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Quasilinear PDEs and forward-backward stochastic differential equations

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Mathematical Sciences



Quasilinear PDEs and Forward-Backward Stochastic Differential Equations

Xince Wang

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of
Doctor of Philosophy of Loughborough University

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Abstract

In this thesis, first we study the unique classical solution of quasi-linear second order parabolic partial differential equations (PDEs). For this, we study the existence and uniqueness of the $L^2_\rho(\mathbb{R}^d; \mathbb{R}^d) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^k) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$ valued solution of forward backward stochastic differential equations (FBSDEs) with finite horizon, the regularity property of the solution of FBSDEs and the connection between the solution of FBSDEs and the solution of quasi-linear parabolic PDEs. Then we establish their connection in the Sobolev weak sense, in order to give the weak solution of the quasi-linear parabolic PDEs. Finally, we study the unique weak solution of quasi-linear second order elliptic PDEs through the stationary solution of the FBSDEs with infinite horizon.

Key words: forward backward stochastic differential equations, weak solutions, partial differential equations, stationary solutions, parabolic, elliptic, infinite horizon.

Contents

| | Page |
|--|-----------|
| Introduction | 3 |
| 1 Forward-Backward Stochastic Differential Equations (FBSDEs) with Finite Horizon | 4 |
| 1.1 Introduction and Preliminaries | 4 |
| 1.2 Local Existence and Uniqueness Result of FBSDEs | 8 |
| 1.3 Global Existence and Uniqueness Result of FBSDEs | 17 |
| 2 Quasi-linear Parabolic PDEs and Finite Horizon FBSDEs: Regular regime | 33 |
| 2.1 Introduction and Preliminaries | 33 |
| 2.2 Regularity of the Solution of The FBSDEs | 36 |
| 2.2.1 Dependence upon the Initial Conditions | 36 |
| 2.2.2 Continuity Version of $Y_s^{t,x}$ and its Derivatives | 50 |
| 2.2.3 Expression of $Z_s^{t,x}$ and Flow Property | 55 |
| 2.3 Main Results | 59 |
| 2.3.1 Probabilistic Interpretation for Quasilinear Parabolic PDEs | 59 |
| 2.3.2 A Unique Classical Solution of Quasilinear Parabolic PDEs | 60 |
| 3 Weak Solutions for Quasi-linear Parabolic PDEs in the Sobolev Space | 62 |
| 3.1 Introduction and Preliminaries | 62 |
| 3.2 Equivalence of Norm Result and Flow Properties | 65 |
| 3.3 Weak Solutions of Quasi-linear Parabolic PDEs in the Sobolev space | 70 |
| 4 Infinite Horizon FBSDEs and Quasi-linear Elliptic PDEs | 80 |
| 4.1 Introduction and Preliminaries | 80 |
| 4.2 Infinite Horizon FBSDEs | 81 |
| 4.3 Stationary Solution of FBSDEs and Quasi-linear Elliptic PDEs | 87 |
| Bibliography | 99 |

Introduction

In this thesis, we study a system of quasi-linear second order parabolic (or elliptic) partial differential equations (PDEs). First we consider the following parabolic type PDEs:

$$\begin{cases} \partial_t u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \sigma^*(t, x, u(t, x))) \nabla u(t, x) = 0, \\ u(T, x) = h(x), \end{cases} \quad (0.0.1)$$

where $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$, and \mathcal{L} is an infinitesimal operator defined by

$$\mathcal{L}u(t, x) = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij}(t, x, u(t, x)) \frac{\partial^2 u(t, x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t, x, u(t, x)) \frac{\partial u(t, x)}{\partial x_i}.$$

One aim in this work is to find a unique classical solution of above PDEs through a forward-backward stochastic differential equations (FBSDEs):

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \quad 0 \leq t \leq T. \end{cases} \quad (0.0.2)$$

The FBSDEs were first studied by Antonelli [1], who obtained an existence and uniqueness result over a small time duration. For an arbitrary time duration, the existence and uniqueness result was given by Ma, Protter and Yong [19], Hu and Peng [12], Peng and Wu [31], Yong [36, 37], Pardoux and Tang [28], Delarue [9].

It is classical that a solution of a linear second order parabolic (or elliptic) PDEs can be formulated as a functional of the solutions of some stochastic differential equations (see [11]). By introducing a kind of backward stochastic differential equations (BSDEs), Peng [29] obtained a probabilistic interpretation for a semi-linear second order parabolic (or elliptic) PDEs, which generalized the linear Feynman-Kac formula to the semi-linear case (see Peng [29], Pardoux and Peng [26], Barles, Buckdahn and Pardoux [6] for further extensions). From this point of view, Pardoux and Tang [28] connected a kind of FBSDEs

with quasi-linear second order parabolic PDEs in order to provide a probabilistic formula for the viscosity solutions of the quasi-linear parabolic PDEs. Recently, Wu and Yu [35] extended this result in the case where the diffusion function σ depends on Z .

The so-called probabilistic interpretation:

$$u(t, x) = Y_t^{t,x}$$

establishes a connection between the solution of PDEs and the solution of BSDEs (or FBSDEs), and provides a new insight in studying the existence and uniqueness result of non-linear PDEs. With the help of the theory of BSDEs, Pardoux and Peng [26] obtained a unique classical solution for semi-linear second order parabolic PDEs. This result generalized the Feynman-Kac formula to the semi-linear case (see Peng [29], Pardoux and Peng [26], Barles, Buckdahn and Pardoux [6]).

For the quasi-linear second order parabolic PDEs, there are only few relevant results. We desire to study the solutions of PDEs in both classical sense and Sobolev weak solution sense. The latter is the main purpose of our work. On the other hand, the FBSDEs will be used as our tool to study the solutions of quasi-linear PDEs. We will prove that the solutions of the corresponding finite horizon FBSDEs (0.0.2) give both classical and weak solutions of quasi-linear second order parabolic PDEs (0.0.1).

Finally, let us consider the following FBSDEs with infinite horizon,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x}, Y_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x}, Y_r^{t,x})dW_r, \\ e^{-Ks}Y_s^{t,x} = \int_s^\infty e^{-Kr} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\ \quad + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr - \int_s^\infty e^{-Kr} Z_r^{t,x} dW_r \end{cases} \quad (0.0.3)$$

The BSDEs case was studied by Zhang and Zhao [38]. We extend their result to FBSDEs (0.0.3) case by the Picard iteration. Moreover, we also study the stationary solution of FBSDEs (0.0.3), in order to give a unique weak solution of the following quasi-linear second order elliptic PDEs,

$$\mathcal{L}u(x) + f(x, u(x), \sigma^*(x, u(x))\nabla u(x)) = 0, \quad (0.0.4)$$

where $u : \mathbb{R}^d \rightarrow \mathbb{R}^k$, and \mathcal{L} is defined as

$$\mathcal{L}u(x) = \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^*)_{ij}(x, u(x)) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x, u(x)) \frac{\partial u(x)}{\partial x_i}.$$

The rest of this thesis is organized as follows. In Chapter 1, we prove the existence and uniqueness result of FBSDEs with finite horizon. This result is one of the necessary intermediate steps to establish their connections with the classical solution (Chapter 2) and weak solution (Chapter 3) of quasi-linear parabolic PDEs. In Chapter 2, we give the regularity properties of FBSDEs, and obtain a unique classical solution of quasi-linear parabolic PDEs through the existence and uniqueness result of FBSDEs with finite horizon. In Chapter 3, we establish a link between FBSDEs and PDEs in Sobolev weak solution sense to provide a probabilistic representation of weak solution of quasi-linear parabolic PDEs. In Chapter 4, we solve the FBSDEs with infinite horizon, and study continuity and stationary property of the solution in order to ensure that it gives the weak solutions of the quasi-linear elliptic PDEs.

Chapter 1

Forward-Backward Stochastic Differential Equations (FBSDEs) with Finite Horizon

1.1 Introduction and Preliminaries

Let (Ω, \mathcal{F}, P) be a probability space, and $T > 0$ be fixed. Let $\{W_t, 0 \leq t \leq T\}$ be a d -dimensional standard Brownian motion, and \mathcal{N} denote the P -null sets of \mathcal{F} . For $t \leq s \leq T$, we define $\mathcal{F}_{t,s} = \sigma\{W_r - W_t; t \leq r \leq s\} \vee \mathcal{N}$, $\mathcal{F}_s = \mathcal{F}_{0,s}$.

Definition 1.1.1. Let \mathbb{S} be a Hilbert space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ -field \mathcal{S} . For $K \in \mathbb{R}^+$, we denote by $M^{2,-K}([0, \infty); \mathbb{S})$ the set of $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}/\mathcal{S}$ -measurable random processes $\{\phi(s)\}_{s \geq 0}$ with values on \mathbb{S} satisfying

(i) $\phi(s) : \Omega \rightarrow \mathbb{S}$ is \mathcal{F}_s -measurable for $s \geq 0$;

(ii) $\|\phi(s)\|_{M^{2,-K}([0, \infty); \mathbb{S})}^2 := \mathbb{E}[\int_0^\infty e^{-Ks} \|\phi(s)\|_{\mathbb{S}}^2 ds] < \infty$.

Also we denote by $S^{2,-K}([0, \infty); \mathbb{S})$ the set of $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}/\mathcal{S}$ -measurable random processes $\{\psi(s)\}_{s \geq 0}$ with values on \mathbb{S} satisfying

(i) $\psi(s) : \Omega \rightarrow \mathbb{S}$ is \mathcal{F}_s -measurable for $s \geq 0$ and $\psi(\cdot, \omega)$ is continuous \mathbb{P} -a.s.;

(ii) $\|\psi(s)\|_{S^{2,-K}([0, \infty); \mathbb{S})}^2 := \mathbb{E}[\sup_{s \geq 0} e^{-Ks} \|\psi(s)\|_{\mathbb{S}}^2] < \infty$.

Definition 1.1.2. Let \mathbb{S} be a Hilbert space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ -field \mathcal{S} . For $K \in \mathbb{R}^+$, we denote by $M^{2,K}([0, \infty); \mathbb{S})$ the set of $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}/\mathcal{S}$ -measurable random processes $\{\phi(s)\}_{s \geq 0}$ with values on \mathbb{S} satisfying

(i) $\phi(s) : \Omega \rightarrow \mathbb{S}$ is \mathcal{F}_s -measurable for $s \geq 0$;

(ii) $\|\phi(s)\|_{M^{2,K}([0, \infty); \mathbb{S})}^2 := \mathbb{E}[\int_0^\infty e^{Ks} \|\phi(s)\|_{\mathbb{S}}^2 ds] < \infty$.

Also we denote by $S^{2,K}([0, \infty); \mathbb{S})$ the set of $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}/\mathcal{S}$ -measurable random processes $\{\psi(s)\}_{s \geq 0}$ with values on \mathbb{S} satisfying

(i) $\psi(s) : \Omega \rightarrow \mathbb{S}$ is \mathcal{F}_s -measurable for $s \geq 0$ and $\psi(\cdot, \omega)$ is continuous \mathbb{P} -a.s.;

(ii) $\|\psi(s)\|_{S^{2,K}([0, \infty); \mathbb{S})}^2 := \mathbb{E}[\sup_{s \geq 0} e^{Ks} \|\psi(s)\|_{\mathbb{S}}^2] < \infty$.

Similarly, for $0 \leq t \leq T < \infty$, we define $M^2([t, T]; \mathbb{S})$ and $S^2([t, T]; \mathbb{S})$ on finite time interval.

Definition 1.1.3. Let \mathbb{S} be a Hilbert space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ -field \mathcal{S} . We denote by $M^2([t, T]; \mathbb{S})$ the set of $\mathcal{B}_{[t, T]} \otimes \mathcal{F}/\mathcal{S}$ -measurable random processes $\{\phi(s)\}_{t \leq s \leq T}$ with values on \mathbb{S} satisfying

(i) $\phi(s) : \Omega \rightarrow \mathbb{S}$ is \mathcal{F}_s -measurable for $t \leq s \leq T$;

(ii) $\|\phi(s)\|_{M^2([t, T]; \mathbb{S})}^2 := \mathbb{E}[\int_t^T \|\phi(s)\|_{\mathbb{S}}^2 ds] < \infty$.

Also we denote by $S^2([t, T]; \mathbb{S})$ the set of $\mathcal{B}_{[t, T]} \otimes \mathcal{F}/\mathcal{S}$ -measurable random processes $\{\psi(s)\}_{t \leq s \leq T}$ with values on \mathbb{S} satisfying

(i) $\psi(s) : \Omega \rightarrow \mathbb{S}$ is \mathcal{F}_s -measurable for $t \leq s \leq T$ and $\psi(\cdot, \omega)$ is continuous \mathbb{P} -a.s.;

(ii) $\|\psi(s)\|_{S^2([t, T]; \mathbb{S})}^2 := \mathbb{E}[\sup_{t \leq s \leq T} \|\psi(s)\|_{\mathbb{S}}^2] < \infty$.

Obviously, one can easily check that the three norms $\|\cdot\|_{M^{2,-K}([0, T]; \mathbb{S})}$, $\|\cdot\|_{M^{2,K}([0, T]; \mathbb{S})}$ and $\|\cdot\|_{M^2([0, T]; \mathbb{S})}$ are equivalent, as well as the norms $\|\cdot\|_{S^{2,-K}([0, T]; \mathbb{S})}$, $\|\cdot\|_{S^{2,K}([0, T]; \mathbb{S})}$ and $\|\cdot\|_{S^2([0, T]; \mathbb{S})}$.

Remark 1.1.4. In this thesis, we always take the Hilbert space \mathbb{S} to be an L_ρ^2 space with the inner product $\langle u_1, u_2 \rangle = \int_{\mathbb{R}^d} u_1(x)u_2(x)\rho^{-1}(x)dx$, a ρ -weighted L^2 space (or weighted Sobolev space). Here $\rho(x) = (1 + |x|^2)^p$, $p \geq 2$, is a weight function. It is easy to see that $\rho(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous positive function satisfying $\int_{\mathbb{R}^d} |x|^q \rho^{-1}(x)dx < \infty$ for any $q \in (2, 2p - 1)$. Note that we can consider more general weights ρ which satisfies the above condition and conditions in [4] and all the results of this thesis still hold.

The theory of backward stochastic differential equations and forward-backward stochastic differential equations have been studied extensively in the last two decades, and their applications have been found in many areas, especially the stochastic control theory and mathematical finance. We consider the following fully coupled FBSDEs

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x})dW_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r, \quad 0 \leq t \leq T. \end{cases} \quad (1.1.1)$$

What we mean by fully coupled is the fact that both solutions of forward and backward equations appear in the coefficients (including the terminal condition) of the backward and forward equations.

Let us take a deeper look at the fully coupled FBSDEs (1.1.1). When the forward equation does not depend on the backward component $(Y_s^{t,x}, Z_s^{t,x})$, the FBSDEs can be solved relatively easily, given the many earlier results. That is, e.g. we can first solve the forward equation (see Øksendal [24] or Kunita [15]), which determines the process $X_s^{t,x}$, and then solve the backward equation, with the process $X_s^{t,x}$ known. The backward equation is a kind of BSDEs, which was pioneered by Pardoux and Peng [25]. The connection between BSDEs and semi-linear PDEs was discovered by Peng in [29], Pardoux and Peng in [26].

FBSDEs like (1.1.1) were first considered by Antonelli [1]. In his work, the functions b, σ, f are independent of Z and satisfy Lipschitz conditions, and he obtained an existence and uniqueness result for the FBSDEs over a small enough time duration by using the *Contraction Mapping Method*. This is followed by a number of articles where the FBSDEs were studied over an arbitrary time duration. For example, inspired by the earlier work of Ma and Yong [20] and widely using the result of Ladyzenskaja et al. [18] on a system of quasi-linear PDEs, Ma, Protter and Yong introduced the so called the *Four Step Scheme* and proved the existence and uniqueness under some regularity assumptions on the coefficients and non-degeneracy of the forward equation in [19]. Later, Hu and Yong relaxed the regularity assumptions and obtained an existence (but not uniqueness) result in [13].

Considering FBSDEs (1.1.1) with random coefficients (non-Markovian FBSDEs), Pardoux and Tang [28] established, by means of a purely probabilistic approach with the *Contraction Mapping Method*, an existence and uniqueness result in the case of a small enough time duration as long as the coefficients satisfy a classical monotone-Lipschitz

condition. Combining the *Contraction Mapping Method* and the *Four Step Scheme*, Delarue [9] first gave an existence and uniqueness result over a small time duration under a non-degeneracy assumption, then extended this local result to a global one by means of a running down induction with the help of some classical results in PDEs (e.g. see [18]).

Several other results on a more general form of FBSDEs (σ allowed to depend on z) are given by Hu and Peng [12], Peng and Wu [31], based on stochastic Hamiltonian systems, under certain monotone conditions. Yong [36] generalized these results by introducing a more flexible type of monotone condition. Using homotopic technique, Yong developed and extended this *Continuation Method* in [37].

Comparing all these works on FBSDEs, the balance between the regularity of the coefficients and the time duration is still a fundamental problem. In fact, under Lipschitz conditions, one can only get the existence and uniqueness result over a small time duration (local solution) by using the *Contraction Mapping Method*. For an arbitrary time duration (global solution), one considered more complicated assumptions by the *Four Step Scheme* or the *Continuation Method*. For an arbitrary time T and a partition: $0 = t_0 < \dots < t_n = T$ of $[0, T]$. In [9], Delarue obtained the local result in $[t_{n-1}, t_n]$ under monotone-Lipschitz conditions by using the *Contraction Mapping Method*. Using the boundedness result of gradients of solutions of the corresponding PDEs, he was able to derive the uniformly Lipschitz condition for the terminal function $u(t_{n-1}, X_{t_{n-1}}) = Y_{t_{n-1}}$ for $[t_{n-2}, t_{n-1}]$, and immediately obtained the local result for $[t_{n-2}, t_{n-1}]$ as well. By this running down induction method, he extended the existence and uniqueness result to the whole interval $[0, T]$.

It is basic to consider Lipschitz conditions on coefficients. In Section 1.2, we give local solution in the space $S^2(L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2(L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2(L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. The Lipschitz conditions are required in the weighted L^2 space, they are weaker than the pointwise Lipschitz conditions. Following Delarue's idea, we can also extend local solution to global solution. But we will not present this result in this thesis. One can see Delarue [9] for details if interested. The solution we consider here is in the weighted L^2 space. Therefore our result is for all $x \in \mathbb{R}^d$, which is hidden in the space, where Delarue's result is for fixed $x \in \mathbb{R}^d$.

However, we prefer using a purely probabilistic method for studying the FBSDEs rather than applying a PDEs approach. The advantage using the purely probabilistic

method is that he can push the probabilistic beyond what the analytic method can not offer, e.g. the infinite horizon case (see Chapter 4) and the non-Markovian FBSDEs (see Pardoux and Tang [28]). In Section 1.3, we give a global result under either of two classes of monotone-Lipschitz conditions. In both cases, the existence and uniqueness result obtained by the *Contraction Mapping Method*, appears also to be new. Our idea is: with the help of the monotonicity and some estimates, we can obtain a unique solution in $M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. After proving the continuity and square integrable of X and Y a.s., we can show that the solution is also in the space $S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. Finally, we extend this result from $[t, T]$ to $[0, T]$.

1.2 Local Existence and Uniqueness Result of FBSDEs

In this section, we use the *Contraction Mapping Method* to solve the finite horizon FBSDEs. Under Lipschitz assumptions, we give a unique solution in $S^2(L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2(L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2(L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ over a small time duration. Now we consider the following FBSDEs with finite horizon $[t, T]$, $0 \leq t \leq T$,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \end{cases} \quad (1.2.1)$$

where the functions $b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^{d \times d}$, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$, $h : \mathbb{R}^d \rightarrow \mathbb{R}^k$. We also assume that b , σ , f and h are measurable functions with respect to the Borelian σ -fields. $X_t^{t,x} = x$ is the initial point in \mathbb{R}^d , and for $0 \leq s \leq t$, we regulate $X_s^{t,x} = x$. Consider following assumptions:

(A.1.1): If we say $\zeta(t, x, y, z)$ is uniformly Lipschitz continuous with respect to (x, y, z) with constant \sqrt{C} , then for any $t \in [0, T]$, $(X_1, Y_1, Z_1) \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^d) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^k) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})$ and $(X_2, Y_2, Z_2) \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^d) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^k) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})$, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} |\zeta(t, X_1(x), Y_1(x), Z_1(x)) - \zeta(t, X_2(x), Y_2(x), Z_2(x))|^2 \rho^{-1}(x) dx \\ & \leq C \int_{\mathbb{R}^d} (|X_1(x) - X_2(x)|^2 + |Y_1(x) - Y_2(x)|^2 + \|Z_1(x) - Z_2(x)\|^2) \rho^{-1}(x) dx. \end{aligned}$$

There exists a constant $L \geq 0$ such that

(i) $b(t, x, y, z)$ is uniformly Lipschitz continuous w.r.t. (x, y, z) with \sqrt{L} ;

- (ii) $\sigma(t, x, y)$ is uniformly Lipschitz continuous w.r.t. (x, y) with \sqrt{L} ;
- (iii) $f(t, x, y, z)$ is uniformly Lipschitz continuous w.r.t. (x, y, z) with \sqrt{L} ;
- (iv) $h(x)$ is uniformly Lipschitz continuous w.r.t. x with \sqrt{L} .

Here the Euclidean norm of a vector $x \in \mathbb{R}^d$ will be denoted by $|x|$, and the matrix norm is denoted by $\|z\| := \sqrt{\text{tr}(zz^*)}$.

(A.1.2): Moreover, the following holds

$$\int_0^T (|b(s, 0, 0, 0)|^2 + \|\sigma(s, 0, 0)\|^2 + |f(s, 0, 0, 0)|^2) ds < \infty.$$

Referring to Definition 1.1.3 and noting that C_c^0 is dense in L_ρ^2 under the norm $(\int_{\mathbb{R}^d} |\cdot|^2 \rho^{-1}(x) dx)^{\frac{1}{2}}$, we can define the solution in L_ρ^2 as follows:

Definition 1.2.1. *The process $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ is called a solution of the Eq (1.2.1) if for any $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^d)$ and $\tilde{\varphi} \in C_c^0(\mathbb{R}^d; \mathbb{R}^k)$,*

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^d} X_s^{t,x} \varphi(x) dx = \int_{\mathbb{R}^d} x \varphi(x) dx + \int_t^s \int_{\mathbb{R}^d} b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx dr \\ \quad + \int_t^s \int_{\mathbb{R}^d} \sigma(r, X_r^{t,x}, Y_r^{t,x}) \varphi(x) dx dW_r, \\ \int_{\mathbb{R}^d} Y_s^{t,x} \tilde{\varphi}(x) dx = \int_{\mathbb{R}^d} h(X_T^{t,x}) \tilde{\varphi}(x) dx + \int_s^T \int_{\mathbb{R}^d} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \tilde{\varphi}(x) dx dr \\ \quad - \int_s^T \int_{\mathbb{R}^d} Z_r^{t,x} \tilde{\varphi}(x) dx dW_r \quad \mathbb{P} - a.s.. \end{array} \right. \quad (1.2.2)$$

Remark 1.2.2. *In this chapter, we will study the FBSDEs solution in L_ρ^2 space, our Lipschitz conditions are in L_ρ^2 space which is weaker than the pointwise one. For example, the function $\sum_{n=2}^\infty H_n(x-n)$ is Lipschitz in L_ρ^2 sense, but not Lipschitz in pointwise sense. Here*

$$H_n(x) = \begin{cases} nx + 1 & x \in [-\frac{1}{n}, 0) \\ -nx + 1 & x \in [0, \frac{1}{n}] \\ 0 & \text{otherwise.} \end{cases}$$

If we strengthen our conditions in L_ρ^2 to a pointwise one, our result still holds. In Chapter 2, we will consider this generality. In fact, we will relate the solution of FBSDEs in pointwise sense with the classical solution of corresponding PDEs.

Theorem 1.2.3. *Under Conditions (A.1.1) and (A.1.2), there exists a constant $C_L > 0$, only depending on Lipschitz constant \sqrt{L} , such that for every $T \leq C_L$, (1.2.1) has a unique solution, i.e. there exist unique process $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ which satisfies (1.2.2).*

Proof. We will prove our theorem by the *Contraction Mapping Method* (see Antonelli [1], Delarue [9], Pardoux and Tang [28]) in six steps.

Step 1: Construct a mapping

$$\begin{aligned} \Xi & : S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})) \\ & \rightarrow S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})), \\ & (X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \mapsto (\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}), \end{aligned}$$

where $(\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot})$ is defined as follows, for any $s \in [t, T]$

$$\bar{X}_s^{t,x} = x + \int_t^s b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_t^s \sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) dW_r, \quad (1.2.3)$$

and

$$\bar{Y}_s^{t,x} = h(\bar{X}_T^{t,x}) + \int_s^T f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) dr - \int_s^T \bar{Z}_r^{t,x} dW_r. \quad (1.2.4)$$

The process $(\bar{X}_s^{t,x})_{t \leq s \leq T}$ is the solution of a forward SDE, whereas the pair processes $(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})_{t \leq s \leq T}$ is the solution of a backward SDE.

Actually, we want to prove that there exists a constant $C_L > 0$, only depending on L , such that for $T \leq C_L$, the map Ξ is a contraction. To this end, we firstly assume that $T \leq 1$, and we consider $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ and $(U^{t,\cdot}, V^{t,\cdot}, \mathcal{W}^{t,\cdot})$ in $S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. We put

$$(\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}) = \Xi(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}), \quad (\bar{U}^{t,\cdot}, \bar{V}^{t,\cdot}, \bar{\mathcal{W}}^{t,\cdot}) = \Xi(U^{t,\cdot}, V^{t,\cdot}, \mathcal{W}^{t,\cdot}).$$

Step 2: For the forward SDE (1.2.3), applying Itô's formula to $|\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2$, taking spatial integration $\rho^{-1}(x)dx$ on both sides for a.e. $x \in \mathbb{R}^d$ and applying stochastic Fubini theorem, then taking sup and expectation, we get

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \\ & \leq \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})\|^2 \rho^{-1}(x) dx dr \end{aligned} \quad (1.2.5)$$

$$\begin{aligned}
 & + 2\mathbb{E} \int_t^T \int_{\mathbb{R}^d} |\langle \bar{X}_r^{t,x} - \bar{U}_r^{t,x}, b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - b(r, \bar{U}_r^{t,x}, V_r^{t,x}, \mathscr{W}_r^{t,x}) \rangle| \rho^{-1}(x) dx dr \\
 & + 2\mathbb{E} \sup_{t \leq s \leq T} \left| \int_t^s \int_{\mathbb{R}^d} (\bar{X}_r^{t,x} - \bar{U}_r^{t,x}) (\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})) \rho^{-1}(x) dx dW_r \right|.
 \end{aligned}$$

By Lipschitz condition **(A.1.1)**, the first term on the RHS of above inequality (1.2.5)

$$\begin{aligned}
 & \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})\|^2 \rho^{-1}(x) dx dr \\
 & \leq L \left(\mathbb{E} \int_t^T \int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr + \mathbb{E} \int_t^T \int_{\mathbb{R}^d} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr \right) \\
 & \leq LT \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx + LT \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x} - V_s^{t,x}|^2 \rho^{-1}(x) dx.
 \end{aligned}$$

Recall that we have assumed $T \leq 1$, so $T \leq T^{\frac{1}{2}} \leq 1 < \frac{1}{T}$. By the Cauchy-Schwarz inequality, Lipschitz condition **(A.1.1)** and Young's inequality, the second term of inequality (1.2.5) is estimated as

$$\begin{aligned}
 & 2\mathbb{E} \int_t^T \int_{\mathbb{R}^d} |\langle \bar{X}_r^{t,x} - \bar{U}_r^{t,x}, b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - b(r, \bar{U}_r^{t,x}, V_r^{t,x}, \mathscr{W}_r^{t,x}) \rangle| \rho^{-1}(x) dx dr \\
 & \leq 2\mathbb{E} \int_t^T \left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \\
 & \quad \left(\int_{\mathbb{R}^d} |b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - b(r, \bar{U}_r^{t,x}, V_r^{t,x}, \mathscr{W}_r^{t,x})|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} dr \\
 & \leq 2L^{\frac{1}{2}} \mathbb{E} \int_t^T \left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \\
 & \quad \left(\int_{\mathbb{R}^d} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2) \rho^{-1}(x) dx \right)^{\frac{1}{2}} dr \\
 & \leq 2L^{\frac{1}{2}} \mathbb{E} \int_t^T \left[\left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 \right. \\
 & \quad \left. + \frac{1}{2} \left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 + \frac{1}{2} \left(\int_{\mathbb{R}^d} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 \right. \\
 & \quad \left. + \frac{1}{2T} \left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 + \frac{1}{2} T \left(\int_{\mathbb{R}^d} \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2 \rho^{-1}(x) dx \right)^2 \right]^{\frac{1}{2}} dr \\
 & \leq \sqrt{2} L^{\frac{1}{2}} \mathbb{E} \int_t^T \left[\frac{4}{T} \left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 + \left(\int_{\mathbb{R}^d} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 \right. \\
 & \quad \left. + T \left(\int_{\mathbb{R}^d} \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2 \rho^{-1}(x) dx \right)^2 \right]^{\frac{1}{2}} dr \\
 & \leq \sqrt{2} L^{\frac{1}{2}} \mathbb{E} \int_t^T \left[\frac{2}{T^{\frac{1}{2}}} \int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx + \int_{\mathbb{R}^d} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx \right. \\
 & \quad \left. + T^{\frac{1}{2}} \int_{\mathbb{R}^d} \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2 \rho^{-1}(x) dx \right] dr
 \end{aligned}$$

$$\begin{aligned} &\leq 2\sqrt{2}L^{\frac{1}{2}}T^{\frac{1}{2}}\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx + \sqrt{2}L^{\frac{1}{2}}T\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x} - V_s^{t,x}|^2 \rho^{-1}(x) dx \\ &\quad + \sqrt{2}L^{\frac{1}{2}}T^{\frac{1}{2}}\mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr. \end{aligned}$$

By Burkholder-Davis-Gundy's inequality (here γ is B-D-G constant), Cauchy-Schwarz inequality, condition **(A.1.1)** and $a(a+b) \leq (a+\frac{1}{2}b)^2$, the third term of inequality (1.2.5)

$$\begin{aligned} &2\mathbb{E} \sup_{t \leq s \leq T} \left| \int_t^s \int_{\mathbb{R}^d} (\bar{X}_r^{t,x} - \bar{U}_r^{t,x}) (\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})) \rho^{-1}(x) dx dW_r \right| \\ &\leq 2\gamma\mathbb{E} \left| \int_t^T \left(\int_{\mathbb{R}^d} (\bar{X}_r^{t,x} - \bar{U}_r^{t,x}) (\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})) \rho^{-1}(x) dx \right)^2 dr \right|^{\frac{1}{2}} \\ &\leq 2\gamma\mathbb{E} \left| \int_t^T \left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right) \right. \\ &\quad \left. \left(\int_{\mathbb{R}^d} \|\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})\|^2 \rho^{-1}(x) dx \right) dr \right|^{\frac{1}{2}} \\ &\leq 2\gamma T^{\frac{1}{2}} L^{\frac{1}{2}} \mathbb{E} \left| \sup_{t \leq s \leq T} \left[\left(\int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \right) \right. \right. \\ &\quad \left. \left. \left(\int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx + \int_{\mathbb{R}^d} |Y_s^{t,x} - V_s^{t,x}|^2 \rho^{-1}(x) dx \right) \right] \right|^{\frac{1}{2}} \\ &\leq 2\gamma T^{\frac{1}{2}} L^{\frac{1}{2}} \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx + \gamma T^{\frac{1}{2}} L^{\frac{1}{2}} \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x} - V_s^{t,x}|^2 \rho^{-1}(x) dx. \end{aligned}$$

Hence, for the $\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx$ term in inequality (1.2.5),

$$\begin{aligned} &\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \\ &\leq (LT + 2\sqrt{2}L^{\frac{1}{2}}T^{\frac{1}{2}} + 2\gamma T^{\frac{1}{2}}L^{\frac{1}{2}})\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \\ &\quad + (LT + \sqrt{2}L^{\frac{1}{2}}T + \gamma T^{\frac{1}{2}}L^{\frac{1}{2}})\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x} - V_s^{t,x}|^2 \rho^{-1}(x) dx \\ &\quad + \sqrt{2}L^{\frac{1}{2}}T^{\frac{1}{2}}\mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \end{aligned}$$

We can choose T small enough such that $LT + 2\sqrt{2}L^{\frac{1}{2}}T^{\frac{1}{2}} + 2\gamma T^{\frac{1}{2}}L^{\frac{1}{2}} < 1$, then there exists a constant $C_L^{(1)}$, only depending on L , such that $C_L^{(1)}T^{\frac{1}{2}} < 1$ and the above inequality become

$$\begin{aligned} &(1 - C_L^{(1)}T^{\frac{1}{2}})\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \tag{1.2.6} \\ &\leq C_L^{(1)}T^{\frac{1}{2}} \left(\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x} - V_s^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \right). \end{aligned}$$

Step 3: For the backward SDEs (1.2.4), applying Itô formula to $|\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2$. Similarly we have

$$\begin{aligned}
 & \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx + \int_s^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 = & \int_{\mathbb{R}^d} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\
 & + 2 \int_s^T \int_{\mathbb{R}^d} \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x}) \rangle \rho^{-1}(x) dx dr \\
 & - 2 \int_s^T \int_{\mathbb{R}^d} (\bar{Y}_r^{t,x} - \bar{V}_r^{t,x})(\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}) \rho^{-1}(x) dx dW_r. \tag{1.2.7}
 \end{aligned}$$

First we will calculate the $\mathbb{E} \int_t^T \int_{\mathbb{R}^d} |\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}|^2 \rho^{-1}(x) dx dr$ term. Due to the fact that the expectation of stochastic integral is zero, we have

$$\begin{aligned}
 & \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 \leq & \mathbb{E} \int_{\mathbb{R}^d} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \tag{1.2.8} \\
 & + 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \left| \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x}) \rangle \right| \rho^{-1}(x) dx dr.
 \end{aligned}$$

Using Lipschitz condition **(A.1.1)**, the first term on the RHS of inequality (1.2.8) have following estimates

$$\begin{aligned}
 \mathbb{E} \int_{\mathbb{R}^d} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx & \leq L \mathbb{E} \int_{\mathbb{R}^d} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\
 & \leq L \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx.
 \end{aligned}$$

By Cauchy-Schwarz inequality, Lipschitz condition **(A.1.1)** and Young's inequality, the last term of inequality (1.2.8)

$$\begin{aligned}
 & 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \left| \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x}) \rangle \right| \rho^{-1}(x) dx dr \\
 \leq & 2 \mathbb{E} \int_t^T \left(\int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \\
 & \left(\int_{\mathbb{R}^d} |f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x})|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} dr \\
 \leq & \mathbb{E} \int_t^T \left[4L \left(\int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx \right) \right. \\
 & \left. \left(\int_{\mathbb{R}^d} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 + \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2) \rho^{-1}(x) dx \right) \right]^{\frac{1}{2}} dr \\
 \leq & \mathbb{E} \int_t^T \left[2L \left(\int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx \right) \right]^2
 \end{aligned}$$

$$\begin{aligned}
 & +2L \left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 + 4L \left(\int_{\mathbb{R}^d} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 \\
 & + 256L^2 \left(\int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 + \frac{1}{64} \left(\int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx \right)^2 \Big]^{1/2} dr \\
 \leq & \mathbb{E} \int_t^T \left[2L \left(\int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 + (6L + 256L^2) \left(\int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^2 \right. \\
 & \left. + \frac{1}{64} \left(\int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx \right)^2 \right]^{1/2} dr \\
 \leq & (2L)^{1/2} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & + (6L + 256L^2)^{1/2} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & + \frac{1}{8} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 \leq & (2L)^{1/2} T \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \\
 & + (6L + 256L^2)^{1/2} T \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x} - V_s^{t,x}|^2 \rho^{-1}(x) dx \\
 & + \frac{1}{8} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr.
 \end{aligned}$$

Therefore, from (1.2.8) and above estimates we have

$$\begin{aligned}
 & \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 \leq & \frac{8}{7} (L + (2L)^{1/2} T) \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \\
 & + \frac{8}{7} (6L + 256L^2)^{1/2} T \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx.
 \end{aligned}$$

It turns out that

$$\begin{aligned}
 & \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 \leq & C_L^{(2)} (1 + T) \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \\
 & + C_L^{(2)} T \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx, \tag{1.2.9}
 \end{aligned}$$

where $C_L^{(2)}$ is a constant only depending on L .

