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# LAMÉ EQUATION, QUANTUM TOP AND ELLIPTIC BERNOULLI POLYNOMIALS

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ABSTRACT. A generalisation of the odd Bernoulli polynomials related to the quantum Euler top is introduced and investigated. This is applied to compute the coefficients of the spectral polynomials for the classical Lamé operator.

## 1. INTRODUCTION

The classical Bernoulli polynomials can be defined through the generating function

$$\frac{ze^{zx}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k :$$

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}, \dots$$

(see e.g. [1, 3]). They appear naturally in the calculation of sums of powers of the natural numbers  $S_{k-1}(n) = 1^{k-1} + 2^{k-1} + \dots + n^{k-1}$  in a simple way:

$$S_{k-1}(n) = (B_k(n+1) - B_k)/k,$$

where  $B_k = B_k(0)$  are the *Bernoulli numbers*. All odd Bernoulli numbers except  $B_1 = -\frac{1}{2}$  are known to be zero, so the odd Bernoulli polynomials (up to a multiple  $k$ ) can be thought of as an "analytic continuation" of the sums of powers from natural argument  $n$  to real (or complex)  $x$ .

In this paper we introduce a new class of polynomials, which can be considered as an elliptic generalisation of the *odd* Bernoulli polynomials  $B_{2k+1}(x)$ . They are related to the quantum top and to the classical *Lamé operator*

$$L_s = -\frac{d^2}{dz^2} + s(s+1)\wp(z).$$

where  $\wp$  is the Weierstrass elliptic function [3], satisfying the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

It is well-known after Ince [7] that the Lamé operator (considered on the real line shifted by the imaginary half-period) for integer  $s$  has a remarkable property: its spectrum has exactly  $s$  gaps. The ends of the spectrum  $E_j$  are given by the zeroes of certain polynomials  $R_{2s+1}(E) = \prod_{j=0}^{2s} (E - E_j(s))$ , which we will call *Lamé spectral polynomials*. The computation of the spectral polynomials  $R_{2s+1}(E)$  for given  $s = 1, 2, 3, \dots$  goes back to Hermite and Halphen [20] and by now is pretty well investigated (see [2, 16, 17] for the recent results in this direction).

Here we consider a related but different problem: we would like to find the coefficients  $b_k$  of the spectral polynomials  $R_{2s+1}(E) = E^{2s+1} + b_1 E^{2s} + b_2 E^{2s-1} + \dots + b_{2s}$  as *functions of  $s$*  (and thus for *all* values of parameter  $s$ ). We will show

that in this relation naturally appear some new polynomials, generalising the odd Bernoulli polynomials.

The following remarkable relation between the Lamé equation and the quantum Euler top, going back to Kramers and Ittmann [9], will be crucial for us. Consider the quantum mechanical Hamiltonian of the Euler top (see e.g. [10])

$$\hat{H} = a_1 \hat{M}_1^2 + a_2 \hat{M}_2^2 + a_3 \hat{M}_3^2,$$

where the angular momentum operators  $\hat{M}_j$  satisfy the standard commutation relations  $[\hat{M}_1, \hat{M}_2] = i\hat{M}_3$ ,  $[\hat{M}_2, \hat{M}_3] = i\hat{M}_1$ ,  $[\hat{M}_3, \hat{M}_1] = i\hat{M}_2$  (we put  $\hbar = 1$  for simplicity).

The operator  $\hat{H}$  naturally acts in any representation of the Lie algebra  $so(3)$ . In particular, it acts in the representation space with spin  $s$  of dimension  $2s + 1$  as a finite-dimensional operator  $\hat{H}_s$ . The claim is that if the parameters  $a_i = e_i$  are the roots  $e_1, e_2, e_3$  of the equation  $4\wp^3 - g_2\wp - g_3 = 0$ , then the characteristic polynomial of the operator  $\hat{H}_s$  coincides with the spectral Lamé polynomial:

$$(1) \quad \det(\lambda I - \hat{H}_s) = R_{2s+1}(\lambda).$$

We discuss this in more detail in the next section.

The Weierstrass condition  $e_1 + e_2 + e_3 = 0$  is unnatural from this point of view (and moreover contradicts the "physical" condition of positivity of  $a_i$ ), so we consider the case when the parameters  $a_i$  are arbitrary and define the *elliptic Bernoulli polynomials*  $\mathcal{B}_{2k+1}$  as the coefficients in the expansion of the trace of the resolvent of  $\hat{H}_s$  at infinity:

$$(2) \quad \text{tr}(\lambda I - \hat{H}_s)^{-1} = \sum_{k=0}^{\infty} \frac{\mathcal{B}_{2k+1}(s)}{\lambda^{k+1}}$$

or, equivalently by the relation

$$\mathcal{B}_{2k+1}(s) = \text{tr} \hat{H}_s^k.$$

They depend also on the parameters  $a_1, a_2, a_3$  or more precisely on their symmetric functions of  $g_1, g_2, g_3$  defined by the relation

$$(3) \quad 4(z - a_1)(z - a_2)(z - a_3) = 4z^3 - g_1z^2 - g_2z - g_3,$$

so strictly speaking we should write  $\mathcal{B}_{2k+1}(s; g_1, g_2, g_3)$  rather than  $\mathcal{B}_{2k+1}(s)$ . We will use both notations depending on the context.

