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European Society for Engineering Education
Europäische Gesellschaft für Ingenieur-Ausbildung
Société Européenne pour la Formation des Ingénieurs

European Society for Engineering Education (SEFI)

A Framework for Mathematics Curricula in Engineering Education

A Report of the Mathematics Working Group



“Each tooth of one gear should meet each tooth of the other one for equally distributed abrasion. How does this affect the choice of tooth numbers?”

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SEFI is the largest network of higher engineering education institutions (HEIs) and educators in Europe. It is an international non profit organisation (scientific society) created in 1973 to contribute to the development and improvement of higher engineering education (HEE) in Europe, to reinforce the position of the engineering professionals in society, to promote information about HEE and improve communication between teachers, researchers and students, to reinforce the university-business cooperation and to encourage the European dimension in higher engineering education. Through its membership composed of HEIs, academic staff, students, related associations and companies, SEFI connects over 1 million students and 158000 academic staff members in 48 countries. To fulfill its mission and objectives, SEFI implements diverse activities such as Annual Conferences, Ad hoc seminars/workshops organised by its thematic working groups, organises specific activities for the Engineering Deans, publishes a series of Scientific publications (European Journal of Engineering Education) and Position Papers, is involved in European projects, cooperates with other major European and international associations and international bodies. SEFI also participated in the creation of engineering organisations and networks such as ENAEE, IFEEES, Euro-Pace, IACEE and more recently of the first “European Engineering Deans Council”, EEDC, and of the International Institute for the Development of Engineering Academics, IIDEA. SEFI acts taking into account a series of values: Creativity and professionalism, engagement and responsibility, respect for diversity and different cultures, institutional inclusiveness, multi-disciplinarity and openness, transparency and sustainability.

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Preface – By the President of SEFI

Dear reader,

It is with great pleasure that I could present you this book about a framework for Mathematics Curricula in Engineering Education.

As President of SEFI I would first of all like to emphasize that this report illustrates in many ways what SEFI stands for. This report is published by the Mathematics Working Group. Working Groups are SEFI's vehicle to bring together its members around a particular topic relevant for Engineering Education. The Mathematics Working Group has already a long-time history and highly estimated track record of activities for all those interested in how Mathematics could be integral part of an engineering program. This publication should be seen in the context of this tradition. Of course, the ideas have developed over the years, and this new framework gives insight in the current state-of-the-art in the engineering curriculum design with respect to the integration of Mathematics competence development. I am sure you will find pointers in this book to practical implementation guidelines for your own practice or you will be intrigued by the principles that are put forward.

A publication like this one is only possible thanks to the great efforts of many people, SEFI members active in the Working Group and others. My highest appreciation goes to all contributors. I would especially like to thank our colleague, Prof. Dr. Burkhard Alpers from the Aalen University (Germany), as chairman of the Working Group and driving force behind this report. SEFI is proud to have all these committed members and partners, and wishes to cherish very much what they together achieve to the benefit for the engineering education community in Europe and beyond.

I wish you lots of inspiration by reading this newest SEFI publication!

Prof. Wim Van Petegem,

SEFI President 2011-2013

Executive Summary

The goal of SEFI's Mathematics Working Group (MWG) is to provide a discussion forum and orientation for those who are interested in the mathematical education of engineering students in Europe. An important contribution to this goal is the group's curriculum document which was first issued in 1992. After ten years, in 2002, a second edition was published which brought the document more in line with current curriculum practices by formulating a detailed and structured list of concrete content-related learning outcomes. During the last decade, in many of the MWG's seminars the topics of higher-level learning goals and outcomes have been discussed. It is the intention of this volume, the third edition of the curriculum document, to state, explain and exemplify a framework for systematically including such higher-level learning goals based on state-of-the-art educational research. For this purpose, the competence concept developed in Denmark and later adopted in the famous OECD PISA study is used. Mathematical competence is the ability to understand, judge, do and use mathematical concepts in relevant contexts and situations, which certainly is the predominant goal of the mathematical education for engineers. Therefore, the main message of this new edition is that although content remains important, knowledge should be embedded in a broader view of mathematical competencies.

This document adapts the competence concept to the mathematical education of engineers and explains and illustrates it by giving examples. It also provides information for specifying the extent to which a competency should be acquired. It **does not** prescribe a particular level of progress for competence acquisition in engineering education. There are many different engineering branches and many different job profiles with various needs for mathematical competencies; consequently it is not appropriate to specify a fixed profile. The competence framework serves as an analytical framework for thinking about the current state in one's own institution and also as a design framework for specifying the intended profile. A sketch of an example profile for a practice-oriented study course in mechanical engineering is given in the document. This document retains the list of content-related learning outcomes (slightly modified) that formed the 'kernel' of the previous curriculum document. These are still important because lecturers teaching application subjects want to be sure that students have at least an 'initial familiarity' with certain mathematical concepts and procedures which they need in their application modelling.

In order to offer helpful orientation for designing teaching processes, teaching and learning environments and approaches are outlined which help students to obtain the competencies to an adequate degree. It is clear that such competencies cannot be obtained by simply listening to lectures, so adequate forms of active involvement of students need to be included. Moreover, in a competence-based approach the mathematical education must be integrated in the surrounding engineering study course to really achieve the ability to use mathematics in engineering contexts. The document presents several forms of how this integration can be realized. This integration is essential to the development of competencies and will require close co-operation between mathematics academics and their engineering counterparts. Finally, since assessment procedures determine to a great extent the behaviour of students, it is extremely important to address competency acquisition in assessment schemes. Ideas for doing this are also outlined in the document.

The main purpose of this document is to provide orientation for those who set up concrete mathematics curricula for their specific engineering programme, and for lecturers who think about learning and assessment arrangements for achieving the intended level of competence acquisition. It also serves as a framework for the group's future work and discussions.

“... those who are developing new curricula should, despite reformist zeal, proceed with due caution.”
Alan H. Schoenfeld (1994)

1 Introduction – Goals and Use of the Curriculum Document

When the SEFI Mathematics Working Group set up its first “Core Curriculum” in 1992, Peter Nüesch, one of the co-authors and former SEFI president, wrote in his address preceding the curriculum (Barry & Steele 1992, p.8): “It is hoped that our Core Curriculum answers only the one very essential question: what should be the content of mathematics courses for engineers?” Accordingly, the ‘heart’ of the curriculum document consisted of a list of topics to be dealt with, organised on different levels, although it is fair to state that other issues concerning the educational process were briefly commented on. For the second edition of the curriculum document in 2002 (Mustoe & Lawson 2002), one motivation for change was to bring the curriculum more in line with current curriculum practices and “... phrase a curriculum in terms of learning outcomes rather than a list of topics to be covered” (p.2). This resulted in a quite detailed organised list of content-related learning outcomes. Moreover, other issues like the role of technology, transition problems and other educational goals like communication and modelling were included in a short commentary section.

During the last decade, in many seminars of the group the topic of higher-level learning goals and outcomes arose. This can be found specifically in the contribution by (Booth 2004) on “learning for understanding” and the paper by (Cardella 2008) on using a “broad notion of mathematical thinking”. Although the curriculum document as of 2002 contains some short statements on such goals (chapter 4, p.47), it does not apply a systematic approach which could provide a framework for other didactical issues in the document. It is the intention of the current third edition of the curriculum document to state, explain and exemplify such a framework based on state-of-the-art educational research. Nevertheless, contents and content-related learning outcomes still provide important orientation for what colleagues in application subjects expect from the mathematical education of engineers. Therefore, the main message of this new edition is that although contents are still important, they should be embedded in a broader view of mathematical competencies that the mathematical education of engineers strives to achieve. The history of the curriculum document so far can hence be described as going ‘from contents to outcomes to competencies’.

When trying to set up a framework for specifying higher-level goals based on current insights from educational research, there are several sources available within the general mathematics education community aiming at school mathematics or undergraduate education or both (for an overview of curricular trends in tertiary education see (Hillel 2001)). Cardella (2008) proposes to use the aspects of mathematical thinking identified by Schoenfeld (1992, 1994) to broaden the view of what mathematical education of engineers should strive for. Schoenfeld emphasises that alongside content knowledge, there are problem solving strategies, meta-cognitive processes in using resources, beliefs and affects and mathematical practices which together make up mathematical thinking:

“... mathematical thinking consists of a lot more than knowing facts, theorems, techniques, etc. ... I would characterize the mathematics a person understands by describing what that person can do mathematically, rather than by an inventory of what the person ‘knows.’” (Schoenfeld 1994)

Schoenfeld's aspects can also be found when observing engineering students as well as engineers working on practical tasks (Cardella 2008). Asiala et al. (1996) similarly present a broad perspective of "what it means to learn and know something in mathematics":

"An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations".

In 2004, the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America wrote a "Curriculum Guide" which contains recommendations that argue along similar lines (Barker et al. 2004). Among other items, the recommendations state:

"... Every course should incorporate activities that will help all students progress in developing analytical, critical reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind. ... Promote awareness of connections to other subjects And strengthen each student's ability to apply the course material to these subjects. ... At every level of the curriculum, some courses should incorporate activities that will help all students progress in learning to use technology ..." (p.1 and p.2)

The report is also based on several workshops where members of 'partner disciplines' (including engineering) stated their understanding of the mathematical qualifications needed for being successful in the discipline (Ganter & Barker 2004).

Finally, in the Danish KOM project a group headed by Niss organised their description of what mathematical education intends to achieve around the notion of competence which also strongly influenced the description of educational goals in the famous OECD-PISA study (OECD 2009):

"Mathematical competence (in the original italics are used instead of underlining) then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. Necessary, but certainly not sufficient, prerequisites for mathematical competence are lots of factual knowledge and technical skills ..." (Niss 2003a, p.6/7)

Blomhoj and Jensen (2007, p. 47) put it in a nutshell by defining a mathematical competency (an ingredient of mathematical competence) as

"... someone's insightful readiness to act in response to a certain kind of mathematical challenge (in the original italics are used instead of underlining) of a given situation ...".

In order to be useful, the KOM project identified a list of such mathematical competencies which overlap but have different emphasis: thinking mathematically; posing and solving mathematical problems; modelling mathematically; reasoning mathematically; representing mathematical entities; handling mathematical symbols and formalism; communicating in, with, and about mathematics; making use of aids and tools. This list is explained in more detailed in Niss 2003a,b. It is meant to be a framework (like the aspects stated by Schoenfeld) overarching all stages of education including tertiary education. For a certain educational setting like engineering education, the specific 'mathematical challenges' have to be identified and the competencies must be interpreted in this context. In order to describe progress in obtaining the competencies during different stages of

education the KOM project identified three dimensions (degree of coverage, radius of action, technical level (see Niss 2003a, p. 10)). These can be used to analyse or prescribe in more detail what one wants the students to achieve at a certain level of education or for a certain educational profile.

Even if the above descriptions provide just a glimpse into the concepts that the authors use, they show the large degree of commonality in identifying a broader spectrum of goals mathematical education should strive for, going far beyond a content-based approach. In this curriculum, we use the competence-based framework set up in the Danish KOM project to identify the higher-level goals of the mathematical part of engineering education. This is also in line with current trends in general engineering education where the notion of “competence” has been used to describe educational activities which favour “action-based knowledge over knowledge simply held, in the name of performance and effectiveness” (Lemaitre et al. 2006, p.47). Competence in this sense is contextualised, i.e. related to a “field of activity, a series of specific tasks ... and a given situation” (p.50). However, it should be noted that the term ‘competence’ is used very differently by various authors (even including the meaning of lower-level skill) and that on the other hand other terms like ‘skill’, ‘capability’, ‘capacity’ are used in literature with a meaning similar to the one given for ‘competence’ (for a discussion of the confusing usage of these terms see Lemaitre et al. (2006)).

The second chapter of this document describes the eight competencies and the three dimensions of progress in more detail. Moreover, we give an illustrative engineering mathematics example for the competencies. We do not prescribe a particular level of progress for engineering education. On the surface, this would facilitate student exchange but it would neglect the difference between study courses and hence would have a low probability of being used. The final section in chapter 2 sketches how a curriculum for a practice-oriented study course in mechanical engineering could be specified using the competence framework.

The third chapter deals with content-related competencies (learning outcomes) concerning knowledge and skills. The latter formed the ‘kernel’ of the curriculum document as of 2002. We still think that these are important since colleagues teaching application subjects want to be sure that students have at least an ‘initial familiarity’ with certain mathematical concepts and procedures they need in their application modelling (as Artigue, Batanero & Kent (2007, p.1034) put it: “The right balance must be found”). The content-related learning outcomes are organized according to mathematical domain. In order to foster mathematical sense-making, we also provide some overarching themes like ‘quantity’ and ‘space and shape’ for organising these outcomes. This was also done in the OECD PISA document (OECD 2009).

In order to provide helpful orientation for designing one’s own teaching, the fourth chapter outlines teaching and learning environments which might help students to obtain the competencies to an adequate degree. It is clear that such competencies cannot be obtained by just listening to lectures, so adequate forms of active involvement of students need to be installed. Topics like transition issues, use of technology and integration of mathematics and engineering education are also discussed here. The short competency definition by Blomhoj and Jensen (2007) indicates that mathematical competency is strongly related to attitude towards mathematics since the ‘readiness’ mentioned in the definition can be expected when one has a somewhat positive attitude with respect to its helpfulness. Therefore, this chapter concludes with an outline of the attitude towards mathematics that we wish engineering students to develop.

Quite understandably, students are also oriented towards getting good marks. Therefore the assessment procedures determine to a good extent the behaviour of students and are hence important for really achieving progress in competencies. Chapter 5 outlines different forms of assessment which might be adequate for capturing certain kinds of achievements. It also discusses the role of technology in assessment and the question of identifying requirements for passing.

The current curriculum document does not prescribe a specific degree of progress relating to mathematical competences or a determined set of content-related learning outcomes. The engineering profession and hence engineering study programmes at university are far too heterogeneous to identify one profile for all. The main purpose of this document is to provide orientation for those who set up concrete mathematics curricula for their specific engineering programme. The competence framework should help to avoid an approach that is mainly restricted to contents. It can be used to analyse existing curricula and to design new ones. It helps institutions and lecturers to identify their own profile in that it facilitates the description of the role and importance of different competencies and hence their weighting within a study programme. Having this in addition to the profile concerning the content-related learning outcomes organised in chapter 3, the intention stated for the second edition of this curriculum is still valid but from a much broader perspective: "This curriculum is intended as a benchmark by which higher education institutions in Europe can judge the mathematics provision in their engineering undergraduate degree programmes." (Mustoe & Lawson 2002, p. 2).

In recent investigations (Cardella 2008; Barker et al. 2004; Ganter & Barker 2004) the importance of having close contacts between lecturers in mathematics and engineering departments was emphasised. The competencies can also serve for discussing with engineering lecturers in which ways the mathematical education of engineers is distributed between mathematics and application subjects. The second edition of the curriculum (2002) already states – from a content-related point of view – that many of the topics listed on level 3 of the curriculum will rather be taught "as part of units on the engineering topics to which they directly apply." (Mustoe & Lawson 2002, p. 45). This is not only true with respect to contents but definitely also with respect to mathematical competencies (or mathematical thinking, cf. Cardella 2008, p. 153). Considering for example the modelling competency, setting up models and solving problems within models is certainly an important activity in engineering mechanics and in many other engineering subjects that make heavy use of mathematics. Having experienced the usage of a mathematical concept in different application subjects definitely adds to the mathematical competence of a student in that it makes a concept more meaningful and also helps to develop an attitude towards mathematics where the role of the latter is perceived as potential problem solver.

Finally, this document is not meant to be a 'Handbook for the mathematical education of engineers'. Nevertheless, it intends to give support for thinking about many aspects of mathematics education like learning environments and assessment since these are quite important for achieving the competencies stated in chapter 2. In this document we merely want to give an overview and to provide some guidance. Many of the issues are, and will be, discussed in journal articles and contributions to seminars of the working group. The reader is advised to consult the Mathematics Working Group's webpage for such material and current discussions (sefi.htw-aalen.de).

2 General Mathematical Competencies for Engineers

As was already stated in the introduction, we adopt the definition of mathematical competence used in the Danish KOM project. Hence, we define **mathematical competence** as “*the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role*” (Niss 2003a, p.6/7). In this chapter we first give an overview of the constituents of such competence, the mathematical competencies identified within the Danish KOM project, and explain their meaning. These competencies are not ‘binary variables’ which are present or not. In order to describe the extent to which competencies do or should exist, the KOM project uses three dimensions: degree of coverage, radius of action and technical level. These are explained in the next section. Moreover, the OECD PISA document (OECD 2009) provides a structuring of the competencies into different levels, called ‘clusters’ which will be adapted to the mathematical education of engineers in the next section. The second section provides examples of the competencies in engineering contexts; this serves to illustrate how the competency concept can actually be used in concrete engineering education settings to describe the goals of mathematics education. The third section then gives some information on how to specify educational profiles using the competency framework.

2.1 Competencies, Dimensions, and Clusters

In order to specify in more detail what mathematical competence is, the KOM project set up a list of eight competencies which together constitute the overall competence. We reproduce a slightly modified version of this list and give some short explanations based on Niss 2003a, p. 7-9. For a more detailed description we refer the reader to Niss 2003a,b and the final report of the KOM group (Niss & Højgaard 2011). Moreover, the illustrative example in the next section should provide more insight into the meaning of the competencies and their application to engineering education.

1. Thinking mathematically

This competency comprises a knowledge of the kind of questions that are dealt with in mathematics and the types of answers mathematics can and cannot provide, and the ability to pose such questions. It includes the recognition of mathematical concepts and an understanding of their scope and limitations as well as extending the scope by abstraction and generalisation of results. This also includes an understanding of the certainty mathematical considerations can provide.

2. Reasoning mathematically

This competency includes on the one hand the ability to understand and assess an already existing mathematical argumentation (chain of logical arguments), in particular to understand the notion of proof and to recognise the central ideas in proofs. It also includes the knowledge and ability to distinguish between different kinds of mathematical statements (definition, if-then-statement, iff-statement etc.). On the other hand it includes the construction of chains of logical arguments and hence of transforming heuristic reasoning into own proofs (reasoning logically).

3. Posing and solving mathematical problems

This competency comprises on the one hand the ability to identify and specify mathematical problems (be they pure or applied, open-ended or closed) and on the other hand the ability to solve mathematical problems (including knowledge of the adequate algorithms). What really constitutes a problem is not well defined and it depends on personal capabilities

whether or not a question is considered as a problem. This has to be borne in mind, for example when identifying problems for a certain group of students.

4. Modelling mathematically

This competency also has essentially two components: the ability to analyse and work in existing models (find properties, investigate range and validity, relate to modelled reality) and the ability to ‘perform active modelling’ (structure the part of reality that is of interest, set up a mathematical model and transform the questions of interest into mathematical questions, answer the questions mathematically, interpret the results in reality and investigate the validity of the model, monitor and control the whole modelling process). This competency has been investigated in more detail in Blomhøj & Jensen 2003, 2007.

5. Representing mathematical entities

This competency includes the ability to understand and use mathematical representations (be they symbolic, numeric, graphical and visual, verbal, material objects etc.) and to know their relations, advantages and limitations. It also includes the ability to choose and switch between representations based on this knowledge.

6. Handling mathematical symbols and formalism

This competency includes the ability to understand symbolic and formal mathematical language and its relation to natural language as well as the translation between both. It also includes the rules of formal mathematical systems and the ability to use and manipulate symbolic statements and expressions according to the rules.

7. Communicating in, with, and about mathematics

This competency includes on the one hand the ability to understand mathematical statements (oral, written or other) made by others and on the other hand the ability to express oneself mathematically in different ways.

8. Making use of aids and tools

This competency includes knowledge about the aids and tools that are available as well as their potential and limitations. Additionally, it includes the ability to use them thoughtfully and efficiently.

These competencies are overlapping (i.e. aspects of one competency are also needed within another, for example to express oneself using symbols one needs the competency of handling mathematical symbols) but emphasise different aspects and are therefore separated. They can be organised in two groups. Competencies 1 to 4 make up “the ability to ask and answer questions in and with mathematics” (Niss 2003a, p. 7) whereas competencies 5 to 8 are concerned with “the ability to deal with and *manage mathematical language and tools*” (Niss 2003a, p. 8). The list is not derived from theoretical considerations. Its value lies in leading the thinking process about what we want to achieve in the mathematical education of engineers to abilities that are widely accepted as being important. This value will become evident in the next section when we present examples.

