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Integrable Geodesic flows

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Lecture 1. Integrability of geodesic flows

What are geodesics?

Let M be a manifold with a Riemannian metric g . In local coordinates, $g = \sum g_{ij}(x) dx^i dx^j$.

Consider a point moving on M and let $\gamma(t)$ be its trajectory. According to the second Newton law, its motion is defined by the equation $ma = F$. What is the meaning of the acceleration a in this case? The answer is: a is the derivative of the velocity $v = \frac{d\gamma}{dt}$, but one should consider the covariant derivative. If the point is moving by inertia, then $F = 0$ and we come to the following equation:

$$\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = 0$$

or, in local coordinates,

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk}(x) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad (1)$$

where $\gamma(t) = (x^1(t), \dots, x^n(t))$ and $\Gamma^i_{jk}(x)$ are the Christoffel symbols of the Levi-Civita connection of the metric g .

This is the equation of geodesics on a Riemannian manifold (M, g) .

Properties of geodesics:

- if the metric is Euclidean, then $\Gamma^i_{jk}(x) = 0$ and the geodesics are straight lines, i.e., $x_i(t) = a_i t + b_i$;
- (1) is a non-linear second-order system of ODEs;
- Existence and Uniqueness Theorem;
- geodesic completeness;
- Hopf-Rinow theorem;
- arc-length parametrisation and admissible reparametrisations $s' = as + b$;

- kinetic energy $H = \sum g_{ij}(x)\dot{x}^i\dot{x}^j$ as a first integral (in particular, the unit (co-)tangent bundle is $2n - 1$ invariant submanifold).

Hamiltonian form of the equation of geodesics.

Consider T^*M as a symplectic manifold with canonical coordinates (x, p) and the symplectic form $\omega = \sum dp_i \wedge dx^i$.

In coordinates (x, p) , the equation of geodesics (1) can be rewritten in Hamiltonian form:

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial x^i}, \quad \frac{dx^i}{dt} = -\frac{\partial H}{\partial p_i}, \quad (2)$$

where $H = \frac{1}{2} \sum g^{ij}(x)p_i p_j$.

If we accept this Hamiltonian representation as a fundamental property of geodesics, then we may simply define a geodesic flow to be a Hamiltonian system on the cotangent bundle T^*M whose Hamiltonian is a positive definite quadratic form in momenta p_i . We can naturally generalise this viewpoint:

- indefinite non-degenerate quadratic form \mapsto pseudo-Riemannian geodesic flows
- non-negative quadratic forms $H = \sum g^{ij}p_i p_j \geq 0$ with the additional property that $H = 0$ iff p belongs a totally non-integrable co-distribution \mapsto sub-Riemannian geodesic flows
- convex homogeneous Hamiltonians \mapsto Finsler geodesic flows

Definition of Integrability [1].

Definition 1. We say that a geodesic flow on (M, g) is Liouville integrable, if the Hamiltonian system (2) admits n commuting first integrals f_1, \dots, f_n that are independent almost everywhere on T^*M .

Some issues to discuss: almost everywhere, what kind of integrals, what about the zero-section.

Local and global aspects.

Proposition 1. *Every geodesic flow is locally completely integrable.*

Basically, this follows from the Darboux theorem and homogeneity (locally means “on $T^*U(x_0)$ ”, i.e. “locality” is meant in the sense of coordinates only (in the sense of p , the neighborhood is not local but global).

The best example explaining why integrability should be understood as a global phenomenon is the geodesic flow on the surface M_m^2 of genus m with a constant negative curvature metric. This geodesic flows is known to be chaotic/ergodic, so definitely non-integrable. However, locally this geodesic flows admit a polynomial integral that commutes with H (there are even 3 independent linear integrals so that this geodesic flow is locally super-integrable). In particular, in any reasonable local coordinates the geodesic flow can be integrated explicitly. The problem is that when trying to extend

local polynomial integrals up to global ones we obtain multivalued functions (but multivalued integrals make no sense).

Proposition 2. *Assume that F is a real analytic (commuting with and independent of H) integral of the geodesic flow on (M, g) . Then this geodesic flow admits a non-trivial polynomial integral¹.*

Proof. It is easy to see that for a series of the form

$$F = \sum a_{n_1 n_2 \dots n_k}(x) p_1^{n_1} p_2^{n_2} \dots p_k^{n_k} = F_0 + F_1 + F_k + \dots,$$

where F_k is the homogeneous component of this power series of degree k , the condition $\{H, F\} = 0$ immediately implies $\{H, F_k\} = 0$ for each $k \in \mathbb{N}$. \square

Also, one needs to distinguish between “abstract integrability” and “practical integrability”. Existence of commuting integrals and possibility to describe them explicitly, these are two different stories.