**Theorem 1.** *The trace of the  $k$ -th power of the quantum Euler top Hamiltonian in the representation with spin  $s$  is a polynomial in  $s$  of degree  $2k + 1$  anti-symmetric with respect to  $s = -\frac{1}{2}$ , whose coefficients are polynomials in  $g_1, g_2, g_3$  with rational coefficients. When  $g_2 = g_3 = 0$ , these polynomials reduce to the classical odd Bernoulli polynomials up to a constant factor:*

$$(4) \quad \mathcal{B}_{2k+1}(s; g_1, 0, 0) = \frac{g_1^k}{(2k + 1)2^{2k-1}} B_{2k+1}(s + 1).$$

In the Appendix one can find the explicit form of the first 8 elliptic Bernoulli polynomials.

When  $g_1 = 0$  we have the *reduced elliptic Bernoulli polynomials* denoted as  $\mathcal{B}_{2k+1}^W(s; g_2, g_3)$  (W is for Weierstrass):  $\mathcal{B}_1^W = 2s + 1$ ,  $\mathcal{B}_3^W = 0$ ,

$$\mathcal{B}_5^W = \frac{1}{60} g_2 s(s + 1)(2s - 1)(2s + 1)(2s + 3),$$

$$\mathcal{B}_7^W = \frac{1}{280} g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5),$$

$$\mathcal{B}_9^W = \frac{1}{1680} g_2^2 s(s+1)(2s-1)(2s+1)(2s+3)(4s^4 + 8s^3 - 11s^2 - 15s + 21).$$

The coefficients of  $\mathcal{B}_{2k+1}^W$  are homogeneous polynomials in  $g_2, g_3$  of weight  $2k$  if we assume as usual that the weights of  $g_2$  and  $g_3$  are 4 and 6 respectively (in other words, they are modular forms of weight  $2k$ , see e.g. [11]).

We discuss some effective ways to compute the elliptic Bernoulli polynomials, investigate their properties and then apply them to the calculation of the coefficients of the Lamé spectral polynomials. In particular we prove the following

**Theorem 2.** *The coefficient  $b_k = b_k(s)$  of the Lamé spectral polynomial  $R_{2s+1}(E) = \prod_{j=0}^{2s} (E - E_j(s)) = E^{2s+1} + b_1 E^{2s} + b_2 E^{2s-1} + \dots + b_{2s+1}$  is a polynomial in  $s, g_2, g_3$  with rational coefficients. It can be computed using the reduced elliptic Bernoulli polynomials by the following recurrent relations with  $b_0 = 1$ :*

$$b_k = -\frac{1}{k} \sum_{j=1}^k \mathcal{B}_{2j+1}^W(s) b_{k-j}.$$

The first coefficients are  $b_1 = 0$ ,  $b_2 = -\frac{g_2}{120} s(s+1)(2s-1)(2s+1)(2s+3)$ ,

$$b_3 = -\frac{g_3}{840} s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5),$$

$$b_4 = \frac{g_2^2}{201600} s(s-1)(s+1)(2s-1)(2s+1)(2s+3)(56s^4 + 76s^3 - 94s^2 + 201s + 630)$$

(see more of them below in section 5).

Note that once  $b_k(s)$  are known for  $k = 0, 1, \dots, 2s$  one can find the eigenvalues of the quantum Euler top in the representation with spin  $s$  (integer or half-integer) by solving the corresponding algebraic equation  $R_{2s+1}(E) = 0$ .

We conclude with the discussion of possible relations and further developments.

## 2. LAMÉ EQUATION AND QUANTUM EULER TOP

The observation that the Lamé equation is closely related to the quantum top probably belongs to Kramers and Ittmann [9] (see also [19]), who showed that the corresponding Schrödinger equation is separable in the elliptic coordinate system and that the resulting differential equations are of Lamé form.

More precisely, let us consider the Hamiltonian

$$\hat{H} = a_1 \hat{M}_1^2 + a_2 \hat{M}_2^2 + a_3 \hat{M}_3^2$$

acting in the space of functions on the unit sphere

$$(5) \quad q_1^2 + q_2^2 + q_3^2 = 1,$$

using the standard representation of the angular momenta as the first order differential operators

$$\hat{M}_1 = -i(q_2 \partial_3 - q_3 \partial_2)$$

$$\hat{M}_2 = -i(q_3 \partial_1 - q_1 \partial_3)$$

$$\hat{M}_3 = -i(q_1 \partial_2 - q_2 \partial_1).$$

Let us introduce the *elliptic* (or *sphero-conical*) coordinates  $u_1, u_2$  on this sphere as the roots of the quadratic equation

$$(6) \quad \frac{q_1^2}{a_1 - u} + \frac{q_2^2}{a_2 - u} + \frac{q_3^2}{a_3 - u} = 0,$$

where the parameters  $a_1, a_2, a_3$  are the same as in the top's Hamiltonian. One has then the following expressions for the cartesian coordinates in terms of  $u_1, u_2$ :

$$(7) \quad \begin{aligned} q_1^2 &= \frac{(a_1 - u_1)(a_1 - u_2)}{(a_1 - a_2)(a_1 - a_3)} \\ q_2^2 &= \frac{(a_2 - u_1)(a_2 - u_2)}{(a_2 - a_1)(a_2 - a_3)} \\ q_3^2 &= \frac{(a_3 - u_1)(a_3 - u_2)}{(a_3 - a_1)(a_3 - a_2)}. \end{aligned}$$

The system has an obvious quantum integral (Casimir)  $\hat{M}^2 = \sum \hat{M}_i^2$ , which is the square of the total angular momentum operator:

$$[\hat{M}^2, \hat{M}_i] = 0.$$

One can check that that in the elliptic coordinate system the operators  $\hat{H}$  and  $\hat{M}^2$  have the form

$$(8) \quad \hat{M}^2 = -\frac{4}{u_1 - u_2} \left[ \sqrt{-P(u_1)} \frac{\partial}{\partial u_1} (\sqrt{-P(u_1)} \frac{\partial}{\partial u_1}) + \sqrt{P(u_2)} \frac{\partial}{\partial u_2} (\sqrt{P(u_2)} \frac{\partial}{\partial u_2}) \right]$$

$$(9) \quad \hat{H} = -\frac{4}{u_1 - u_2} \left[ u_2 \sqrt{-P(u_1)} \frac{\partial}{\partial u_1} (\sqrt{-P(u_1)} \frac{\partial}{\partial u_1}) + u_1 \sqrt{P(u_2)} \frac{\partial}{\partial u_2} (\sqrt{P(u_2)} \frac{\partial}{\partial u_2}) \right]$$

where  $P(u) = (u - a_1)(u - a_2)(u - a_3)$ . Note that the operator  $\hat{M}^2$  corresponds to the standard Laplacian  $-\Delta$  on the unit sphere.