If one wants to state for a certain mathematical competency to which extent students should have obtained it at a certain stage of their mathematical education, one needs some criteria or dimensions for specifying this. In the KOM project, three different dimensions for specifying and measuring progress are introduced (Niss 2003a, p. 10):

- Degree of coverage “is the extent to which the person masters the characteristic aspects” of a competency. In the short descriptions given above one can already recognise that a competency consists of or includes a bundle of components. For example, there often is an

‘analytic’ side related to understanding and analysing existing ‘objects’ (expressions, proofs, models etc.) and a ‘constructive’ side related to setting up one’s own ‘objects’ (chains of reasoning, models, texts etc.). The coverage then might be focussed on one of these sides.

- The Radius of action comprises the “contexts and situations in which a person can activate” a competency. If, for example, the modelling competency is restricted to growth or decline situations then this should be stated using the ‘radius of action’. If symbolic manipulation of functions is only possible when the independent variable is x and the dependent one is y , this is also a restriction of the radius of action.
- The Technical level “indicates how conceptually and technically advanced the entities and tools are with which the person can activate the competence”. For example, the modelling of growth can be restricted to linear models, or the usage of symbolic expressions for the computation of areas can be restricted to formulae for simple geometric figures (excluding expressions using integrals).

For the modelling competency, a more comprehensive investigation and exemplification of these dimensions can be found in Blomhøj & Jensen 2007. Having a clear perception of the desired progress regarding the dimensions is an important prerequisite for setting up learning environments (chapter 4) and assessment regimes (chapter 5). In order to specify the desired degree of coverage, one can also use the ‘clusters’ described in the OECD PISA document (OECD 2009). There, three different levels are distinguished: the ‘reproduction’ level where students are able to perform the activities trained before in the same contexts and situations; the ‘connections’ level where students combine pieces of their knowledge and/or apply it to slightly different contexts; and the ‘reflection’ level where students use their knowledge to tackle problems different to those dealt with in former education and/or do this in new contexts, so they have to reflect on what to use and on the possibilities and limitations of using knowledge in different contexts. For example, regarding the competency of mathematical thinking, the reproduction level would include the recognition of mathematical questions which were similarly posed in earlier educational settings and the ability to recall potential answers. Regarding the problem solving competency, this includes the ability to recognise and solve well-practised closed-form problem types (most of which can be found in the list of learning outcomes in chapter 3) where the solution can be obtained by using well-trained procedures. An example for the connections level with respect to the reasoning competency would be the connection of well-known arguments to new chains such as using well-known geometric arguments to justify why a more complex geometric configuration is determined by some given quantities and relations. Finally, an example for the reflection level regarding the modelling competency would be a reflection on adequate modelling means and models and the setting up of more complex non-familiar models. A reflection about the modelling process itself and the ability to describe and justify modelling decisions also belong to this level. A more detailed treatment of the levels for each of the eight competencies for a specific study course can be found in section 2.3.

It is important to have a clear understanding of the relationship between mathematical contents/topics and competencies in order to recognize the role contents play in competency-based curricula. Niss 2003a, p. 10 suggests viewing competencies and mathematical topic areas as “orthogonal”, i.e. to specify “how the corresponding competency manifests itself when dealing with the corresponding topic at the educational level at issue”. Having the dimensions describing the extent to which a competency is present at hand, one can be a bit more specific: Content-related abilities and hence contents appear in the dimension ‘technical level’ where the mathematical entities and operations to which the competency can be applied are to be specified. In some

examples in Niss & Højgaard 2011 the radius of action also includes contents, for example when for the problem posing and solving competency different mathematical areas are named to indicate the radius of action.

2.2 Example

In this section we clarify the competency concept by presenting an example task from mechanical engineering where the competencies are necessary for successful work (for more examples cf. the appendix 10). The example shows on the one hand what we want students to be able and willing to do, and on the other hand what might be an adequate assignment for learning, i.e. for obtaining the competencies.

Example Task: Consider two gears with tooth numbers m and n (see picture below). Each tooth in one gear should meet each tooth of the other one (and not just a subset) in order to have equally distributed abrasion and low noise excitation. How does this affect the choice of tooth numbers?



Reading this task a student should think that it has to do with integers and relationships between integers, so mathematics should provide an answer (*thinking mathematically*). Then, the requirement on the meeting of teeth has to be translated into a mathematical condition including m and n applying a respective chain of arguments (*reasoning mathematically*): Say, tooth 1 of gear one meets first tooth 1 of gear two, then tooth $1 + m$, $1 + 2m$, $1 + 3m$, ..., i.e. $(1 + rm) \bmod n$ for any integer r . Therefore, the meeting condition is equivalent to “for any $s = 1, \dots, n$ there is an integer r such that $s = (1 + rm) \bmod n$ ”. Having this condition, one has to solve the problem for which m and n the condition is fulfilled. The condition is equivalent to “for any $s = 1, \dots, n$ there are integers r, k such that $1 + rm = s + kn$ ”. This is equivalent to “for any $s = 0, \dots, n - 1$ there are integers r, k such that $rm = s + kn$ ” which in turn is equivalent to “there are integers r, k such that $rm = 1 + kn$ ”. Therefore, m and n must be relatively prime, i.e. they have no common factor except 1 (*reasoning mathematically, posing and solving mathematical problems*). Another way of tackling this task might be to get a book on machine elements, find and understand the respective information in this book (*making use of aids and tools, communication in, with, about mathematics*).

2.3 Profiles

Since mathematical competencies are concerned with the ability to master the mathematical challenges of given situations, it is a reasonable starting point for specifying a competence profile to identify the contexts and situations where students of a certain study course meet mathematical challenges. These then determine the envisaged radius of action for the competencies to be listed later. One can then specify in more detail the mathematical concepts and procedures occurring in

the challenges identified before as well as the corresponding abilities (*technical level*) and finally elaborate in more detail the aspects of the general competencies which are involved (*degree of coverage, level*). As an example, we roughly sketch below how such a profile could look like for the mathematical education of practice-oriented mechanical engineers aiming at a Bachelor degree at a university of applied sciences.

Concerning the mathematical challenges such students of mechanical engineering meet, it seems reasonable to inspect the application subjects occurring in the study course and search for those challenges. For taking into account later challenges showing up at the workplace, workplace studies are required but this is a field where much more research is needed (cf. Alpers 2010). In the following we present some contexts and situations containing mathematical challenges which mainly occur in engineering mechanics, CAD, measurement and control, and machine elements and dynamics:

- Determination of loads (forces, torques) and the resulting stress and strains in machine elements or other mechanical configurations (the respective models can already be found in text books)
- Varying the dimensions of machine elements or other mechanical configurations in order to improve or even optimise certain properties (stress, weight, costs, ...)
- Analysis of motion and design of motion of machines or machine parts
- Analysis of vibrations
- Modelling of controlled devices and design of controllers
- Processing of measurement data, computation of descriptive quantities and error analysis, model fitting for measured data.
- ...

A systematic investigation is required for achieving a good coverage. Such an investigation can also be of great value later on when trying to find good example tasks or themes for mathematical application projects. Note that we do not assume that all mathematical challenges occurring in application subjects are handled in the mathematical part of engineering education. Nevertheless, for providing an integrated study course it is very advantageous to have a clear view of the split of responsibility.

The mathematical concepts and algorithms occurring in the identified contexts include:

- Functions and functional dependencies, construction of functions with desired properties
- Using functions for modelling behaviour (growth/decay, vibrations, logistic behaviour, ...)
- Systems of equations, solution types and algorithms
- Iterative improvement and optimisation algorithms
- Geometric descriptions using classical and free-form geometries and their computation
- Differentiation and integration
- Differential equations, solution types and algorithms
- Laplace transforms and working in the complex variable domain
- Fourier analysis
- Stochastic concepts like distribution, mean, variance, confidence intervals, ...
- ...

In the next step on the technical level one could specify in more detail the abilities chosen from those described in chapter 3.

Finally, one has to specify to which degree the eight competencies have to be covered for successfully handling the challenges. This is certainly a ‘non-trivial’ task needing in-depth reasoning. As a first approach it is useful to specify in a coarse way the importance of each competency level (or *cluster*) as has been done in the table below. Note that such an importance specification should provide information on what one wants to achieve for all students. It should not prevent institutions from offering particularly talented students additional learning experiences for acquiring higher levels.

Competency \ Level	Reproduction	Connections	Reflection
Thinking math.	+	+	O
Reasoning math.	+	O	-
Problem solving	+	+	O
Modelling math.	+	+	O
Communication	+	+	O
Representation	+	+	O
Symbols and formalism	+	O	-
Aids and tools	+	+	+

Meaning of signs: +: very important, O: medium important, -: less important

As will probably be the case in any other study course, one wants students to be able to master the reproduction level completely, so what makes the difference between study courses will be the emphasis on the connections and the reflection level. In the following we provide some additional explanatory material just for these two levels within the profile:

- Regarding the competency of thinking mathematically, students should recognise the usefulness of mathematical concepts in situations similar to those encountered before (for example in the computation of stress for a different machine element). The recognition of the potential of mathematical work in totally new situations (for example recognition that a problem in a totally new context could be formulated as a mathematical optimization problem) is of medium importance.
- Regarding the competency of reasoning mathematically, the connections level is of medium importance. Students should understand a well-known, not too-complicated mathematical argumentation in an application context. They should be able to connect simple geometric arguments for determining whether a geometric configuration is fully specified by giving some data or deduce consequences from programme results in order to perform plausibility checks. The reflection level is of less importance since the engineering students in this profile are not required to set up chains of advanced mathematical reasoning (for example to write an article in theoretical mechanics).
- Regarding the competency of problem solving, students should be able to recognise and solve problems similar to those learnt before in different contexts (for example solve a linear DE with constant coefficients in a different context to the one encountered before). They should also be able to work on more open design-type mathematical questions, for example the design of a motion function with certain properties or the design of a machine element fulfilling certain restrictions regarding stress and geometry. Students should also have – to a moderate extent – reflective capabilities concerning problem solving strategies, for example

see strategies for parameter variation in order to improve output values. They are not required to work on harder mathematical problems for which new strategies are necessary.

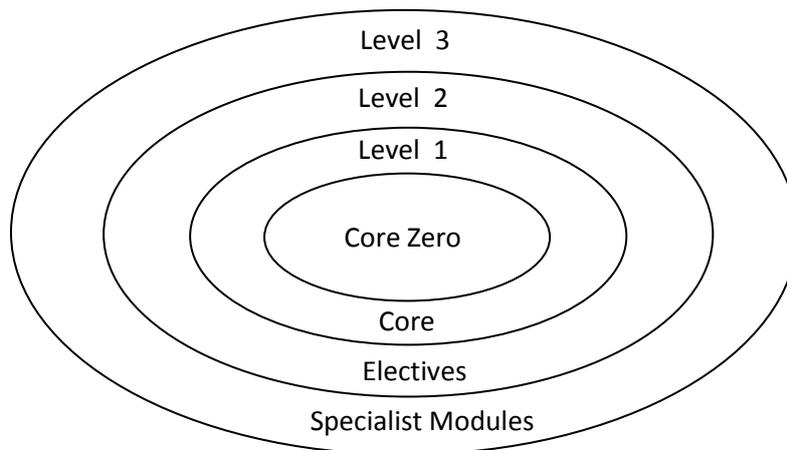
- Regarding the competency of mathematical modelling, students should be able to use well-known modelling means (like forces, torques, equilibrium principle, cutting principle) learnt before (mostly in application subjects) to set up models in different situations (for example model a mechanical configuration by identifying important forces and using equilibrium equations; use the function concept for modelling motion). Students are not expected to develop new modelling means but they should be able to reflect on useful simplifications and on the validity of a simplified model.
- Regarding the competency of communication, students should be able to use simple, mostly informal mathematical language encountered before in their mathematical education and they should be able to understand such language in application text books and formularies. They should be able to informally describe orally and in written form their argumentation or procedure for solving a mathematical problem (like parameter variation to improve a property) or for setting up and working within a mathematical model. It is less important to communicate using formal mathematical language or to communicate more complex logical argumentations.
- Regarding the competency of representing mathematical entities, students should be able to understand and use standard representations in the new context of application text books (different representations of functions, of geometric entities, but also more advanced representations of signal functions in the frequency domain). They should be able to extract information from such representations and switch to a particularly meaningful one. Students should be able to reflect about advantages and disadvantages of representations but they do not have to invent new ones.
- Regarding the competency of handling mathematical symbols and formalism, students should be able to handle symbols and formalism in contexts that are not totally familiar to them and where different notations are used (for example $s(t)$ for motion instead of $y(x)$). They are not required to have a deeper understanding of logical formalism (using implication and equivalent symbols is sufficient). There also is no necessity of being able to perform large and complex formal computations by hand. Invention of new symbols and formalism (as usually done by mathematicians) is also of no importance.
- Regarding the competency of using aids and tools, this is of particular importance in engineering work environments where computer programmes are ubiquitous. Students should be able to use mathematics programmes in new contexts to solve problems similar to those handled before (for example solve a new differential equation with a mathematics programme). They should also be able to use books or other text sources (for example internet pages) to look up computations or mathematical results that are helpful for working on their application problems (for example look up the relationship between tooth numbers in the example presented in the previous section). They should also learn to reflect on what one can expect from a programme based on a certain mathematical model for the application situation. Moreover, they should be able to check the correct working of a programme by using simple examples computed by hand.

The above sketch is meant to provide a rough impression of how a concrete curriculum based on this document can look like. Considerably more effort is necessary to complete this to a full specification (cf. Alpers 2013).

3 Content-related competencies, knowledge, and skills

In the first OECD PISA study and in national or regional curricula developed in response to the results of the study (for example (KMK 2003), (Ministry of Education Baden-Württemberg 2004)), more detailed content-related competencies, knowledge and skills have been identified since the general competencies are not sufficient for guiding teaching in the secondary education environment. In this chapter we apply a similar approach for engineering education.

The content-related learning outcomes have been arranged in a structure which has four levels. These levels represent an attempt to reflect the hierarchical structure of mathematics and the way in which mathematics can be linked to real applications of ever-greater sophistication as the student progresses through the engineering degree programme.



Schematic diagram of the proposed structure

A schematic diagram of the proposed structure is shown in the figure above. Note that there is a central core of material essential for all engineering graduates. Typically this core material would be covered by the end of the first year but the teaching of some of it might extend into the second year of a course. Realistic pre-requisite knowledge (Core Zero) would be assumed. Core Zero, as set out in Section 3.1, does not represent the minimum which can be assumed in every European country. Instead it covers those topics which make up an essential foundation for Core Level 1 and beyond. It is likely that most institutions will need to teach explicitly some of Core Zero topics whilst other institutions may have a parallel programme of support classes or clinics to help students who may be deficient in some areas.

Core Zero is specified in Section 3.1 in considerable detail. It comprises such essential material that only minor omissions are acceptable. The knowledge and skills in mathematics of a student entering tertiary education is not easily predicted from the qualifications achieved prior to entry and some kind of diagnostic testing and additional support may be needed. This support may well be needed throughout the first year of an engineering degree programme.

Core level 1 comprises the knowledge and skills which are necessary in order to underpin the general Engineering Science that is assumed to be essential for most engineering graduates. Items of basic knowledge will be linked together and simple illustrative examples will be used. It should be pointed out that the mathematical needs of Computer Science and Software Engineering are markedly different from all other branches of engineering. This core curriculum is only of limited use for such courses.

Level 2 comprises specialist or advanced knowledge and skills which are considered essential for individual engineering disciplines. Synoptic elements will link together items of knowledge and the use of simple illustrative examples from real-life engineering.

Level 3 comprises highly specialist knowledge and skills which are associated with advanced levels of study and incorporates synoptic mathematical theory and its integration with real-life engineering examples. Students would progress from the core in mathematics by studying more subject-specific compulsory modules (electives). These would normally build upon the core modules and be expected to correspond to the outcomes associated with level 2 material. Such electives may build additionally on level 1, requiring knowledge of more advanced skills, and may link level 1 skills or introduce additional more engineering-specific related topics. An example of the first mode is that Mechanical Engineering students may need to study the vibration of mechanical systems through the applied use of systems of ordinary differential equations. Here, topics that build on foundations of differential calculus, complex numbers and matrix analysis might be expected to be covered in level 1, as these are topics probably learnt in isolation and without reference to specific engineering application. Alternatively an Electronic Engineering student may be required to learn elements of discrete mathematics directly relevant to the design and study of computer systems; these would probably be unsuitable for core material for all engineers. This is not to say that discrete mathematics should not be taught at level 1, but the context and outcomes need to be clearly discernible within the level. Typically level 2 modules would be distributed within the second or third year of an Engineering course due to the logistics of level 1 prerequisites.

Students within the more numerate Engineering disciplines might be expected to take further more specialised modules incorporating mathematics on an optional basis, aimed to help match their career aspirations with appropriate theoretical formation. These modules will be at an advanced level, making use of appropriate technology, and heavily influenced with examples from engineering. Teaching of these level 3 modules would be most appropriate in year 3 or 4 of a degree course. It is likely that many of these topics already exist within specialist engineering courses and typically the mathematics is embedded and taught by engineers, mathematicians or both. For some programmes meeting the highest requirements, students might be expected to study some topics close to current areas of research where the available techniques and tools may well be mathematically based.

Within the three main levels the material has been arranged under five sub-headings: analysis and calculus, discrete mathematics, geometry, linear algebra, statistics and probability. There is no intention to prescribe how the topics in the Core Curriculum should be ordered: what is offered here is a convenient grouping of items. In many cases a topic could have been placed under one of the other sub-headings. The curriculum is specified in terms of content and learning objectives. This makes the document longer, but it makes more explicit exactly what is required and is therefore more transparent for both teacher and learner.

In order to foster 'overarching' sense making, in the OECD PISA study and in other documents the content-related competencies have not been organized according to the traditional areas of mathematics but rather along some general themes, called 'overarching ideas' in OECD 2009. The ideas stated there are 'quantity', 'space and shape', 'change and relationships', and 'uncertainty' (OECD 2009, p. 93-104). A similar rearrangement can be made regarding engineering education. In the following, we reuse themes stated in (Ministry of Education Baden-Württemberg 2004) for the grades 6 to 12 and add some new ones that are specific to engineering education:

- Quantity: In university education, the means of quantification are extended in several ways. The set of complex numbers is introduced as extension of the reals where 'more' operations are possible. Another extension are vectors as a concept for modelling quantities which cannot just be described by one number since they also have a direction or are multi-valued. Thirdly, the ubiquitous work with computers makes it necessary to be aware of the restrictions of computer-representable numbers and the corresponding numerical problems.
- Measuring: Measuring of geometric objects is extended to objects with curved boundary curves or surfaces by considering them as 'infinite sum of infinitely small quantities'. Moreover, the measurement is not restricted to measuring geometric properties but also application quantities like moment of inertia or electric charge. The measurement can also be done in an approximative way when discrete data representations of the curved objects are used.
- Space and shape: Engineering students have to extend their understanding of space to multi-dimensional vector spaces and recognize which properties of the three dimensional real space are retained. Regarding shape, formal mathematical description is no longer restricted to a set of simple geometries (lines, surface, bodies) but can be extended to arbitrary geometries (free form geometries).
- Functional dependency: The understanding of functional dependency acquired at school is extended to functional dependency of multidimensional quantities which might also be complex. Another aspect of the theme of functional dependency is the design of such dependencies in different engineering areas, e.g. in motion design.
- Relations between functions: There are relationships between functions and their derivatives (rates of change) leading to differential equations. There are relationships between functions which can be understood as transformations (building derivative and integral, Laplace or Fourier transforms). Moreover, functions that are finite or infinite combinations of basic functions can be considered.
- Data and chance: Handling of data, deterministic or based on random processes, is ubiquitous in engineering. Often data have to be fitted to mathematical models by using interpolation or approximation methods. Quantities with values influenced by chance are modeled as random variables and adequate probability models based on properties of the quantity must be found.
- Algorithms: Algorithms as methods for computing and constructing mathematical objects are important for getting an understanding of what kind of problems can be solved when working within a mathematical model. There are symbolic algorithms and numerical algorithms, both having their specific advantages and disadvantages which need to be known for making an adequate choice of what to use to tackle a specific problem.
- Modelling: Modelling application behaviour is a general theme in engineering. This comprises functional dependency models, equation models, differential equation models, differential models (using 'infinitely small objects' as limits of finitely small objects). Engineers should recognise the same mathematical model (structure) in different applications and value the advantage of abstraction and reuse of formalism, knowledge of properties and algorithms.

