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1 **M-PERT. A manual project duration estimation technique**
2 **for teaching scheduling basics**

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6 **ABSTRACT**

7 The Program Evaluation and Review Technique (PERT) has become a classic Project
8 Management tool for estimating project duration when the activities have uncertain durations.
9 However, despite its simplicity and widespread adoption, the original PERT, in neglecting
10 the merge event bias, significantly underestimated the duration average and overestimated the
11 duration variance of real-life projects. To avoid these and other shortcomings, many authors
12 have worked over the last 60 years at producing interesting alternative PERT extensions. This
13 paper proposes joining the most relevant of those to create a new reformulated PERT, named
14 M-PERT.

15 M-PERT is quite accurate when estimating real project duration, while also allowing for a
16 number of interesting network modelling features the original PERT lacked: probabilistic
17 alternative paths, activity self-loops, minima of activity sets and correlation between
18 activities. However, unlike similar scheduling methods, M-PERT allows manual calculation
19 through a recursive merging procedure that downsizes the network until the last standing
20 activity represents the whole (or remaining) project duration. Hence, M-PERT constitutes an
21 attractive tool for teaching scheduling basics to engineering students in a more intuitive way,
22 with or without the assistance of computer-based simulations or software. One full case study
23 will also be proposed and future research paths suggested.

24 **KEYWORDS**

25 Scheduling, PERT; GERT; Stochastic Network Analysis; Project duration.

26 **Introduction**

27 In 1959, an Operations Research team formed by D.G. Malcom, J.H. Roseboom and C.E.
28 Clark from the company *Booz, Allen and Hamilton* in Chicago, along with W. Fazar, from
29 the *US Navy Special Projects Office* in Washington, published a technique for measuring and
30 controlling the progress of the Polaris Fleet Ballistic Missile (FBM) program (Malcolm et al.
31 1959). This technique was initially named the *Program Evaluation Research Task* and later
32 renamed the *Program Evaluation and Review Technique* (PERT). It has become one of the
33 most popular Project Management (PM) scheduling tools since then.

34 The initial idea was to provide the project manager with an integrated and quantitative
35 management methodology that allowed him/her to evaluate ‘(a) [the] progress to date and
36 [the] outlook for accomplishing the objectives of the FBM program, (b) [the] validity of
37 established plans and schedules for accomplishing the program objectives, and (c) [the]
38 effects of changes proposed in established plans’ (Malcolm et al., 1959, p.646). These three
39 aims were quite ambitious indeed, and despite one imagining PERT doing all of these with a
40 leap of faith, what PERT actually and basically does is to estimate the probabilistic duration
41 of any project or the remaining part of a project when the activities have uncertain (not
42 deterministic) durations. Also, PERT can be implemented for cost estimation purposes with
43 some technical variations (e.g. (Asmar et al. 2011)); however, for the sake of clarity we will
44 just deal with the time dimension in this paper.

45 In its favor, this technique is very easy to implement and was the first of its kind for
46 dealing with projects whose activity durations are modeled by probability distributions, or
47 Stochastic Network Analysis (SNA) as it known as nowadays. Against it, the procedure
48 proposed by PERT underestimates the project duration when there are multiple parallel paths,
49 which unfortunately is the norm in construction projects. Indeed, the authors were aware of
50 this shortcoming since they very briefly mentioned in the original paper that ‘this

51 simplification gives biased estimates such that the estimated expected time of events are
52 always too small' (Malcolm et al. 1959). Two publications by C.E. Clark indeed followed the
53 publication of PERT (Clark 1961, 1962), but the solution proposed was not easy to
54 implement, so it opened interesting avenues of research for the scientific community that
55 have continued to this day, mostly in the Operations Research discipline.

56 Since the inception of PERT almost 60 years ago, the number of fields in which this
57 technique has been applied has been quite varied. It is not only applied to military, research &
58 development and civil engineering projects as was originally intended (Stauber et al. 1959),
59 but also to fields as diverse as medicine (e.g. Woolf et al. (1968)), exports (e.g. Tatterson
60 (1974)), contracting (e.g. Mummolo (1997)) and rural development (e.g. Tavares (2002)), to
61 cite just a few. In fact, its reach and application are virtually unlimited since it is useful for
62 any project duration estimation purposes where activities have non-deterministic durations.

63 However, another collateral problem has been that despite PERT being a celebrated PM
64 technique, it also probably is one of the most widely misunderstood. In particular, many
65 engineering and scheduling students, as well as professionals, still think that PERT is either a
66 type of network representation, as one of the first versions of the Project Management Body
67 of Knowledge (PMBok) pointed out explicitly (PMI 1996), or that it is just a three-point time
68 and/or cost estimation technique (PMI 2008). The latter misrepresentation is the consequence
69 of the second half of the PERT method, which deals with calculating the expected project
70 duration and its variance by means of critical path project activities, being left out of this
71 publication.

72 The purpose of this paper is to update PERT and propose a newly redefined technique
73 named M-PERT, which deals with the most relevant shortcoming of the original technique
74 (the merge event bias), allows manual calculation, and adds a series of new interesting
75 features that the original technique lacked. M-PERT will allow the modelling of real-life

76 projects far more representatively and help scheduling students to understand more intuitively
77 basic concepts of scheduling when activities have uncertain durations.

78 The number of PERT-related publications might be counted in the thousands nowadays,
79 so we cannot reference them all. Instead, as in any other mature fields, we will undertake a
80 systematic review of the most relevant sources that have shaped this new technique. This
81 systematic review will be presented in the *Background* section under three thematic
82 subsections. Then, an outline of the model will be presented followed by an application in the
83 *M-PERT outline* and *Case study* sections. A *Results* section will gather the application
84 outcomes while comparing the accuracy improvements achieved, whereas a final and brief
85 *Discussions and conclusions* section will summarize the most important aspects and possible
86 research directions for the model proposed.

87

88 **Background**

89 An overview of PERT and its limitations

90 In a nutshell, PERT is a scheduling tool that is applied in two stages. First, after
91 identifying all the activities involved in a project and stating their precedence relationships,
92 the first stage comprises modelling their probable durations. For that purpose, the original
93 PERT authors proposed a three-point estimation procedure. The user (or another expert on
94 hand) is asked to state the Optimistic (O), most Likely (L) and Pessimistic (P) durations that
95 each activity is foreseen to have before those activities take place. Then, the mean duration
96 (μ) and the standard deviation (σ) from each activity are calculated by using these three
97 duration estimates (O , L and P) by means of these straightforward expressions:

$$98 \qquad \qquad \qquad = (O + 4L + P)/6 \qquad \qquad \qquad (1)$$

$$99 \qquad \qquad \qquad = (P - O)/6 \qquad \qquad \qquad (2)$$

100 With the values of μ and σ for each activity, a Beta distribution which varies between
 101 $[O, P]$ is fitted for modelling the activity duration. However, despite the highly controversial
 102 and still ongoing debate about the accuracy of both the three-point estimate procedure and the
 103 choice of the Beta distribution to model the activity durations, the PERT method is only
 104 interested in the μ and σ values of each activity for the second stage of the application. This
 105 second stage of PERT (not mentioned by any Project Management standard) comprises: (a)
 106 identifying the activities that are in the critical path, and (b) assuming that the Project
 107 duration mean (μ_p) and the standard deviation (σ_p) are equal:

$$108 \quad \mu_p = \sum_{i \text{ critical path}} \mu_i \quad (3)$$

$$109 \quad \sigma_p = \sqrt{\sum_{i \text{ critical path}} \sigma_i^2} \quad (4)$$

110 Now, if the project scheduler wants to know what the probabilities of ending a project in
 111 X days are, he/she just needs to know that the project total duration is supposed to be
 112 following a Normal distribution with mean μ_p and standard deviation σ_p . Then, the specific
 113 probability of X happening can be calculated by means of either the mathematical expressions
 114 of the cumulative Normal distribution, by resorting to a spreadsheet or, when the calculations
 115 are made by hand, by standardizing the project duration as in Equation 5 and then looking up
 116 the cumulative probability value of z in a standard Normal table.

$$117 \quad z = \frac{(X - \mu_p)}{\sigma_p} \quad (5)$$

118 This is the essence of the PERT method. The logical lack of trust displayed by the
 119 research community since PERT was published has manifested itself as a determination to
 120 see what kind of errors are involved by implementing it. In this regard, MacCrimmon and
 121 Ryavec (1964) broke down PERT into every possible piece and developed the most thorough
 122 analytical review that anyone has made of PERT since its publication. This piece of research

123 was actually outsourced by the US Air Force to the Rand Corporation due to the consistent
124 and systematic errors that PERT was evidencing in their project forecasts, and they divided
125 their memorandum into two main sections. In the first section, they studied the activity
126 duration-level errors, that is, '(1) the assumption of the Beta distribution, (2) imprecise time
127 estimates, and (3) the assumption of the standard deviation (one-sixth of the range) and the
128 approximation formula for calculating the mean time' (MacCrimmon and Ryavec 1964). The
129 second section checked the accuracy of the project duration mean, variance and the (Normal)
130 probability statements.

131 The results summary from the first section was that using the three-point estimates could
132 cause a range estimation error of between 10% and 20% of the range $[O, P]$, but, despite the
133 error of estimation of the mean (μ) and standard deviation (σ) being likely to be bigger (from
134 15% to 30%), a degree of cancellation was expected to occur depending on the network.
135 Indeed, there is generally a high degree of cancellation as will be proven later. But the other
136 noteworthy statement that the authors made at that time was that, to a certain extent, it was
137 pointless having a discussion about what the right kind of distribution is – Beta or another –
138 for modelling an activity duration. Indeed, they reinforced this idea by saying that if PERT
139 had used triangular distributions instead of Beta distributions, their analysis would have given
140 almost identical results (MacCrimmon and Ryavec 1964). A recent study only focusing on
141 determining the importance of the specific distribution selection in PERT also supported this
142 claim and stated that the maximum deviation by using alternative distributions in the project
143 duration an estimation is generally well below 10% (Hajdu and Bokor 2014).

144 Nevertheless, the results from the second section of the PERT model analysis evidenced
145 the real problem. They confirmed that unless there is one clear dominant critical path (a chain
146 of activities whose total duration is significantly longer than the second longest path), the
147 PERT-calculated mean (equation 3) will be biased optimistically (underestimating the real

148 project duration), while the PERT-calculated variance (equation 4) will be biased in the other
149 direction (overestimating the actual duration variance). At the time this study was published,
150 Extreme Value theory had not been properly developed, but the authors clearly identified that
151 the higher the number of parallel paths, the more positively skewed the distribution modelling
152 the project duration would become, making the Normally-distributed assumption of the
153 ‘total’ project duration of questionable validity with multiple parallel paths.

154

155 What matters and what does not matter in PERT

156 The first part of the systematic review by MacCrimmon and Ryavec (1964) focused on
157 the activity level duration modelling. They stated that the particular choice of the distribution
158 was hardly relevant. Instead, it was the distribution (duration) mean and variance that really
159 matters. This was an important outcome that, apparently, a big part of the research
160 community had tried to ignore due to the extraordinary amount of research focused on
161 improving the particular statistical distribution for better modelling of the activity duration.

162 In a similar vein, a recent study by Herrerías-Velasco et al. (2011), seeking to close the
163 unending debate about whether the approximations originally proposed by the PERT authors
164 to estimate whether the activity duration mean and variance were accurate enough, concluded
165 that, in general, the expression of the mean (Equation 1) is very accurate, whereas the
166 expression for the variance (Equation 2) requires to be multiplied by a correction factor K
167 whose expression is:

$$168 \quad K = \sqrt{\frac{5}{7} + \frac{16}{7} \frac{(L - O)(P - L)}{(P - O)^2}} \quad (6)$$

169 Concerning the (un)importance of the specific distribution chosen for modelling the
170 activity duration, Figure 1 is divided into two vertical blocks. The top half shows how the

171 statistical distribution resulting from one, two and three activities in series evolves when the
172 activity duration is modeled by a Uniform distribution (first row) or a Beta (PERT-like)
173 distribution (second row). The case with just one activity (first column) illustrates indeed the
174 original distribution, but when more independent and identically distributed (*iid*) activities are
175 added in series, the resulting distribution quickly changes its shape to resemble a Normal
176 distribution. In fact, if we observed the result for five activities in series, it would be quite
177 difficult to tell them apart from a Normal distribution. Therefore, the particular statistical
178 distribution chosen hardly matters since it is difficult to find more extreme examples than a
179 Uniform and a Beta distribution, both equally and quickly converging to a Normal
180 distribution.

181 **<Figure 1>**

182 However, what is relevant is the original activity duration mean and standard deviation.
183 Both the Uniform and the Beta distribution had the same duration mean of $\mu=5$ (time units)
184 and it is easy to see that as we add more activities in series, the resulting mean is $n\cdot\mu$, where n
185 is the number of activities in the series ($n\cdot\mu=10$ for two activities, 15 for three activities). On
186 the other hand, the Uniform distribution above has a higher variance (higher standard
187 deviation) compared to the Beta distribution below which, on top of that, is positively skewed
188 and has a range of variation from 0 to 15. It is again evident that as we add more activities in
189 series, the resulting distribution of the Uniform is sparser (since it originally had a bigger
190 variance), whereas the result from the Beta-distributed activities has almost removed the
191 skewness (degree of asymmetry) while it is narrower (duration values are more concentrated
192 near the mean).

193 Therefore, in any project that has three or more activities in series (basically any real
194 project), the selection of the specific statistical distribution is not relevant. What matters is to

195 specify as accurately as possible the activity mean and standard deviation. The original PERT
196 technique proposed doing this by a three-point estimate, but this caused an error of between
197 15 and 35%, particularly concentrated in the variance estimation (when departing from the O ,
198 L and P estimates). The obvious solution is to get rid of the intermediate step and ask the
199 scheduler (or the field expert) to directly provide a mean and a standard deviation for each
200 activity duration, and if he/she feels unsure about quantifying particularly the variance, then
201 resort to the original PERT expressions (equations 1 and 2) with the correction factor stated
202 in Equation 6.

203 With the activity-level duration issues clarified, it is time to address another PERT
204 shortcoming: the result of having multiple parallel paths in a network. To exemplify this, we
205 will make use of the bottom half of Figure 1. But first, let us propose an example.

206 Imagine that the activity duration can be modeled with a fair six-sided die. Then, the
207 activity duration might be $\{1, 2, 3, 4, 5, 6\}$, with equal probability each of $1/6$ (a discrete
208 Uniform distribution). The average of these six possible outcomes would be 3.5 (time units),
209 So, if we cast two dice, the average duration would be two times 3.5, that is, 7. If we cast
210 three dice the average duration would be three times 3.5, that is, 10.5. This is the effect of
211 adding activities in series; we are just ‘adding’ distributions (convolution of distributions), as
212 in the upper half of Figure 1. But when there are several parallel paths, we are not doing sums
213 anymore: we are taking the ‘maxima’ (computing the Cumulative Distribution Function of a
214 maximum) of the different path durations. In our dice example, imagine that we have two
215 parallel paths the duration of which are modeled by a six-sided die. The next activity after
216 these two paths merge in one path (gray nodes in Figure 1) will start only after the path with
217 the maximum duration is finished (the one with the maximum die cast value). If two dice are
218 cast, the maximum of both dice will not be 3.5 anymore, it will be higher. Analogously, if we
219 had more and more parallel paths (imagine 10 paths with one activity each, for instance), the

220 probabilities of getting at least one six in one of the casts would be quite high. Therefore,
221 what is happening is that when several parallel paths converge, the mean project duration is
222 shifted to the right (the project ends later than expected) and the variance contracts.
223 Additionally, the skewness and kurtosis also momentarily deviate from the Normal
224 distribution (resembling more at that point an Extreme Value distribution). But unlike the
225 mean and the variance, they will dissipate again when more activities in the series are added
226 after the activities merger point or before the paths are separated.

227 This same effect can be visualized in the bottom half of Figure 1. The same two
228 distributions were chosen to illustrate how as more and more paths converge, the resulting
229 distribution (which actually is the highest order statistic of several distributions) shifts its
230 mean towards the right (it is increasingly higher than 5, which was the original mean duration
231 value), whereas the dispersion (variance) also decreases (the distribution becomes more
232 compressed). Overall, this phenomenon has been named *merge bias* or *merge-event bias*
233 (Khamooshi and Cioffi 2013; Vanhoucke 2012), and it was the biggest problem that the
234 original PERT had and the one that the scientific community has struggled most to solve.
235 Indeed, this interesting phenomenon is hardly known by most professional construction
236 schedulers nowadays.

