm g}\right).$$

Example 4.12 Let $f(x,\xi)=e^{i(\gamma\cdot x)+i\xi(h)}$. We have

$$\begin{aligned}&[H_0,e^{i(\gamma\cdot x)+i\xi(h)}]=[-\frac{1}{2}\Delta+H_{\rm f},e^{i(\gamma\cdot x)+i\xi(h)}]=[-\frac{1}{2}\Delta,e^{i(\gamma\cdot x)}]e^{i\xi(h)}+[H_{\rm f},e^{i\xi(h)}]e^{i(\gamma\cdot x)}\\&=e^{i(\gamma\cdot x)+i\xi(h)}(\frac{1}{2}|\gamma|^2-i\gamma\cdot\nabla)+e^{i(\gamma\cdot x)+i\xi(h)}(\Pi(\omega h)+\frac{1}{2}\|\sqrt{\omega}h\|^2)\end{aligned}$$

Thus

$$\begin{aligned}\sigma^2 &= 2\left(e^{i(\gamma\cdot x)+i\xi(h)}\varphi_{\mathrm{g}}, [H_0, e^{i(\gamma\cdot x)+i\xi(h)}]\varphi_{\mathrm{g}}\right)\\ &= |\gamma|^2 - 2\left(\varphi_{\mathrm{g}}, i\gamma\cdot\nabla\varphi_{\mathrm{g}}\right) + 2\left(\varphi_{\mathrm{g}}, \Pi(\omega h)\varphi_{\mathrm{g}}\right) + \|\sqrt{\omega}h\|^2.\end{aligned}$$

Let $X = (\varphi_{\mathrm{g}}, i\gamma\cdot\nabla\varphi_{\mathrm{g}})$. Note that $\varphi_{\mathrm{g}} > 0$. Since $X \in \mathbb{R}$ and $-iX \in \mathbb{R}$, we have $X = 0$ and thus

$$\sigma^2 = |\gamma|^2 + 2\left(\varphi_{\mathrm{g}}, \Pi(\omega h)\varphi_{\mathrm{g}}\right) + \|\sqrt{\omega}h\|^2.$$

4.3 Relativistic Nelson model

The previous constructions can be extended to the relativistic case. Let

$$\tilde{M}_t = f(\tilde{X}_t) - f(\tilde{X}_0) + \int_0^t \tilde{L}_{\mathrm{N}} f(\tilde{X}_s)ds, \quad t \geqslant 0. \tag{4.23}$$

Theorem 4.13 *$(\tilde{M}_t)_{t\geqslant 0}$ is a martingale with stationary increments under $\tilde{\mathcal{P}}$, and*

$$\lim_{s\to\infty}\frac{1}{\sqrt{s}}\tilde{M}_{st} \stackrel{\mathrm{d}}{=} \tilde{\sigma}^2 B_t,$$

where $\tilde{\sigma}^2 = 2\left(f\varphi_{\mathrm{g}}, [\tilde{H}_0, f]\varphi_{\mathrm{g}}\right)$.

Proof: The proof is an analogue of Lemma 4.1 and Theorem 4.4. □

We conclude by some explicit cases of variances $\tilde{\sigma}^2$.

Example 4.14 Let $g(x,\xi) = \gamma\cdot x$. We have

$$[\tilde{H}_0, (\gamma\cdot x)] = [\sqrt{-\Delta+m^2}, (\gamma\cdot x)] = \frac{-(\gamma\cdot\nabla)}{\sqrt{-\Delta+m^2}}.$$

Then

$$\begin{aligned}\tilde{\sigma}^2 &= 2\left((\gamma\cdot x)\tilde{\varphi}_{\mathrm{g}}, \frac{-(\gamma\cdot\nabla)}{\sqrt{-\Delta+m^2}}\tilde{\varphi}_{\mathrm{g}}\right)\\ &= -2\sum_{1\leq j,k\leq d}\gamma_j\gamma_k\left(\frac{-\nabla_k}{\sqrt{-\Delta+m^2}}x_j\tilde{\varphi}_{\mathrm{g}}, \tilde{\varphi}_{\mathrm{g}}\right)\\ &= -2\sum_{1\leq j,k\leq d}\gamma_j\gamma_k\left(\tilde{\varphi}_{\mathrm{g}}, [x_j, \frac{\nabla_k}{\sqrt{-\Delta+m^2}}]\tilde{\varphi}_{\mathrm{g}}\right) - 2\left(\frac{-\nabla_k}{\sqrt{-\Delta+m^2}}\tilde{\varphi}_{\mathrm{g}}, x_j\tilde{\varphi}_{\mathrm{g}}\right)\\ &= -2\sum_{1\leq j,k\leq d}\gamma_j\gamma_k\left(\tilde{\varphi}_{\mathrm{g}}, [x_j, \frac{\nabla_k}{\sqrt{-\Delta+m^2}}]\tilde{\varphi}_{\mathrm{g}}\right) - \overline{\tilde{\sigma}^2}.\end{aligned} \tag{4.24}$$