Step 4: Next, we consider the $\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx$ term. Note from (1.2.7) we have

$$\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx$$

$$\begin{aligned}
 &\leq \mathbb{E} \int_{\mathbb{R}^d} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\
 &\quad + 2\mathbb{E} \int_t^T \int_{\mathbb{R}^d} \left| \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x}) \rangle \right| \rho^{-1}(x) dx dr \\
 &\quad + 2\mathbb{E} \sup_{t \leq s \leq T} \left| \int_s^T \left\langle \int_{\mathbb{R}^d} (\bar{Y}_r^{t,x} - \bar{V}_r^{t,x})(\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}) \rho^{-1}(x) dx, dW_r \right\rangle \right|. \tag{1.2.10}
 \end{aligned}$$

Obviously, the first two terms in above inequality (1.2.10) are the same as the terms in inequality (1.2.8), so the rest of the work is to deal with the last stochastic integral. By Burkholder-Davis-Gundy's inequality (here γ is B-D-G constant), Cauchy-Schwarz inequality, condition **(A.1.1)**, Young's inequality and estimate (1.2.9) for $\mathbb{E} \int_t^T \int_{\mathbb{R}^d} |\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}|^2 \rho^{-1}(x) dx dr$, we have

$$\begin{aligned}
 &2\mathbb{E} \sup_{t \leq s \leq T} \left| \int_s^T \int_{\mathbb{R}^d} (\bar{Y}_r^{t,x} - \bar{V}_r^{t,x})(\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}) \rho^{-1}(x) dx dW_r \right| \\
 &\leq 2\gamma \mathbb{E} \left| \int_t^T \left(\int_{\mathbb{R}^d} (\bar{Y}_r^{t,x} - \bar{V}_r^{t,x})(\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}) \rho^{-1}(x) dx \right)^2 dr \right|^{\frac{1}{2}} \\
 &\leq 2\gamma \mathbb{E} \left| \int_t^T \left(\int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx \right) \left(\int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx \right) dr \right|^{\frac{1}{2}} \\
 &\leq 2\gamma \mathbb{E} \left| \left(\sup_{t \leq r \leq T} \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx \right) \left(\int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \right) \right|^{\frac{1}{2}} \\
 &\leq 2\gamma \mathbb{E} \left[\frac{1}{2M_1} \left(\sup_{t \leq r \leq T} \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx \right) + \frac{M_1}{2} \left(\int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \right) \right] \\
 &\leq \frac{\gamma}{M_1} \mathbb{E} \sup_{t \leq r \leq T} \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx + M_1 \gamma \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 &\leq \frac{1}{4} \mathbb{E} \sup_{t \leq r \leq T} \int_{\mathbb{R}^d} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx + C_L^{(2')} (1+T) \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \\
 &\quad + C_L^{(2')} T \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx.
 \end{aligned}$$

Here $C_L^{(2')} = M_1 \gamma C_L^{(2)}$, and we can choose M_1 such that $\frac{\gamma}{M_1} \leq \frac{1}{4}$. Note that, by the estimate (1.2.9) for $\mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr$, we can also calculate the first three terms in (1.2.10) by a similar method in Step 3.

Therefore, for the $\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx$ term, there exists a constant $C_L^{(3)}$, only depending on L , such that

$$\begin{aligned}
 &\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx \\
 &\leq C_L^{(3)} (1+T) \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx \\
 &\quad + C_L^{(3)} T \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx
 \end{aligned}$$

$$+\frac{1}{4}\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx.$$

Here we can choose T small enough such that $C_L^{(3)}T < 1$ and it turns out that

$$\begin{aligned} & (1 - C_L^{(3)}T)\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx \\ & \leq C_L^{(3)}(1+T)\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx. \end{aligned} \quad (1.2.11)$$

Step 5: Now we denote

$$\begin{aligned} \bar{A} &= \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2 \rho^{-1}(x) dx, & A &= \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |X_s^{t,x} - U_s^{t,x}|^2 \rho^{-1}(x) dx \\ \bar{B} &= \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2 \rho^{-1}(x) dx, & B &= \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_s^{t,x} - V_s^{t,x}|^2 \rho^{-1}(x) dx, \\ \bar{C} &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr, & C &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr. \end{aligned}$$

Observe that, the estimates (1.2.6), (1.2.9) and (1.2.11) follow

$$\begin{aligned} (1 - C_L^{(1)}T^{\frac{1}{2}})\bar{A} &\leq C_L^{(1)}T^{\frac{1}{2}}(B + C), \\ (1 - C_L^{(3)}T)\bar{B} &\leq C_L^{(3)}(1+T)\bar{A}, \\ \bar{C} &\leq C_L^{(2)}(1+T)\bar{A} + C_L^{(2)}T\bar{B}. \end{aligned}$$

\Rightarrow

$$\begin{aligned} (1 - C_L^{(1)}T^{\frac{1}{2}})\bar{A} &\leq C_L^{(1)}T^{\frac{1}{2}}(B + C), \\ (C_L^{(2)}T + 1)\bar{B} &\leq \frac{C_L^{(3)}(1+T)}{1 - C_L^{(3)}T}\bar{A}(C_L^{(2)}T + 1), \\ \bar{C} &\leq C_L^{(2)}(1+T)\bar{A} + C_L^{(2)}T\bar{B}. \end{aligned}$$

\Rightarrow

$$\begin{aligned} \bar{B} + \bar{C} &\leq \frac{C_L^{(3)}(1+T)}{1 - C_L^{(3)}T}\bar{A}(C_L^{(2)}T + 1) + C_L^{(2)}(1+T)\bar{A}, \\ &\leq \frac{(1+T)(C_L^{(2)} + C_L^{(3)})}{1 - C_L^{(3)}T} \frac{C_L^{(1)}T^{\frac{1}{2}}}{1 - C_L^{(1)}T^{\frac{1}{2}}}(B + C). \end{aligned}$$

\Rightarrow

$$\begin{aligned} \bar{A} + \bar{B} + \bar{C} &\leq \frac{(1 + C_L^{(2)} + C_L^{(3)} + C_L^{(2)}T)C_L^{(1)}T^{\frac{1}{2}}}{(1 - C_L^{(3)}T)(1 - C_L^{(1)}T^{\frac{1}{2}})}(B + C) \\ &\leq \frac{(1 + C_L^{(2)} + C_L^{(3)} + C_L^{(2)}T)C_L^{(1)}T^{\frac{1}{2}}}{(1 - C_L^{(3)}T)(1 - C_L^{(1)}T^{\frac{1}{2}})}(A + B + C) \end{aligned}$$

To construct the contraction mapping Ξ , we have

$$\frac{(1 + C_L^{(2)} + C_L^{(3)} + C_L^{(2)}T)C_L^{(1)}T^{\frac{1}{2}}}{(1 - C_L^{(3)}T)(1 - C_L^{(1)}T^{\frac{1}{2}})} < 1.$$

By solving above inequality, there exists a constant $C_L > 0$ only depending on L , such that for $T \leq C_L$, the map Ξ is contractive from $S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ into itself. Consequently, the Picard's fixed point theorem shows that, for every $T \leq C_L$, (1.2.2) has a unique solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ taking values in $S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

Step 6: Finally, for $s \in [0, t]$, we regulate $X_s^{t,x} = x$, and (1.2.1) is equivalent to the following FBSDEs

$$\begin{cases} X_s^{t,x} = x, \\ Y_s^x = Y_t^{t,x} + \int_s^t f(r, x, Y_r^x, 0) dr, \\ Z_s^x = 0. \end{cases} \quad (1.2.12)$$

Here $Y_t^{t,x}$ is an \mathcal{F}_t measurable random vector, and therefore is deterministic. In this case, the FBSDEs turns into a simple BSDE. By a similar method, we can obtain process $(X_\cdot, Y_\cdot, Z_\cdot) \in S^2([0, t]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, t]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, t]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, is the unique solution of (1.2.12). To unify the notation, we define $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) = (X_s^x, Y_s^x, Z_s^x)$ when $s \in [0, t]$.

Therefore, it turns out that there exists a constant $C_L > 0$, only depending on Lipschitz constance L , such that for every $T \leq C_L$, there exist unique processes $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfies (1.2.2). \square

1.3 Global Existence and Uniqueness Result of FBSDEs

In Section 1.2, Theorem 1.2.3 shows the local existence and uniqueness because the time duration T is restricted by a constant C_L . We can use an idea of Delarue [9], and extend the local result to a global one by a PDE approach. But from the probabilistic viewpoint, it is natural to consider a purely probalistic method to deal with the global problem.

In this section, we consider two classes of monotone-Lipschitz conditions and study the FBSDEs over an arbitrary time duration. In both cases, the existence and uniqueness

result will be proved by the *Contraction Mapping Method*. For the FBSDEs (1.2.1), we consider following assumptions:

(A.2.1): For any $t \in [0, T]$, $b(t, x, y, z)$, $\sigma(t, x, y)$ and $h(x)$ satisfy the Lipschitz condition in **(A.1.1)**. $f(t, x, y, z)$ is uniformly Lipschitz continuous w.r.t. (x, z) with \sqrt{L} in the sense as **(A.1.1)**.

(A.2.2): There exists a constant $\mu > 0$ with $2\mu - K - 2L^2 - 7L - 1 > 0$ where $K > 2L^3 + L^2 + 5L + 1$ such that for any $t \in [0, T]$, $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)$, $X \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$, $Z \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$, we have the monotonicity

$$\begin{aligned} & \int_{\mathbb{R}^d} \langle Y_1(x) - Y_2(x), f(t, X(x), Y_1(x), Z(x)) - f(t, X(x), Y_2(x), Z(x)) \rangle \rho^{-1}(x) dx \\ & \leq -\mu \int_{\mathbb{R}^d} |Y_1(x) - Y_2(x)|^2 \rho^{-1}(x) dx. \end{aligned}$$

Moreover, for any $t \in [0, T]$, $Y \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)$, we have

$$\|f(t, 0, Y(x), 0)\|_{L^2_\rho(\mathbb{R}^k)}^2 \leq L \left(1 + \|Y(x)\|_{L^2_\rho(\mathbb{R}^k)}^2 \right).$$

And $\forall t \in [0, T], \forall (x, z) \in \mathbb{R}^d \times \mathbb{R}^{k \times d}$, the function $v \mapsto f(t, x, v, z)$ is continuous;

Note that in **(A.2.1)** f is Lipschitz continuous with respect to $X(\cdot)$, $Z(\cdot)$ and together with **(A.2.2)**, we can get that f is in $L^2_\rho(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}; \mathbb{R}^k)$. Alternatively, we can also use the following two assumptions **(A.2.3)**, **(A.2.4)** to derive the same result as well.

(A.2.3): $\sigma(t, x, y)$, $f(t, x, y, z)$ and $h(x)$ satisfy the Lipschitz condition in **(A.1.1)**. $b(t, x, y, z)$ is uniformly Lipschitz continuous w.r.t. (y, z) with \sqrt{L} in the sense as **(A.1.1)**.

(A.2.4): Moreover, there exists a constant $\mu > 0$ with $2\mu - K - L - \max\{4L + 1, 2L^2\} > 0$ where $K > 2L^2 + 7L + 1$ such that for any $t \in [0, T]$, $X_1, X_2 \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$, $Y \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)$, $Z \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$, we have the monotonicity

$$\begin{aligned} & \int_{\mathbb{R}^d} \langle X_1(x) - X_2(x), b(t, X_1(x), Y(x), Z(x)) - b(t, X_2(x), Y(x), Z(x)) \rangle \rho^{-1}(x) dx \\ & \leq -\mu \int_{\mathbb{R}^d} |X_1(x) - X_2(x)|^2 \rho^{-1}(x) dx. \end{aligned}$$

Moreover, for any $t \in [0, T]$, $X \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$, we have

$$\|b(t, X(x), 0, 0)\|_{L^2_\rho(\mathbb{R}^d)}^2 \leq L \left(1 + \|X(x)\|_{L^2_\rho(\mathbb{R}^d)}^2 \right).$$

And $\forall t \in [0, T], \forall (y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$, the function $u \mapsto b(t, u, y, z)$ is continuous;

(A.2.5): Moreover, the following holds

$$\int_0^T (|b(s, 0, 0, 0)|^2 + \|\sigma(s, 0, 0)\|^2 + |f(s, 0, 0, 0)|^2) ds < \infty.$$

Notation 1.3.1. To simplify our notation, we denote these two classes of conditions as follows

$$(\mathbf{A.2. Class 1}) := (\mathbf{A.2.1}) + (\mathbf{A.2.2}) + (\mathbf{A.2.5})$$

$$(\mathbf{A.2. Class 2}) := (\mathbf{A.2.3}) + (\mathbf{A.2.4}) + (\mathbf{A.2.5})$$

First we have the following lemma for preparation.

Lemma 1.3.2. Under Condition **(A.2. Class 1)** (resp. **(A.2. Class 2)**), if there exists $(X(\cdot), Y(\cdot), Z(\cdot)) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfying the spatial integral form of (1.2.2) for $t \leq s \leq T$, then $(X(\cdot), Y(\cdot)) \in S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$. Therefore $(X_s(x), Y_s(x), Z_s(x))$ is a solution of (1.2.1).

Proof. In the following, we only prove our result under the condition **(A.2. Class 1)**, the other one can be done similarly. The proof is similar as in Zhang and Zhao [38].

Step 1: Let us first see $Y_s(\cdot)$ is continuous with respect to s in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$. Since $(X_s(x), Y_s(x), Z_s(x))$ satisfies (1.2.2) for $t \leq s \leq T$, therefore

$$\begin{aligned} & \int_{\mathbb{R}^d} |Y_{s+\Delta s}(x) - Y_s(x)|^2 \rho^{-1}(x) dx \\ & \leq C_p \int_{\mathbb{R}^d} \int_s^{s+\Delta s} |f(r, X_r(x), Y_r(x), Z_r(x))|^2 dr \rho^{-1}(x) dx \\ & \quad + C_p \int_{\mathbb{R}^d} \left| \int_s^{s+\Delta s} Z_r(x) dW_r \right|^2 \rho^{-1}(x) dx. \end{aligned}$$

For the forward stochastic integral part, it is trivial to see that for $0 \leq \Delta s \leq T - s$

$$\left| \int_s^{s+\Delta s} Z_r(x) dW_r \right|^2 \leq \sup_{0 \leq \Delta s \leq T-s} \left| \int_s^{s+\Delta s} Z_r(x) dW_r \right|^2 a.s.$$

And we can deduce that $\int_{\mathbb{R}^d} \sup_{0 \leq \Delta s \leq T-s} \left| \int_s^{s+\Delta s} \langle Z_r(x), dW_r \rangle \right|^2 \rho^{-1}(x) dx < \infty$ a.s. by Burkholder-Davis-Gundy's inequality and $Z(\cdot) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. So by the dominated convergence theorem, $\lim_{\Delta s \rightarrow 0^+} \int_{\mathbb{R}^d} \left| \int_s^{s+\Delta s} \langle Z_r(x), dW_r \rangle \right|^2 \rho^{-1}(x) dx = 0$. Similarly we can prove $\lim_{\Delta s \rightarrow 0^-} \int_{\mathbb{R}^d} \left| \int_s^{s+\Delta s} \langle Z_r(x), dW_r \rangle \right|^2 \rho^{-1}(x) dx = 0$ for $t < s \leq T$. So $Y_s(\cdot)$ is continuous w.r.t. s in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$. Secondly

$$\int_{\mathbb{R}^d} |X_{s+\Delta s}(x) - X_s(x)|^2 \rho^{-1}(x) dx$$

$$\begin{aligned} &\leq C_p \int_{\mathbb{R}^d} \int_s^{s+\Delta s} |b(r, X_r(x), Y_r(x), Z_r(x))|^2 dr \rho^{-1}(x) dx \\ &\quad + C_p \int_{\mathbb{R}^d} \left| \int_s^{s+\Delta s} \sigma(r, X_r(x), Y_r(x)) dW_r \right|^2 \rho^{-1}(x) dx. \end{aligned}$$

We can check $X_s(\cdot)$ is continuous w.r.t. s in $L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$ by similar method.

Step 2: From condition **(A.2.Class 1)** and $(X(\cdot), Y(\cdot), Z(\cdot)) \in M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, we have that for a.e. $x \in \mathbb{R}^d$

$$\begin{aligned} &\mathbb{E} \int_t^T |b(r, X_r(x), Y_r(x), Z_r(x))|^2 dr < \infty, \\ &\mathbb{E} \int_t^T \|\sigma(r, X_r(x), Y_r(x))\|^2 dr < \infty, \\ &\mathbb{E} \int_t^T |f(r, X_r(x), Y_r(x), Z_r(x))|^2 dr < \infty. \end{aligned}$$

Referring to Lemma 3.3 in [38], we use the generalized Itô's formula to $\psi_M(X_r(x))$ and $\psi_M(Y_r(x))$, $\psi_M(x) = x^2 I_{\{-M \leq x < M\}} + 2M(x - M) I_{\{x \geq M\}} - 2M(x + M) I_{\{x < -M\}}$, then

$$\begin{aligned} \psi_M(X_s(x)) &= x^2 + \int_t^s \psi'_M(X_r(x)) b(r, X_r(x), Y_r(x), Z_r(x)) dr \\ &\quad + \int_t^s I_{\{-M \leq X_r(x) < M\}} \|\sigma(r, X_r(x), Y_r(x))\|^2 dr \\ &\quad + \int_t^s \langle \psi'_M(X_r(x)) \sigma(r, X_r(x), Y_r(x)), dW_r \rangle \end{aligned}$$

and

$$\begin{aligned} &\psi_M(Y_s(x)) + \int_s^T I_{\{-M \leq Y_r(x) < M\}} \|Z_r(x)\|^2 dr \\ &= \psi_M(h(X_T(x))) + \int_s^T \psi'_M(Y_r(x)) f(r, X_r(x), Y_r(x), Z_r(x)) dr \\ &\quad - \int_s^T \langle \psi'_M(Y_r(x)) Z_r(x), dW_r \rangle. \end{aligned}$$

We take the spatial integration $\rho^{-1}(x) dx$ on both sides and apply stochastic Fubini theorem. Then we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \psi_M(X_s(x)) \rho^{-1}(x) dx \\ &\leq \int_{\mathbb{R}^d} x^2 \rho^{-1}(x) dx + \int_t^s \int_{\mathbb{R}^d} \psi'_M(X_r(x)) b(r, 0, 0, 0) \rho^{-1}(x) dx dr \\ &\quad + \int_t^s \int_{\mathbb{R}^d} \psi'_M(X_r(x)) (b(r, X_r(x), Y_r(x), Z_r(x)) - b(r, 0, 0, 0)) \rho^{-1}(x) dx dr \\ &\quad + C_p \int_t^s \int_{\mathbb{R}^d} \|\sigma(r, X_r(x), Y_r(x)) - \sigma(r, 0, 0)\|^2 \rho^{-1}(x) dx dr + C_p \int_t^s \int_{\mathbb{R}^d} \|\sigma(r, 0, 0)\|^2 \rho^{-1}(x) dx dr \end{aligned}$$

$$+ \int_t^s \langle \int_{\mathbb{R}^d} \psi'_M(X_r(x)) \sigma(r, X_r(x), Y_r(x)) \rho^{-1}(x) dx, dW_r \rangle$$

and

$$\begin{aligned} & \int_{\mathbb{R}^d} \psi_M(Y_s(x)) \rho^{-1}(x) dx \\ \leq & \int_{\mathbb{R}^d} \psi_M(h(X_T(x))) \rho^{-1}(x) dx + \int_s^T \int_{\mathbb{R}^d} \psi'_M(Y_r(x)) f(r, 0, Y_r(x), 0) \rho^{-1}(x) dx dr \\ & + \int_s^T \int_{\mathbb{R}^d} \psi'_M(Y_r(x)) (f(r, X_r(x), Y_r(x), Z_r(x)) - f(r, 0, Y_r(x), 0)) \rho^{-1}(x) dx dr \\ & - \int_s^T \langle \int_{\mathbb{R}^d} \psi'_M(Y_r(x)) Z_r(x) \rho^{-1}(x) dx, dW_r \rangle. \end{aligned}$$

Note that $|\psi'_M(X_r(x))|^2 \leq 4|X_r(x)|^2$. Using condition **(A.2.Class 1)** and Burkholder-Davis-Gundy's inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} \psi_M(X_s(x)) \rho^{-1}(x) dx \\ \leq & \int_{\mathbb{R}^d} x^2 \rho^{-1}(x) dx + C_p \mathbb{E} \int_t^T \int_{\mathbb{R}^d} |b(r, 0, 0, 0)|^2 \rho^{-1}(x) dx dr + C_p \mathbb{E} \int_t^T \int_{\mathbb{R}^d} |X_r(x)|^2 \rho^{-1}(x) dx dr \\ & + C_L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2 + \|Z_r(x)\|^2) \rho^{-1}(x) dx dr \\ & + C_L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2) \rho^{-1}(x) dx dr + C_p \mathbb{E} \int_t^T \int_{\mathbb{R}^d} \|\sigma(r, 0, 0)\|^2 \rho^{-1}(x) dx dr \\ & + C_p \mathbb{E} \left| \int_t^T \left(\int_{\mathbb{R}^d} |\psi'_M(X_s(x))|^2 \rho^{-1}(x) dx \right) \left(\int_{\mathbb{R}^d} \|\sigma(r, X_r(x), Y_r(x))\|^2 \rho^{-1}(x) dx \right) dr \right|^{\frac{1}{2}} \\ \leq & \int_{\mathbb{R}^d} x^2 \rho^{-1}(x) dx + C_p \mathbb{E} \int_t^T (|b(r, 0, 0, 0)|^2 + \|\sigma(r, 0, 0)\|^2) dr \\ & + C_L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2 + \|Z_r(x)\|^2) \rho^{-1}(x) dx dr \\ & + \frac{1}{5} \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\psi'_M(X_s(x))|^2 \rho^{-1}(x) dx. \end{aligned}$$

Since $(X(\cdot), Y(\cdot), Z(\cdot)) \in M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ and condition **(A.2.Class 1)**, taking the limit as $M \rightarrow \infty$ and applying the monotone convergence theorem, we have $\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |X_s(x)|^2 \rho^{-1}(x) dx < \infty$.

Due to $\psi_M(h(X_T(x))) \leq |h(X_T(x))|^2$ and $|\psi'_M(Y_r(x))|^2 \leq 4|Y_r(x)|^2$, by the similar estimate we have

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} \psi_M(Y_s(x)) \rho^{-1}(x) dx \\ \leq & C_p \mathbb{E} \int_{\mathbb{R}^d} |h(X_T(x)) - h(0)|^2 \rho^{-1}(x) dx + C_p |h(0)|^2 \end{aligned}$$

$$\begin{aligned}
 & +C_\lambda \left(1 + \mathbb{E} \int_t^T \int_{\mathbb{R}^d} |Y_r(x)|^2 \rho^{-1}(x) dx dr \right) \\
 & +C_L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2 + \|Z_r(x)\|^2) \rho^{-1}(x) dx dr \\
 & +C_p \mathbb{E} \left| \int_t^T \left(\int_{\mathbb{R}^d} |\psi'_M(Y_s(x))|^2 \rho^{-1}(x) dx \right) \left(\int_{\mathbb{R}^d} \|Z_r(x)\|^2 \rho^{-1}(x) dx \right) dr \right|^{\frac{1}{2}} \\
 \leq & C_L \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |X_s(x)|^2 \rho^{-1}(x) dx + C_p |h(0)|^2 \\
 & +C_{L,\lambda} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2 + \|Z_r(x)\|^2) \rho^{-1}(x) dx dr \\
 & +\frac{1}{5} \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\psi'_M(Y_s(x))|^2 \rho^{-1}(x) dx + C_\lambda.
 \end{aligned}$$

Similarly, taking $M \rightarrow \infty$, we can see that $\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_r(x)|^2 \rho^{-1}(x) dx < \infty$. So $(X(\cdot), Y(\cdot)) \in S^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k))$ follows. That is to say $(X(\cdot), Y(\cdot), Z(\cdot))$ is a solution of (1.2.1). \square

Next, we will present our main results of existence and uniqueness solutions to FBDS-DEs with two different methods.

Theorem 1.3.3. *Under Condition (A.2.Class 1), (1.2.1) has a unique solution, i.e. there exist unique processes $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfying the (1.2.2).*

Proof. The proof is different from that in Theorem 1.2.3. First we construct a contraction mapping from $M^{2,-K}([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^{2,-K}([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ to itself. Since the two norms $\|\cdot\|_{M^{2,-K}([t, T]; \mathbb{S})}$ and $\|\cdot\|_{M^2([t, T]; \mathbb{S})}$ are equivalent, we obtain a unique $M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ solution. By Lemma 1.3.2 we know that the solution is in $S^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as well. Finally, we extend our result from $[t, T]$ to $[0, T]$. Rather than using the Lipschitz conditions as in Theorem 1.2.3, our proof is based on monotone-Lipschitz assumptions, which work well in the contraction mapping argument. The proof will be splitted to three steps.

Before we prove the theorem, let us introduce the method to construct the solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. Consider the following BSDE

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \quad (1.3.1)$$

where $X_s^{t,x}$ is a diffusion process given by the solution of the following SDE

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r. \quad (1.3.2)$$

Observe that the coefficients b and σ are time-dependent, so the forward SDE (1.3.2) is different from those in [27], [4], [38]. However, there exists a unique solution for SDE (1.3.2) (see Øksendal [24] or Kunita [15]). For the BSDEs (1.3.1), we can use a similar method as in the proof of Theorem 3.5 in [38] to prove that there exists a unique solution $(Y^{\cdot,t}, Z^{\cdot,t}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

Now we set up the following iterative procedure: Given $(Y_s^{t,x,N-1}, Z_s^{t,x,N-1}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ and define $(X_s^{t,x,N}, Y_s^{t,x,N}, Z_s^{t,x,N})$ as follows

$$\begin{cases} X_s^{t,x,N} = x + \int_t^s b(r, X_r^{t,x,N}, Y_r^{t,x,N-1}, Z_r^{t,x,N-1}) dr + \int_t^s \sigma(r, X_r^{t,x,N}, Y_r^{t,x,N-1}) dW_r \\ Y_s^{t,x,N} = h(X_T^{t,x,N}) + \int_s^T f(r, X_r^{t,x,N}, Y_r^{t,x,N}, Z_r^{t,x,N}) dr - \int_s^T Z_r^{t,x,N} dW_r. \end{cases} \quad (1.3.3)$$

For $N = 1$, let $(Y_s^{t,x,0}, Z_s^{t,x,0}) = (0, 0)$, the forward SDE (1.3.3) above will turn to

$$X_s^{t,x,1} = x + \int_t^s b(r, X_r^{t,x,1}, 0, 0) dr + \int_t^s \sigma(r, X_r^{t,x,1}, 0) dW_r.$$

Since $b(r, X_r^{t,x,1}, 0, 0)$ and $\sigma(r, X_r^{t,x,1}, 0)$ satisfy conditions in **(A.2.Class 1)**, it is easy to see that the forward SDE has a unique solution $X^{\cdot,t,1} \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. Then we substitute $X_r^{t,x,1}$ into BSDEs (1.3.3) to have

$$Y_s^{t,x,1} = h(X_T^{t,x,1}) + \int_s^T f(r, X_r^{t,x,1}, Y_r^{t,x,1}, Z_r^{t,x,1}) dr - \int_s^T Z_r^{t,x,1} dW_r.$$

Note this equation is a kind of BSDEs. By Theorem 3.5 in [38], the BSDEs admits a unique solution $(Y^{\cdot,t,1}, Z^{\cdot,t,1}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. For $N = 2$, we substitute $(Y_s^{t,x,1}, Z_s^{t,x,1})$ into the SDE in (1.3.3) to solve $X_s^{t,x,2}$. Following the same procedure, we obtain $(X^{\cdot,t,2}, Y^{\cdot,t,2}, Z^{\cdot,t,2}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. In general, we see (1.3.3) is an iterated mapping from $M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ to itself and obtain a sequence $(X_s^{t,x,i}, Y_s^{t,x,i}, Z_s^{t,x,i})_{i=0,1,2,\dots}$. We will prove that (1.3.3) is a contraction mapping.

Step 1: Construct the following mapping

$$\begin{aligned} \Xi & : M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})) \\ & \rightarrow M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})), \end{aligned}$$

$$(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \mapsto (\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}),$$

where $(\bar{X}_s^{t,x}, \bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})$ is defined as follows, for any $s \in [t, T]$

$$\bar{X}_s^{t,x} = x + \int_t^s b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_t^s \sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x})dW_r, \quad (1.3.4)$$

and

$$\bar{Y}_s^{t,x} = h(\bar{X}_T^{t,x}) + \int_s^T f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x})dr - \int_s^T \bar{Z}_r^{t,x}dW_r. \quad (1.3.5)$$

The process $(\bar{X}_s^{t,x})_{t \leq s \leq T}$ is a solution of a forward SDE, whereas the pair process $(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})_{t \leq s \leq T}$ is a solution of a backward SDE.