237 The reason why this has been an enduring problem (indeed, no exact analytical solution
238 has been found to date) is that despite apparently any distribution could be chosen to model
239 the activity durations, that distribution would need to be sum-stable and max-stable. Sum-
240 stable means that the distribution of the convolution (sum) of several activity durations
241 (activities in series) should again belong to the same kind of distribution. An example is the
242 Normal distribution, in which, after being summed, the result is again Normal and with a
243 mean and standard deviation as in equations 3 and 4. Max-stable means that after computing
244 the maximum from some distributions of the same type (activities in parallel), the result is

245 again another distribution of the same type. The only two existing max-stable distributions
246 are the Gumbel distribution and the Fréchet distribution, and neither one is sum-stable.
247 Hence, the ideal situation in which both operations can be performed with just one type of
248 distribution is not feasible.

249 To sum up, it is not possible to resort to an exact analytical approach for solving the
250 PERT major shortcoming, and the only alternative is to work with mathematical
251 approximations that keep the mean and variance relatively intact towards the end of the
252 project. The logical alternative will be to resort to the Normal distribution, since at least it is
253 sum-stable (which is the most frequent operation in any network due to the generally
254 dominant number of activities in series), it is very well documented in the PERT-related
255 literature, and it is still easy to handle.

256

257 Review of the most relevant PERT-related extensions

258 The study of project completion times when activities have uncertain durations mostly
259 started attracting interest when PERT was published by Malcolm et al. (1959), but it has
260 continued up to the present. Seminal works that followed PERT publication have been
261 Anklesaria and Drezner's (1986) multivariate approach to estimating the project completion
262 time for stochastic networks, Elmaghraby's (1989) study, review and critique of the
263 estimation of activity network parameters, Kamburowski's (1989) network analysis in
264 situations when probabilistic information is incomplete, and, more recently, the
265 computational studies by Ludwig et al. (2001) on bounding the makespan distribution of
266 stochastic project networks. These works are considered, without exception, classics in the
267 stochastic network analysis (SNA) field and have established the foundations of most of the
268 later PERT-related research.

269 However, computer-based (Monte Carlo) simulations have been to date probably the best
270 alternative for obtaining a highly accurate representation of the actual project duration
271 distribution no matter what the SNA topology and size are (e.g. (Douglas 1978; Hajdu 2013;
272 Khamooshi and Cioffi 2013; Nelson et al. 2016)). Simulations will also be used here for
273 comparing M-PERT outputs with the actual solution. But, obviously, the main reason why
274 approximations like M-PERT are developed is because implementing computer simulations
275 in schedule networks with special attributes (e.g., self-loops, alternative paths, correlated
276 activities) is not usually straightforward for most construction students and practitioners
277 (Ballesteros-Pérez et al. 2015). Also, manual procedures like M-PERT allow step-by-step
278 incremental calculations, offering a more intuitive vision than simulation results.

279 Likewise, there have been other attempts to create pieces of software to develop PERT-
280 like approaches for Stochastic Network Analysis (SNA) (e.g., Pontrandolfo (2000); Trietsch
281 and Baker (2012)) that basically have the same aim as the Monte Carlo simulation, but allow
282 a higher user interaction mostly oriented to enhancing the Decision Support process and
283 progress monitoring, but with some accuracy loss.

284 In parallel, many recent algorithms (e.g., Mouhoub et al. (2011)) and other more
285 advanced statistical techniques (Cho 2009; Węglarz et al. 2011), mostly involving Markov
286 chains (e.g., Creemers et al. (2010); Magott and Skudlarski (1993); Xiangxing et al. (2010)),
287 have also been developed for handling (optimizing) some project outcomes like activity time-
288 cost trade-offs (e.g., Azaron and Tavakkoli-Moghaddam (2007)), the total project duration
289 (e.g., Baradaran et al. (2010)) or the project Net Present Value (NPV) (Creemers et al. 2010)
290 while implementing PERT, but particularly in Resource-Constrained Project Schedules (e.g.,
291 Azaron et al. (2006); Baradaran et al. (2012); Yaghoubi et al. (2015)). These extensions,
292 while worth noting, deal with computer implementations, unlike M-PERT, which adopts a
293 manual approach.

294 Furthermore, other techniques like fuzzy logic (e.g., Chen (2007)) and Artificial Neural
295 Network (ANN) analysis (e.g., Lu (2002)) have also been applied to improve activity
296 duration estimation and critical path(s) determination for later ranking purposes. We think
297 that these last two techniques and others in a similar fashion (e.g., Kuklan et al. (1993)),
298 despite having proved to be successful in other fields, do not offer a significant advantage
299 versus Monte Carlo simulations – commonly materialized in Schedule Risk Analysis (SRA)
300 (Vanhoucke 2012) – which are equally (or less) computer-intensive and give almost exact
301 results.

302 Also, concerning the monitoring and control dimension of PERT, some research has been
303 carried out to improve PERT for use as a project progress tool (e.g., Castro et al. (2007);
304 Wenying and Xiaojun (2011)), for example, by means of intersecting buffers after key
305 activities in networks considering both the time and cost dimensions (Khamooshi and Cioffi
306 2013), sometimes referred as Dynamic Planning and Scheduling (Azaron and Tavakkoli-
307 Moghaddam 2007; Yaghoubi et al. 2015). Also in this vein, some research has been
308 developed connected to crashing the PERT activities in order to fast-track project execution
309 (e.g., Abbasi and Mukattash (2001); Foldes and Soumis (1993)). The scope of these PERT
310 extensions, however, is related to control and schedule compression, which deviates from the
311 aim of the current paper (proposing a technique for estimating the total or remaining project
312 duration).

313 Interesting, but certainly minority research has also been devoted to studying the possible
314 effects of assuming independence versus the existence of (partial) correlation between
315 activities (e.g., Banerjee and Paul (2008); Cho (2009); Mehrotra et al. (1996); Sculli and
316 Shum (1991)). Some of their principles will be used here in M-PERT too, and two brief
317 examples are provided later in Figure 5.

318 Concerning the application of PERT to situations where the activities are of a highly
319 repetitive nature, other PERT extensions have been published (e.g., RPERT (Aziz 2014)).
320 Repetition in activities will also be included among the M-PERT modelling capabilities later
321 by means of activities (or groups of activities) self-loops.

322 Concerning the number of point estimates to obtain the activity duration average and
323 standard deviation, as in expressions 1 and 2, extensive research has also been conducted. In
324 particular, the ‘three-point estimates’ from expressions 1 and 2, as this approach is commonly
325 known, make use of the minimum (optimistic), most likely (mode) and maximum
326 (pessimistic) durations that an activity duration following a Beta distribution can have. Other
327 attempts, which are actually as complementary to M-PERT as the original three-point
328 estimate proposal, have been proposed using two parameters (e.g., the mode with either the
329 minimum or the maximum duration (Mohan et al. 2006)), three parameters but with some
330 degree of reparametrization (e.g., Herrerías-Velasco et al. 2011), or even up to four
331 parameters (e.g., (Hahn 2008)). However, normally, when the parameters have been changed,
332 the distribution type modelling the activity duration changes too.

333 Another minor but relevant subfield of research to this study is the work that deals with
334 the PERT network reducibility problem. Essentially, M-PERT is a technique that recursively
335 downsizes the schedule network by merging activities in series and in parallel. With a similar
336 aim, but with a strong emphasis on the computational point of view, the original works by
337 Ringer (1969), Elmaghraby (1989) and Bein et al. (1992) studied this challenging problem in
338 extensive detail. As M-PERT lends itself to a manual reduction approach, the reader
339 interested in computer implementations is referred to these remarkable works.

340 Finally, as was anticipated earlier, an overwhelming amount of research has been devoted
341 to trying to improve the original Beta distribution fitness (Hahn 2008; Premachandra 2001),
342 or just try to find another probability distribution that better fits the activity durations (e.g.,

343 Lau and Somarajan (1995); López Martín et al. (2012)). The variety of distributions that can
344 be found as an alternative to the Beta distribution is overwhelming. They include the doubly
345 truncated normal distribution (Kotiah and Wallace 1973), the triangular distribution (Johnson
346 1997), the log-normal distribution (Mohan et al. 2006), the mixed beta and uniform
347 distribution (Hahn 2008), and the Parkinson distribution (Trietsch et al. 2012), to cite a few.

348 As expected, however, this line of research has caused quite a lot of controversy, with
349 some researchers in favor of pursuing the fit-for-all distribution (e.g., Clark (1962); Golenko-
350 Ginzburg (1989); Grubbs (1962); Healy (1961); Sasieni (1986)) and others against (e.g.,
351 Herrerías-Velasco et al. (2011); Kamburowski (1997); Pleguezuelo et al. (2003)). As
352 MacCrimmon and Ryavec (1964) anticipated and we again proved in Figure 1, the particular
353 distribution is hardly relevant, so no more references will be made to this discussion.

354 With the most relevant PERT-related lines of research identified, it is time to identify the
355 subset of works upon which M-PERT has been built, which are those that deal with the
356 merge event bias in some way or another. These works are summarized in Table 1 by
357 columns, and all of them are approximate techniques for solving some flaws that the original
358 PERT technique had. Unfortunately, these works are not widely known by the research
359 community. Even more surprisingly, most of them were unaware of the others, even though
360 all but the first one were contemporaneous. For the sake of brevity, some details have been
361 given by the rows in Table 1 and some more will be specified later in the *M-PERT outline*
362 section.

363 **<Table 1>**

364 Of particular interest perhaps, is the work by Pritsker (1966), who developed the
365 Graphical Evaluation and Review Technique (GERT), which would have fulfilled the
366 promise of devising a new PERT without most of its problems if it had not been for the

367 mathematical complexity that did not allow the authors to finally implement it without
368 resorting to Monte Carlo simulations. However, this technique introduced very interesting
369 features, some of which M-PERT has inherited, like self-loops and probabilistic (alternative)
370 paths.

371 Last of all, another small subset of approximate PERT variants sought to tackle the merge
372 bias problem by obtaining the upper and/or lower bounds of the project total duration.
373 Without seeking to be exhaustive, maybe some of the most relevant findings were the Clark's
374 (one of the PERT creators) bias correction procedure (Clark 1961, 1962), the 'f' estimate
375 (Fulkerson 1962) and the Modified Network Evaluation Technique (PNET) (Ang et al. 1975).
376 A recent exhaustive study by González et al. (2014) comparing the accuracy achieved by
377 these methods when approximating the Project duration mean and variance in networks with
378 a varied array of topological indicators showed that the best methods were the 'f' estimate for
379 the mean (with an average error of 2.6%) and the PNET for the variance (with an average
380 error of 29.8%). Anticipating some results presented later, the methods summarized in Table
381 1, on which M-PERT was built, exhibited smaller errors in the benchmark networks tested,
382 which has allowed M-PERT to currently have average errors below 2% for the mean and
383 below 10% for the variance. Therefore, these methods, despite deserving acknowledgment,
384 will not be referred to any more.

385

386 **M-PERT outline**

387 M-PERT has been built upon the five models presented in Table 1, most of which shared
388 some common traits (evidenced in lines with several '✓' marks). All five methods made use
389 of Activity-on-Arc (AoA) networks, whereas M-PERT resorts to Activity-on-Node (AoN)
390 representation since nowadays it is more user-friendly for practitioners and more commonly
391 found in software (e.g., Microsoft Project, Oracle Primavera).

417 was hardly important, the Normal assumption was also made by three out of the five methods
418 stated in Table 1, and this obeys the need of simplifying as much as possible the merging
419 operation of activities in series, which is the most frequent in any network. The merger
420 operations in the serial activities are represented in the first row of Figure 3.

421 **<Figure 3>**

422 Figure 3 is, in general, quite self-explanatory and basically represents the most common
423 operations (mergers) that can be found in any network – mostly serial activities (first row)
424 and maxima of multiple paths (bottom row) – along with other interesting operations that
425 GERT (Pritsker 1966) proposed (probabilistic paths, self-loops and activity minima) but was
426 not able to implement without ad hoc computational programmes. For illustrative purposes,
427 the case study shown later in Figure 4 will provide examples of these logic operations.

428 Particularly in construction projects, operations like probabilistic (alternative) paths might
429 seem quite straightforward for modelling uncertain courses of action that will mostly depend
430 on information that is still unknown, or at least inaccurate, at the conceptual stage. Examples
431 are the choice between different types of foundations depending on the ground conditions or
432 the soil-bearing capacity, or just the need to take special measures if the project uncovers
433 archeological remains.

434 Self-loops, on the other hand, are helpful for representing activities being repeated after
435 an unsuccessful outcome. These activities can correspond to isolated activities (e.g., a load
436 test, a calibration) or even groups of activities (e.g., a whole project drawing up process, a
437 series of defective structural elements, a non-compliant subnetwork within a water supply
438 system). It is up to the scheduler, however, to decide how many loops, or even how many
439 multiple nested loops, could be feasibly implemented before the construction manager or the
440 client stops the iterations.

441 Similarly, activity minima, despite certainly being much less frequent than activity
442 maxima, can also be useful at times to allow the project continuation after one (or some) of
443 the predecessors are finished. A distinctive quality of activity minima versus probabilistic
444 paths is that all the activities involved in the minima merger must be executed at some point
445 before the project finishes, whereas in probabilistic paths only one (or a subset of) path(s)
446 will be finally carried out. Examples of activity minima are the supply of any
447 electromechanical equipment among a series of any other equipment that needs to be
448 received in a worksite before the mechanical engineers can start working, or the clearance of
449 any stocking areas of a linear worksite before a new pipeline section can be supplied.

450 Overall, for every operation by rows in Figure 3, their representation (second column) is
451 shown; the input parameters (third column) necessary for defining the activities and
452 performing the merger into a single activity (which is generically named k) are provided; the
453 output parameters (fourth column) that define the new duration mean and variance of the
454 resulting merged activity k are provided; a representation of the result (fifth column) is
455 shown; and, finally, some observations (last column on the right) that clarify how to apply the
456 merger operations in particular (more complex) contexts.

457 It is worth highlighting that M-PERT requires that the activity duration means (μ_i) and
458 variances (σ_i) have been specified for each activity beforehand, a requirement that forces the
459 student or scheduler to elicit them or, if he/she wanted to start from the three-point estimates
460 (O , L and P), it would require a first stage of application of equations 1 and 2 (with the
461 correction proposed in equation 6) for each activity, like the original PERT.

462 Concerning the origin of the mathematical expressions stated for all the mergers (stated in
463 the *Output parameters* column), the serial activities merger just corresponds to the sum of
464 means and variances of several Normal distributions (as in equations 3 and 4), whereas the
465 expressions for the probabilistic alternative paths and the self-loops are easily derived from

466 the mathematical expressions of the mixture (Union) of n non-overlapping (Normal)
 467 population samples ($X_i=X_1 \dots X_n$), which are:

$$468 \quad k = \frac{W_{X_i} X_i}{W_{X_i}} \quad (7)$$

$$469 \quad \frac{2}{k} = \frac{W_{X_i} X_i^2}{W_{X_i}} + \frac{\sum_{i < j} W_{X_i} W_{X_j} (X_i X_j)^2}{\left(\sum_{i} W_{X_i} \right)^2} \quad (8)$$

470 Where the W_{X_i} are the weights of each Normal population merged, that is, the proportion
 471 by which that Normal distribution (representing a set of previously merged activities'
 472 duration) will be present in the resulting distribution after the next merger. For example, for
 473 the probabilistic paths, the weights are the probabilities of each path occurring, that is, $W_{X_i}=p$
 474 and $W_{X_2}=1-p$ for two paths; or $W_{X_i}=p_x$ for $i=x=1,2,\dots,n$ paths, with $\sum_{x=1}^n W_x = 1$ always. For the
 475 self-loops, an activity i is merged with itself – activity i 's mean $\mu_{X_1}=\mu_i$ and standard deviation
 476 $\sigma_{X_1}=\sigma_i$ when the activity happens just once with probability $W_{X_1}=1-p_i$, and $\mu_{X_2}=2\mu_i$ and $\sigma_{X_2}=\sqrt{2}$
 477 σ_i when it is repeated with probability $W_{X_2}=p_i$.

478 Finally, the case of the maximum from two Normal distributions (the durations of two
 479 activities) was coincidentally proposed by Clark (1961) but at that time it required intensive
 480 use of tables and did not allow for correlation (ρ) between both activities. In particular, the
 481 expression for the maximum was taken from Sculli and Shum (1991) and was also used by
 482 Cox (1995), whereas the expression for the minimum was taken from Nadarajah and Kotz
 483 (2008). Both expressions, despite looking long, are very easy to calculate.

