

Supplementary materials for Roy, S., Inglis, M., & Alcock, L., When research-based interventions fail: Multimedia resources designed to support learning from written proofs.

Study 1

Theorem and Proof used for Immediate Post-Test

Theorem

Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) .

Suppose also that $\forall x \in (a, b), g'(x) \neq 0$.

Then $\exists c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Proof

- 1) Note that if $g(a) = g(b)$ then by Rolle's Theorem $\exists c \in (a, b)$ s.t. $g'(c) = 0$.
- 2) This contradicts the theorem premise so $g(a) \neq g(b)$.
- 3) Define $h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}g(x)$.
- 4) h is continuous on $[a, b]$ and differentiable on (a, b) (sum and constant multiple rules).
- 5) Also $h(a) = f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}g(a) = \frac{f(a)g(b) - g(a)f(b)}{g(b) - g(a)}$.
- 6) and $h(b) = f(b) - \frac{f(b) - f(a)}{g(b) - g(a)}g(b) = \frac{f(a)g(b) - g(a)f(b)}{g(b) - g(a)}$, so $h(a) = h(b)$.
- 7) Hence, by Rolle's Theorem, $\exists c \in (a, b)$ s.t. $h'(c) = 0$.
- 8) But $h'(c) = 0 \Rightarrow f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(c) = 0 \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.
- 9) So $\exists c \in (a, b)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Theorem and Proof used for Delayed Post-Test

Theorem

Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) .
Suppose also that $\forall x \in (a, b), g'(x) \neq 0$.

Then $\exists c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Proof

- 1) If $g(a) = g(b)$ then $\exists c \in (a, b)$ s.t. $g'(c) = 0$.
- 2) This contradicts the theorem premise so $g(a) \neq g(b)$.
- 3) Define $h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}g(x)$.
- 4) Then h is continuous on $[a, b]$ and differentiable on (a, b) .
- 5) Also $h(a) = h(b)$.
- 6) Hence $\exists c \in (a, b)$ s.t. $h'(c) = 0$.
- 7) But $h'(c) = 0 \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.
- 8) So $\exists c \in (a, b)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Comprehension Test

- (a) How does line (1) contradict the theorem premise? **1 mark**
(b) Why do we need the conclusion from the contradiction? **1 mark**
- Which of the following properties are used in the proof?

If functions p and q are continuous on $[a, b]$ and differentiable on (a, b) then

- $p + q$ is continuous on $[a, b]$ and differentiable on (a, b) . **1 mark (yes)**
 - pq is continuous on $[a, b]$ and differentiable on (a, b) . **1 mark (no)**
 - mp is continuous on $[a, b]$ and differentiable on (a, b) , where m is a constant. **1 mark (yes)**
 - p/q is continuous on $[a, b]$ and differentiable on (a, b) . **1 mark (no)**
- Write a short paragraph to explain why we set

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}g(x).$$

3 marks

- With h defined as in the proof, what is $h'(x)$? **1 mark**
- Where does the proof show that h satisfies the conditions for Rolle's Theorem? **2 marks**

- (a) Imagine that the premises of Cauchy's GMVT read:
Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) and that $\forall x \in (a, b), f'(x) \neq 0$.
What would the conclusion say? **1 mark**

- If you were proving the new version (i.e. (a)) directly, how would you define $h(x)$? **1 mark**

- Find an interval where $f(x) = 1 + x^2$ and $g(x) = x^2 - 1$ satisfy the premises of Cauchy's Generalised MVT. **2 marks**

- Show that

$$f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}g(a) = \frac{f(a)g(b) - g(a)f(b)}{g(b) - g(a)}.$$

2 marks

Additional Information Sheet: Definitions and Theorems

Definition: $f : A \rightarrow \mathbb{R}$ is continuous at $a \in A$ if and only if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $x \in A$ and $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

Definition: $f : A \rightarrow \mathbb{R}$ is differentiable at $a \in A$ if and only if $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists.

Rolle's Theorem Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) and that $f(a) = f(b)$. Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) . Then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Theorem (algebra for continuous functions) Suppose that $f, g : A \rightarrow \mathbb{R}$ are both continuous at $a \in A$ and that $c \in \mathbb{R}$. Then:

1. $cf : A \rightarrow \mathbb{R}$ is continuous at a .
2. $f + g : A \rightarrow \mathbb{R}$ is continuous at a .
3. $fg : A \rightarrow \mathbb{R}$ is continuous at a .
4. $f/g : A \rightarrow \mathbb{R}$ is continuous at a , provided $g(a) \neq 0$.