Since  $\hat{M}^2$  and  $\hat{H}$  commute, one can look for joint eigenfunctions. The spectral problem  $\hat{M}^2 \psi = \mu \psi$  is well-known in the theory of spherical harmonics (see e.g. [14]). It is known that the spectrum has the form  $\mu = s(s + 1)$  for non-negative integer values of  $s$ . The dimension of the corresponding eigenspace  $V_s$  is  $2s + 1$  and  $V_s$  is an irreducible representation of dimension  $2s + 1$  of the rotation group  $SO_3$  called *representation with spin  $s$* .

It turns out that the joint eigenvalue problem

$$\hat{M}^2 \phi = s(s + 1) \phi$$

$$\hat{H} \phi = E \phi$$

is separable in the elliptic coordinates  $u_1, u_2$  (see [9, 19]). Namely, if we substitute  $\phi(u_1, u_2) = \phi_1(u_1) \phi_2(u_2)$  into this system we find that each of the functions  $\phi_1(u_1), \phi_2(u_2)$  satisfies the same differential equation:

$$[4P(u)]^{\frac{1}{2}} \frac{d}{du} \left( [P(u)]^{\frac{1}{2}} \frac{d}{du} \right) - s(s + 1)u + E \psi = 0,$$

which can be rewritten as

$$(10) \quad \frac{d^2}{du^2} \psi + \frac{1}{2} \left[ \frac{1}{u - a_1} + \frac{1}{u - a_2} + \frac{1}{u - a_3} \right] \frac{d}{du} \psi = \frac{1}{4} \frac{s(s + 1)u - E}{(u - a_1)(u - a_2)(u - a_3)} \psi.$$

A remarkable fact is that this is an algebraic form of the following slightly generalised version of the Lamé differential equation

$$(11) \quad -\frac{d^2}{dz^2}\psi + s(s+1)\wp_*(z)\psi = E\psi$$

where  $\wp_*(z)$  is a solution of the differential equation

$$(12) \quad (\wp_*')^2 = 4P(\wp_*) = 4(\wp_* - a_1)(\wp_* - a_2)(\wp_* - a_3).$$

Indeed, after the change of variables  $u = \wp_*(z)$  the equation (11) coincides with (10) (see [20]). When the sum  $a_1 + a_2 + a_3 = 0$  the equation (12) determines the Weierstrass elliptic function  $\wp(z)$ , otherwise it differs from it by adding a constant.

It is well-known (see e.g. [3]) that for  $\phi$  to be a regular solution on the sphere the corresponding  $\psi$  must be doubly-periodic, which implies that  $s$  is integer and  $E$  must have one of the  $2s + 1$  characteristic values  $E_m(s)$ . For each  $E_m(s)$  there exists exactly one (up to a factor) doubly-periodic solution to the Lamé equation, which is the Lamé polynomial  $\mathcal{E}_s^m(u)$ . Therefore the basis of the eigenfunctions of the operator  $\hat{H}$  in the invariant subspace  $V_s$  consists of  $2s + 1$  solutions  $\phi(u_1, u_2)$  of the form  $\mathcal{E}_s^m(u_1)\mathcal{E}_s^m(u_2)$ . They are called sometime *ellipsoidal harmonics* (see [20]).

Thus, we came to the following result (cf. [9, 19]):

**Theorem 3.** *The characteristic polynomial of the quantum top Hamiltonian  $\hat{H}_s$  in the representation space with integer spin  $s$  coincides with the spectral polynomial  $R_{2s+1}(\lambda) = \prod_{j=0}^{2s} (\lambda - E_j(s))$  of the generalised Lamé operator (11).*

*Remark.* A simple relation between the quantum Euler top and the Lamé equation mentioned above is a bit misleading. Indeed there are several spectral problems related to the Lamé equation. We have considered only smooth real periodic version related to real  $x$  shifted by the imaginary half-period. If we would consider  $x$  just real, we would have a singular version (since  $\wp$  has poles on the real line), whose spectrum has nothing to do with the quantum top. In its turn, the quantum Euler top in the representation with half-integer spin  $s$  has eigenvalues which are just some special double eigenvalues of the periodic Lamé operator, which in this case has infinitely many gaps.

### 3. ELLIPTIC BERNOULLI POLYNOMIALS

We define now the *elliptic Bernoulli polynomials*  $\mathcal{B}_{2k+1}(s)$  as the traces of the powers of  $\hat{H}_s$ , where  $\hat{H}_s$  is as before the quantum top operator  $\hat{H}$  in the representation with spin  $s$ :

$$(13) \quad \mathcal{B}_{2k+1}(s; g_1, g_2, g_3) = \text{tr} \hat{H}_s^k, \quad k = 0, 1, 2, \dots$$

Here the parameters  $g_1 = 4(a_1 + a_2 + a_3)$ ,  $g_2 = -4(a_1a_2 + a_2a_3 + a_1a_3)$ ,  $g_3 = 4a_1a_2a_3$  are defined by the relation (3).