Actually, we want to prove that the map Ξ is a contraction. To this end, we consider $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ and $(U^{t,\cdot}, V^{t,\cdot}, \mathscr{W}^{t,\cdot})$ in $M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. We put

$$(\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}) = \Xi(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}), \quad (\bar{U}^{t,\cdot}, \bar{V}^{t,\cdot}, \bar{\mathscr{W}}^{t,\cdot}) = \Xi(U^{t,\cdot}, V^{t,\cdot}, \mathscr{W}^{t,\cdot}).$$

Step 2: For the forward SDE (1.3.4), applying Itô's formula to $e^{-Ks}|\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2$, taking spatial integration $\rho^{-1}(x)dx$ on both sides for a.e. $x \in \mathbb{R}^d$, applying stochastic Fubini theorem and taking expectation we get

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\ & + K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ = & \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})\|^2 \rho^{-1}(x) dx dr \\ & + 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \langle \bar{X}_r^{t,x} - \bar{U}_r^{t,x}, \\ & \quad b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - b(r, \bar{U}_r^{t,x}, V_r^{t,x}, \mathscr{W}_r^{t,x}) \rangle \rho^{-1}(x) dx dr. \end{aligned} \quad (1.3.6)$$

By condition **(A.2.Class 1)**, the first term on the RHS of above (1.3.6) is estimated as

$$\begin{aligned} & \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})\|^2 \rho^{-1}(x) dx dr \\ \leq & L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & + L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr. \end{aligned}$$

By the Cauchy-Schwarz inequality, Lipschitz condition **(A.2.Class 1)** and Young's inequality, the second term of (1.3.6) is estimated as

$$2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \left| \langle \bar{X}_r^{t,x} - \bar{U}_r^{t,x}, \right.$$

$$\begin{aligned}
 & b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - b(r, \bar{U}_r^{t,x}, V_r^{t,x}, \mathscr{W}_r^{t,x}) \Big| \rho^{-1}(x) dx dr \\
 \leq & 2\mathbb{E} \int_t^T \left(\int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \\
 & \left(\int_{\mathbb{R}^d} e^{-Kr} |b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - b(r, \bar{U}_r^{t,x}, V_r^{t,x}, \mathscr{W}_r^{t,x})|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} dr \\
 \leq & 2\mathbb{E} \int_t^T \left| \left(L \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right) \right. \\
 & \left. \left(\int_{\mathbb{R}^d} e^{-Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2) \rho^{-1}(x) dx \right) \right|^{\frac{1}{2}} dr \\
 \leq & 2\mathbb{E} \int_t^T \left| \frac{4}{2} \left(L \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx \right) \right. \\
 & \left. + \frac{1}{2 \times 4} \left(\int_{\mathbb{R}^d} e^{-Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2) \rho^{-1}(x) dx \right) \right| dr \\
 \leq & (4L + \frac{1}{4}) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & + \frac{1}{4} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2) \rho^{-1}(x) dx dr.
 \end{aligned}$$

Hence, for the forward SDE (1.3.6), we have

$$\begin{aligned}
 & \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\
 & + K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 \leq & L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & + L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & + (4L + \frac{1}{4}) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & + \frac{1}{4} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2) \rho^{-1}(x) dx dr. \quad (1.3.7)
 \end{aligned}$$

For the BSDE (1.3.5), applying *Itô's* formula to $e^{-Ks} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2$. Similarly we have

$$\begin{aligned}
 & \mathbb{E} \int_{\mathbb{R}^d} e^{-Kt} |\bar{Y}_t^{t,x} - \bar{V}_t^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\bar{Z}_r^{t,x} - \mathscr{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 & - K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 = & \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\
 & + 2\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x},
 \end{aligned}$$

$$f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x}) \rangle \rho^{-1}(x) dx dr. \quad (1.3.8)$$

Note that, we can also have following estimate from (1.3.6)

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\ & \leq L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |Y_r^{t,x} - V_r^{t,x}|^2) \rho^{-1}(x) dx dr \\ & \quad + (2L^2 + \frac{1}{2L}) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & \quad + \frac{1}{2L} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2) \rho^{-1}(x) dx dr. \end{aligned}$$

By **(A.2.Class 1)** and above result, the first term on the RHS of (1.3.8) can be written as

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\ & \leq L \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\ & \leq L^2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |Y_r^{t,x} - V_r^{t,x}|^2) \rho^{-1}(x) dx dr \\ & \quad + (2L^3 + \frac{1}{2}) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & \quad + \frac{1}{2} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2) \rho^{-1}(x) dx dr. \end{aligned}$$

Similarly, the second term is estimated as

$$\begin{aligned} & 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, \\ & \quad f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x}) \rangle \rho^{-1}(x) dx dr \\ & \leq 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, \\ & \quad f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{X}_r^{t,x}, \bar{V}_r^{t,x}, \bar{Z}_r^{t,x}) \rangle \rho^{-1}(x) dx dr \\ & \quad + 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, \\ & \quad f(r, \bar{X}_r^{t,x}, \bar{V}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x}) \rangle \rho^{-1}(x) dx dr \\ & \leq -2\mu \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & \quad + 5L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & \quad + \frac{1}{5} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2) \rho^{-1}(x) dx dr. \end{aligned}$$

Therefore, for the BDSDEs (1.3.8)

$$\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr - K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr$$

$$\begin{aligned}
 &\leq L^2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |Y_r^{t,x} - V_r^{t,x}|^2) \rho^{-1}(x) dx dr \\
 &+ (2L^3 + \frac{1}{2}) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 &+ \frac{1}{2} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2) \rho^{-1}(x) dx dr \\
 &- 2\mu \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 &+ 5L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 &+ \frac{1}{5} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2) \rho^{-1}(x) dx dr. \tag{1.3.9}
 \end{aligned}$$

Step 3: Now let us construct the contraction mapping. For easy notation, we denote

$$\begin{aligned}
 \bar{A} &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 A &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x} - U_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 \bar{B} &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 B &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 \bar{C} &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr, \\
 C &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr.
 \end{aligned}$$

So the estimates (1.3.7) and (1.3.9) become:

$$K\bar{A} \leq L\bar{A} + LB + (4L + \frac{1}{4})\bar{A} + \frac{1}{4}(B + C).$$

and

$$\begin{aligned}
 \bar{C} - K\bar{B} &\leq L^2(\bar{A} + B) + (2L^3 + \frac{1}{2})\bar{A} + \frac{1}{2}(B + C) \\
 &\quad - 2\mu\bar{B} + 5L\bar{B} + \frac{1}{5}(\bar{A} + \bar{C}).
 \end{aligned}$$

This leads to

$$\begin{aligned}
 &(K - 2L^3 - L^2 - 5L - \frac{19}{20})\bar{A} + (2\mu - K - 5L)\bar{B} + \frac{4}{5}\bar{C} \\
 &\leq (\frac{3}{4} + L + L^2)B + \frac{3}{4}C.
 \end{aligned}$$

In fact,

$$\frac{4}{5} \left\{ \left(\frac{K - 2L^3 - L^2 - 5L - \frac{19}{20}}{\frac{4}{5}} \right) \bar{A} + \left(\frac{2\mu - K - 5L}{\frac{4}{5}} \right) \bar{B} + \bar{C} \right\}$$

$$\leq \frac{3}{4} \left\{ \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) B + C \right\}.$$

It turns out that

$$\begin{aligned} & \left(\frac{K - 2L^3 - L^2 - 5L - \frac{19}{20}}{\frac{4}{5}} \right) \bar{A} + \left(\frac{2\mu - K - 5L}{\frac{4}{5}} \right) \bar{B} + \bar{C} \\ & \leq \frac{15}{16} \left\{ \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) B + C \right\}. \end{aligned}$$

We assume $1 + \frac{4}{3}L + \frac{4}{3}L^2 \leq \frac{2\mu - K - 5L}{\frac{4}{5}}$ and $K - 2L^3 - L^2 - 5L - \frac{19}{20} > 0$, then we have,

$$\begin{aligned} & \left(\frac{K - 2L^3 - L^2 - 5L - \frac{19}{20}}{\frac{4}{5}} \right) \bar{A} + \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) \bar{B} + \bar{C} \\ & \leq \left(\frac{K - 2L^3 - L^2 - 5L - \frac{19}{20}}{\frac{4}{5}} \right) \bar{A} + \left(\frac{2\mu - K - 5L}{\frac{4}{5}} \right) \bar{B} + \bar{C} \\ & \leq \frac{15}{16} \left\{ \left(\frac{K - 2L^3 - L^2 - 5L - \frac{19}{20}}{\frac{4}{5}} \right) \bar{A} + \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) \bar{B} + \bar{C} \right\}. \end{aligned}$$

Thus the map Ξ is a contraction from $M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ into itself. Note that the two norms $\|\cdot\|_{M^{2,-K}([t, T]; \cdot)}$ and $\|\cdot\|_{M^2([t, T]; \cdot)}$ are equivalent. Consequently, Picard's fixed point theorem shows that, (1.2.1) has a unique solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

By Lemma 1.3.2, the solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ is in $S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as well.

Finally, following a similar procedure as in Step 6 of the proof of Theorem 1.2.3, we obtain a unique solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

□

Theorem 1.3.4. *Under Condition (A.2.Class 2), (1.3.3) has a unique solution, i.e. there exist unique processes $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfies (1.2.2).*

Proof. It is natural to consider $\|\cdot\|_{M^{2,K}([t, T]; \cdot)}$ norm to approach our contraction mapping since the norms $\|\cdot\|_{M^{2,K}([t, T]; \cdot)}$ and $\|\cdot\|_{M^2([t, T]; \cdot)}$ are equivalent as well. In this case, after

applying $It\hat{o}$'s formula to the forward solution, the coefficient of \bar{A} is $-K - 5L - \frac{1}{5}$ which is definitely negative. So we should give a monotonicity condition **(A.2.4)** to cover this negative part, which could be positive if μ is big enough. On the other hand, the way we treat $|h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x})|^2$ is also different from that in the proof of Theorem 1.3.3. In fact, we put $2\mu - K - 2L^2 - L \geq 0$ to guarantee $|h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x})|^2$ being small enough, so that the contraction can be well obtained. The sketch of the proof is similar to that of Theorem 1.3.3.

Step 1: Construct following mapping

$$\begin{aligned} \Xi & : M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})) \\ & \rightarrow M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})), \\ & (X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \mapsto (\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}), \end{aligned}$$

And the rest of mapping structure is exactly the same as that in the proof of Theorem 1.3.3.

Step 2: For the forward SDE (1.3.4), applying $It\hat{o}$'s formula to $e^{Ks}|\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2$, taking spatial integration $\rho^{-1}(x)dx$ on both sides, applying stochastic Fubini theorem and taking expectation we get

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^d} e^{KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\ & - K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ = & \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \|\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})\|^2 \rho^{-1}(x) dx dr \\ & + 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \langle \bar{X}_r^{t,x} - \bar{U}_r^{t,x}, \\ & \quad b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - b(r, \bar{U}_r^{t,x}, V_r^{t,x}, \mathcal{W}_r^{t,x}) \rangle \rho^{-1}(x) dx dr. \quad (1.3.10) \end{aligned}$$

By the Cauchy-Schwarz inequality, Lipschitz condition **(A.2.Class 2)** and Young's inequality, for the forward SDE (1.3.10)

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^d} e^{KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\ & - K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ \leq & L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & + L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr \end{aligned}$$

$$\begin{aligned}
 & -2\mu\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & +4L\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & +\frac{1}{4}\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} (|Y_r^{t,x} - V_r^{t,x}|^2 + \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2) \rho^{-1}(x) dx dr. \quad (1.3.11)
 \end{aligned}$$

For the BSDE (1.2.4), applying Itô's formula to $e^{Ks}|\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2$. Similarly we have

$$\begin{aligned}
 & \mathbb{E} \int_{\mathbb{R}^d} e^{Kt} |\bar{Y}_t^{t,x} - \bar{V}_t^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \|\bar{Z}_r^{t,x} - \bar{\mathscr{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 & +K\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 = & \mathbb{E} \int_{\mathbb{R}^d} e^{KT} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\
 & +2\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, \\
 & \quad f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathscr{W}}_r^{t,x}) \rangle \rho^{-1}(x) dx dr. \quad (1.3.12)
 \end{aligned}$$

From (1.3.10), we have the following estimate which is a little different from (1.3.11).

$$\begin{aligned}
 & \mathbb{E} \int_{\mathbb{R}^d} e^{KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\
 & (2\mu - K - 2L^2 - L)\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 \leq & \left(\frac{1}{2L} + L\right)\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & +\frac{1}{2L}\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr.
 \end{aligned}$$

We assume $2\mu - K - 2L^2 - L \geq 0$, by **(A.2.Class 2)**, the first term on the RHS of (1.3.12) can be written as

$$\begin{aligned}
 & \mathbb{E} \int_{\mathbb{R}^d} e^{KT} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\
 \leq & L\mathbb{E} \int_{\mathbb{R}^d} e^{KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\
 \leq & \left(\frac{1}{2} + L^2\right)\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 & +\frac{1}{2}\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \|Z_r^{t,x} - \mathscr{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr.
 \end{aligned}$$

Similarly, the second term

$$\begin{aligned}
 & 2\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, \\
 & \quad f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathscr{W}}_r^{t,x}) \rangle \rho^{-1}(x) dx dr
 \end{aligned}$$

$$\begin{aligned}
 &\leq 5L\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 &\quad + \frac{1}{5} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 + \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2) \rho^{-1}(x) dx dr.
 \end{aligned}$$

Therefore, (1.3.12) gives

$$\begin{aligned}
 &\mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr + K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 &\leq \left(\frac{1}{2} + L^2\right) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 &\quad + \frac{1}{2} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
 &\quad + 5L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \tag{1.3.13} \\
 &\quad + \frac{1}{5} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} (|\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 + |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 + \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2) \rho^{-1}(x) dx dr.
 \end{aligned}$$

Step 3: Now let us construct the contraction mapping. For easy notation, we denote

$$\begin{aligned}
 \bar{A} &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 A &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |X_r^{t,x} - U_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 \bar{B} &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 B &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 \bar{C} &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr, \\
 C &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{Kr} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr.
 \end{aligned}$$

So the estimates (1.3.12) and (1.3.13) are equivalent to

$$-K\bar{A} \leq L\bar{A} + LB - 2\mu\bar{A} + 4L\bar{A} + \frac{1}{4}(B + C).$$

and

$$\bar{C} + K\bar{B} \leq \left(\frac{1}{2} + L^2\right)B + \frac{1}{2}C + 5L\bar{B} + \frac{1}{5}(\bar{A} + \bar{B} + \bar{C}).$$

This leads to

$$\begin{aligned}
 &(2\mu - K - 5L - \frac{1}{5})\bar{A} + (K - 5L - \frac{1}{5})\bar{B} + \frac{4}{5}\bar{C} \\
 &\leq \left(\frac{3}{4} + L + L^2\right)B + \frac{3}{4}C.
 \end{aligned}$$

In fact,

$$\begin{aligned} & \frac{4}{5} \left\{ \left(\frac{2\mu - K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) \bar{A} + \left(\frac{K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) \bar{B} + \bar{C} \right\} \\ & \leq \frac{3}{4} \left\{ \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) B + C \right\}. \end{aligned}$$

It turns out that

$$\begin{aligned} & \left(\frac{2\mu - K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) \bar{A} + \left(\frac{K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) \bar{B} + \bar{C} \\ & \leq \frac{15}{16} \left\{ \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) B + C \right\}. \end{aligned}$$

We assume $1 + \frac{4}{3}L + \frac{4}{3}L^2 \leq \frac{K-5L-\frac{1}{5}}{\frac{4}{5}}$ and $2\mu - K - 5L - \frac{1}{5} > 0$, then we have,

$$\begin{aligned} & \left(\frac{2\mu - K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) \bar{A} + \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) \bar{B} + \bar{C} \\ & \leq \left(\frac{2\mu - K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) \bar{A} + \left(\frac{K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) \bar{B} + \bar{C} \\ & \leq \frac{15}{16} \left\{ \left(\frac{2\mu - K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) A + \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) B + C \right\}. \end{aligned}$$

Thus the map Ξ is a contraction from $M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ into itself. Note that the two norms $\|\cdot\|_{M^{2,K}([t, T]; \cdot)}$ and $\|\cdot\|_{M^2([t, T]; \cdot)}$ are equivalent. Consequently, the Picard's fixed point theorem shows that, (1.3.3) has a unique solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

Finally, following a similar procedure as in Step 3 of the proof of Theorem 1.3.3, we obtain a unique solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. \square

Remark 1.3.5. *In either (A.2.Class 1) or (A.2.Class 2), only one of functions b and f satisfies monotone-Lipschitz conditions, and the other one is Lipschitz continuous. Moreover, if we strengthen our assumptions that both b and f satisfy monotone-Lipschitz conditions our result also holds, and the monotone coefficient μ can be more flexible. This can be verified by using either of the methods in the proof of Theorem 1.3.3 or Theorem 1.3.4. We will not give any detail here.*

Chapter 2

Quasi-linear Parabolic PDEs and Finite Horizon FBSDEs: Regular regime

2.1 Introduction and Preliminaries

We consider the following backward stochastic differential equations (BSDEs) with its forward stochastic differential equation (SDE):

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r, \quad 0 \leq t \leq T, \end{cases} \quad (2.1.1)$$

introduced by Peng [29] in order to give a probabilistic interpretation for the solution of semi-linear parabolic partial differential equation (PDEs) of the form

$$u(t, x) = h(x) + \int_t^T [\mathcal{L}u(s, x) + f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))]ds, \quad (2.1.2)$$

where $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$ and \mathcal{L} is an infinitesimal operator defined by

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}.$$

The following so-called probabilistic interpretation:

$$Y_t^{t,x} = u(t, x)$$

establishes a link between the solution of BSDEs (2.1.1) and the solution of semi-linear parabolic PDEs (2.1.2). By this link, Pardoux and Peng [26] deduce a converse result of

[29], a given function expressed in terms of the solution of the BSDE (2.1.1) solves a certain system of semi-linear parabolic PDEs (2.1.2). This means, a unique classical solution for semi-linear parabolic PDEs can be obtained through the existence and uniqueness result of BSDEs (see [25]). This result also extends the Feynman-Kac formula to the semi-linear parabolic PDEs case.

Inspired by this idea, we will study the following quasi-linear second order parabolic PDE,

$$\begin{cases} \partial_t u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x), \sigma^*(t, x, u(t, x))) \nabla u(t, x) = 0, \\ u(T, x) = h(x), \end{cases} \quad (2.1.3)$$

where $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$, and \mathcal{L} is an infinitesimal operator defined by

$$\mathcal{L}u(t, x) = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij}(t, x, u(t, x)) \frac{\partial^2 u(t, x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t, x, u(t, x)) \frac{\partial u(t, x)}{\partial x_i}.$$

This type of quasi-linear PDEs is very general, where the nonlinear functions b and σ depend on $(t, u(t, x))$. The main purpose in this chapter is to find a unique classical solution for such PDEs through the corresponding FBSDEs result.

This type of quasi-linear second order parabolic PDEs should be related to the following FBSDEs,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \quad 0 \leq t \leq T. \end{cases} \quad (2.1.4)$$

Here the functions $b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^{d \times d}$, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$, $h : \mathbb{R}^d \rightarrow \mathbb{R}^k$. We also assume that b , σ , f and h are measurable functions with respect to the Borelian σ -fields. Note that the function b is independent of process Z . The existence and uniqueness result of a general FBSDEs (1.1.1) has been studied in Theorem 1.3.3 and Theorem 1.3.4 in Chapter 1.

In this chapter, we study the connection between FBSDEs (2.1.4) and quasi-linear parabolic PDEs (2.1.3), in order to obtain a unique classical solution of PDEs. The rest of this chapter is organised as follows. In Section 2.2, we establish some estimates and regularity results for the solution of the FBSDEs. In Section 2.3, we will relate it to a system of quasi-linear parabolic PDEs to give a probabilistic representation for such PDEs, and use it to prove an existence and uniqueness result of such quasi-linear parabolic PDEs.

Let us first repeat some notation, for $k \geq 0$, $C^k(\mathbb{R}^p; \mathbb{R}^q)$ and $C_{l,b}^k(\mathbb{R}^p; \mathbb{R}^q)$ will denote respectively the set of functions of class C^k from \mathbb{R}^p into \mathbb{R}^q , the set of C^k -functions whose partial derivatives of order less than or equal to k are bounded (and hence the function itself grows at most linearly at infinity). And we consider following assumptions:

(B.0): For any $s \in [0, T]$, $b(s, \cdot, \cdot) \in C_{l,b}^3(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)$; $\sigma(s, \cdot, \cdot) \in C_{l,b}^3(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^{d \times d})$; $f(s, \cdot, \cdot, \cdot) \in C_{l,b}^3(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}; \mathbb{R}^k)$; $h \in C_{l,b}^3(\mathbb{R}^d; \mathbb{R}^k)$.

(B.1): There exists a constant $L \geq 0$ such that for any $t \in [0, T]$, $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\begin{aligned} |b(t, x_1, y_1) - b(t, x_2, y_2)|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ \|\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)\|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ |f(t, x_1, y, z_1) - f(t, x_2, y, z_2)|^2 &\leq L(|x_1 - x_2|^2 + \|z_1 - z_2\|^2) \\ |h(x_1) - h(x_2)|^2 &\leq L|x_1 - x_2|^2. \end{aligned}$$

Here the Euclidean norm of a vector $x \in \mathbb{R}^d$ will be denoted by $|x|$, and the matrix norm is denoted by $\|z\| := \sqrt{\text{tr}(zz^*)}$.

(B.2): There exists a constant $\mu > 0$ with $2\mu - K - 2L^2 - 7L - 1 > 0$, where $K > 2L^3 + L^2 + 5L + 1$ such that for any $t \in [0, T]$, $y_1, y_2, y \in \mathbb{R}^k$, $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{k \times d}$

$$\begin{aligned} \langle y_1 - y_2, f(t, x, y_1, z) - f(t, x, y_2, z) \rangle &\leq -\mu|y_1 - y_2|^2, \\ |f(t, 0, y, 0)|^2 &\leq L(1 + |y|^2). \end{aligned}$$

And the function $v \mapsto f(t, x, v, z)$ is continuous;

(B.3): There exists a constant $L \geq 0$ such that for any $t \in [0, T]$, $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\begin{aligned} |b(t, x, y_1) - b(t, x, y_2)|^2 &\leq L|y_1 - y_2|^2 \\ \|\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)\|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ |f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2 + \|z_1 - z_2\|^2) \\ |h(x_1) - h(x_2)|^2 &\leq L|x_1 - x_2|^2. \end{aligned}$$

(B.4): There exists a constant $\mu > 0$ with $2\mu - K - L - \max\{4L + 1, 2L^2\} > 0$, where $K > 2L^2 + 7L + 1$ such that for any $t \in [0, T]$, $x_1, x_2, x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{k \times d}$

$$\langle x_1 - x_2, b(t, x_1, y) - b(t, x_2, y) \rangle \leq -\mu|x_1 - x_2|^2,$$

$$|b(t, x, 0, 0)|^2 \leq L(1 + |x|^2).$$

And the function $u \mapsto b(t, u, y, z)$ is continuous;

(B.5): Moreover, the following holds

$$\int_0^T (|b(s, 0, 0)|^p + \|\sigma(s, 0, 0)\|^p + |f(s, 0, 0, 0)|^p) ds < \infty.$$

Notation 2.1.1. To simplify our notation, we denote these two classes of conditions as follows

$$(\mathbf{B. Class 1}) := (\mathbf{B.0}) + (\mathbf{B.1}) + (\mathbf{B.2}) + (\mathbf{B.5})$$

$$(\mathbf{B. Class 2}) := (\mathbf{B.0}) + (\mathbf{B.3}) + (\mathbf{B.4}) + (\mathbf{B.5})$$

Remark 2.1.2. In this chapter, we strengthen our conditions in L^2_ρ space (in Section 1.3) to pointwise ones. The corresponding existence and uniqueness results still hold: for any $t \in [0, T]$, $x \in \mathbb{R}^d$, FBSDEs (2.1.4) has a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x}) \in S^2([0, T]; \mathbb{R}^d) \otimes S^2([0, T]; \mathbb{R}^k) \otimes M^2([0, T]; \mathbb{R}^{k \times d})$.

2.2 Regularity of the Solution of The FBSDEs

2.2.1 Dependence upon the Initial Conditions

Lemma 2.2.1. Under Conditions **(B.1)**, **(B.2)** and **(A.2.5)** (or **(B.3)**, **(B.4)** and **(A.2.5)**), (2.1.4) has a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$. Moreover, there exists a constant $p > 2$, and a constant $C_{p,L,\mu,T}$ only depending on p , L , μ and T such that

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} \leq C_{p,L,\mu,T} (1 + |x|^p). \quad (2.2.1)$$

Proof. In this chapter, we strengthen the condition from L^2_ρ space to L^2 space. Therefore, by using a similar method to that of the proof of Theorem 1.3.3 (or Theorem 1.3.4), it is easy to see that, for any $t \in [0, T]$, $x \in \mathbb{R}^d$, FBSDEs (2.1.4) has a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x}) \in S^2([0, T]; \mathbb{R}^d) \otimes S^2([0, T]; \mathbb{R}^k) \otimes M^2([0, T]; \mathbb{R}^{k \times d})$. In the following, we only consider Conditions **(B.1)**, **(B.2)** and **(A.2.5)**. But our result still holds under Conditions **(B.3)**, **(B.4)** and **(A.2.5)**.

Step 1: For any $p > 2$, we apply Itô's formula to $(|X_r^{t,x}|^2)^{\frac{p}{2}}$, yielding

$$\begin{aligned}
 |X_s^{t,x}|^p &= |x|^p + p \int_t^s |X_r^{t,x}|^{p-2} \langle X_r^{t,x}, b(r, X_r^{t,x}, Y_r^{t,x}) \rangle dr \\
 &\quad + \frac{p}{2} \int_t^s |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &\quad + \frac{p}{2}(p-2) \int_t^s |X_r^{t,x}|^{p-4} \langle \sigma \sigma^*(r, X_r^{t,x}, Y_r^{t,x}) X_r^{t,x}, X_r^{t,x} \rangle dr \\
 &\quad + p \int_t^s |X_r^{t,x}|^{p-2} \langle X_r^{t,x}, \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \rangle. \tag{2.2.2}
 \end{aligned}$$

As the stochastic integral has zero expectation, so

$$\begin{aligned}
 \mathbb{E}|X_s^{t,x}|^p &\leq |x|^p + p \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} |\langle X_r^{t,x}, b(r, X_r^{t,x}, Y_r^{t,x}) \rangle| dr \\
 &\quad + \frac{p}{2} \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &\quad + \frac{p}{2}(p-2) \mathbb{E} \int_t^T |X_r^{t,x}|^{p-4} |\langle \sigma \sigma^*(r, X_r^{t,x}, Y_r^{t,x}) X_r^{t,x}, X_r^{t,x} \rangle| dr. \tag{2.2.3}
 \end{aligned}$$

Moreover, it is easy to see that

$$\begin{aligned}
 \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p &\leq |x|^p + p \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} |\langle X_r^{t,x}, b(r, X_r^{t,x}, Y_r^{t,x}) \rangle| dr \\
 &\quad + \frac{p}{2} \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &\quad + \frac{p}{2}(p-2) \mathbb{E} \int_t^T |X_r^{t,x}|^{p-4} |\langle \sigma \sigma^*(r, X_r^{t,x}, Y_r^{t,x}) X_r^{t,x}, X_r^{t,x} \rangle| dr \\
 &\quad + p \mathbb{E} \sup_{t \leq s \leq T} \left| \int_t^s |X_r^{t,x}|^{p-2} \langle X_r^{t,x}, \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \rangle \right|. \tag{2.2.4}
 \end{aligned}$$

By Young's inequality, the second term on the RHS in inequality (2.2.4) is estimated as

$$\begin{aligned}
 &p \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} |\langle X_r^{t,x}, b(r, X_r^{t,x}, Y_r^{t,x}) \rangle| dr \\
 &\leq p \mathbb{E} \int_t^T (|X_r^{t,x}|^{p-2} |\langle X_r^{t,x}, b(r, X_r^{t,x}, Y_r^{t,x}) - b(r, 0, 0) \rangle| + |X_r^{t,x}|^{p-2} |\langle X_r^{t,x}, b(r, 0, 0) \rangle|) dr \\
 &\leq p \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \left(|X_r^{t,x}|^2 |b(r, X_r^{t,x}, Y_r^{t,x}) - b(r, 0, 0)|^2 \right)^{\frac{1}{2}} dr \\
 &\quad + p \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \left(|X_r^{t,x}|^2 |b(r, 0, 0)|^2 \right)^{\frac{1}{2}} dr \\
 &\leq p \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \left(L |X_r^{t,x}|^2 (|X_r^{t,x}|^2 + |Y_r^{t,x}|^2) \right)^{\frac{1}{2}} dr \\
 &\quad + p \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \left(|X_r^{t,x}|^2 |b(r, 0, 0)|^2 \right)^{\frac{1}{2}} dr \\
 &\leq p \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \left(\frac{L}{2} |X_r^{t,x}|^2 + \frac{1}{2} |X_r^{t,x}|^2 + \frac{1}{2} |Y_r^{t,x}|^2 \right) dr \\
 &\quad + C_p \mathbb{E} \int_t^T \left(|X_r^{t,x}|^p + |b(r, 0, 0)|^p \right) dr
 \end{aligned}$$

$$\begin{aligned}
 &\leq \mathbb{E} \int_t^T \left(\frac{pL}{2} |X_r^{t,x}|^p + \frac{p}{2} |X_r^{t,x}|^p + \frac{p-2}{2} |X_r^{t,x}|^p + |Y_r^{t,x}|^p \right) dr \\
 &\quad + C_p \mathbb{E} \int_t^T \left(|X_r^{t,x}|^p + |b(r, 0, 0)|^p \right) dr \\
 &\leq C_{p,L} \mathbb{E} \int_t^T \left(|X_r^{t,x}|^p + |Y_r^{t,x}|^p \right) dr + C_p \mathbb{E} \int_t^T |b(r, 0, 0)|^p dr.
 \end{aligned}$$

Similarly, the third and the fourth terms in (2.2.4) can also be estimated as

$$\begin{aligned}
 &\frac{p}{2} \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &\quad + \frac{p}{2} (p-2) \mathbb{E} \int_t^T |X_r^{t,x}|^{p-4} |\langle \sigma \sigma^*(r, X_r^{t,x}, Y_r^{t,x}) X_r^{t,x}, X_r^{t,x} \rangle| dr \\
 &= \frac{p(p-1)}{2} \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &\leq p(p-1) \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, 0, 0)\|^2 dr \\
 &\quad + p(p-1) \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, 0, 0)\|^2 dr \\
 &\leq 2(p-1)^2 L \mathbb{E} \int_t^T |X_r^{t,x}|^p dr + 2(p-1)L \mathbb{E} \int_t^T |Y_r^{t,x}|^p dr \\
 &\quad + (p-1)(p-2) \mathbb{E} \int_t^T |X_r^{t,x}|^p dr + 2(p-1) \mathbb{E} \int_t^T \|\sigma(r, 0, 0)\|^p dr \\
 &\leq C_{p,L} \mathbb{E} \int_t^T \left(|X_r^{t,x}|^p + |Y_r^{t,x}|^p \right) dr + C_p \mathbb{E} \int_t^T \|\sigma(r, 0, 0)\|^p dr.
 \end{aligned}$$

For the last term in (2.2.4), we use the Burkholder-Davis-Gundy inequality and estimate of $\mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr$ above

$$\begin{aligned}
 &p \mathbb{E} \sup_{t \leq s \leq T} \left| \int_t^s |X_r^{t,x}|^{p-2} \langle X_r^{t,x}, \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \rangle \right| \\
 &\leq C_p \mathbb{E} \left| \int_t^T |X_r^{t,x}|^p |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \right|^{\frac{1}{2}} \\
 &\leq C_p \mathbb{E} \left| \sup_{0 \leq r \leq T} |X_r^{t,x}|^p \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \right|^{\frac{1}{2}} \\
 &\leq \frac{C_p}{N} \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + \frac{C_p N}{4} \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &\leq \frac{C_p}{N} \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + C_{L,p,N} \mathbb{E} \int_t^T \left(|X_r^{t,x}|^p + |Y_r^{t,x}|^p \right) dr + C_p \mathbb{E} \int_t^T \|\sigma(r, 0, 0)\|^p dr.
 \end{aligned}$$

Here we can choose N such that $\frac{C_p}{N} < \frac{1}{2}$. Therefore, from (2.2.3) and (2.2.4) we have

$$\begin{aligned}
 \mathbb{E} |X_s^{t,x}|^p &\leq |x|^p + C_L \mathbb{E} \int_t^T \left(|X_r^{t,x}|^p + |Y_r^{t,x}|^p \right) dr \\
 &\quad + C_p \mathbb{E} \int_t^T \left(|b(r, 0, 0)|^p + \|\sigma(r, 0, 0)\|^p \right) dr
 \end{aligned}$$

$$\leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \quad (2.2.5)$$

And

$$\mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \quad (2.2.6)$$

Step 2: We apply Itô's formula to $(|Y_r^{t,x}|^2)^{\frac{p}{2}}$ then we have

$$\begin{aligned} & |Y_s^{t,x}|^p + \frac{p}{2} \int_s^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr + \frac{p}{2}(p-2) \int_s^T |Y_r^{t,x}|^{p-4} \langle Z_r^{t,x} (Z_r^{t,x})^* Y_r^{t,x}, Y_r^{t,x} \rangle dr \\ = & |h(X_T^{t,x})|^p + p \int_s^T |Y_r^{t,x}|^{p-2} \langle Y_r^{t,x}, f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \rangle dr \\ & - p \int_s^T |Y_r^{t,x}|^{p-2} \langle Y_r^{t,x}, Z_r^{t,x} dW_r \rangle. \end{aligned} \quad (2.2.7)$$

Since the stochastic integral has zero expectation, we have

$$\begin{aligned} & \mathbb{E} |Y_s^{t,x}|^p + \frac{p}{2} \mathbb{E} \int_s^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr + \frac{p}{2}(p-2) \mathbb{E} \int_s^T |Y_r^{t,x}|^{p-4} \langle Z_r^{t,x} (Z_r^{t,x})^* Y_r^{t,x}, Y_r^{t,x} \rangle dr \\ \leq & \mathbb{E} |h(X_T^{t,x})|^p + p \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} |\langle Y_r^{t,x}, f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \rangle| dr. \end{aligned} \quad (2.2.8)$$

The first term on the RHS in (2.2.8) is estimated as

$$\begin{aligned} \mathbb{E} |h(X_T^{t,x})|^p & \leq \mathbb{E} \left(2|h(X_T^{t,x}) - h(0)|^2 + 2|h(0)|^2 \right)^{\frac{p}{2}} \\ & \leq \mathbb{E} \left(2L|X_T^{t,x}|^2 + 2|h(0)|^2 \right)^{\frac{p}{2}} \\ & \leq \mathbb{E} \left(2L \sup_{t \leq s \leq T} |X_s^{t,x}|^2 + 2|h(0)|^2 \right)^{\frac{p}{2}} \\ & \leq C_{p,L} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p + C_p |h(0)|^p \\ & \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right), \end{aligned}$$

where the last inequality is from (2.2.6). Similarly, the second term can be estimated as

$$\begin{aligned} & p \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} |\langle Y_r^{t,x}, f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \rangle| dr \\ \leq & p \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} |\langle Y_r^{t,x}, f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - f(r, X_r^{t,x}, 0, Z_r^{t,x}) \rangle| dr \\ & + p \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} |\langle Y_r^{t,x}, f(r, X_r^{t,x}, 0, Z_r^{t,x}) - f(r, 0, 0, 0) \rangle| dr \\ & + p \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} |\langle Y_r^{t,x}, f(r, 0, 0, 0) \rangle| dr \\ \leq & -p\mu \mathbb{E} \int_t^T |Y_r^{t,x}|^p dr + p \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \left(2L|Y_r^{t,x}|^2 + \frac{1}{8} (|X_r^{t,x}|^2 + \|Z_r^{t,x}\|^2) \right) dr \end{aligned}$$

$$\begin{aligned}
 & +(p-1)\mathbb{E} \int_t^T |Y_r^{t,x}|^p dr + \frac{1}{p}\mathbb{E} \int_t^T |f(r, 0, 0, 0)|^p dr \\
 \leq & 2pL\mathbb{E} \int_t^T |Y_r^{t,x}|^p dr + \frac{p-2}{8}\mathbb{E} \int_t^T |Y_r^{t,x}|^p dr + \frac{1}{4}\mathbb{E} \int_t^T |X_r^{t,x}|^p dr \\
 & + \frac{p}{8}\mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr + (p-1)\mathbb{E} \int_t^T |Y_r^{t,x}|^p dr + C_p\mathbb{E} \int_t^T |f(r, 0, 0, 0)|^p dr \\
 \leq & C_{p,L} \left(1 + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right) + \frac{p}{8}\mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr.
 \end{aligned}$$

From (2.2.8) we have,

$$\mathbb{E}|Y_s^{t,x}|^p \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right) + \frac{p}{8}\mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr.$$

Taking $s = t$ in (2.2.8) and by the same estimate we have,

$$\begin{aligned}
 & \frac{p}{2}\mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \\
 \leq & C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right) + \frac{p}{8}\mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr.
 \end{aligned}$$

From above two inequalities,

$$\begin{aligned}
 & \mathbb{E}|Y_s^{t,x}|^p + \frac{p}{4}\mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \\
 \leq & 2C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \tag{2.2.9}
 \end{aligned}$$

Then we calculate $\mathbb{E} \sup_{t \leq s \leq T} |Y_r^{t,x}|^p$ from (2.2.7).

$$\begin{aligned}
 & \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p \\
 \leq & \mathbb{E}|h(X_T^{t,x})|^p + p\mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} |\langle Y_r^{t,x}, f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \rangle| dr \\
 & + p\mathbb{E} \sup_{t \leq s \leq T} \left| \int_s^T |Y_r^{t,x}|^{p-2} \langle Y_r^{t,x}, Z_r^{t,x} dW_r \rangle \right|. \tag{2.2.10}
 \end{aligned}$$

It is easy to see that the first two terms in (2.2.10) are the same as those in inequality (2.2.8). So the rest of the work is to deal with the last two stochastic integrals. By the Burkholder-Davis-Gundy inequality, the Young inequality and estimate (2.2.9) we have

$$\begin{aligned}
 & p\mathbb{E} \sup_{t \leq s \leq T} \left| \int_s^T |Y_r^{t,x}|^{p-2} \langle Y_r^{t,x}, Z_r^{t,x} dW_r \rangle \right| \\
 \leq & C_p\mathbb{E} \left| \int_t^T |Y_r^{t,x}|^p |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \right|^{\frac{1}{2}} \\
 \leq & C_p\mathbb{E} \left| \sup_{t \leq s \leq T} |Y_s^{t,x}|^p \int_t^T |Y_s^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \right|^{\frac{1}{2}} \\
 \leq & \frac{C_p}{N}\mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p + \frac{C_p N}{4}\mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr
 \end{aligned}$$

$$\leq \frac{C_p}{N} \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p + C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right).$$