484 The obvious problem with the maxima and minima from several activities is that the
 485 formulae only allow for the merger of two of them at the same time. Then these expressions
 486 can be applied recursively until there is just one activity left, a moment in which the activity
 487 will be in series again and can keep being merged with its neighbor activities in series. This
 488 recursive merging procedure is not new either, as it was successfully applied by Cox (1995),

489 Gong and Hugsted (1993) and Sculli and Shum (1991). However, it is also true that when
490 more than two activities are merged, there is a small error in the final mean and variance
491 calculation, which is also dependent (but to a minimum extent) on the exact merger order too.
492 Simulations performed that try to estimate the maximum error magnitude with M-PERT
493 identified how this error is maximized when the different paths' duration mean is the same,
494 and variances are also identical between the parallel paths. However, even in that case, the
495 error obtained when merging "eight" activities under the same node (which is a higher
496 number than most of the real projects have under one converging node) causes an error of
497 6.7% in the mean and 7.3% in the standard deviation, which are considered, in general, good
498 approximations.

499 Therefore, M-PERT consists of merging activities until there is one activity left that
500 models the total project duration by means of μ_p and σ_p , which correspond to the project
501 duration mean and standard deviation, respectively. The order in which merger operations are
502 performed is relevant too. For example, if a self-loop comprised several activities, those inner
503 activities should first be merged, and then the self-loop resolved; or, if there are several
504 activities in series within a parallel path, those activities need to be merged before the path
505 can be merged with other paths. Overall, there must be an overriding use of common sense
506 when performing the mergers, and unless there is correlation among activities (a situation that
507 will be treated separately later), the merging procedure is quite straightforward.

508 There is just one last step left in the application of M-PERT. Obviously, the resulting
509 distribution modelling the total project duration is forced to be Normal in M-PERT, but, in
510 reality, this distribution might be more similar to an Extreme Value distribution when there is
511 a "high number of parallel paths converging all of them to the same node (activity)". This is
512 not a secret, but it was not properly incorporated in a PERT model until Dodin and Sirvanci

513 (1990). He proposed using the Gumbel distribution (despite it being generically named the
514 Extreme Value distribution) for modelling the total project duration.

515 In M-PERT, however, we think the Normal distribution constitutes a reasonable
516 approximation. This is because most construction projects, despite usually including multiple
517 paths, they do not “all converge in the same single activity”, rather they have multiple paths
518 which converge at different activities. In other words, there is not a dominance of maxima
519 computation, but some maxima calculations mixed with a high number of activities in series
520 or activity maxima in series, both of which quickly degenerate (as shown in Figure 1) in
521 Normal distributions. Furthermore, students who are exposed to learning M-PERT are much
522 more familiar with the Normal distribution than with Extreme Value distributions.

523

524 **Case study**

525 A fictitious and simplified bridge project

526 M-PERT can be equally applied to simple and complex networks, the only difference
527 being the number of mergers that are required to be performed before the network is reduced
528 to a single activity representing the project duration. In this regard, Figure 4 presents the
529 application in a simple but representative bridge project construction with three piles (three
530 parallel paths), in which, purposefully, all possible kinds of merging operations have been
531 included. The activities ‘Start’ and ‘End’ are just dummy activities (null duration).

532 **<Figure 4>**

533 On the lower part of Figure 4, the sequential merging procedure has been exemplified.
534 Note, however, that the activities represent the standard deviation, not the variance, and this
535 is to be considered when applying all the expressions stated in Figure 3. Furthermore, the
536 type of merger operation has been stated in the five-step (solution) reduction procedure,
537 replacing the original activity descriptions and allowing the reader to understand where each

538 newly merged activity came from. All these indications however, are not necessary when we
539 try to expedite the manual calculation process.

540 Few explanations are needed at this stage, since the merger procedure just requires the
541 application of simple mathematical expressions already introduced mostly in Figure 3. It is
542 worth highlighting that, unlike the original PERT with fixed paths (in which what was
543 described in the network had to be necessarily carried out and excluded the possibility of
544 going backwards), the introduction of probabilistic paths, minima and self-loops allows for a
545 closer representation of construction projects. In this sense, M-PERT allows for some paths
546 happening while others do not, and that a group (or even the whole project if need be) is
547 repeated thanks to the inclusion of a self-loop. Overall, the modelling possibilities of M-
548 PERT are significantly more powerful than the original PERT.

549

550 Networks with correlation between activities

551 Sometimes there are networks (project schedules) in which the merger procedure ends in
552 a knot that cannot be untangled. This is quite usual indeed when there are several parallel
553 paths that do not depart from the same node and/or do not converge in the same node
554 (Mehrotra et al. 1996). In those situations, the partially reduced network can end up in
555 (sub)networks similar to the ones shown in Figure 5, although maybe with more nodes.

556 **<Figure 5>**

557 For solving these subnetworks that usually represent a small portion of the whole
558 network, authors like Mehrotra et al. (1996) and Sculli and Shum (1991) proposed a
559 sequential recursive procedure that involves calculating all the variance and covariance
560 values between the ‘untangled’ paths. Implementing this procedure basically requires
561 identifying all the possible paths and then merging them, first, two of them, and then one by

562 one with the already merged group. However, the user needs to be aware that there is a
 563 correlation ($\rho \neq 0$) between activities when applying the maxima (or minima) expressions from
 564 Figure 3, that is, between paths that have some activities in common (as the covariance
 565 between those two paths will not generally be zero).

566 As can be seen in Figure 5, identifying all the possible paths is not difficult unless the
 567 network knot size involves a lot of activities and is very dense in precedence relationships
 568 (number of arrows). The only tricky part is to calculate the correct value of ρ between the
 569 paths. However, Sculli and Shum (1991) derived the generic expression for obtaining it, but
 570 since that expression is somehow difficult to understand for the beginner, the ρ expressions
 571 for up to four parallel paths (normally ordered from higher to lower mean duration μ_k and
 572 sequentially named A , B , C and D) are given here:

$$573 \quad \rho_{AB} = \frac{COV_{AB}}{\sigma_A \sigma_B} \quad (9)$$

$$574 \quad \rho_{ABC} = \frac{COV_{AC} \Phi(\frac{\mu_A - \mu_B}{\sigma_A}) + COV_{BC} (1 - \Phi(\frac{\mu_A - \mu_B}{\sigma_A}))}{\sigma_{AB} \sigma_C} \quad (10)$$

$$575 \quad \rho_{ABCD} = \frac{[COV_{AD} \Phi(\frac{\mu_A - \mu_B}{\sigma_A}) + COV_{BD} (1 - \Phi(\frac{\mu_A - \mu_B}{\sigma_A}))] \Phi(\frac{\mu_{AB} - \mu_C}{\sigma_{AB}}) + COV_{CD} (1 - \Phi(\frac{\mu_{AB} - \mu_C}{\sigma_{AB}}))}{\sigma_{ABC} \sigma_D} \quad (11)$$

576 Where ρ_{AB} , ρ_{ABC} , ρ_{ABCD} representing, respectively, the correlation coefficients as paths A
 577 with B, AB with C, and ABC with D are merged one by one with the previously merged
 578 paths. COV_{XY} simply corresponds to the sum of variances of the activities that paths X and Y
 579 have in common, and σ_X and σ_Y , the standard deviations of the paths X and Y respectively,
 580 with $X, Y = A, B, C, D$. Examples of calculation of the variance and covariance values have
 581 been provided in Figure 5 in the covariance matrices.

582 Finally, for the sake of clarity, the complete step-by-step procedure for implementing
 583 expressions 9 to 11 in a project network with four paths with correlation has been described

584 in the supplemental data under Alternative 2 in the example project 0, along with other
585 multiple examples of M-PERT exercises.

586

587 **Results**

588 The five methods from which M-PERT has been built were tested in several benchmark
589 networks. There is no reason to believe that M-PERT, which makes use of their most
590 meritorious contributions, even including new features for better modelling real-life projects,
591 will have a lower accuracy. Anyhow, M-PERT has been applied to one network in this paper
592 and its accuracy comparison is presented in Figure 6.

593 **<Figure 6>**

594 It must be borne in mind, though, that Figure 6 shows a comparison of four statistical
595 distributions of which the distribution parameters are perfectly known, not a comparison of
596 datasets against another dataset or a known distribution, as is more usual in research analyses.
597 Hence, we will not be testing how ‘significant’ the difference between any two distributions
598 is, as it is already known that they are different distributions. Instead, a graphical approach is
599 adopted here for illustrating the location and dispersion deviations between the distributions.

600 Figure 6, in particular, shows four curves. The two continuous gray curves show the
601 exact solution of the project duration distribution (obtained through computer simulations) by
602 modelling the activity durations with Normal distributions and Uniform distributions
603 respectively. As stated before, the choice of one or other distribution hardly has any
604 relevance, an outcome proven again by observing how close both gray curves are. The
605 average (project duration) of these two simulated distributions is close to 147.5 days despite
606 they being lightly positively skewed and leptokurtic.

607 The dashed curve, on the other hand, represents the approximation offered by M-PERT
608 by using the Normal distribution modelling the project duration distribution, whose

609 parameters were obtained at the bottom of Figure 4. The bridge project, with three dominant
610 parallel paths, does not deviate much from the Normal approximation. It is also easy to
611 appreciate how the M-PERT project duration mean is close to 147 days (estimation error
612 below 1% when compared to the simulated curves, whereas the variance or dispersion error is
613 also below 10%).

614 Last of all, the continuous black curve representing the classical PERT approximation
615 significantly falls behind, with an average project duration of 133 days (error above 10%).
616 Again, this evidences how inaccurate the original PERT is when there are parallel paths and
617 the merge event bias is neglected.

618 Finally, for reassurance, results in other networks exemplifying step-by-step calculations,
619 along with the (simulated) actual project duration distribution, can be found as supplemental
620 data.

621

622 **Discussion and conclusions**

623 A new scheduling technique named M-PERT has been proposed. This technique takes
624 advantage of several methods that were proposed since PERT was devised for correcting
625 several shortcomings that the original technique had. In particular, in situations with multiple
626 parallel paths (the norm in construction projects), the original PERT resulted in a project
627 duration underestimation and a variance overestimation.

628 A systematic literature review has been developed in order to justify why M-PERT
629 concentrates on the really significant weaknesses of PERT and why, by neglecting other
630 accessorial aspects, the resulting tool is still accurate, but not complex. Indeed, the method
631 proposed is easy to implement and easy to learn due to its intuitive nature and simplifying
632 assumptions. This makes M-PERT an attractive tool for teaching scheduling basics to
633 construction and project management students, especially since its calculations can be

634 developed manually or by means of very simple spreadsheets. Finally, an application has
635 been included in which the new features offered by M-PERT, such as the minima of
636 activities, probabilistic (alternative) paths or activity or groups of activities self-loops
637 evidence a higher resemblance between scheduled networks and real construction projects.

638 However, there is still a long way to go. Indeed, despite this tool really departing from the
639 original PERT conception, it is still open to improvement. The cost dimension has not been
640 included in M-PERT, but it should follow a very similar approach. However, the inclusion of
641 resources in the analysis really represents a challenging route that should probably be
642 incorporated in the next versions of this scheduling technique.

643

644 **Data availability**

645 All data generated or analyzed during the study are included in the submitted article or
646 supplemental materials files.

647

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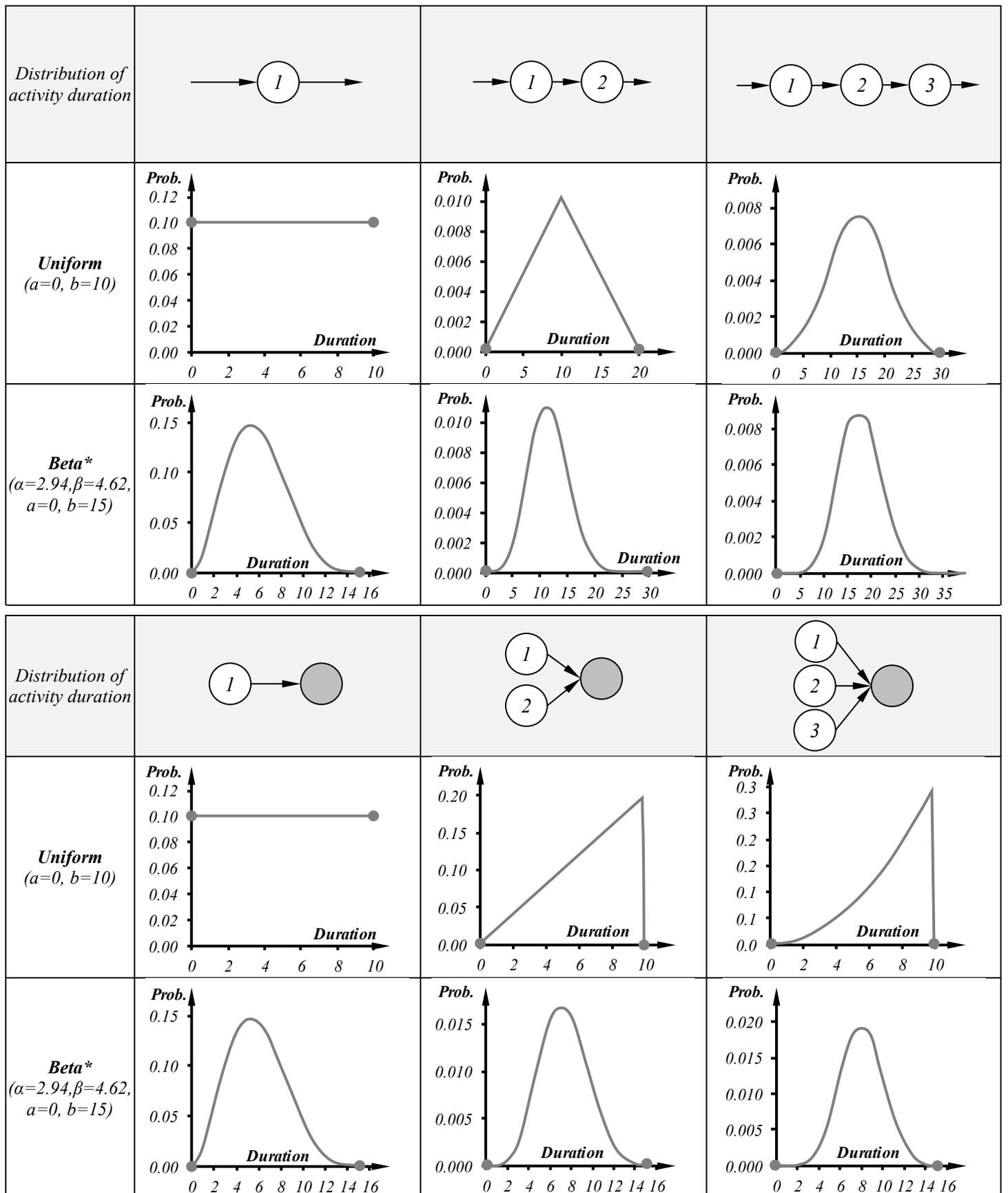
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▼ Highlights Work ►	Pritsker (1966)	Sculli and Shum (1991)	Gong & Hugsted (1993)	Cox (1995)	Mehrotra et al. (1996)
<i>Avoids having to calculate all the possible paths (critical and non-critical) (allows big networks)</i>	✓		✓	✓	
<i>Allows for recursive application of activity maxima and/or minima (time-saving simplification)</i>		✓	✓	✓	
<i>Normal distributions for modelling the activity durations (only mean and variance matter)</i>		✓	✓		✓
<i>Allows for correlation between different activities (allows complex projects)</i>	✓	✓	✓		✓
<i>Use of simple expressions for solving the activity mergers (calculation simplicity)</i>				✓	✓
<i>Allows for extra features (activity minima, self-loops or probabilistic paths, etc) (resemblance to reality)</i>	✓				
<i>Number of benchmark networks in which the model was tested (Number of validation tests)</i>	4	2	3	2	10

Table 1. Key works on which PERT 2.0 has been based



*According to the original PERT, with Optimistic duration: 0, Most likely: 5, and Pessimistic: 15 time units.