Zoll metrics is an example of metrics whose geodesic flows are integrable in abstract but not practical sense.

The most natural and strongest type of integrability is polynomial integrability. Let’s discuss this property in $\dim = 2$.

- A generic Riemannian metric does not admit any non-trivial polynomial integrals² (except for the powers of the Hamiltonian).
- For any $n \in \mathbb{N}$, there exist (local) examples of geodesic flows that admit polynomial integrals of degree n and do not admit any non-trivial integrals of smaller degree (V. Ten, V. Kozlov, K. Kiyohara).
- All such metrics are defined by an integrable system of PDE’s and, in principle, can be described “explicitly” (M. Bialy, A. Mironov, S. Tsarev, M. Pavlov).

Lecture 2: Integrable geodesic flows on two-dimensional surfaces

Examples: Sphere, flat torus, surface of revolution, ellipsoid [3]

Proposition 3. 1. *Assume that the geodesic flow of g admits a linear integral $F = a_1(x)p_1 + a_2(x)p_2$. If F is not zero at a point $x_0 \in M$, then there exists a local coordinate system (u, v) such that*

$$g = du^2 + G(u)dv^2 \quad (\text{equivalently, } H = \frac{1}{2}(p_u^2 + G^{-1}(u)p_v^2)) \quad \text{and} \quad F = p_v.$$

¹In this theory, by saying “polynomial integrals” we always mean polynomiality in momenta p , i.e. functions of the form $\sum a_{n_1 n_2 \dots n_k}(x) p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ with coefficients being some functions on M . This functions will be automatically smooth or real analytic depending on the properties of g .

²The statement is more or less obvious as the existence of a polynomial integral for a given g is defined by an overdetermined system of PDEs and solutions exist only under some special assumptions on g . However, rigorous proof requires serious efforts, see [2].

2. Assume that the geodesic flow of g admits a quadratic integral $F = A(x)p_1^2 + 2B(x)p_1p_2 + C(x)p_2^2$. If F (as a quadratic form) is not proportional to the Hamiltonian at a point $x_0 \in M$, then there exists a local coordinate system (u, v) such that

$$g = (f(u) + g(v))(du^2 + dv^2)$$

(equivalently, $H = \frac{p_u^2 + p_v^2}{2(f(u) + g(v))}$) and $F = \frac{g(v)p_u^2 - f(u)p_v^2}{f(u) + g(v)}$).

Similar statements can be formulated for singular points.

From now on, we assume that M is a closed 2-dimensional surface. Consider the unit (co)-tangent bundle $Q^3 = \{H(x, p) = \frac{1}{2}\} \subset T^*M^2$.

Theorem 1. *There are only 4 closed surfaces which admit integrable geodesic flows with polynomial (real-analytic on Q^3 , geometrically simple, tame) integrals, namely, S^2 , $\mathbb{R}P^2$, T^2 and the Klein bottle.*

Lecture 3: Integrability of geodesic flows, homogeneous spaces and bi-quotients of Lie groups

The theory of integrable geodesic flows on Lie groups and their homogeneous spaces is based on some quite natural and simple constructions from Poisson Geometry. I will follow the classical approach developed by Sophus Lie: *Poisson algebras* and “*groups of functions*”.

We consider a symplectic manifold (M, ω) and $C^\infty(M)$ as a Poisson algebra.

By saying “Poisson (sub)algebra” \mathcal{F} , I mean basically three things:

- elements of \mathcal{F} are functions of some kind;
- \mathcal{F} is closed under a Poisson bracket, i.e. $f, g \in \mathcal{F}$ implies $\{f, g\} \in \mathcal{F}$;
- if $f_1, \dots, f_k \in \mathcal{F}$, then $h(f_1, \dots, f_k) \in \mathcal{F}$.