Here we can choose N such that $\frac{C_p}{N} < \frac{1}{3}$. Then (2.2.10) immediate leads to

$$\mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \quad (2.2.11)$$

Step 3: From (2.2.6) and (2.2.11)

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p \\ & \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \end{aligned} \quad (2.2.12)$$

Note we can not use Gronwall's inequality here. To estimate $\mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr$, we let $\varphi_{N,p}(y) = y^{\frac{p}{2}} I_{\{0 \leq y \leq N\}} + \frac{p}{2} N^{\frac{p-2}{2}} (y-N) I_{\{y \geq N\}}$. We apply *Itô's* formula to $e^{-Kr} \varphi_{N,p}(\psi_M(X_r^{t,x}))$ and $e^{-Kr} \varphi_{N,p}(\psi_M(Y_r^{t,x}))$ to have the following estimation, where $\psi_M(x) = x^2 I_{\{-M \leq x < M\}} + 2M(x-M) I_{\{x \geq M\}} - 2M(x+M) I_{\{x < -M\}}$.

$$\begin{aligned} & e^{-KT} \varphi_{N,p}(\psi_M(X_T^{t,x})) + K \int_t^T e^{-Kr} \varphi_{N,p}(\psi_M(X_r^{t,x})) dr \\ & = e^{-Kt} \varphi_{N,p}(\psi_M(X_t^{t,x})) + \int_t^T e^{-Kr} \langle \varphi'_{N,p}(\psi_M(X_r^{t,x})) \psi'_M(X_r^{t,x}), b(r, X_r^{t,x}, Y_r^{t,x}) \rangle dr \\ & \quad + \int_t^T e^{-Kr} \langle \varphi'_{N,p}(\psi_M(X_r^{t,x})) \psi'_M(X_r^{t,x}), \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \rangle \\ & \quad + \frac{1}{2} \int_t^T e^{-Kr} \varphi''_{N,p}(\psi_M(X_r^{t,x})) |\psi'_M(X_r^{t,x})|^2 \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\ & \quad + \int_t^T e^{-Kr} \varphi'_{N,p}(\psi_M(X_r^{t,x})) I_{\{-M \leq X_r^{t,x} \leq M\}} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr. \end{aligned}$$

And

$$\begin{aligned} & e^{-Kt} \varphi_{N,p}(\psi_M(Y_t^{t,x})) - K \int_t^T e^{-Kr} \varphi_{N,p}(\psi_M(Y_r^{t,x})) dr \\ & \quad + \frac{1}{2} \int_t^T e^{-Kr} \varphi''_{N,p}(\psi_M(Y_r^{t,x})) |\psi'_M(Y_r^{t,x})|^2 \|Z_r^{t,x}\|^2 dr \\ & \quad + \int_t^T e^{-Kr} \varphi'_{N,p}(\psi_M(Y_r^{t,x})) I_{\{-M \leq Y_r^{t,x} \leq M\}} \|Z_r^{t,x}\|^2 dr \\ & = e^{-KT} \varphi_{N,p}(\psi_M(h(X_t^{t,x}))) + \int_t^T e^{-Kr} \langle \varphi'_{N,p}(\psi_M(Y_r^{t,x})) \psi'_M(Y_r^{t,x}), f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \rangle dr \\ & \quad - \int_t^T e^{-Kr} \langle \varphi'_{N,p}(\psi_M(Y_r^{t,x})) \psi'_M(Y_r^{t,x}), Z_r^{t,x} dW_r \rangle. \end{aligned}$$

Note that the stochastic integral has zero expectation. We can take the limit as $M \rightarrow \infty$ first, then the limit as $N \rightarrow \infty$, by monotone convergence theorem, we have

$$\mathbb{E} e^{-KT} |X_T^{t,x}|^p + K \mathbb{E} \int_t^T e^{-Kr} |X_r^{t,x}|^p dr$$

$$\begin{aligned}
 &= e^{-Kt}|x|^p + p\mathbb{E} \int_t^T e^{-Kr}|X_r^{t,x}|^{p-2} \langle X_r^{t,x}, b(r, X_r^{t,x}, Y_r^{t,x}) \rangle dr \\
 &\quad + \frac{1}{2}p(p-1)\mathbb{E} \int_t^T e^{-Kr}|X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr, \tag{2.2.13}
 \end{aligned}$$

and

$$\begin{aligned}
 &\mathbb{E}e^{-Kt}|Y_t^{t,x}|^p - K\mathbb{E} \int_t^T e^{-Kr}|Y_r^{t,x}|^p dr \\
 &\quad + \frac{1}{2}p(p-1)\mathbb{E} \int_t^T e^{-Kr}|Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \tag{2.2.14} \\
 &= \mathbb{E}e^{-KT}|h(X_T^{t,x})|^p + p\mathbb{E} \int_t^T e^{-Kr}|Y_r^{t,x}|^{p-2} \langle Y_r^{t,x}, f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \rangle dr.
 \end{aligned}$$

Now we denote

$$\begin{aligned}
 \gamma &:= p\mu - K - 4pL - \frac{p-2}{16} - \varepsilon - L(p-1)^2(1+\varepsilon) - \frac{1}{8} - L(p-1)(1+\varepsilon) \\
 &\quad - \left[\frac{1}{4L} + L(p-1)(1+\varepsilon) \right] (1+\varepsilon)L^{\frac{p}{2}}, \\
 \beta &:= K - 4pL - \frac{p}{8} + \frac{1}{8} - \varepsilon - L(p-1)^2(1+\varepsilon) - \frac{1}{8} - L(p-1)(1+\varepsilon) \\
 &\quad - \left[2pL^2 + \frac{p}{4L} - \frac{1}{4L} + \varepsilon + L(p-1)^2(1+\varepsilon) \right] (1+\varepsilon)L^{\frac{p}{2}},
 \end{aligned}$$

For both (2.2.13) and (2.2.14), using a similar method as in the proof of Theorem 1.3.3, we have

$$\begin{aligned}
 &\gamma\mathbb{E} \int_t^T e^{-Kr}|X_r^{t,x}|^p dr + \beta\mathbb{E} \int_t^T e^{-Kr}|Y_r^{t,x}|^p dr \\
 &\quad + \left(\frac{1}{2}p(p-1) - \frac{p}{16} \right) \mathbb{E} \int_t^T e^{-Kr}|Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \\
 &\quad + \mathbb{E}e^{-KT}|X_T^{t,x}|^p + \mathbb{E}e^{-Kt}|Y_t^{t,x}|^p \\
 &\leq C_{p,L}e^{-Kt}|x|^p + C_{p,L} \int_t^T e^{-Kr} dr. \tag{2.2.15}
 \end{aligned}$$

Here $\frac{1}{2}p(p-1) - \frac{p}{16} > 0$. Moreover, if we assume that $2\mu - K - L^2 - 10L - 1 > 0$, where $K - 4L^3 - L^2 - 10L - 1 > 0$, then there exists a constant $p \in (2, \infty)$ such that $\gamma, \beta > 0$.

Note that (2.2.15) immediate leads to

$$\mathbb{E} \int_t^T e^{-Kr}|X_r^{t,x}|^p dr + \mathbb{E} \int_t^T e^{-Kr}|Y_r^{t,x}|^p dr \leq C_{p,L,\mu}e^{-Kt}|x|^p + C_{p,L,\mu} \int_t^T e^{-Kr} dr.$$

Note that

$$e^{-KT}\mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \leq \mathbb{E} \int_t^T e^{-Kr} (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr.$$

So we have

$$\mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \leq C_{p,L,\mu}e^{-K(t-T)}|x|^p + C_{p,L,\mu}e^{KT} \int_t^T e^{-Kr} dr$$

$$\leq C_{p,L,\mu,T}(1 + |x|^p). \quad (2.2.16)$$

From (2.2.12) and (2.2.16) we have

$$\begin{aligned} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p &\leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right) \\ &\leq C_{p,L,\mu,T}(1 + |x|^p). \end{aligned}$$

Next, following a similar procedure as in Step 6 of the proof of Theorem 1.2.3, we can extend our result from $s \in [t, T]$ to $s \in [0, T]$ so that

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x}|^p \leq C_{p,L,\mu,T}(1 + |x|^p).$$

Step 4: Moreover,

$$\begin{aligned} &\int_0^T \|Z_r^{t,x}\|^2 dr \\ &= |h(X_T^{t,x})|^2 - |Y_0^{t,x}|^2 + 2 \int_0^T \langle f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), Y_r^{t,x} \rangle dr - 2 \int_0^T \langle Z_r^{t,x}, Y_r^{t,x} dW_r \rangle. \end{aligned}$$

Hence

$$\begin{aligned} &\left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} \\ &\leq C_p \left(|h(X_T^{t,x})|^p + |Y_0^{t,x}|^p + \left| \int_0^T \langle Z_r^{t,x}, Y_r^{t,x} dW_r \rangle \right|^{\frac{p}{2}} \right. \\ &\quad \left. + \left| \int_0^T \langle f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), Y_r^{t,x} \rangle dr \right|^{\frac{p}{2}} \right). \end{aligned}$$

Note that

$$\begin{aligned} &C_p \mathbb{E} \left| \int_0^T \langle f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), Y_r^{t,x} \rangle dr \right|^{\frac{p}{2}} \\ &\leq C_p \mathbb{E} \left| \int_0^T \left(-\mu |Y_r^{t,x}|^2 + [L |Y_r^{t,x}|^2 (|X_r^{t,x}|^2 + \|Z_r^{t,x}\|^2)]^{\frac{1}{2}} + |Y_r^{t,x}|^2 + |f(r, 0, 0, 0)|^2 \right) dr \right|^{\frac{p}{2}} \\ &\leq C_{p,L,\mu} \int_0^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr + \frac{1}{4} \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} + C_p \int_0^T |f(r, 0, 0, 0)|^p dr. \end{aligned}$$

And

$$\begin{aligned} &C_p \mathbb{E} \left| \int_0^T \langle Z_r^{t,x}, Y_r^{t,x} dW_r \rangle \right|^{\frac{p}{2}} \\ &\leq C_p \mathbb{E} \left| \int_0^T \|Z_r^{t,x}\|^2 |Y_r^{t,x}|^2 dr \right|^{\frac{p}{4}} \end{aligned}$$

$$\begin{aligned}
 &\leq C_p \mathbb{E} \left| \sup_{0 \leq s \leq T} |Y_s^{t,x}|^2 \int_0^T \|Z_r^{t,x}\|^2 dr \right|^{\frac{p}{4}} \\
 &\leq C_p \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x}|^p + \frac{1}{4} \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}}
 \end{aligned}$$

As a result, we have

$$\mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} \leq C_{p,L,\mu,T} (1 + |x|^p)$$

Eventually, we have our claim

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} \leq C_{p,L,\mu,T} (1 + |x|^p).$$

□

Remark 2.2.2. We can also use the Gronwall inequality to obtain the same result of (2.2.1). But the key estimate to make it work is (2.2.15). We can rewrite the FBSDEs as follows,

$$\begin{cases} X_T^{t,x} = X_s^{t,x} + \int_s^T b(r, X_r^{t,x}, Y_r^{t,x}) dr + \int_s^T \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \quad 0 \leq s \leq T. \end{cases}$$

Note that the forward SDE is from s to T . We apply Itô's formula to $(|X_r^{t,x}|^2)^{\frac{p}{2}}$ and $(|Y_r^{t,x}|^2)^{\frac{p}{2}}$ from s to T , and use a similar approach as in the proof in Lemma 2.2.1 to obtain

$$\begin{aligned}
 &\mathbb{E}|X_s^{t,x}|^p + \mathbb{E}|Y_s^{t,x}|^p + \mathbb{E} \int_s^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \\
 &\leq C_{p,L} \left(1 + |x|^p + \mathbb{E}|X_T^{t,x}|^p + \mathbb{E} \int_s^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right).
 \end{aligned}$$

To estimate $\mathbb{E}|X_T^{t,x}|^p$, following (2.2.15) we have

$$\mathbb{E} e^{-KT} |X_T^{t,x}|^p \leq C_{p,L,\mu} e^{-Kt} |x|^p + C_{p,L,\mu} \int_t^T e^{-Kr} dr,$$

which leads to

$$\mathbb{E}|X_T^{t,x}|^p \leq C_{p,L,\mu} e^{-K(t-T)} |x|^p + C_{p,L,\mu} e^{KT} \int_t^T e^{-Kr} dr \leq C_{L,\mu} (1 + |x|^p).$$

Therefore we have

$$\mathbb{E}|X_s^{t,x}|^p + \mathbb{E}|Y_s^{t,x}|^p \leq C_{p,L,\mu,T} \left(1 + |x|^p + \mathbb{E} \int_s^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right).$$

By the Gronwall inequality, we have

$$\mathbb{E}|X_s^{t,x}|^p + \mathbb{E}|Y_s^{t,x}|^p \leq C_{p,L,\mu,T} (1 + |x|^p).$$

And the rest of the proof is exactly the same as that in Lemma 2.2.1.

Lemma 2.2.3. *Under Conditions (B.1), (B.2) and (A.2.5) (or (B.3), (B.4) and (A.2.5)), for any $t, t' \in [0, T]$, $x, x' \in \mathbb{R}^d$, $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$ and $(X_s^{t',x'}, Y_s^{t',x'}, Z_s^{t',x'})_{0 \leq s \leq T}$ stand for the solutions of (2.1.4) associated to the initial conditions (t, x) and (t', x') . Moreover, there exists a constant $p > 2$, and a constant $C_{p,L,\mu,T}$ only depending on p, L, μ and T such that*

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr \right)^{\frac{p}{2}} \\ & \leq C_{p,L,\mu,T} |x - x'|^p + C_{p,L,\mu,T} (1 + |x|^p + |x'|^p) |t - t'|^{\frac{p}{2}}. \end{aligned} \quad (2.2.17)$$

Proof. In the following, we only consider Conditions (B.1), (B.2) and (A.2.5). The result also holds for Conditions (B.3), (B.4) and (A.2.5) as well. From Lemma 2.2.1, it is clear that $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$ and $(X_s^{t',x'}, Y_s^{t',x'}, Z_s^{t',x'})_{0 \leq s \leq T}$ are solutions to the FBSDEs (2.1.4) associated to the initial conditions (t, x) and (t', x') respectively. Now we assume $t \leq t'$, then we consider the difference,

$$\left\{ \begin{aligned} X_s^{t,x} - X_s^{t',x'} &= x - x' + \int_t^{t'} b(r, X_r^{t,x}, Y_r^{t,x}) dr + \int_t^{t'} \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \\ &\quad + \int_{t'}^s (b(r, X_r^{t,x}, Y_r^{t,x}) - b(r, X_r^{t',x'}, Y_r^{t',x'})) dr \\ &\quad + \int_{t'}^s (\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'})) dW_r \\ Y_s^{t,x} - Y_s^{t',x'} &= h(X_T^{t,x}) - h(X_T^{t',x'}) - \int_s^T (Z_r^{t,x} - Z_r^{t',x'}) dW_r \\ &\quad + \int_s^T (f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - f(r, X_r^{t',x'}, Y_r^{t',x'}, Z_r^{t',x'})) dr. \end{aligned} \right.$$

Step 1: For any $p > 2$, we apply Itô's formula to $(|X_r^{t,x} - X_r^{t',x'}|^2)^{\frac{p}{2}}$ then we have

$$|X_s^{t,x} - X_s^{t',x'}|^p$$

$$\begin{aligned}
 &= |x - x'|^p + p \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, b(r, X_r^{t,x}, Y_r^{t,x}) \right\rangle dr \\
 &+ p \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, b(r, X_r^{t,x}, Y_r^{t,x}) - b(r, X_r^{t',x'}, Y_r^{t',x'}) \right\rangle dr \\
 &+ \frac{p}{2}(p-1) \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &+ \frac{p}{2}(p-1) \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'})\|^2 dr \\
 &+ p \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \right\rangle \\
 &+ p \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, \left(\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'}) \right) dW_r \right\rangle. \tag{2.2.18}
 \end{aligned}$$

We want to estimate $\mathbb{E}|X_s^{t,x} - X_s^{t',x'}|^p$ and $\mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p$ from above equation. First we estimate the third, the fifth and the seventh terms on the RHS of (2.2.18). By a similar method in the proof of Lemma 2.2.1, it is trivial to see that the third term can be written as

$$\begin{aligned}
 &p \mathbb{E} \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, b(r, X_r^{t,x}, Y_r^{t,x}) - b(r, X_r^{t',x'}, Y_r^{t',x'}) \right\rangle dr \\
 &\leq C_{p,L} \mathbb{E} \int_t^T \left(|X_r^{t,x} - X_r^{t',x'}|^p + |Y_r^{t,x} - Y_r^{t',x'}|^p \right) dr.
 \end{aligned}$$

Similarly, the fifth term has following estimate

$$\begin{aligned}
 &\frac{p(p-1)}{2} \mathbb{E} \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'})\|^2 dr \\
 &\leq C_{p,L} \mathbb{E} \int_t^T \left(|X_r^{t,x} - X_r^{t',x'}|^p + |Y_r^{t,x} - Y_r^{t',x'}|^p \right) dr.
 \end{aligned}$$

Moreover, the seventh term can be estimated as

$$\begin{aligned}
 &p \mathbb{E} \sup_{t \leq s \leq T} \left| \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \right. \\
 &\quad \left. \left\langle X_r^{t,x} - X_r^{t',x'}, \left(\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'}) \right) dW_r \right\rangle \right| \\
 &\leq \frac{C_p}{N} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \frac{C_p N}{4} \mathbb{E} \int_t^T |X_r^{t,x} - X_r^{t',x'}|^{p-2} \\
 &\quad \|\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'})\|^2 dr \\
 &\leq \frac{C_p}{N} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_{L,p} \mathbb{E} \int_t^T \left(|X_r^{t,x} - X_r^{t',x'}|^p + |Y_r^{t,x} - Y_r^{t',x'}|^p \right) dr.
 \end{aligned}$$

Here we can choose N such that $\frac{C_p}{N} < \frac{1}{8}$. For the the second, the fourth and the sixth terms on the RHS in (2.2.18), we need the following estimates. Applying Lemma 2.2.1 and Cauchy-Schwarz inequality, we have

$$\mathbb{E} \left(\int_t^{t'} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \right)^{\frac{p}{2}}$$

$$\begin{aligned}
 &\leq \mathbb{E} \left(\int_t^{t'} 2\|\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, 0, 0)\|^2 dr + \int_t^{t'} 2\|\sigma(r, 0, 0)\|^2 dr \right)^{\frac{p}{2}} \\
 &\leq \mathbb{E} \left(\int_t^{t'} C_L(1 + |X_r^{t,x}|^2 + |Y_r^{t,x}|^2) dr \right)^{\frac{p}{2}} \\
 &\leq C_{p,L} \mathbb{E} \left(1 + \sup_{t \leq s \leq T} |X_s^{t,x}|^2 + \sup_{t \leq s \leq T} |Y_s^{t,x}|^2 \right)^{\frac{p}{2}} \left(\int_t^{t'} dr \right)^{\frac{p}{2}} \\
 &\leq C_{p,L} \left(1 + \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p \right) |t - t'|^{\frac{p}{2}} \\
 &\leq C_{p,L,\mu,T}(1 + |x|^p)|t - t'|^{\frac{p}{2}}. \tag{2.2.19}
 \end{aligned}$$

Similarly

$$\mathbb{E} \left(\int_t^{t'} |b(r, X_r^{t,x}, Y_r^{t,x})| dr \right)^p \leq C_{p,L,\mu,T}(1 + |x|^p)|t - t'|^{\frac{p}{2}}. \tag{2.2.20}$$

Now we estimate the second, the fourth and the sixth terms on the RHS in (2.2.18). By using Young's inequality, results of (2.2.19) and (2.2.20), the second term can be estimated as

$$\begin{aligned}
 &p \mathbb{E} \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, b(r, X_r^{t,x}, Y_r^{t,x}) \right\rangle dr \\
 &\leq p \mathbb{E} \left(\int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-1} |b(r, X_r^{t,x}, Y_r^{t,x})| dr \right) \\
 &\leq p \mathbb{E} \left(\left[\frac{1}{N} \sup_{t \leq r \leq T} |X_r^{t,x} - X_r^{t',x'}|^{p-1} \right] \left[N \int_t^{t'} |b(r, X_r^{t,x}, Y_r^{t,x})| dr \right] \right) \\
 &\leq (p-1) \left(\frac{1}{N} \right)^{\frac{p}{p-1}} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + N^p \mathbb{E} \left(\int_t^{t'} |b(r, X_r^{t,x}, Y_r^{t,x})| \right)^p \\
 &\leq \frac{1}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_{p,L,\mu,T}(1 + |x|^p)|t - t'|^{\frac{p}{2}}.
 \end{aligned}$$

Here we can choose N big enough such that $(p-1)\left(\frac{1}{N}\right)^{\frac{p}{p-1}} < \frac{1}{8}$. Similarly, the fourth term is

$$\begin{aligned}
 &\frac{p}{2}(p-1) \mathbb{E} \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &\leq \frac{1}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_p \mathbb{E} \left(\int_t^{t'} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \right)^{\frac{p}{2}} \\
 &\leq \frac{1}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_{p,L,\mu,T}(1 + |x|^p)|t - t'|^{\frac{p}{2}}.
 \end{aligned}$$

And also the sixth term can be written as

$$p \mathbb{E} \sup_{t \leq t' \leq T} \left| \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \right\rangle \right|$$

$$\begin{aligned}
 &\leq \frac{1}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_p \mathbb{E} \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\
 &\leq \frac{1}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \frac{1}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_p \mathbb{E} \left(\int_t^{t'} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \right)^{\frac{p}{2}} \\
 &\leq \frac{2}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_{p,L,\mu,T} (1 + |x|^p) |t - t'|^{\frac{p}{2}}.
 \end{aligned}$$

Finally, from (2.2.18) we have

$$\begin{aligned}
 \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p &\leq |x - x'|^p + C_{p,L,\mu,T} (1 + |x|^p + |x'|^p) |t - t'|^{\frac{p}{2}} \quad (2.2.21) \\
 &\quad + C_{p,L} \mathbb{E} \int_t^T \left(|X_r^{t,x} - X_r^{t',x'}|^p + |Y_r^{t,x} - Y_r^{t',x'}|^p \right) dr.
 \end{aligned}$$

Similarly, since the stochastic integrals have zero expectation, from (2.2.18) and estimates all above we also have

$$\begin{aligned}
 \mathbb{E} |X_s^{t,x} - X_s^{t',x'}|^p &\leq |x - x'|^p + C_{p,L,\mu,T} (1 + |x|^p + |x'|^p) |t - t'|^{\frac{p}{2}} \\
 &\quad + C_{p,L} \mathbb{E} \int_t^T \left(|X_r^{t,x} - X_r^{t',x'}|^p + |Y_r^{t,x} - Y_r^{t',x'}|^p \right) dr. \quad (2.2.22)
 \end{aligned}$$

Step 2: We want to estimate $\mathbb{E} |Y_s^{t,x} - Y_s^{t',x'}|^p + \mathbb{E} \int_t^T |Y_r^{t,x} - Y_r^{t',x'}|^{p-2} \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr$ and $\mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p$. We apply Itô's formula to $(|Y_r^{t,x} - Y_r^{t',x'}|^2)^{\frac{p}{2}}$ then we have

$$\begin{aligned}
 &|Y_s^{t,x} - Y_s^{t',x'}|^p + \frac{p}{2} \int_s^T |Y_r^{t,x} - Y_r^{t',x'}|^{p-2} \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr \\
 &+ \frac{p}{2} (p-2) \int_s^T |Y_r^{t,x} - Y_r^{t',x'}|^{p-4} \\
 &\quad \left\langle (Z_r^{t,x} - Z_r^{t',x'}) (Z_r^{t,x} - Z_r^{t',x'})^* (Y_r^{t,x} - Y_r^{t',x'}), (Y_r^{t,x} - Y_r^{t',x'}) \right\rangle dr \\
 &= |h(X_T^{t,x}) - h(X_T^{t',x'})|^p \\
 &+ p \int_s^T |Y_r^{t,x} - Y_r^{t',x'}|^{p-2} \left\langle Y_r^{t,x} - Y_r^{t',x'}, \right. \\
 &\quad \left. f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - f(r, X_r^{t',x'}, Y_r^{t',x'}, Z_r^{t',x'}) \right\rangle dr \\
 &- p \int_s^T |Y_r^{t,x} - Y_r^{t',x'}|^{p-2} \left\langle Y_r^{t,x} - Y_r^{t',x'}, (Z_r^{t,x} - Z_r^{t',x'}) dW_r \right\rangle. \quad (2.2.23)
 \end{aligned}$$

Since the procedure is almost the same as the proof of Lemma 2.2.1, we will not give any detail here, but the briefly sketch of the proof and results. From (2.2.23)

$$\begin{aligned}
 &\mathbb{E} |Y_s^{t,x} - Y_s^{t',x'}|^p + \mathbb{E} \int_t^T |Y_r^{t,x} - Y_r^{t',x'}|^{p-2} \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr \\
 &\leq C_{p,L} |x - x'|^p + C_{p,L,\mu,T} (1 + |x|^p + |x'|^p) |t - t'|^{\frac{p}{2}}
 \end{aligned}$$

$$+C_{p,L}\mathbb{E} \int_t^T \left(|X_r^{t,x} - X_s^{t',x'}|^p + |Y_r^{t,x} - Y_s^{t',x'}|^p \right) dr. \quad (2.2.24)$$

Moreover,

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p \\ & \leq C_{p,L}|x - x'|^p + C_{L,\mu,T}(1 + |x|^p + |x'|^p)|t - t'|^{\frac{p}{2}} \\ & \quad + C_{p,L}\mathbb{E} \int_t^T \left(|X_r^{t,x} - X_s^{t',x'}|^p + |Y_r^{t,x} - Y_s^{t',x'}|^p \right) dr. \end{aligned} \quad (2.2.25)$$

Step 3: From (2.2.21) and (2.2.25)

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p \\ & \leq C_{p,L}|x - x'|^p + C_{p,L,\mu,T}(1 + |x|^p + |x'|^p)|t - t'|^{\frac{p}{2}} \\ & \quad + C_{p,L}\mathbb{E} \int_t^T \left(|X_r^{t,x} - X_s^{t',x'}|^p + |Y_r^{t,x} - Y_s^{t',x'}|^p \right) dr. \end{aligned} \quad (2.2.26)$$

Following the similar procedure as in Step 3 of the proof of Lemma 2.2.1, there exists a constant $p \in (2, \infty)$ such that

$$\begin{aligned} & \mathbb{E} \int_t^T e^{-Kr} \left(|X_r^{t,x} - X_s^{t',x'}|^p + |Y_r^{t,x} - Y_s^{t',x'}|^p \right) dr \\ & \quad + \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x} - Y_s^{t',x'}|^{p-2} \|Z_r^{t,x} - Z_s^{t',x'}\|^2 dr \\ & \leq C_{p,L,\mu}|x - x'|^p + C_{L,\mu,T}(1 + |x|^p + |x'|^p)|t - t'|^{\frac{p}{2}}. \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \mathbb{E} \int_t^T \left(|X_r^{t,x} - X_s^{t',x'}|^p + |Y_r^{t,x} - Y_s^{t',x'}|^p \right) dr \\ & \leq C_{p,L,\mu}e^{KT}|x - x'|^p + C_{p,L,\mu,T}e^{KT}(1 + |x|^p + |x'|^p)|t - t'|^{\frac{p}{2}}. \end{aligned} \quad (2.2.27)$$

From (2.2.26) and (2.2.27)

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p \\ & \leq C_{p,L,\mu,T}|x - x'|^p + C_{p,L,\mu,T}(1 + |x|^p + |x'|^p)|t - t'|^{\frac{p}{2}}. \end{aligned}$$

Next, following a similar procedure as in Step 6 of the proof of Theorem 1.2.3, we can extend our result from $s \in [t, T]$ to $s \in [0, T]$ such that

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p$$

$$\leq C_{p,L,\mu,T}|x - x'|^p + C_{p,L,\mu,T}(1 + |x|^p + |x'|^p)|t - t'|^{\frac{p}{2}}.$$

Step 4: Following a similar procedure as in Step 4 of the proof of Lemma 2.2.1, we have following estimate

$$\mathbb{E} \left(\int_0^T \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr \right)^{\frac{p}{2}} \leq C_{p,L,\mu,T}|x - x'|^p + C_{p,L,\mu,T}(1 + |x|^p + |x'|^p)|t - t'|^{\frac{p}{2}}$$

Finally, we have our claim

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr \right)^{\frac{p}{2}} \\ & \leq C_{p,L,\mu,T}|x - x'|^p + C_{p,L,\mu,T}(1 + |x|^p + |x'|^p)|t - t'|^{\frac{p}{2}}. \end{aligned}$$

□

2.2.2 Continuity Version of $Y_s^{t,x}$ and its Derivatives

In this subsection, we study the regularity of $Y_t^{t,x}$ with respect to x , including the continuity with respect to t and differentiability with respect to x . The continuity result for BSDE (2.1.1) was studied by Pardoux and Peng [26]. The following corresponding results for FBSDEs (2.1.4) appears to be new.

Theorem 2.2.4. *Under Condition (B.Class 1) (or (B.Class 2)), for $t \in [0, T]$, $x \in \mathbb{R}^d$, (2.1.4) has a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$. Moreover, $\{Y_s^{t,x}; (s, t) \in [0, T]^2, x \in \mathbb{R}^d\}$ has a version whose trajectories belong to $C^{0,0,2}([0, T]^2 \times \mathbb{R}^d)$.*

Proof. We only consider condition (B.Class 1) in our proof, the case with condition (B.Class 2) can be proved similarly.

First, by Lemma 2.2.3 and Kolmogorov continuity theorem, we have $(t, x) \rightarrow Y_s^{t,x}$ is a.s. continuous for $t \in [0, T], x \in \mathbb{R}^d$. Moreover, Since $Y_s^{t,x} \in S^2([0, T]; \mathbb{R}^k)$, so $s \rightarrow Y_s^{t,x}$ is a.s. continuous for $s \in [0, T], x \in \mathbb{R}^d$. Finally, we conclude that $\{Y_s^{t,x}; s, t \in [0, T]^2, x \in \mathbb{R}^d\}$ has an a.s. continuous version.

Next, we will consider the continuity of derivative of $Y_s^{t,x}$ w.r.t. x . Without losing generality in the following proof we assume $t' \geq t$. For $t' \leq s \leq T$, by mean value theorem we have

$$Y_s^{t,x} - Y_s^{t',x'} = \left[\int_0^1 h' \left(X_T^{t',x'} + \lambda(X_T^{t,x} - X_T^{t',x'}) \right) d\lambda \right] [X_T^{t,x} - X_T^{t',x'}]$$

$$\begin{aligned}
 & + \int_s^T \left(\phi_r^f(t, x; t', x') [X_r^{t,x} - X_r^{t',x'}] + v_r^f(t, x; t', x') [Y_r^{t,x} - Y_r^{t',x'}] \right. \\
 & \left. + \chi_r^f(t, x; t', x') [Z_r^{t,x} - Z_r^{t',x'}] \right) dr - \int_s^T [Z_r^{t,x} - Z_r^{t',x'}] dW_r.
 \end{aligned}$$

Here

$$\begin{aligned}
 \phi_r^f(t, x; t', x') &= \int_0^1 f'_x(\Sigma_{r,\lambda}^{t,x;t',x'}) d\lambda, \\
 v_r^f(t, x; t', x') &= \int_0^1 f'_y(\Sigma_{r,\lambda}^{t,x;t',x'}) d\lambda, \\
 \chi_r^f(t, x; t', x') &= \int_0^1 f'_z(\Sigma_{r,\lambda}^{t,x;t',x'}) d\lambda,
 \end{aligned}$$

$$\Sigma_{r,\lambda}^{t,x;t',x'} = \left(r, X_r^{t',x'} + \lambda(X_r^{t,x} - X_r^{t',x'}), Y_r^{t',x'} + \lambda(Y_r^{t,x} - Y_r^{t',x'}), Z_r^{t',x'} + \lambda(Z_r^{t,x} - Z_r^{t',x'}) \right).$$

And

$$\begin{aligned}
 X_s^{t,x} - X_s^{t',x'} &= x - x' + \int_{t'}^s \left(\phi_r^b(t, x; t', x') [X_r^{t,x} - X_r^{t',x'}] + v_r^b(t, x; t', x') [Y_r^{t,x} - Y_r^{t',x'}] \right. \\
 & \left. + \int_{t'}^s \left(\phi_r^\sigma(t, x; t', x') [X_r^{t,x} - X_r^{t',x'}] + v_r^\sigma(t, x; t', x') [Y_r^{t,x} - Y_r^{t',x'}] \right) dW_r \right. \\
 & \left. + \int_t^{t'} b(r, X_r^{t,x}, Y_r^{t,x}) dr + \int_t^{t'} \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r. \right.
 \end{aligned}$$

$\phi_r^\sigma, v_r^\sigma, \phi_r^b$ and v_r^b are defined similarly. Now we define

$$\Delta_l^i X_s^{t,x} \triangleq \frac{X_s^{t,x+le_i} - X_s^{t,x}}{l},$$

where $l \in \mathbb{R} \setminus \{0\}$, $\{e_1, e_2, \dots, e_d\}$ is an orthonormal basis of \mathbb{R}^d . And $\Delta_l^i Y_s^{t,x}$ and $\Delta_l^i Z_s^{t,x}$ can be defined similarly. Then we have

$$\left\{ \begin{aligned}
 \Delta_l^i X_s^{t,x} &= e_i + \int_t^s \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + (b'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x}) \right) d\lambda dr \\
 & \quad + \int_t^s \int_0^1 \left(\sigma'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + (\sigma'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x}) \right) d\lambda dW_r \\
 \Delta_l^i Y_s^{t,x} &= \int_0^1 h' \left(X_T^{t,x} + \lambda l \Delta_l^i X_T^{t,x} \right) \Delta_l^i X_T^{t,x} d\lambda \\
 & \quad + \int_s^T \int_0^1 \left(f'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + (f'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x} + (f'_z(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Z_r^{t,x}) \right) d\lambda dr \\
 & \quad - \int_s^T \Delta_l^i Z_r^{t,x} dW_r,
 \end{aligned} \right. \tag{2.2.28}$$

where $\Sigma_{r,\lambda}^{t,x,l} = \left(r, X_r^{t,x} + \lambda l \Delta_l^i X_r^{t,x}, Y_r^{t,x} + \lambda l \Delta_l^i Y_r^{t,x}, Z_r^{t,x} + \lambda l \Delta_l^i Z_r^{t,x} \right)$.