Fig. 1. Comparison of two probability distributions alteration as more activities are added

<i>Precedence type</i>	<i>Activity-on-Node (AoN) representation</i>	<i>Finish-Start equivalent</i>
Finishes <i>i</i> , Starts <i>j</i> with a time lag <i>t</i> (omit if $t=0$)		
Starts <i>i</i> , Starts <i>j</i> with a time lag <i>t</i>		
Finishes <i>i</i> , Finishes <i>j</i> with a time lag <i>t</i>		
Starts <i>i</i> , Finishes <i>j</i> with a time lag <i>t</i>		

Fig. 2. Array of activity precedences in M-PERT

Operation	AoN Representation	Input par.	Output parameters	Result	Observations
Serial activities		μ_i, σ_i^2 μ_j, σ_j^2 ... μ_n, σ_n^2	$\mu_k = \sum_{x=i,j,\dots,n} \mu_x$ $\sigma_k^2 = \sum_{x=i,j,\dots,n} \sigma_x^2$		When there is a time lag t between any two activities, then $\mu_k = \mu_i + \mu_j + t$ and $\sigma_k^2 = \sigma_i^2 + \sigma_j^2$
Probabilistic (alternative) paths		μ_i, σ_i^2, p $\mu_j, \sigma_j^2, q=1-p$	$\mu_k = p\mu_i + (1-p)\mu_j$ $\sigma_k^2 = p\sigma_i^2 + (1-p)\sigma_j^2 + p(1-p)(\mu_i - \mu_j)^2$		If there are more activities after any of the first activities from the paths, resolve those mergers first, then merge into a single activity k
		μ_x, σ_x^2, p_x for $x=1, 2, 3 \dots n$, with $\sum_{x=1}^n p_x = 1$	$\mu_k = \sum_{x=1}^n p_x \mu_x$ $\sigma_k^2 = \sum_{x=1}^n p_x (\sigma_x^2 + \mu_x^2) - \mu_k^2$		
Self-loops		μ_i, σ_i^2, p_i	$\mu_k = (1+p_i)\mu_i$ $\sigma_k^2 = (1+p_i)\sigma_i^2 + p(1-p)\mu_i^2$		When the self-loop comprises more than one activity, resolve (merge) the activities within the loop first. If there are nested (or multiple) self-loops, resolve inner self-loops first.
(maximum or minimum) Parallel paths	<p>Maximum (all activities are to finish before continuing)</p> <p>or</p> <p>Minimum (just one activity needs to finish before continuing)</p>	μ_i, σ_i^2 μ_j, σ_j^2 ρ (if i and j are correlated, otherwise $\rho=0$)	<p><u>Maximum:</u> $\mu_k = \mu_i\Phi(\delta) + \mu_j(1-\Phi(\delta)) + \theta\phi(\delta)$ $\sigma_k^2 = (\sigma_i^2 + \mu_i^2)\Phi(\delta) + (\sigma_j^2 + \mu_j^2)(1-\Phi(\delta)) + (\mu_i + \mu_j)\theta\phi(\delta) - \mu_k^2$</p> <p><u>Minimum:</u> $\mu_k = \mu_i(1-\Phi(\delta)) + \mu_j\Phi(\delta) - \theta\phi(\delta)$ $\sigma_k^2 = (\sigma_i^2 + \mu_i^2)(1-\Phi(\delta)) + (\sigma_j^2 + \mu_j^2)\Phi(\delta) - (\mu_i + \mu_j)\theta\phi(\delta) - \mu_k^2$</p> <p>Where $\delta = (\mu_i - \mu_j)/\theta$; $\theta = \sqrt{\sigma_i^2 + \sigma_j^2 - 2\rho\sigma_i\sigma_j}$</p> <p>and $\phi(\cdot)$ and $\Phi(\cdot)$ are the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the Standard Normal distribution (use Supplemental Figure 1 from the Supplemental Online material for quick calculation)</p>		When there are several activities on, at least, one path, resolve (merge) the activities within that/those path/s first, then apply the maximum/minimum merger. When there are more than two alternative paths, keep recursively resolving (merging) the paths by groups of two activities, until finishing with just one path (activity k will be then in series)

Fig. 3. Array of activity mergers in M-PERT

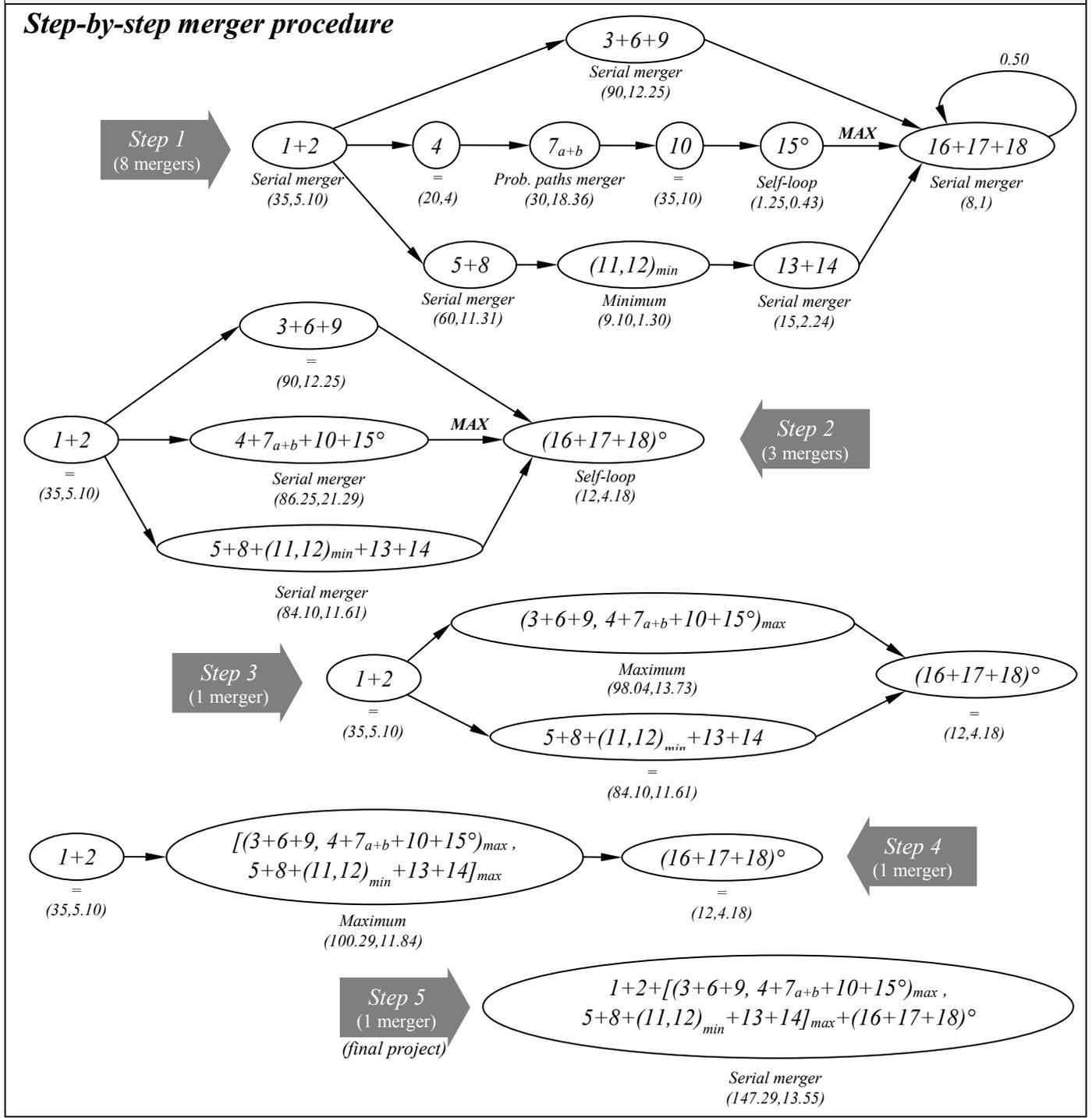
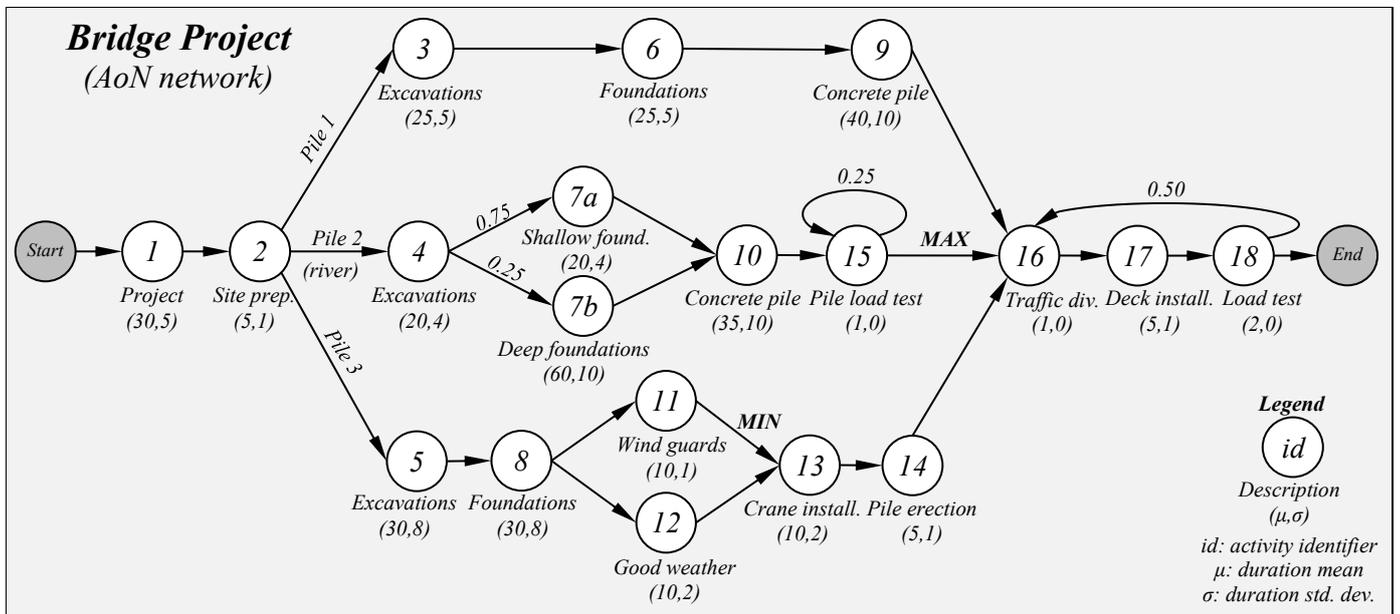


Fig. 4. M-PERT application in a fictitious simplified bridge project

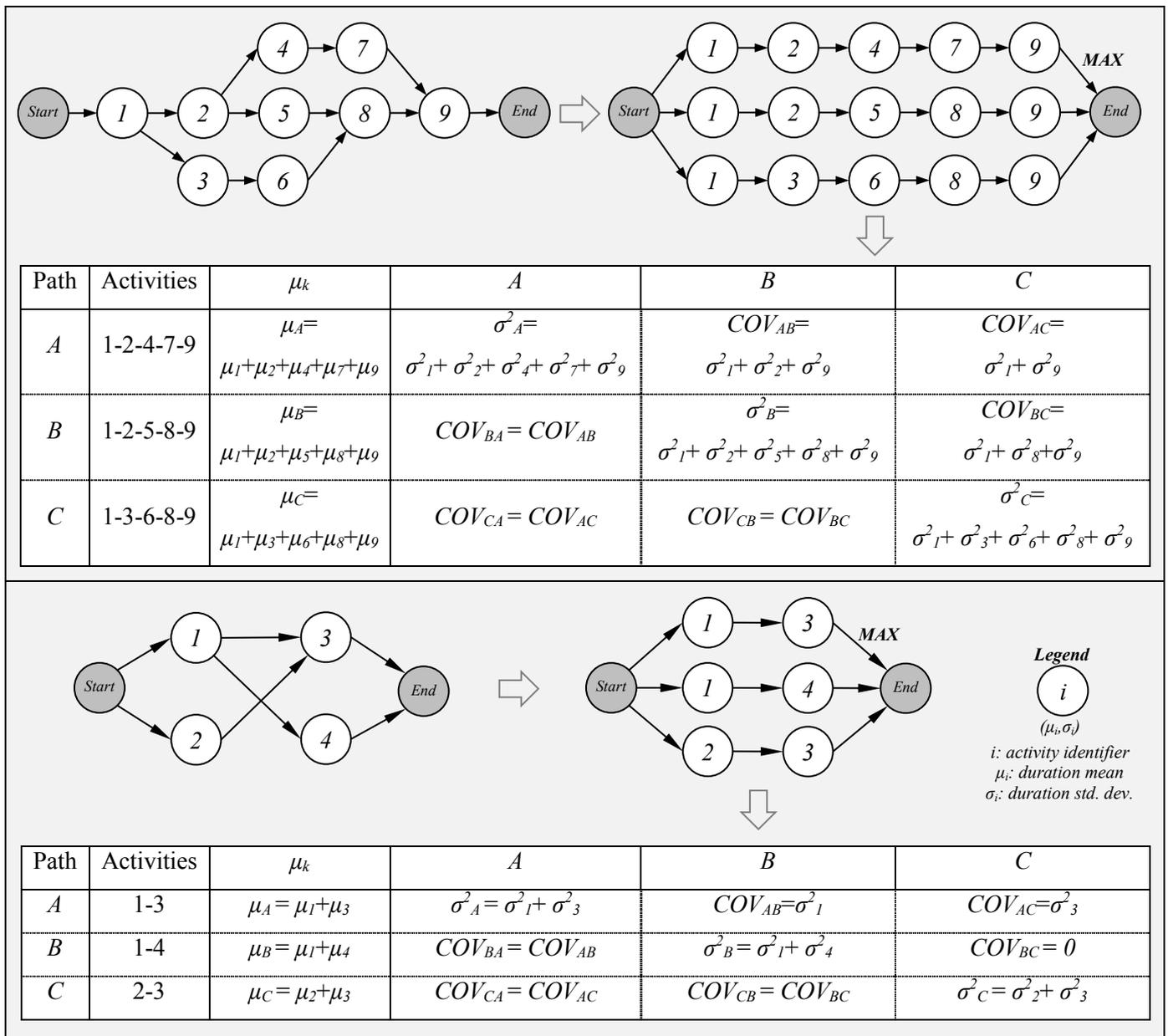


Fig. 5. Two examples of AoN networks with correlation and the calculation of their covariance matrix

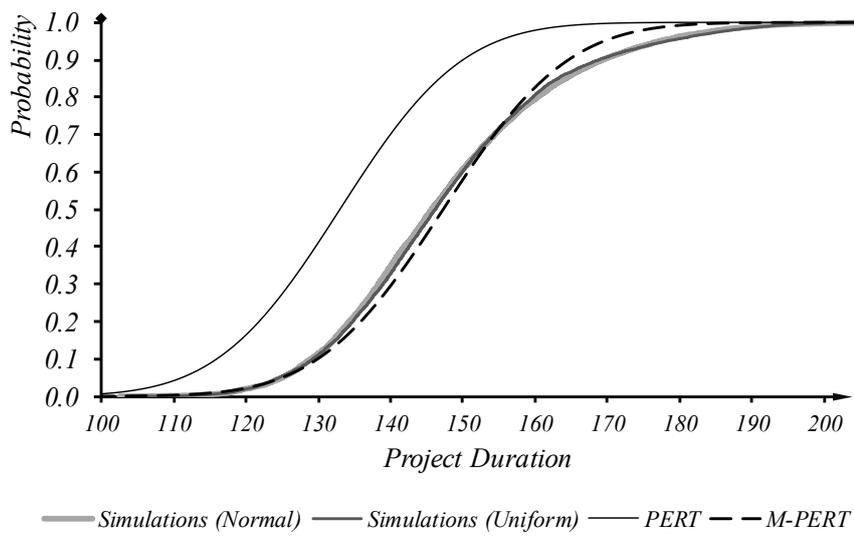


Fig. 6. Comparison of project duration estimation accuracy between PERT and M-PERT for the simplified bridge project

Supplemental data

Contents

Detailed step-by-step calculations of the (Sub)network with correlation

Example Project 0 (**page 2**)

Alternative 1 (**page 3**)

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Sample of 7 beginner-level exercises with increasing difficulty for learning M-PERT

Example Project 1 (**page 7**)

Example Project 2 (**page 8**)

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Example Project 4 (**page 10**)

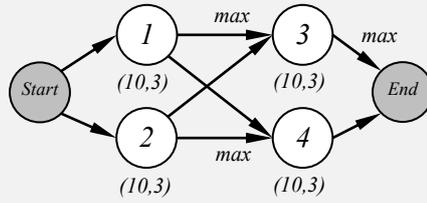
Example Project 5 (**page 12**)

Example Project 6 (**page 14**)

Example Project 7 (**page 16**)

Detailed step-by-step calculations for a (Sub)network with correlation.