Let $\mathcal{F} \in C^\infty(M)$ be a Poisson algebra. Consider a generic point $x \in M$. This means that in a neighbourhood of x there are generators f_1, \dots, f_k such that they are independent and any function $h \in \mathcal{F}$ can be written as $h = h(f_1, \dots, f_k)$ (the number k can be understood as the (differential) dimension of \mathcal{F}). Also, we assume that the matrix of brackets $(P_{ij}) = (\{f_i, f_j\})$ is of (locally) constant rank.

Theorem 2 (Classification of groups of functions (S. Lie)). *There exist functions $p_1, \dots, p_r, q_1, \dots, q_r, z_1, \dots, z_s \in \mathcal{F}$, $2r + s = \text{ddim } \mathcal{F}$, such that the matrix of brackets is of the form*

$$\begin{pmatrix} 0 & \text{id} & 0 \\ -\text{id} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Moreover, these functions can be completed up to a canonical coordinate system $p_1, \dots, p_n, q_1, \dots, q_n$ on M^{2n} so that $z_i = p_{s+i}$.

If the rank is not locally constant then we get the famous Weinstein splitting theorem:

$$\begin{pmatrix} 0 & \text{id} & 0 \\ -\text{id} & 0 & 0 \\ 0 & 0 & P(z) \end{pmatrix}$$

The next theorem is even more interesting. For each Poisson algebra \mathcal{F} we can define a dual (polar, reciprocal) Poisson algebra $\tilde{\mathcal{F}} = \{f \in C^\infty(M), \{f, h\} = 0 \text{ for all } h \in \mathcal{F}\}$. It is obvious that $\tilde{\mathcal{F}}$ is indeed a Poisson algebra.

Can we classify such pairs? In local setting there is no problem at all.

Corollary 1. *There is a canonical coordinate system on M such that in coordinates $P_I, P_{II}, P_{III}, Q_I, Q_{II}, Q_{III}$ such that \mathcal{F} is generated by P_I, Q_I, P_{II} , its dual $\tilde{\mathcal{F}}$ is generated by P_{II}, P_{III}, Q_{III} and the Poisson matrix in coordinates $Q_I, P_I, P_{II}, P_{III}, Q_{III}, Q_{II}$ (order is slightly changed!) takes the form:*

$$\begin{pmatrix} 0 & -\text{id}_r & 0 & 0 & 0 & 0 \\ \text{id}_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{id}_s \\ 0 & 0 & 0 & 0 & \text{id}_m & 0 \\ 0 & 0 & 0 & -\text{id}_m & 0 & 0 \\ 0 & 0 & -\text{id}_s & 0 & 0 & 0 \end{pmatrix}$$

Different types of Poisson (sub)algebras.

We will need several natural definitions. For each Poisson algebra \mathcal{F} and each point $x \in M$ we define $d\mathcal{F}(x) = \text{span}\{df(x), f \in \mathcal{F}\} \subset T_x^*M$.

Important: the idea of a generic point

Definition 2. A commutative subalgebra $\mathcal{A} \subset C^\infty(M)$ is called complete if at a generic point $x \in M$, the subspace $d\mathcal{A}(x) \subset T_x^*M$ is maximal isotropic.

Similarly, we say that a commutative subalgebra $\mathcal{A} \subset \mathcal{F}$ is complete in \mathcal{F} if $d\mathcal{A}(x)$ is a maximal isotropic subspace of $d\mathcal{F}(x)$.

- commutative, i.e. $d\mathcal{F}(x)$ is isotropic (equivalently, $\mathcal{F} \subset \tilde{\mathcal{F}}$);
- coisotropic (complete in the non-commutative sense) $\tilde{\mathcal{F}} \subset \mathcal{F}$;
- complete commutative $\mathcal{F} = \tilde{\mathcal{F}}$.

Casimirs of $\mathcal{F} = \text{centre of } \mathcal{F}$ (Lie: ausgezeichnet).

Liouville integrability: the algebra of integral is complete commutative.

Non-commutative integrability: the algebra of integrals is co-isotropic.