Now we investigate the new type of Δ_l^i FBSDEs (2.2.28). Note that, **(B.0)** and **(B.1)** implies that $b'_x, b'_y, \sigma'_x, \sigma'_y, f'_x, f'_z$ are all bounded by \sqrt{L} . Hence we have the corresponding

Lipschitz condition. On the other hand, **(B.2)** implies that $f'_y \leq -\mu$, therefore the monotone condition also holds. This is to say that the Δ_l^i FBSDEs (2.2.28) satisfies the corresponding monotone-Lipschitz assumptions. By using a similar method as in the proof of Theorem 1.3.4, (2.2.28) has a unique solution $(\Delta_l^i X_s^{t,x}, \Delta_l^i Y_s^{t,x}, \Delta_l^i Z_s^{t,x})_{0 \leq s \leq T}$. And also by Lemma 2.2.3 we have that, there exists a constant $p > 2$ and C only depending on p , L , μ and T such that

$$\mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i Y_s^{t,x}|^p + \mathbb{E} \left(\int_0^T \|\Delta_l^i Z_s^{t,x}\|^2 ds \right)^{\frac{p}{2}} \leq C. \quad (2.2.29)$$

Here we only give a brief proof:

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i Y_s^{t,x}|^p + \mathbb{E} \left(\int_0^T \|\Delta_l^i Z_s^{t,x}\|^2 ds \right)^{\frac{p}{2}} \\ = & \mathbb{E} \sup_{0 \leq s \leq T} \left| \frac{X_s^{t,x+le_i} - X_s^{t,x}}{l} \right|^p + \mathbb{E} \sup_{0 \leq s \leq T} \left| \frac{Y_s^{t,x+le_i} - Y_s^{t,x}}{l} \right|^p + \mathbb{E} \left(\int_0^T \frac{\|Z_s^{t,x+le_i} - Z_s^{t,x}\|^2}{l^2} ds \right)^{\frac{p}{2}} \\ = & |l|^{-p} \left(\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x+le_i} - X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x+le_i} - Y_s^{t,x}|^p \right. \\ & \left. + \mathbb{E} \left(\int_0^T \|Z_s^{t,x+le_i} - Z_s^{t,x}\|^2 ds \right)^{\frac{p}{2}} \right) \\ \leq & |l|^{-p} (C_{L,\mu,T} |x + le_i - x|^p) \\ \leq & C. \end{aligned}$$

Finally, we consider

$$\begin{aligned} \Delta_l^i X_s^{t,x} - \Delta_{l'}^i X_s^{t',x'} &= \int_{t'}^s \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\ &+ \int_{t'}^s \int_0^1 \left(b'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_y(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\ &+ \int_t^{t'} \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + b'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x} \right) d\lambda dr \\ &+ \int_{t'}^s \int_0^1 \left(\sigma'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - \sigma'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dW_r \\ &+ \int_{t'}^s \int_0^1 \left(\sigma'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - \sigma'_y(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dW_r \\ &+ \int_t^{t'} \int_0^1 \left(\sigma'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + \sigma'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x} \right) d\lambda dW_r, \end{aligned}$$

and

$$\begin{aligned} \Delta_l^i Y_s^{t,x} - \Delta_{l'}^i Y_s^{t',x'} &= \int_0^1 h' \left(X_T^{t,x} + \lambda l \Delta_l^i X_T^{t,x} \right) \Delta_l^i X_T^{t,x} d\lambda \\ &- \int_0^1 h' \left(X_T^{t',x'} + \lambda l' \Delta_{l'}^i X_T^{t',x'} \right) \Delta_{l'}^i X_T^{t',x'} d\lambda \end{aligned}$$

$$\begin{aligned}
 & + \int_s^T \int_0^1 \left(f'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - f'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\
 & + \int_s^T \int_0^1 \left(f'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - f'_y(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\
 & + \int_s^T \int_0^1 \left(f'_z(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - f'_z(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\
 & - \int_s^T \left(\Delta_l^i Z_r^{t,x} - \Delta_{l'}^i Z_r^{t',x'} \right) dW_r.
 \end{aligned}$$

By **(B.0)** and Lemma 2.2.3. we have that, there exists a constant $p > 2$ and C only depending on p, L, μ and T such that

$$\begin{aligned}
 & \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i X_s^{t,x} - \Delta_{l'}^i X_s^{t',x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i Y_s^{t,x} - \Delta_{l'}^i Y_s^{t',x'}|^p \\
 & + \mathbb{E} \left(\int_{t \wedge t'}^T \|\Delta_l^i Z_s^{t,x} - \Delta_{l'}^i Z_s^{t',x'}\|^2 ds \right)^{\frac{p}{2}} \\
 & \leq C|x - x'|^p + C|l - l'|^p + C(1 + |x|^p + |x'|^p + |l|^p + |l'|^p)|t - t'|^{\frac{p}{2}}. \quad (2.2.30)
 \end{aligned}$$

Here we give a brief proof: For the SDE, we apply Itô's formula

$$\begin{aligned}
 & |\Delta_l^i X_s^{t,x} - \Delta_{l'}^i X_s^{t',x'}|^p \\
 = & p \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2} \left\langle \Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}, \right. \\
 & \left. \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda \right\rangle dr \\
 & + p \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2} \left\langle \Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}, \right. \\
 & \left. \int_0^1 \left(b'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_y(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda \right\rangle dr \\
 & + p \int_t^{t'} |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2} \left\langle \Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}, \right. \\
 & \left. \int_0^1 b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + b'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x} \right\rangle dr \\
 & + p \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2} \left\langle \Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}, \right. \\
 & \left. \int_0^1 \left(\sigma'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - \sigma'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dW_r \right\rangle \\
 & + p \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2} \left\langle \Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}, \right. \\
 & \left. \int_0^1 \left(\sigma'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - \sigma'_y(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dW_r \right\rangle \\
 & + p \int_t^{t'} |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2} \left\langle |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2}, \right. \\
 & \left. \int_0^1 \left(\sigma'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + \sigma'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x} \right) d\lambda dW_r \right\rangle.
 \end{aligned}$$

We only calculate the first term, others can be estimated similarly.

$$\begin{aligned}
 & p\mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2} \left\langle \Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}, \right. \\
 & \quad \left. \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda \right\rangle dr \\
 & \leq C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr \\
 & \quad + C_p \mathbb{E} \int_{t'}^s \left| \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda \right|^p dr \\
 & \leq C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr + C_p \mathbb{E} \int_{t'}^s \left| \int_0^1 b'_x(\Sigma_{r,\lambda}^{t,x,l}) (\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}) d\lambda \right|^p dr \\
 & \quad + C_p \mathbb{E} \int_{t'}^s \left| \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) - b'_x(\Sigma_{r,\lambda}^{t',x',l'}) \right) \Delta_{l'}^i X_r^{t',x'} d\lambda \right|^p dr \\
 & \leq C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr + C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr \\
 & \quad + C_p \sqrt{\mathbb{E} \int_{t'}^s \left(\int_0^1 |\Sigma_{r,\lambda}^{t,x,l} - \Sigma_{r,\lambda}^{t',x',l'}|^2 d\lambda \right)^p} dr \sqrt{\mathbb{E} \int_{t'}^s |\Delta_{l'}^i X_r^{t',x'}|^{2p} dr} \\
 & \leq C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr \\
 & \quad + C_p \mathbb{E} \sup_{t' \leq r \leq s} |X_r^{t,x} - X_r^{t',x'}|^p + C_p \mathbb{E} \sup_{t' \leq r \leq s} |X_r^{t,x+le_i} - X_r^{t',x'+l'e_i}|^p \\
 & \quad + C_p \mathbb{E} \sup_{t' \leq r \leq s} |Y_r^{t,x} - Y_r^{t',x'}|^p + C_p \mathbb{E} \sup_{t' \leq r \leq s} |Y_r^{t,x+le_i} - Y_r^{t',x'+l'e_i}|^p \\
 & \leq C_p \mathbb{E} \int_t^T |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr \\
 & \quad + C|x - x'|^p + C|l - l'|^p + C(1 + |x|^p + |x'|^p + |l|^p + |l'|^p)|t - t'|^{\frac{p}{2}}.
 \end{aligned}$$

Here $\mathbb{E} \int_{t'}^s \left(\int_0^1 |\Sigma_{r,\lambda}^{t,x,l} - \Sigma_{r,\lambda}^{t',x',l'}|^2 d\lambda \right)^p dr$ can be estimated with the help of (2.2.17). And due to (2.2.29), $\mathbb{E} \int_{t'}^s |\Delta_{l'}^i X_r^{t',x'}|^{2p} dr$ is dominated by a constant. The backward-SDE can also be estimated. By using a similar procedure as in the proof of Lemma 2.2.3, we obtain (2.2.30).

Using Kolmogorov continuity theorem, it immediately follows from (2.2.30) that: For any $t, s \in [0, T]^2$, $x \in \mathbb{R}^d$, the mapping $x \rightarrow Y_s^{t,x}$ is a.s. differentiable, and the partial derivatives with respect to x , denoted by $\frac{\partial Y_s^{t,x}}{\partial x_i} = \lim_{l \rightarrow 0} \Delta_l^i Y_s^{t,x}$, has a version which is a.s. continuous with respect to (s, t, x) . The existence of a continuous second derivative of $Y_s^{t,x}$ with respect to x is proved in a similar way. \square

Theorem 2.2.4 immediately leads to the following corollary.

Corollary 2.2.5. *Under Condition (B.Class 1) (or (B.Class 2)), for any $t \in [0, T]$, the mapping $x \rightarrow Y_t^{t,x}$, is of class C^2 a.s., the function and its derivatives of order one*

and two being a.s. continuous in (t, x) .

As a by-product of the proof of Theorem 2.2.4, we also have

Corollary 2.2.6. *Under Condition (B. Class 1) (or (B. Class 2)), $(\nabla X_s^{t,x} = \frac{\partial X_s^{t,x}}{\partial x}, \nabla Y_s^{t,x} = \frac{\partial Y_s^{t,x}}{\partial x}, \nabla Z_s^{t,x} = \frac{\partial Z_s^{t,x}}{\partial x})_{0 \leq s \leq T}$ is the unique solution of the following ∇ FBSDEs,*

$$\left\{ \begin{array}{l} \nabla X_s^{t,x} = 1 + \int_t^s \sigma'_x(r, X_r^{t,x}, Y_r^{t,x}) \nabla X_r^{t,x} dW_r + \int_t^s \sigma'_y(r, X_r^{t,x}, Y_r^{t,x}) \nabla Y_r^{t,x} dW_r \\ \quad + \int_t^s b'_x(r, X_r^{t,x}, Y_r^{t,x}) \nabla X_r^{t,x} dr + \int_t^s b'_y(r, X_r^{t,x}, Y_r^{t,x}) \nabla Y_r^{t,x} dr \\ \nabla Y_s^{t,x} = h'(X_T^{t,x}) \nabla X_T^{t,x} - \int_s^T \nabla Z_r^{t,x} dW_r \\ \quad + \int_s^T f'_x(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla X_r^{t,x} dr + f'_y(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla Y_r^{t,x} dr \\ \quad + \int_s^T f'_z(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla Z_r^{t,x} dr. \end{array} \right. \quad (2.2.31)$$

Proof. This follows easily by the result of Theorem 2.2.4 and the definition of partial derivatives,

$$\frac{\partial X_s^{t,x}}{\partial x_i} = \lim_{l \rightarrow 0} \Delta_l^i X_s^{t,x}, \quad \frac{\partial Y_s^{t,x}}{\partial x_i} = \lim_{l \rightarrow 0} \Delta_l^i Y_s^{t,x}, \quad \frac{\partial Z_s^{t,x}}{\partial x_i} = \lim_{l \rightarrow 0} \Delta_l^i Z_s^{t,x}.$$

It is easy to check that (2.2.31) satisfies the corresponding monotone-Lipschitz assumptions, therefore $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})_{0 \leq s \leq T}$ is the unique solution. □

2.2.3 Expression of $Z_s^{t,x}$ and Flow Property

In this subsection, we give the main result in the Proposition 2.2.8. By using Malliavin calculus, $Z_s^{t,x}$ can be expressed as the Malliavin derivative of Y . Then we compare the ∇ FBSDEs (2.2.31) with the Malliavin differential form of FBSDEs (2.2.32), to give a formula relating Z with the gradients of Y and X . The corresponding result for BSDEs case is given by Pardoux and Peng in [26] [27]. For the convenience of the reader, in the following we will give a complete proof. First we give the following preparations.

Let us now recall the notion of the derivation on Wiener space. We denote by \mathbb{S} the set of random variables ξ of the form:

$$\xi = \varphi(W(h_1), \dots, W(h_n)),$$

where $\varphi \in C_b^\infty(\mathbb{R}^n)$ is a polynomial function, $h_1, \dots, h_n \in L^2([0, T], \mathbb{R}^d)$ and

$$W(h_i) \triangleq \int_0^T \langle h_i(t), dW_t \rangle.$$

The random variable ξ has a derivative $\{\mathcal{D}_r \xi; r \in [0, T]\}$ (see [23]) defined as

$$\mathcal{D}_t \xi = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i(t), \quad 0 \leq t \leq T.$$

For such an ξ , we define its 1,2-norm as

$$\|\xi\|_{1,2}^2 = \mathbb{E}(\xi^2) + \mathbb{E} \int_0^T |\mathcal{D}_r \xi|^2 dr.$$

And we define the Sobolev space:

$$\mathbb{D}^{1,2} \triangleq \overline{\mathbb{S}} \|\cdot\|_{1,2}.$$

From Øksendal [23], we know that the "derivation operator" \mathcal{D} . extends as an operator from $\mathbb{D}^{1,2}$ into $L^2(\Omega; L^2([0, T], \mathbb{R}^d))$. It turns out that the components $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ take values in $\mathbb{D}^{1,2}$ under Condition **(B.Class 1)** (or **(B.Class 2)**). The detail will be given in Proposition 2.2.7. Moreover, we will use Malliavin calculus to express Z in terms of the Malliavin derivative of Y in Proposition 2.2.8.

Proposition 2.2.7. *Under Condition **(B.Class 1)** (or **(B.Class 2)**), for any $0 \leq t \leq s \leq T$, $x \in \mathbb{R}^d$, $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \in L^2([t, T]; (\mathbb{D}^{1,2})^d) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^k) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^{k \times d})$, and a version of $\{\mathcal{D}_r X_s^{t,x}, \mathcal{D}_r Y_s^{t,x}, \mathcal{D}_r Z_s^{t,x}; t \leq r \leq T, t \leq s \leq T\}$ is given by*

- (i) $\mathcal{D}_r X_s^{t,x} = 0, \mathcal{D}_r Y_s^{t,x} = 0, \mathcal{D}_r Z_s^{t,x} = 0. r \in [0, T] \setminus (t, s];$
- (ii) *For any $t < r \leq T$, $\{\mathcal{D}_r X_s^{t,x}, \mathcal{D}_r Y_s^{t,x}, \mathcal{D}_r Z_s^{t,x}; r \leq s \leq T\}$ is the unique solution of the following differential form of FBSDEs with respect to Wiener process.*

$$\left\{ \begin{array}{l} \mathcal{D}_r X_s^{t,x} = \sigma(r, X_r^{t,x}, Y_r^{t,x}) \\ \quad + \int_r^s b'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \mathcal{D}_r X_\tau^{t,x} d\tau + \int_r^s b'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \mathcal{D}_r Y_\tau^{t,x} d\tau \\ \quad + \int_r^s \sigma'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \mathcal{D}_r X_\tau^{t,x} dW_\tau + \int_r^s \sigma'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \mathcal{D}_r Y_\tau^{t,x} dW_\tau \\ \mathcal{D}_r Y_s^{t,x} = h'(X_T^{t,x}) \mathcal{D}_r X_T^{t,x} - \int_s^T \mathcal{D}_r Z_\tau^{t,x} dW_\tau \\ \quad + \int_s^T f'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r X_\tau^{t,x} d\tau + \int_s^T f'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r Y_\tau^{t,x} d\tau \\ \quad + \int_s^T f'_z(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r Z_\tau^{t,x} d\tau. \end{array} \right. \quad (2.2.32)$$

Moreover, $\{\mathcal{D}_s Y_s^{t,x}, t \leq s \leq T\}$ is a version of $\{(Z_s^{t,x}), t \leq s \leq T\}$.

Proof. First, we will show that $(X^{t,x}, Y^{t,x}, Z^{t,x}) \in L^2([t, T]; (\mathbb{D}^{1,2})^d) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^k) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^{k \times d})$. Recall the iteration procedure for FBSDEs

$$\begin{cases} X_s^{t,x,N} = x + \int_t^s b(r, X_r^{t,x,N}, Y_r^{t,x,N-1})dr + \int_t^s \sigma(r, X_r^{t,x,N}, Y_r^{t,x,N-1})dW_r \\ Y_s^{t,x,N} = h(X_T^{t,x,N}) + \int_s^T f(r, X_r^{t,x,N}, Y_r^{t,x,N}, Z_r^{t,x,N})dr - \int_s^T Z_r^{t,x,N}dW_r. \end{cases}$$

When $N=1$, and we let $Y_s^{t,x,0} = 0$, then above FBSDEs became the BSDEs in [26]. From results in [26] and [27], $(X^{t,x,1}, Y^{t,x,1}, Z^{t,x,1}) \in L^2([t, T]; (\mathbb{D}^{1,2})^d) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^k) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^{k \times d})$ and (2.2.32) holds. Since the derivatives of coefficients are all bounded, we can easily show that $(\mathcal{D}_r X^{t,x,N}, \mathcal{D}_r Y^{t,x,N}, \mathcal{D}_r Z^{t,x,N})$ is a Cauchy sequence in L^2 sense, and its limit denoted by $(\mathcal{D}_r X_s^{t,x}, \mathcal{D}_r Y_s^{t,x}, \mathcal{D}_r Z_s^{t,x})$ satisfies (2.2.32) for any $r \leq s \leq T$.

Finally, we consider the following equation

$$Y_s^{t,x} = Y_t^{t,x} - \int_t^s f(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})d\mu + \int_t^s Z_\tau^{t,x}dW_\tau.$$

For $t \leq r \leq s \leq T$, we have

$$\begin{aligned} \mathcal{D}_r Y_s^{t,x} &= Z_r^{t,x} - \int_r^s f'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})\mathcal{D}_r X_\tau^{t,x}d\tau - \int_r^s f'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})\mathcal{D}_r Y_\tau^{t,x}d\tau \\ &\quad - \int_r^s f'_z(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})\mathcal{D}_r Z_\tau^{t,x}d\tau + \int_r^s \mathcal{D}_r Z_\tau^{t,x}dW_\tau. \end{aligned}$$

It is easy to see that $\mathcal{D}_r Y_s^{t,x} = Z_s^{t,x}$ a.s. at $r = s$, this means exactly that

$$\mathcal{D}_s Y_s^{t,x} \triangleq \lim_{r \rightarrow s} \mathcal{D}_r Y_s^{t,x} = Z_s^{t,x}, \quad a.s..$$

□

Proposition 2.2.8. *Under Condition (B.Class 1) (or (B.Class 2)), the random field $\{Z_s^{t,x}; 0 \leq t \leq s \leq T, x \in \mathbb{R}^d\}$ has an a.s. continuous version which is given by:*

$$Z_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(s, X_s^{t,x}, Y_s^{t,x})$$

and in particular

$$Z_t^{t,x} = \nabla Y_t^{t,x} \sigma(t, x, Y_t^{t,x}).$$

Proof. First, we will show that $\{\mathcal{D}_s Y_s^{t,x}\}$ processes an a.s. continuous version. For this, recall Corollary 2.2.6, we have $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})_{0 \leq s \leq T}$ solves ∇ FBSDEs (2.2.31)

which can be written as

$$\left\{ \begin{array}{l} \nabla X_s^{t,x} = \nabla X_r^{t,x} + \int_r^s \sigma'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \nabla X_\tau^{t,x} dW_\tau \\ \quad + \int_r^s \sigma'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \nabla Y_\tau^{t,x} dW_\tau + \int_r^s b'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \nabla X_\tau^{t,x} d\tau \\ \quad + \int_r^s b'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \nabla Y_\tau^{t,x} d\tau \\ \nabla Y_s^{t,x} = h'(X_T^{t,x}) \nabla X_T^{t,x} - \int_s^T \nabla Z_\tau^{t,x} dW_\tau \\ \quad + \int_s^T f'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \nabla X_\tau^{t,x} d\tau + f'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \nabla Y_\tau^{t,x} d\tau \\ \quad + \int_s^T f'_z(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \nabla Z_\tau^{t,x} d\tau. \end{array} \right.$$

From the uniqueness of the solution of (2.2.32), we have following expressions

$$\mathcal{D}_r X_s^{t,x} = \nabla X_s^{t,x} (\nabla X_r^{t,x})^{-1} \sigma(r, X_r^{t,x}, Y_r^{t,x}),$$

and

$$\mathcal{D}_r Y_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_r^{t,x})^{-1} \sigma(r, X_r^{t,x}, Y_r^{t,x}), \quad t \leq r \leq s \leq T. \quad (2.2.33)$$

So $\mathcal{D}_s Y_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(s, X_s^{t,x}, Y_s^{t,x})$, and the continuity of $\mathcal{D}_s Y_s^{t,x}$ follows from that of $\nabla Y_s^{t,x}$, $\nabla X_s^{t,x}$, $X_s^{t,x}$ and $Y_s^{t,x}$. Finally, using the result of Proposition 2.2.7 and (2.2.33), we have

$$Z_s^{t,x} = \mathcal{D}_s Y_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(s, X_s^{t,x}, Y_s^{t,x}).$$

And the continuity follows from the continuity of $\{\mathcal{D}_s Y_s^{t,x}; t \leq s \leq T\}$. This gives the first part of the proposition. The second part easily follows when $s = t$. \square

Now we will give the following flow property, which plays an important role in the next section.

Proposition 2.2.9. *Under conditions (B.1), (B.2) and (A.2.5) (resp. (B.3), (B.4) and (A.2.5)), (2.1.4) has a unique solution $(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$, then for any $t \leq s \leq T$, $X_r^{s, X_s^{t,x}} = X_r^{t,x}$, $Y_r^{s, X_s^{t,x}} = Y_r^{t,x}$ and $Z_r^{s, X_s^{t,x}} = Z_r^{t,x}$ for any $r \in [s, T]$ a.s..*

Proof. By using a similar method as in the proof of Theorem 1.3.3 or Theorem 1.3.4, we can prove that (2.1.4) has a unique solution $(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \in S^2([t, T]; \mathbb{R}^d) \otimes S^2([t, T]; \mathbb{R}^k) \otimes M^2([t, T]; \mathbb{R}^{k \times d})$. For $t \leq s \leq r \leq T$, above equations can be rewritten as follows

$$\left\{ \begin{array}{l} X_r^{t,x} = X_s^{t,x} + \int_s^r b(\mu, X_\mu^{t,x}, Y_\mu^{t,x}) d\mu + \int_s^r \sigma(\mu, X_\mu^{t,x}, Y_\mu^{t,x}) dW_\mu, \\ Y_r^{t,x} = h(X_T^{t,x}) + \int_r^T f(\mu, X_\mu^{t,x}, Y_\mu^{t,x}, Z_\mu^{t,x}) d\mu - \int_r^T Z_\mu^{t,x} dW_\mu. \end{array} \right. \quad (2.2.34)$$

Here

$$X_s^{t,x} = x + \int_t^s b(\mu, X_\mu^{t,x}, Y_\mu^{t,x})d\mu + \int_t^s \sigma(\mu, X_\mu^{t,x}, Y_\mu^{t,x})dW_\mu.$$

On the other hand, for $t \leq s \leq r \leq T$, it is easy to check that $(X_r^{s, X_s^{t,x}}, Y_r^{s, X_s^{t,x}}, Z_r^{s, X_s^{t,x}})$ is the solution of following

$$\begin{cases} X_r^{s, X_s^{t,x}} = X_s^{t,x} + \int_s^r b(\mu, X_\mu^{s, X_s^{t,x}}, Y_\mu^{s, X_s^{t,x}})d\mu + \int_s^r \sigma(\mu, X_\mu^{s, X_s^{t,x}}, Y_\mu^{s, X_s^{t,x}})dW_\mu, \\ Y_r^{s, X_s^{t,x}} = h(X_r^{s, X_s^{t,x}}) + \int_r^T f(\mu, X_\mu^{s, X_s^{t,x}}, Y_\mu^{s, X_s^{t,x}}, Z_\mu^{s, X_s^{t,x}})d\mu - \int_r^T Z_\mu^{s, X_s^{t,x}} dW_\mu. \end{cases} \quad (2.2.35)$$

By the uniqueness of the solution of FBSDEs, it follows from comparing (2.2.34) and (2.2.35) that for any $s \in [t, T]$, $X_r^{s, X_s^{t,x}} = X_r^{t,x}$, $Y_r^{s, X_s^{t,x}} = Y_r^{t,x}$ and $Z_r^{s, X_s^{t,x}} = Z_r^{t,x}$ for any $r \in [s, T]$ a.s.. \square

2.3 Main Results

In this section, we link our FBSDEs (2.1.4) to the system of quasilinear second order parabolic partial differential equations (2.1.3). Note that, the FBSDEs and PDEs are different from those in Pardoux and Peng [26], where the non-linear functions (b, σ) depends on $(s, u(s, x))$. But the idea of the proofs below is almost the same as theirs after we proved the regularity of solution. We nevertheless include a complete proof for the convenience of reader.

2.3.1 Probabilistic Interpretation for Quasilinear Parabolic PDEs

First, we give one of our main results in this chapter, the probabilistic representation of solution of quasilinear parabolic PDEs in terms of solution of FBSDEs.

Theorem 2.3.1. *Under Conditions (B.1), (B.2) and (A.2.5) (or (B.3), (B.4) and (A.2.5)), if $u \in C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$ solves PDEs (2.1.3), then $u(t, x) = Y_t^{t,x}$, $t \geq 0$, $x \in \mathbb{R}^d$, where $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$ is the unique solution of the FBSDEs (2.1.4).*

Proof. It suffices to show that $(u(t, X_s^{t,x}), \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))) \nabla u(s, X_s^{t,x}); t \leq s \leq T)$ solves the FBSDEs (2.1.4).

Let $t = t_0 < t_1 < t_2 < \dots < t_n = T$

$$\sum_{i=0}^{n-1} \left[u(t_i, X_{t_i}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x}) \right]$$

$$\begin{aligned}
 &= \sum_{i=0}^{n-1} \left[u(t_i, X_{t_i}^{t,x}) - u(t_i, X_{t_{i+1}}^{t,x}) \right] + \sum_{i=0}^{n-1} \left[u(t_i, X_{t_{i+1}}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x}) \right] \\
 &= - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathcal{L}u(t_i, X_s^{t,x}) ds - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sigma^* (t_i, X_s^{t,x}, u(t_i, X_s^{t,x})) \nabla u(t_i, X_s^{t,x}) dW_s \\
 &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[\mathcal{L}u(s, X_{t_{i+1}}^{t,x}) \right. \\
 &\quad \left. + f \left(s, X_{t_{i+1}}^{t,x}, u(s, X_{t_{i+1}}^{t,x}), \sigma^* \left(s, X_{t_{i+1}}^{t,x}, u(s, X_{t_{i+1}}^{t,x}) \right) \nabla u(s, X_{t_{i+1}}^{t,x}) \right) \right] ds.
 \end{aligned}$$

Here we applied Itô's formula to $u(t_i, \cdot)$ to estimate $u(t_i, X_{t_i}^{t,x}) - u(t_i, X_{t_{i+1}}^{t,x})$ (note the fact that $u(t_i, \cdot) \in C^2(\mathbb{R}^d; \mathbb{R}^k)$), and $u(t_i, X_{t_i}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x})$ satisfied the PDE (2.1.3). Finally, by the fact that $u \in C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$ and the monotone-Lipschitz assumptions, we let the mesh size go to zero and obtain

$$\begin{aligned}
 u(t, x) - h(X_T^{t,x}) &= \int_t^T f \left(s, X_s^{t,x}, u(s, X_s^{t,x}), \sigma^* \left(s, X_s^{t,x}, u(s, X_s^{t,x}) \right) \nabla u(s, X_s^{t,x}) \right) ds \\
 &\quad - \int_t^T \sigma^* \left(s, X_s^{t,x}, u(s, X_s^{t,x}) \right) \nabla u(s, X_s^{t,x}) dW_s,
 \end{aligned}$$

where $\left(u(s, X_s^{t,x}), \sigma^* \left(s, X_s^{t,x}, u(s, X_s^{t,x}) \right) \nabla u(s, X_s^{t,x}); t \leq s \leq T \right)$ solves the FBSDEs (2.1.4). By the uniqueness of the solution of FBSDEs, we have $\left(u(s, X_s^{t,x}), \sigma^* \left(s, X_s^{t,x}, u(s, X_s^{t,x}) \right) \nabla u(s, X_s^{t,x}) \right) = (Y_s^{t,x}, Z_s^{t,x})$. In particular, $u(t, x) = Y_t^{t,x}$. \square

Remark 2.3.2. Conditions (B.1), (B.2) and (A.2.5) (resp. (B.3), (B.4) and (A.2.5)) guarantee a unique solution to the FBSDEs (2.1.4).

2.3.2 A Unique Classical Solution of Quasilinear Parabolic PDEs

We can also prove the converse result to Theorem 2.3.1. From the regularity properties, flow property and the expression of process Z , the solution of FBSDEs (2.1.4) give a unique solution of a quasilinear parabolic PDEs. (2.1.3).

Theorem 2.3.3. Under Condition (B.Class 1) (or (B.Class 2)), then $\{u(t, x) \triangleq Y_t^{t,x}; 0 \leq t \leq T, x \in \mathbb{R}^d\}$ is of class $C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$, and solves the PDE (2.1.3).

Proof. From Theorem 2.2.4, $u(t, x) \in C^{0,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$. Let $h > 0$ be such that $t + h \leq T$. By flow property in Proposition 2.2.9, $Y_{t+h}^{t,x} = Y_{t+h}^{t+h, X_{t+h}^{t,x}}$. Hence

$$u(t+h, x) - u(t, x) = u(t+h, x) - u(t+h, X_{t+h}^{t,x}) + u(t+h, X_{t+h}^{t,x}) - u(t, x)$$

$$\begin{aligned}
 &= - \int_t^{t+h} \mathcal{L}u(t+h, X_s^{t,x}) ds \\
 &\quad - \int_t^{t+h} \sigma^*(t+h, X_s^{t,x}, u(t+h, X_s^{t,x})) \nabla u(t+h, X_s^{t,x}) dW_s \\
 &\quad - \int_t^{t+h} f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds + \int_t^{t+h} Z_s^{t,x} dW_s.
 \end{aligned}$$

Here we applied Itô's formula to $u(t+h, \cdot)$ to estimate $u(t+h, x) - u(t+h, X_{t+h}^{t,x})$. Note here $u(t_i, \cdot) \in C^2(\mathbb{R}^d, \mathbb{R}^k)$, and $u(t+h, X_{t+h}^{t,x}) - u(t, x) = Y_{t+h}^{t+h, X_{t+h}^{t,x}} - Y_t^{t,x} = Y_{t+h}^{t,x} - Y_t^{t,x}$ satisfies the FBSDEs (2.1.4). Now let $t = t_0 < t_1 < \dots < t_n = T$. We have

$$\begin{aligned}
 u(T, x) - u(t, x) &= - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\mathcal{L}u(t_{i+1}, X_s^{t_i, x}) + f(s, X_s^{t_i, x}, Y_s^{t_i, x}, Z_s^{t_i, x})) ds \\
 &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (Z_s^{t_i, x} - \sigma^*(t_{i+1}, X_s^{t_i, x}, u(t_{i+1}, X_s^{t_i, x})) \nabla u(t_{i+1}, X_s^{t_i, x})) dW_s.
 \end{aligned}$$

If we take a sequence of meshes $t = t_0^n < t_1^n < \dots < t_n^n = T$ such that $\lim_{n \rightarrow \infty} \sup_{i \leq n-1} (t_{i+1}^n - t_i^n) = 0$, with Proposition 2.2.8 and the fact that $Y_s^{t,x}$ and the derivative of $Y_s^{t,x}$ w.r.t. x are uniformly continuous w.r.t. (s, t, x) a.s. we obtain

$$u(t, x) = h(x) + \int_t^T [\mathcal{L}u(s, x) + f(s, x, u(s, x), \sigma^*(s, x, u(s, x)) \nabla u(s, x))] ds.$$

Hence $u(t, x) \in C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$ and satisfies the PDE (2.1.3). \square

Chapter 3

Weak Solutions for Quasi-linear Parabolic PDEs in the Sobolev Space

3.1 Introduction and Preliminaries

Under rather strong smoothness assumptions on coefficients, Pardoux and Peng ([26],[27]) proved that the solution of semi-linear parabolic PDEs (2.1.2) can be interpreted as the solution of BSDEs (2.1.1). This result gives a unique classical solution for the semi-linear parabolic PDEs. But if the coefficients (f and h) are just Lipschitz continuous one has to consider weak solutions. Pardoux and Peng [26] considered viscosity solution for PDEs. Later, Barles and Lesigne [5] studied the connection between weak solutions of PDEs and BSDEs. Follow this idea, Bally and Matoussi [4] found a weak formulation for PDEs and established the link with BSDEs. Meanwhile, they also proved that the PDE has a unique weak solution in the Sobolev space under weaker hypotheses on the coefficients (coefficients in backward equations are only Lipschitz continuous). On the other hand, Zhang and Zhao [38] proved the existence and uniqueness of solutions of BSDEs in L^2_ρ sense (independent of any initial value) with both finite and infinite horizons.

For FBSDEs and corresponding quasi-linear PDEs case, there are only a few relevant works. Pardoux and Tang [28] obtained a probabilistic interpretation of viscosity solution of quasi-linear PDEs in terms of FBSDEs (1.2.1). Recently, Wu and Yu [35] generalized this result in the case where the diffusion function σ depend on Z .

In Chapter 2, for the smoothness coefficients, we proved the quasi-linear parabolic PDEs (2.1.3) has a unique classical solution (see Theorem 2.3.3). In this chapter, we will relax our assumptions and study the weak solution in the Sobolev space. This chapter is organized as follows. In Section 3.2, we introduce a useful tool, the norm equivalence result, which is the key to relate the weak solution of PDEs to the solution of FBSDEs. In Section 3.3, we construct a smotherized FBSDEs (mollify the coefficients) and use its corresponding smotherized PDEs to find the weak solution of PDEs (2.1.3).

First, we will give the definition of weak solution of PDE (2.1.3). Now following Definition 1.1.3 we write down the solution spaces needed in our thesis: $M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k))$, $M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ and $S^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k))$.