These themes can be used by a lecturer for offering a wider perspective to the students by embedding a specific mathematical topic into an overarching mathematical context. They can also be

used in projects where students investigate the basic concepts within a theme and their extension by using more advanced concepts.

3.1 Core Zero

The material in this section is the material which ideally should have been studied before entry to an undergraduate engineering degree programme. However, it is recognised that whilst there is some commonality across Europe over what is studied in pre-university mathematics, there are also a number of areas of difference. Core Zero does not consist of just those topics which are taught in school in all European countries, rather it contains material which together forms a solid platform on which to build a study of engineering mathematics at university. A consequence of this is that in many countries it will be necessary to cover some Core Zero material during the first year of a university engineering course.

The material in Core Zero has been grouped into five areas: Algebra, Analysis & Calculus, Discrete Mathematics, Geometry & Trigonometry and Statistics & Probability. These relate to the five areas in each of the three main levels of the curriculum: Analysis & Calculus, Discrete Mathematics, Geometry, Linear Algebra and Statistics & Probability.

Algebra

Arithmetic of real numbers

As a result of learning this material you should be able to

- carry out the operations add, subtract, multiply and divide on both positive and negative numbers
- express an integer as a product of prime factors
- calculate the highest common factor and lowest common multiple of a set of integers
- obtain the modulus of a number
- understand the rules governing the existence of powers of a number
- combine powers of a number
- evaluate negative powers of a number
- express a fraction in its lowest form
- carry out arithmetic operations on fractions
- represent roots as fractional powers
- express a fraction in decimal form and vice-versa
- carry out arithmetic operations on numbers in decimal form
- round numerical values to a specified number of decimal places or significant figures
- understand the concept of ratio and solve problems requiring the use of ratios
- understand the scientific notation form of a number
- manipulate logarithms
- understand how to estimate errors in measurements and how to combine them.

Algebraic expressions and formulae

As a result of learning this material you should be able to

- add and subtract algebraic expressions and simplify the result
- multiply two algebraic expressions, removing brackets
- evaluate algebraic expressions using the rules of precedence
- change the subject of a formula
- distinguish between an identity and an equation
- obtain the solution of a linear equation
- recognise the kinds of solution for two simultaneous equations

- understand the terms direct proportion, inverse proportion and joint proportion
- solve simple problems involving proportion
- factorise a quadratic expression
- carry out the operations add, subtract, multiply and divide on algebraic fractions
- interpret simple inequalities in terms of intervals on the real line
- solve simple inequalities, both geometrically and algebraically
- interpret inequalities which involve the absolute value of a quantity.

Linear laws

As a result of learning this material you should be able to

- understand the Cartesian co-ordinate system
- plot points on a graph using Cartesian co-ordinates
- understand the terms 'gradient' and 'intercept' with reference to straight lines
- obtain and use the equation $y = mx + c$
- obtain and use the equation of a line with known gradient through a given point
- obtain and use the equation of a line through two given points
- use the intercept form of the equation of a straight line
- use the general equation $ax + by + c = 0$
- determine algebraically whether two points lie on the same side of a straight line
- recognise when two lines are parallel
- recognise when two lines are perpendicular
- obtain the solution of two simultaneous equations in two unknowns using graphical and algebraic methods
- interpret simultaneous linear inequalities in terms of regions in the plane
- reduce a relationship to linear form.

Quadratics, cubics, polynomials

As a result of learning this material you should be able to

- recognise the graphs of $y = x^2$ and of $y = -x^2$
- understand the effects of translation and scaling on the graph of $y = x^2$
- rewrite a quadratic expression by completing the square
- use the rewritten form to sketch the graph of the general expression $ax^2 + bx + c$
- determine the intercepts on the axes of the graph of $y = ax^2 + bx + c$
- determine the highest or lowest point on the graph of $y = ax^2 + bx + c$
- sketch the graph of a quadratic expression
- state the criterion that determines the number of roots of a quadratic equation
- solve the equation $ax^2 + bx + c = 0$ via factorisation, by completing the square and by the formula
- recognise the graphs of $y = x^3$ and of $y = -x^3$
- recognise the main features of the graph of $y = ax^3 + bx^2 + cx + d$
- recognise the main features of the graphs of quartic polynomials
- state and use the remainder theorem
- derive the factor theorem
- factorise simple polynomials as a product of linear and quadratic factors.

Analysis and Calculus

Functions and their inverses

As a result of learning this material you should be able to

- define a function, its domain and its range
- use the notation $f(x)$

- determine the domain and range of simple functions
- relate a pictorial representation of a function to its graph and to its algebraic definition
- determine whether a function is injective, surjective, bijective
- understand how a graphical translation can alter a functional description
- understand how a reflection in either axis can alter a functional description
- understand how a scaling transformation can alter a functional description
- determine the domain and range of simple composite functions
- use appropriate software to plot the graph of a function
- obtain the inverse of a function by a pictorial representation, graphically or algebraically
- determine the domain and range of the inverse of a function
- determine any restrictions on $f(x)$ for the inverse to be a function
- obtain the inverse of a composite function
- recognise the properties of the function $1/x$
- understand the concept of the limit of a function.

Sequences, series, binomial expansions

As a result of learning this material you should be able to

- define a sequence and a series and distinguish between them
- recognise an arithmetic progression and its component parts
- find the general term of an arithmetic progression
- find the sum of an arithmetic series
- recognise a geometric progression and its component parts
- find the general term of a geometric progression
- find the sum of a finite geometric series
- interpret the term 'sum' in relation to an infinite geometric series
- find the sum of an infinite geometric series when it exists
- find the arithmetic mean of two numbers
- find the geometric mean of two numbers
- obtain the binomial expansions of $(a + b)^s$, $(1 + x)^s$ for s a rational number
- use the binomial expansion to obtain approximations to simple rational functions.

Logarithmic and exponential functions

As a result of learning this material you should be able to

- recognise the graphs of the power law function
- define the exponential function and sketch its graph
- define the logarithmic function as the inverse of the exponential function
- use the laws of logarithms to simplify expressions
- solve equations involving exponential and logarithmic functions
- solve problems using growth and decay models.

Rates of change and differentiation

As a result of learning this material you should be able to

- define average and instantaneous rates of change of a function
- understand how the derivative of a function at a point is defined
- recognise the derivative of a function as the instantaneous rate of change
- interpret the derivative as the gradient at a point on a graph
- distinguish between 'derivative' and 'derived function'
- use the notations $\frac{dy}{dx}$, $f'(x)$, $y'(x)$ etc.
- use a table of the derived functions of simple functions

- recall the derived function of each of the standard functions
- use the multiple, sum, product and quotient rules
- use the chain rule
- relate the derivative of a function to the gradient of a tangent to its graph
- obtain the equation of the tangent and normal to the graph of a function.

Stationary points, maximum and minimum values

As a result of learning this material you should be able to

- use the derived function to find where a function is increasing or decreasing
- define a stationary point of a function
- distinguish between a turning point and a stationary point
- locate a turning point using the first derivative of a function
- classify turning points using first derivatives
- obtain the second derived function of simple functions
- classify stationary points using second derivatives.

Indefinite integration

As a result of learning this material you should be able to

- reverse the process of differentiation to obtain an indefinite integral for simple functions
- understand the role of the arbitrary constant
- use a table of indefinite integrals of simple functions
- understand and use the notation for indefinite integrals
- use the constant multiple rule and the sum rule
- use indefinite integration to solve practical problems such as obtaining velocity from a formula for acceleration or displacement from a formula for velocity.

Definite integration, applications to areas and volumes

As a result of learning this material you should be able to

- understand the idea of a definite integral as the limit of a sum
- realise the importance of the Fundamental Theorem of the Calculus
- obtain definite integrals of simple functions
- use the main properties of definite integrals
- calculate the area under a graph and recognise the meaning of a negative value
- calculate the area between two curves
- calculate the volume of a solid of revolution
- use trapezium and Simpson's rules to approximate the value of a definite integral.

Complex numbers

As a result of learning this material you should be able to

- define a complex number and identify its component parts
- represent a complex number on an Argand diagram
- carry out the operations of addition and subtraction
- write down the conjugate of a complex number and represent it graphically
- identify the modulus and argument of a complex number
- carry out the operations of multiplication and division in both Cartesian and polar form
- solve equations of the form $z^n = a$, where a is a real number.

Proof

As a result of learning this material you should be able to

- understand how a theorem is deduced from a set of assumptions
- appreciate how a corollary is developed from a theorem
- follow a proof of Pythagoras' theorem
- follow proofs of theorems for example, the concurrency of lines related to triangles and/or the equality of angles related to circles.

Discrete Mathematics

Sets

As a result of learning this material you should be able to

- understand the concepts of a set, a subset and the empty set
- determine whether an item belongs to a given set or not
- use and interpret Venn diagrams
- find the union and intersection of two given sets
- apply the laws of set algebra.

Geometry and Trigonometry

Geometry

As a result of learning this material you should be able to

- recognise the different types of angle
- identify the equal angles produced by a transversal cutting parallel lines
- identify the different types of triangle
- state and use the formula for the sum of the interior angles of a polygon
- calculate the area of a triangle
- use the rules for identifying congruent triangles
- know when two triangles are similar
- state and use Pythagoras' theorem
- understand radian measure
- convert from degrees to radians and vice-versa
- state and use the formulae for the circumference of a circle and the area of a disc
- calculate the length of a circular arc
- calculate the areas of a sector and of a segment of a circle
- quote formulae for the area of simple plane figures
- quote formulae for the volume of elementary solids: a cylinder, a pyramid, a tetrahedron, a cone and a sphere
- quote formulae for the surface area of elementary solids: a cylinder, a cone and a sphere
- sketch simple orthographic views of elementary solids
- understand the basic concept of a geometric transformation in the plane
- recognise examples of a metric transformation (isometry) and affine transformation (similitude)
- obtain the image of a plane figure in a defined geometric transformation: a translation in a given direction, a rotation about a given centre, a symmetry with respect to the centre or to the axis, scaling to a centre by a given ratio.

Trigonometry

As a result of learning this material you should be able to

- define the sine, cosine and tangent of an acute angle

- define the reciprocal ratios cosecant, secant and cotangent
- state and use the fundamental identities arising from Pythagoras' theorem
- relate the trigonometric ratios of an angle to those of its complement
- relate the trigonometric ratios of an angle to those of its supplement
- state in which quadrants each trigonometric ratio is positive (the CAST rule)
- state and apply the sine rule
- state and apply the cosine rule
- calculate the area of a triangle from the lengths of two sides and the included angle
- solve a triangle given sufficient information about its sides and angles
- recognise when there is no triangle possible and when two triangles can be found.

Co-ordinate geometry

As a result of learning this material you should be able to

- calculate the distance between two points
- find the position of a point which divides a line segment in a given ratio
- find the angle between two straight lines
- calculate the distance of a given point from a given line
- calculate the area of a triangle knowing the co-ordinates of its vertices
- give simple examples of a locus
- recognise and interpret the equation of a circle in standard form and state its radius and centre
- convert the general equation of a circle to standard form
- recognise the parametric equations of a circle
- derive the main properties of a circle, including the equation of the tangent at a point
- define a parabola as a locus
- recognise and interpret the equation of a parabola in standard form and state its vertex, focus, axis, parameter and directrix
- recognise the parametric equation of a parabola
- derive the main properties of a parabola, including the equation of the tangent at a point
- understand the concept of parametric representation of a curve
- use polar co-ordinates and convert to and from Cartesian co-ordinates.

Trigonometric functions and applications

As a result of learning this material you should be able to

- define the term *periodic function*
- sketch the graphs of $\sin(x)$, $\cos(x)$, and $\tan(x)$ and describe their main features
- deduce the graphs of the reciprocal functions cosec, sec and cot
- deduce the nature of the graphs of $a \cdot \sin(x)$, $a \cdot \cos(x)$, $a \cdot \tan(x)$
- deduce the nature of the graphs of $\sin(ax)$, $\cos(ax)$, $\tan(ax)$
- deduce the nature of the graphs of $\sin(x + a)$, $a + \sin(x)$, etc.
- solve the equations $\sin(x) = c$, $\cos(x) = c$, $\tan(x) = c$
- use the expression $a \cdot \sin(\omega t + \varphi)$ to represent an oscillation and relate the parameters to the motion
- rewrite the expression $a \cdot \cos(\omega t) + b \cdot \sin(\omega t)$ as a single cosine or sine formula.

Trigonometric identities

As a result of learning this material you should be able to

- obtain and use the compound angle formulae and double angle formulae
- obtain and use the product formulae

- solve simple problems using these identities.

Statistics and Probability

Data Handling

As a result of learning this material you should be able to

- interpret data presented in the form of line diagrams, bar charts, pie charts
- interpret data presented in the form of stem and leaf diagrams, box plots, histograms
- construct line diagrams, bar charts, pie charts, stem and leaf diagrams, box plots, histograms for suitable data sets
- calculate the mode, median and mean for a set of data items.

Probability

As a result of learning this material you should be able to

- define the terms 'outcome', 'event' and 'probability'.
- calculate the probability of an event by counting outcomes
- calculate the probability of the complement of an event
- calculate the probability of the union of two mutually-exclusive events
- calculate the probability of the union of two events
- calculate the probability of the intersection of two independent events.

3.2 Core Level 1

The material at this level builds on Core Zero and is regarded as basic to all engineering disciplines in that it provides the fundamental understanding of many mathematical principles. However, it is recognised that the emphasis given to certain topics within Core level 1 may differ according to the engineering discipline. So, for example, electrical and electronic engineers may cover some of the topics in Discrete Mathematics in greater depth than, say, Mechanical Engineers.

The material in Core level 1 can be used by engineers in the understanding and the development of theory and in the sensible selection of tools for analysis of engineering problems. This material will be taught in the early stages of a university programme. Noting the comment made in Section 3.1, it is possible that some of this material will be taught alongside or immediately after coverage of missing topics from Core Zero.

Analysis and Calculus

The material in this section covers the basic development of analysis and calculus consequent on the material in Core Zero.

Hyperbolic functions

As a result of learning this material you should be able to

- define and sketch the functions \sinh , \cosh , \tanh
- sketch the reciprocal functions cosech , sech and coth
- state the domain and range of the inverse hyperbolic functions
- recognise and use basic hyperbolic identities
- apply the functions to a practical problem (for example, a suspended cable)
- understand how the functions are used in simplifying certain standard integrals.

Rational functions

As a result of learning this material you should be able to

- sketch the graph of a rational function where the numerator is a linear expression and the denominator is either a linear expression or the product of two linear expressions
- obtain the partial fractions of a rational function, including cases where the denominator has a repeated linear factor or an irreducible quadratic factor.

Complex numbers

As a result of learning this material you should be able to

- state and use Euler's formula
- state and understand De Moivre's theorem for a rational index
- find the roots of a complex number
- link trigonometric and hyperbolic functions
- describe regions in the plane by restricting the modulus and / or the argument of a complex number.

Functions

As a result of learning this material you should be able to

- define and recognise an odd function and an even function
- understand the properties 'concave' and 'convex'
- identify, from its graph, where a function is concave and where it is convex
- define and locate points of inflection on the graph of a function
- evaluate a function of two or more variables at a given point
- relate the main features, including stationary points, of a function of 2 variables to its 3D plot and to a contour map
- obtain the first partial derivatives of simple functions of several variables
- use appropriate software to produce 3D plots and/or contour maps.

Differentiation

As a result of learning this material you should be able to

- understand the concepts of continuity and smoothness
- differentiate inverse functions
- differentiate functions defined implicitly
- differentiate functions defined parametrically
- locate any points of inflection of a function
- find greatest and least values of physical quantities.

Sequences and series

As a result of learning this material you should be able to

- understand convergence and divergence of a sequence
- know what is meant by a partial sum
- understand the concept of a power series
- apply simple tests for convergence of a series
- find the tangent and quadratic approximations to a function
- understand the idea of radius of convergence of a power series
- recognise Maclaurin series for standard functions
- understand how Maclaurin series generalise to Taylor series
- use Taylor series to obtain approximate percentage changes in a function.

Methods of integration

As a result of learning this material you should be able to

- obtain definite and indefinite integrals of rational functions in partial fraction form
- apply the method of integration by parts to indefinite and definite integrals
- use the method of substitution on indefinite and definite integrals
- solve practical problems which require the evaluation of an integral
- recognise simple examples of improper integrals
- use the formula for the maximum error in a trapezoidal rule estimate
- use the formula for the maximum error in a Simpson's rule estimate.

Applications of integration

As a result of learning this material you should be able to

- find the length of part of a plane curve
- find the curved surface area of a solid of revolution
- obtain the mean value and root-mean-square (RMS) value of a function in a closed interval
- find the first and second moments of a plane area about an axis
- find the centroid of a plane area and of a solid of revolution.

Solution of non-linear equations

As a result of learning this material you should be able to

- use intersecting graphs to help locate approximately the roots of non-linear equations
- use Descartes' rules of signs for polynomial equations
- understand the distinction between point estimation and interval reduction methods
- use a point estimation method and an interval reduction method to solve a practical problem
- understand the various convergence criteria
- use appropriate software to solve non-linear equations.

Discrete Mathematics

The material in this section covers the basic development of discrete mathematics consequent on the material in Core Zero.

Mathematical logic

As a result of learning this material you should be able to

- recognise a proposition
- negate a proposition
- form a compound proposition using the connectives AND, OR, IMPLICATION
- construct a truth table for a compound proposition
- construct a truth table for an implication
- verify the equivalence of two propositions using a truth table
- identify a contradiction and a tautology
- construct the converse of a proposition
- obtain the contrapositive form of an implication
- understand the universal quantifier 'for all'
- understand the existential quantifier 'there exists'
- negate propositions with quantifiers
- follow simple examples of direct and indirect proof
- follow a simple example of a proof by contradiction.

Sets

As a result of learning this material you should be able to

- understand the notion of an ordered pair
- find the Cartesian product of two sets
- define a characteristic function of a subset of a given universe
- compare the algebra of switching circuits to that of set algebra and logical connectives
- analyse simple logic circuits comprising AND, OR, NAND, NOR and EXCLUSIVE OR gates
- understand the concept of a countable set.

Mathematical induction and recursion

As a result of learning this material you should be able to

- understand (weak) mathematical induction
- follow a simple proof which uses mathematical induction
- define a set by induction
- use structural induction to prove some simple properties of a set which is given by induction.
- understand the concept of recursion
- define the factorial of a positive integer by recursion (any other suitable example will serve just as well).

Graphs

As a result of learning this material you should be able to

- recognise a graph (directed and/or undirected) in a real situation
- understand the notions of a path and a cycle
- understand the notion of a tree and a binary tree

Geometry

The material in this section covers the basic development of geometry consequent on the material in Core Zero.

Conic sections

As a result of learning this material you should be able to

- recognise the equation of an ellipse in standard form and state its foci, semiaxes and directrices
- recognise the parametric equations of an ellipse
- derive the main properties of an ellipse, including the equation of the tangent at a point
- recognise the equation of a hyperbola in standard form and find its foci, semiaxes and asymptotes
- recognise the parametric equations of a hyperbola
- derive the main properties of a hyperbola, including the equation of the tangent at a point
- recognise the equation of a conic section in the general form and classify the type of conic section

3D co-ordinate geometry

As a result of learning this material you should be able to

- recognise and use the standard equation of a straight line in 3D

- recognise and use the standard equation of a plane
- find the angle between two straight lines
- find where two straight lines intersect
- find the angle between two planes
- find the intersection line of two planes
- find the intersection of a line and a plane
- find the angle between a line and a plane
- calculate the distance between two points, a point and a line, a point and a plane
- calculate the distance between two lines, a line and a plane, two planes
- recognise and use the standard equation of a singular quadratic surface (cylindrical, conical)
- recognise and use the standard equation of a regular quadratic surface (ellipsoid, paraboloid, hyperboloid).

Linear Algebra

The material in this section covers the basic development of linear algebra consequent on the material in Core Zero.

Vector arithmetic

As a result of learning this material you should be able to

- distinguish between vector and scalar quantities
- understand and use vector notation
- represent a vector pictorially
- carry out addition and scalar multiplication and represent them pictorially
- determine the unit vector in a specified direction
- represent a vector in component form (two and three components only).

Vector algebra and applications

As a result of learning this material you should be able to

- solve simple problems in geometry using vectors
- solve simple problems using the component form (for example, in mechanics)
- define the scalar product of two vectors and use it in simple applications
- understand the geometric interpretation of the scalar product
- define the vector product of two vectors and use it in simple applications
- understand the geometric interpretation of the vector product
- define the scalar triple product of three vectors and use it in simple applications
- understand the geometric interpretation of the scalar triple product.