Example Project 0



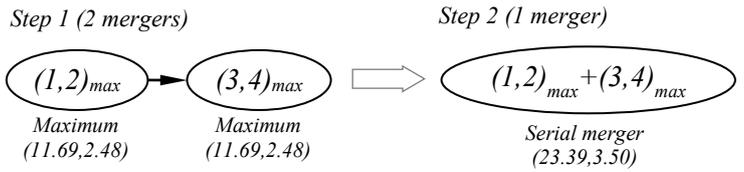
Legend



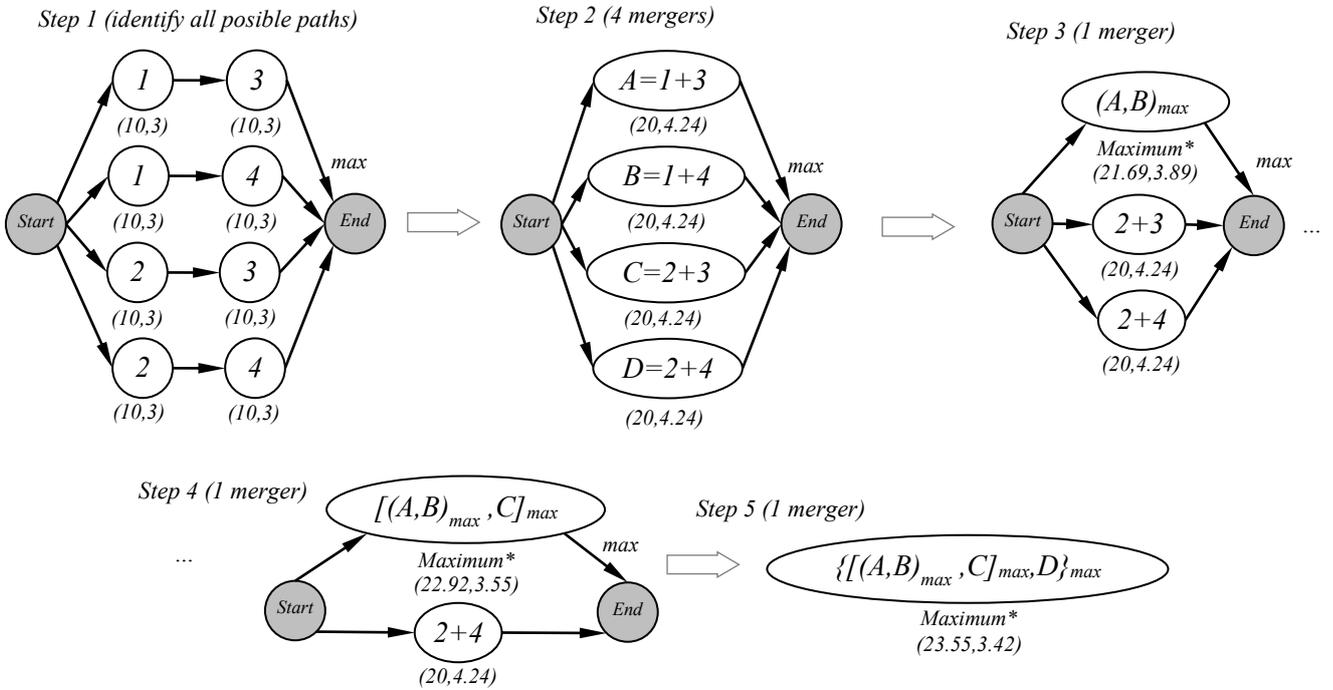
id: activity identifier
 μ : duration mean
 σ : duration std. dev.

Step-by-step merger procedure

Alternative 1 (sometimes possible):

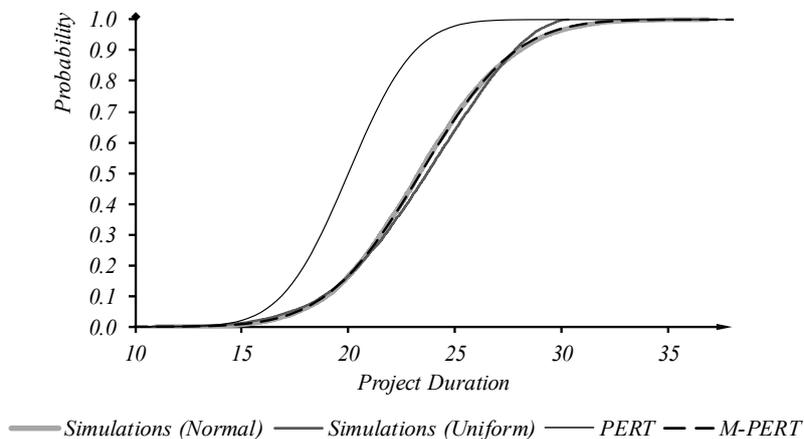


Alternative 2 (general procedure):



* merger with correlation between paths

Results



Alternative 1

This alternative is possible due to the problem symmetry. If, for instance, the Finish-Start precedence relationship between nodes 2 and 4 had not existed, this alternative would have not been possible, only Alternative 2. However, when Alternative 1 is possible is always preferable since it is quicker and more accurate (since normally it entails fewer recursive applications of the maximum merger formulae)

Step 1 (2 mergers)

■ Maximum merger between activities $i=1$ (with $\mu_i=10$ and $\sigma_i=3$) and $j=2$ (with $\mu_j=10$ and $\sigma_j=3$), then $k=(1,2)_{max}$. Values of $\Phi(\cdot)$ and $\phi(\cdot)$ can be calculated with the Supplemental Figure 1.

$$\theta = \sqrt{\sigma_i^2 + \sigma_j^2 - 2\rho\sigma_i\sigma_j} = \sqrt{3^2 + 3^2 - 2 \cdot 0 \cdot 3 \cdot 3} = \sqrt{18} = 4.243 \text{ (there is no correlation between 1 and 2, i.e. } \rho=0\text{)}$$

$$\delta = (\mu_i - \mu_j) / \theta = (10 - 10) / 4.243 = 0$$

$$\mu_k = \mu_{(1,2)_{max}} = \mu_i \Phi(\delta) + \mu_j (1 - \Phi(\delta)) + \theta \phi(\delta) = 10 \Phi(0) + 10 (1 - \Phi(0)) + 4.243 \phi(0) = 10 \cdot 0.5 + 10 \cdot (1 - 0.5) + 4.243 \cdot 0.399 = 11.693$$

$$\begin{aligned} \sigma_k^2 &= \sigma_{(1,2)_{max}}^2 = (\sigma_i^2 + \mu_i^2) \Phi(\delta) + (\sigma_j^2 + \mu_j^2) (1 - \Phi(\delta)) + (\mu_i + \mu_j) \theta \phi(\delta) - \mu_k^2 = (3^2 + 10^2) \Phi(0) \\ &+ (3^2 + 10^2) (1 - \Phi(0)) + (10 + 10) 4.243 \phi(0) - 11.693^2 = 109 \cdot 0.5 + 109 \cdot (1 - 0.5) + 84.86 \cdot 0.399 - 136.726 \\ &= 6.135 \rightarrow \sigma_k = \sqrt{6.135} = 2.477 \end{aligned}$$

■ Maximum merger between activities $i=3$ (with $\mu_i=10$ and $\sigma_i=3$) and $j=4$ (with $\mu_j=10$ and $\sigma_j=3$), then $k=(3,4)_{max}$.

In this case, it is obvious that since the means and variances of all the activities are the same, the result coincides with the maximum merger for $i=1$ and $j=2$. Therefore, $\mu_k = \mu_{(3,4)_{max}} = 11.70$ and $\sigma_k = \sigma_{(3,4)_{max}} = 2.46$

Step 2 (1 mergers)

■ Serial merger between activities $i=(1,2)_{max}$. (with $\mu_i=11.693$ and $\sigma_i=2.477$) and $j=(3,4)_{max}$ (with $\mu_j=11.693$ and $\sigma_j=2.477$), then $k=(1,2)_{max}+(3,4)_{max}$.

$$\mu_k = \mu_{(1,2)_{max}+(3,4)_{max}} = \sum_{x=i,j,\dots,n} \mu_x = \mu_i + \mu_j = 11.693 + 11.693 = 23.385 \rightarrow \text{Project Duration mean}$$

$$\sigma_k^2 = \sigma_{(1,2)_{max}+(3,4)_{max}}^2 = \sum_{x=i,j,\dots,n} \sigma_x^2 = \sigma_i^2 + \sigma_j^2 = 2.477^2 + 2.477^2 = 12.270 \rightarrow \sigma_k = \sqrt{12.270} = 3.503 \rightarrow \text{Project}$$

Duration standard deviation

Now that $\mu_p=23.385$ and $\sigma_p=3.503$ are known, the probabilities of any possible project duration (p) can be calculated by using the $\Phi(z)$ curve in Supplemental Figure 1 with $z = (D - \mu_p) / \sigma_p$. Or the other way around, that is, given a known probability $\Phi(z)=P$ (Y axis value), obtain the respective z value and then $D = \mu_p + z\sigma_p$.

Alternative 2 (general approach)

When alternative 1 option is not possible, a recursive (maximum) merger procedure for untangling the network is always possible between paths with shared activities. The only caution needed is to account for the correlation among the different paths when they are merged one by one until there is no one left to merge.

Step 1 (identify all possible paths)

■ There are four possible paths in this (sub)network 1-3, 1-4, 2-3 and 2-4, which we will name for the sake of brevity for later calculations as *A*, *B*, *C* and *D*, respectively.

It is also necessary to calculate all the activities means and variances, as well as their covariances. The covariance between any two paths is just the sum of the variances of those activities which are common between those two paths (unless another additional source of correlation is stated).

Path (<i>i</i>)	Activities	μ_i	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>A</i>	1,3	10+10=20	3 ² +3 ² =18	3 ² =9	3 ² =9	0
<i>B</i>	1,4	10+10=20	3 ² =9	3 ² +3 ² =18	0	3 ² =9
<i>C</i>	2,3	10+10=20	3 ² =9	0	3 ² +3 ² =18	3 ² =9
<i>D</i>	2,4	10+10=20	0	3 ² =9	3 ² =9	3 ² +3 ² =18

Values in the diagonal correspond to the variances, and those outside to the covariances between couples of paths. Finally, when the time comes later, the coefficient of correlation between two paths *i* and *j* will be calculated as $\rho = COV_{ij} / \sigma_i \sigma_j$ where all the values are taken from the table above.

Step 2 (4 mergers)

■ There are four serial mergers, as indicated in the table above. Detailed calculations are given for path *A* applying the serial activities merger from Figure 3:

$$\mu_A = \mu_1 + \mu_3 = 20 \quad \sigma_A = \sqrt{\sigma_1^2 + \sigma_3^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 4.243$$

Since the rest of the paths (in this example) have the same duration means and standard deviations, then: $\mu_A = \mu_B = \mu_C = \mu_D = 20$ and $\sigma_A = \sigma_B = \sigma_C = \sigma_D = 4.243$.

Step 3 (1 merger)

As a general rule all paths will be ordered from longest to shortest mean project duration and then merged. First, those two longest paths will be merged, and then one by one, in a recursive fashion, the rest will also be merged/added. This one-by-one order rule (unlike the quickest approach by couples of paths first, then by couples of couples of paths, and so on, when there is no correlation between paths) is necessary due to how the mathematical expressions of the correlation coefficients ρ between maxima of paths are stated, since it is not possible (not easily at least) to calculate covariance between two different groups with multiple paths each.

■ Maximum merger between paths *i*=*A* (with $\mu_i=20$ and $\sigma_i=4.243$) and *j*=*B* (with $\mu_j=20$ and $\sigma_j=4.243$), then $k=(A,B)_{max}$. Values of $\Phi(\cdot)$ and $\phi(\cdot)$ can be calculated with the Supplemental Figure 1.

$$\rho_{AB} = \frac{COV_{ij}}{\sigma_i \sigma_j} = \frac{COV_{AB}}{\sigma_A \sigma_B} = \frac{9}{4.243 \cdot 4.243} = 0.5$$

$$\theta_{AB} = \sqrt{\sigma_i^2 + \sigma_j^2 - 2\rho\sigma_i\sigma_j} = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B} = \sqrt{4.243^2 + 4.243^2 - 2 \cdot 0.5 \cdot 4.243 \cdot 4.243} = \sqrt{18} = 4.243$$

$$\delta_{AB} = (\mu_i - \mu_j)/\theta = (\mu_A - \mu_B)/\theta_{AB} = (20 - 20)/4.243 = 0$$

$$\begin{aligned} \mu_k &= \mu_{(A,B)\max} = \mu_i\Phi(\delta) + \mu_j(1 - \Phi(\delta)) + \theta\phi(\delta) = \mu_A\Phi(\delta_{AB}) + \mu_B(1 - \Phi(\delta_{AB})) + \theta_{AB}\phi(\delta_{AB}) = \\ &= 20\Phi(0) + 20(1 - \Phi(0)) + 4.243\phi(0) = 20 \cdot 0.5 + 20 \cdot (1 - 0.5) + 4.243 \cdot 0.399 = 21.693 \rightarrow \mu_{AB} \end{aligned}$$

$$\begin{aligned} \sigma_k^2 &= \sigma_{(A,B)\max}^2 = (\sigma_i^2 + \mu_i^2)\Phi(\delta) + (\sigma_j^2 + \mu_j^2)(1 - \Phi(\delta)) + (\mu_i + \mu_j)\theta\phi(\delta) - \mu_k^2 = \\ &= (\sigma_A^2 + \mu_A^2)\Phi(\delta_{AB}) + (\sigma_B^2 + \mu_B^2)(1 - \Phi(\delta_{AB})) + (\mu_A + \mu_B)\theta_{AB}\phi(\delta_{AB}) - \mu_{AB}^2 = \\ &= (4.243^2 + 20^2)\Phi(0) + (4.243^2 + 20^2)(1 - \Phi(0)) + (20 + 20)4.243\phi(0) - 21.693^2 = \\ &= 418 \cdot 0.5 + 418 \cdot (1 - 0.5) + 169.72 \cdot 0.399 - 470.586 = 15.132 \rightarrow \sigma_k = \sqrt{15.132} = 3.890 \rightarrow \sigma_{AB} \end{aligned}$$

Note that, for ease of notation, both results have been renamed as μ_{AB} and σ_{AB} .

Step 4 (1 merger)

■ Maximum merger between paths $i=(A,B)_{\max}=AB$ (with $\mu_i=21.693$ and $\sigma_i=3.890$) and $j=C$ (with $\mu_j=20$ and $\sigma_j=4.243$), then $k=[(A,B)_{\max}, C]=ABC$. The expression for the coefficient of correlation calculation is derived from Sculli & Shum (1991):

$$\rho_{ABC} = \frac{COV_{AC}\Phi(\delta_{AB}) + COV_{BC}(1 - \Phi(\delta_{AB}))}{\sigma_{AB}\sigma_C} = \frac{9\Phi(0) + 0(1 - \Phi(0))}{3.890 \cdot 4.243} = \frac{9 \cdot 0.5 + 0}{16.505} = 0.273$$

$$\theta_{ABC} = \sqrt{\sigma_{AB}^2 + \sigma_C^2 - 2\rho_{ABC}\sigma_{AB}\sigma_C} = \sqrt{3.890^2 + 4.243^2 - 2 \cdot 0.273 \cdot 3.890 \cdot 4.243} = \sqrt{24.120} = 4.913$$

$$\delta_{ABC} = (\mu_{AB} - \mu_C)/\theta_{ABC} = (21.693 - 20)/4.913 = 0.345$$

$$\begin{aligned} \mu_{ABC} &= \mu_{AB}\Phi(\delta_{ABC}) + \mu_C(1 - \Phi(\delta_{ABC})) + \theta_{ABC}\phi(\delta_{ABC}) = 21.693\Phi(0.345) + 20(1 - \Phi(0.345)) \\ &+ 4.913\phi(0.345) = 21.693 \cdot 0.635 + 20 \cdot (1 - 0.635) + 4.913 \cdot 0.376 = 22.921 \rightarrow \mu_{ABC} \end{aligned}$$

$$\begin{aligned} \sigma_{ABC}^2 &= (\sigma_{AB}^2 + \mu_{AB}^2)\Phi(\delta_{ABC}) + (\sigma_C^2 + \mu_C^2)(1 - \Phi(\delta_{ABC})) + (\mu_{AB} + \mu_C)\theta_{ABC}\phi(\delta_{ABC}) - \mu_{ABC}^2 = \\ &= (3.890^2 + 21.693^2)\Phi(0.345) + (4.243^2 + 20^2)(1 - \Phi(0.345)) + (21.693 + 20)4.913\phi(0.345) \\ &- 22.921^2 = 485.718 \cdot 0.635 + 418 \cdot (1 - 0.635) + 204.754 \cdot 0.376 - 525.372 = 12.617 \rightarrow \\ \sigma_k &= \sqrt{12.617} = 3.548 \rightarrow \sigma_{ABC} \end{aligned}$$

Step 5 (1 merger)