Definition 3.1.1. *A process u is called a weak solution (solution in $L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)$) of PDEs (2.1.3) if $(u, \sigma^* \nabla u) \in M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ and for an arbitrary $\Psi \in C_c^{1, \infty}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$,*

$$\begin{aligned}
 & \int_t^T \int_{\mathbb{R}^d} u(s, x) \partial_s \Psi(s, x) dx ds + \int_{\mathbb{R}^d} u(t, x) \Psi(t, x) dx - \int_{\mathbb{R}^d} u(T, x) \Psi(T, x) dx \\
 & + \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} \sigma^*(s, x, u(s, x)) \nabla u(s, x) \sigma^*(s, x, u(s, x)) \nabla \Psi(s, x) dx ds \quad (3.1.1) \\
 & + \int_t^T \int_{\mathbb{R}^d} u(s, x) \operatorname{div} \left((b - \tilde{A}) \Psi(s, x) \right) dx ds \\
 & = \int_t^T \int_{\mathbb{R}^d} f(s, x, u(s, x), \sigma^*(s, x, u(s, x)) \nabla u(s, x)) \Psi(s, x) dx ds.
 \end{aligned}$$

Here $\tilde{A}_j \triangleq \sum_{i=1}^d \frac{\partial(\sigma \sigma^*)_{i,j}(s, x, u(s, x))}{\partial x_i}$, and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_d)^*$.

Note that, this definition can be easily understood if we note the following integration by parts formula: for $\varphi_1, \varphi_2 \in C^2(\mathbb{R}^d)$,

$$- \int_{\mathbb{R}^d} \mathcal{L} \varphi_1(x) \varphi_2(x) dx = \frac{1}{2} \int_{\mathbb{R}^d} (\sigma^* \nabla \varphi_1)(x) (\sigma^* \nabla \varphi_2)(x) dx + \int_{\mathbb{R}^d} \varphi_1(x) \operatorname{div} \left((b - \tilde{A}) \varphi_2 \right) (x) dx.$$

The main purpose of this chapter is to find the weak solution of PDEs (2.1.3) via the solution of FBSDEs (2.1.4). For the weak solution of PDEs, we consider following assumptions.

(C.0): For any $s \in [0, T]$, $b(s, \cdot, \cdot) \in C_{l,b}^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)$; $f(s, \cdot, \cdot, \cdot) \in C_{l,b}^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}; \mathbb{R}^k)$; $h \in C_{l,b}^{1,\alpha}(\mathbb{R}^d; \mathbb{R}^k)$; $\sigma(s, \cdot, \cdot) \in C_{l,b}^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^{d \times d})$ for some $\alpha \in (0, 1]$. $C_{l,b}^{1,\alpha}$ denote the set of $C_{l,b}^1$ -functions whose first derivative is Hölder continuous of order α .

(C.1): There exists a constant $L \geq 0$ such that for any $t \in [0, T]$, $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\begin{aligned} |b(t, x_1, y_1) - b(t, x_2, y_2)|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ \|\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)\|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ |f(t, x_1, y, z_1) - f(t, x_2, y, z_2)|^2 &\leq L(|x_1 - x_2|^2 + \|z_1 - z_2\|^2) \\ |h(x_1) - h(x_2)|^2 &\leq L|x_1 - x_2|^2. \end{aligned}$$

Here the Euclidean norm of a vector $x \in \mathbb{R}^d$ will be denoted by $|x|$, and the matrix norm is denoted by $\|z\| := \sqrt{\text{tr}(zz^*)}$.

(C.2): For any $p \in [2, \infty)$, there exists positive constants $\mu, C_{p,L}$ and $C'_{p,L}$, where $C_{p,L}, C'_{p,L}$ only depending on p and L , such that $p\mu > K + C_{p,L}$ and $K > C'_{p,L}$. And for any $t \in [0, T]$, $y_1, y_2, y \in \mathbb{R}^k, x \in \mathbb{R}^d, z \in \mathbb{R}^{k \times d}$

$$\begin{aligned} \langle y_1 - y_2, f(t, x, y_1, z) - f(t, x, y_2, z) \rangle &\leq -\mu|y_1 - y_2|^2, \\ |f(t, 0, y, 0)|^2 &\leq L(1 + |y|^2). \end{aligned}$$

(C.3): There exists a constant $L \geq 0$ such that for any $t \in [0, T]$, $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\begin{aligned} |b(t, x, y_1) - b(t, x, y_2)|^2 &\leq L|y_1 - y_2|^2 \\ \|\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)\|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ |f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2 + \|z_1 - z_2\|^2) \\ |h(x_1) - h(x_2)|^2 &\leq L|x_1 - x_2|^2. \end{aligned}$$

(C.4): For any $p \in [2, \infty)$, there exists positive constants $\mu, C_{p,L}$ and $C'_{p,L}$, where $C_{p,L}, C'_{p,L}$ only depending on p and L , such that $p\mu > K + C_{p,L}$ and $K > C'_{p,L}$. And for any $t \in [0, T]$, $x_1, x_2, x \in \mathbb{R}^d, y \in \mathbb{R}^k, z \in \mathbb{R}^{k \times d}$

$$\begin{aligned} \langle x_1 - x_2, b(t, x_1, y) - b(t, x_2, y) \rangle &\leq -\mu|x_1 - x_2|^2, \\ |b(t, x, 0, 0)|^2 &\leq L(1 + |x|^2) \end{aligned}$$

Notation 3.1.2. To simplify our notation, we denote these two classes of conditions as follows

$$(\mathbf{C. Class 1}) := (\mathbf{C.0}) + (\mathbf{C.1}) + (\mathbf{C.2}) + (\mathbf{B.5})$$

$$(\mathbf{C. Class 2}) := (\mathbf{C.0}) + (\mathbf{C.3}) + (\mathbf{C.4}) + (\mathbf{B.5})$$

Remark 3.1.3. For the weak solution of PDEs, we replace the condition **(B.0)** by a weaker condition **(C.0)** rather than in C^3 class and bounded first and second derivatives, The coefficients b , f and h are only in $C^{1,\alpha}$ class rather than in C^3 class and bounded first and second order derivatives.

In section 3.2, we will give an important result, the norm equivalence result (Lemma 3.2.3), which is the key to link the weak solution of PDEs and the solution of FBSDEs. Since the constants c and C in inequality (3.2.4) depend on the bounds of the first order derivatives of b, σ, f and h . And also the first derivative of the coefficients should be Hölder continuous, in order to guarantee the existence of $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})_{0 \leq s \leq T}$ in Lemma 3.2.2. So we assume that the coefficients satisfy smooth conditions.

3.2 Equivalence of Norm Result and Flow Properties

The norm equivalence result plays an important role in this chapter. The corresponding results in BSDEs case are given in [4], [5], [17] and [38]. For the FBSDEs case, we will give the relevant result which is the main purpose of this section. Before that, we need following preparations.

Lemma 2.2.1 and Lemma 2.2.1 immediately lead to

Lemma 3.2.1. Under Condition **(C.Class 1)** (or **(C.Class 2)**) without condition **(C.0)**, for any $p \in [2, \infty)$, there exists a constant $C_{p,L,\mu,T}$ only depending on p, L, μ and T such that

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} \leq C_{p,L,\mu,T} (1 + |x|^p), \quad (3.2.1)$$

and

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr \right)^{\frac{p}{2}} \\ & \leq C_{p,L,\mu,T} |x - x'|^p + C_{p,L,\mu,T} (1 + |x|^p + |x'|^p) |t - t'|^{\frac{p}{2}}. \end{aligned} \quad (3.2.2)$$

Proof. The proof is similar to those in Lemma 2.2.1 and Lemma 2.2.1. Note from (2.2.15) in the proof of Lemma 2.2.1, we need $\gamma, \beta > 0$ to estimate (2.2.1). In Chapter 2, we assume $2\mu - K - L^2 - 10L - 1 > 0$ and $K - 4L^3 - L^2 - 10L - 1 > 0$, then we can find a constant $p \in (2, \infty)$ such that $\gamma, \beta > 0$. This is enough for regularity properties in Lemma 2.2.1 and Lemma 2.2.3.

In this chapter, we need (3.2.1) and (3.2.2), in order to estimate the weighted function $\rho(\hat{X}) := (1 + |\hat{X}_s^{t,y}|^2)^p$ for any $p \geq 2$ in Lemma 3.2.3. Therefore, from above inequalities, for any $p \in [2, \infty)$, there exists two constants $C_{p,L}$ and $C'_{p,L}$ only depending on p and L , such that $p\mu > K + C_{p,L}$ and $K > C'_{p,L}$. That is why we strengthen our assumption for μ in **(C.2)** and **(C.4)**. □

Lemma 3.2.2. *Under Condition **(C.Class 1)** (or **(C.Class 2)**), for $p = 2$, there exists a constant $C_{L,\mu,T}$ only depending on L , μ and T such that*

$$\mathbb{E} \sup_{0 \leq s \leq T} \|\nabla X_s^{t,x}\|^2 + \mathbb{E} \sup_{0 \leq s \leq T} \|\nabla Y_s^{t,x}\|^2 + \mathbb{E} \int_0^T \|\nabla Z_r^{t,x}\|^2 dr \leq C_{L,\lambda,T}, \quad (3.2.3)$$

here $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})_{0 \leq s \leq T}$ are the derivatives of $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$ with respect to x .

Proof. Since coefficients are in $C_{l,b}^{1,\alpha}$ class and monotone-Lipschitz, then we have (2.2.30) as in the proof of Theorem 2.2.4. By Kolmogorov continuity theorem, for any $t, s \in [0, T]^2$, $x \in \mathbb{R}^d$, the mapping $x \rightarrow (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ is a.s. differentiable. By Corollary 2.2.6, $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})_{0 \leq s \leq T}$ is the unique solution of ∇ FBSDEs (2.2.31). Immediately, we have (3.2.3) by using a similar procedure as in the proof of Lemma 2.2.1. □

The following norm equivalence result plays an important role in the analysis in this chapter. The relevant works for BSDE (2.1.1) case (where the forward equation is independent of Y) were studied by Barles and Lesigne [5], Kunita [17], Bally and Matoussi [4], and extended by Zhang and Zhao [38] as well. Here we consider the FBSDEs case.

Lemma 3.2.3. *(Norm Equivalence Result) Under Condition **(B.Class 1)** (or **(B.Class 2)**), let $X_s^{t,x}$ be the solution of the forward SDE defined in FBSDEs (2.1.4), ρ be a weighted function. For every $s \in [t, T]$, $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^k$ and $\varphi \in L^1_\rho(\mathbb{R}^d; \mathbb{R}^k)$, then there exist two constants $c > 0$ and $C > 0$ such that*

$$c \int_{\mathbb{R}^d} |\varphi(x)| \rho(x) dx \leq \mathbb{E} \left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})| \rho(x) dx \right] \leq C \int_{\mathbb{R}^d} |\varphi(x)| \rho(x) dx. \quad (3.2.4)$$

Moreover for every $\Psi : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$ and $\Psi \in L^1_\rho([t, T] \otimes \mathbb{R}^d; \mathbb{R}^k)$, then

$$c \int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)| \rho(x) dx ds \leq \mathbb{E} \left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, X_s^{t,x})| \rho(x) dx ds \right]$$

$$\leq C \int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)| \rho(x) dx ds. \quad (3.2.5)$$

Here constants c and C depend on T, L, μ, ρ and on the bounds of the first order derivatives of b, σ, h and f , and do not depend on the initial value x .

Proof. First, we take $\rho(x) := (1 + |x|^2)^p$ for any $p \in [2, \infty)$. And we claim that there exist two constants $c > 0$ and $C > 0$ such that

$$c \leq \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y}) \rho(\hat{X}_s^{t,y})}{\rho(x)} \right] \leq C, \quad \forall y \in \mathbb{R}^d, \quad t \leq s \leq T. \quad (3.2.6)$$

Here $\hat{X}_s^{t,y}$ is the inverse flow of $X_s^{t,x}$. $J(\hat{X}_s^{t,y}) := \det \nabla \hat{X}_s^{t,y}$ is the determinant of the Jacobian matrix of $\hat{X}_s^{t,y}$. Using a similar procedure as in the proof of Lemma 3.2.1, we have (3.2.1) and (3.2.2) for negative power q . By these results, we can verify that $X_s^{t,\cdot} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is injective and surjective a.s.. The proof is almost the same as that of Kunita (see [15], pp. 224-227). Therefore the inverse flow $\hat{X}_s^{t,y}$ exists.

Now we prove (3.2.6). We assume that $T - h \leq t \leq T$ for some small $h > 0$. And we substitute $x = \hat{X}_s^{t,y}$ into FBSDEs (2.1.4) (see Kunita [15], pp. 234-237), with $X_s^{t,\hat{X}_s^{t,y}} = X_s^{t,\cdot} \circ \hat{X}_s^{t,y} = y$, we have the following

$$\begin{cases} \hat{X}_s^{t,y} = y - \int_t^s b(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) dr - \int_t^s \sigma(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) dW_r, \\ Y_s^{t,\hat{X}_s^{t,y}} = h(X_T^{t,\hat{X}_s^{t,y}}) + \int_s^T f(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}, Z_r^{t,\hat{X}_s^{t,y}}) dr - \int_s^T Z_r^{t,\hat{X}_s^{t,y}} dW_r. \end{cases} \quad (3.2.7)$$

Here we define the integral $\int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r|_{x=\hat{X}_s^{t,y}} := \int_t^s \sigma(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) dW_r$. Others can be treated similarly. We differentiate with respect to y in (3.2.7) in order to get

$$\begin{aligned} & \nabla \hat{X}_s^{t,y} \\ = & I - \int_t^s b'_x(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) \nabla X_r^{t,\hat{X}_s^{t,y}} dr - \int_t^s b'_y(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) \nabla Y_r^{t,\hat{X}_s^{t,y}} dr \\ & - \int_t^s \sigma'_x(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) \nabla X_r^{t,\hat{X}_s^{t,y}} dW_r - \int_t^s \sigma'_y(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) \nabla Y_r^{t,\hat{X}_s^{t,y}} dW_r \\ := & I + J_s^t(y). \end{aligned} \quad (3.2.8)$$

For the upper bound, by Cauchy-Schwarz inequality,

$$\begin{aligned} & \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y}) \rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \\ \leq & \sqrt{\mathbb{E} |J(\hat{X}_s^{t,y})|^2} \sqrt{\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2} \end{aligned}$$

$$\leq \sqrt{C + C\mathbb{E}\|J_s^t(y)\|^2} \sqrt{\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2},$$

where $\|z\| := \sqrt{\text{tr}(zz^*)}$ and

$$\begin{aligned} \left| J(\hat{X}_s^{t,y}) \right|^2 &= \left| \det(I + J_s^t(y)) \right|^2 \\ &\leq \left| 1 + \text{Tr}(J_s^t(y)) + o(\|J_s^t(y)\|) \right|^2 \\ &\leq 3 \left(1 + |\text{Tr}(J_s^t(y))|^2 + o(\|J_s^t(y)\|^2) \right) \\ &\leq C \left(1 + \text{Tr}[J_s^t(y)(J_s^t(y))^*] \right). \end{aligned}$$

For the lower bound, we have $J(\hat{X}_s^{t,y}) \geq 1 - c\|J_s^t(y)\|$. Similarly,

$$\mathbb{E} \left[\frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \geq \mathbb{E} \left[\frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] - c \sqrt{\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2} \sqrt{\mathbb{E}\|J_s^t(y)\|^2}.$$

It is obvious that the upper and lower bounds depending on the estimates of $\mathbb{E}\|J_s^t(y)\|^2$, $\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2$ and $\mathbb{E} \left[\frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right]$.

Now we first calculate $\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,x})}{\rho(x)} \right|^2$ and $\mathbb{E} \left[\frac{\rho(\hat{X}_s^{t,x})}{\rho(x)} \right]$. From (3.2.7), weighted function $\rho(x) := (1 + |x|^2)^p$ and applying Itô's formula to $(1 + |\hat{X}_s^{t,y}|^2)^p$,

$$\begin{aligned} (1 + |\hat{X}_s^{t,y}|^2)^p &= (1 + |y|^2)^p \\ &\quad - 2p \int_t^s (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^{p-1} X_r^{t,\hat{X}_s^{t,y}} b(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) dr \\ &\quad - 2p \int_t^s (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^{p-1} X_r^{t,\hat{X}_s^{t,y}} \sigma(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) dW_r \\ &\quad - p(2p-1) \int_t^s (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^{p-2} |X_r^{t,\hat{X}_s^{t,y}}|^2 \|\sigma(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}})\|^2 dr \\ &:= (1 + |y|^2)^p + S_s^t(y). \end{aligned} \tag{3.2.9}$$

And

$$1 - \frac{|S_s^t(y)|}{(1 + |y|^2)^p} \leq \frac{(1 + |\hat{X}_s^{t,y}|^2)^p}{(1 + |y|^2)^p} \leq 1 + \frac{|S_s^t(y)|}{(1 + |y|^2)^p}. \tag{3.2.10}$$

From (3.2.7), using a similar method as in the proof of Lemma 3.2.1, for any $p \in [2, \infty)$, $r \in [s, T]$, there exists a constant c only depending on p, L, μ and T such that

$$\mathbb{E} \sup_{s \leq r \leq T} (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^p + \mathbb{E} \sup_{s \leq r \leq T} (1 + |Y_r^{t,\hat{X}_s^{t,y}}|^2)^p \leq c(1 + |y|^2)^p.$$

Similarly

$$\mathbb{E} \sup_{t \leq r \leq s} (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^p + \mathbb{E} \sup_{t \leq r \leq s} (1 + |Y_r^{t,\hat{X}_s^{t,y}}|^2)^p \leq c(1 + |y|^2)^p + \mathbb{E}|Y_s^{t,\hat{X}_s^{t,y}}|^2 p \leq c(1 + |y|^2)^p.$$

So

$$\mathbb{E}|S_s^t(y)| \leq c\mathbb{E} \int_t^s \left(1 + |X_r^{t, \hat{X}_s^{t,y}}|^2 + |Y_r^{t, \hat{X}_s^{t,y}}|^2\right)^p dr \leq (s-t)c(1 + |y|^2)^p.$$

Therefore (3.2.10) leads to

$$1 - c(s-t) \leq \mathbb{E} \left[\frac{(1 + |\hat{X}_s^{t,y}|^2)^p}{(1 + |y|^2)^p} \right] = \mathbb{E} \left[\frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \leq 1 + c(s-t). \quad (3.2.11)$$

Using similar estimates, $\mathbb{E} \left| \frac{\rho(\hat{X}_s^{\tau,x})}{\rho(x)} \right|^2$ easily follows

$$1 - c(s-t) \leq \mathbb{E} \left[\frac{(1 + |\hat{X}_s^{t,y}|^2)^{2p}}{(1 + |y|^2)^{2p}} \right] = \mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2 \leq 1 + c(s-t). \quad (3.2.12)$$

For $\mathbb{E}\|J_t^s(x)\|^2$, we consider (3.2.8), apply Itô's formula and use a similar method as in the proof of Lemma 3.2.2. Then there exists a constant c only depending on L, μ, T and on the bounds of the first order derivatives of b, σ, h and f such that

$$\mathbb{E} \sup_{t \leq r \leq T} (\|\nabla X_r^{t, \hat{X}_s^{t,y}}\|^2 + \|\nabla Y_r^{t, \hat{X}_s^{t,y}}\|^2) \leq c.$$

So

$$\mathbb{E}\|J_s^t(y)\|^2 \leq c(s-t). \quad (3.2.13)$$

From the result of (3.2.11), (3.2.12) and (3.2.13), the upper bound and the lower bound can be estimated as

$$B_{low} \leq \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \leq B_{up}.$$

Here $B_{low} = 1 - c(s-t) - c\sqrt{c(s-t)}\sqrt{1 + c(s-t)}$ and $B_{up} = \sqrt{C + c(s-t)}\sqrt{1 + c(s-t)}$. If $s-t$ small enough, the lower bound $1 - c(s-t) - c\sqrt{c(s-t)}\sqrt{1 + c(s-t)} > 0$. Therefore, we can take h small enough such that (3.2.6) holds for $T - h \leq s \leq T$. Note that the constants c and C depend on T, L, μ, ρ and on the bounds of the first order derivatives of b, σ, h and f , and does not depend on the initial value y . So we use the flow property $\hat{X}_s^{t,y} = \hat{X}_r^{t,\cdot} \circ \hat{X}_s^{r,y}, \forall t \leq r \leq s \leq T$ (using Proposition 2.2.9) in order to drop the restriction $T - h \leq t \leq T$ and so extend the inequality (3.2.6) to the whole of $[t, T]$.

Finally, we prove (3.2.4), using the change of variable $y = X_s^{t,x}$, conditional expectation with respect to $\mathcal{F}_{t,s}$, and noting that $\frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)}$ is $\mathcal{F}_{t,s}$ measurable, we get

$$\mathbb{E} \left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})| \rho(x) dx \right]$$

$$\begin{aligned}
&= \int_{\mathbb{R}^d} \mathbb{E} \left[\mathbb{E} \left[|\varphi(y)| \rho(y) \frac{J(\hat{X}_s^{t,y}) \rho(\hat{X}_s^{t,y})}{\rho(y)} \middle| \mathcal{F}_{t,s} \right] \right] dy \\
&= \int_{\mathbb{R}^d} |\varphi(y)| \rho(y) \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y}) \rho(\hat{X}_s^{t,y})}{\rho(y)} \right] dy.
\end{aligned}$$

By (3.2.6), $c \leq \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y}) \rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \leq C$, $\forall x \in \mathbb{R}^d$, $t \leq s \leq T$ for any $y \in \mathbb{R}^d$, $s \in [t, T]$, we prove (3.2.4). Moreover, for $x \rightarrow \Psi(s, x)$ and integrating with respect to $s \in [t, T]$ we get (3.2.5) as well so the lemma is proved. \square

Next we will give our flow properties. From Proposition 2.2.9, we have,

Proposition 3.2.4. *Under Condition (C.Class 1) (or (C.Class 2)) without condition (C.0), FBSDEs (2.1.4) has a unique solution $(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \in M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, then for any $t \leq s \leq T$, $X_r^{s, X_s^{t,x}} = X_r^{t,x}$, $Y_r^{s, X_s^{t,x}} = Y_r^{t,x}$ and $Z_r^{s, X_s^{t,x}} = Z_r^{t,x}$ for any $r \in [s, T]$ and a.e. $x \in \mathbb{R}^d$ a.s..*

3.3 Weak Solutions of Quasi-linear Parabolic PDEs in the Sobolev space

Next, we will use the idea of Bally and Matoussi [4], Zhang and Zhao [38] to give the correspondence between the weak solutions of PDEs (2.1.3) and FBSDEs (2.1.4). The outline of the proof is on follows: First, we construct a smotherized FBSDEs (3.3.1), with C^∞ functions $(b^m, \sigma^m, f^m, h^m) \rightarrow (b, \sigma, f, h)$ as $m \rightarrow \infty$. Then we prove that $(X_{s,m}^{t,x}, Y_{s,m}^{t,x}, Z_{s,m}^{t,x}) \rightarrow (X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$. Here $(X_{s,m}^{t,x}, Y_{s,m}^{t,x}, Z_{s,m}^{t,x})$ is the solution of the smotherized FBSDEs.

Since the coefficients b^m, σ^m, f^m, h^m satisfy Condition (B.Class 1) (or (B.Class 2)), by Theorem 2.3.3, we have $u^m(t, x) = Y_{t,m}^{t,x}$ is the classical solution of the corresponding smotherized PDEs (3.3.2). Meanwhile, $u^m(t, x)$ also satisfies the weak formulation of smotherized PDEs (3.3.4).

By using the equivalence of norm result, flow property and relations: $u^m(s, X_s^{t,x}) = Y_{s,m}^{t,x} \rightarrow Y_s^{t,x} = u(s, X_s^{t,x})$ as $m \rightarrow \infty$, we can show that the weak formulation of smotherized PDEs (3.3.4) converge to the weak formulation of PDEs (3.1.1). Therefore we prove that $u(t, x)$ is the weak solution of PDEs (2.1.3).

3.3. WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC PDES IN THE SOBOLEV SPACE

Now we define the smoothness coefficients. Suppose that $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Hölder-continuous function with exponent γ and let us define for each $m > 0$, the smooth function

$$\phi^m(x) := \int_{\mathbb{R}^d} K_d^m(x - x')\phi(x')dx'.$$

Here we define the mollifier:

$$K_d(x) := \begin{cases} C_d \exp\left(\frac{-1}{1 - |x|^2}\right) & , |x| < 1; \\ 0 & , |x| \geq 1, \end{cases}$$

C_d is chosen so that $\int_{\mathbb{R}^d} K_d(x)dx = 1$. And we let $K_d^m(x) := m^d K_d(mx)$. As a result, ϕ^m is a C^∞ function, and Hölder-continuous with exponent γ . Moreover, $\phi^m \rightarrow \phi$ uniformly on \mathbb{R} as $m \rightarrow \infty$. Similarly, we define

$$\begin{aligned} h^m(x) &= \int_{\mathbb{R}^d} K_d^m(x - x')h(x')dx', \\ \sigma^m(r, x, y) &= \int_{\mathbb{R}^d \times \mathbb{R}^k} K_d^m(x - x')K_k^m(y - y')\sigma(r, x', y')dx'dy' \\ b^m(r, x, y) &= \int_{\mathbb{R}^d \times \mathbb{R}^k} K_d^m(x - x')K_k^m(y - y')b(r, x', y')dx'dy' \\ f^m(r, x, y, z) &= \int_{\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}} K_d^m(x - x')K_k^m(y - y')K_{k \times d}^m(z - z')f(r, x', y', z')dx'dy'dz'. \end{aligned}$$

It is easy to see that, $(b^m, \sigma^m, f^m, h^m)_{m \in \mathbb{N}}$ are C^∞ smooth functions such that for any $t \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{k \times d}$, $(b^m, \sigma^m, f^m, h^m)(t, x, y, z) \rightarrow (b, \sigma, f, h)(t, x, y, z)$ in L_ρ^2 sense as $m \rightarrow \infty$. From the definition, one can easily check that h^m , b^m , σ^m and f^m also satisfy the monotone-Lipschitz condition which is independent of m . From Theorem 1.3.3 or Theorem 1.3.4, the smotherized FBSDE

$$\begin{cases} X_{s,m}^{t,x} = x + \int_t^s b^m(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x})dr + \int_t^s \sigma^m(r, X_{r,m}^{t,x}, Y_{s,m}^{t,x})dW_r, \\ Y_{s,m}^{t,x} = h^m(X_{T,m}^{t,x}) + \int_s^T f^m(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}, Z_{r,m}^{t,x})dr - \int_s^T Z_{r,m}^{t,x}dW_r, \end{cases} \quad (3.3.1)$$

has a unique solution $(X_{s,m}^{t,x}, Y_{s,m}^{t,x}, Z_{s,m}^{t,x})_{t \leq s \leq T} \in M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. Moreover, the corresponding regularity properties in Chapter 2 still hold which is also independent of m .

Lemma 3.3.1. *Under Conditions (B.1), (B.2) and (C.2) (resp. (B.3), (B.4) and (C.2)), there exists a sequence of C^∞ functions $(b^m, \sigma^m, f^m, h^m)_{m \in \mathbb{N}}$ such that for any $t \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{k \times d}$, $(b^m, \sigma^m, f^m, h^m)(t, x, y, z) \rightarrow (b, \sigma, f, h)(t, x, y, z)$*

3.3. WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC PDES IN THE SOBOLEV SPACE

as $m \rightarrow \infty$. Moreover, $(X_{\cdot,m}^{t,\cdot}, Y_{\cdot,m}^{t,\cdot}, Z_{\cdot,m}^{t,\cdot}) \rightarrow (X_{\cdot,\cdot}^{t,\cdot}, Y_{\cdot,\cdot}^{t,\cdot}, Z_{\cdot,\cdot}^{t,\cdot})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$, where $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ is the solution of FBSDEs (2.1.4) and $(X_{s,m}^{t,x}, Y_{s,m}^{t,x}, Z_{s,m}^{t,x})$ is the solution of the smotherized FBSDEs (3.3.1).

Proof. The first part of the lemma was given above. Now we prove $(X_{\cdot,m}^{t,\cdot}, Y_{\cdot,m}^{t,\cdot}, Z_{\cdot,m}^{t,\cdot}) \rightarrow (X_{\cdot,\cdot}^{t,\cdot}, Y_{\cdot,\cdot}^{t,\cdot}, Z_{\cdot,\cdot}^{t,\cdot})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$, where $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ is the solution of FBSDEs (2.1.4). Applying Itô's formula to $e^{-Ks}|X_{s,m}^{t,x} - X_s^{t,x}|^2$ and $e^{-Ks}|Y_{s,m}^{t,x} - Y_s^{t,x}|^2$, using a similar estimate as in the proof of Theorem 1.3.3 we have

$$\begin{aligned}
& (K - 2L^3 - L^2 - 5L - \frac{19}{20})\mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |X_{r,m}^{t,x} - X_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
& + (2\mu - K - L^2 - 6L - \frac{3}{4})\mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |Y_{r,m}^{t,x} - Y_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
& + \frac{4}{5}\mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} \|Z_{r,m}^{t,x} - Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
\leq & 2\mathbb{E} \int_{\mathbb{R}^d} e^{-KT} \left| h^m(X_{T,m}^{t,x}) - h^m(X_T^{t,x}) \right| \left| h^m(X_T^{t,x}) - h(X_T^{t,x}) \right| \rho^{-1}(x) dx \\
& + \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} \left| h^m(X_T^{t,x}) - h(X_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\
& + 2\mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} \left\langle X_{r,m}^{t,x} - X_r^{t,x}, b^m(r, X_r^{t,x}, Y_r^{t,x}) - b(r, X_r^{t,x}, Y_r^{t,x}) \right\rangle \rho^{-1}(x) dx dr \\
& + 2\mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} \left\langle Y_{r,m}^{t,x} - Y_r^{t,x}, \right. \\
& \quad \left. f^m(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \right\rangle \rho^{-1}(x) dx dr \\
& + 2\mathbb{E} \int_{\mathbb{R}^d} e^{-Kr} \left\| \sigma^m(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}) - \sigma^m(r, X_r^{t,x}, Y_r^{t,x}) \right\| \\
& \quad \left\| \sigma^m(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t,x}, Y_r^{t,x}) \right\| \rho^{-1}(x) dx \\
& + \mathbb{E} \int_{\mathbb{R}^d} e^{-Kr} \left\| \sigma^m(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t,x}, Y_r^{t,x}) \right\|^2 \rho^{-1}(x) dx \\
\leq & \frac{1}{N_1} \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} \left(|X_{T,m}^{t,x} - X_T^{t,x}|^2 \right) \rho^{-1}(x) dx \\
& + N_1 \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} \left| h^m(X_T^{t,x}) - h(X_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\
& + \frac{1}{N_2} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} (|X_{r,m}^{t,x} - X_r^{t,x}|^2) \rho^{-1}(x) dx dr \\
& + N_2 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |b^m(r, X_r^{t,x}, Y_r^{t,x}) - b(r, X_r^{t,x}, Y_r^{t,x})|^2 \rho^{-1}(x) dx dr \\
& + \frac{1}{N_3} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} (|Y_{r,m}^{t,x} - Y_r^{t,x}|^2) \rho^{-1}(x) dx dr \\
& + N_3 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |f^m(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})|^2 \rho^{-1}(x) dx dr
\end{aligned}$$

3.3. WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC PDES IN THE SOBOLEV SPACE

$$\begin{aligned}
& + \frac{1}{N_4} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} (|X_{r,m}^{t,x} - X_r^{t,x}| + |Y_{r,m}^{t,x} - Y_r^{t,x}|^2) \rho^{-1}(x) dx dr \\
& + N_4 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} \|\sigma^m(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 \rho^{-1}(x) dx dr
\end{aligned}$$

From Step 3 in the proof of Theorem 1.3.3 we have

$$\begin{aligned}
& \frac{1}{N_1} \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} (|X_{T,m}^{t,x} - X_T^{t,x}|^2) \rho^{-1}(x) dx \\
& \leq \frac{1}{N_1} (4L^2 + L + \frac{1}{4L}) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |X_{r,m}^{t,x} - X_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
& \quad + \frac{1}{N_1} (L + \frac{1}{4L}) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |Y_{r,m}^{t,x} - Y_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
& \quad + \frac{1}{N_1} \frac{1}{4L} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} \|Z_{r,m}^{t,x} - Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
& \leq \frac{1}{N_5} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |X_{r,m}^{t,x} - X_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
& \quad + \frac{1}{N_5} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |Y_{r,m}^{t,x} - Y_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
& \quad + \frac{1}{N_5} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} \|Z_{r,m}^{t,x} - Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr
\end{aligned}$$

Here we can choose $\frac{1}{N_1}$ small enough such that above inequality holds. Finally, we can choose $\frac{1}{N_2}, \frac{1}{N_3}, \frac{1}{N_4}, \frac{1}{N_5}$ small enough such that $\frac{1}{N_5} \leq \frac{1}{20}, \frac{1}{N_3} + \frac{1}{N_4} + \frac{1}{N_5} \leq \frac{1}{4}$ and $\frac{1}{N_2} + \frac{1}{N_4} + \frac{1}{N_5} \leq \frac{1}{20}$. Eventually we have

$$\begin{aligned}
& (K - 4L^3 - L^2 - 9L - \sum_{j=1}^{\infty} L_j - 1) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |X_{s,m}^{t,x} - X_s^{t,x}|^2 \rho^{-1}(x) dx dr \\
& + (2\mu - K - L^2 - 9L - \sum_{j=1}^{\infty} L_j - 1) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |Y_{s,m}^{t,x} - Y_s^{t,x}|^2 \rho^{-1}(x) dx dr \\
& + \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} \|Z_{s,m}^{t,x} - Z_s^{t,x}\|^2 \rho^{-1}(x) dx dr \\
& \leq N_1 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |h^m(X_T^{t,x}) - h(X_T^{t,x})|^2 \rho^{-1}(x) dx dr \\
& \quad + N_2 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |b^m(r, X_s^{t,x}, Y_s^{t,x}) - b(r, X_s^{t,x}, Y_s^{t,x})|^2 \rho^{-1}(x) dx dr \\
& \quad + N_3 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} |f^m(r, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) - f(r, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})|^2 \rho^{-1}(x) dx dr \\
& \quad + N_4 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} e^{-Kr} \|\sigma^m(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 \rho^{-1}(x) dx dr \\
& \rightarrow 0 \quad \text{as } m \rightarrow \infty
\end{aligned}$$

Note that, for any $t \in [0, T], x \in \mathbb{R}^d, y \in \mathbb{R}^k, z \in \mathbb{R}^{k \times d}, (b^m, \sigma^m, f^m, h^m) \rightarrow (b, \sigma, f, h)$ in L^2_ρ