Theorem (algebra for derivatives) Suppose that $f, g : (a, b) \rightarrow \mathbb{R}$ are both differentiable at $x \in (a, b)$ and that $c \in \mathbb{R}$. Then:

1. $cf : (a, b) \rightarrow \mathbb{R}$ is differentiable at x .
2. $f + g : (a, b) \rightarrow \mathbb{R}$ is differentiable at x .
3. $fg : (a, b) \rightarrow \mathbb{R}$ is differentiable at x .
4. $f/g : (a, b) \rightarrow \mathbb{R}$ is differentiable at x , provided $g'(x) \neq 0$.

Study 2

Theorem, Proof and Comprehension Test 1

Theorem 1

If $n \in \mathbb{Z}$ is composite, then n has a prime factor not exceeding \sqrt{n} .

Proof 1

There are integers a, b such that $1 < a \leq b < n$ and $n = a.b$

If $a > \sqrt{n}$, then $b \geq a > \sqrt{n}$ and $a.b > \sqrt{n}.\sqrt{n} = n$ which is inconsistent with the line above. Therefore, $a \leq \sqrt{n}$.

We know that every positive integer greater than 1 has a prime factor.

Therefore, a has a prime factor p

This prime factor p also divides n , therefore p is also a prime factor of n .

Therefore, $p \leq \sqrt{n}$. This proves the theorem.

Comprehension Test 1

Q1. What method is used to show that $a \leq \sqrt{n}$?

- No particular method.
- Contradiction.
- Induction.
- A direct method.

Q2. Why is showing $a \leq \sqrt{n}$ not enough to prove the theorem?

- Because it was important to show that n has more than one factor, so we considered p .
- Showing $a \leq \sqrt{n}$ is enough, but showing $p \leq \sqrt{n}$ makes the argument stronger.
- Because we don't know whether or not a is prime, but we do know that p is prime.
- If we declare a to be prime, we do not need to show $p \leq \sqrt{n}$.

Q3. Can we write n in terms of p and b ?

- It is impossible to write n in terms of p and b .
- It is only possible if $a = p$, then $n = pb$.
- We can write n in terms of p and b in many different ways.
- None of these is true.

Theorem, Proof and Comprehension Test 2

Theorem 2

There are infinitely many primes.

Proof 2

Let $n > 1$ be a positive integer.

Since n and $n + 1$ consecutive integers, they have no factor in common except 1; therefore they are relatively prime.

Consider the number $K_1 = n(n + 1)$.

We know that every integer greater than 1 must have a prime factor.

Therefore K_1 must have at least two different prime factors.

Similarly, the integers $K_1 = n(n + 1)$ and $K_1 + 1 = n(n + 1) + 1$ are consecutive and therefore relatively prime.

Consider the number $K_2 = K_1(K_1 + 1) = [n(n + 1)][n(n + 1) + 1]$, K_2 must have at least three distinct prime factors.

This process can be continued indefinitely. This proves the theorem.

Comprehension Test 2

Q1. Which of the justifications best explains why “two consecutive numbers are relatively prime”?

- One of them is always odd and the other one is even. Therefore they cannot have a common factor.
- Because this is an accepted rule of mathematics.
- Because the difference between them is 1, therefore the only possible common factor is 1.
- None of these is true.

Q2. Which of the following statements is correct?

- If K_5 has at least 6 different prime factors, then K_6 will have exactly one more but not necessarily different prime factor.
- If K_5 has at least 6 different prime factors, then K_6 must have at least one more different prime factor.
- If K_5 has at least 6 different prime factors, then K_6 will have exactly one more different prime factor.

Q3. Which of the following discussions do you think justifies the proving method of the theorem?

- This is an incomplete proof because it only showed two steps and then stated that “this process can be continued indefinitely”, which is not enough to complete theorem.
- This is a valid proof even though it showed only two steps, because an inductive method was used in the proof.
- This is an invalid proof because it only showed two steps and then stated that “this process can be continued indefinitely”, but anything can happen while continuing the process which we do not know.

Theorem, Proof and Comprehension Test 3

Rolle’s Theorem (Theorem 3)

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) and that $f(a) = f(b)$. Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof 3

Suppose that $f(a) = f(b)$ and f is continuous on $[a, b]$ and differentiable on (a, b) .

Then f is bounded and attains its maximum and minimum values by the EVT.

So $\exists x_1, x_2 \in [a, b]$ s. t. $\forall x \in [a, b], f(x_1) \leq f(x) \leq f(x_2)$.

If x_1 and x_2 are both endpoints of $[a, b]$, then one is equal to a and the other is equal to b .

Hence $f(x_1) = f(x_2)$ so f is constant.

In this case, $\forall c \in (a, b), f'(c) = 0$.

If x_1 and x_2 are not both endpoints, then $x_1 \in (a, b)$ or $x_2 \in (a, b)$ (or both).