We are going to prove now our Theorem 1 claiming that the trace  $\text{tr} \hat{H}_s^k$  is a polynomial in all variables  $s, g_1, g_2, g_3$  with rational coefficients, which reduces to the standard odd Bernoulli polynomial  $B_{2k+1}$  when  $g_2 = g_3 = 0$ .

To prove this, consider the standard basis in  $V_s$  consisting of the eigenvectors  $|j\rangle$  of  $\hat{M}_3$ :  $\hat{M}_3|j\rangle = j|j\rangle$ ,  $j = -s, -s+1, \dots, s-1, s$ . In this basis, the Hamiltonian  $\hat{H}$  is a tri-diagonal symmetrical matrix  $H = H_s$  with the following elements (see

e.g. Landau-Lifshitz [10], page 417):

$$(14) \quad \begin{aligned} \langle j|H|j \rangle &= \frac{1}{2}(a_1 + a_2)[s(s+1) - j^2] + a_3j^2 \\ \langle j|H|j+2 \rangle &= \langle j+2|H|j \rangle = \frac{1}{4}(a_1 - a_2)\sqrt{(s-j)(s-j-1)(s+j+1)(s+j+2)}. \end{aligned}$$

Note that both expressions are symmetrical with respect to  $s = -\frac{1}{2}$ ; they are also homogeneous polynomials of degree 1 in  $a_1, a_2, a_3$ . Now, consider any diagonal element of  $H^k$ ; it has the form:

$$\langle j|H^k|j \rangle = \sum_{i_1, i_2, \dots, i_{k-1}} \langle j|H|i_1 \rangle \langle i_1|H|i_2 \rangle \dots \langle i_{k-1}|H|j \rangle$$

where the distance between 2 consecutive indices  $i_l, i_{l+1}$  is either 0 or  $\pm 2$ . Since the starting point and the ending point coincide, if the matrix element  $\langle i_l|H|i_l+2 \rangle$  appears along the path so does the element  $\langle i_l+2|H|i_l \rangle$ . This proves that the diagonal matrix elements of  $H^k$  are polynomials of degree  $2k$  in both  $s$  and  $j$ . From (14) they are symmetric with respect to  $s = -\frac{1}{2}$  and homogeneous symmetric polynomials of degree  $k$  in  $a_1, a_2, a_3$ . Now summing over  $j = -s, -s+1, \dots, s-1, s$  and taking into account that the sums of the odd powers of  $j$  are zero while the sums of even powers  $2l$  are the odd Bernoulli polynomials  $B_{2l+1}(s+1)$  (multiplied by  $\frac{2}{2l+1}$ ) we have the first statement of the theorem. The anti-symmetry of  $\mathcal{B}_{2k+1}(s)$  with respect to  $s = -\frac{1}{2}$  follows from the well-known property of the Bernoulli polynomials:  $B_m(1-s) = (-1)^m B_m(s)$ .

In the case when  $a_1 = a_2 = 0$ , we have  $g_2 = g_3 = 0$ ,  $g_1 = 4a_3$  and  $\hat{H} = a_3 \hat{M}_3^2$ . The spectrum of  $H_s$  is then very simple:  $\lambda_j = a_3 j^2$  for  $j = -s, -s+1, \dots, s-1, s$ . Since the sum  $\sum_{j=1}^s j^{2k} = \frac{1}{2k+1} B_{2k+1}(s+1)$ , we thus obtain:

$$\mathcal{B}_{2k+1}(s; g_1, 0, 0) = \frac{g_1^k}{(2k+1)2^{2k-1}} B_{2k+1}(s+1).$$

This completes the proof of Theorem 1.

Note that from the point of view of the elliptic curve  $\Gamma$  given by the equation

$$y^2 = 4x^3 - g_1x^2 - g_2x - g_3,$$

the last case corresponds to the limit when one of the periods goes to infinity ("trigonometric limit"). There are two more interesting special cases: *lemniscatic* when  $g_1 = g_3 = 0$  and *equianharmonic* when  $g_1 = g_2 = 0$ , corresponding to the elliptic curves with additional symmetries.

It is natural also to consider the Weierstrass reduction  $g_1 = 0$ ; we will call the corresponding polynomials  $\mathcal{B}_{2k+1}^W(s; g_2, g_3) = \mathcal{B}_{2k+1}(s; 0, g_2, g_3)$  the *reduced elliptic Bernoulli polynomials*.

**Theorem 4.** *The elliptic Bernoulli polynomial  $\mathcal{B}_{2k+1}$  has the following properties:*

- (1) *as a polynomial in  $g_1, g_2, g_3$   $\mathcal{B}_{2k+1}$  is homogeneous of weight  $2k$ , where the weights of  $g_1, g_2$  and  $g_3$  are assumed to be 2, 4 and 6 respectively*
- (2)  *$\mathcal{B}_{2k+1}$  for  $k \geq 1$  is divisible by  $s(s+1)(2s+1)$*
- (3) *in the reduced case  $\mathcal{B}_{2k+1}^W$  is divisible by  $s(s+1)(2s-1)(2s+1)(2s+3)$  for all  $k$  and by  $s(s+1)(2s-1)(2s+1)(2s+3)(2s-3)(2s+5)$  for odd  $k$*
- (4) *in the lemniscatic case  $\mathcal{B}_{2k+1}(s; 0, g_2, 0) = 0$  for odd integer  $k$*
- (5) *in the equianharmonic case  $\mathcal{B}_{2k+1}(s; 0, 0, g_3) = 0$  if  $k$  is not divisible by 3*

- (6) *in the isotropic case*  $a_1 = a_2 = a_3 = a$  i.e  $g_1 = 12a, g_2 = 12a^2, g_3 = 4a^3$   
 $\mathcal{B}_{2k+1}(s) = a^k(2s+1)s^k(s+1)^k$ .