Corollary 2. *Let \mathcal{F} be a Poisson subalgebra of $C^\infty(M)$ and $\tilde{\mathcal{F}}$ be its dual. Then*

- $\mathcal{F} + \tilde{\mathcal{F}}$ is a Poisson subalgebra;

- $\mathcal{F} + \tilde{\mathcal{F}}$ is co-isotropic;
- the Casimirs of \mathcal{F} coincides with Casimirs of $\tilde{\mathcal{F}}$ and also with Casimirs of $\mathcal{F} + \tilde{\mathcal{F}}$;
- if \mathcal{A} is complete commutative subalgebra of \mathcal{F} , then $\mathcal{A} + \tilde{\mathcal{F}}$ is coisotropic, the Casimirs of $\mathcal{A} + \tilde{\mathcal{F}}$ is \mathcal{A} ;
- if \mathcal{B} is complete commutative subalgebra of $\tilde{\mathcal{F}}$, then $\mathcal{B} + \mathcal{F}$ is coisotropic, the Casimirs of $\mathcal{B} + \mathcal{F}$ is \mathcal{B} ;
- $\tilde{\tilde{\mathcal{F}}} = \mathcal{F}$;
- $\mathcal{A} + \mathcal{B}$ is complete commutative subalgebra of $C^\infty(M)$;
- for any $f \in \mathcal{F}$, the dual algebra $\tilde{\mathcal{F}}$ consists of the first integrals of X_f ;
- for any $h \in \tilde{\mathcal{F}}$, the algebra \mathcal{F} consists of the first integrals of X_h .

Theorem 3. *Let \mathcal{F} be a co-isotropic algebra of integrals of a certain Hamiltonian system. Consider a common level of the integrals passing through a point $x_0 \in M$, i.e. $X = \{f_i = f_i(x_0)\}$. Assume that this level is generic in the sense that we can choose generators f_1, \dots, f_m of \mathcal{F} , $m = \dim \mathcal{F}$, in such a way that they are independent and the Poisson matrix is of constant rank nearby X (notice that this condition is local but not semi-local).*

Then X (and all neighbouring fibers) are tori with quasi-periodic dynamics and there exist a natural action-angle coordinate system $p_1, \dots, p_r, q_1, \dots, q_r, I_1, \dots, I_s, \phi_1, \dots, \phi_s$ (here $p_1, \dots, p_r, q_1, \dots, q_r, I_1, \dots, I_s \in \mathcal{F}$ and are local generators of it).

This means that such a system can be understood as the direct product of an integrable system with s degrees of freedom and $2r$ -dimensional symplectic manifold which does not really play any essential role.

Example. Assume that G acts on a symplectic manifold in a Hamiltonian way, which means that for every $\xi \in \mathfrak{g}$ the corresponding generator $\hat{\xi}$ is a Hamiltonian vector field with a Hamiltonian H_ξ and we have $\{H_\xi, H_\eta\} = H_{[\xi, \eta]}$ (symplectic action with a good momentum mapping).

This action naturally defines two natural subalgebras \mathcal{F} , the algebra of Noether integrals generated by H_ξ , $\xi \in \mathfrak{g}$ and the algebra of G -invariant functions which is nothing else by $\tilde{\mathcal{F}}$.

We have a natural map $\mathfrak{g} \rightarrow C^\infty(M)$ and more generally $C^\infty(g^*) \rightarrow C^\infty(M)$. The dual map $M \rightarrow \mathfrak{g}^*$ is known as the momentum mapping associated with the action of G on M . In fact for our purposes the map between the Poisson algebras is more important than the momentum mapping between the manifolds.

If G -invariant functions distinguish generic orbits³, then (at a generic point) we are in the situation described by the above Lie theorem.

³This always happens if G is compact

One more remark: the algebra \mathcal{F} is “isomorphic” to the algebra of polynomials on \mathfrak{g}^* with the standard Lie-Poisson bracket. We can simply identify H_ξ as a function on M with ξ as a function on \mathfrak{g}^* .

Immediately, we get the following result.

Theorem 4. *Let G act on M as above and G -invariant functions distinguish generic orbits. Assume that H is a Casimir of \mathfrak{g} . Then the Hamiltonian system with the Hamiltonian H is completely integrable in the non-commutative sense. The algebra of integral is $\mathcal{F} + \tilde{\mathcal{F}}$. More generally, let $H \in C^\infty(\mathfrak{g}^*)$ be a function that defines a completely integrable Hamiltonian system on \mathfrak{g}^* (more precisely on those orbits which belong to the image of the momentum mapping). Then H (considered as a (lifted) function on T^*M) defines a completely integrable system on T^*M .*

Corollary 3. *Let $M = G/H$ be a homogeneous space of a compact Lie group H and g be the normal metric on M . Then the geodesic flow on (M, g) is always completely integrable in non-commutative sense. The same is true for any bi-quotient $K \backslash G/H$.*

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