If $x_1 \in (a, b)$ then $f'(x_1) = 0$ by the IET.

If $x_2 \in (a, b)$ then $f'(x_2) = 0$ by the IET.

So in all cases, $\exists c \in (a, b)$, such that $f'(c) = 0$.

Comprehension Test 3

Q1. Let $f(x)$ be a function continuous on $[a, b]$ and differentiable on (a, b) . If there exists c in (a, b) such that $f'(c) = 0$, then $f(a) = f(b)$.

- True.
- False.

Q2. It follows from Rolle’s theorem that there exists only one point such that $f'(c) = 0$.

- True.
- False.

Q3. Do all the assumptions of Rolle's Theorem hold for $f(x) = |x|$ on the interval $[-5, 5]$?

- Yes.
- No.

Q4. Do all the assumptions of Rolle's Theorem hold for $f(x) = \tan(x)$ on the interval $[-\pi/2, \pi/2]$?

- Yes.
- No.

Q5. Is the assumption that $f(a) = f(b)$ necessary to find c in (a, b) such that $f'(c) = 0$?

- Yes necessary, because otherwise we would not be able to find c in (a, b) such that $f'(c) = 0$.
- Not necessary, because all the three assumptions are sufficient conditions but not necessary.
- It depends on the function.
- It depends on the interval.

Theorem, Proof and Comprehension Test 4

Cauchy's GMVT (Theorem 4)

Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) . Suppose also that $\forall x \in (a, b), g'(x) \neq 0$. Then $\exists c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Proof 4

Note that if $g(a) = g(b)$ then by Rolle's Theorem $\exists c \in (a, b)$ s.t. $g'(c) = 0$.

This contradicts the theorem premise so $g(a) \neq g(b)$.

Define $h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}g(x)$.

h is continuous on $[a, b]$ and differentiable on (a, b) (sum and constant multiple rules).

Also $h(a) = f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}g(a) = \frac{f(a)g(b) - g(a)f(b)}{g(b) - g(a)}$.

and $h(b) = f(b) - \frac{f(b) - f(a)}{g(b) - g(a)}g(b) = \frac{f(a)g(b) - g(a)f(b)}{g(b) - g(a)}$, so $h(a) = h(b)$.

Hence, by Rolle's Theorem, $\exists c \in (a, b)$ s.t. $h'(c) = 0$.

But $h'(c) = 0 \Rightarrow f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(c) = 0 \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

So $\exists c \in (a, b)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Comprehension Test 4

Q1. Choose an interval (with a correct justification) where the functions $f(x) = 1 + x^2$ and $g(x) = x^2 - 1$ satisfy the premises of Cauchy's GMVT.

- On the interval $[-1, 1]$ because $f(x)$ and $g(x)$ cross the y -axis at 1 and -1 respectively.
- On the interval $[1, \infty)$ because interval must be continuous and differentiable.
- On the interval $[1, 12]$ because it does not contain 0.
- On the interval $(0, 2)$ because $f'(x) = 2x$ and $g'(x) = 2x$.

Q2. Which of the following properties are used in the proof? If functions f and g are continuous on $[a, b]$ and differentiable on (a, b) then

- $f + g$ and kg are continuous on $[a, b]$ and differentiable on (a, b) ; where k is a constant.
- $f + g$ and fg are continuous on $[a, b]$ and differentiable on (a, b) .
- kg and f/g are continuous on $[a, b]$ and differentiable on (a, b) ; where $g \neq 0$ and k is a constant.
- $f + g$ and f/g are continuous on $[a, b]$ and differentiable on (a, b) ; where $g \neq 0$.

Q3. Choose the best short summary of Cauchy's GMVT from the following short summaries.

- The quotient of two derivatives of continuous functions f and g is equal to a constant belonging to (a, b) .
- Establish $g(a) \neq g(b)$ then we can divide by $g(b) - g(a)$. Define h in terms of f and g and prove that h satisfies Rolle's theorem on $[a, b]$. Apply Rolle's theorem to conclude that there exists $c \in (a, b)$ s.t. $h'(c) = 0$. We rewrite $h'(c) = 0$ in terms of f and g and rearrange to obtain the desired conclusion.
- On $[a, b]$ with continuous functions f and g , there is a point c , where the ratio between the gradients $f'(x)$ and $g'(x)$ at this point c is equal to the ratio between the straight lines $f(b) - f(a)$ and $g(b) - g(a)$.
- Make sure denominator is never zero. Find a function h by manipulating s.t. it satisfies Rolle's theorem. Prove continuity and differentiability of h . $h(a) = h(b)$ implies Rolle's theorem which implies $h'(c) = 0$. Manipulate $h'(c)$ to obtain the desired result.