The proof of the first two claims follows from the definition and anti-symmetry property. To prove the third one consider the representation with spin  $s = \frac{1}{2}$ . It is easy to check that  $\hat{H}$  acts as the 2 by 2 scalar matrix  $\frac{1}{4}(a_1 + a_2 + a_3)Id$ , which is zero in the reduced case. Therefore  $\mathcal{B}_{2k+1}^W(\frac{1}{2}) = 0$  for all  $k$ . By anti-symmetry with respect to  $-\frac{1}{2}$  we also have  $\mathcal{B}_{2k+1}^W(-\frac{3}{2}) = 0$ . For half integer  $s$ , we know from Kramers' theorem (see [10], paragraph 60) that the eigenvalues are no longer distinct but are double roots. For the particular case  $s = 3/2$ , these eigenvalues take the values  $\pm\sqrt{[3(a_1^2 + a_2^2 + a_3^2)/2]}$  (see [10], page 419) therefore for odd  $k$ ,  $\mathcal{B}_{2k+1}^W(3/2) = 0$  and again by anti-symmetry  $\mathcal{B}_{2k+1}^W(-5/2) = 0$ . The lemniscatic and equianharmonic cases follow from the first claim. In the isotropic case  $\hat{H}_s = as(s+1)Id$ , which implies the last statement.

In the general case the elliptic Bernoulli polynomials are not zero and their highest coefficients are described by the following

**Theorem 5.** *The highest coefficient  $A_0$  of the elliptic Bernoulli polynomial  $\mathcal{B}_{2k+1}(s) = A_0s^{2k+1} + A_1s^{2k} + \dots + A_{2s}$  can be written*

$$(15) \quad A_0 = 2 \int_0^s \text{Res} \quad \xi^{-1}[\gamma(s^2 - j^2)\xi + (\alpha s^2 + \beta j^2) + \gamma(s^2 - j^2)\xi^{-1}]^k dj,$$

where  $\alpha = \frac{1}{2}(a_1 + a_2)$ ,  $\beta = \frac{2a_3 - a_1 - a_2}{2}$ ,  $\gamma = \frac{1}{4}(a_1 - a_2)$ .

Indeed, for large  $s$  and  $j$  the leading behaviour of the matrix elements of  $\hat{H}$  is

$$\begin{aligned} \langle j|\hat{H}|j \rangle &= \frac{1}{2}(a_1 + a_2)[s^2 - j^2] + a_3j^2 = \alpha s^2 + \beta j^2, \\ \langle j|\hat{H}|j+2 \rangle &= \langle j+2|\hat{H}|j \rangle = \frac{1}{4}(a_1 - a_2)(s^2 - j^2) = \gamma(s^2 - j^2). \end{aligned}$$

Therefore the leading term of the diagonal element  $\langle j|\hat{H}^k|j \rangle$  coincide with the constant term of the Laurent polynomial  $[\gamma(s^2 - j^2)\xi + (\alpha s^2 + \beta j^2) + \gamma(s^2 - j^2)\xi^{-1}]^k$  in auxiliary variable  $\xi$ . Replacing the summation over  $j$  by the integration, which is fine in the leading order, we come to our formula.

Note that from this formula the fact that the final result is a symmetric function of  $a_1, a_2, a_3$  (and thus is a polynomial in  $g_1, g_2, g_3$ ) is not obvious at all.

#### 4. EFFECTIVE WAY TO COMPUTE THE ELLIPTIC BERNOULLI POLYNOMIALS

Although the definition of the elliptic Bernoulli polynomials themselves gives a way to compute them as traces of powers of the given matrices  $H_s$ , it does not seem to be as effective as the following procedure based on the fact that the matrix  $H_s$  is tri-diagonal.

Indeed, in the basis  $u_{s,m}$  the eigenvalue problem  $\hat{H}\psi = \lambda\psi$  in the space  $V_s$  leads to the following difference equation:

$$(16) \quad c_{n-2}\psi_{n-2} + v_n\psi_n + c_n\psi_{n+2} = \lambda\psi_n,$$

where

$$\begin{aligned} c_n &= \frac{a_1 - a_2}{4} \sqrt{(s-n)(s-n-1)(s+n+1)(s+n+2)}, \\ v_n &= \frac{1}{2}(a_1 + a_2)^2[s(s+1) - n^2] + a_3k^2. \end{aligned}$$



For such an equation one can use the standard procedure (see e.g. [4]) from the theory of solitons to find the local spectral densities, which are difference analogues of the famous KdV densities [13]. In our case it works as follows.

Let  $\chi_n = \frac{c_n \psi_{n+2}}{\psi_n}$ , then the equation (16) becomes

$$(17) \quad c_{n-2}^2 + (v_n - \lambda)\chi_{n-2} + \chi_n \chi_{n-2} = 0$$

We look for a solution in the form  $\chi_n = \lambda - \sum_{k=0}^{\infty} \chi_{n,k} \lambda^{-k}$ . Substitution of this expression into the equation (17) gives  $\chi_{n,0} = v_n$ ,  $\chi_{n,1} = c_{n-2}^2$ ,  $\chi_{n,2} = c_{n-2}^2 v_{n-2}$ , and for general  $k \geq 1$  the recurrence relation:

$$(18) \quad \chi_{n,k+1} = \sum_{i=1}^k \chi_{n,i} \chi_{n-2,k-i}.$$

Let  $X = \sum_{k=0}^{\infty} \chi_{n,k} \lambda^{-(k+1)}$  so that  $\chi_n = \lambda(1-X)$  and  $\log \chi_n = \log \lambda - \sum_{i=1}^{\infty} \frac{X^i}{i}$ . Thus we have

$$(19) \quad \log \chi_n - \log \lambda = - \sum_{i=1}^{\infty} \frac{\mathcal{I}_{n,i}}{\lambda^i},$$

where  $\mathcal{I}_{n,1} = v_n$ ,  $\mathcal{I}_{n,2} = c_{n-2}^2 + \frac{v_n^2}{2}$ ,  $\mathcal{I}_{n,3} = c_{n-2}^2 v_{n-2} + v_n c_{n-2}^2 + \frac{v_n^3}{3}$ , .....