Moreover, we have

$$\begin{aligned}[x_j, \frac{\nabla_k}{\sqrt{-\Delta+m^2}}] &= [x_j, \nabla_k]\frac{1}{\sqrt{-\Delta+m^2}} + \nabla_k[x_j, \frac{1}{\sqrt{-\Delta+m^2}}] \\ &= -\frac{\delta_{jk}}{\sqrt{-\Delta+m^2}} - \frac{\nabla_k\nabla_j}{(-\Delta+m^2)^{\frac{3}{2}}}.\end{aligned}$$

By (4.24), we obtain

$$\tilde{\sigma}^2 = 2\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\tilde{\varphi}_{\mathrm{g}}, \frac{1}{\sqrt{-\Delta+m^2}}\tilde{\varphi}_{\mathrm{g}}\right)\delta_{jk} + 2\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\tilde{\varphi}_{\mathrm{g}}, \frac{\nabla_k\nabla_j}{(-\Delta+m^2)^{\frac{3}{2}}}\tilde{\varphi}_{\mathrm{g}}\right) - \overline{\tilde{\sigma}^2}.$$

Hence we conclude that

$$\tilde{\sigma}^2 = \sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\tilde{\varphi}_{\mathrm{g}}, \frac{1}{\sqrt{-\Delta+m^2}}\tilde{\varphi}_{\mathrm{g}}\right)\delta_{jk} + \sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\tilde{\varphi}_{\mathrm{g}}, \frac{\nabla_k\nabla_j}{(-\Delta+m^2)^{\frac{3}{2}}}\tilde{\varphi}_{\mathrm{g}}\right).$$

Example 4.15 Let $g(x,\xi) = (\gamma\cdot x)\xi(h)$. We have

$$[\tilde{H}_0, (\gamma\cdot x)\xi(h)] = [\sqrt{-\Delta+m^2}, (\gamma\cdot x)]\xi(h) + (\gamma\cdot x)[H_{\mathrm{f}}, \xi(h)].$$

Thus

$$\begin{aligned}\tilde{\sigma}^2 &= 2\left((\gamma\cdot x)\xi(h)\tilde{\varphi}_{\mathrm{g}}, [\sqrt{-\Delta+m^2}, (\gamma\cdot x)]\xi(h)\tilde{\varphi}_{\mathrm{g}}\right) + 2\left((\gamma\cdot x)\xi(h)\tilde{\varphi}_{\mathrm{g}}, [H_{\mathrm{f}}, \xi(h)](\gamma\cdot x)\tilde{\varphi}_{\mathrm{g}}\right) \\ &= \sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\xi(h)\tilde{\varphi}_{\mathrm{g}}, \frac{1}{\sqrt{-\Delta+m^2}}\xi(h)\tilde{\varphi}_{\mathrm{g}}\right)\delta_{jk} + \sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\xi(h)\tilde{\varphi}_{\mathrm{g}}, \frac{\nabla_k\nabla_j}{(-\Delta+m^2)^{\frac{3}{2}}}\xi(h)\tilde{\varphi}_{\mathrm{g}}\right) \\ &\qquad + 2\|(\gamma\cdot x)\tilde{\varphi}_{\mathrm{g}}\|\|\sqrt{\omega}h\|^2.\end{aligned}$$

Example 4.16 Let $g(x,\xi) = (\gamma\cdot x)e^{i\xi(h)}$. We have

$$[\tilde{H}_0, (\gamma\cdot x)e^{i\xi(h)}] = [\sqrt{-\Delta+m^2}, (\gamma\cdot x)]e^{i\xi(h)} + (\gamma\cdot x)[H_{\mathrm{f}}, e^{i\xi(h)}].$$

Similarly, we obtain

$$\begin{aligned}\tilde{\sigma}^2 &= 2\left((\gamma\cdot x)e^{i\xi(h)}\tilde{\varphi}_{\mathrm{g}}, [\sqrt{-\Delta+m^2}, (\gamma\cdot x)]e^{i\xi(h)}\tilde{\varphi}_{\mathrm{g}}\right) + 2\left((\gamma\cdot x)e^{i\xi(h)}\tilde{\varphi}_{\mathrm{g}}, [H_{\mathrm{f}}, e^{i\xi(h)}](\gamma\cdot x)\tilde{\varphi}_{\mathrm{g}}\right) \\ &= \sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\tilde{\varphi}_{\mathrm{g}}, \frac{1}{\sqrt{-\Delta+m^2}}\tilde{\varphi}_{\mathrm{g}}\right)\delta_{jk} + \sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\tilde{\varphi}_{\mathrm{g}}, \frac{\nabla_k\nabla_j}{(-\Delta+m^2)^{\frac{3}{2}}}\tilde{\varphi}_{\mathrm{g}}\right) \\ &\qquad + \|(\gamma\cdot x)\tilde{\varphi}_{\mathrm{g}}\|^2\|\sqrt{\omega}h\|^2 + 2\left((\gamma\cdot x)\tilde{\varphi}_{\mathrm{g}}, \Pi(\omega h)(\gamma\cdot x)\tilde{\varphi}_{\mathrm{g}}\right).\end{aligned}$$