Matrices and determinants

As a result of learning this material you should be able to

- understand what is meant by a matrix
- recall the basic terms associated with matrices (for example, diagonal, trace, square, triangular, identity)
- obtain the transpose of a matrix
- determine any scalar multiple of a matrix
- recognise when two matrices can be added and find, where possible, their sum
- recognise when two matrices can be multiplied and find, where possible, their product
- calculate the determinant of 2 x 2 and 3 x 3 matrices

- understand the geometric interpretation of 2 x 2 and 3 x 3 determinants
- use the elementary properties of determinants in their evaluation
- state the criterion for a square matrix to have an inverse
- write down the inverse of a 2 x 2 matrix when it exists
- determine the inverse of a matrix, when it exists, using row operations
- calculate the rank of a matrix
- use appropriate software to determine inverse matrices.

Solution of simultaneous linear equations

As a result of learning this material you should be able to

- represent a system of linear equations in matrix form
- understand how the general solution of an inhomogeneous linear system of m equations in n unknowns is obtained from the solution of the homogeneous system and a particular solution
- recognise the different possibilities for the solution of a system of linear equations
- give a geometrical interpretation of the solution of a system of linear equations
- understand how and why the rank of the coefficient matrix and the augmented matrix of a linear system can be used to analyse its solution
- use the inverse matrix to find the solution of 3 simultaneous linear equations when possible
- understand the term 'ill-conditioned'
- apply the Gauss elimination method and recognise when it fails
- understand the Gauss-Jordan variation
- use appropriate software to solve simultaneous linear equations.

Least squares curve fitting

As a result of learning this material you should be able to

- define the least squares criterion for fitting a straight line to a set of data points
- understand how and why the criterion is satisfied by the solution of $A^T Ax = A^T b$
- understand the effect of outliers
- modify the method to deal with polynomial models
- use appropriate software to fit a straight line to a set of data points.

Linear spaces and transformations

As a result of learning this material you should be able to

- define a linear space
- define and recognise linear independence
- define and obtain a basis for a linear space
- define a subspace of a linear space and find a basis for it
- define scalar product in a linear space
- understand the concept of norm
- define the Euclidean norm
- define a linear transformation between two spaces; define the image space and the null space for the transformation
- derive the matrix representation of a linear transformations
- understand how to carry out a change of basis
- define an orthogonal transformation

- apply the above matrices of linear transformations in the Euclidean plane and Euclidean space
- recognise matrices of Euclidean and affine transformations: identity, translation, symmetry, rotation and scaling.

Statistics and Probability

The material in this section covers the basic development of statistics and probability consequent on the material in Core Zero.

Data Handling

As a result of learning this material you should be able to

- calculate the range, inter-quartile range, variance and standard deviation for a set of data items
- distinguish between a population and a sample
- know the difference between the characteristic values (moments) of a population and of a sample
- construct a suitable frequency distribution from a data set
- calculate relative frequencies
- calculate measures of average and dispersion for a grouped set of data
- understand the effect of grouping on these measures.

Combinatorics

As a result of learning this material you should be able to

- evaluate the number of ways of arranging unlike objects in a line
- evaluate the number of ways of arranging objects in a line, where some are alike
- evaluate the number of ways of arranging unlike objects in a ring
- evaluate the number of ways of permuting r objects from n unlike objects
- evaluate the number of combinations of r objects from n unlike objects
- use the multiplication principle for combinations.

Simple probability

As a result of learning this material you should be able to

- interpret probability as a degree of belief
- understand the distinction between *a priori* and *a posteriori* probabilities
- use a tree diagram to calculate probabilities
- know what conditional probability is and be able to use it (Bayes' theorem)
- calculate probabilities for series and parallel connections.

Probability models

As a result of learning this material you should be able to

- define a random variable and a discrete probability distribution
- state the criteria for a binomial model and define its parameters
- calculate probabilities for a binomial model
- state the criteria for a Poisson model and define its parameters
- calculate probabilities for a Poisson model

- state the expected value and variance for each of these models
- understand when a random variable is continuous
- explain the way in which probability calculations are carried out in the continuous case.

Normal distribution

As a result of learning this material you should be able to

- handle probability statements involving continuous random variables
- convert a problem involving a normal variable to the area under part of its density curve
- relate the general normal distribution to the standardised normal distribution
- use tables for the standardised normal variable
- solve problems involving a normal variable using tables.

Sampling

As a result of learning this material you should be able to

- define a random sample
- know what a sampling distribution is
- understand the term 'mean squared error' of an estimate
- understand the term 'unbiasedness' of an estimate

Statistical inference

As a result of learning this material you should be able to

- apply confidence intervals to sample estimates
- follow the main steps in a test of hypothesis.
- understand the difference between a test of hypothesis and a significance test (pvalue)
- define the level of a test (error of the first kind)
- define the power of a test (error of the second kind)
- state the link between the distribution of a normal variable and that of means of samples
- place confidence intervals around the sample estimate of a population mean
- test claims about the population mean using results from sampling
- recognise whether an alternative hypothesis leads to a one-tail or a two-tail test
- compare the approaches of using confidence intervals and hypothesis tests.

3.3 Level 2

The material at this level builds on Core Level 1. The material is now advanced enough for simple real engineering problems to be addressed. The material in this level can no longer be regarded as essential for every engineer (hence the omission of 'Core' from the title of this level). Different disciplines will select different topics from the material outlined here. Furthermore, different disciplines may well select different amounts of material from Level 2. Those engineering disciplines which are more mathematically based, such as electrical and chemical engineering, will require their students to study more Level 2 topics than other disciplines, such as manufacturing and production engineering which are less mathematically-based.

Analysis and Calculus

The material in this section covers the basic development of analysis and calculus consequent on the material in Core Level 1.

Ordinary differential equations

As a result of learning this material you should be able to

- understand how rates of change can be modelled using first and second derivatives
- recognise the kinds of boundary condition which apply in particular situations
- distinguish between boundary and initial conditions
- distinguish between general solution and particular solution
- understand how existence and uniqueness relate to a solution
- classify differential equations and recognise the nature of their general solution
- understand how substitution methods can be used to simplify ordinary differential equations
- use an appropriate software package to solve ordinary differential equations.

First order ordinary differential equations

As a result of learning this material you should be able to

- recognise when an equation can be solved by separating its variables
- obtain general solutions of equations by applying the method
- obtain particular solutions by applying initial conditions
- recognise the common equations of the main areas of application
- interpret the solution and its constituent parts in terms of the physical problem
- understand the term 'exact equation'
- obtain the general solution to an exact equation
- solve linear differential equations using integrating factors
- find and interpret solutions to equations describing standard physical situations
- use a simple numerical method for estimating points on the solution curve.

Second order equations - complementary function and particular integral

As a result of learning this material you should be able to

- distinguish between free and forced oscillation
- recognise linear second-order equations with constant coefficients and how they arise in the modelling of oscillation
- obtain the types of complementary function and interpret them in terms of the model
- find the particular integral for simple forcing functions
- obtain the general solution to the equation
- apply initial conditions to obtain a particular solution
- identify the transient and steady-state response
- apply boundary conditions to obtain a particular solution, where one exists
- recognise and understand the meaning of 'beats'
- recognise and understand the meaning of resonance.

Functions of several variables

As a result of learning this material you should be able to

- define a stationary point of a function of several variables
- define local maximum, local minimum and saddle point for a function of two variables
- locate the stationary points of a function of several variables
- obtain higher partial derivatives of simple functions of two or more variables
- understand the criteria for classifying a stationary point of a function of two variables
- obtain total rates of change of functions of two variables

- approximate small errors in a function using partial derivatives.

Fourier series

As a result of learning this material you should be able to

- understand the effects of superimposing sinusoidal waves of different frequencies
- recognise that a Fourier series approximation can be derived by a least squares approach
- understand the idea of orthogonal functions
- use the formulae to find Fourier coefficients in simple cases
- appreciate the effect of including more terms in the approximation
- interpret the resulting series, particularly the constant term
- comment on the usefulness of the series obtained.
- state the simplifications involved in approximating odd or even functions
- sketch odd and even periodic extensions to a function defined on a restricted interval
- obtain Fourier series for these extensions
- compare the two series for relative effectiveness
- obtain a Fourier series for a function of general period.

Double integrals

As a result of learning this material you should be able to

- interpret the components of a double integral
- sketch the area over which a double integral is defined
- evaluate a double integral by repeated integration
- reverse the order of a double integral
- convert a double integral to polar co-ordinates and evaluate it
- find volumes using double integrals.

Further multiple integrals

As a result of learning this material you should be able to

- express problems in terms of double integrals
- interpret the components of a triple integral
- sketch the region over which a triple integral is defined
- evaluate a simple triple integral by repeated integration
- formulate and evaluate a triple integral expressed in cylindrical polar co-ordinates
- formulate and evaluate a triple integral expressed in spherical polar co-ordinates
- use multiple integrals in the solution of engineering problems.

Vector calculus

As a result of learning this material you should be able to

- obtain the gradient of a scalar point function
- obtain the directional derivative of a scalar point function and its maximum rate of change at a point
- understand the concept of a vector field
- obtain the divergence of a vector field
- obtain the curl of a vector field
- apply simple properties of the operator ∇

- know that the curl of the gradient of a scalar is the zero vector
- know that the divergence of the curl of a vector is zero
- define and use the Laplacian operator ∇^2 .

Line and surface integrals, integral theorems

As a result of learning this material you should be able to

- evaluate line integrals along simple paths
- apply line integrals to calculate work done
- apply Green's theorem in the plane to simple examples
- evaluate surface integrals over simple surfaces
- use the Jacobian to transform a problem into a new co-ordinate system
- apply Gauss' divergence theorem to simple problems
- apply Stokes' theorem to simple examples.

Linear optimisation

As a result of learning this material you should be able to

- recognise a linear programming problem in words and formulate it mathematically
- represent the feasible region graphically
- solve a maximisation problem graphically by superimposing lines of equal profit
- carry out a simple sensitivity analysis
- represent and solve graphically a minimisation problem
- explain the term 'redundant constraint'
- understand the meaning and use of slack variables in reformulating a problem
- understand the concept of duality and be able to formulate the dual to a given problem.

The simplex method

As a result of learning this material you should be able to

- convert a linear programming problem into a simplex tableau
- solve a maximisation problem by the simplex method
- interpret the tableau at each stage of the journey round the simplex
- recognise cases of failure
- write down the dual to a linear programming problem
- use the dual problem to solve a minimisation problem.

Non-linear optimisation

As a result of learning this material you should be able to

- solve an unconstrained optimisation problem in two variables
- use information in a physically-based problem to help obtain the solution
- use the method of Lagrange multipliers to solve constrained optimisation problems
- solve practical problems such as minimising surface area for a fixed enclosed volume or minimising enclosed volume for a fixed surface area.

Laplace transforms

As a result of learning this material you should be able to

- use tables to find the Laplace transforms of simple functions
- use the property of linearity to find the Laplace transforms
- use the first shift theorem to find the Laplace transforms
- use the 'multiply by t ' theorem to find the Laplace transforms
- obtain the transforms of first and second derivatives
- invert a transform using tables and partial fractions
- solve initial-value problems using Laplace transforms
- compare this method of solution with the method of complementary function / particular integral.
- use the unit step function in the definition of functions
- know the Laplace transform of the unit step function
- use the second shift theorem to invert Laplace transforms
- obtain the Laplace transform of a periodic function
- know the Laplace transform of the unit impulse function
- obtain the transfer function of a simple linear time-invariant system
- obtain the impulse response of a simple system
- apply initial-value and final-value theorems
- obtain the frequency response of a simple system.

z transforms

As a result of learning this material you should be able to

- recognise the need to sample continuous-time functions to obtain a discrete-time signal
- obtain the z transforms of simple sequences
- use the linearity and shift properties to obtain z transforms
- know the 'multiply by a^k ' and 'multiply by k^n ' theorems
- use the initial-value and final-value theorems
- invert a transform using tables and partial fractions
- solve initial-value problems using z transforms
- compare this method of solution with the method using Laplace transforms.

Complex functions

As a result of learning this material you should be able to

- define a complex function and an analytic function
- determine the image path of a linear mapping
- determine the image path under the inversion mapping
- determine the image path under a bilinear mapping
- determine the image path under the mapping $w = z^2$
- understand the concept of conformal mapping and know and apply some examples
- verify that a given function satisfies the Cauchy-Riemann conditions
- recognise when complex functions are multi-valued
- define a harmonic function
- find the conjugate to a given harmonic function.

Complex series and contour integration

As a result of learning this material you should be able to

- obtain the Taylor series of simple complex functions
- determine the radius of convergence of such series

- obtain the Laurent series of simple complex functions
- recognise the need for different series in different parts of the complex plane
- understand the terms 'singularity', 'pole'
- find the residue of a complex function at a pole
- understand the concept of a contour integral
- evaluate a contour integral along simple linear paths
- use Cauchy's theorem and Cauchy's integral theorem
- state and use the residue theorem to evaluate definite real integrals

Introduction to partial differential equations

As a result of learning this material you should be able to

- recognise the three main types of second-order linear partial differential equations
- appreciate in outline how each of these types is derived
- state suitable boundary conditions to accompany each type
- understand the nature of the solution of each type of equation.

Solving partial differential equations

As a result of learning this material you should be able to

- understand the main steps in the separation of variables method
- apply the method to the solution of Laplace's equation
- interpret the solution in terms of the physical problem.

Discrete Mathematics

The material in this section covers the basic development of discrete mathematics consequent on the material in Core Level 1.

Number systems

As a result of learning this material you should be able to

- carry out arithmetic operations in the binary system
- carry out arithmetic operations in the hexadecimal system
- use Euclid's algorithm for finding the greatest common divisor

Algebraic operations

As a result of learning this material you should be able to

- understand the notion of a group
- establish the congruence of two numbers modulo n
- understand and carry out arithmetic operations in Z_n , especially in Z_2
- carry out arithmetic operations on matrices over Z_2
- understand the Hamming code as an application of the above (any other suitable code will serve just as well).

Recursion and difference equations

As a result of learning this material you should be able to

- define a sequence by a recursive formula
- obtain the general solution of a linear first-order difference equation with constant coefficients
- obtain the particular solution of a linear first-order difference equation with constant coefficients which satisfies suitable given conditions
- obtain the general solution of a linear second-order difference equation with constant coefficients
- obtain the particular solution of a linear second-order difference equation with constant coefficients which satisfies suitable given conditions

Relations

As a result of learning this material you should be able to

- understand the notion of binary relation
- find the composition of two binary relations
- find the inverse of a binary relation
- understand the notion of a ternary relation
- understand the notion of an equivalence relation on a set
- verify whether a given relation is an equivalence relation or not
- understand the notion of a partition on a set
- view an equivalence either as a relation or a partition
- understand the notion of a partial order on a set
- understand the difference between maximal and greatest element, and between minimal and smallest element.

Graphs

As a result of learning this material you should be able to

- recognise an Euler trail in a graph and / or an Euler graph
- recognise a Hamilton cycle (path) in a graph
- find components of connectivity in a graph
- find components of strong connectivity in a directed graph
- find a minimal spanning tree of a given connected graph.

Algorithms

As a result of learning this material you should be able to

- understand when an algorithm solves a problem
- understand the 'big O' notation for functions
- understand the worst case analysis of an algorithm
- understand one of the sorting algorithms
- understand the idea of depth-first search
- understand the idea of breadth-first search
- understand a multi-stage algorithm (for example, finding the shortest path, finding the minimal spanning tree or finding maximal flow)
- understand the notion of a polynomial-time-solvable problem
- understand the notion of an NP problem (as a problem for which it is 'easy' to verify an affirmative answer)
- understand the notion of an NP-complete problem (as a hardest problem among NP problems).

Geometry

The material in this section covers the basic development of geometry consequent on the material in Core Level 1.

Helix

As a result of learning this material you should be able to

- recognise the parametric equation of a helix
- derive the main properties of a helix, including the equation of the tangent at a point, slope and pitch.

Geometric spaces and transformations

As a result of learning this material you should be able to

- define Euclidean space and state its general properties
- understand the Cartesian co-ordinate system in the space
- apply the Euler transformations of the co-ordinate system
- understand the polar co-ordinate system in the plane
- understand the cylindrical co-ordinate system in the space
- understand the spherical co-ordinate system in the space
- define Affine space and state its general properties
- understand the general concept of a geometric transformation on a set of points
- understand the terms 'invariants' and 'invariant properties'
- know and use the non-commutativity of the composition of transformations
- understand the group representation of geometric transformations
- classify specific groups of geometric transformations with respect to invariants
- derive the matrix form of basic Euclidean transformations
- derive the matrix form of an affine transformation
- calculate coordinates of an image of a point in a geometric transformation
- apply a geometric transformation to a plane figure.

Linear Algebra

The material in this section covers the basic development of linear algebra consequent on the material in Core Level 1.

Matrix methods

As a result of learning this material you should be able to

- define a banded matrix
- recognise the notation for a tri-diagonal matrix
- use the Thomas algorithm for solving a system of equations with a tri-diagonal coefficient matrix
- partition a matrix
- carry out addition and multiplication of suitably-partitioned matrices
- find the inverse of a matrix in partitioned form.

Eigenvalue problems

As a result of learning this material you should be able to

- interpret eigenvectors and eigenvalues of a matrix in terms of the transformation it represents
- convert a transformation into a matrix eigenvalue problem
- find the eigenvalues and eigenvectors of 2x2 and 3x3 matrices algebraically
- determine the modal matrix for a given matrix
- reduce a matrix to diagonal form
- reduce a matrix to Jordan form
- state the Cayley-Hamilton theorem and use it to find powers and the inverse of a matrix
- understand a simple numerical method for finding the eigenvectors of a matrix
- use appropriate software to compute the eigenvalues and eigenvectors of a matrix
- apply eigenvalues and eigenvectors to the solution of systems of linear difference and differential equations
- understand how a problem in oscillatory motion can lead to an eigenvalue problem
- interpret the eigenvalues and eigenvectors in terms of the motion
- define a quadratic form and determine its nature using eigenvalues.

Statistics and Probability

The material in this section covers the basic development of statistics and probability consequent on the material in Core Level 1.

One-dimensional random variables

As a result of learning this material you should be able to

- compare empirical and theoretical distributions
- apply the exponential distribution to simple problems
- apply the normal distribution to simple problems
- apply the Weibull distribution to simple problems
- apply the gamma distribution to simple problems.

Two-dimensional random variables

As a result of learning this material you should be able to

- understand the concept of a joint distribution
- understand the terms 'joint density function', 'marginal distribution functions'
- define independence of two random variables
- solve problems involving linear combinations of random variables
- determine the covariance of two random variables
- determine the correlation of two random variables.

Small sample statistics

As a result of learning this material you should be able to

- realise that the normal distribution is not reliable when used with small samples
- use tables of the t -distribution
- solve problems involving small-sample means using the t -distribution
- use tables of the F -distribution
- use pooling of variances where appropriate
- use the method of pairing where appropriate.

Small sample statistics: chi-square tests

As a result of learning this material you should be able to

- use tables for chi-squared distributions
- decide on the number of degrees of freedom appropriate to a particular problem
- use the chi-square distribution in tests of independence (contingency tables)
- use the chi-square distribution in tests of goodness of fit.

Analysis of variance

As a result of learning this material you should be able to

- set up the information for a one-way analysis of variance
- interpret the ANOVA table
- solve a problem using one-way analysis of variance
- set up the information for a two-way analysis of variance
- interpret the ANOVA table
- solve a problem using two-way analysis of variance.

Simple linear regression

As a result of learning this material you should be able to

- derive the equation of the line of best fit to a set of data pairs
- calculate the correlation coefficient
- place confidence intervals around the estimates of slope and intercept
- place confidence intervals around values estimated from the regression line
- carry out an analysis of variance to test goodness of fit of the regression line
- interpret the results of the tests in terms of the original data
- describe the relationship between linear regression and least squares fitting.

Multiple linear regression and design of experiments

As a result of learning this material you should be able to

- understand the ideas involved in a multiple regression analysis
- appreciate the importance of experimental design
- recognise simple statistical designs.

3.4 Level 3

This level is the one at which the mathematical techniques covered should be applied to a range of problems encountered in industry by practising engineers. These advanced methods build on the foundations laid by Levels 1 and 2 of the curriculum. It is quite possible that much of this material will be taught not within the context of dedicated mathematical units but as part of units on the engineering topics to which they directly apply. It is expected that significant use will be made of industry standard mathematical software tools. The specialised nature of these techniques and the importance of their application in an engineering setting makes detailed learning outcomes (as given for the other levels of the curriculum) less straightforward to define. For this reason only a list of general topic headings will be given. This material will be taught only towards the end of a degree programme.