3.3. WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC PDES IN THE SOBOLEV SPACE

sense as $m \rightarrow \infty$. So $(X_{\cdot,m}^{t,\cdot;n}, Y_{\cdot,m}^{t,\cdot;n}, Z_{\cdot,m}^{t,\cdot;n}) \rightarrow (X_{\cdot}^{t,\cdot;n}, Y_{\cdot}^{t,\cdot;n}, Z_{\cdot}^{t,\cdot;n})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$. \square

Lemma 3.3.2. *Under Condition (C.Class 1) (or (C.Class 2)), $(\nabla X_{\cdot,m}^{t,\cdot}, \nabla Y_{\cdot,m}^{t,\cdot}, \nabla Z_{\cdot,m}^{t,\cdot}) \rightarrow (\nabla X_{\cdot}^{t,\cdot}, \nabla Y_{\cdot}^{t,\cdot}, \nabla Z_{\cdot}^{t,\cdot})$ in L_ρ^2 sense as $m \rightarrow \infty$, where $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})$ is the solution of ∇ FBSDEs (2.2.31) and $(\nabla X_{s,m}^{t,x}, \nabla Y_{s,m}^{t,x}, \nabla Z_{s,m}^{t,x})$ is the solution of the following smoothened ∇ FBSDEs.*

$$\left\{ \begin{array}{l} \nabla X_{s,m}^{t,x} = 1 + \int_t^s (\sigma^m)'_x(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}) \nabla X_{r,m}^{t,x} dW_r + \int_t^s (\sigma^m)'_y(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}) \nabla Y_{r,m}^{t,x} dW_r \\ \quad + \int_t^s (b^m)'_x(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}) \nabla X_{r,m}^{t,x} dr + \int_t^s (b^m)'_y(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}) \nabla Y_{r,m}^{t,x} dr \\ \nabla Y_{s,m}^{t,x} = (h^m)'(X_{T,m}^{t,x}) \nabla X_{T,m}^{t,x} - \int_s^T \nabla Z_{r,m}^{t,x} dW_r \\ \quad + \int_s^T (f^m)'_x(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}, Z_{r,m}^{t,x}) \nabla X_{r,m}^{t,x} dr + (f^m)'_y(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}, Z_{r,m}^{t,x}) \nabla Y_{r,m}^{t,x} dr \\ \quad + \int_s^T (f^m)'_z(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}, Z_{r,m}^{t,x}) \nabla Z_{r,m}^{t,x} dr. \end{array} \right.$$

Proof. Since $(b^m, \sigma^m, f^m, h^m)_{m \in \mathbb{N}}$ are C^∞ smooth functions, it is easy to check that $(\nabla X_{s,m}^{t,x}, \nabla Y_{s,m}^{t,x}, \nabla Z_{s,m}^{t,x})$ is the unique solution of above linear FBSDEs. Following the same procedure of the proof of Lemma 3.3.1, we can also show that $(\nabla X_{\cdot,m}^{t,\cdot}, \nabla Y_{\cdot,m}^{t,\cdot}, \nabla Z_{\cdot,m}^{t,\cdot}) \rightarrow (\nabla X_{\cdot}^{t,\cdot}, \nabla Y_{\cdot}^{t,\cdot}, \nabla Z_{\cdot}^{t,\cdot})$ in L_ρ^2 sense as $m \rightarrow \infty$. \square

Theorem 3.3.3. *Under Condition (C.Class 1) (or (C.Class 2)), Let $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ be the solution of FBSDEs (2.1.4). If we define $u(t, x) = Y_t^{t,x}$, then $\sigma^*(t, x, u(t, x)) \nabla u(t, x)$ exists for a.e. $t \in [0, T]$, $x \in \mathbb{R}^d$, and $u(s, X_s^{t,x}) = Y_s^{t,x}$, $\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s..*

Proof. From Lemma 3.3.1, we know that $(X_{s,m}^{t,x}, Y_{s,m}^{t,x}, Z_{s,m}^{t,x})$ is the unique solution of the smoothened FBSDEs (3.3.1), and f^m, b^m, σ^m, h^m are in C^∞ . Following the result of Theorem 2.3.3, the following smoothened PDEs has a unique solution $u^m(t, x)$:

$$\left\{ \begin{array}{l} \partial_t u^m(t, x) + \mathcal{L}^m u^m(t, x) + f^m(t, x, u^m(t, x), (\sigma^m)^*(t, x, u^m(t, x))) \nabla u^m(t, x) = 0, \\ u^m(T, x) = h^m(x), \end{array} \right. \quad (3.3.2)$$

3.3. WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC PDES IN THE SOBOLEV SPACE

where

$$\mathcal{L}^m = \frac{1}{2} \sum_{i,j=1}^d (\sigma^m(\sigma^m)^*)_{ij}(t, x, u^m(t, x)) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i^m(t, x, u^m(t, x)) \frac{\partial}{\partial x_i}.$$

Let $u^m(t, x) = Y_{t,m}^{t,x}$, we have $Z_{t,m}^{t,x} = (\sigma^m)^*(t, x, u^m(t, x)) \nabla u^m(t, x)$. From Proposition ?? we have

$$u^m(s, X_{s,m}^{t,x}) = Y_{s,m}^{t,x}, \quad (\sigma^m)^*(s, X_{s,m}^{t,x}, u^m(s, X_{s,m}^{t,x})) \nabla u^m(s, X_{s,m}^{t,x}) = Z_{s,m}^{t,x}. \quad (3.3.3)$$

Moreover, $u^m(t, x)$ also satisfies the following weak formulation of PDEs (3.3.4). For any smooth test function $\Psi \in C_c^{1,\infty}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$, we have

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} u^m(s, x) \partial_s \Psi(s, x) dx ds + \int_{\mathbb{R}^d} u^m(t, x) \Psi(t, x) dx - \int_{\mathbb{R}^d} h^m(x) \Psi(T, x) dx \\ & + \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} \sigma^*(s, x, u^m(s, x)) \nabla \Psi(s, x) dx ds \\ & + \int_t^T \int_{\mathbb{R}^d} u^m(s, x) \operatorname{div} \left((b^m - \tilde{A}^m) \Psi(s, x) \right) dx ds \\ & = \int_t^T \int_{\mathbb{R}^d} f^m(s, x, u^m(s, x), (\sigma^m)^*(s, x, u^m(s, x)) \nabla u^m(s, x)) \Psi(s, x) dx ds. \end{aligned} \quad (3.3.4)$$

Here $\tilde{A}_j^m \triangleq \sum_{i=1}^d \frac{\partial (\sigma^m(\sigma^m)^*)_{i,j}(s, x, u(s, x))}{\partial x_i}$, and $\tilde{A}^m = (\tilde{A}_1^m, \tilde{A}_2^m, \dots, \tilde{A}_d^m)^*$.

For any $m_1, m_2 \in \mathbb{N}$, by Lemma 3.2.3 and the fact that $u^{m_1}(s, x), u^{m_2}(s, x) \in C^{1,2}$ (see Theorem 2.3.3), we have the following estimate

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} (c|u^{m_1}(s, x) - u^{m_2}(s, x)|^2 - c' \mathbb{E}|X_{s,m_1}^{0,x} - X_{s,m_2}^{0,x}|^2) \rho(x) dx ds \\ & \leq \int_0^T \int_{\mathbb{R}^d} \mathbb{E} (|u^{m_1}(s, X_{s,m_1}^{0,x}) - u^{m_2}(s, X_{s,m_2}^{0,x})|^2) \rho(x) dx ds \\ & \leq C \int_0^T \int_{\mathbb{R}^d} (|u^{m_1}(s, x) - u^{m_2}(s, x)|^2 + \mathbb{E}|X_{s,m_1}^{0,x} - X_{s,m_2}^{0,x}|^2) \rho(x) dx ds. \end{aligned} \quad (3.3.5)$$

Now we prove $u^m(\cdot, \cdot)$ is a Cauchy sequence in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$. For this, by (3.3.5), (3.3.3) and Lemma 3.3.1, as $m_1, m_2 \rightarrow \infty$, we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} (|u^{m_1}(s, x) - u^{m_2}(s, x)|^2) \rho^{-1}(x) dx ds \\ & \leq C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (|u^{m_1}(s, X_{s,m_1}^{0,x}) - u^{m_2}(s, X_{s,m_2}^{0,x})|^2 + |X_{s,m_1}^{0,x} - X_{s,m_2}^{0,x}|^2) \rho^{-1}(x) dx ds \\ & = C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (|Y_{s,m_1}^{0,x} - Y_{s,m_2}^{0,x}|^2 + |X_{s,m_1}^{0,x} - X_{s,m_2}^{0,x}|^2) \rho^{-1}(x) dx ds \\ & \leq C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (|Y_{s,m_1}^{0,x} - Y_s^{0,x}|^2 + |Y_{s,m_2}^{0,x} - Y_s^{0,x}|^2 \\ & \quad + |X_{s,m_1}^{0,x} - X_s^{0,x}|^2 + |X_{s,m_2}^{0,x} - X_s^{0,x}|^2) \rho^{-1}(x) dx ds \end{aligned}$$

$\rightarrow 0$.

So there exists $\hat{u} \in M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k))$ as the limit of u^m such that

$$\int_0^T \int_{\mathbb{R}^d} (|u^m(s, x) - \hat{u}(s, x)|^2) \rho^{-1}(x) dx ds \rightarrow 0, \quad \text{as } m \rightarrow \infty. \quad (3.3.6)$$

We define $u(t, x) = Y_t^{t, x}$, then by Proposition 3.2.4, we have $u(s, X_s^{t, x}) = Y_s^{s, X_s^{t, x}} = Y_s^{t, x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s.. Again by (3.3.5), (3.3.3), Lemma 3.3.1, (3.3.6) and Lemma 3.3.1,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} |u(s, x) - \hat{u}(s, x)|^2 \rho^{-1}(x) dx ds \\ & \leq 2 \int_0^T \int_{\mathbb{R}^d} (|u(s, x) - u^m(s, x)|^2 + |u^m(s, x) - \hat{u}(s, x)|^2) \rho^{-1}(x) dx ds \\ & \leq C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (|u(s, X_s^{0, x}) - u^m(s, X_{s, m}^{0, x})|^2 + |X_s^{0, x} - X_{s, m}^{0, x}|^2 \\ & \quad + |u^m(s, x) - \hat{u}(s, x)|^2) \rho^{-1}(x) dx ds \\ & = C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (|Y_s^{0, x} - Y_{s, m}^{0, x}|^2 + |X_s^{0, x} - X_{s, m}^{0, x}|^2 + |u^m(s, x) - \hat{u}(s, x)|^2) \rho^{-1}(x) dx ds \\ & \rightarrow 0, \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Hence $u(t, x) = \hat{u}(t, x)$ for a.e. $t \in [0, T]$, $x \in \mathbb{R}^d$.

By Lemma 3.2.3 and the fact that $u^{m_1}(s, x), u^{m_2}(s, x) \in C^{1,2}$ we have the following estimate

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} (c \|\nabla u^{m_1}(s, x) - \nabla u^{m_2}(s, x)\|^2 - c' \mathbb{E} |X_{s, m_1}^{0, x} - X_{s, m_2}^{0, x}|^2) \rho(x) dx ds \\ & \leq \int_0^T \int_{\mathbb{R}^d} \mathbb{E} (\|\nabla u^{m_1}(s, X_{s, m_1}^{0, x}) - \nabla u^{m_2}(s, X_{s, m_2}^{0, x})\|^2) \rho(x) dx ds \\ & \leq C \int_0^T \int_{\mathbb{R}^d} (\|\nabla u^{m_1}(s, x) - \nabla u^{m_2}(s, x)\|^2 + \mathbb{E} |X_{s, m_1}^{0, x} - X_{s, m_2}^{0, x}|^2) \rho(x) dx ds. \end{aligned} \quad (3.3.7)$$

By (3.3.7) and Lemma 3.3.2, as $m_1, m_2 \rightarrow \infty$, we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} (\|\nabla u^{m_1}(s, x) - \nabla u^{m_2}(s, x)\|^2) \rho^{-1}(x) dx ds \\ & \leq C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (\|\nabla u^{m_1}(s, X_{s, m_1}^{0, x}) - \nabla u^{m_2}(s, X_{s, m_2}^{0, x})\|^2 + |X_{s, m_1}^{0, x} - X_{s, m_2}^{0, x}|^2) \rho^{-1}(x) dx ds \\ & = C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (\|\nabla Y_{s, m_1}^{0, x} - \nabla Y_{s, m_2}^{0, x}\|^2 + |X_{s, m_1}^{0, x} - X_{s, m_2}^{0, x}|^2) \rho^{-1}(x) dx ds \\ & \leq C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (\|\nabla Y_{s, m_1}^{0, x} - \nabla Y_s^{0, x}\|^2 + \|\nabla Y_s^{0, x} - \nabla Y_{s, m_2}^{0, x}\|^2 \\ & \quad + |X_{s, m_1}^{0, x} - X_s^{0, x}|^2 + |X_{s, m_2}^{0, x} - X_s^{0, x}|^2) \rho^{-1}(x) dx ds \\ & \rightarrow 0. \end{aligned}$$

3.3. WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC PDES IN THE SOBOLEV SPACE

So there exists $\nabla \hat{u} \in M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as the limit of ∇u^m . And it is easy to verify that $\nabla u(t, x) = \nabla \hat{u}(t, x)$ for a.e. $t \in [0, T]$, $x \in \mathbb{R}^d$. Therefore $\sigma^*(t, x, u(t, x))\nabla u(t, x)$ exists for a.e. $t \in [0, T]$, $x \in \mathbb{R}^d$. And it is easy to prove that $\sigma^m(s, x, u^m(s, x)) \rightarrow \sigma(s, x, u(s, x))$ in L^2_ρ sense as $m \rightarrow \infty$.

Finally, we show that $\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s.. For this, we use the Cauchy-Schwarz inequality and the convergences of $\sigma^m(s, x, u^m(s, x))$ and $\nabla u^m(s, x)$, we have

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} (|(\sigma^m)^*(s, x, u^m(s, x))\nabla u^m(s, x) - \sigma^*(s, x, u(s, x))\nabla u(s, x)|) \rho(x) dx ds \\
\leq & \int_0^T \int_{\mathbb{R}^d} (|(\sigma^m)^*(s, x, u^m(s, x))\nabla u^m(s, x) - \sigma^*(s, x, u(s, x))\nabla u^m(s, x)|) \rho(x) dx ds \\
& + \int_0^T \int_{\mathbb{R}^d} (|\sigma^*(s, x, u(s, x))\nabla u^m(s, x) - \sigma^*(s, x, u(s, x))\nabla u(s, x)|) \rho(x) dx ds \\
\leq & \int_0^T \sqrt{\|(\sigma^m)^*(s, x, u^m(s, x)) - \sigma^*(s, x, u(s, x))\|_{L^2_\rho}^2} \sqrt{\|\nabla u^m(s, x)\|_{L^2_\rho}^2} ds \\
& + \int_0^T \sqrt{\|\sigma^*(s, x, u(s, x))\|_{L^2_\rho}^2} \sqrt{\|\nabla u^m(s, x) - \nabla u(s, x)\|_{L^2_\rho}^2} ds \\
\rightarrow & 0, \quad \text{as } m \rightarrow \infty.
\end{aligned} \tag{3.3.8}$$

Here $\|\nabla u^m(s, x)\|_{L^2_\rho}^2 \leq C\mathbb{E}\|\nabla u^m(s, X_{s,m}^{t,x})\|_{L^2}^2 = C\mathbb{E}\|\nabla Y_{s,m}^{t,x}\|_{L^2}^2 \leq C$ (by Lemma 3.2.3 and Lemma 3.2.2). Moreover, using Lemma 3.2.3, the Cauchy-Schwarz inequality, (3.3.8), Lemma 3.3.1, Lemma 3.3.2 and the fact that $\mathbb{E}\|\nabla Y_{s,m}^{t,x}\|_{L^2}^2$ being bounded (see Lemma 3.2.2), we have

$$\begin{aligned}
& \mathbb{E} \int_t^T \|(\sigma^m)^*(s, X_{s,m}^{t,x}, u^m(s, X_{s,m}^{t,x}))\nabla u^m(s, X_{s,m}^{t,x}) - \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\nabla u(s, X_s^{t,x})\|_{L^2_\rho} ds \\
\leq & \mathbb{E} \int_t^T \left(\|(\sigma^m)^*(s, X_{s,m}^{t,x}, u^m(s, X_{s,m}^{t,x}))\nabla u^m(s, X_{s,m}^{t,x}) - \sigma^*(s, X_{s,m}^{t,x}, u(s, X_{s,m}^{t,x}))\nabla u(s, X_{s,m}^{t,x})\|_{L^2_\rho} \right. \\
& \left. + \|\sigma^*(s, X_{s,m}^{t,x}, u(s, X_{s,m}^{t,x}))\nabla u(s, X_{s,m}^{t,x}) - \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\nabla u(s, X_s^{t,x})\|_{L^2_\rho} \right) ds \\
\leq & \int_0^T \left(C\|(\sigma^m)^*(s, x, u^m(s, x))\nabla u^m(s, x) - \sigma^*(s, x, u(s, x))\nabla u(s, x)\|_{L^2_\rho} \right. \\
& + \mathbb{E}\|\sigma^*(s, X_{s,m}^{t,x}, u(s, X_{s,m}^{t,x}))\nabla u(s, X_{s,m}^{t,x}) - \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\nabla u(s, X_s^{t,x})\|_{L^2_\rho} \\
& \left. + \mathbb{E}\|\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\nabla u(s, X_s^{t,x}) - \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\nabla u(s, X_s^{t,x})\|_{L^2_\rho} \right) ds \\
\leq & \int_0^T \left(C\|(\sigma^m)^*(s, x, u^m(s, x))\nabla u^m(s, x) - \sigma^*(s, x, u(s, x))\nabla u(s, x)\|_{L^2_\rho} \right. \\
& + \sqrt{\mathbb{E}\|\sigma^*(s, X_{s,m}^{t,x}, u(s, X_{s,m}^{t,x})) - \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\|_{L^2_\rho}^2} \sqrt{\mathbb{E}\|\nabla u(s, X_{s,m}^{t,x})\|_{L^2_\rho}^2} \\
& \left. + \sqrt{\mathbb{E}\|\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\|_{L^2_\rho}^2} \sqrt{\mathbb{E}\|\nabla u(s, X_{s,m}^{t,x}) - \nabla u(s, X_s^{t,x})\|_{L^2_\rho}^2} \right) ds \\
\leq & \int_0^T \left(C\|(\sigma^m)^*(s, x, u^m(s, x))\nabla u^m(s, x) - \sigma^*(s, x, u(s, x))\nabla u(s, x)\|_{L^2_\rho} \right.
\end{aligned}$$

3.3. WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC PDES IN THE SOBOLEV SPACE

$$\begin{aligned}
& + \sqrt{\mathbb{E} \left(|X_{s,m}^{t,x} - X_s^{t,x}|_{L_\rho^2}^2 + |Y_{s,m}^{t,x} - Y_s^{t,x}|_{L_\rho^2}^2 \right)} \sqrt{\mathbb{E} \|\nabla Y_{s,m}^{t,x}\|_{L_\rho^2}^2} \\
& + \sqrt{\mathbb{E} \|\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x}))\|_{L_\rho^2}^2} \sqrt{\mathbb{E} \|\nabla Y_{s,m}^{t,x} - \nabla Y_s^{t,x}\|_{L_\rho^2}^2} ds \\
& \rightarrow 0, \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

As a result, we get

$$\begin{aligned}
& \mathbb{E} \int_t^T \|\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) - Z_s^{t,x}\|_{L_\rho^2} ds \\
& \leq \mathbb{E} \int_t^T \left(\|\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) - (\sigma^m)^*(s, X_{s,m}^{t,x}, u^m(s, X_{s,m}^{t,x})) \nabla u^m(s, X_{s,m}^{t,x})\|_{L_\rho^2} \right. \\
& \quad \left. + \|(\sigma^m)^*(s, X_{s,m}^{t,x}, u^m(s, X_{s,m}^{t,x})) \nabla u^m(s, X_{s,m}^{t,x}) - Z_{s,m}^{t,x}\|_{L_\rho^2} + \|Z_{s,m}^{t,x} - Z_s^{t,x}\|_{L_\rho^2} \right) ds \\
& \rightarrow 0, \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

Note that above convergence hold in L_ρ^1 sense, but $Z_s^{t,x}$ is in L_ρ^2 sense as well. Therefore $\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s.. \square

Theorem 3.3.4. *Under Condition (C.Class 1) (or (C.Class 2)), if we define $u(t, x) = Y_t^{t,x}$, where $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ is the solution of FBSDE (2.1.4). Then $u(t, x)$ is the unique weak solution of PDEs (2.1.3) with $u(T, x) = h(x)$. Moreover, $u(s, X_s^{t,x}) = Y_s^{t,x}$, $\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s..*

Proof. From Theorem 3.3.3, we only need to verify that this u is the unique weak solution of PDE (2.1.3) with $u(T, x) = h(x)$. By Lemma 3.2.3,

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} (|u(s, x)|^2 + \|\sigma^*(s, x, u(s, x)) \nabla u(s, x)\|^2) \rho^{-1}(x) dx ds \\
& \leq C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (|u(s, X_s^{0,x})|^2 + \|(\sigma^*)^*(s, X_s^{0,x}, u(s, X_s^{0,x})) \nabla u(s, X_s^{0,x})\|^2) \rho^{-1}(x) dx ds \\
& = C \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (|Y_s^{0,x}|^2 + \|Z_s^{0,x}\|^2) \rho^{-1}(x) dx ds \\
& < \infty.
\end{aligned}$$

So $(u(s, x), \sigma^*(s, x, u(s, x)) \nabla u(s, x)) \in M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. Now we verify that $u(t, x)$ satisfies (3.1.1) with $u(T, x) = h(x)$ by passing the limit in L^2 to (3.3.4). We only show the convergence of the last term. By Lipschitz condition, the fact that $f^m(t, x, y, z) \rightarrow f(t, x, y, z)$ in L_ρ^2 sense as $m \rightarrow \infty$, and the convergences in Theorem 3.3.3,

$$\left| \int_t^T \int_{\mathbb{R}^d} f^m(s, x, u^m(s, x), (\sigma^m)^*(s, x, u^m(s, x)) \nabla u^m(s, x)) \Psi(s, x) dx ds \right.$$

3.3. WEAK SOLUTIONS OF QUASI-LINEAR PARABOLIC PDES IN THE SOBOLEV SPACE

$$\begin{aligned}
& \left| - \int_t^T \int_{\mathbb{R}^d} f(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x) \Psi(s, x) dx ds \right|^2 \\
\leq & C_p \int_t^T \int_{\mathbb{R}^d} |f^m(s, x, u^m(s, x), (\sigma^m)^*(s, x, u^m(s, x))) \nabla u^m(s, x) \\
& - f(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x)|^2 \rho^{-1}(x) dx ds \\
\leq & C_p \int_t^T \int_{\mathbb{R}^d} |f^m(s, x, u^m(s, x), (\sigma^m)^*(s, x, u^m(s, x))) \nabla u^m(s, x) \\
& - f^m(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x)|^2 \rho^{-1}(x) dx ds \\
& + C_p \int_t^T \int_{\mathbb{R}^d} |f^m(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x) \\
& - f(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x)|^2 \rho^{-1}(x) dx ds \\
\leq & C_{p,L} \int_t^T \int_{\mathbb{R}^d} \left(|u^m(s, x) - u(s, x)| \right. \\
& \left. + |(\sigma^m)^*(s, x, u^m(s, x)) \nabla u^m(s, x) - \sigma^*(s, x, u(s, x)) \nabla u(s, x)|^2 \right) \rho^{-1}(x) dx ds \\
& + C_p \int_t^T \int_{\mathbb{R}^d} |f^m(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x) \\
& - f(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x)|^2 \rho^{-1}(x) dx ds \\
\rightarrow & 0, \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

Therefore $u(t, x)$ satisfy (3.1.1), so is a weak solution of (2.1.3) with $u(T, x) = h(x)$. The uniqueness follows from Lemma 3.2.3 and the uniqueness of FBSDEs. \square

Chapter 4

Infinite Horizon FBSDEs and Quasi-linear Elliptic PDEs

4.1 Introduction and Preliminaries

The stationary solution for stochastic partial differential equations (SPDEs) is a fundamental concept in the study of the long time behaviour of random dynamical systems (RDS) driven by SPDEs [7, 10, 21, 32, 33, 38]. Zhang and Zhao [38] gave the stationary solution for type of SPDEs through the stationary solution of BDSDEs with infinite horizon.

For the deterministic case, it is easy to see that the stationary solutions of parabolic type partial differential equations (PDEs) give the solutions of elliptic type PDEs. On the other hand, we desire to prove the stationary solution of FBSDEs with infinite horizon and use the connection between the weak solution of quasi-linear parabolic PDEs and the solution of FBSDEs, in order to get the stationary solution of quasi-linear parabolic PDEs. Eventually, we obtain the unique weak solution of quasi-linear elliptic type PDEs.

In this chapter, we will study the unique weak solution of elliptic type PDEs through the stationary solution of FBSDEs with infinite horizon. This chapter is organized as follows: In Section 4.2, we study the infinite horizon FBSDEs (4.2.1). In Section 4.3, we give the stationary solution of the FBSDEs and relate it with a type of quasi-linear elliptic PDEs (4.3.1).

4.2 Infinite Horizon FBSDEs

In this section, we consider the following FBSDEs with infinite horizon,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x})dW_r, \\ e^{-Ks}Y_s^{t,x} = \int_s^\infty e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\ \quad + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr - \int_s^\infty e^{-Kr} Z_r^{t,x} dW_r, \end{cases} \quad (4.2.1)$$

for $s \geq t$. Here the functions $b : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^d$, $\sigma : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^{d \times d}$, $f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$. We also assume that b , σ and f are measurable functions with respect to the Borelian σ -fields. Equivalently we have the differential form,

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + \sigma(s, X_s^{t,x}, Y_s^{t,x})dW_s, \\ dY_s^{t,x} = -f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + Z_s^{t,x}dW_s, \\ X_t^{t,x} = x, \quad \lim_{T \rightarrow \infty} e^{-KT} Y_T^{t,x} = 0 \quad a.s.. \end{cases} \quad (4.2.2)$$

We consider following assumptions.

(D.1.1): Change $t \in [0, T]$ to $t \geq 0$ in **(A.2.1)**.

(D.1.2): Change $t \in [0, T]$ to $t \geq 0$, and $2\mu - K - 2L^2 - 7L - 1 > 0$ with $K > 2L^3 + L^2 + 5L + 1$ to $2\mu > K + 8\frac{1}{5}L + 1$ with $K > 5L + 1$ in **(A.2.2)**.

(D.1.3): Moreover, the following holds

$$\int_0^\infty e^{-Ks} (|b(s, 0, 0, 0)|^p ds + \|\sigma(s, 0, 0)\|^p + |f(s, 0, 0, 0)|^p) ds < \infty.$$

Definition 4.2.1. The process $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ is called a solution of (4.2.1) if for any $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^d)$ and $\tilde{\varphi} \in C_c^0(\mathbb{R}^d; \mathbb{R}^k)$,

$$\begin{cases} \int_{\mathbb{R}^d} X_s^{t,x} \varphi(x) dx = \int_{\mathbb{R}^d} x \varphi(x) dx + \int_t^s \int_{\mathbb{R}^d} b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \varphi(x) dx dr \\ \quad + \int_t^s \int_{\mathbb{R}^d} \sigma(r, X_r^{t,x}, Y_r^{t,x}) \varphi(x) dx dW_r, \\ \int_{\mathbb{R}^d} e^{-Ks} Y_s^{t,x} \tilde{\varphi}(x) dx = \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \tilde{\varphi}(x) dx dr \\ \quad + \int_s^\infty \int_{\mathbb{R}^d} K e^{-Kr} Y_r^{t,x} \tilde{\varphi}(x) dx dr \\ \quad - \int_s^\infty \int_{\mathbb{R}^d} e^{-Kr} Z_r^{t,x} \tilde{\varphi}(x) dx dW_r \quad \mathbb{P} - a.s.. \end{cases} \quad (4.2.3)$$

Before we study infinite horizon FBSDEs (4.2.1), we recall some results for BSDEs case.

Remark 4.2.2. *Zhang and Zhao ([38]) considered the following finite horizon BSDEs with $h = 0$,*

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r \\ Y_s^{t,x,n} = \int_s^n f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})dr - \int_s^n Z_r^{t,x,n}dW_r. \end{cases} \quad (4.2.4)$$

For each $n \in \mathbb{N}$, they proved that $(Y^{\cdot,t,\cdot,n}, Z^{\cdot,t,\cdot,n}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, the unique solution of above finite BSDEs (4.2.4), is also a Cauchy sequence in $S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ space. Therefore the limit of this sequence, denoted by $(Y_s^{t,x}, Z_s^{t,x})$, is a unique solution of following BSDEs with infinite horizon,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, \\ e^{-Ks}Y_s^{t,x} = \int_s^\infty e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\ \quad + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr - \int_s^\infty e^{-Kr} Z_r^{t,x} dW_r. \end{cases} \quad (4.2.5)$$

Here the coefficients satisfy the same conditions as those in FBSDEs (4.2.1).

For the infinite horizon BSDEs (4.2.5) we have,

Theorem 4.2.3. *Under Conditions (D.1.1)–(D.1.3) the BSDE (4.2.5) has a unique solution, i.e. there exist unique process $(X^{\cdot,t,\cdot}, Y^{\cdot,t,\cdot}, Z^{\cdot,t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfying (4.2.4).*

Proof. Note that the SDE in (4.2.5) is slightly different from that in Zhang and Zhao [38]. In both cases the SDE can be solved (see Øksendal [24] or Kunita [15]).

For the infinite horizon BSDEs (4.2.5), we can use a similar method as in the proof of Theorem 5.1 in [38] to prove that the BSDEs has a unique solution $(Y_s^{t,x}, Z_s^{t,x}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

It is easy to see that SDE in (4.2.5) has a unique solution $X_s^{t,x} \in M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. Then applying Itô's formula to $e^{-Ks}|X_s^{t,x}|^2$, taking the spatial integration $\rho^{-1}(x)dx$ on both sides and applying stochastic Fubini theorem, we have

$$\mathbb{E} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx + (K - L) \mathbb{E} \int_t^s \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x}|^2 \rho^{-1}(x) dx dr$$

$$\leq \int_{\mathbb{R}^d} e^{-Kt} x^2 \rho^{-1}(x) dx + C \int_t^s e^{-Kr} dr.$$

As $s \rightarrow \infty$, we have

$$\mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x}|^2 \rho^{-1}(x) dx dr < \infty.$$

By B-D-G inequality,

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx \\ & \leq C_p \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x}|^2 \rho^{-1}(x) dx dr + \int_{\mathbb{R}^d} e^{-Kt} x^2 \rho^{-1}(x) dx + C \int_t^T e^{-Kr} dr. \end{aligned}$$

As $T \rightarrow \infty$, we have

$$\mathbb{E} \sup_{s \geq t} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx < \infty.$$

Therefore we have,

$$\mathbb{E} \sup_{s \geq t} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x}|^2 \rho^{-1}(x) dx dr < \infty.$$

So $X_s^{t,x} \in S^{2,-K} \cap M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. Following a similar procedure as in Step 6 of the proof of Theorem 1.2.3, we can extend our result from $[t, \infty)$ to $[0, \infty)$. \square

Now we consider the infinite horizon FBSDEs (4.2.1),

Theorem 4.2.4. *Under Conditions (D.1.1) – (D.1.3), infinite horizon FBSDEs (4.2.1) has a unique solution, i.e. there exists a unique process $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfying (4.2.3).*

Proof. Step 1: First, recall the iterative procedure for finite horizon FBSDEs with $h = 0$. Given $(Y_s^{t,x,N-1}, Z_s^{t,x,N-1}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ and define $(X_s^{t,x,N}, Y_s^{t,x,N}, Z_s^{t,x,N})$ as follows

$$\begin{cases} X_s^{t,x,N} = x + \int_t^s b(r, X_r^{t,x,N}, Y_r^{t,x,N-1}, Z_r^{t,x,N-1}) dr + \int_t^s \sigma(r, X_r^{t,x,N}, Y_r^{t,x,N-1}) dW_r \\ Y_s^{t,x,N} = \int_s^T f(r, X_r^{t,x,N}, Y_r^{t,x,N}, Z_r^{t,x,N}) dr - \int_s^T Z_r^{t,x,N} dW_r. \end{cases} \quad (4.2.6)$$

For $N = 1$, let $(Y_s^{t,x,0}, Z_s^{t,x,0}) = (0, 0)$, (4.2.6) leads to

$$\begin{cases} X_s^{t,x,1} = x + \int_t^s b(r, X_r^{t,x,1}, 0, 0) dr + \int_t^s \sigma(r, X_r^{t,x,1}, 0) dW_r \\ Y_s^{t,x,1} = \int_s^T f(r, X_r^{t,x,1}, Y_r^{t,x,1}, Z_r^{t,x,1}) dr - \int_s^T Z_r^{t,x,1} dW_r. \end{cases} \quad (4.2.7)$$

Here (4.2.7) is a finite horizon BSDEs, which has the same form of (4.2.4). By Remark 4.2.2 and Theorem 4.2.3, we obtain a unique solution $(X_s^{t,x,1}, Y_s^{t,x,1}, Z_s^{t,x,1}) \in S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ for (4.2.7), and a unique solution $(X_s^{t,x,1}, Y_s^{t,x,1}, Z_s^{t,x,1}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ for following infinite horizon FBSDEs.

$$\begin{cases} X_s^{t,x,1} = x + \int_t^s b(r, X_r^{t,x,1}) dr + \int_t^s \sigma(r, X_r^{t,x,1}) dW_r, \\ e^{-Ks} Y_s^{t,x,1} = \int_s^\infty e^{-Kr} f(r, X_r^{t,x,1}, Y_r^{t,x,1}, Z_r^{t,x,1}) dr \\ \quad + \int_s^\infty K e^{-Kr} Y_r^{t,x,1} dr - \int_s^\infty e^{-Kr} Z_r^{t,x,1} dW_r. \end{cases}$$

For $N = 2$, (4.2.6) leads to

$$\begin{cases} X_s^{t,x,2} = x + \int_t^s b(r, X_r^{t,x,2}, Y_r^{t,x,1}, Z_r^{t,x,1}) dr + \int_t^s \sigma(r, X_r^{t,x,2}, Y_r^{t,x,1}) dW_r \\ Y_s^{t,x,2} = \int_s^T f(r, X_r^{t,x,2}, Y_r^{t,x,2}, Z_r^{t,x,2}) dr - \int_s^T Z_r^{t,x,1} dW_r. \end{cases} \quad (4.2.8)$$

We substitute $Y_s^{t,x,1}$ into the SDE in (4.2.8) to solve $X_s^{t,x,2}$. Therefore (4.2.8) is BSDE again. By Remark 4.2.2 and Theorem 4.2.3, we obtain a unique solution $(X_s^{t,x,2}, Y_s^{t,x,2}, Z_s^{t,x,2}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ for (4.2.8), and a unique solution $(X_s^{t,x,2}, Y_s^{t,x,2}, Z_s^{t,x,2}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ for the following infinite horizon FBSDEs.

$$\begin{cases} X_s^{t,x,2} = x + \int_t^s b(r, X_r^{t,x,2}, Y_r^{t,x,1}, Z_r^{t,x,1}) dr + \int_t^s \sigma(r, X_r^{t,x,2}, Y_r^{t,x,1}) dW_r, \\ e^{-Ks} Y_s^{t,x,2} = \int_s^\infty e^{-Kr} f(r, X_r^{t,x,2}, Y_r^{t,x,2}, Z_r^{t,x,2}) dr \\ \quad + \int_s^\infty K e^{-Kr} Y_r^{t,x,2} dr - \int_s^\infty e^{-Kr} Z_r^{t,x,2} dW_r. \end{cases}$$

Following the same procedure, we have constructed another iterative procedure as following:

$$\begin{cases} X_s^{t,x,N} = x + \int_t^s b(r, X_r^{t,x,N}, Y_r^{t,x,N-1}, Z_r^{t,x,N-1}) dr + \int_t^s \sigma(r, X_r^{t,x,N}, Y_r^{t,x,N-1}) dW_r, \\ e^{-Ks} Y_s^{t,x,N} = \int_s^\infty e^{-Kr} f(r, X_r^{t,x,N}, Y_r^{t,x,N}, Z_r^{t,x,N}) dr \\ \quad + \int_s^\infty K e^{-Kr} Y_r^{t,x,N} dr - \int_s^\infty e^{-Kr} Z_r^{t,x,N} dW_r. \end{cases} \quad (4.2.9)$$

In general, we see (4.2.9) is an iterated mapping from $S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ to itself and obtain

a sequence $(X_s^{t,x,i}, Y_s^{t,x,i}, Z_s^{t,x,i})_{i=0,1,2,\dots}$. We will prove that (4.2.9) is a contraction mapping.