■ Maximum merger between paths $i=[(A,B)_{\max}, C]_{\max}=ABC$ (with $\mu_i=22.921$ and $\sigma_i=3.548$) and $j=D$ (with $\mu_j=20$ and $\sigma_j=4.243$), then $k=\{[(A,B)_{\max}, C]_{\max}, D\}_{\max}=ABCD$. The expression for the coefficient of correlation calculation is derived from Sculli & Shum (1991):

$$\begin{aligned} \rho_{ABCD} &= \frac{[COV_{AD}\Phi(\delta_{AB}) + COV_{BD}(1 - \Phi(\delta_{AB}))]\Phi(\delta_{ABC}) + COV_{CD}(1 - \Phi(\delta_{ABC}))}{\sigma_{ABC}\sigma_D} = \\ &= \frac{[0\Phi(0) + 9(1 - \Phi(0))]\Phi(0.345) + 9(1 - \Phi(0.345))}{3.548 \cdot 4.243} = \frac{9(1 - 0.5) \cdot 0.635 + 9(1 - 0.635)}{15.054} = 0.408 \end{aligned}$$

$$\theta_{ABCD} = \sqrt{\sigma_{ABC}^2 + \sigma_D^2 - 2\rho_{ABCD}\sigma_{ABC}\sigma_D} = \sqrt{3.548^2 + 4.243^2 - 2 \cdot 0.408 \cdot 3.548 \cdot 4.243} = \sqrt{18.305} = 4.278$$

$$\delta_{ABCD} = (\mu_{ABC} - \mu_D)/\theta_{ABCD} = (22.921 - 20)/4.278 = 0.683$$

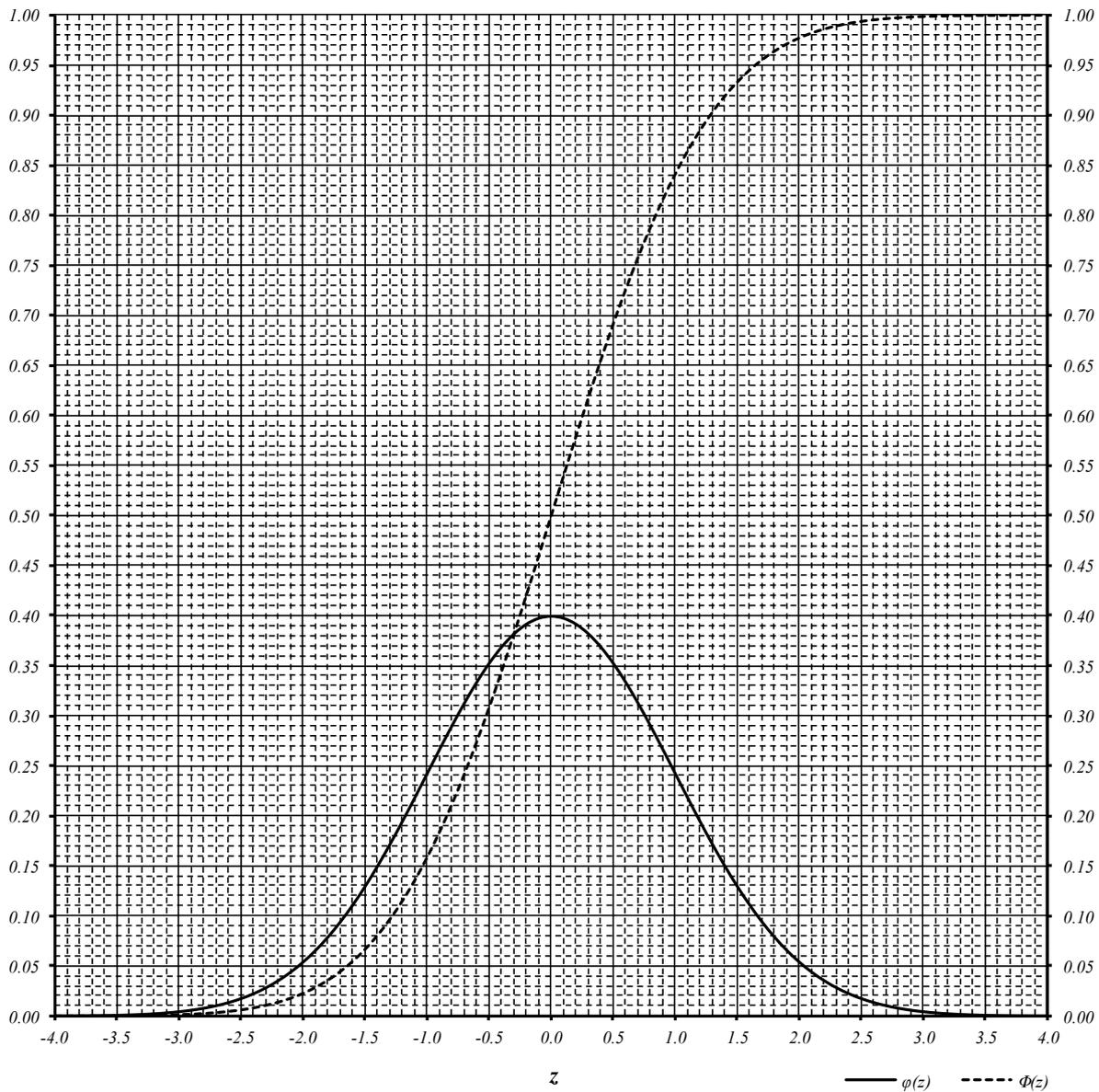
$$\mu_{ABCD} = \mu_{ABC}\Phi(\delta_{ABCD}) + \mu_D(1 - \Phi(\delta_{ABCD})) + \theta_{ABCD}\phi(\delta_{ABCD}) = 22.921\Phi(0.683) + 20(1 - \Phi(0.683)) + 4.278\phi(0.683) = 22.921 \cdot 0.753 + 20(1 - 0.753) + 4.278 \cdot 0.316 = 23.551 \rightarrow \text{Project Duration mean}$$

$$\sigma_{ABCD}^2 = (\sigma_{ABC}^2 + \mu_{ABC}^2)\Phi(\delta_{ABCD}) + (\sigma_D^2 + \mu_D^2)(1 - \Phi(\delta_{ABCD})) + (\mu_{ABC} + \mu_D)\theta_{ABCD}\phi(\delta_{ABCD}) - \mu_{ABCD}^2 = (3.548^2 + 22.921^2)\Phi(0.683) + (4.243^2 + 20^2)(1 - \Phi(0.683)) + (22.921 + 20)4.278\phi(0.683) - 23.551^2 = 537.960 \cdot 0.753 + 418(1 - 0.753) + 183.616 \cdot 0.316 - 554.650 = 11.695$$

$$\rightarrow \sigma_k = \sqrt{11.695} = 3.420 \rightarrow \text{Project Duration standard deviation}$$

It is easy to see that, due to the recursive application of the maximum merger for the four paths, these results ($\mu_p=23.551$ and $\sigma_p=3.420$) do not exactly coincide with the previous from alternative 1 ($\mu_p=23.385$ and $\sigma_p=3.503$), but they are very close. Again, the probabilities of any possible project duration (p) can be calculated exactly as was explained for the Alternative 1.

Prob.

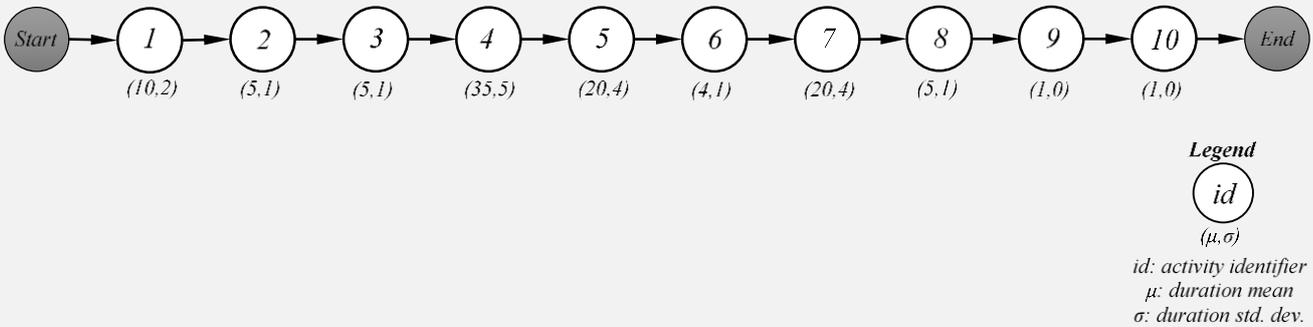


Supplemental Figure 1. Abacus for calculating the probability density function $\phi(z)$ and the cumulative distribution function $\Phi(z)$ of the Standard Normal distribution

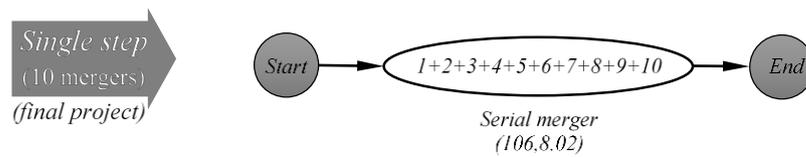
Sample of 7 exercises (project schedules) of increasing difficulty for learning M-PERT

Note: for the calculations of the original PERT technique, self-loops are ignored (as they cannot be calculated with PERT) and it has been assumed that the Critical Path goes through the path with higher probabilities of happening in presence of probabilistic paths.

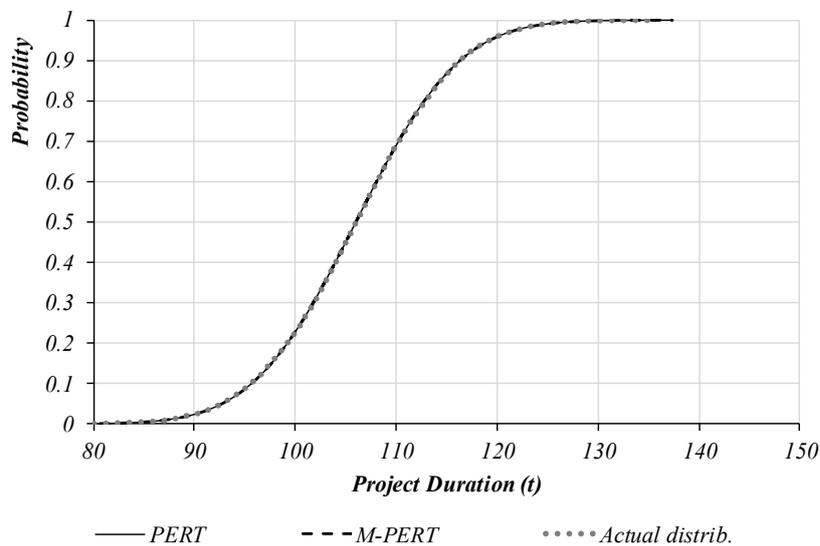
Example Project 1



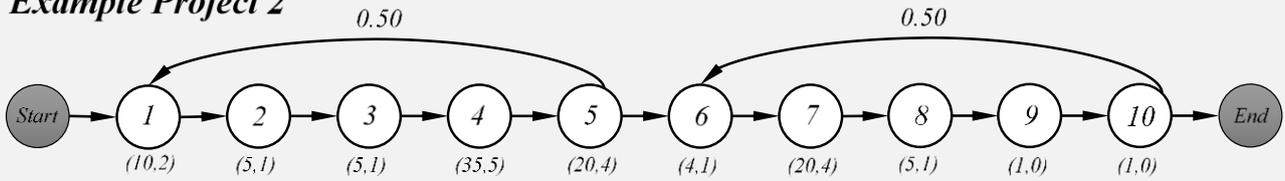
Step-by-step merger procedure



Results



Example Project 2

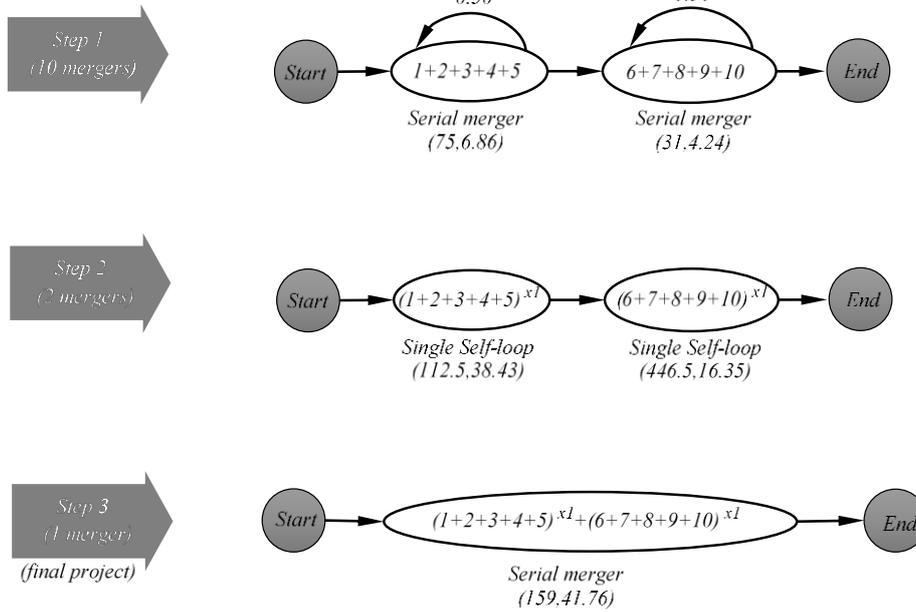


Legend

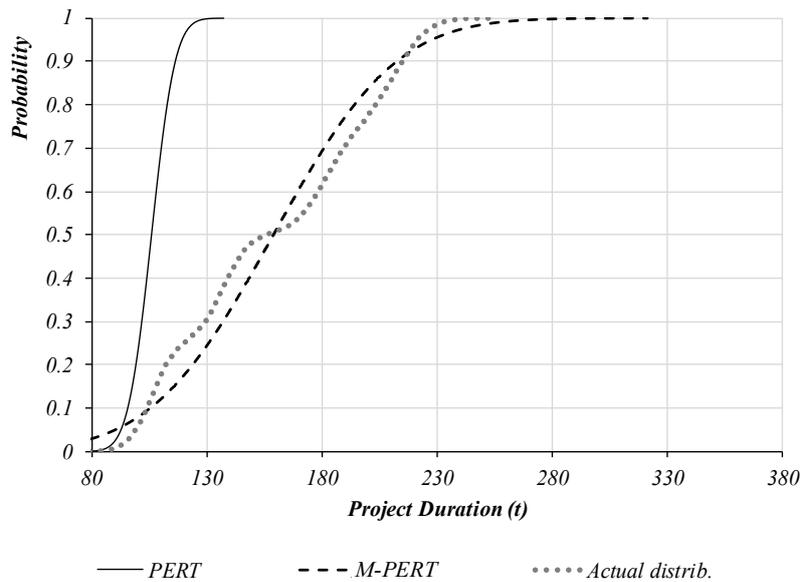


id: activity identifier
 μ : duration mean
 σ : duration std. dev.