Analysis and calculus

- Numerical solution of ordinary differential equations
- Fourier analysis
- Solution of partial differential equations, including the use of Fourier series
- Fourier transforms
- Finite element method

Discrete mathematics

- Combinatorics
- Graph theory
- Algebraic structures
- Lattices and Boolean algebra
- Grammars and languages

Geometry

- Differential geometry
- Geometric modelling of curves and surfaces
- Geometric methods in solid modelling
- Non-Euclidean geometry
- Computer geometry
- Fractal geometry
- Geometric core of Computer Graphics

Linear Algebra

- Matrix decomposition
- Further numerical methods

Statistics and probability

- Stochastic processes
- Statistical quality control
- Reliability
- Experimental design
- Queueing theory and discrete simulation
- Filtering and control
- Markov processes and renewal theory
- Statistical inference
- Multivariate analysis

4 Teaching and learning environments

A subject-specific curriculum sets the educational goals for a part of the study course. The previous two chapters presented a framework for specifying a mathematics curriculum for an engineering study course based on the concept of mathematical competence. In order to implement such a curriculum, the whole teaching and learning environment has to be taken into account. In this chapter we intend to discuss some issues which are related to the provision of such an environment for a competence-based mathematics curriculum. Section 4.1 investigates the suitability of different teaching and learning arrangements like lectures, e-learning scenarios, tutorials or projects. Section 4.2 on transition addresses the problem that many content-related learning outcomes listed in the core zero part of chapter 3 are missing on entrance to the study course such that support has to be provided. Section 4.3 deals with the use of mathematics technology which is ubiquitous in engineering and engineering education, mostly in the form of mathematics or application programs. Section 4.4 sheds light on various aspects of integrating the mathematics curriculum into an engineering study course and section 4.5 briefly addresses the attitude of students towards the value of mathematics for their field of study and the consequences for their learning behavior.

4.1 Teaching and learning arrangements¹

In this section, traditional and more recent learning arrangements are investigated regarding their potential for competence acquisition. Learning and teaching arrangements appear to have changed little over the years. The predominant form of delivery remains the lecture, albeit this is now often backed up with supporting materials on a Virtual Learning Environment and may be delivered using much more modern methods than chalk and blackboard. There are many challenges facing the teacher. These include the challenge of teaching to a large cohort of students, often with widely varying levels of prior mathematical knowledge. There are challenges in motivating and engaging engineering students in their study of mathematics, in particular in incorporating engineering applications into the mathematics presented. There are also challenges in determining what mathematics is relevant in today's fast-changing society and how the introduction of computer algebra packages and other software impacts upon this. Moreover, as assessment often drives learning, there is the challenge of ensuring that assessment is relevant and assesses the required skills and competencies.

The learning arrangements considered here include lectures, assignments, tutorials, projects, laboratories and technology enhanced learning (which includes, but is much broader than, e-learning); these will be related to the eight competencies. The contributions that the different learning arrangements can make to competency acquisition are discussed.

Lectures

We start with the most traditional, and probably also the most widespread, form of mathematics teaching – the lecture. Even in problem-based learning settings as described below, lectures still play a certain role (Christensen 2008). Lectures can take many different forms. Traditionally, giving a lecture meant a one-directional presentation of material, during which student activity is primarily restricted to taking notes, although occasionally a student may ask the lecturer a question. Recently, ways of increasing the level of student involvement in lectures have been explored (active learning components). These methods include the use of 'paired discussion' and the use of 'clickers', or

¹ Most of the following is essentially a reproduction of Alpers & Demlova 2012.

personal response systems (Robinson 2010). It should be noted that the size of the lecture plays an important role – in smaller lectures (up to 50 students) it is considerably easier to promote an active role of students than in larger ones (in excess of 100 students).

The main reason for lectures is to introduce a larger audience to certain mathematical concepts and procedures. The goal is to give students a ‘first familiarity’ with the material; subsequent individual or group activities, carried out by the learners, are usually necessary to increase understanding of the material, to recognise when it should be used and to be able to apply the concepts and procedures in both mathematical and applications contexts. A good lecture should motivate the material, relate it to previous concepts and provide the “overall picture” (Slomson 2010).

In what follows, we outline how lectures can contribute to the acquisition of the eight mathematical competencies. We restrict ourselves to traditional introductory mathematics lectures for engineers. More advanced mathematical lectures (for example courses in discrete mathematics or mathematical logic) may contribute in a slightly different way to acquiring competencies. There might also be specific lectures on mathematical modelling or problem solving which are dedicated to addressing specific competencies but – given the usual curricular restrictions – most lectures cover the principal concepts and procedures in analysis and linear algebra.

Mathematical thinking: In order to enhance the mathematical thinking competency, lecturers should emphasise in their lectures what mathematics is able to contribute to engineering work. For example, by arguing logically it is possible to show that a certain geometrical construction in a technical drawing is fixed by certain data, or that an ODE modelling a damped mass-spring system can only behave in a small number of ways. Moreover, in some circumstances, using a mathematical model can enable the determination of reasonable or even optimal configurations in advance, thereby avoiding the need for costly experimentation.

Mathematical reasoning: In a lecture, the lecturer demonstrates correct mathematical reasoning when proving results, justifying certain assumptions or selecting a method of solving a problem. If the theory is laid out as a finished piece of mathematics, students do not see the process of creation and thinking behind the theory (as is also often the case with mathematical articles). Therefore, the lecturer should explain the reasoning behind setting up definitions and theorems and should not just present the formal definitions and arguments but should provide a considerable amount of explanatory material.

Mathematical problem solving: Again, in a lecture the students do not see the real problem-solving process but merely the ‘polished’ final version (which often gives the wrong impression, that everything in mathematics is straightforward once you have learned the correct procedure). Therefore, a lecture is quite restricted here. Nevertheless, the lecturer should explicitly outline the problem-solving strategies that are applied, for example analogy (do it as in the case of ...); transforming into a familiar domain; ‘divide and conquer’ (split up into special cases); try to make use of information/properties you have (relate them to things you want to know or understand).

Mathematical modelling: As stated above regarding problem solving, the modelling process can only be shown in simple examples (not the real going back and forth in the modelling cycle). One can explain and emphasise which kind of situation or behaviour can be modelled with a certain mathematical concept (e.g. vibration with sine functions, certain kinds of growth and decay with exponential functions, static behaviour with equations, etc.). When the students carry out their own

modelling activities in other learning arrangements, they then have at least ‘material’ with which to experiment. As Niss (2010) stated, if one wants to set up a model one has to anticipate what might work, and the lecture might help in the process of anticipation (real experience with many modelling activities will be of greater help, though).

Representing mathematical entities: In lectures the value of different representations can be, and should be, demonstrated (and therefore the necessity to switch between representations). There are many places in undergraduate mathematics where this can be done (different representations of lines: parameter form and equation form; graphical and algebraic representations of equations and inequalities; representations of functions; time domain and frequency domain). Therefore, the ‘theme’ of different representations can be explicitly emphasised at several places in a lecture, enlarging the probability that students see and retain the value for later use.

Handling mathematical symbols and formalism: The lecture provides examples of the correct use of symbols and formalism in mathematics. This need not be as formal as in lectures for mathematics students (which would be too formal for most engineering students); but a semi-formal presentation should also serve as an example for students of computation and logical argument. For example, the use of set notation or short notations such as Σ for a sum at several places in the lecture should help students to familiarise themselves with this formal notation and language.

Communicating in, with and about mathematics: In a classical lecture, the receptive side of this competency is emphasised. Students are required to listen and follow the oral (in the lecture theatre) and written (in accompanying scripts) argument of the lecturer. Here again, the lecturer should provide good examples of mathematical presentation appropriate to the audience (for example explain your reasoning, make the structure of your argument clear, try to make connections to the previous experience of the audience, emphasise important topics and de-emphasise technicalities). The students should try to relate the new concepts and procedures to their previous knowledge base and gain a preliminary understanding that should be enhanced in their own active studies later.

Making use of aids and tools: The lecturer can provide demonstrations of the reasonable use of tools and other aids (e.g. visualisation of complex concepts; animation of processes; choice of adequate representation; quick computation of larger examples). These examples can then be used by students later when working on assignments or projects.

There have been several attempts to make the classroom scenario more interactive, even in larger classrooms (Mason 2002; Gavalcova 2008; Robinson 2010). In smaller groups (20 to 50) it is possible to create a “guided, directed dialogue” (Gavalcova 2008) by asking questions and letting students give and explain answers. Students can also give answers by using electronic voting systems (EVS, see Robinson 2010), which provide the lecturer with an overall picture of the current understanding of the audience. One can also include student activities by giving them small problems to discuss with each other in pairs or to make individual computations using their own technology. These active learning methods can enhance the acquisition of additional aspects of competencies compared to the classical unidirectional situation. If students are given questions that go beyond mere facts and require some sort of mathematical reasoning, the acquisition of the respective competency is being developed. If students are to exchange their arguments in pairs, the active side of the communication competency is also addressed. There are many conceptual questions (for a bank of such questions for use with EVS see Robinson 2010), for example regarding different forms of representations and their relationships, which can be given to students to discuss in lectures.

Moreover, when questions require the use of technology (normally pocket calculators) then the respective competency is also included in an active way. In summary, there are several ways of involving students actively even within a lecture scenario which help them acquire the 'active' side of mathematical competencies.

Assignments

By assignments we mean all kinds of 'smaller' tasks that students have to undertake on their own, be it in groups or individually. These include standard computational tasks that serve to develop more familiarity with notation, formalism and procedures but also more open and investigative assignments, with or without technology. Larger problems or projects are not included here but are dealt with separately below.

Mathematical thinking: Mathematical thinking could be fostered in more open application tasks where students have to work with application models and solve questions that are of practical interest. This would demonstrate to students that having a mathematical model is helpful when working on practical tasks like machine dimensioning or choosing adequate parameters in control devices. On the other hand, a complete restriction to standard procedural computation tasks could lead the students to think that mathematics has nothing to do with real engineering work and hence is just an obstacle to be overcome during the early semesters.

Mathematical reasoning: In standard tasks very restricted forms of procedural reasoning can be exercised but in more open assignments the development of chains of logical arguments can be developed (for example show that a certain geometric configuration is uniquely determined by certain data; or even more open: by which data is the configuration uniquely determined). Advanced mathematical courses provide even more material for exercising mathematical reasoning, for example courses on discrete mathematics or mathematical logic.

Mathematical problem solving: Standard problems (for example how to integrate a function using one of the standard methods) can be learned using standard tasks (for example integrate a rational function using the partial fraction method). More open assignments (like 'construct a function to move from A to B given certain restrictions') can serve to reflect on the principal procedure to tackle such a problem. It is a question, though, whether many students are able to work on such a problem without tutorial support. So, the problems in such a learning arrangement are still likely to be rather 'well-formulated'.

Mathematical modelling: In standard tasks, only that part of mathematical modelling is practised where mathematically-formulated problems are solved using given mathematical models. In more open assignments there will still be a well-defined application situation but the 'translation task' (as in word problems) to be performed might be more challenging.

Representing mathematical entities: In standard tasks, one can train students to switch between different representations (the computational part). In more open assignments, one can also train them to choose an adequate representation for a particular problem.

Handling mathematical symbols and formalism: Standard tasks are necessary and important to enable students to become familiar with fundamental concepts and procedures. A certain fluency in dealing with symbols and formalism needs more or less permanent training (like fitness in sports), depending on individual abilities.

Communicating in, with and about mathematics: If students have to hand in written assignments, they have to state clearly their arguments and in this way learn to actively communicate mathematically. If students work on assignments in groups, the oral component of this competency is also taken into account.

Making use of aids and tools: If the assignments include the use of aids like formulae books, pocket calculators or even mathematical programs, then the competency of making adequate use of aids and tools is also addressed.

Tutorials

By tutorials we mean learning arrangements where a tutor (teaching assistant or, possibly, a student) works with students in order to improve their understanding related to a lecture. Such tutorials can differ significantly with regard to the method of teaching and learning. There are tutorials where the tutor mainly performs example computations leading to a situation that is not much different from the lecture. But there are also forms of tutorial with active involvement of students who work on standard tasks or on more open assignments with the help of tutors. Students might also give presentations of their solutions on the blackboard.

Since, in tutorials, similar tasks are dealt with as in assignments, the statements made in the previous section on assignments also hold for tutorials. In addition to this, tutorials provide the opportunity to have group discussions and presentations by students such that the communication competency can be better addressed. Moreover, because tutorial support is available, tasks can be more open since students can ask the tutor for help.

Projects

By projects we mean learning arrangements where students work – mostly in groups – on problems which are larger, more open and investigative in nature (for guidelines, see Alpers 2002). Usually, students have to document and present their work at the end. In problem-based learning settings (Niss 2001; Christensen 2008) this is the predominant way of learning although even there mixed forms including lectures can be found.

Mathematical thinking, reasoning, problem solving and modelling: In projects, particularly in application projects, students can extend their understanding of what mathematics can do for them as prospective engineers. Students have to think about how to proceed, which steps to take in tackling the given problem and to check how far they got in the process, and what still needs to be done. This planning, monitoring and control work is of a general nature but when it comes to the mathematical kernel of a project it also addresses the mathematical problem-solving competency. Larger projects allow students to experience the full modelling cycle. Students set up and work with mathematical models reflecting an application situation which allows them to make variations and to experiment in order to get a better understanding of the situation and/or to achieve certain properties. This reflects real engineering work with programs implementing a mathematical model. Critical mathematical thinking (What are the restrictions of what mathematics can do for you?) can also be fostered when students think about the assumptions in models and parameters of models. When students do not simply experiment randomly but rather reason mathematically about the influence of parameters and dependency on assumptions, the respective competency is also

developed. Note that whether this potential can actually be put into reality depends strongly on the quality of the project tasks and the tutorial support.

Communication in, with, and about mathematics: When students have to read mathematical texts on their own (including short web pages on mathematical concepts) and when they have to understand the mathematical explanations of a project group member, the passive side of the communication competency is addressed. When they explain themselves, write project documentation and make an oral presentation to other students, the active side is also taken into account. Moreover, in documentation and presentation, questions of adequate representation very often arise in the need to get a clear message across to the audience.

Making use of aids and tools: More realistic problems usually require the use of mathematical software, so students also improve their competency of using tools properly. When they create their own experimentation environment for an application situation and try to use it in a goal-directed way (by making informed changes and interpreting the effect), they become accustomed to the way engineers use mathematically-based software programs in their work.

Mathematics laboratories

By mathematics laboratories, we mean learning scenarios where students work in a PC laboratory on tasks requiring the use of mathematical software such as numerical programs (Matlab®), CAS (Maple®, Mathematica®) or spreadsheets (Excel®). In such laboratory sessions, students practise the usage of the programs and see how they can be used for standard tasks. They might also be used for experimenting with more open tasks of an investigative nature.

The same competency potential that is outlined in the earlier section on assignments can also be claimed for laboratories. In addition, the tool usage competency is specifically addressed. Moreover, since mathematical programs require mathematical notation and formalism as input, the respective competency is also developed. Regarding the representation competency, work with mathematical programs in laboratories also has high potential since students can switch flexibly between different representations. This must be embedded in adequate tasks to be meaningful and not just 'playing around'.

Technology enhanced learning

There are many ways in which technology can be used to enhance the learning process. These are given different names, including e-learning, blended learning, on-line learning, etc., which often suggest that the learning activity may be carried out remotely from the presence of a member of academic staff. However, whilst a substantial amount of such materials is available, to restrict thinking about enhancing learning only to material for distance learning is an overly narrow view of using technology to enhance learning. As noted in the preceding section, technology can be used to enhance face-to-face learning experiences.

For remote learning, presentation material, potentially using multimedia, can be made available for students to use to re-visit certain content in order to gain better understanding. This material can be prepared by, for example, using lecture capture technology, or Tablet computers. Animated worked examples can be particularly effective as they allow students to see solutions being developed in real-time (and the audio of the animations allows for explanation of the more difficult steps). Such worked solutions could be prepared using technology such as Livescribe pens (see

<http://www.livescribe.com/uk/>) or screen capture from a Tablet PC using software such as Camtasia (see <http://www.camtasia.com>).

There is also a wealth of supporting material on the web, such as mathcentre (www.mathcentre.ac.uk) and Khan Academy (www.khanacademy.org) which provide explanatory text, videos and self-assessment that is not directly related to a specific course but which is topic-based. Computer-assisted assessment, for example, as implemented in testing systems like STACK (www.stack.bham.ac.uk) or Maple TA[®] (www.maplesoft.com/products/mapleta), can be used to allow lecturers and students to check procedural accuracy and, in some cases, understanding.

Technology can also enable more interactive learning scenarios: For example, applets or other small learning objects can be produced which allow students to make changes (e.g. parameter variations) and determine their effect, or to work on tasks to achieve certain properties by making variations. There are also more sophisticated intelligent tutoring systems which allow the insertion of single steps and provide tutorial help. An electronic forum might also be used as a means of collaboration and communication between students, or between students and tutors/lecturers. More recently there has been the advent of massive open online courses (MOOCs) as part of the open educational resource movement. Some universities in the US and elsewhere have partnered with companies, such as Coursera, to make some of their courses freely available online to a large audience.

Technology-enhanced learning offers competence acquisition opportunities similar to the other learning arrangements discussed above. The passive side of the communication competency is addressed when students have to read and understand mathematical material presented electronically. The active side is particularly taken into account when students work in on-line discussion groups (forum) and explain mathematical material to each other. Using electronic aids and tools can also be trained in an e-learning environment. Working on larger problems or projects needs human interaction and tutoring which could in theory also be provided via electronic communication channels but personal dialogue is still stronger here.

In summary, one can state that the classical 'lecture theatre' arrangement still has its potential, in particular when it is enhanced by components of active learning, but it is certainly not sufficient. It can be considered as an example of 'cognitive apprenticeship' where students see mathematical competence in action as performed by the lecturer; but students still have to work on mathematical tasks and problems themselves to become really competent. A blended approach, containing a mixture of several learning arrangements seems to be appropriate, where the particular offering certainly depends on circumstances like group size and available resources. Moreover, mathematical competencies are also acquired in application subjects, such as engineering mechanics, where the setting up of and working with mathematical models play an important role (cf. section 4.4).

4.2 Transition issues

The move from secondary education to university can be a challenging time for many students. It is well-known that the 'drop-out' rate is highest during the first few weeks of the first term. This is an issue that faces all subjects and is one that exercises academics from across Europe and indeed the world. In Europe a network of interested academics has been created, the European First Year Experience Network (<http://www.efye.eu>). Since 2006, the Network has organised an annual conference, details of which can be found on their website. In the USA, work on the first-year experience is even more firmly established. The National Resource Center for the First Year Experience and Students in Transition is hosted by the University of South Carolina

(<http://www.sc.edu/fye/>) and in 2013 the 32nd annual First Year Experience conference was held, following the 19th annual Students in Transition conference which took place in October 2012.

Engineering students face the same transition issues as students of other disciplines but, in addition, they face some subject specific issues. Most notable amongst these is the issue of mathematics. The study of mathematics is essential for all aspiring engineers. However, for many undergraduates this is viewed as a chore, as a necessary evil to be endured. Engineering undergraduates have chosen to go to university to study engineering not mathematics and it is often the practical, problem-solving elements of engineering that inspire them. Many undergraduates find the mathematics that they study too abstract and theoretical to be enthused by the subject. A pedagogy different from that used with mathematics undergraduates is needed to motivate and inspire engineering undergraduates. It is for this reason that recent seminars of the SEFI Mathematics Working Group have focused on 'active learning'. Papers which have addressed this theme include Gavalcova (2008), Healy, Marjoram, O'Sullivan, Reilly and Robinson (2010), Janilionis and Valantinas (2008) and Robinson (2010).

There is another crucial issue concerning mathematics and incoming engineering undergraduates – and that is the issue of basic mathematical competence. For many years now there has been considerable discussion in the United Kingdom about the so-called *Mathematics Problem* – this is the gap between the level of mathematical competency that higher education wishes incoming undergraduate engineers to possess and the level that they actually do possess. A seminal report, published in 2000 by the Engineering Council, *Measuring the Mathematics Problem* (Hawkes and Savage 2000), showed how the level of basic mathematical skills amongst students entering university with the same level of secondary qualification in mathematics had declined drastically during the 1990s.