Step 2: Here we use the *Contraction Mapping Method*. Consider the map

$$\begin{aligned} \Xi & : M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})) \\ & \rightarrow M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})), \\ & (X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \mapsto (\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}), \end{aligned}$$

where $(\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot})$ is defined as follows, for any $s \geq 0$

$$\bar{X}_s^{t,x} = x + \int_t^s b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_t^s \sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) dW_r. \quad (4.2.10)$$

and

$$\begin{aligned} e^{-Ks} \bar{Y}_s^{t,x} & = \int_s^\infty e^{-Kr} f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) dr \\ & \quad + K \int_t^\infty e^{-Kr} \bar{Y}_r^{t,x} dr - \int_s^\infty e^{-Kr} \bar{Z}_r^{t,x} dW_r. \end{aligned} \quad (4.2.11)$$

The process $(\bar{X}_s^{t,x})_{s \geq t}$ is a solution of a forward SDE, whereas the coupled process $(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})_{s \geq t}$ is a solution of a backward SDE.

Actually, we want to prove that the map Ξ is a contraction. To this end, we consider $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ and $(U^{t,\cdot}, V^{t,\cdot}, \mathcal{W}^{t,\cdot})$ in $M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. We put

$$(\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}) = \Xi(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}), (\bar{U}^{t,\cdot}, \bar{V}^{t,\cdot}, \bar{\mathcal{W}}^{t,\cdot}) = \Xi(U^{t,\cdot}, V^{t,\cdot}, \mathcal{W}^{t,\cdot}).$$

For the SDE (4.2.10) and BSDEs (4.2.11), applying Itô's formula to $e^{-Ks} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2$ and $e^{-Ks} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2$, taking spatial integration $\rho^{-1}(x) dx$ on both sides for a.e. $x \in \mathbb{R}^d$, applying stochastic Fubini theorem and taking expectation, following the similar procedure as in the proof of Theorem 1.3.3, we have

$$\begin{aligned} & (K - 5L - \frac{9}{20}) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & + (2\mu - K - 5L) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & + \frac{4}{5} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\ & + \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \int_{\mathbb{R}^d} e^{-Kt} |\bar{Y}_t^{t,x} - \bar{V}_t^{t,x}|^2 \rho^{-1}(x) dx \end{aligned}$$

$$\begin{aligned}
 &\leq \left(\frac{1}{4} + L\right) \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 &\quad + \frac{1}{4} \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr.
 \end{aligned} \tag{4.2.12}$$

Now let us construct the contraction mapping. To simplify notation, we denote

$$\begin{aligned}
 \bar{A} &= \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 A &= \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x} - U_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
 \bar{B} &= \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 B &= \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} |Y_r^{t,x} - V_r^{t,x}|^2 \rho^{-1}(x) dx dr, \\
 \bar{C} &= \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr, \\
 C &= \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} \|Z_r^{t,x} - \mathcal{W}_r^{t,x}\|^2 \rho^{-1}(x) dx dr.
 \end{aligned}$$

Now we take the limit as $T \rightarrow \infty$ in (4.2.12), we have

$$\begin{aligned}
 &(K - 5L - \frac{9}{20})\bar{A} + (2\mu - K - 5L)\bar{B} + \frac{4}{5}\bar{C} \\
 &\leq \left(\frac{1}{4} + L\right)B + \frac{1}{4}C.
 \end{aligned}$$

In fact,

$$\left(\frac{K - 5L - \frac{9}{20}}{\frac{4}{5}}\right)\bar{A} + \left(\frac{2\mu - K - 5L}{\frac{4}{5}}\right)\bar{B} + \bar{C} \leq \frac{5}{16}((1 + 4L)B + C).$$

We assume $1 + 4L < \frac{2\mu - K - 5L}{\frac{4}{5}}$ and $K - 5L - \frac{9}{20} > 0$, then we have,

$$\begin{aligned}
 &\left(\frac{K - 5L - \frac{9}{20}}{\frac{4}{5}}\right)\bar{A} + (1 + 4L)\bar{B} + \bar{C} \\
 &\leq \left(\frac{K - 5L - \frac{9}{20}}{\frac{4}{5}}\right)\bar{A} + \left(\frac{2\mu - K - 5L}{\frac{4}{5}}\right)\bar{B} + \bar{C} \\
 &\leq \frac{5}{16} \left\{ \left(\frac{K - 5L - \frac{9}{20}}{\frac{4}{5}}\right)\bar{A} + (1 + 4L)B + C \right\}.
 \end{aligned}$$

So the map Ξ is a contraction from $M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ into itself. Consequently, (4.2.1) has a unique solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ in $M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

Step 3: For the FBSDEs (4.2.1), we have proved that there exists a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ in $M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([t, \infty);$

4.3. STATIONARY SOLUTION OF FBSDES AND QUASI-LINEAR ELLIPTIC PDES

$L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$). Now applying Itô's formula to $e^{-Ks}|X_s^{t,x}|^2$ and $e^{-Ks}|Y_s^{t,x}|^2$, taking spatial integration $\rho^{-1}(x)dx$ on both sides for a.e. $x \in \mathbb{R}^d$, applying stochastic Fubini theorem and using B-D-G inequality, we have

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x}|^2 \rho^{-1}(x) dx \\ & \leq C_p \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|X_r^{t,x}|^2 + |Y_r^{t,x}|^2 + \|Z_r^{t,x}\|^2) \rho^{-1}(x) dx dr \\ & \quad + \int_{\mathbb{R}^d} e^{-Kt} x^2 \rho^{-1}(x) dx + C_{L,\mu} \int_t^T e^{-Kr} dr. \end{aligned}$$

As $T \rightarrow \infty$,

$$\mathbb{E} \sup_{s \geq t} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq t} \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x}|^2 \rho^{-1}(x) dx < \infty.$$

Therefore, $(X_s^{t,x}, Y_s^{t,x}) \in S^{2,-K} \cap M^{2,-K}([t, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([t, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^k))$. Following a similar procedure as in Step 6 of the proof of Theorem 1.2.3, we can extend our result from $[t, \infty)$ to $[0, \infty)$. □

4.3 Stationary Solution of FBSDEs and Quasi-linear Elliptic PDEs

In this section, we desire to study the following quasi-linear elliptic PDEs

$$\mathcal{L}u(x) + f(x, u(x), \sigma^*(x, u(x)) \nabla u(x)) = 0, \quad (4.3.1)$$

where $u : \mathbb{R}^d \rightarrow \mathbb{R}^k$, and \mathcal{L} is an infinitesimal operator defined by

$$\mathcal{L}u(x) = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij}(x, u(x)) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x, u(x)) \frac{\partial u(x)}{\partial x_i}.$$

To find the weak solution of PDEs (4.3.1), we consider its corresponding infinite horizon FBSDEs,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x}, Y_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}, Y_r^{t,x}) dW_r, \\ e^{-Ks} Y_s^{t,x} = \int_s^\infty e^{-Kr} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\ \quad + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr - \int_s^\infty e^{-Kr} Z_r^{t,x} dW_r. \end{cases} \quad (4.3.2)$$

4.3. STATIONARY SOLUTION OF FBSDES AND QUASI-LINEAR ELLIPTIC PDES

In Section 4.2, we study a more general form of the above infinite horizon FBSDEs (4.3.2) with time variable dependent coefficients, and give the existence and uniqueness result. If (4.3.2) has an unique solution, then for an arbitrary T , we have

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x}, Y_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x}, Y_r^{t,x})dW_r, \\ Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r, \quad 0 \leq s \leq T. \end{cases} \quad (4.3.3)$$

In Chapter 3, we deduce the following PDEs associated with FBSDEs (4.3.3)

$$u(t, x) = u(T, x) + \int_t^T [\mathcal{L}u(s, x) + f(x, u(s, x), \sigma^*(x, u(s, x))\nabla u(s, x))]ds. \quad (4.3.4)$$

Here \mathcal{L} is given by (4.3.1), $u(T, x) = Y_T^{T,x}$.

In this section, we will prove that there exists a unique weak solution for quasi-linear elliptic PDEs (4.3.4). First we study the stationary property of the infinite horizon FBSDEs (4.3.2). By the connection between FBSDEs (4.3.3) and PDEs (4.3.4) such that $u(t, \cdot) = Y_t^{t,\cdot}$, we can transfer the stationary property from $Y_t^{t,\cdot}$ to $u(t, \cdot)$. Since $u(t, \cdot)$ is a deterministic function, together with the stationary property, immediately we have that $u(t, \cdot)$ is independent of t . Therefore, (4.3.4) turns into a quasi-linear elliptic type PDE (4.3.1), where u is the unique weak solution of such PDE. Now we consider the following conditions:

(D.2.0): For any $s \in [0, T]$, $b(s, \cdot, \cdot) \in C^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)$; $f(s, \cdot, \cdot, \cdot) \in C^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}; \mathbb{R}^k)$; $\sigma(s, \cdot, \cdot) \in C_{l,b}^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^{d \times d})$ for some $\alpha \in (0, 1]$.

(D.2.1): There exists a constant $L \geq 0$ such that for any $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\begin{aligned} |b(x_1, y_1) - b(x_2, y_2)|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ \|\sigma(x_1, y_1) - \sigma(x_2, y_2)\|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ |f(x_1, y, z_1) - f(x_2, y, z_2)|^2 &\leq L(|x_1 - x_2|^2 + \|z_1 - z_2\|^2). \end{aligned}$$

(D.2.2): For any $p \in (2, \infty)$, there exist positive constants $\mu, C_{p,L}$ and $C'_{p,L}$, where $C_{p,L}, C'_{p,L}$ only depend on p and L , such that $p\mu > K + C_{p,L}$ and $K > C'_{p,L}$. And for any $y_1, y_2, y \in \mathbb{R}^k, (x, z) \in \mathbb{R}^d \times \mathbb{R}^{k \times d}$

$$\langle y_1 - y_2, f(x, y_1, z) - f(x, y_2, z) \rangle \leq -\mu|y_1 - y_2|^2,$$

$$|f(t, 0, y, 0)|^2 \leq \lambda(1 + |y|^2).$$

(D.2.3): The following holds

$$|b(0, 0)|^2 ds + \|\sigma(0, 0)\|^2 + |f(0, 0, 0)|^2 < \infty.$$

Theorem 4.3.1. *Under conditions (D.2.1)-(D.2.3), (4.3.2) has a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$. Moreover*

$$\mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |X_s^{t,x}|^p \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x}|^p \rho^{-1}(x) dx < \infty.$$

Proof. Since the conditions (D.2.1)-(D.2.3) are stronger than conditions (D.1.1) – (D.1.3) in Theorem 4.2.4, so there exists a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ to (4.3.2). We only need to prove $\mathbb{E}[\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} (|X_s^{t,x}|^p + |Y_s^{t,x}|^p) \rho^{-1}(x) dx] < \infty$. Let $\varphi_{N,p}(y) = y^{\frac{p}{2}} I_{\{0 \leq y \leq N\}} + \frac{p}{2} N^{\frac{p-2}{2}} (y - N) I_{\{y \geq N\}}$. We apply Itô's formula to $e^{-pKr} \varphi_{N,p}(\psi_M(X_r^{t,x}))$, $e^{-pKr} \varphi_{N,p}(\psi_M(Y_r^{t,x}))$ for a.e. $x \in \mathbb{R}^d$ to have the following estimation, where $\psi_M(x) = x^2 I_{\{-M \leq x < M\}} + 2M(x - M) I_{\{x \geq M\}} - 2M(x + M) I_{\{x < -M\}}$.

$$\begin{aligned} & e^{-pKT} \varphi_{N,p}(\psi_M(X_T^{t,x})) + pK \int_t^T e^{-pKr} \varphi_{N,p}(\psi_M(X_r^{t,x})) dr \\ = & e^{-pKt} \varphi_{N,p}(\psi_M(X_t^{t,x})) + \int_t^T e^{-pKr} \varphi'_{N,p}(\psi_M(X_r^{t,x})) \psi'_M(X_r^{t,x}) b(X_r^{t,x}, Y_r^{t,x}) dr \\ & + \int_t^T e^{-pKr} \langle \varphi'_{N,p}(\psi_M(X_r^{t,x})) \psi'_M(X_r^{t,x}), \sigma(X_r^{t,x}, Y_r^{t,x}) dW_r \rangle \\ & + \frac{1}{2} \int_t^T e^{-pKr} \varphi''_{N,p}(\psi_M(X_r^{t,x})) |\psi'_M(X_r^{t,x})|^2 \|\sigma(X_r^{t,x}, Y_r^{t,x})\|^2 dr \\ & + \int_t^T e^{-pKr} \varphi'_{N,p}(\psi_M(X_r^{t,x})) I_{\{-M \leq X_r^{t,x} \leq M\}} \|\sigma(X_r^{t,x}, Y_r^{t,x})\|^2 dr. \end{aligned} \quad (4.3.5)$$

And

$$\begin{aligned} & e^{-pKt} \varphi_{N,p}(\psi_M(Y_t^{t,x})) - pK \int_t^T e^{-pKr} \varphi_{N,p}(\psi_M(Y_r^{t,x})) dr \\ & + \frac{1}{2} \int_t^T e^{-pKr} \varphi''_{N,p}(\psi_M(Y_r^{t,x})) |\psi'_M(Y_r^{t,x})|^2 \|Z_r^{t,x}\|^2 dr \\ & + \int_t^T e^{-pKr} \varphi'_{N,p}(\psi_M(Y_r^{t,x})) I_{\{-M \leq Y_r^{t,x} \leq M\}} \|Z_r^{t,x}\|^2 dr \\ = & e^{-pKT} \varphi_{N,p}(\psi_M(Y_T^{t,x})) + \int_t^T e^{-pKr} \varphi'_{N,p}(\psi_M(Y_r^{t,x})) \psi'_M(Y_r^{t,x}) f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\ & - \int_t^T e^{-pKr} \langle \varphi'_{N,p}(\psi_M(Y_r^{t,x})) \psi'_M(Y_r^{t,x}), Z_r^{t,x} dW_r \rangle. \end{aligned} \quad (4.3.6)$$

4.3. STATIONARY SOLUTION OF FBSDES AND QUASI-LINEAR ELLIPTIC PDES

From above estimations, note that $\lim_{T \rightarrow \infty} e^{-pKT} \varphi_{N,p} \left(\psi_M(Y_T^{t,x}) \right) = 0$, after taking limit as $T \rightarrow \infty$, we take the integration on $\Omega \times \mathbb{R}^d$. As $(X^t, Y^t, Z^t) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ and $\varphi'_{N,p} \left(\psi_M(X_r^{t,x}) \right) \psi'_M(X_r^{t,x}), \varphi'_{N,p} \left(\psi_M(Y_r^{t,x}) \right) \psi'_M(Y_r^{t,x})$ are bounded, we can use the stochastic Fubini theorem and taking the limit as $M \rightarrow \infty$ first, then the limit as $N \rightarrow \infty$, by the monotone convergence theorem, we have

$$\begin{aligned}
& \left\{ K - 4pL - \frac{p}{8} + \frac{1}{8} - \varepsilon - L(p-1)^2(1+\varepsilon) - \frac{1}{8} \right\} \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |X_r^{t,x}|^p \rho^{-1}(x) dx dr \\
& + \left\{ p\mu - K - 4pL - \frac{p-2}{16} - \varepsilon - L(p-1)^2(1+\varepsilon) - \frac{1}{8} \right\} \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^p \rho^{-1}(x) dx dr \\
& + \left\{ \frac{1}{2}p(p-1) - \frac{p}{16} \right\} \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
& \leq C_p \int_{\mathbb{R}^d} (|b(0,0)|^p + \|\sigma(0,0)\|^p + |f(0,0,0)|^p) \rho^{-1}(x) dx \\
& \quad + C_p \int_{\mathbb{R}^d} e^{-pKt} |x|^p \rho^{-1}(x) dx.
\end{aligned}$$

From conditions conditions **(D.2.1)**-**(D.2.3)**, there exists a constant $C_{L,\mu}$ only depending on L and μ such that

$$\begin{aligned}
& \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |X_r^{t,x}|^p \rho^{-1}(x) dx dr + \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^p \rho^{-1}(x) dx dr \\
& + \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
& \leq C_{L,\mu} \int_{\mathbb{R}^d} (|b(0,0)|^p + \|\sigma(0,0)\|^p + |f(0,0,0)|^p) \rho^{-1}(x) dx \\
& \quad + C_{L,\mu} \int_{\mathbb{R}^d} e^{-Kt} |x|^p \rho^{-1}(x) dx < \infty. \tag{4.3.7}
\end{aligned}$$

Next, by the B-D-G inequality, the Cauchy-Schwarz inequality, the Young's inequality and (4.3.7), we can obtain another estimation from (4.3.5) and (4.3.6):

$$\begin{aligned}
& \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |X_s^{t,x}|^p \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x}|^p \rho^{-1}(x) dx \\
& \leq C_{L,\mu} \int_{\mathbb{R}^d} (|b(0,0)|^p + \|\sigma(0,0)\|^p + |f(0,0,0)|^p) \rho^{-1}(x) dx \\
& \quad + C_{L,\mu} \mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} e^{-pKr} |X_r^{t,x}|^p \rho^{-1}(x) dx dr + C_{L,\mu} \mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^p \rho^{-1}(x) dx dr \\
& \quad + C_{L,\mu} \mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr + C_{L,\mu} \int_{\mathbb{R}^d} e^{-Kt} |x|^p \rho^{-1}(x) dx \\
& < \infty.
\end{aligned}$$

□

4.3. STATIONARY SOLUTION OF FBSDES AND QUASI-LINEAR ELLIPTIC PDES

Theorem 4.3.2. *Under conditions (D.2.0)-(D.2.3), let $u(t, \cdot) \triangleq Y_t^{t, \cdot}$, where $(X^{t, \cdot}, Y^{t, \cdot}, Z^{t, \cdot})$ is the solution of (4.3.2). Then for an arbitrary T and $t \in [0, T]$, $u(t, \cdot)$ is a weak solution for (4.3.4). Moreover, $u(t, \cdot)$ is a.s. continuous with respect to t in $L_p^2(\mathbb{R}^d; \mathbb{R}^1)$.*

Proof. Let $(Y_s^{t,x})_{s \geq 0}$, $(Y_s^{t',x})_{s \geq 0}$ be the solutions of (4.3.2). First we claim that, for an arbitrary $T > 0$, $t, t' \in [0, T]$,

$$\mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x} - Y_s^{t',x}|^p \rho^{-1}(x) dx \leq C_p |t' - t|^{\frac{p}{2}}.$$

To see this,

$$\begin{aligned} \bar{X}_s &= X_s^{t',x} - X_s^{t,x}, \quad \bar{Y}_s = Y_s^{t',x} - Y_s^{t,x}, \quad \bar{Z}_s = Z_s^{t',x} - Z_s^{t,x}, \\ \bar{b}(s) &= b(X_s^{t',x}, Y_s^{t',x}) - b(X_s^{t,x}, Y_s^{t,x}), \\ \bar{\sigma}(s) &= \sigma(X_s^{t',x}, Y_s^{t',x}) - \sigma(X_s^{t,x}, Y_s^{t,x}), \\ \bar{f}(s) &= f(X_s^{t',x}, Y_s^{t',x}, Z_s^{t',x}) - f(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}), \quad s \geq 0. \end{aligned}$$

Then

$$\begin{cases} d\bar{X}_s = \bar{b}(s) ds + \bar{\sigma}(s) dW_s, \\ d\bar{Y}_s = -\bar{f}(s) ds + \bar{Z}_s dW_s, \\ \lim_{T \rightarrow \infty} e^{-KT} \bar{Y}_T = 0 \quad \text{for a.e. } x \in \mathbb{R}^d \quad \text{a.s.} \end{cases}$$

From Theorem 4.3.1, we have

$$\mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |X_s^{t,x}|^p \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x}|^p \rho^{-1}(x) dx < \infty.$$

Applying Itô's formula to $e^{-pKr} |\bar{X}_r|^p$ and $e^{-pKr} |\bar{Y}_r|^p$ for a.e. $x \in \mathbb{R}^d$ and following a similar procedure as in the proof of Lemma 2.2.3, we have

$$\begin{aligned} & \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |\bar{X}_r|^p \rho^{-1}(x) dx dr + \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |\bar{Y}_r|^p \rho^{-1}(x) dx dr \\ & + \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |\bar{Y}_r|^{p-2} \|\bar{Z}_r\|^2 \rho^{-1}(x) dx dr \leq C_{L,\mu} |t' - t|^{\frac{p}{2}}. \end{aligned}$$

Also by the B-D-G inequality, we have

$$\mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |\bar{X}_s|^p \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |\bar{Y}_s|^p \rho^{-1}(x) dx \leq C_{L,\mu} |t' - t|^{\frac{p}{2}}.$$

As a result, we have

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |Y_s^{t',x} - Y_s^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{p}{2}} \\ & \leq C_{L,\mu} \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t',x} - Y_s^{t,x}|^p \rho^{-1}(x) dx \left(\int_{\mathbb{R}^d} \rho^{-1}(x) dx \right)^{\frac{p-2}{2}} \end{aligned}$$

$$\leq C_{L,\mu}|t' - t|^{\frac{p}{2}}.$$

Noting $p > 2$, by the Kolmogorov continuity theorem, we have $t \rightarrow Y_s^{t,x}$ is a.s. continuous for $t \in [0, T]$ under the norm $(\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |\cdot|^2 \rho^{-1}(x) dx)^{\frac{1}{2}}$. Without losing any generality, assume that $t' > t$. Then we have

$$\begin{aligned} & \lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} e^{-2Kt'} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \\ & \leq \lim_{t' \rightarrow t} \left(\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |Y_s^{t',x} - Y_s^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} = 0 \quad a.s.. \end{aligned}$$

Notice $t' \in [0, T]$, so

$$\lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} = 0 \quad a.s.. \quad (4.3.8)$$

Since $Y^{t,\cdot} \in S^{2,-K} \cap M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$, $Y_{t'}^{t,\cdot}$ is continuous w.r.t. t' in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$.

This is to say for each t ,

$$\lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t,x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} = 0 \quad a.s.. \quad (4.3.9)$$

By (4.3.8) and (4.3.9)

$$\begin{aligned} & \lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \\ & \leq \lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} + \lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t,x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \\ & = 0 \quad a.s.. \end{aligned}$$

Therefore, for an arbitrary $T > 0$, $0 \leq t \leq T$, define $u(t, \cdot) = Y_t^{t,\cdot}$, then $u(t, \cdot)$ is a.s. continuous w.r.t. t in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$. Moreover, recall the results in Chapter 2 and Chapter 3, it is easy to check that $u(t, x)$ is a weak solution of (4.3.4). \square

Definition 4.3.3. Let $u : [0, \infty) \times U \times \Omega \rightarrow U$ is a measurable random dynamical system on a measurable space (U, \mathcal{B}) over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \geq 0})$, where U is a Hilbert space and $\theta_t : \Omega \rightarrow \Omega$ is a P -preserving transformation. Then a stationary solution is an \mathcal{F} -measurable random variable $Y : \Omega \rightarrow U$ such that (see Arnold [2]) for all $t \geq 0$

$$u(t, Y(\omega), \omega) = Y(\theta_t \omega).$$

4.3. STATIONARY SOLUTION OF FBSDES AND QUASI-LINEAR ELLIPTIC PDES

We now construct the measurable metric dynamical system through defining a measurable and measure preserving shift. Let $\theta_t : \Omega \rightarrow \Omega, t \geq 0$, be a measurable mapping on (Ω, \mathcal{F}, P) , defined by $\theta_t \circ W_s = W_{s+t} - W_t$, Then for any $s, t \geq 0$,

(i) $P \cdot \theta_t^{-1} = P$;

(ii) $\theta_0 = I$, where I is the identity transformation on Ω ;

(iii) $\theta_s \circ \theta_t = \theta_{s+t}$.

Also for an arbitrary \mathcal{F} -measurable $\phi : \Omega \rightarrow H$ where H is a Hilbert space, set

$$\theta \circ \phi(\omega) = \phi(\theta(\omega)).$$

Theorem 4.3.4. *Under conditions (D.2.0)-(D.2.3), let $u(t, \cdot) \triangleq Y_t^{t, \cdot}$, where $(X^{t, \cdot}, Y^{t, \cdot}, Z^{t, \cdot})$ is the solution of (4.3.2). Then $u(t, \cdot)$ has an indistinguishable version which is a "perfect" stationary weak solution of (4.3.4).*

Proof. Note that (4.3.2) is equivalent to following FBSDEs

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x}, Y_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x}, Y_r^{t,x})dW_r, \\ Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r, \\ \lim_{T \rightarrow \infty} e^{-KT} Y_T = 0 \quad a.s.. \end{cases} \quad (4.3.10)$$

First we will prove that $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{s \geq 0}$ is a "perfect" stationary solution of (4.3.10), i.e.

$$\theta_r \circ X_s^{t,x} = X_{s+r}^{t+r,x}, \quad \theta_r \circ Y_s^{t,x} = Y_{s+r}^{t+r,x}, \quad \theta_r \circ Z_s^{t,x} = Z_{s+r}^{t+r,x}.$$

Here the integral w.r.t. W is a standard Itô's integral. Recall the Definition ??, θ_t is a shift with respect to W . We want to apply the operator θ_r on Itô's integral. For any $s, r \geq 0$,

$$\theta_r \circ dW_s = \theta_r \circ (W_{s+\Delta s} - W_s) = W_{s+\Delta s+r} - W_r - (W_{s+r} - W_r) = dW_{s+r}.$$

Let $\{h(s, \cdot)\}_{s \geq 0}$ being a \mathcal{F}_s -measurable and locally square integrable stochastic process with values in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^l)$. For an arbitrary $T > 0$ and $0 \leq t \leq T$,

$$\theta_r \circ \int_t^T h(s, \cdot) dW_s = \int_t^T \theta_r \circ h(s, \cdot) dW_{s+r} = \int_{t+r}^{T+r} \theta_r \circ h(s-r, \cdot) dW_s, \quad (4.3.11)$$

and

$$\theta_r \circ \int_t^T h(s, \cdot) ds = \int_t^T \theta_r \circ h(s, \cdot) ds = \int_{t+r}^{T+r} \theta_r \circ h(s-r, \cdot) ds. \quad (4.3.12)$$

From conditions **(D.2.1)**-**(D.2.3)** and $(X^{\cdot, \cdot}, Y^{\cdot, \cdot}, Z^{\cdot, \cdot}) \in S^{2, -K} \cap M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2, -K} \cap M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ is the unique solution of (4.3.10), it is easy to see that b, σ, f are locally square integrable. Now applying θ_r on both sides of (4.3.10), by (4.3.11) and (4.3.12), we know that $(\theta_r \circ X_s^{t,x}, \theta_r \circ Y_s^{t,x}, \theta_r \circ Z_s^{t,x})$ satisfies the following equations

$$\left\{ \begin{array}{l} \theta_r \circ X_s^{t,x} = x + \int_{t+r}^{s+r} b(\theta_r \circ X_{u-r}^{t,x}, \theta_r \circ Y_{u-r}^{t,x}) du + \int_{t+r}^{s+r} \sigma(\theta_r \circ X_{u-r}^{t,x}, \theta_r \circ Y_{u-r}^{t,x}) dW_u, \\ \theta_r \circ Y_s^{t,x} = \theta_r \circ Y_T^{t,x} + \int_{s+r}^{T+r} f(\theta_r \circ X_{u-r}^{t,x}, \theta_r \circ Y_{u-r}^{t,x}, \theta_r \circ Z_{u-r}^{t,x}) du - \int_{s+r}^{T+r} \theta_r \circ Z_{u-r}^{t,x} dW_u \\ \lim_{T \rightarrow \infty} e^{-K(T+r)} \theta_r \circ Y_T^{t,x} = 0 \quad a.s. \end{array} \right. \quad (4.3.13)$$

On the other hand, from (4.3.10), it follows that

$$\left\{ \begin{array}{l} X_{s+r}^{t+r,x} = x + \int_{t+r}^{s+r} b(X_u^{t+r,x}, Y_u^{t+r,x}) du + \int_{t+r}^{s+r} \sigma(X_u^{t+r,x}, Y_u^{t+r,x}) dW_u, \\ Y_{s+r}^{t+r,x} = Y_{T+r}^{t+r,x} + \int_{s+r}^{T+r} f(X_u^{t+r,x}, Y_u^{t+r,x}, Z_u^{t+r,x}) du - \int_{s+r}^{T+r} Z_u^{t+r,x} dW_u, \\ \lim_{T \rightarrow \infty} e^{-K(T+r)} Y_{T+r}^{t+r,x} = 0 \quad a.s. \end{array} \right. \quad (4.3.14)$$

By the uniqueness of the solution of (4.3.10) in the space $S^{2, -K} \cap M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2, -K} \cap M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2, -K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, it follows from comparing (4.3.13) and (4.3.14) that for any $r \geq 0$ and $t \geq 0$, in the space $L_\rho^2(\mathbb{R}^d; \mathbb{R}^d) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^k) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})$, for all $s \geq t$,

$$\theta_r \circ X_s^{t, \cdot} = X_{s+r}^{t+r, \cdot}, \quad \theta_r \circ Y_s^{t, \cdot} = Y_{s+r}^{t+r, \cdot}, \quad \theta_r \circ Z_s^{t, \cdot} = Z_{s+r}^{t+r, \cdot} \quad a.s. \quad (4.3.15)$$

Next, we will prove that $u(t, \cdot)$ is a "perfect" stationary weak solution of (4.3.4). By the perfection procedure ([3],[2]), we can prove above identities (4.3.15) are true for all $s \geq t, r \geq 0$, but fixed $t \geq 0$ a.s. In particular, for any $t \geq 0$, in the space $L_\rho^2(\mathbb{R}^d; \mathbb{R}^d) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^k) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})$

$$\theta_r \circ Y_t^{t, \cdot} = Y_{t+r}^{t+r, \cdot}, \quad \text{for all } r \geq 0 \quad a.s. \quad (4.3.16)$$

From Theorem 4.3.2, we know that $u(t, \cdot) \triangleq Y_t^{t, \cdot}$ is the continuous weak solution of (4.3.4). So we get from (4.3.16) that for any $t \geq 0$, in space $L_\rho^2(\mathbb{R}^d; \mathbb{R}^d) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$

$$\otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$$

$$\theta_r \circ u(t, \cdot) = u(t+r, \cdot), \quad \text{for all } r \geq 0.$$

Until now, we know "crude" stationary property for $u(t, \cdot)$, but due to the continuity of $u(t, \cdot)$ w.r.t. t , we can obtain an indistinguishable version of $u(t, \cdot)$ still denoted by $u(t, \cdot)$, such that

$$\theta_r \circ u(t, \cdot) = u(t+r, \cdot), \quad \text{for all } t, r \geq 0.$$

So we proved that $u(t, \cdot)$ is a "perfect" stationary weak solution of (4.3.4). □

Then we will give our main result in this section.

Theorem 4.3.5. *Under conditions (D.2.0)-(D.2.3), the quasi-linear elliptic PDEs (4.3.1) has a unique weak solution, i.e. there exist unique processes $(u, \sigma^* \nabla u) \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^k) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$ and for an arbitrary $\Psi \in C^\infty(\mathbb{R}^d; \mathbb{R}^k)$,*

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^d} \sigma^*(x, u(x)) \nabla u(x) \sigma^*(x, u(x)) \nabla \Psi(x) dx + \int_{\mathbb{R}^d} u(x) \operatorname{div} \left((b - \tilde{A}) \Psi(x) \right) dx \\ &= \int_{\mathbb{R}^d} f(x, u(x), \sigma^*(s, x, u(x)) \nabla u(x)) \Psi(x) dx. \end{aligned}$$

Here $\tilde{A}_j \triangleq \sum_{i=1}^d \frac{\partial(\sigma \sigma^*)_{i,j}(x, u(x))}{\partial x_i}$, and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_d)^*$.

Proof. By Theorem 4.3.4, we know (4.3.4) has a unique weak solution $u(t, \cdot)$, with stationary property,

$$\theta_r \circ u(t, \cdot) = u(t+r, \cdot), \quad \text{for all } t, r \geq 0.$$

Note that $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$ is a deterministic function, so $\theta_r \circ u(t, \cdot) = u(t, \cdot)$ for all $t, r \geq 0$, together with stationary property, $u(t+r, \cdot) = u(t, \cdot)$ for all $t, r \geq 0$, so $u(t, \cdot) = u(\cdot)$ is independent of time t . So (4.3.4) turns into the quasi-linear elliptic PDEs (4.3.1), therefore $u(\cdot)$ is the unique weak solution of the quasi-linear elliptic PDEs (4.3.1). □

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