On the other hand one can check that  $\prod_n \frac{\chi_n}{\lambda} = \prod_m (1 - \frac{E_m(s)}{\lambda})$  where  $E_m(s)$  are the eigenvalues of  $\hat{H}_s$ . Thus

$$\sum_n (\log \chi_n - \log \lambda) = - \sum_n \sum_{i=1}^{\infty} \frac{\lambda_n^i}{i \lambda^i} = \sum_{i=1}^{\infty} \frac{Tr \hat{H}_s^i}{i \lambda^i}.$$

Comparing this with the equations (19), we obtain

$$Tr \hat{H}_s^k = k \sum_n \mathcal{I}_{n,k} = k \sum_{n=-s}^s \mathcal{I}_{n,k}.$$

**Theorem 6.** *The elliptic Bernoulli polynomials  $\mathcal{B}_{2k+1}$  can be computed as*

$$(20) \quad \mathcal{B}_{2k+1} = k \sum_{n=-s}^s \mathcal{I}_{n,k},$$

where  $\mathcal{I}_{n,k}$  are the local densities determined by the relations (18, 19).

This gives a very effective way to compute the elliptic Bernoulli polynomials since the local densities are polynomials in  $c_n^2$  and  $v_n$  (and hence in  $n$ ) and thus the summation over  $n$  can be done with the use of the standard Bernoulli polynomials. We had applied this procedure to find the first 10 elliptic Bernoulli polynomials using Mathematica (see 8 of them in the Appendix).

## 5. APPLICATION: COEFFICIENTS OF THE LAMÉ SPECTRAL POLYNOMIALS

We will consider again the generalised version of the Lamé operator (11). The coefficients  $b_k = b_k(s)$  of the corresponding spectral polynomial

$$R_{2s+1}(E) = \prod_{i=0}^{2s} (E - E_i) = E^{2s+1} + b_1 E^{2s} + b_2 E^{2s-1} + \dots + b_k E^{2s-k+1} + \dots + b_{2s+1}$$

up to a sign are the elementary symmetric functions of the eigenvalues:  $b_k = (-1)^k e_k$ , where  $e_1 = \sum E_i$ ,  $e_2 = \sum_{i<j} E_i E_j$ ,  $e_3 = \sum_{i<j<k} E_i E_j E_k \dots$ . The elementary symmetric functions are related to power sums  $\mathcal{B}_{2k+1}(s) = \sum E_i^k$  by the following well-known relations:

$$k e_k = \sum_{j=1}^k (-1)^{j-1} \mathcal{B}_{2j+1} e_{k-j}$$

with  $e_0 = b_0 = 1$  (see e.g. [12]). This implies the following

**Theorem 7.** *The coefficients  $b_k$  of the Lamé spectral polynomial  $R_{2s+1}(E)$  are related to the elliptic Bernoulli polynomials  $\mathcal{B}_{2j+1}(s)$  by the recurrent relations*

$$b_k = -\frac{1}{k} \sum_{j=1}^k \mathcal{B}_{2j+1}(s) b_{k-j}.$$

*The coefficient  $b_k$  is a polynomial in  $s, g_1, g_2, g_3$  with rational coefficients. As a polynomial in  $s$  it has a degree  $3k$  and is divisible by  $(s+1)s(s-1)\dots(s - [\frac{k-2}{2}])$ .*

One can apply this result also to the case of half-integer spin  $s$ : in that case all the roots of the polynomial  $R_{2s+1}(E)$  are double and corresponds to the doubly-periodic solutions of the Lamé equation.

In the reduced case ( $g_1 = 0$ ) the degree of  $b_k$  drops to  $[\frac{5k}{2}]$  (for  $k > 1$ ). Using the explicit form of the elliptic Bernoulli polynomials given in the Appendix one can find the first seven coefficients  $b_k$ , which in reduced case are:  $b_1 = 0$ ,

$$b_2 = -\frac{g_2}{120} s(s+1)(2s-1)(2s+1)(2s+3)$$

$$b_3 = -\frac{g_3}{840} s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)$$

$$b_4 = \frac{g_2^2}{201600} s(s-1)(s+1)(2s-1)(2s+1)(2s+3)(56s^4 + 76s^3 - 94s^2 + 201s + 630)$$

$$b_5 = +\frac{g_2 g_3}{1108800} s(s-1)(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)(88s^4 + 68s^3 - 302s^2 + 663s + 1890)$$

$$b_6 = \frac{g_3^2}{201801600} (s-2)(s-1)s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \times$$

$$(4576s^5 + 12944s^4 - 20720s^3 + 48312s^2 + 597150s + 779625) -$$

$$\frac{g_2^3}{10378368000} (s-2)(s-1)s(s+1)(2s-5)(2s-3)(2s-1)(2s+1)(2s+3) \times$$

$$(16016s^6 + 89232s^5 + 197160s^4 + 544280s^3 + 2033829s^2 + 3858813s + 2619540)$$

$$b_7 = -\frac{g_2^2 g_3}{24216192000} (s-3)(s-2)(s-1)s(s+1)(2s-5)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \times$$

$$(32032s^6 + 189072s^5 + 463440s^4 + 1682920s^3 + 7301418s^2 + 15249213s + 11351340)$$

## 6. CONCLUDING REMARKS.

We have shown that for any given  $k$  the coefficient  $b_k(s)$  of the spectral Lamé polynomial  $R_{2s+1}$  can be computed effectively for all values of parameter  $s$ . In particular, for fixed  $s$  it gives an alternative way to compute the whole polynomial. It would be interesting to compare this approach with the classical one going back to Halphen and Hermite [20] further developed recently by Belokolos and Enolski [2] and Takemura [16, 17].