Although a great deal of attention, including a major Government inquiry into post-14 mathematics education (the Smith Inquiry which produced the report *Making Mathematics Count* (Smith 2004)), has been devoted to the 'mathematics problem' it is clear that the problem remains unsolved. In 2007, a report prepared for the House of Commons (National Audit Office 2007) stated that

Many students require some additional academic support, especially in the mathematical skills required in science, mathematics, engineering and technology. (op.cit. paragraph 3.16, page 32).

The persistent nature of the mathematics problem was highlighted in a report by the House of Lords (the upper legislative chamber) Select Committee on Science and Technology (House of Lords 2012):

In 2006, the Royal Society argued that the gap between the mathematical skills of students when they entered HE and the mathematical skills needed for STEM [Science, Technology, Engineering and Mathematics] first degrees was a problem which had become acute ... The evidence we received suggested that the problem remains. (op. cit. paragraph 25, page 15).

Whilst the United Kingdom has paid the most attention to the 'mathematics problem', these issues are affecting a range of countries throughout Europe and the world. In recent years, several papers highlighting the mathematical under-preparedness of incoming engineering undergraduates have been presented at SEFI Mathematics Working Group seminars. These papers cover a range of nations including Germany (Cramm 2012, Kurz 2010, Schwenk and Kalus 2012), Hungary (Csakany 2012), Ireland (Carr, Murphy, Bowe and Ni Fhloinn 2012), Spain (Nieto 2012). In a discussion at the

14th SEFI Mathematics Working Group Seminar entitled *What are the major problems facing engineering maths education in Europe*, the delegates concluded that “the lack of basic skills of university freshmen is well-known and seems to be Europe-wide”(Alpers 2008).

Whilst the vast majority of universities would undoubtedly prefer incoming engineering undergraduates to have greater mathematical skills than they do, such universities must accept the realities of the students they enrol. As a House of Commons report (Public Accounts Committee 2008) pointed out

There is much that universities can do to improve retention ... They can provide additional academic support for students, for example those struggling with the mathematical elements of their course. (op. cit. page 3).

One mechanism to provide this additional academic support that has been commonly adopted at universities throughout the United Kingdom has been the establishment of mathematics support centres. Mathematics support is the provision of extra-curricular assistance for students of any discipline (most frequently engineering and the physical sciences) who are encountering difficulties with the mathematical elements of their courses. The most common model of mathematics support is that of the ‘drop-in centre’. In this approach, students can drop in, that is, attend without appointment at a time suitable to them (within the designated opening hours) and ask for assistance with any areas of mathematics which are causing them difficulties.

A key element of the provision of successful mathematics support is the friendly, welcoming, supportive atmosphere of the drop-in centre. For many students, the most significant problem with regard to their mathematical attainment within their course is not their current level of competence but their confidence. Many students arrive in higher education having had bad experiences in their mathematical education to date and they believe that they “cannot do mathematics”. If this lack of confidence goes unaddressed, for many students it will result in them performing poorly in the mathematical elements of their university course. Mathematics Support Centres can be an effective way of improving student confidence. However, in order to ensure that as many students as possible engage with the services offered it is necessary to provide a safe, non-threatening, non-judgemental environment in which these students can address their lack of confidence and gaps in their background knowledge.

The guide *Setting Up a Maths Support Centre* (Lawson 2012) presents a series of five case studies of mathematics support centres at different universities across England. Although the drop-in centre model is the most prevalent, other models of mathematics support are used and descriptions of some of these can be found in *Responding to the Mathematics Problem: The Implementation of Institutional Support Mechanisms* (Marr and Grove, 2010). The report *How to set up a mathematics and statistics support provision* (Mac an Bhaird and Lawson, 2012) provides step-by-step information for colleagues wishing to establish mathematics support in their own institution.

The most valuable resource provided in mathematics support centres is the staff who work with students on a one-to-one basis or in small groups. However, most centres also provide a range of resources which students can use for self-study and for support at times when the centre is not open. The **mathcentre** web-site (www.mathcentre.ac.uk) contains several hundred resources of different types (including short help leaflets, longer self-study guides, video tutorials and interactive

exercises) which are now used by students from universities around the world. These quality assured resources (all in English) are freely available for students and staff to download and use.

Two of the leading mathematics support centres are located at Coventry University and Loughborough University and these two institutions have collaborated to establish **sigma**, a Centre for Excellence in University wide mathematics and statistics support. In 2005, **sigma** was designated by the Higher Education Funding Council for England as a Centre for Excellence in Teaching and Learning (CETL) (<http://www.hefce.ac.uk/whatwedo/lt/enh/cetl/>). Having been acknowledged as one of the most successful CETLs, **sigma** was commissioned by the National HE STEM Programme (www.hestem.ac.uk) to promote mathematics support, to assist in the establishment of new mathematics support centres and to set up a network of mathematics support providers throughout England and Wales. It has done this with great success, helping to establish 22 new mathematics support centres, developing six **sigma** regional hubs and creating the **sigma-network**, a free association of those involved in the provision of mathematics and statistics support throughout England and Wales (www.sigma-network.ac.uk). Since 2006, **sigma** has organised an annual conference (CETL-MSOR). The proceedings of these conferences contain many useful papers relating to mathematics support. These proceedings are available at <http://www.mathstore.ac.uk/?q=node/2049>.

A recent survey (Perkin et al, 2012) showed that around 85% of universities in the UK have some form of mathematics support provision. However, mathematics support is not confined to the United Kingdom. Such provision is now widespread in Australia (MacGillivray, 2008) and Ireland (Gill, O'Donoghue and Johnson, 2008), where there is a national network of support providers (the Irish Mathematics Learning Support Network <http://supportcentre.maths.nuim.ie/mathsnetwork/>). In addition, in recent years, mathematics support centres have been opened in Germany, Switzerland and Sweden.

In addition to the need to adopt a pedagogy that motivates engineering students to study mathematics and which accommodates incoming undergraduates whose mathematical skills are not at the desired level, academic staff must also deal with an increasing inhomogeneity amongst the students that they teach. There are several causes of this inhomogeneity. As noted previously, the mathematical skills of many new undergraduates are not the same as their contemporaries in previous year – however the skills of the best students remain at a very high level. In addition, in many European countries there has been an increase in the number of students entering higher education. At the very least, this means that a larger proportion of the age cohort enters higher education which inevitably increases the spread of student ability (if only the top 5% of the cohort enters higher education then the spread of ability will be quite limited but if the top 40% go to university then inevitably there is a much broader spread). In addition to the increased spread of ability there is often also an increase in the diversity of pre-university mathematics education. In other words, students entering the same university engineering course have studied different mathematics qualifications prior to entering university (Carr et al, 2012). A variety of approaches have been adopted to deal with the greater diversity of mathematical preparation and ability amongst the engineering cohort. Mathematics support centres, as outlined above, can play a useful role. Other initiatives introduced to address the issue include diagnostic testing (MathsTEAM, 2003), bridging courses (Bamforth et al, 2007) and streaming (Steele, 2000).

4.3 Mathematics technology

The effect of computer technology on education seems to be greater in mathematics than in any other subject. There are two distinct ways in which developments in technology affect learning and teaching in mathematics. The first is that new technology provides opportunities for new approaches to teaching and learning (this applies to all disciplines not only to mathematics); the second is that advances in technology impact not only on how mathematics is taught but also on what mathematics is taught (in this area mathematics is probably unique). Advances in the capabilities and user-friendliness of mathematical software mean that a whole range of problems which previously would have needed graduate level skills to solve can now be accessed by first year undergraduates. In this section, we do not cover the use of general computer technology in mathematics education rather, we restrict ourselves to what we will call ‘mathematics technology’, by which we mean technology whose specific purpose is ‘doing mathematics’. General technology aspects related to learning scenarios have already been treated in section 4.1 and technology-supported assessment will be discussed in chapter 5. The term ‘mathematics technology’ covers a wide range of different artifacts including:

- Pocket calculators with different symbolic and/or numerical and/or graphical capabilities
- Mathematical computer programs:
 - symbolic and numerical ones, e.g. Computer Algebra Systems (CAS) like Maple®, Mathematica® or MathCad®;
 - numerical programs like Matlab®;
 - dynamic geometry programs like Cabri and Geogebra;
 - spreadsheet programs.
- Engineering programs based on mathematical models which ‘shine through’ to a certain extent (CAD, FEM, mechanism design, multi-body dynamics, CFD, ...).

There are several potential educational advantages of using such mathematics technology which have been identified in intensive research particularly in general mathematics education during the past 20 years but risks have also been recognised and discussed in several seminars of the Mathematics Working Group (see e.g. Alpers 2006). We give a brief overview of the opportunities and risks as far as the mathematical education of engineers is concerned (for further reading see (Oates 2009) and the references therein). We start with potential advantages:

- Visualisation/demonstration: In some topics computer animation can greatly increase the effectiveness of the teaching process, for example, in calculus or multivariable calculus (cf. Velichova 2008), in the theory of differential equations or in geometry, where CAS can be used as demonstration and visualization tools for better conceptual understanding.
- Explorative approach to learning: CAS can be used as cognitive tools, as they facilitate the technical dimension of mathematical activity and allow the user to take action on mathematical objects or representations of those objects. This feature can be utilised to enable students to explore objects and structures and to discover properties and connections e.g. by performing parameter variations.
- Experimental approach to problem solving: Mathematics programs provide new ways of problem solving. In classical paper and pencil work students had to know a certain procedure in order to solve a problem and they could not advance once they got stuck in the process. Mathematics programs allow students to select different ways of investigating a problem (for example finding an approximate solution by looking at the graph of a function instead of

getting an exact solution; investigating several related examples to derive a hypothesis or to discover a counter-example) and thus increase the student's likelihood of making progress with a problem. This is particularly helpful in design problems, e.g. for designing a motion function fulfilling certain conditions regarding maximum velocity and acceleration. The worksheet interface of CAS allows the lecturer to create easily an experimental environment for such problems and the thoughtful (or guided) variation of parameters can enhance the understanding of function properties considerably.

- Realistic modeling: Mathematics and engineering programs allow the earlier introduction of more interesting modelling tasks since some parts of the computation can be delegated to the program (for example solution of a non-linear differential equation). This might also enable re-sequencing of mathematics instruction since students do not need to have as many prerequisites as previously, because they can 'outsource' some parts of a problem to a program (often the time-consuming but cognitively lower level tasks such as routine calculation) and concentrate on other parts (such as the more cognitively demanding interpretation of results).
- Experiencing the work in a real engineering environment: Engineers working in industry face, on a daily basis, many problems that either cannot be solved using classical analytical methods, or where analytical methods will produce a solution but their implementation will be excessively time-consuming, for example in the control and optimisation of a particular industrial process. Therefore, the use of application programs which have a mathematical basis is ubiquitous throughout the engineering industry. Training students to use such programs in a thoughtful manner is already an essential element of preparing engineering students to work in real engineering environments.
- Change of roles: Using mathematics programs like CAS can help to bring about changes in the way classes are conducted, as their usage requires student active participation and autonomous activity. Such activity can also be designed to require interaction among students. The result is that the process of acquiring and developing mathematical knowledge becomes more student-centred. This also changes the role of teachers, who become tutors and instructors rather than lecturers (see the analysis of the feedback from students and their opinion of the on-line Pilot course in Differential and Integral Calculus in (Norstein et al. 2004), or the evaluation of students' reactions to the project utilising graphing calculators in teaching linear algebra at secondary schools, which can be found in (Verweij 2004)).
- Motivational aspects: Most students are accustomed to using technology such as smart phones in their daily life; consequently simply having technology involved can make a huge difference in students' attitudes and feelings towards mathematics. Therefore increased use of mathematics technology may help to improve student motivation. Nevertheless, the perception of the students of their achievements might be different: (Galan Garcia et al. 2005) observed that most students were not aware of the improvements in their knowledge and skills, and in their assimilation of the contents presented in class.

The following risks have been identified and should be addressed when using mathematics technology:

- Loss of basic capabilities: When adapting the mathematical educational process to make use of new technological tools one must be aware of the risk that this computer-based learning environment may cause an unexpected reduction in students' grasp of the 'traditional

mathematical culture'. This is not just a loss of fluency in carrying out procedural mathematical tasks brought about by a reduced amount of practice (due to using computer programs to carry out these tasks), but can also be a more limited understanding of core mathematical concepts as the reduced practice may bring with it reduced need to think about the basic concepts thereby impacting on overall mathematical reasoning skills.

- Loss of connection between procedures and understanding: Extensive and exclusive usage of CAS can potentially prevent students from making proper connections between the techniques used for calculations and conceptual understanding, for example the Gauss algorithm for solving a linear system of equations also provides insight into the possible solution types; simply using a “solve” command does not give this insight.
- Pure trial and error working style without thinking: There is a danger that students may use mathematics and application programs in a largely thoughtless trial and error mode, making variations without any particular strategy in the hope that somehow they will achieve what is required without having any idea why what they did solved the problem ('mere button pushing'). Problems must be found where such a strategy is not productive so that students are forced to think about the effects of possible variations.
- Tool dependence and 'faith': When students are no longer able to compute even simple examples by hand, they depend totally on what the tool they are using provides. They also have no idea of what to do when a program fails to give them an answer to a problem because they do not know what the program is able to do. Although the students do not need to know in detail what a program does, they should know which model(s) a program is based upon 'in principle' so that they are able to judge when it is, and when it is not, appropriate to use that program.

The risks show that a naïve introduction of mathematics technology might have detrimental effects. It is a challenge to strike the right balance between thoughtful tool usage and training in mathematical procedures using paper and pencil. Adequate tasks have to be found to avoid the potential risks of using mathematics technology.

The use of mathematics technology can be meaningfully connected to the competence approach. In what follows, we state how the mathematical competencies explained in chapter 2 can be affected by the introduction of mathematics technology in the education of engineers.

- Thinking mathematically: This competency should also include the ability to recognize that for some problems there is either a program based on a mathematical algorithm or that it is possible to implement one's own routine to solve the problem within an appropriate technical environment.
- Reasoning mathematically: Technology allows an explorative working style where one makes variations, for example to investigate the influence of parameters. At first sight, trial and error could replace the need for mathematical reasoning, but in a huge design space it is still important to apply mathematical reasoning (for example investigating the influence of symbols within a formula) in order to make variations efficiently. Technology also allows students to perform simulation experiments to find patterns or find counter-examples for assumptions. It is vitally important that students should know the difference between proof and experimental plausibility.
- Posing and solving mathematical problems: Technology allows for an experimental problem solving style using heuristics, using knowledge about the influence of factors and using pre-

written numerical solution procedures. With technology learners can set up their own experimental environment in which to solve problems.

- Modelling mathematically: Technology allows students to work with more complex and realistic models since work within the model is supported by technology and, indeed, even the setup of models can be facilitated by technology (for example, through using simulation packages such as Simulink®). In engineering programs, models are often only partially visible, so students need to learn to work with technology where the underlying model is not known to them. This requires knowledge about strategies for checking one's understanding of the workings of the program and also for checking the results.
- Representing mathematical entities: Technology provides new representations which can be used as a cognitive aid for understanding a mathematical concept (for example a 3D plot which can be rotated and geometric representations of algebraic expressions). In particular, dynamic representations are available which in former times were only possible by constructing mechanical devices. Moreover, the possibility of interactive manipulation of representations enables the exploration of relations between different representations.
- Handling mathematical symbols and formalism: Programs still require mathematical symbols and formalism as input; sometimes this may be with a program-specific syntax or it may be facilitated by pallet-style input. The same is true for the output – this may be mathematically rendered or it may be in program-specific syntax (or even a combination of both) – whatever the format, students need to learn to understand the program output.
- Communicating in, with, and about mathematics: Mathematical programs can be used as means for communication, when the user documents and presents solutions to problems within the program (e.g. CAS or spreadsheet). For this the user has to encode the mathematical ideas, objects and procedures with the expressive means which the program offers. Students also have to decode such documentation when they use annotated worksheets that have been set up by others.
- Making use of aids and tools: Being able to use efficiently and effectively mathematical and mathematics-based application programs is an essential requirement of the engineering workplace. Therefore, students should learn at university about the capabilities and limitations of such programs, and they should be able to check the plausibility of program output in order to use them properly.

The relationship between technology usage and competence acquisition has (at least) two facets: on the one hand using technology can help in the acquisition of competencies, on the other hand knowledgeable technology usage requires special additional aspects of each competency.

The degree of integration of mathematics technology in the mathematical education within a study course can be quite different. The taxonomy developed in the thesis of Oates (2009) can be used to analyse or to specify the degree of integration. Oates uses the six characteristics presented in the table below. The taxonomy gives ideas of where and how integration can take place and it can be used to check whether all the necessary measures have been taken in a study course to achieve the desired degree of integration.

Characteristics		Example of questions asked to examine the degree of integration
A	Access	To what extent do students have access to technology tools, e.g. is it compulsory? Do they use their own, or access it in computer labs?
B	Student Facility	How proficient are students with the use of the technology, and what assistance is provided to help them?
C	Assessment	Is technology expected and/or permitted in assessment?
D	Pedagogy	How and when do the staff and students interact with the technology? For example, is it used mainly as a complex calculation device and demonstration tool, or to develop and explain concepts?
E	Curriculum	Has the course curriculum, for example content, order of teaching, changed to reflect the use of technology?
F	Staff Facility	Are staff familiar with the use and capabilities of the technology, both mathematically and pedagogically?

A Taxonomy for Integrated Technology Characteristic (Oates 2009)

4.4 Integrating the mathematics curriculum into the engineering study course

In the competence-based approach that is advocated in this document, it is quite obvious that the mathematics curriculum should be strongly linked with the application subjects taught in the engineering study course under consideration. The contexts and situations where mathematics plays a role during the study course predominantly appear in those application subjects which are ‘heavy users’ of mathematical concepts and procedures. The linkage has several aspects that have to be taken into account:

- **What to do?** Chapter 3 shows that there is a vast amount of possible content-related competencies such that a suitable subset has to be defined for a study course and this procedure should be driven by the needs of application subjects.
- **When to do it?** This question has at least two facets: On the one hand this is related to an appropriate sequencing within the mathematics modules such that the competencies are available in time for concurrently running application modules (like engineering mechanics or physics); on the other hand the question is concerned with sustainability and repetition when mathematical concepts and procedures are needed in application subjects later on in the study course (**when to do it again?**).
- **Where to do it?** When it comes to refreshing basic mathematical concepts and developing new application-specific, mostly more-advanced mathematical concepts, this often takes place within application subjects. More generally, in a competence-based approach which strives for a broader view of mathematics education, the question comes up how the acquisition of mathematical competence is distributed over mathematics and application modules.

In this section we will briefly discuss the above aspects and give some hints to related work.

For specifying a mathematics curriculum for a concrete engineering study course the mathematical challenges provided by the application subjects in the course have to be identified. Willcox & Bounova (2004) observed that mathematics staff often do not know about the usage of mathematical concepts and procedures in later application classes and staff teaching these application subjects do not know about the contents of former mathematics classes, particularly when they belong to different departments. This very likely leads to a mismatch (regarding contents and notation) between the needs of application subjects and the provision of mathematics classes. As a consequence, they advise a strong communication link between both groups. In particular, the needs of application subjects should be systematically collected and analysed, be it by questionnaires or by investigation of lecture manuscripts, in order to elicit what they call the 'implicit mathematics curriculum'.

If the mathematics modules are based on known application needs, then these might also lead to enhance motivation in mathematics classes by relating mathematical concepts to interesting applications questions or introducing the concepts 'in context. McCartan et al. (2010) report about very positive effects on student motivation and engagement by using contexts and activating learning strategies. Moreover, the relationships can be investigated in mathematical case studies or projects (Mustoe & Croft 1999; Wilkinson & Earnshaw 2000; Alpers 2002; Härterich et al. 2012) which might form an obligatory, or at least credited, part of the module. Therefore, the results of the rather time-consuming process of identifying connections between mathematics and application subjects can also be exploited for such more demanding learning scenarios where many mathematical competencies can be addressed (cf. section 4.1).

It should be mentioned that mathematics has a coherent structure on its own which also has to be taken into account to avoid the impression that mathematics is just a set of unrelated 'chunks' which might be useful in certain models. Therefore, due caution is called for when basing the mathematics education on application needs.