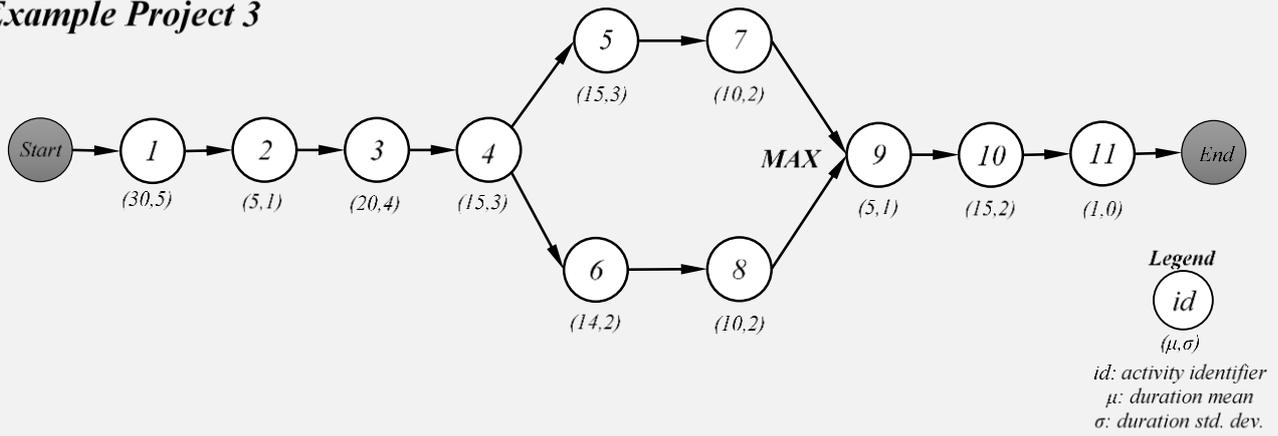
Step-by-step merger procedure



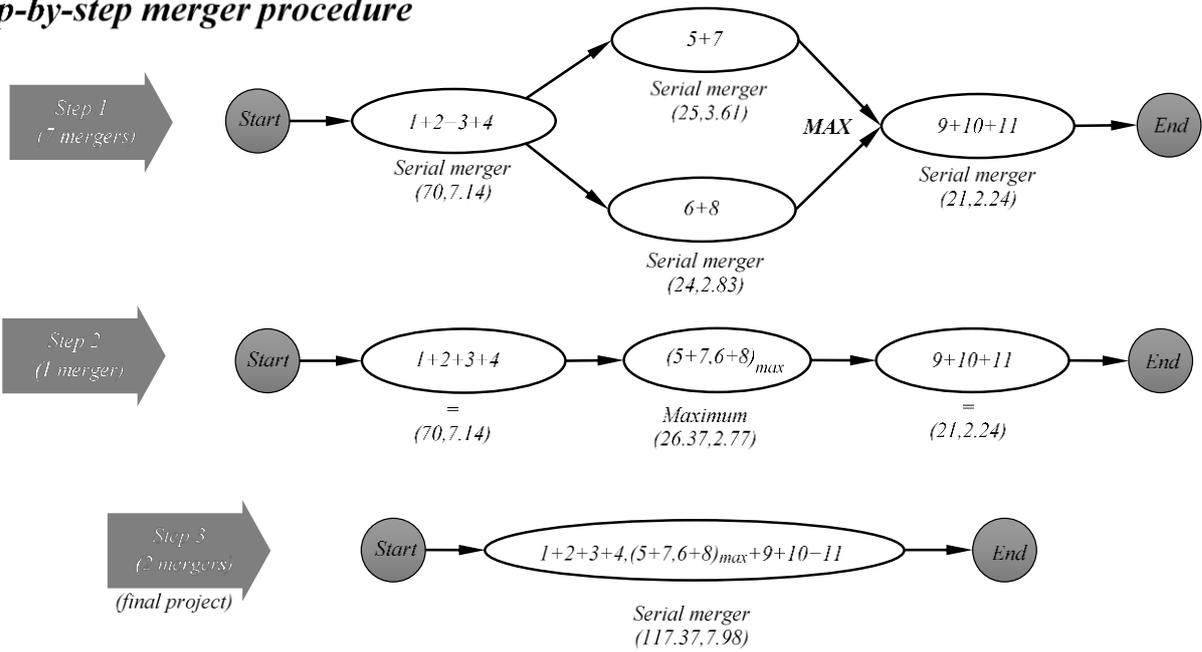
Results



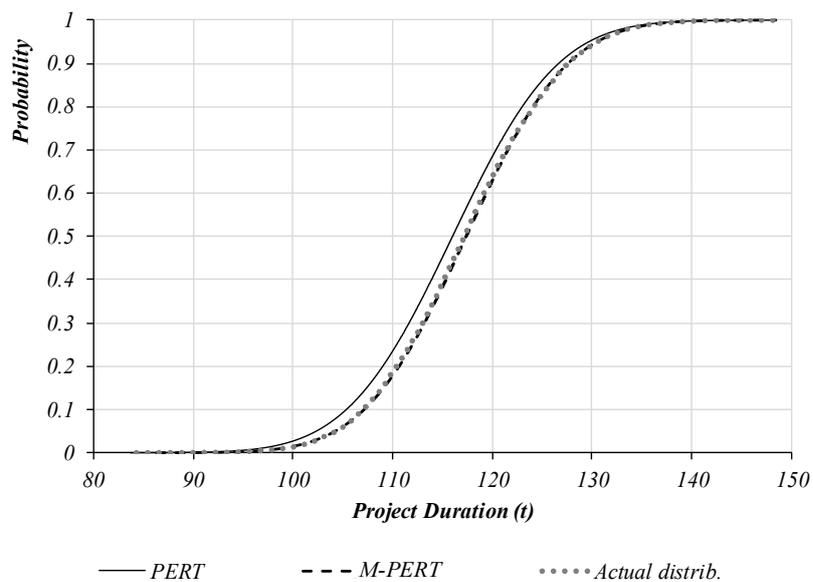
Example Project 3



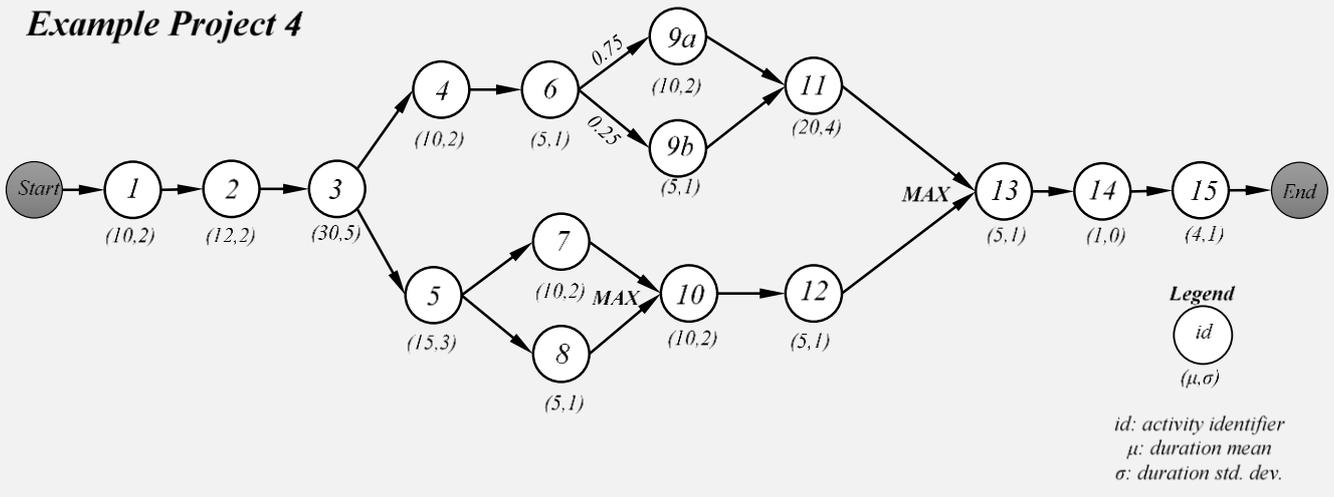
Step-by-step merger procedure



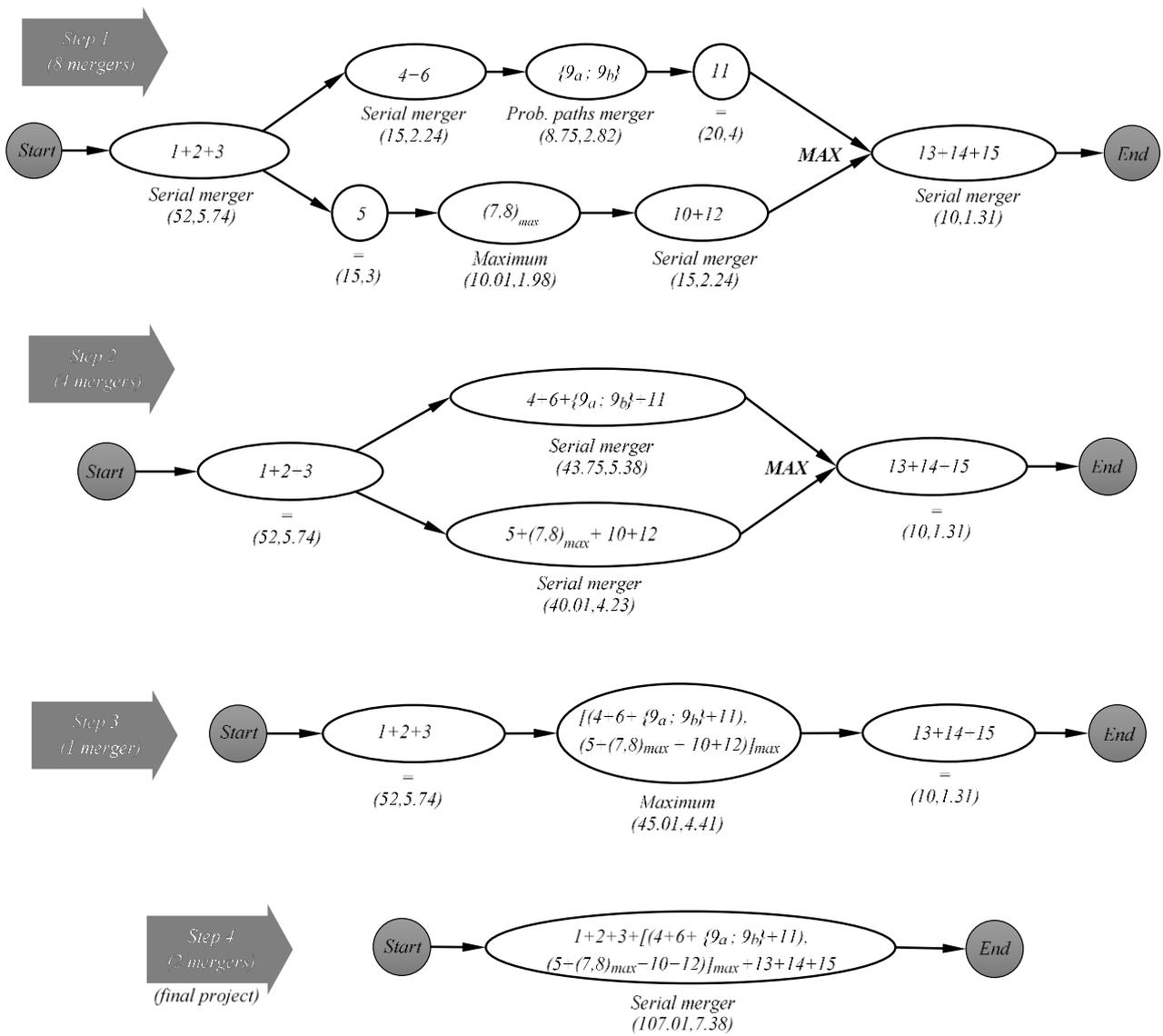
Results



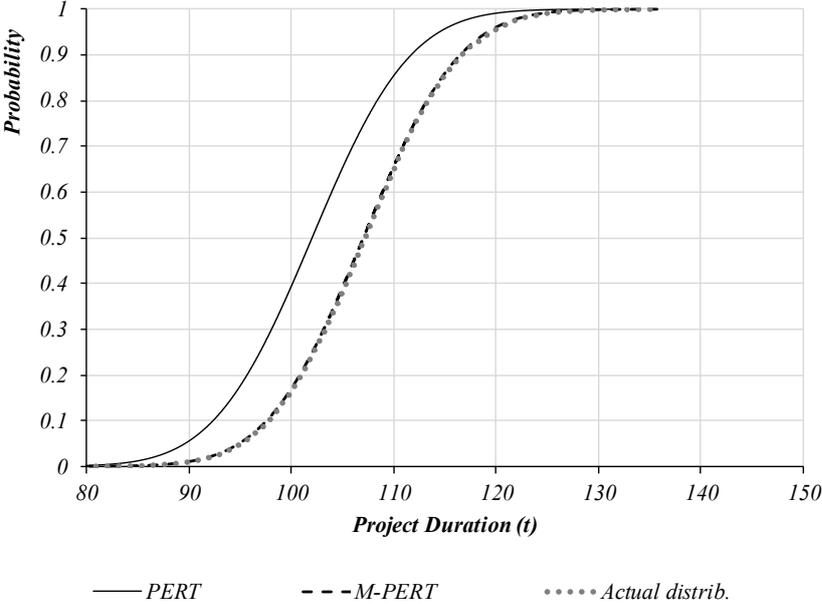
Example Project 4



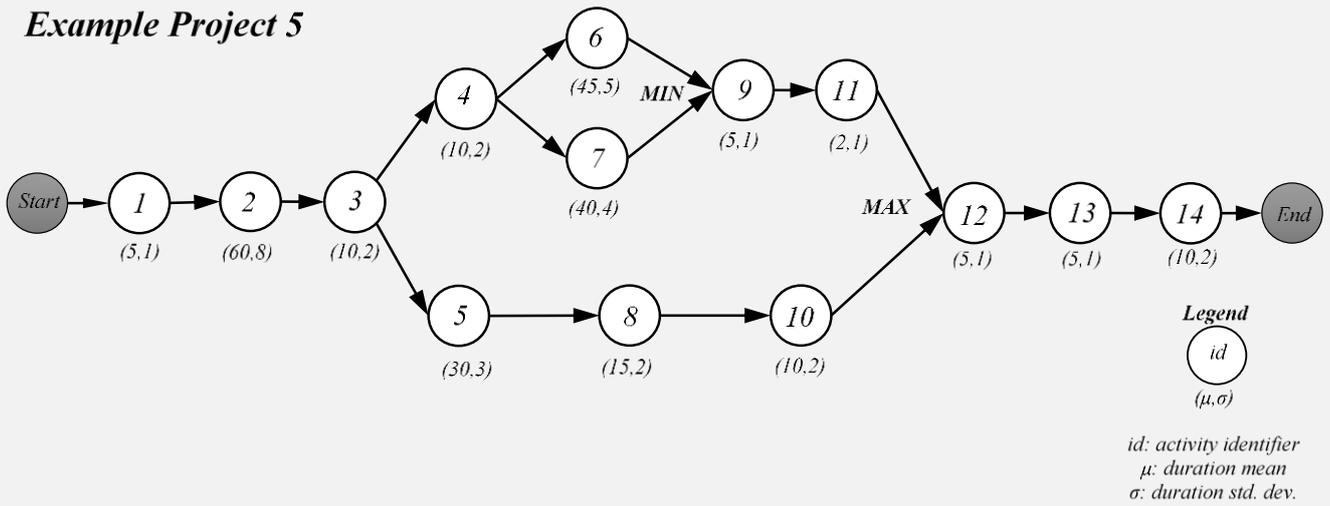
Step-by-step merger procedure