However we believe that the elliptic Bernoulli polynomials are of interest by themselves. In particular one can expect interesting relations with the arithmetic of the corresponding elliptic curves and the representation theory. In this relation we would like to mention the elliptic generalisation of the Bernoulli numbers - the so-called *Bernoulli-Hurwitz numbers*  $BH_{2k}$ , whose arithmetic was investigated in [8, 15].

Another interesting possible relation is with the zeta-function  $\zeta_H(z) = \text{tr} \hat{H}^{-z}$  of the quantum top and its special values. A lemniscatic case  $a_3 = \frac{a_1 + a_2}{2}$  could be particularly interesting from the arithmetic point of view.

Recall that the parameter  $s$  was originally integer or half-integer (spin). A natural question is the role of these values in the theory of elliptic Bernoulli polynomials. We conjecture that like in the case of the usual Bernoulli polynomials (see e.g. [18]) these values are the asymptotic positions of the real roots of the polynomials  $\mathcal{B}_{2k+1}$  for large  $k$ . More precisely, we conjecture that for real  $s$  in the bounded interval the ratio

$$\frac{\mathcal{B}_{2k+1}(s)}{\mathcal{B}'_{2k+1}(0)} \rightarrow \frac{\sin 2\pi s}{2\pi}$$

as  $k$  tends to infinity. Actually, we believe that this is true for each component of  $\mathcal{B}_{2k+1}$ , which is a coefficient at monomial  $g_1^p g_2^q g_3^r$ .

It is interesting to look at the graphs. In the Fig 1 we show the graphs of the coefficients of the polynomial  $\mathcal{B}_{15}(s)$  at: (a)  $g_1^7$ , (b)  $g_1^3 g_2^2$ , (c)  $g_1^2 g_2 g_3$ , and (d)  $g_2^2 g_3$ . Each polynomial has been normalised by dividing by its first derivative at zero and then multiplied by  $2\pi$ . The sinusoidal behavior for small  $s$  looks quite plausible.

We would like to mention that the even Bernoulli polynomials (or more precisely closely related Faulhaber polynomials) also have elliptic versions related to the Lamé operator. They were introduced in our recent paper [6] motivated by [5] as certain complete elliptic integrals of second kind and have quite different properties. The fact that the theory of the Lamé equation leads to two different classes of polynomials, both related to Bernoulli polynomials (one to odd, another to even) seems to be remarkable. To make the picture even more intriguing we would like to mention that the integrals in the definition of the elliptic Faulhaber polynomials are coming from the formal expansion of the trace of the resolvent of the Lamé operator (cf. our formula (2)).

Finally, one can consider our results from the general point of view of the quantisation of integrable systems. Usually one can find the spectrum in a closed form only if the classical system is integrable in elementary functions. The Euler top is probably the most natural classical problem integrable in elliptic functions. The question about the nature of its integrability in the quantum case seems to be not as easy as it may look. We hope that our paper adds something in this direction.

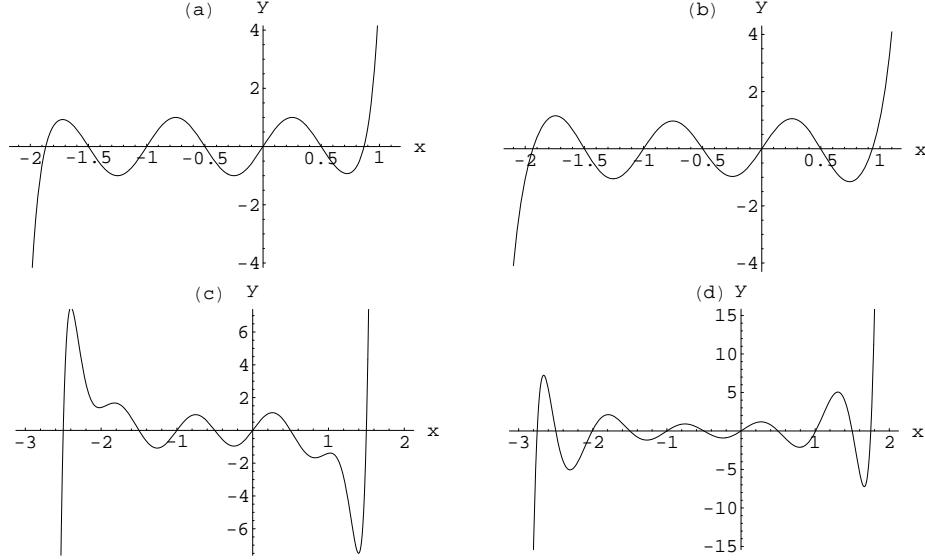


FIGURE 1

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## 8. APPENDIX - THE FIRST 8 ELLIPTIC BERNOULLI POLYNOMIALS

$$\mathcal{B}_1 = 2s + 1$$

$$\mathcal{B}_3 = \frac{1}{12} g_1 s(s+1)(2s+1)$$

$$\mathcal{B}_5 = + \frac{1}{240} g_1^2 s(s+1)(2s+1)(3s^2+3s-1) \\ + \frac{1}{60} g_2 s(s+1)(2s-1)(2s+1)(2s+3)$$