The second important issue regarding the integration aspects is concerned with when certain mathematical concepts and procedures should be learnt by students. Usually, the mathematics classes take place during the first two to four semesters (like Mathematics I-IV). This often results in problems of availability of mathematical concepts in concurrently running application subjects like engineering mechanics or fundamentals of electronics. Rossiter (2008) and Patel & Rossiter (2011) advocate a sequencing of mathematical concepts such that the concepts are available shortly before they are used in engineering modules. As already stated in the discussion of the first aspect, this requires a thorough analysis of the application subjects and close cooperation between those responsible. They observed that the approach helped to foster a positive attitude among the students regarding the usefulness of the mathematics education (cf. the corresponding reasoning in section 4.5 on attitudes). It better enabled students to interconnect mathematical and application issues when similar notation and examples were used in mathematics and engineering classes. There are certainly limits to this approach as already recognised by Patel & Rossiter (2011) since it might lead to unwanted fragmentation of the mathematics curriculum or the mathematical concepts needed in an application subject are simply too advanced for early introduction. Then the mathematical foundation must be provided – at least in a preliminary way – within the application subject, which we deal with below when treating the third aspect.

A different problem comes up when there is a larger time interval between the first learning of a mathematical concept or procedure in a mathematics class and its later use in engineering classes for example on control theory or machine dynamics. Colleagues in application classes often report that the required concepts are no longer available, and sometimes students even claim that they have not encountered them before. Since it seems quite natural that knowledge fades out to a certain degree over time when it is not used, there should be some way to deal with refreshing mathematical concepts and procedures, although the simplest answer still is that the responsibility for refreshing knowledge lies with the students. There are some approaches to facilitate the refreshing. Allaire & Willcox (2004) thoroughly analysed their class on “Principles of Automatic Control” lecture by lecture, identified the mathematical concepts and procedures needed for the different lectures and provided links to basic mathematics courses where the topics had been dealt with. Moreover, they also provided refreshing material for those topics which they identified as particularly problematic (like vector calculus and linearisation). The information on necessary mathematics and remedial material is distributed before the lectures, and during the lectures sometimes short ‘flashbacks’ are inserted. Students made limited use of the material. Alpers (2000) also analysed the mathematics used in a control class for mechanical engineering students and provided interactive refreshing material implemented in Maple®. This gave students the opportunity for tailor-made refreshment of the material needed in the control class. Yet, the use by the students was also very limited since students envisaged the material rather as an ‘add-on’ to the application class than as an essential part of the course. It certainly needs strong involvement of the engineering colleague giving the application class to counter this impression.

There is also a more rigorous approach to ensure that students refresh their former mathematical knowledge as is reported from Ireland. Carr et al. (2012) describe an approach where students in later years of their study course (honours degree) take a mathematical “advanced core skills” test on topics and procedures dealt with in earlier years (differentiation, integration, 1st and 2nd order differential equations and others). They can sit the test several times but they need a very high score (at least 90%) to get any marks for the test (as part of the total marks for the mathematics module in that year). This way, an incentive for continuous repetition is created which is also likely to enhance sustainability of mathematics education as far as core computational and procedural skills are concerned.

The question where the acquisition of mathematical competence should take place seems to have an obvious answer: in mathematics modules of the engineering curriculum. But a more thorough analysis reveals that this is certainly not a sufficient answer for several reasons. First, we have the timing problem already stated above. Even if one tries to sequence the mathematics modules in a way to provide the needed mathematical concepts and procedures ‘just in time’ this is not possible for more advanced concepts when they are needed early. Hennig & Mertsching (2012) describe a situation which is quite usual in German engineering study courses: for bachelor students of electrical engineering and related fields a course on “Fundamentals of Electrical Engineering” is offered in the first semester which needs several mathematical concepts for example from vector calculus which are dealt with later in mathematics education (as part of multivariate calculus). Therefore, Hennig & Mertsching developed a concept for incorporating the teaching of this material in the application course. They integrated ‘short mathematical digressions’ into the application lecture in order to introduce key mathematical concepts needed for application modelling. An essential advantage of this approach is the situated introduction of mathematical concepts which relates them directly to authentic problems for electrical engineers. This provides application

meaning for the concepts and gives students insight into the usefulness of these concepts for their professional work. Since for time reasons the digressions can only be short, the authors provide a web-based learning resource where the concepts and procedures are dealt with more deeply. The authors concentrated on those mathematical topics which according to a questionnaire turned out to be the most difficult ones for students. They recognise that their approach is restricted such that a more comprehensive treatment of the issues in mathematics modules is still required. The latter might be facilitated when students are already able to attach application meaning to the mathematical concepts.

In engineering classes that take place after the mathematical education, i.e. in later years of the study course, the refreshing of necessary mathematical concepts and procedures usually takes place within the application subjects themselves. When mathematical concepts additional to those provided in the mathematical education are needed, they also have to be taught in application classes. This might for example be the case for mathematical procedures in signal processing or in using the finite element method (cf. section 3.4 in this document).

A broader view of where the acquisition of mathematical competence takes place must not be restricted to the treatment of mathematical concepts and procedures but has to include the definition of mathematical competence as stated in chapter 2 of this document: *“the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role”* (Niss 2003a, p.6/7). It is quite clear that – even with the foundation laid in the mathematics modules of the engineering curriculum – the major part of understanding and using mathematics in extra-mathematical contexts takes place in application subjects. To make this more specific, it is appropriate to use the dimensions for specifying progress in competence acquisition which have been described in section 2.1: ‘degree of coverage’, ‘radius of action’, and ‘technical level’. First of all, the radius of action, i.e. the set of contexts and situations where a competency can be activated, should definitely be extended when mathematical concepts and procedures are used in application subjects. Regarding the ‘degree of coverage’ dimension, there are differences with respect to the single competencies stated in section 2.1 which make up the overall mathematical competence. The subsequent statements are meant to present some ideas on possible extensions within this dimension in engineering courses but are definitely not a comprehensive treatment of the issue:

- **Thinking mathematically:** When mathematical concepts are used in an engineering course to answer practical questions (for example find suitable dimensions in the design of a machine element), students see better what kind of questions can be treated mathematically, i.e. how a mathematical approach can help. It is also easier to see the value of abstraction if students recognise the same mathematical concept in different application scenarios. On the other hand, students see also the limits of mathematical approaches (importance of experience to set up an initial design, make decisions with vague conditions for reasoning).
- **Reasoning mathematically:** The main aspects of this competency are probably already covered in mathematics education but they are trained again in model development and in trying to solve problems or to achieve properties in application models.
- **Posing and solving mathematical problems:** It might be the case that students learn new aspects of this competency, e.g. by learning new problem solving strategies for dealing with uncertainty or with large design spaces.

- Modeling mathematically: This competency is definitely the one which is covered to a large extent in application subjects. There, modeling principles are developed and used to set up real models where finding an adequate modelling granularity is a major issue. Students also have to interpret the results of working within the mathematical models from an application perspective and have to validate the models, e.g. by making experiments and taking measurements.
- Representing mathematical entities: The main aspects of this competency are probably already covered in mathematics education but they are covered again in applications where finding adequate representations for conveying messages to a certain audience is an important task.
- Handling mathematical symbols and formalism: There will probably be no new aspects turning up in engineering subjects. The competency will be just used and developed.
- Communicating in, with and about mathematics: The new aspect of this competency in an engineering course will be the (oral and written) understanding of and own presentation of mathematical reasoning and procedures in context whereas in the mathematical part this often takes place in isolation. Students have to explain and justify engineering decisions by making oral or written mathematical statements.
- Making use of aids and tools: Whereas in mathematics education the coverage of this competency is mainly restricted to mathematical tools and aids, in engineering courses adequate usage of application tools is dealt with which are based on mathematical models but display only a restricted visibility of mathematical concepts at the user interface. Therefore, the new aspect here is handling such a tool where the underlying mathematics is not fully understood. This includes e.g. the ability to test the tool usage with small examples that can be computed by hand for control.

With respect to the ‘technical level’ dimension it was already stated above that in some advanced application subjects mathematical concepts and procedures can be introduced which have not been treated before in the proper mathematics education part. Therefore, an extension in this dimension may also occur in the proper engineering part of the programme.

Regarding all aspects described above it is quite obvious that a strong link between those responsible for the mathematical education and those who are in charge of proper engineering courses is decisive for achieving a good integration of mathematics into the engineering curriculum. This is more likely to happen if there is a certain personal continuity in the delivery of math and application education since then lecturers might rather be interested in creating a linkage and having a coherent education. This would be helpful in avoiding the dangers of modularisation which might be of organisational value but can be very detrimental from a didactical point of view if it leads to a compartmentalisation of knowledge.

4.5 Attitudes

If – as proposed in this document – it is the ultimate goal of mathematical education of engineers to make them mathematically competent, and if this competence is defined as “insightful readiness to act in response to *a certain kind of mathematical challenge* of a given situation” (Blomhoj & Jensen 2007, p.47), then such readiness is strongly related to the attitude a student has towards mathematics. In their studies on attitudes of engineering students to mathematics in a few British universities, Shaw & Shaw (1997, 1999) found out that only about one third of the students were motivated, about 75% had the desire to improve their mathematical abilities and a broad range from

20% to 66% perceived mathematics as being difficult. According to the authors, the attitude of students towards mathematics is more positive when the environment provided by universities is perceived as being supportive. This can be achieved by organising support in additional tutorials, foundation classes, online materials or mathematical support centres, for example (cf. section 4.2 of this document). This can at least prevent students with difficulties in mathematics to turn into mathematics “haters” (Shaw & Shaw 1999).

Booth (2004, p. 18) investigated the different perceptions of mathematics with engineering students in more detail and distinguished between three views of mathematics:

- Mathematics is “a part of the degree programme”.
- Mathematics is a “basis for other subjects”.
- Mathematics is a “tool for analysing problems that occur in the world ...”.

In the first view, mathematics is just an ‘isolated subject’, whereas in the second it is ‘integrated into the programme of study’ and in the third view also ‘into the world it describes’. These different kinds of perception have a considerable influence on the students’ view of their own responsibility for the learning process and their approach to mathematical learning. Booth distinguishes between a ‘surface approach’ where students focus on the ‘sign’, on the demands of the course and on the reproduction of course material, and a ‘deep approach’ where students focus on meaning, construct relations between mathematics and engineering subjects and also to their wider experience. If students see mathematics (or a certain part of mathematics since this can differ from topic to topic) as an isolated subject they are likely to apply a surface approach to learning whereas a view relating mathematics to other subjects and the world will rather lead to a deep approach. Therefore, inducing in students a realistic perspective of the role of mathematics in the study programme as well as in later engineering life is important for achieving a deep learning approach.

In her investigation of the “Mathematical Disposition of Structural Engineers”, Gainsburg (2007) proposes that mathematical education should strive for a similar “mathematical disposition” as she found with structural engineers and which she termed “skeptical reverence”: “mathematics is a powerful and necessary tool that must be used judiciously and skeptically” (p. 498). One could denote such an attitude also as critical appreciation: mathematics can be of help in many engineering situations but it is not the only constituent of engineering work since there are many other aspects to be taken into account which are different from those that can be stated and treated in a mathematical way.

In general mathematics education, the topic of attitude is discussed under the heading of “beliefs and affects” (Schoenfeld 1992, Cardella 2008). Here, it is also emphasised that the beliefs about mathematics are largely shaped by the experience in school education and that – on the other hand – these beliefs shape the mathematical behaviour shown by students. Therefore, it is quite important to create experiences where mathematical thinking is seen as a process which helps in capturing and solving real problems and not just a five-minute activity to work on an isolated exercise. Only by having made such experience will the students in their later engineering life be willing to use mathematical thinking for solving their engineering problems.

How can mathematical education encourage and strengthen a perception of mathematics which is integrated in the engineering world (be it educational or a real work environment) and where mathematics is critically appreciated as relevant part of the problem solving process? The

competence approach already intends to make students see what mathematics can do for them (mathematical thinking) and it emphasizes the 'action character' and the contextualisation since students should be enabled to master the mathematical challenges they meet in engineering contexts. Therefore, the competence approach seems to be particularly suitable for creating and supporting a desirable attitude towards mathematics. There will still be large differences regarding mathematical abilities but having a good understanding of what mathematics can do in engineering contexts and a realistic perception of own abilities (What can I do myself, where do I need an expert?) should lead to a realistic und helpful attitude for a professional engineer.

5 Assessment

The constructive alignment principle, as presented by John Biggs and Catherine Tang in the book “Teaching for Quality Learning at University” in 1999, has had an increasing impact on the teaching – learning – assessing cycle at many universities. There is now a fourth edition of the book available (Biggs & Tang, 2011). Furthermore many short articles on the subject can be found on the internet. One guideline in constructive alignment (there is of course a lot more to it) is that the planning of a course or a module must give answers to three questions:

- What will the students learn?
- What will the students do to learn?
- How can the students’ knowledge be evaluated?

In this chapter we discuss the third question, that of assessment. Assessing and grading are extremely important parts of the teacher’s work. The grade achieved by a student, in relation to what other students have achieved, can determine his/her future, the first job, a PhD education for instance. The students know this and find it in general extremely annoying – it may even have a strong negative impact on the interest in the subject – if the assessment is considered unfair or if it seems to be safe to cheat to get a better grade.

We begin with an overview of the different forms of assessment that are in use around Europe and were identified in a SEFI MWG project, the assessment project, reported at the SEFI MWG seminar in Vienna 2004 (Lawson 2004a).

5.1 Forms of assessment

The most common assessment method is a written examination, with closed books, at the end of the course. Less common is a written examination with open books or computer facilities to support the problem solving. One argument for allowing computers and/or advanced calculators is that the assessment situation should be as realistic as possible, i.e. it should mirror the engineers’ future work. This argument applies in particular at a later stage of the education. However, modern advanced electronic equipment can communicate wirelessly over long distances. Therefore, we cannot claim legal certainty in these assessment situations unless other assessment methods, such as oral presentations, are added. The work done with support of computers etc. is then more to be seen as a *part* of the learning process. Still it can be reasonable that this work is done under some time pressure as that indeed is a part of the future work situation.

We have to take into account that modern advanced calculators are more or less equivalent to open source assessment. To best support our students, legal certainty should never be neglected.

Another difference worth mentioning is the duration of the written exam. In some countries four or five hours are standard, in others only one or two. It is in general not possible to evaluate every aspect of a student’s knowledge, all we can do is to spot-check. But the shorter the duration of a written exam the less of the contents can be covered and the more it is open to gambling strategies. The shorter duration can be compensated by additional assessed activities, for instance, as in many central European institutions, by a follow-up oral examination, either for all students or for those that scored well enough to get a higher grade. Teachers comment on oral examination that it is highly staff intensive but gives the best opportunity to test in-depth understanding.

Take-away assignments are used at several institutions, but always as one amongst a number of methods of assessment and never as the only or primary method. They give students an opportunity to explore more realistic problems than they can in an ordinary written examination and for this reason often require the use of computer software to complete the assessment task. Some teachers have reservations about this method of assessment because it is impossible to be certain that the student submitting the work actually did it for him/her-self. When the take-away assignment is followed up with an oral presentation of the work the legal certainty is stronger.

Only a few institutions use multiple-choice tests and those that do use them do so only occasionally. Such tests can be cheap to administer as they can be computer delivered and marked. They can be useful in giving formative feedback during the course. There are reasons to believe that the use of this kind of computer-supported assessment is increasing. We will discuss this in detail later in this chapter. Again it is impossible to be certain that the student submitting the work actually did it for him/her-self, unless the test is implemented under invigilation and on computers that are not connected to any net. Furthermore, as all that is marked is the student's final answer, they have limitations when being used for summative assessment.

Other methods of assessment such as project work, group work and oral presentations are not widely used. However, when it comes to examination of mathematical competencies, these methods can be more interesting. It is difficult to give individual grading of group work, but individual time-logs, progress-logs and contribution reports together with the project report can support the grading. We will return to this matter later in the chapter.

5.2 Requirements for passing

In the previous section we recalled the findings of the SEFI MWG assessment survey; the major part of the assessment is based on a traditional final written exam with closed books. Also the construction of these exams is similar across Europe, possibly around the world. Most written exams consist of a number of problems more or less similar to the problems in the textbooks, each given a certain maximum score and together covering a major part of the intended learning outcome. When marking an exam the examiner gives the student a score for each problem depending on how successful the student's attempt to solve the problem turned out to be. The examiner then decides whether the student has failed or passed and whether to award a better grade than just 'passed'. Traditionally this decision is entirely depending on the student's total score. The limit between fail and pass is very often set to a percentage of the maximum score. This percentage varies between 40 and 60. The grading systems vary from country to country, sometimes between universities in the same country. When the ETCS grading system is adopted across Europe it will be of interest to investigate the equality of the grading. But even then the differences between the course modules at different universities will make the comparison very problematic. The aim of this section is to discuss the requirements for passing related to the expected learning outcome.

Expected learning outcomes specify depth and what students should be able to do at the end of the module. This specification consists of a series of statements of the type: 'On successful completion of this module students will be able to' followed by a verb like *calculate*, *solve*, *explain* or *prove*. Not only content-related competencies, knowledge and skills should be included here. If we expect the students to achieve a certain level of one of the eight competencies, then we have to state that in a way that can be understood by the students. If we cannot communicate to the students what we expect them to learn, we cannot demand or expect that they learn it.

To improve quality in teaching and learning, if we adapt to the constructive alignment principle, we have to state the expected learning outcome in a way that supplies the students with a proper guidance for their learning. We tell the students: “This is what we want you to learn to do and when you can do it, you will pass”. But the alignment should not be within the courses only, but also between courses that together form a programme. The expected learning outcome statement also informs our engineering colleagues of what they can expect the students to know; “a student who passed this course module is, or has been, able to do what is stated here”. But an improved quality in the education is not achieved automatically just by applying constructive alignment thoughts. It is of course heavily dependent on what the students actually learn. Thus, there is a strong argument for aligning the expected learning outcome and the requirements for passing in a way that leads to equivalence between ‘passing the exam’ and ‘being able to do all that is stated’. Stating the expected learning outcome and aligning this to the course work and to the requirements for passing is a delicate task. We have to decide what all students must be able to do after the course module, design the course work so it leads to this ability and also assess the skills and knowledge in a way that distinguishes between those that can and those that cannot. But still we shall inspire and help all students to go deeper into the stuff and also assess this deeper understanding.

The prevailing principle mentioned above: “A student who is given a certain percentage of all possible points on a written final exam will pass”, should not be applied if we want to assure quality in the education. In *Aligning teaching and assessment to curriculum objectives*, John Biggs states: “The logic of awarding a pass to a student on a section of a course in which that student has already failed is difficult to grasp” REF. To pass, the student should instead have demonstrated an acceptable level of skill, knowledge or competence for every part of the expected learning outcome. We have to rethink the assessment methods in order to ensure that the students’ achievement in every part of the expected learning outcome is assessed.

In the rest of this section we will deal with content-related competencies, knowledge and skills which can be assessed by traditional final written exams, or technology supported assessments like multiple choice tests, or a combination of these. We assume that a pass-level is set for every part of the course/module and discuss how the assessment can be designed.

The most basic skills and knowledge can preferably be assessed by computer-supported tests. Using a platform similar to MapleTA[®] the test can consist of a number of problems/questions picked at random from a large question-bank. The students can do the test several times and it is then reasonable to demand that they can answer all questions correctly. As these tests are an essential part of the requirements for passing they should be inviolated, for the sake of legal certainty. This part of the expected learning outcome can also be assessed by written tests during the course. In this case the tests cannot be done repeatedly and minor ‘numerical’ errors may be accepted.

The final written exam then can focus on the not so basic skills and knowledge and a deeper understanding of the subject. The exam can be split into two parts, one that only covers methods and procedures which can be rather complicated but standardised and theoretical questions which require a limited understanding and a second part covering problem solving and a higher level of understanding. To pass the student must score well on every item of the first part. The second part is used only for grades above ‘passed’. Or the exam can consist of a number of items where the student can show both low and high level of understanding or capability to apply either standard techniques

or genuine problem solving in the same field. The criterion for pass is then to score reasonably well on all items.

When other competencies are added to the expected learning outcome the assessment must consist of several different parts. In the next section we will discuss how to assess some other competencies.

5.3 Assessing competencies

The competencies provide a framework for our discussions and thoughts about what we expect our students to be able to do with the mathematics they have learnt, not directly related to a specific field of mathematics. Most competencies are developed when the student study different subjects, not only mathematics. For instance the student's competence in mathematical modelling can be improved in any subject where mathematical models are in use. Therefore the competencies are also to be seen as expected learning outcomes of the programme - the entire education. It would be a benefit for the education if at least some mathematical competencies were included among the general competencies that are, or ought to be, included in the description of the expected learning outcomes of the engineering programme. We could then discuss with our engineering colleagues how each competence is best developed, what the student must do to obtain the competence and how we shall assess it. Some of the competencies are best developed in project work similar to bachelor or master thesis projects. Others are mainly developed in studies of mathematics. Thus, in the very near future we must broaden this discussion and include engineering colleagues and programme managers. There is much to be done in this field.