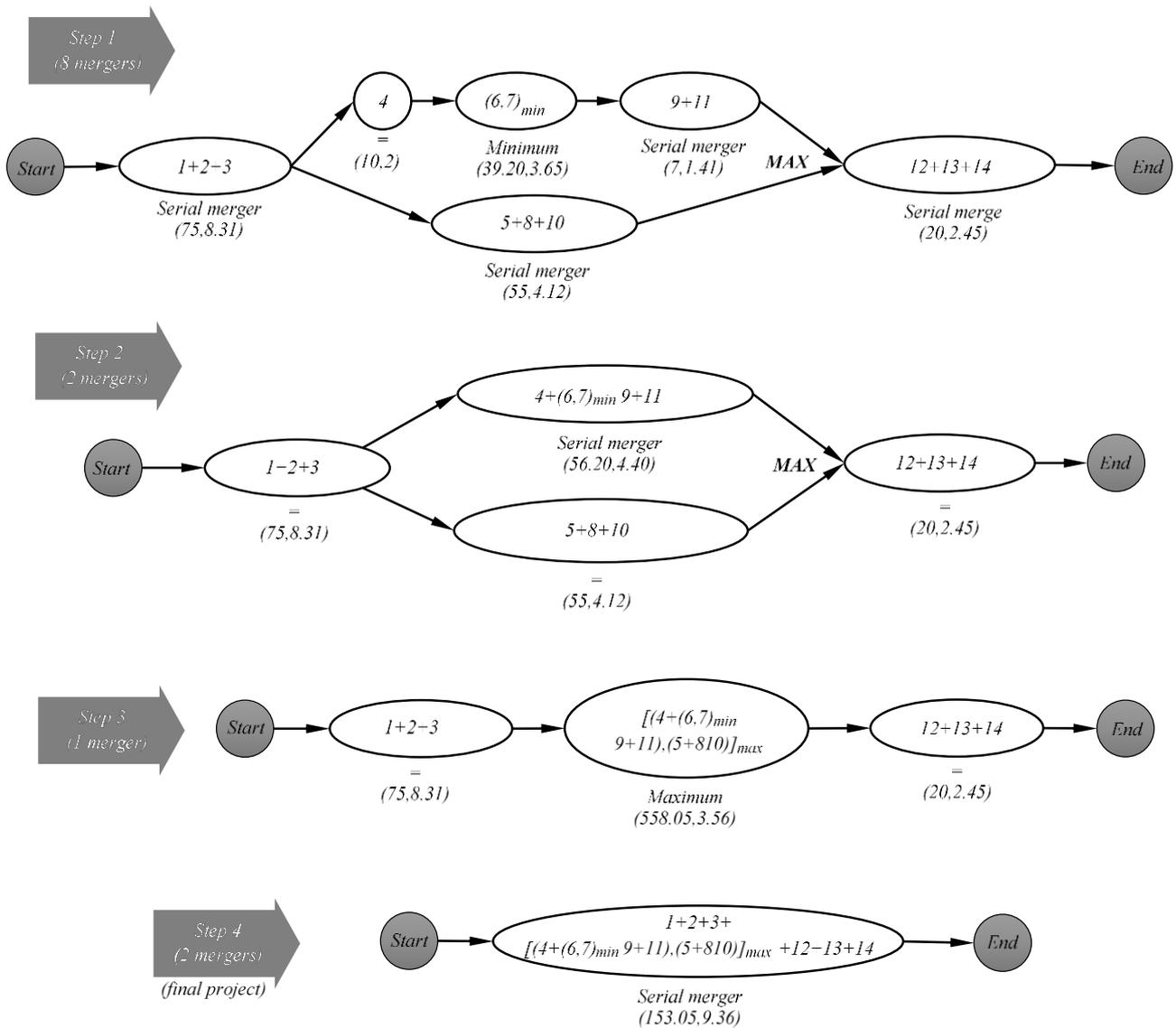
Results



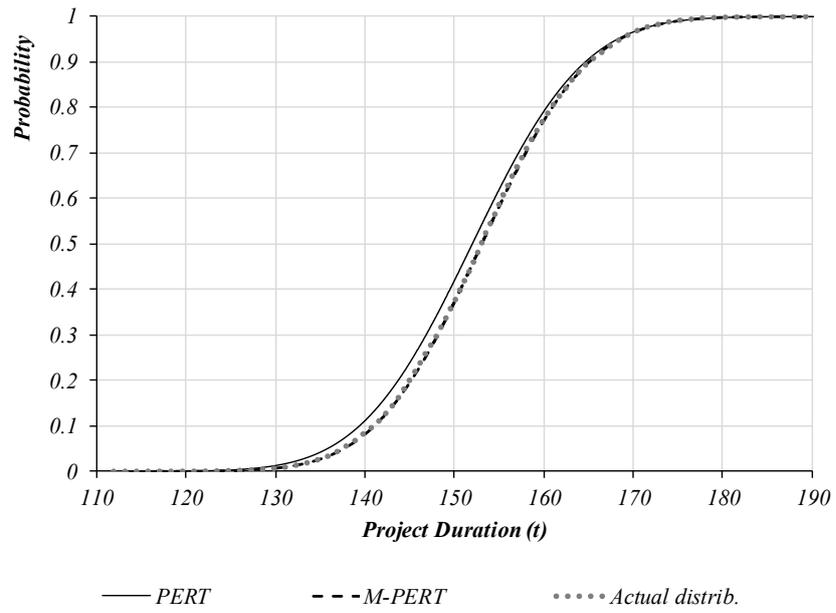
Example Project 5



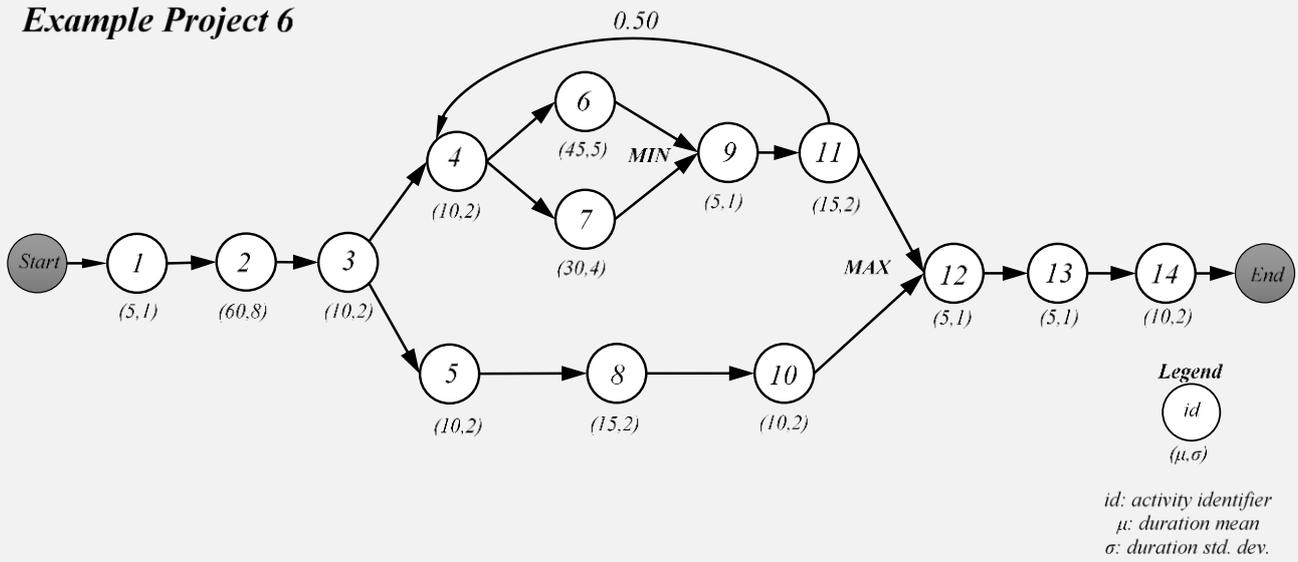
Step-by-step merger procedure



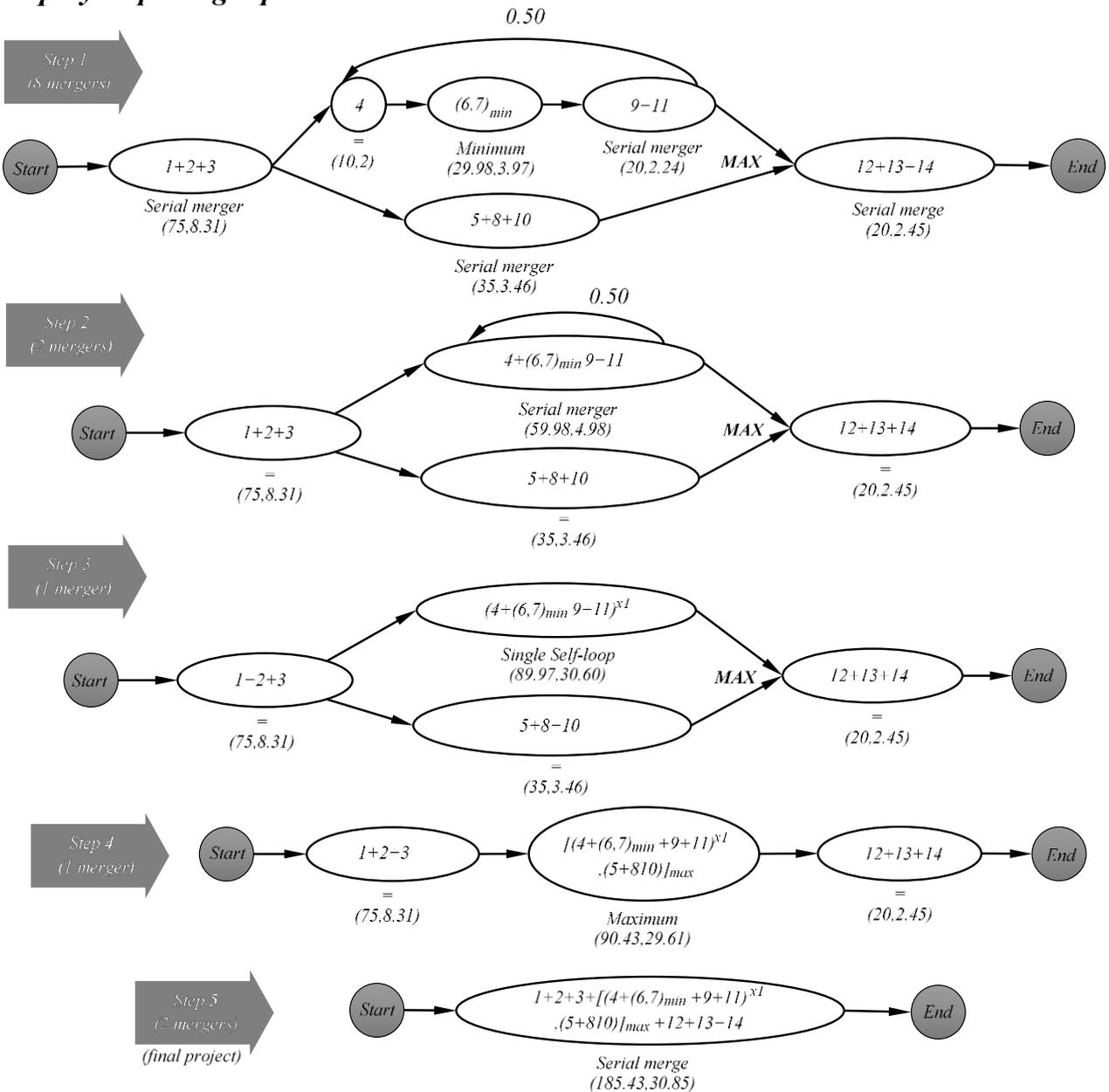
Results



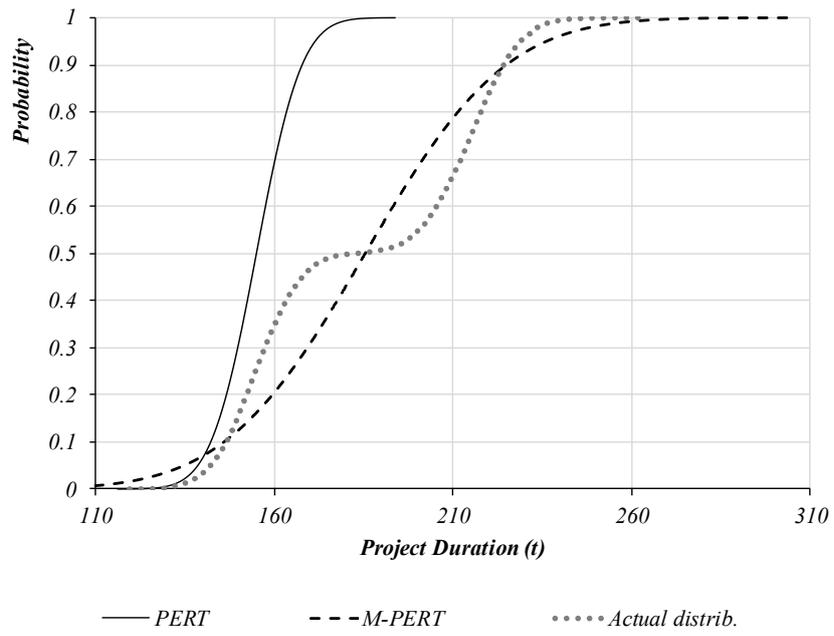
Example Project 6



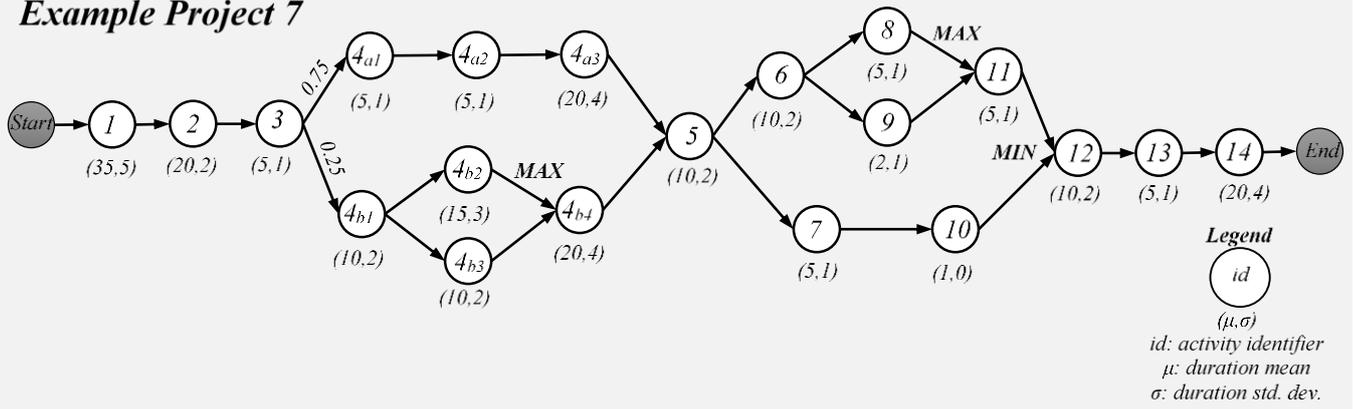
Step-by-step merger procedure



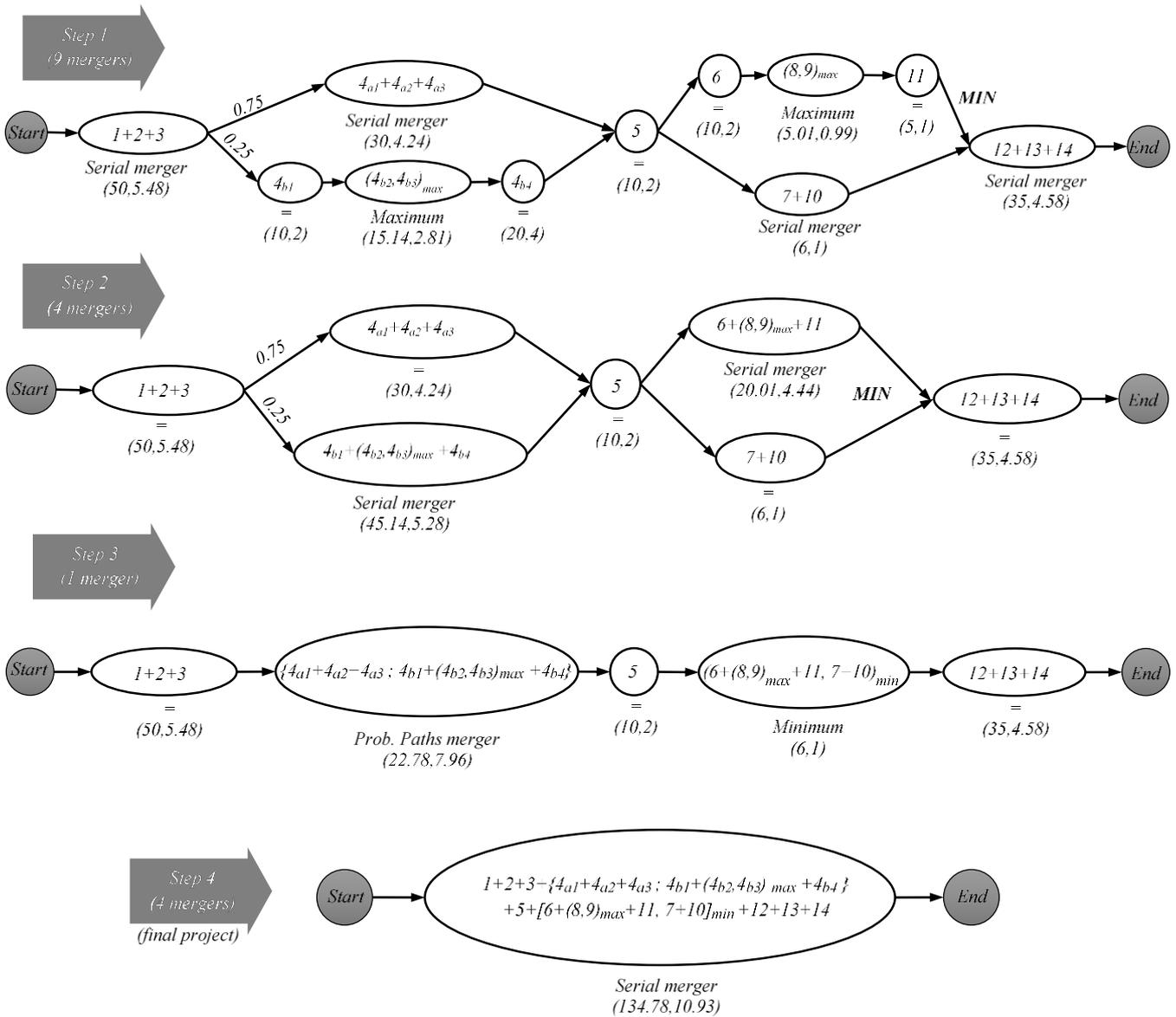
Results



Example Project 7



Step-by-step merger procedure



Results