$$\mathcal{B}_7 = + \frac{1}{1344} g_1^3 s(s+1)(2s+1)(3s^4+6s^3-3s+1) \\ + \frac{1}{1120} g_1 g_2 s(s+1)(2s-1)(2s+1)(2s+3)(6s^2+6s-5) \\ + \frac{1}{280} g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)$$

$$\mathcal{B}_9 = + \frac{1}{11520} g_1^4 s(s+1)(1+2s)(5s^6+15s^5+5s^4-15s^3-s^2+9s-3) \\ + \frac{1}{3360} g_1^2 g_2 s(s+1)(2s-1)(2s+1)(2s+3)(5s^4+10s^3-5s^2-10s+7) \\ + \frac{1}{840} g_1 \tilde{g}_3 s^2(s+1)^2(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\ + \frac{1}{1680} g_2^2 s(s+1)(2s-1)(2s+1)(2s+3)(4s^4+8s^3-11s^2-15s+21)$$

$$\begin{aligned}
\mathcal{B}_{11} = & + \frac{1}{33792} g_1^5 s(s+1)(2s+1)(s^2+s-1)(3s^6+9s^5+2s^4-11s^3+3s^2+10s-5) \\
& + \frac{1}{50688} g_1^3 g_2 s(s+1)(2s-1)(2s+1)(2s+3)(20s^6+60s^5-10s^4-120s^3+44s^2+114s-75) \\
& + \frac{1}{29568} g_1^2 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)(10s^4+20s^3-4s^2-14s+21) \\
& + \frac{1}{29568} g_1 g_2^2 s(s+1)(2s-1)(2s+1)(2s+3)(40s^6+120s^5-86s^4-372s^3+242s^2+448s-315) \\
& + \frac{1}{7392} g_2 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)(8s^4+16s^3-34s^2-42s+63) \\
\\
& + \frac{1}{5591040} g_1^6 s(s+1)(2s+1)(105s^{10}+525s^9+525s^8-1050s^7-1190s^6 \\
& \quad +2310s^5+1420s^4-3285s^3-287s^2+2073s-691) \\
& + \frac{1}{5125120} g_1^4 g_2 s(s+1)(2s-1)(2s+1)(2s+3) \\
& \quad (525s^8+2100s^7+350s^6-6300s^5-70s^4+12810s^3-4105s^2-11910s+7601) \\
& + \frac{1}{2842840} g_1^3 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
& \quad (350s^6+1050s^5-100s^4-1950s^3+1433s^2+2583s-1650) \\
\mathcal{B}_{13} = & + \frac{1}{2562560} g_1^2 g_2^2 s(s+1)(2s-1)(2s+1)(2s+3)(1400s^8+5600s^7-1450s^6-23950s^5 \\
& \quad +5438s^4+57326s^3-24627s^2-58215s+41481) \\
& + \frac{1}{320320} g_1 g_2 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
& \quad (200s^6+600s^5-670s^4-2340s^3+1922s^2+3192s-2475) \\
& + \frac{1}{960960} g_2^3 s(s+1)(2s-1)(2s+1)(2s+3)(400s^8+1600s^7-1640s^6-10520s^5 \\
& \quad +8193s^4+35786s^3-28282s^2-48195s+43659) \\
& + \frac{1}{160160} g_3^2 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
& \quad (80s^6+240s^5-840s^4-2080s^3+4401s^2+5481s-7425) \\
\\
& + \frac{1}{737280} g_1^7 s(s+1)(2s+1)(3s^12+18s^11+24s^10-45s^9-81s^8+144s^7+182s^6 \\
& \quad -345s^5-217s^4+498s^3+44s^2-315s+105) \\
& + \frac{1}{1597440} g_1^5 \tilde{g}_2 s(s+1)(2s-1)(2s+1)(2s+3)(42s^{10}+210s^9+105s^8-840s^7 \\
& \quad -364s^6+2730s^5+205s^4-5540s^3+1650s^2+5078s-3185) \\
& + \frac{1}{13178880} g_1^4 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)(315s^8+1260s^7 \\
& \quad +140s^6-3990s^5+1265s^4+10650s^3-5152s^2-11352s+9009) \\
& + \frac{1}{13178880} g_1^3 g_2^2 s(s+1)(2s-1)(2s+1)(2s+3)(2520s^{10}+12600s^9+1750s^8-68600s^7 \\
& \quad -13130s^6+253630s^5-14558s^4-557066s^3+206601s^2+542619s-360360) \\
\mathcal{B}_{15} = & + \frac{1}{1098240} g_1^2 g_2 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)(280s^8+1120s^7-670s^6 \\
& \quad -5930s^5+3047s^4+17284s^3-11237s^2-21054s+18018) \\
& + \frac{1}{3294720} g_1 g_2^3 s(s+1)(2s-1)(2s+1)(2s+3)(1120s^{10}+5600s^9-2400s^8-43200s^7 \\
& \quad -8814s^6+201162s^5-60127s^4-517124s^3+256797s^2+557766s-405405) \\
& + \frac{1}{274560} g_1 g_3^2 s^2(s+1)^2(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
& \quad (80s^6+240s^5-840s^4-2080s^3+4401s^2+5481s-7425) \\
& + \frac{1}{274560} g_2^2 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)(80s^8+320s^7 \\
& \quad -600s^6-2920s^5+4037s^4+13314s^3-16959s^2-24156s+27027)
\end{aligned}$$

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