Example 4.17 Let $g(x,\xi) = e^{i(\gamma\cdot x)+i\xi(h)}$. We have

$$[\tilde{H}_0, e^{i\gamma\cdot x+i\xi(h)}] = [\sqrt{-\Delta+m^2}, e^{i\gamma\cdot x}]e^{i\xi(h)} + e^{\gamma\cdot x}[H_{\mathrm{f}}, e^{i\xi(h)}].$$

Since $e^{-i\gamma\cdot x}\sqrt{-\Delta+m^2}e^{i\gamma\cdot x}=\sqrt{-(\nabla-i\gamma)^2+m^2}$, we obtain

$$[\sqrt{-\Delta+m^2},e^{i\gamma\cdot x}]=e^{i\gamma\cdot x}\sqrt{-(\nabla-i\gamma)^2+m^2}-e^{i\gamma\cdot x}\sqrt{-\Delta+m^2}.$$

Finally, we deduce that

$$\begin{aligned}\tilde{\sigma}^2&=2\left(\tilde{\varphi}_{\rm g},\sqrt{-(\nabla-i\gamma)^2+m^2}\tilde{\varphi}_{\rm g}\right)-2\left(\tilde{\varphi}_{\rm g},\sqrt{-\Delta+m^2}\tilde{\varphi}_{\rm g}\right)+2\left(\tilde{\varphi}_{\rm g},\Pi(\omega h)\tilde{\varphi}_{\rm g}\right)+\|\sqrt{\omega}h\|^2\\&=2\left(\tilde{\varphi}_{\rm g},\sqrt{(-i\nabla-\gamma)^2+m^2}\tilde{\varphi}_{\rm g}\right)-2\left(\tilde{\varphi}_{\rm g},\sqrt{-\Delta+m^2}\tilde{\varphi}_{\rm g}\right)+2\left(\tilde{\varphi}_{\rm g},\Pi(\omega h)\tilde{\varphi}_{\rm g}\right)+\|\sqrt{\omega}h\|^2.\end{aligned}$$

5 Concluding remarks

Although in this paper we focused on the Nelson model, $P(\phi)_1$ processes and an FCLT can further be constructed also for related models. We briefly mention two cases.

Nelson model with fixed total momentum P. Let $V=0$. Then $H_{\rm N}$ is translation invariant, i.e., $[H_{\rm N},T_{\rm tot}]=0$, where $T_{\rm tot}=p\otimes\mathbb{1}+\mathbb{1}\otimes T_{\rm f}$ denotes the total momentum and $T_{{\rm f}\mu}=d\Gamma(k_\mu)$. Thus $H_{\rm N}$ can be decomposed as

$$H_{\rm N}=\int_{\mathbb{R}^d}^{\oplus}H_{\rm N}(P)dP,$$

where

$$H_{\rm N}(P)=\frac{1}{2}(P-T_{\rm f})^2+\phi(0)+H_{\rm f}$$

is a self-adjoint operator in $\mathscr{F}_{\rm b}$, called Nelson Hamiltonian with total momentum $P\in\mathbb{R}^d$. It is known that for sufficiently small $|P|$ the operator $H_{\rm N}(P)$ has a ground state [7].

Pauli-Fierz model. The Pauli-Fierz Hamiltonian in non-relativistic quantum electrodynamics is defined by

$$H_{\rm PF}=\frac{1}{2m}(-i\nabla\otimes\mathbb{1}+\sqrt{\alpha}A)^2+V\otimes\mathbb{1}+\mathbb{1}\otimes H_{\rm f},$$

where A denotes the quantized radiation filed given by

$$A_\mu(x)=\frac{1}{\sqrt{2}}\sum_{j=1,2}\int_{\mathbb{R}^d}\left(\frac{\widehat{\varphi}(k)}{\sqrt{\omega(k)}}e_\mu(k,j)e^{ikx}a^*(k,j)+\frac{\widehat{\varphi}(-k)}{\sqrt{\omega(k)}}e_\mu(k,j)e^{-ikx}a(k,j)\right)dk.$$

Here $e(k,1)$ and $e(k,2)$ denote polarisation vectors such that $k\cdot e(k,j)=0$ for $j=1,2$, and $[a(k,j),a^*(k',j')]=\delta_{jj'}\delta(k-k')$ is satisfied. Then $H_{\rm PF}$ is a self-adjoint operator in $L^2(\mathbb{R}^3)\otimes\mathscr{F}_{\rm b}(L^2(\mathbb{R}^3\times\{1,2\}))$. The existence of the ground state is studied in [1, 8, 10].

Acknowledgements: FH is financially supported by JSPS Open Partnership Joint Projects between Japan and Tunisia “Non-commutative infinite dimensional harmonic analysis: A unified approach from representation theory and probability theory”, and is also supported by Grant-in-Aid for Science Research (B)16H03942 from JSPS. JL thanks IHES, Bures-sur-Yvette, for a visiting fellowship, where part of this paper has been written.

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