When the expected learning outcomes are stated in terms of both knowledge and skills and competencies we have to rethink the assessment. For most of the competencies the traditional end-of-course assessment is not enough. For some, it is. The competence *handling mathematical symbols and formalism* is, to some extent, assessed in any written exam. *Reasoning mathematically* can be assessed by theory questions in the exam; true-false questions in particular are suited for this: 'Conclude whether the following statement is true or false and prove your conclusion'.

Other competencies can be assessed while they are learned and practised. Using Biggs' words: "The learner shall in a sense be 'trapped', and find it difficult to escape without learning what is intended should be learned" (Biggs 2003, p. 2). The assessment then can consist of an observation of the learning process and a judgment of the final result of that process. We do not have to arrange special assessment sessions.

The competencies *posing and solving mathematical problems* or *modelling mathematically* together with all the others, in particular Communicating in, with, and about mathematics and Making use of aids and tools can be practised and assessed by asking individual students or groups of 2 – 4 students to solve genuine mathematical problems or implement mathematical models and then present their solution orally and/or in a written report to a teacher-student audience. The problem solving or modelling can include numerical calculations or experiments using software, the presentation can include graphical representations of the result. All this can be done as a minor part of a single course module or as a larger project. When the project is a joint work by two or more students it is important that we observe the process as well. We will have to decide whether all students in a group will fail or pass whether they get the same grade. This decision can be grounded on individual time-logs, showing the individual effort, progress-logs, showing the progress of the group, and a contribution report, showing how the individuals have contributed to the joint work. In the case where the students are only supposed to work with a given mathematical model and improve their

ability to use software in the implementation of the model, it can be sufficient to observe that it works and at the end check the implementation.

The communication competence can be practised and assessed in class, working with the ordinary exercises from the text book. One such method is often called ‘ticking’. The teacher selects a number of exercises or theoretical questions such as true-false statements. Every student solves all or some of the problems, perhaps together with other students. They then prepare to present their solutions to the other students in class, using for instance the black board. They tick-mark which exercises or questions they are prepared to talk about and the teacher decides who will present what. This activity is good not only for communication skills; it also activates the learners, which many students comment on in course evaluations. Ticking can be a mandatory part of the assessment or an optional part for instance giving bonus points to the final exam, mainly depending on how the expected learning outcomes have been stated. The student’s performance and the quality of the explanations can be graded, but at an early stage of the education it may be enough to reward the willingness to try to explain.

In a ‘ticking’ activity also the competence in mathematical reasoning can be developed and tested, in particular when true-false statements are used. In a written exam students sometimes feel cheated when they give an incorrect answer to such a question. The statement reminds them of a true statement but some word or minor part is altered to make it false. It takes quite a good understanding of both the concepts and the logic to give a correct answer and to prove it: perhaps more than we can expect from an average student. Thus, we have to be careful when selecting statements for written exams. In a ticking activity it is not that crucial, as an incorrect answer from one student can benefit the entire group by the discussion that the mistake may lead to.

In this section we have just presented a rough overview of important aspects and some ideas related to the assessment of competencies. For more information we refer the reader to Højgaard 2009.

5.4 Technology-supported assessment

In this section we will discuss how technology may support formative assessment during the course and summative assessment after the course (or course module).

In its most primitive setting a test suitable for a computer-supported assessment system consists of a number of multiple-choice questions comprising a question together with one correct answer and a number of incorrect answers (distractors). The distractors must be close to the correct answer. The student has to select the correct answer to all or most questions in order to pass the test or to get a positive feedback. The student’s work is not simplified or improved; he/she could do the same with paper and pen, so long as the questions are similar to those in the textbook. The advantage for the teacher is that once the system is there and a suitable set of exercises or questions are imported to the system, the system will do the work. The advantage for the student is that he/she can often do the test anywhere and at any time. If the test is created by a randomised selection of questions out of a large question bank the student can do the test many times. He/she will get immediate feedback and the teacher will get immediate information about the student’s progress. The need for multiple-choice question decreases if the test system is supported by a computer algebra system since such programs are able to check mathematical input. But yet there can be specific demands on how the answer is given or formulated (for example the use of certain variable names). A correct answer in a ‘wrong’ format is considered to be an incorrect answer by the program, confusing the student, of course. One challenge for the teacher is to find questions that assess a deeper understanding of the

subject and still have only one correct answer. Another challenge is to rethink the assessment and find questions that could not be asked when only paper and pen were available.

The feedback to the student in a simple system consists only of a mark of 'correct' or 'incorrect'; the student has to find out what to do to improve. A more advanced system includes also learning support for the students. There are many reasons, and also attempts, still visionary, to strive towards complete, computer-supported systems, the so-called Intelligent Tutoring Systems (ITS), where the student gets information not only about his/her errors or mistakes but also about underlying misconceptions or lack of knowledge, together with support to fill the gaps. Currently, the student needs help from a human teacher to figure out the nature of the misconceptions and what to do to improve. Some of this can be e-support linked to the test; some may be personal given by the teacher or a support centre on request from the student. The nature of the support can also vary from 'read this example' or 'view this explanation' to 'read again chapter X in your textbook'.

There are specific problems with the legal certainty when computers are used in summative assessment or when students get some kind of credit (bonus points) for the performance in a formative assessment. In general, the computers at universities are connected to a network and to the internet. To prevent cheating the network connections must be closed, perhaps some other programs must be blocked and the students' work must be invigilated. If the number of students is greater than the number of available computers, the need for randomised tests is obvious. The tests cannot be too similar otherwise the last students can have some help from the first, since all tests must be of the same difficulty. If the test is done out of campus or out of office hours then the examiner does not know who actually took the test. For these reasons at most a minor part of the entire assessment should be computer-supported and not invigilated.

In the paper '*A review of computer-assisted assessment*', by Gráinne Conole and Bill Warburton (2005), the authors give a survey of both the use of technology for assessment and of the research on this use. There is an obvious need for a thorough survey of today's use of technology for assessment in mathematics and a deep discussion concerning the consequences of that use.

6 Conclusions and future developments

This document adapts the competence concept to the mathematical education of engineers and explains and illustrates it by giving examples. It also provides information for specifying the extent to which a competency should be acquired. It **does not** prescribe a particular level of progress for competence acquisition in engineering education. There are many different engineering branches and many different job profiles with various needs for mathematical competencies such that it does not make sense to specify a fixed profile. The competence framework serves as an analytical framework for thinking about the current state in one's own institution and also as a design framework for specifying the intended profile. A sketch of an example profile for a practice-oriented study course in mechanical engineering is given in the document. The document retained the slightly changed list of content-related learning outcomes that formed the 'kernel' of the previous curriculum document. These are still important since lecturers teaching application subjects want to be sure that students have at least an 'initial familiarity' with certain mathematical concepts and procedures they need in their application modeling. In order to provide sense-making beyond the purely mathematical structure, overarching themes like 'measuring' or 'functional dependency' were identified as was also done in the OECD PISA document.

In order to offer helpful orientation for designing teaching processes, teaching and learning environments are outlined which help students to obtain the competencies to an adequate degree. It is clear that such competencies cannot be obtained by just listening to lectures, so adequate forms of active involvement of students need to be installed. Topics such as the use of technology and integration of mathematics and engineering education are also discussed. Since assessment procedures determine to a good extent the behaviour of students and are hence important for really achieving progress in competencies, different forms of assessment which are adequate for capturing certain kinds of achievements are discussed.

The main purpose of this document is to provide orientation for those who set up concrete mathematics curricula for their specific engineering programme, and for lecturers who think about learning and assessment arrangements for achieving the intended level of competence acquisition. We envisage future work based on this curriculum framework document to include the following issues:

- Specification of different mathematics curricula for different kinds of engineering study courses. An example for such a specification for a practice-oriented study course in mechanical engineering can be found in Alpers 2013.
- Investigation of assessment of competencies: Only if there are satisfactory ways of assessing competencies, will they become an integral part of curriculum design.
- Investigation of the competence acquisition in different learning arrangements: In particular, changes in technology are enabling new forms of learning and teaching to take place and change will continue into the future.
- Specification of further example tasks for competency acquisition that serve to improve the understanding of the competency concept by providing 'best-practice' examples.
- Studies on workplace mathematics in order to obtain more information on the mathematical challenges engineers meet at their later workplaces and not just in the application subjects.

Future seminars of the Mathematics Working Group will explore these aspects.

7 Glossary

In this section we summarise the definition of those terms that are essential for this document. We also state in which section or chapter the reader can find more information.

Competence cluster: *This term is used in the OECD PISA Assessment Framework (OECD 2009) in order to specify the level of progress in mathematical competence. Three levels are distinguished: the ‘reproduction cluster’, the ‘connections cluster’, and the ‘reflection cluster’. (section 2.1)*

Competence profile: *A well-specified level of mathematical competence that makes up a mathematical curriculum for a certain study course. It contains the specification of the desired progress in mathematical competence in the dimensions ‘degree of coverage’, ‘radius of action’, and ‘technical level’. (section 2.3)*

Connections cluster: *This is a level of progress in gaining mathematical competence. It consists of abilities where students have to connect knowledge acquired before or they have to apply it to situations and contexts which are at least slightly different from those where they first used it. (section 2.1)*

Core Zero: *This is a part of the content-related learning outcomes specified in chapter 3. Core Zero comprises learning outcomes regarding essential material that no engineering student can afford to be deficient in these topics. (section 3.1)*

Core Level 1: *This is a part of the content-related learning outcomes specified in chapter 3. Core level 1 comprises the knowledge and skills which are necessary in order to underpin the general Engineering Science that is assumed to be essential for most engineering graduates. Items of basic knowledge will be linked together and simple illustrative examples will be used. (section 3.2)*

Degree of coverage: *This is one of the dimensions in which progress in mathematical competence is measured. It is “the extent to which the person masters the characteristic aspects” of a competency (Niss 2003a, p. 10) (section 2.1)*

Mathematical Competence: *“The ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role” (Niss 2003a, p.6/7). (chapter 2)*

Mathematical Competency: *Mathematical competence is split up into eight distinguishable but overlapping mathematical competencies which are thinking mathematically, reasoning mathematically, posing and solving mathematical problems, modeling mathematically, representing mathematical entities, handling mathematical symbols and formalism, communicating in, with and about mathematics, making use of aids and tools. (section 2.1)*

Level 2: *This is a part of the content-related learning outcomes specified in chapter 3. Level 2 comprises specialist or advanced knowledge and skills which are considered essential for individual engineering disciplines. Synoptic elements will link together items of knowledge and the use of simple illustrative examples from real-life engineering. (section 3.3)*

Level 3: *This is a part of the content-related learning outcomes specified in chapter 3. Level 3 comprises highly specialist knowledge and skills which are associated with advanced levels of study and incorporates synoptic mathematical theory and its integration with real-life engineering*

examples. Students would progress from the core in mathematics by studying more subject-specific compulsory modules (electives). These would normally build upon the core modules and be expected to correspond to the outcomes associated with level 2 material. Such electives may build additionally on level 1, requiring knowledge of more advanced skills, and may link level 1 skills or introduce additional more engineering-specific related topics. (section 3.4)

Overarching theme: *In order to foster ‘overarching’ sense making, in the OECD PISA and other documents the content-related competencies have not been organized according to the traditional areas of mathematics but rather along some general themes, called “overarching ideas” in (OECD 2009). Such themes are for example “quantity”, “functional dependency” or “data and chance”. (chapter 3)*

Radius of action: *This is one of the dimensions in which progress in mathematical competence is measured. It comprises the “contexts and situations in which a person can activate” a competency. (Niss 2003a, p. 10) (section 2.1)*

Reflection cluster: *This is a level of progress in gaining mathematical competence. It is concerned with abilities where students have to apply mathematics in new contexts and situations, so to reflect upon which mathematical concepts to use and to combine, how to formulate a mathematical problem and how to combine existing or new concepts to solve them. (section 2.1)*

Reproduction cluster: *This is a level of progress in gaining mathematical competence. It comprises the ability to work on tasks where students are required to recall or reproduce facts, procedures, manipulations, tool usage patterns learned and practiced before in familiar contexts and situations. (section 2.1)*

Technical level: *This is one of the dimensions in which progress in mathematical competence is measured. It “indicates how conceptually and technically advanced the entities and tools are with which the person can activate the competence”. (Niss 2003a, p. 10) (section 2.1)*

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10 Appendix

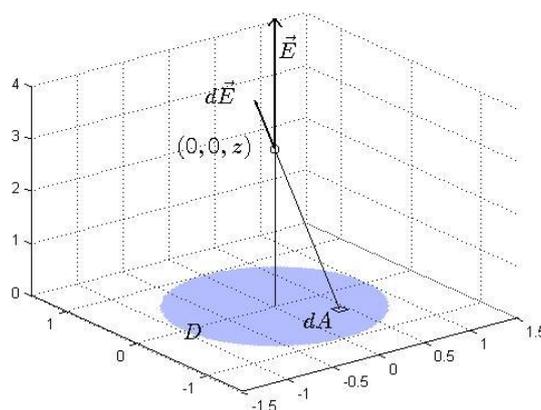
Section 2.2 of this document contains an example task from mechanical engineering where the competencies are necessary for successful work. In this appendix we provide additional examples which come from a purely mathematical environment, from the area of electrical engineering and from civil engineering such that we cover the other main areas of application. We consider such examples as very important for understanding the competence concept and for getting ideas to implement it in one's own teaching.

1. On a slide there are the graph of a function and several candidates for the graph of the derivative. Discuss with your neighbor which one is the correct one and give your vote in a voting system.

Here, students have to reason about the properties of the function and corresponding properties of the derivative and its graph (*reasoning*) in order to find the correct candidate. They also have to communicate their line of argumentation to their neighbour (*communicating*). Moreover, for solving the problem they can think about strategies like "look for simple properties which should be there but which are not (exclusion principle) in order to remove candidates from the list" (*problem solving*).

2. A thin circular disc has an evenly distributed charge. Find the electrostatic field at an arbitrary point above the centre of the disc.

To solve this problem, the student first must understand that mathematics can do the job. First the real-world problem should be transformed into a mathematical one. The real object, a thin disc, is represented by a mathematical object, the set D of points subject to the conditions: $x^2 + y^2 \leq R^2, z = 0$, where R is the unspecified radius of the disc. The, also unspecified, charge Q is evenly distributed which implies that the surface charge density is constant $\sigma = \frac{Q}{\pi R^2} \left[\frac{C}{m^2} \right]$ (*thinking mathematically*).



Then some mathematical modelling should take place; Coulomb's law $F = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r^2}$ implies that the electrostatic field at a point P, caused by the charge in a small area, dA , is $\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\sigma dA}{d^2} \vec{e}$, where \vec{e} is a unit vector pointing from the small surface area towards the point and d is the distance

between the small area and the given point. The superposition principle in physics implies that the electrostatic field is then given by an integral $\vec{E} = \iint_D \frac{1}{4\pi\epsilon_0} \cdot \frac{\sigma dA}{d^2} \vec{e}$.

By rotational symmetry the field at a point on the z-axis is directed along the axis. Thus we only have to take the vertical component of \vec{e} into account and the magnitude of the field is

$|\vec{E}| = \frac{\sigma}{4\pi\epsilon_0} \cdot \iint_D \frac{z}{\sqrt{z^2+r^2}} \cdot \frac{dA}{z^2+r^2}$, where r is the distance from the point in the disc to the origin (*reasoning mathematically, posing and solving mathematical problems*).

Polar coordinates and a straightforward calculation then give the answer

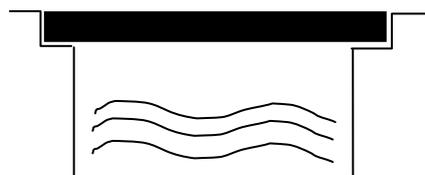
$|\vec{E}| = \frac{Q}{2\pi R^2 \epsilon_0} \cdot \left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$ (*handling mathematical symbols and formalism*).

The problem can be altered in order to add more competencies. Moving the point P away from the axis or altering the distribution of the charge may lead to integrals which cannot easily be calculated by hand (*making use of aids and tools*).

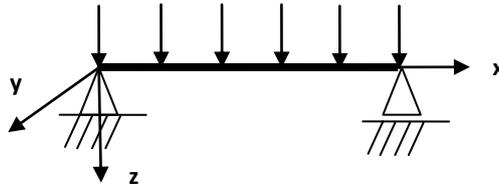
The problem presented here can be found in any textbook in electrostatics, but altered in a suitable way as suggested above or by altering the charged surface to some surface in space (for instance a spherical shell or a torus), the problem turns into a project which can be reported in a student group. The report may well include a presentation using graphics (*communicating in, with, and about mathematics, representing mathematical entities*).

3. A water channel is to be covered by wooden beams such that pedestrians can cross it. What kind of beams should be used?

This is quite an open task, therefore a sketch of a cross section like the one below is helpful to clarify the question. The requirement stated in the task is that the set of beams has to carry people, so they have to be sufficiently dimensioned for this purpose. In order to achieve this there should be a mathematical model to link the loads with the dimensions of the beams (*mathematical thinking*), so it is not necessary to make several trials in order to answer the question.



To set up or choose a model (*mathematical modelling*) one has to identify the important quantities where already a simplification takes place. Beams are modelled as cuboids having length l , breadth b and height h . The span of the channel be L . The stiffness of beams can be described by the Young's modulus E and by the maximum bearable stress σ_{\max_b} . The most uncertain quantity is the load. Here, one has to make some assumptions on the number and weight of the people which are at the same time on one beam (or on the average weight and space for one person), and one has again to make some simplification, i.e. assume an equal distribution of load over the surface of a beam. This gives a value for the surface load p (weight over surface, units N/m^2). Now, one should set up or look for an already existing model for the stress caused by the load (*mathematical modelling*). A standard model that is available from engineering statics is the idealised model of a line-loaded beam which is depicted below.



In the model the usual assumption is made that one bearing is fixed (such that the beam cannot move) whereas the other one is not fixed in the longitudinal direction such that it can extend (for example because of changing temperature). Note that in this model $l = L$ since the overlaying part of a beam is of no relevance to the stress determination. The problem-solving strategy now is to compute how the maximum occurring stress depends on the model quantities and then to decide how to choose them in order to get below the maximum bearable stress (*mathematical problem solving*).

Since from the assumptions one gets a surface load but the model works with a line load q (force over length, units N/m) a transformation is necessary: $q = p \cdot b$. The next step consists of determining the bending moment depending on the place on the x -axis. This can be computed from the transversal force function by integration, or it can be found in standard text books or formularies on statics: $M_y(x) = \frac{q}{2} \cdot (xL - x^2)$. This has its maximum at $\frac{L}{2}$, since it is a quadratic function with zeros at 0 and L (*mathematical reasoning*). The maximum is $M_{max} = \frac{qL^2}{8}$ (*handling symbols*). As can be found in any engineering mechanics book (and is developed in an engineering statics class), the bending stress depends on the bending moment and varies linearly in the z -direction (in the so-called neutral fibre $z = 0$ the stress is 0):

$$\sigma(z) = \frac{M_y(x)}{I_y} \cdot z, \text{ where } -\frac{h}{2} \leq z \leq \frac{h}{2}.$$

Here, I_y is the moment of inertia for the rectangle (cross section of the beam) with breadth b and height h which can also be found in any formulary on mechanics $I_y = \frac{bh^3}{12}$. By evaluating the stress at $h/2$ and inserting the maximum bending moment and the moment of inertia one obtains for the maximum occurring stress

$$\sigma_{max} = \frac{\frac{qL^2}{8}}{\frac{bh^3}{12}} \cdot \frac{h}{2} = \frac{3qL^2}{4bh^2}.$$

With $q = p \cdot b$ this becomes $\sigma_{max} = \frac{3pL^2}{4h^2}$ (*handling symbols*).

As a result one observes that one has to make the height h of a beam large enough such that the maximum occurring stress is below the maximum bearable stress (or choose a kind of beam with higher maximum bearable stress). An engineer might also use an application programme for computing the stress. This would also be valuable when the load assumptions are varied in order to see how this influences the result. Such a programme also needs correct input which is particularly important regarding the load input (line load, surface load). It is important to be able to do a small control computation like the above in order to check whether the programme usage is correct (*making use of aids and tools*). Usually, an engineer has to write up the reasoning stated above for justifying her/his choice of dimensions (*communication competence*).